



On Quotient Subtraction Algebra

Zenebe Lakew Degefu
Department Mathematics
College of Natural and Computational Sciences
Addis Ababa University

July 22, 2016

Addis Ababa University
Department of Mathematics

The undersigned hereby certify that we have read and recommend to the school of graduate studies for acceptance of a project entitled **Quotient Subtraction Algebra** by Zenebe Lakew in partial fulfillment of the requirements for the degree of master of Science in Mathematics.

July, 2016

Advisor:

Tilahun Abebaw(Ph.D)

Examining committees:

Berhanu Bekele(Ph.D)

Yibeltal Yitayew(Ph.D)

Contents

Acknowledgement	i
Abstract	ii
1 Preliminaries	1
1.1 Meet Semilattice, Semilattice and Lattice	1
1.2 Boolean Algebra	2
1.3 Equivalence and Order Relations	3
2 Subtraction Algebras	5
2.1 Some Properties of Subtraction Algebras	8
3 Ideal of Subtraction Algebra	26
3.1 Definition and Some properties Ideal of Subtraction Algebra	26
4 Quotient Subtraction Algebra	32
4.1 Equivalence Relation	32
4.2 Quotient Subtraction Algebras	42
5 Isomorphism Theorems of Subtraction Algebras	45
5.1 Definitions and Some basic Properties of Subtraction Algebra Homomorphisms	45
5.2 Isomorphism Theorems on Subtraction Algebras	49
Conclusion	52
Bibliography	53

Acknowledgement

First of all, I would like to thank Almighty God for helping me to reach where I am today. I couldn't have made it without him and His Holy Mothers Help and Blessing. Then, I would like to extend my gratitude and deep appreciation to my advisor, Dr. Tilahun Abebaw for his superb, guidance, understanding, and most constructive advice, which built my confidence and helped me get through the study. Next my deeply heartfelt thanks and gratitude goes to my wife Woynshet Wondimu for her continuous support and encouragement. I would also like to thank all the staff members of the department and especially Dr Berhau Bekele for his constructive advice, which built my confidence. Finally, my special thanks goes to my family and friends who encouraged me and had been there for me when I needed help throughout my study.

Abstract

Certain techniques fundamental to the study of algebra. One of such techniques is the construction of a quotient set of an algebraic structure by a means of an equivalence relation on the given set. Based on this, we construct a quotient subtraction algebra, which plays a crucial role in the study of algebra. In this project work, we will introduce the notion of subtraction algebra, which is an algebraic structure on a set with a binary operation, usually called subtraction. To define subtraction algebra, we use the the concept of equivalence relation, subtraction algebra and ideal of subtraction algebra to define a quotient subtraction algebras and study its properties. The idea of isomorphism theorems of other algebraic structures is used to define isomorphism theorems of quotient subtraction algebras.

Chapter 1

Preliminaries

In this part of the project we will define some important concepts that are useful in our subsequent discussions. Let us start by defining a meet semi lattice, semi lattice and a lattice and then a Boolean algebra. In the last part of this chapter we will discuss about equivalence and order relations, which are important in defining our main object discussion, quotient subtraction algebra.

1.1 Meet Semilattice, Semilattice and Lattice

Let us start this section by defining what is meant by meet semilattice.

Definiton 1. *Let \wedge be a binary operation on a non-empty set X . Then the algebraic structure (X, \wedge) is called a meet semilattice for all $x, y, z \in X$, if it satisfies the following conditions:*

- i. $x \wedge x = x$, (Idempotent law)*
- ii. $x \wedge y = y \wedge x$, (Commutative law)*
- iii. $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, (Associative law)*

A semilattice is a non-empty set X together with two binary operations on which it is a meet semilattice with each of these two operations.

Definiton 2. *Let \vee and \wedge be two binary operations on a non-empty set X . Then the algebraic structure (X, \vee, \wedge) is called semilattice, if for all $x, y, z \in X$, and it satisfies the following properties:*

- i. $x \vee x = x$ and $x \wedge x = x$. (Idempotent law)*

ii. $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$. (Commutative law)

iii. $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$. (Associative law)

The following is the most common example for a semilattice.

Example 1. Given a set X , then the power set $P(X)$ together with the two binary operations: intersection " \cap " and union " \cup " of elements of $P(X)$ is semilattice. In this case we define \vee by \cup and \wedge by \cap .

If a semilattice satisfies one more property, called the Absorption law, then it is called a Lattice.

Definiton 3. Let \vee and \wedge be two binary operations on a non-empty set X . Then the algebraic structure (X, \vee, \wedge) is called Lattice, if (X, \vee, \wedge) is a semilattice and for all $x, y \in X$,

$$x \vee (x \wedge y) = x \text{ and } x \wedge (x \vee y) = x,$$

(which is called the Absorption law)

1.2 Boolean Algebra

In this section we will introduce the meaning of Boolean Algebra and give an example, as our object of discussion can be made to be a Boolean Algebra.

Definiton 4. Let " \vee " and " \wedge " be two binary operations on a non-empty set X . Then the algebraic structure $(X, \vee, \wedge, ', 0, 1)$ is called a Boolean algebra, if for arbitrary element $x, y, z \in X$. And the following conditions hold true.

1. It is Lattice on a set X with two binary operations. i.e (X, \vee, \wedge) .
2. Distributive laws: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
3. Involution law: $x'' = x$.
4. Compliment laws: $y' \vee y = 1$ and $y' \wedge y = 0$.
5. Identity laws: There is a zero element 0 in X such that $y \vee 0 = y$, and there is a unit element 1 in X such that $y \wedge 1 = y$.
6. Both $y \vee 1 = 1$ and $y \wedge 0 = 0$ are satisfied
7. De' Morgan's laws: $(x \vee y)' = x' \wedge y'$ and $(x \wedge y)' = x' \vee y'$.

The two operations " \wedge " and " \vee " are called meet and join, respectively.

Example 2. Let X be a non-empty set. Then the algebraic structure

$$(P(X), \vee, \wedge, ', 0, 1),$$

where $P(X)$ is the the power set of X , \vee and \wedge are the set operations \cup and \cap respectively, the unary operation $'$ is the set compliment and the elements 0 for \emptyset and 1 for X , is a Boolean algebra.

1.3 Equivalence and Order Relations

In defining quotient algebra we use the concept of equivalence relation, as we define operations on equivalence classes.

Definiton 5. Let X be a non-empty set.

A. A relation R on X is called an equivalence relation, if the following conditions are satisfied:

- i. Reflexive: That is for all $x \in X, xRx$.
- ii. Symmetry: If xRy , then yRx , for $x, y \in X$.
- iii. Transitive: If xRy and yRz , then xRz , for $x, y, z \in X$

B. An order relation " \leq " on a non-empty set X is called partial ordering, if the following conditions are true:

- a. Reflexive: If $x \leq x$, for all $x \in X$
- b. Antisymmetry: If $x \leq y$ and $y \leq x$, then $x = y$ for all $x, y \in X$.
- c. Transitive: If $x \leq y$ and $y \leq z$, then $x \leq z$, for all $x, y, z \in X$.

A set X together with a partial order " \leq " on X is denoted by $\langle X, \leq \rangle$ and we say it is a Partially Ordered Set (Poset).

Example 3. Let " \leq " be a an order relation on \mathbb{N} , the set of natural numbers, defined by

$$a \leq b$$

if and only if a divides b . Then " \leq " is a partial ordering on \mathbb{N} and (\mathbb{N}, \leq) is a poset.

C. A partially order set is called a totally order set if for any two elements $x, y \in X$, either $x \leq y$ or $y \leq x$. A set X with a total ordering is called totally ordered set or a chain.

The most common examples of a totally ordered set are the different sets of numbers with the usual ordering, as we can see in the following example.

Example 4. *The set of natural numbers, the set of integers, the set of rational numbers and real numbers with the usual ordering " \leq " are all chains.*

Remark 1. *Given a poset, $\langle X, \leq \rangle$,*

- (a) if there is some element $m \in X$ so that $m \leq x$, for all $x \in X$, then m is unique and such element m is called the smallest or the least element of X .*
- (b) if there exists an element $b \in X$, so that $x \leq b$, for all $x \in X$, then b is unique and is called the greatest element of X .*

Chapter 2

Subtraction Algebras

The most important objects of discussions in Mathematics are algebraic structures. We are interested on one algebraic structure for our discussion in this project, which is subtraction algebra. A subtraction algebra is a set X together with the single binary operation, which is called subtraction that is denoted by " $-$ ". In this chapter we will define a subtraction algebra, study some properties of subtraction algebra and discuss some concepts related with subtraction algebra.

Definiton 6. *A non-empty set X together with a binary operation " $-$ " is said to be a subtraction algebra, if it satisfies the following properties:*

$$S_1. x - (y - x) = x, \text{ for all } x, y \in X$$

$$S_2. x - (x - y) = y - (y - x), \text{ for all } x, y \in X$$

$$S_3. (x - y) - z = (x - z) - y, \text{ for all } x, y, z \in X$$

Let us see some examples of algebraic structures that are subtraction algebras and then algebraic structures that is not subtraction algebras.

Example 5. *Let $X = \{0, a, b\}$. Define " $-$ " on X by the following table.*

-	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Then $(X, -)$ is a subtraction algebra .

Proof. Let us check all the properties of a subtraction algebra for each element in X .

- (i) Let $x = 0, y = a$. Then $x - (y - x) = 0 - (a - 0) = 0 - a = 0 \in X$.
Hence $0 - (a - 0) = 0$.
- (ii) Let $x = 0, y = b$. Then $x - (y - x) = 0 - (b - 0) = 0 - b = 0 \in X$.
Hence $0 - (b - 0) = 0$.
- (iii) Let $x = a, y = 0$. Then $x - (y - x) = a - (0 - a) = a - 0 = a \in X$.
Hence $a - (0 - a) = a$.
- (iv) Let $x = a, y = b$. Then $x - (y - x) = a - (b - a) = a - b = a \in X$.
Hence $a - (b - a) = a$.
- (v) Let $x = b, y = 0$. Then $x - (y - x) = b - (0 - b) = b - 0 = b \in X$.
Hence $b - (0 - b) = b$.
- (vi) Let $x = b, y = a$. Then $x - (y - x) = b - (a - b) = b - a = b \in X$.
Hence $b - (a - b) = b$.
- (vii) Let $x = b, y = b$. Then $x - (y - x) = b - (b - b) = b - 0 = b \in X$.
Hence $b - (b - b) = b$. Thus property S_1 holds true.
Similarly property S_2 and S_3 hold true.

Therefore, $(X, -)$ is subtraction algebra. □

Example 6. Let X be a set and $P(X)$ be the power set of X . Then $(P(X), -)$, where " $-$ " is the set theoretic subtraction, is a subtraction algebra.

Proof. Let $x, y, z \in P(X)$. Then $x - y := x \cap y'$.

$$\begin{aligned}
 1. \quad x - (y - x) &= x \cap (y - x)' \\
 &= x \cap (y \cap x')' \\
 &= x \cap (y' \cup x''), \text{ by De'Morgan's law} \\
 &= x \cap (y' \cup x), \text{ by Involution's law } x'' = x \\
 &= (x \cap y') \cup (x \cap x), \text{ by Distributive law} \\
 &= x \cup x, \text{ since } x \cap y' \subseteq x \\
 &= x
 \end{aligned}$$

Thus $x - (y - x) = x$

$$\begin{aligned} 2. \quad x - (x - y) &= x \cap (x - y)' \\ &= x \cap (x \cap y')' \\ &= x \cap (x' \cup y''), \text{ by De'Morgan's law} \\ &= x \cap (x' \cup y), \text{ by Involution's law } y'' = y \\ &= (x \cap x') \cup (x \cap y), \text{ by Distributive law} \\ &= \emptyset \cup x \cap y \\ &= x \cap y = y \cap x, \text{ by commutative property} \\ &= (y \cap y') \cup (y \cap x) \\ &= y \cap (y' \cup x) \\ &= y \cap (y \cap x')' \\ &= y - (y \cap x') \\ &= y - (y - x) \end{aligned}$$

Hence $x - (x - y) = y - (y - x)$

$$\begin{aligned} 3. \quad (x - y) - z &= (x - y) \cap z' \\ &= (x \cap y') \cap z' \\ &= x \cap (y' \cap z'), \text{ by Associative property} \\ &= x \cap (z' \cap y'), \text{ by Commutative property} \\ &= (x \cap z') \cap y', \text{ by Associative property} \\ &= (x \cap z') - y \\ &= (x - z) - y \end{aligned}$$

Thus $(x - y) - z = (x - z) - y$.

Therefore, $(P(X), -)$ is subtraction algebra □

Example 7. *The set of integers together with the usual subtraction, $(\mathbb{Z}, -)$, is not a subtraction algebra.*

Proof. Take $x = 1$ and $y = 2$ from \mathbb{Z} . Then

$$x - (y - x) = 1 - (2 - 1) = 1 - 1 = 0.$$

This implies S_1 in the definition of a subtraction algebra, which states that $x - (y - x) = x$, is not satisfied. Hence, as a result $(\mathbb{Z}, -)$ is not subtraction algebra. □

2.1 Some Properties of Subtraction Algebras

Now let us see some properties of subtraction algebra.

Theorem 1. *If $(X, -)$ is a subtraction algebra, then the following conditions hold true.*

i. $(x - y) - y = x - y.$

ii. $x - x = y - y,$ for all $x, y \in X.$

Proof. i. For $x, y \in X:$

$$\begin{aligned}(x - y) - y &= (x - y) - [y - (x - y)], \text{ since } y = [y - (x - y)] \\ &= (x - y), \text{ by } S_1\end{aligned}$$

Hence $(x - y) - y = x - y.$

ii. For $x, y \in X.$

$$\begin{aligned}x - x &= [x - (y - x)] - [x - (y - x)], \text{ since } x - (y - x) = x \\ &= [x - [x - (y - x)]] - (y - x), \text{ by } S_3 \\ &= [(y - x) - [(y - x) - x]] - (x - y), \text{ by } S_2 \\ &= [(y - x) - (y - x)] - (y - x), \text{ by theorem 1(i)} \\ &= [(y - (y - x)) - x] - (y - x), \text{ by } S_3 \\ &= [(x - (x - y)) - x] - (y - x), \text{ by } S_2 \\ &= [(x - (x - y)) - (y - x)] - x, \text{ by } S_3 \\ &= [(x - (y - x)) - (x - y)] - x, \text{ by } S_3 \\ &= [x - (x - y)] - x, \text{ since } x - (y - x) = x \\ x - x &= [x - (x - y)] - x \\ x - x &= [y - (y - x)] - x.\end{aligned}$$

Similarly, $y - y = [y - (y - x)] - y$. Then

$$\begin{aligned}
x - x &= [y - (y - x)] - x \\
&= [(y - (x - y)) - [(y - (x - y)) - x]] - x, \text{ by } S_1 \\
&= [(y - (x - y)) - [(y - x) - (x - y)]] - x, \text{ by } S_3 \\
&= [[(y - (x - y)) - x] - [(y - x) - (x - y)], \text{ by } S_3 \\
&= [(y - x) - (x - y)] - [(y - x) - (x - y)], \text{ by } S_3 \\
&= [(y - x) - [(y - x) - (x - y)]] - (x - y), \text{ by } S_3 \\
&= [(x - y) - [(x - y) - (y - x)]] - (x - y), \text{ by } S_2 \\
&= [(x - y) - [(x - (y - x)) - y]] - (x - y), \text{ by } S_3 \\
&= [(x - y) - (x - y)] - (x - y), \text{ since } x - (y - x) = x \\
&= [(x - (x - y)) - y] - (x - y), \text{ by } S_3 \\
&= [(y - (y - x)) - (x - y)] - y, \text{ by } S_2 \\
&= [(y - (x - y)) - (y - x)] - y, \text{ by } S_3 \\
&= [y - (y - x)] - y, \text{ by } S_1 \\
&= y - y
\end{aligned}$$

Therefore

$$x - x = y - y.$$

□

From the above theorem, we can see that there is a unique element in X that is denoted by $x - x = y - y := 0$, which does not depend on the choice of $x, y \in X$.

The following Lemma is very important through out our work, so we give attention for each part listed in the given Lemma.

Lemma 1. *Let $(X, -)$ be a subtraction algebra. Then the following conditions are true. For all $x, y, z \in X$.*

$$b_1. x - 0 = x$$

$$b_2. 0 - x = 0$$

$$b_3. (x - y) - x = 0$$

$$b_4. (x - (x - (x - y))) = x - y$$

$$b_5. (x - y) - z = (x - z) - (y - z)$$

Proof. Using the concept of subtraction algebra, we prove Lemma(1) in the following way.

b₁. For $x \in X$, we have:

$$\begin{aligned} x - 0 &= x - (x - x) \text{ since } x - x = 0 \\ &= x \text{ (by } S_1). \end{aligned}$$

Hence $x - 0 = x$.

b₂ For $x \in X$, we have:

$$\begin{aligned} 0 - x &= 0 - (x - 0), \text{ (by } b_1 \text{ since } x - 0 = x) \\ &= 0 \text{ (by } S_1). \end{aligned}$$

Hence $0 - x = 0$.

b₃. For $x, y \in X$, we have:

$$\begin{aligned} (x - y) - x &= (x - x) - y, \text{ by } S_3 \\ &= 0 - y, \text{ since } x - x = 0 \\ &= 0, \text{ (by } b_2). \end{aligned}$$

Hence $(x - y) - x = 0$.

b₄. For $x, y \in X$, we have:

$$\begin{aligned} b_4. (x - (x - (x - y))) &= [(x - y) - [(x - y) - x]], \text{ by } S_2 \\ &= (x - y) - 0, \text{ since } (x - y) - x = 0. \\ &= x - y. \end{aligned}$$

Hence

$$(x - (x - (x - y))) = x - y.$$

b₅. For $x, y \in X$, we have: To prove this we need to show that the following axioms.

- i. $[(x - y) - z] - [(x - z) - (y - z)] = 0$
- ii. $[(x - z) - (y - z)] - [(x - y) - z] = 0.$

Then using the concept of subtraction algebra and Lemma1 above we proof b_5 in the following way:

$$\begin{aligned}
i. \quad [(x - y) - z] - [(x - z) - (y - z)] &= [(x - z) - y] - [(x - z) - (y - z)], \text{ by } S_3 \\
&= [(x - z) - [(x - z) - (y - z)] - y], \text{ by } S_3 \\
&= [(y - z) - [(y - z) - (x - z)] - y], \text{ by } S_2 \\
&= [(y - z) - y] - [(y - z) - (y - z)], \text{ by } S_3 \\
&= [(y - y) - z] - [(y - z) - (y - z)], \text{ by } S_3 \\
&= [0 - y] - [(y - z) - (y - z)], \text{ by Lemma1 } (b_2) \\
&= 0 - [(y - z) - (y - z)] \\
&= 0, \text{ by } b_2
\end{aligned}$$

which implies

$$\begin{aligned}
[(x - y) - z] - [(x - z) - (y - z)] &= 0 \\
[(x - y) - z] &\leq [(x - z) - (y - z)] \tag{2.1}
\end{aligned}$$

Similarly using the same properties given in (i).

$$\begin{aligned}
ii. \quad [(x - z) - (y - z)] - [(x - y) - z] &= [(x - z) - (y - z)] - [(x - z) - y], \text{ by } S_3 \\
&= [(x - z) - [(x - z) - y] - (y - z)], \text{ by } S_3 \\
&= [(x - z) - (x - z)] - (y - z), \\
&\quad \text{since } (x - z) - y \leq (x - z) \\
&= 0 - (y - z) \\
&= 0, \text{ by } b_2
\end{aligned}$$

which implies

$$\begin{aligned}
[(x - z) - (y - z)] - [(x - y) - z] &= 0 \\
[(x - z) - (y - z)] &\leq [(x - y) - z] \tag{2.2}
\end{aligned}$$

Hence from equation (2.1) and (2.2) we have,

$$(x - y) - z = (x - z) - (y - z).$$

□

In the following theorem we will see that subtraction algebra determines an order relation on the given set X .

Theorem 2. Let $(X, -)$ be a subtraction algebra. Then " $-$ " determine an order relation on X defined by:

$$x \leq y \Leftrightarrow x - y = 0 \text{ for } x, y \in X.$$

Then " \leq " is a partial ordering on set X .

Proof. Let $(X, -)$ be a subtraction algebra, for all $x, y \in X$, which is defined by

$$x \leq y \Leftrightarrow x - y = 0.$$

Then we want to show that " \leq " is a partial ordering on set X .

- i. Reflexive: For all $x \in X$, we have $x - x = 0$ if and only if $x \leq x$. Hence " \leq " is Reflexive.
- ii. Antisymmetry: For all $x, y \in X$, such that $x \leq y \Leftrightarrow x - y = 0$ and $y \leq x \Leftrightarrow y - x = 0$. Then we want to show that $x = y$.

$$\begin{aligned} x &= x - (y - x), \text{ by } S_1 \\ x &= x - (x - y), \text{ since } x - y = 0 = y - x. \\ x &= y - (y - x), \text{ by } S_2 \\ x &= y - (x - y), \text{ since } x - y = y - x = 0. \\ x &= y - 0 \\ x &= y, \text{ by Lemmal } (b_1). \end{aligned}$$

Hence $x = y$.

Thus " \leq " is Antisymmetric.

- iii. Transitive: Suppose $x \leq y$ and $y \leq z$, for $x, y, z \in X$. Then we want to show that $x \leq z \Leftrightarrow x - z = 0$. Now:

$$\begin{aligned} x - y &= [x - (y - x)] - y, \text{ by } S_1 \\ &= [(x - y)] - (y - x), \text{ by } S_3 \\ &= [(y - z)] - (y - x), \text{ since } x - y = 0 \text{ and } y - z = 0. \\ &= [y - (y - x)] - z, \text{ by } S_3 \\ &= [x - (x - y)] - z, \text{ by } S_2 \\ &= [(x - z)] - (x - y), \text{ by } S_3 \end{aligned}$$

which implies

$$(x - z) - (x - y) = 0$$

and

$$(x - z) - 0 = 0, \text{ since } x - y = 0$$

Thus

$$x - z = 0$$

Hence $x \leq z$. Which implies " \leq " is Transitive

Therefore, " \leq " is a partial ordering on set X. \square

In the following theorem, we shall see that a partial ordering set X is a meet semi lattice and a boolean algebra.

Theorem 3. Let $(X, -)$ be a subtraction algebra and " $-$ " determine an order relation " \leq " on X is defined by

$$x \leq y \Leftrightarrow x - y = 0, \text{ for } x, y \in X.$$

Then $\langle X, \leq \rangle$ is a partial ordering on set X. Let $x \in X$, and define on

$$[0, x] = \{a \in X : 0 \leq a \leq x\}$$

by:

- a. $x \wedge y = x - (x - y)$;
- b. The complement of an element $y, \in [0, x]$, is denoted by $y' = x - y$;
- c. Let $y, z \in [0, x]$, then

$$\begin{aligned} y \vee z &= (y' \wedge z')', \\ &= ((x - y) \wedge (x - z))' \text{ by theorem 3(ii)} \\ &= x - ((x - y) \wedge (x - z)), \text{ by theorem 3(i)} \\ &= x - [(x - y) - [(x - y) - (x - z)]]. \end{aligned}$$

Then

- i. $([0, x], \wedge)$ is a meet semi lattice.
- ii. $([0, x], \vee, \wedge, ', 0, x)$ is a Boolean algebra.

Proof. Then applying the concept of subtraction algebra and Lemma 1 above, we give the proof as follows.

- i. Let $y, z \in X$, then we want to show that $([0, x], \wedge)$ is a meet semilattice. To show this, we will check the following three conditions:

1. Idempotent Law: we want to show that $y \wedge y = y$.

$$\begin{aligned} \Rightarrow y \wedge y &= y - (y - y), \text{ by theorem 3(i)} \\ &= y - 0, \text{ since } y - y = 0 \\ &= y, \text{ by Lemma1 (} b_1 \text{)}. \end{aligned}$$

Hence

$$y \wedge y = y.$$

2. Commutative law: we want to show that $y \wedge z = z \wedge y$.

$$\begin{aligned} \Rightarrow y \wedge z &= y - (y - z), \text{ by theorem 3(i)} \\ &= z - (z - y), \text{ by } S_2. \end{aligned}$$

Hence

$$y \wedge z = z \wedge y.$$

3. Associative Law: we want to show that $(x \wedge y) \wedge z = x \wedge (y \wedge z)$

$$\begin{aligned} (x \wedge y) \wedge z &= (x \wedge y) - [(x \wedge y) - z], \text{ by theorem 3(i)} \\ &= [x - (x - y)] - [(x - (x - y)) - z], \text{ by theorem 3(i)} \\ &= [y - (y - x)] - [(y - (y - x)) - z], \text{ by } S_2 \\ &= [y - (y - x)] - [(y - z) - (y - x)], \text{ by } S_3 \\ &= [y - (y - z)] - (y - x), \text{ by Lemma1 (} b_5 \text{)} \\ &= [y - (y - x)] - (y - z), \text{ by } S_3 \\ &= [y - (y - z)] - [(y - x) - (y - z)], \text{ by Lemma1 (} b_5 \text{)} \\ &= [y - (y - z)] - [(y - (y - z)) - x], \text{ by } S_3 \\ &= [y \wedge z] - [(y \wedge z) - x], \text{ by theorem 3(i)} \\ &= (y \wedge z) \wedge x = x \wedge (y \wedge z), \text{ by theorem 3(i)} \\ &= x \wedge (y \wedge z), \text{ by commutative property} \end{aligned}$$

Hence $(x \wedge y) \wedge z = x \wedge (y \wedge z)$

Therefore, $([0, x], \wedge)$ is a meet semi lattice.

ii. Let $y, z \in [0, x]$ in which 0 and x are the identities elements of " \vee " and " \wedge " respectively. Then we want to show that $([0, x], \vee, \wedge, ', 0, x)$ is a Boolean algebra.

1. Idempotent law: we want to show that:

A. $x \vee x = x$

$$B. x \wedge x = x$$

Then by applying definition of subtraction algebra and Lemma 1 above, we can prove in the following way:

$$\begin{aligned}
A. x \vee x &= (x' \wedge x')', \text{ by theorem 3(iii)} \\
&= ((x - x) \wedge (x - x))', \text{ by theorem 3(ii)} \\
&= x - (0 \wedge 0), \text{ since } x - x = 0 \\
&= x - (0 - (0 - 0)), \text{ by theorem 3(i)} \\
&= x - (0 - 0), \text{ since } 0 - 0 = 0 \\
&= x - 0, \text{ since } 0 - 0 = 0 \\
&= x \text{ by Lemma 1 } (b_1)
\end{aligned}$$

$$x \vee x = x.$$

$$\begin{aligned}
B. x \wedge x &= x - (x - x), \text{ by theorem 3(i)} \\
&= x - 0, \text{ since } x - x = 0.
\end{aligned}$$

$$\text{Thus } x \wedge x = x$$

Hence $x \vee x = x$ and $x \wedge x = x$

2. Commutative law: We want to show that:

$$A. x \vee y = y \vee x$$

$$B. x \wedge y = y \wedge x.$$

$$\begin{aligned}
A. x \vee y &= (x' \wedge y')', \text{ by theorem 3(iii)} \\
&= (y' \wedge x')', \text{ by commutative property} \\
&= y \vee x, \text{ by theorem 3(iii)}
\end{aligned}$$

Hence $x \vee y = y \vee x$

B. $x \wedge y = y \wedge x$ is proved in Meet semi-Lattice ₂

3. Associative Law: we want to show that:

$$A. x \vee (y \vee z) = (x \vee y) \vee z$$

$$B. (x \wedge y) \wedge z = x \wedge (y \wedge z).$$

$$\begin{aligned}
A. \quad x \vee (y \vee z) &= [x' \wedge (y \vee z)']', \text{ by theorem 3(i)} \\
&= [(y \vee z)' - ((y \vee z)' - x')]', \text{ by theorem 3(i)} \\
&= [(y \vee z)' \wedge x']', \text{ by theorem 3(i)} \\
&= (y \vee z) \vee x, \text{ by theorem 3(iii)} \\
&= (y \vee x) \vee z, \text{ by } S_3 \\
&= (x \vee y) \vee z, \text{ since } x \vee y = y \vee x \text{ by } 2A
\end{aligned}$$

Therefore, $x \vee (y \vee z) = (x \vee y) \vee z$.

B. $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ is proved in Meet semi-Lattice $\mathfrak{3}$
Therefore, $([0, \mathbf{x}], \vee, \wedge, ', 0, \mathbf{x})$ holds Associative property.

4. Absorption Law: We want to show that:

$$\begin{aligned}
A. \quad x \vee (x \wedge y) &= \mathbf{x} \\
B. \quad x \wedge (x \vee y) &= \mathbf{x}
\end{aligned}$$

$$\begin{aligned}
A. \quad x \vee (x \wedge y) &= x \vee (x - (x - y)), \text{ by theorem 3(i)} \\
&= [x' \wedge (x - (x - y))]', \text{ by theorem 3(iii)} \\
&= [(x - x) \wedge (x - (x - y))]', \text{ by theorem 3(ii)} \\
&= [0 \wedge (x - (x - y))]', \text{ since } x - x = 0 \\
&= x - [0 - [0 - [(x - (x - y))]]], \text{ by theorem 3(i)} \\
&= x - [0 - 0], \text{ by Lemma 1 } (b_2) \\
&= x - 0, \text{ since } 0 - 0 = 0 \\
&= x, \text{ by } b_1
\end{aligned}$$

which implies

$$x \vee (x \wedge y) = x \tag{2.3}$$

$$\begin{aligned}
B. \quad x \wedge (x \vee y) &= x - (x - (x \vee y)), \text{ by theorem 3(i)} \\
&= (x \vee y), \text{ since } x - (x - (x \vee y)) = (x \vee y) \\
&= (x' \wedge y')', \text{ by theorem 3(iii)} \\
&= x - ((x - x) \wedge (x - y)), \text{ by theorem 3(ii)} \\
&= x - (0 \wedge (x - y)), \text{ since } x - x = 0 \\
&= x - (0 - (0 - (x - y))), \text{ by theorem 3(i)} \\
&= x - (0 - 0), \text{ by Lemma 1 } (b_2) \\
&= x - 0, \text{ since } 0 - 0 = 0 \\
&= x, \text{ by Lemma 1 } (b_1)
\end{aligned}$$

which implies

$$x \wedge (x \vee y) = x \quad (2.4)$$

Therefore, as we see from equation (2.3) and (2.4) ($[0, x], \vee, \wedge, ', 0, x$) holds Absorption property.

5. Distributive Law: We want to show that:

- A. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
- B. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

$$\begin{aligned}
A. \quad x \wedge (y \vee z) &= x \wedge (y' \wedge z')', \text{ by theorem 3(iii)} \\
&= x \wedge [(x - y) \wedge (x - z)]', \text{ by theorem 3(ii)} \\
&= x \wedge [x - [(x - y) \wedge (x - z)]], \text{ by theorem 3(i)} \\
&= x - [x - [x - (x - y)] - [(x - y) - (x - z)]], \text{ by theorem 3(i)} \\
&= x - [(x - y) - [(x - y) - (x - z)]], \text{ by Lemma 1 } (b_4) \\
&= x - [(x - y) \wedge (x - z)], \text{ by theorem 3(i)} \\
&= x - [y' \wedge z'], \text{ by theorem 3(ii)} \\
&= [y' \wedge z']', \text{ by theorem 3(ii)} \\
&= y \vee z, \text{ by theorem 3(iii)}.
\end{aligned}$$

which implies

$$x \wedge (\vee z) = y \vee z \quad (2.5)$$

$$\begin{aligned}
(x \wedge y) \vee (x \wedge z) &= [x - (x - y)] \vee [x - (x - z)], \text{ by theorem 3(i)} \\
&= [[x - (x - y)]' \wedge [x - (x - z)]']', \text{ by theorem 3(iii)} \\
&= [[x - [x - (x - y)]] \wedge [x - [x - (x - z)]]]', \text{ by theorem 3(ii)} \\
&= x - [(x - y) \wedge (x - z)], \text{ by Lemma 1 } (b_4) \\
&= x - [y' \wedge z'], \text{ by theorem 3(ii)} \\
&= [y' \wedge z']', \text{ by theorem 3(ii)} \\
&= y \vee z, \text{ by definition}.
\end{aligned}$$

which implies

$$(x \wedge y) \vee (x \wedge z) = y \vee z \quad (2.6)$$

Thus from equation (2.5) and (2.6), we get

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

$$\begin{aligned}
B. \quad x \vee (y \wedge z) &= x \vee [y - (y - z)], \text{ by theorem 3(i)} \\
&= [x' \wedge [y - (y - z)]]', \text{ by theorem 3(iii)} \\
&= [(x - x) \wedge [x - [y - (y - z)]]]', \text{ by theorem 3(ii)} \\
&= [0 \wedge [x - [y - (y - z)]]]', \text{ since } x - x = 0 \\
&= [0 - [0 - [x - [y - (y - z)]]]]', \text{ by theorem 3(i)} \\
&= [0 - 0]', \text{ by Lemma1 } b_2 \\
&= [0]' = x - 0, \text{ by theorem 3(ii)} \\
&= x, \text{ by Lemma1 } (b_1).
\end{aligned}$$

which implies

$$x \vee (y \wedge z) = x \quad (2.7)$$

$$\begin{aligned}
(x \vee y) \wedge (x \vee z) &= (x' \wedge y')' \wedge (x' \wedge z')', \text{ by theorem 3(iii)} \\
&= [(x - x) \wedge (x - y)]' \wedge [(x - x) \wedge (x - z)]', \text{ by theorem 3(ii)} \\
&= [0 \wedge (x - y)]' \wedge [0 \wedge (x - z)]', \text{ since } x - x = 0 \\
&= x - [0 \wedge (x - y)] \wedge x - [0 \wedge (x - z)], \text{ by theorem 3(ii)} \\
&= x - [0 - [0 - (x - y)]] \wedge x - [0 - [0 - (x - z)]], \text{ by theorem 3(i)} \\
&= x - [0 - 0] \wedge x - [0 - 0], \text{ by Lemma1 } (b_2) \\
&= x - 0 \wedge x - 0, \text{ since } 0 - 0 = 0 \\
&= x \wedge x, \text{ by Lemma1 } (b_1) \\
&= x - (x - x), \text{ by theorem 3(i)} \\
&= x - 0, \text{ since } x - x = 0 \\
&= x, \text{ by } b_1
\end{aligned}$$

which implies

$$(x \vee y) \wedge (x \vee z) = x \quad (2.8)$$

From equation (2.7) and (2.8), we have

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

Hence $([0, x], \vee, \wedge, ', 0, x)$ holds Distributive property.

6. Involution Law: we want to show that: $x'' = x$.

$$\begin{aligned}
x'' &= (x')', \text{ by theorem 3(ii)} \\
&= (x - x)', \text{ by theorem 3(ii)} \\
&= x - (x - x), \text{ by theorem 3(i)} \\
&= x - 0, \text{ since } x - x = 0 \\
&= x, \text{ by Lemma 1 } (b_1)
\end{aligned}$$

Hence $x'' = x$.

Therefore, $([0, x], \vee, \wedge, ', 0, x)$ holds Involution property.

7. Complement Law: We want to show that:

A. $y' \vee y = x$

B. $y' \wedge y = 0$.

$$\begin{aligned}
 \text{A. } y' \vee y &= (y'' \wedge y')', \text{ by theorem 3(iii)} \\
 &= (y \wedge y')', \text{ since by (6) } y'' = y \\
 &= (y \wedge (x - y))', \text{ by theorem 3(ii)} \\
 &= x - [(y - (y - (x - y)))], \text{ by theorem 3(i)} \\
 &= x - (y - y), \text{ since } (y - (x - y)) = y \\
 &= x - 0, \text{ since } y - y = 0 \\
 &= x, \text{ by Lemma 1 (} b_1 \text{)}.
 \end{aligned}$$

Hence $y' \vee y = x$

$$\begin{aligned}
 \text{B. } y' \wedge y &= (x - y) \wedge y, \text{ by theorem 3(ii)} \\
 &= (x - y) - [(x - y) - y], \text{ by theorem 3(i)} \\
 &= (x - y) - (x - y), \text{ since } (x - y) - y = (x - y) \\
 &= 0.
 \end{aligned}$$

Hence $y' \wedge y = 0$.

Therefore, $([0, x], \vee, \wedge, ', 0, x)$ holds Compliment property.

8. Identities Law: We want to show that:

A. There is a unit element 0 in X such that $y \vee 0 = y$.

B. There is a unit element x in X such that $y \wedge x = y$.

$$\begin{aligned}
 \text{A. } y \vee 0 &= [y' \wedge 0']', \text{ by theorem 3(iii)} \\
 &= x - [(x - y) \wedge (x - 0)], \text{ by theorem 3(ii)} \\
 &= x - [(x - y) \wedge x], \text{ by Lemma 1 (} b_1 \text{)} \\
 &= x - [(x - y) - [(x - y) - x]], \text{ by theorem 3(i)} \\
 &= x - [(x - y) - 0], \text{ since } [(x - y) - x] = 0. \\
 &= x - (x - y), \text{ by } b_1 \\
 &= x - (y'), \text{ by theorem 3(ii)} \\
 &= y'' = y, \text{ since by (6) } y'' = y.
 \end{aligned}$$

Hence $y \vee 0 = y$.

$$\begin{aligned} B. \quad y \wedge x &= y - (y - x), \text{ by theorem 3(i)} \\ &= x - (x - y), \text{ by } S_2 \\ &= x - (y'), \text{ by theorem 3(i)} \\ &= y'' = y, \text{ since by (6) } y'' = y. \end{aligned}$$

Hence $y \wedge x = y$.

Therefore, $([0, x], \vee, \wedge, ', 0, x)$ holds Identities property

9. We want to show that:

1. $y \vee x = x$
2. $y \wedge 0 = 0$

$$\begin{aligned} 1. \quad y \vee x &= (y' \wedge x')', \text{ by theorem 3(iii)} \\ &= [(x - y) \wedge (x - x)]', \text{ by theorem 3(ii)} \\ &= x - [(x - y) \wedge 0], \text{ since } x - x = 0 \\ &= x - [(x - y) - [(x - y) - 0]], \text{ by theorem 3(i)} \\ &= x - [(x - y) - (x - y)], \text{ by } b_2 \\ &= x - 0, \text{ since } (x - y) - (x - y) = 0 \\ &= x, \text{ by } b_1 \end{aligned}$$

Hence $y \vee x = x$.

$$\begin{aligned} 2. \quad y \wedge 0 &= y - (y - 0), \text{ by theorem 3(i)} \\ &= y - y, \text{ since } y - 0 = y \\ &= 0, \text{ since } y - y = 0. \end{aligned}$$

Hence $y \wedge 0 = 0$.

10. De'Morgan's Law: We want to show that:

- i. $(x \vee y)' = x' \wedge y'$
- ii . $(x \wedge y)' = x' \vee y'$.

$$\begin{aligned}
i. (x \vee y)' &= x - (x \vee y), \text{ by theorem 3(ii)} \\
&= x - (x' \wedge y)', \text{ by theorem 3(iii)} \\
&= x - [(x - x) \wedge (x - y)], \text{ by theorem 3(ii)} \\
&= x - [x - [0 \wedge (x - y)]], \text{ since } x - x = 0 \\
&= x - [x - [0 - (0 - (x - y))]], \text{ by theorem 3(i)} \\
&= x - [x - [0 - 0]], \text{ by Lemma 1 } (b_2) \\
&= x - [x - 0], \text{ since } 0 - 0 = 0 \\
&= x - x, \text{ by Lemma 1 } (b_1) \\
&= 0, \text{ since } x - x = 0
\end{aligned}$$

which implies

$$(x \vee y)' = 0 \quad (2.9)$$

$$\begin{aligned}
x' \wedge y' &= [(x - x) \wedge (x - y)], \text{ by theorem 3(i)} \\
&= [0 \wedge (x - y)], \text{ since } x - x = 0 \\
&= [0 - (0 - (x - y))], \text{ by theorem 3(i)} \\
&= 0 - 0, \text{ by Lemma 1 } (b_2) \\
&= 0
\end{aligned}$$

which implies

$$x' \wedge y' = 0 \quad (2.10)$$

Now from equation (9) and (10), we get $(x \vee y)' = x' \wedge y'$

$$\begin{aligned}
ii. (x \wedge y)' &= x - [x - (x - y)], \text{ by theorem 3(i)} \\
&= (x - y), \text{ by lemma 1 } (b_4) \\
&= y'
\end{aligned}$$

which implies

$$(x \wedge y)' = y' \quad (2.11)$$

$$\begin{aligned}
x' \vee x' &= (x - x) \vee (x - y), \text{ by theorem 3(ii)} \\
&= (0 \vee (x - y)), \text{ since } x - x = 0 \\
&= [0' \wedge (x - y)]', \text{ by theorem 3(iii)} \\
&= x - [(x - 0) \wedge [x - (x - y)]], \text{ by theorem 3(ii)} \\
&= x - [x \wedge [x - (x - y)]], \text{ by Lemma 1 } (b_1) \\
&= x - [x - [x - [x - (x - y)]]], \text{ by theorem 3(i)} \\
&= x - [x - (x - y)], \text{ by Lemma 1 } (b_4) \\
&= x - [x - (x - y)], \text{ by Lemma 1 } (b_4) \\
&= (x - y) = y'
\end{aligned}$$

which implies

$$x' \vee y' = y' \quad (2.12)$$

From equation (11) and (12) above we have

$$(x \wedge y)' = x' \vee y'$$

Hence $([0, x], \vee, \wedge, ', 0, x)$ holds De'Morgan's property.

Therefore, $([0, x], \vee, \wedge, ', 0, x)$ is Boolean Algebra.

□

Lemma 2. *Let $(X, -)$ be a subtraction algebra. Then following conditions are true, for all $x, y, z \in X$.*

a. $(x - (x - y)) \leq y$

b. $(x - y) - (y - x) = (x - y)$

c. $(x - z) - (z - y) \leq (x - z)$

d. $x \leq y \Leftrightarrow x = y - w$, for some $w \in X$.

e. $x \leq y \Rightarrow x - z \leq y - z$ and $z - y \leq z - x$, for all $z \in X$

f. $x, y \leq z \Rightarrow x - y = x \wedge (z - y)$

h. $(x \wedge y) - (x \wedge z) \leq x \wedge (z - y)$.

Proof. a. We want to show that: $(x - (x - y)) - y = 0$

$$\begin{aligned} (x - (x - y)) - y &= (y - (y - x)) - y, \text{ by } S_2 \\ &= (y - y) - (y - x), \text{ by } S_3 \\ &= 0 - (y - x), \text{ since } y - y = 0 \\ &= 0, \text{ by Lemma 1 (b}_2\text{)} \end{aligned}$$

$$\text{implies } (x - (x - y)) - y = 0$$

Hence

$$(x - (x - y)) \leq y.$$

$$\begin{aligned} \text{b. } (x - y) - (y - x) &= (x - (y - x)) - y, \text{ by } S_3 \\ &= x - y, \text{ since } (x - (y - x)) = x \end{aligned}$$

Hence

$$(x - y) - (y - x) = (x - y).$$

- c. If $x - z \leq x$ and $z - y \leq z$, then $(x - z) - (z - y) \leq (x - z)$.
- d. (\Rightarrow) If $x \leq y$, then $x - y = 0$. Let $w = y - x$, for some $w \in X$. We want to show that $x = y - w$.

$$\begin{aligned}
 x &= x - 0, \text{ by Lemma 1 } (b_1) \\
 &= x - (x - y), \text{ since } x - y = 0 \\
 &= y - (y - x), \text{ by } S_2 \\
 &= y - w, \text{ since } w = y - x
 \end{aligned}$$

Hence

$x = y - w$, for some $w \in X$.

(\Leftarrow) Suppose $x = y - w$, for some $w \in X$. We want to show that: $x \leq y$.

$$\begin{aligned}
 x &= y - w \\
 x - y &= (y - w) - y, \text{ by subtracting 'y' from both sides .} \\
 x - y &= (y - y) - w, \text{ by } S_3 \\
 x - y &= 0 - w = 0, \text{ since } y - y = 0 \\
 x - y &= 0, \text{ by } b_2.
 \end{aligned}$$

Hence $x \leq y$.

e. Using 'd' we prove 'e' in the following way.

- i. $x \leq y$ if and only if $x = y - w$, for some $w \in X$, then

$$\begin{aligned}
 x &= y - w \\
 x - z &= (y - w) - z, \text{ by subtracting 'z' from both sides} \\
 x - z &= (y - z) - w \text{ by } S_3 \\
 x - z &\leq (y - z), \text{ since } (y - z) - w \leq (y - z).
 \end{aligned}$$

which implies $x - z \leq (y - z)$.

- ii. If $x \leq y$ then $x - y = 0$ which implies

$$\begin{aligned}
 (z - y) - (z - x) &= [z - (z - x)] - y, \text{ by } S_3 \\
 &= [x - (x - z)] - y, \text{ by } S_2 \\
 &= (x - y) - (x - z), \text{ by } S_3 \\
 &= 0 - (x - z), \text{ since } x - y = 0 \\
 &= 0, \text{ by Lemma 1 } (b_2)
 \end{aligned}$$

, which implies

$$(z - y) - (z - x) = 0 \text{ it gives } (z - y) \leq (z - x).$$

Hence from i and ii we get,

$$x \leq y \text{ implies } x - z \leq y - z \text{ and } z - y \leq z - x, \text{ for all } z \in X.$$

f. $x, y \leq z \Rightarrow x - y = x \wedge (z - y),$

If $x \leq z$, then $x - y \leq z - y$, using 'e' above. But $x - y \leq x$ which implies

$$x - y \leq x \wedge (z - y) \tag{2.13}$$

$$\begin{aligned} \text{Let } w &= x \wedge (z - y), \text{ for some } w \in X. \\ &= x - (x - (z - y)), \text{ by theorem 3(i) .} \\ &= (z - y) - [(z - y) - x], \text{ by } S_2. \\ &\leq x, \text{ by Lemma 2(a)} \end{aligned}$$

Then $w \leq x$, so $w = x \wedge w = x - (x - w)$ and also,

$$\begin{aligned} \text{Let } w &= y \wedge (z - y), \text{ for some } w \in X. \\ &= y - (y - (z - y)), \text{ by theorem 3(i) .} \\ &= (z - y) - [(z - y) - y], \text{ by } S_2. \\ &= [(z - y) - (z - y)], \text{ since } (z - y) - y = (z - y) \\ &= 0, \text{ since } (z - y) - (z - y) = 0. \end{aligned}$$

implies $w = y \wedge (z - y) = y \wedge w = 0$

$$\begin{aligned} \text{implies } y \wedge w &= y \wedge x \wedge [(z - y)], \text{ since } w = y \wedge (z - y). \\ &= (x \wedge y) \wedge [(z - y)], \text{ by commutativity.} \\ &= x \wedge [y - [y - (z - y)]], \text{ by theorem 3(i)} \\ &= x \wedge [(z - y) - [(z - y) - y]], \text{ by } S_2 \\ &= x \wedge [(z - y) - (z - y)], \text{ by theorem 1(i)} \\ &= x \wedge 0, \text{ since } (z - y) - (z - y) = 0 \\ &= x - (x - 0), \text{ by theorem 3(i) .} \\ &= x - x, \text{ by Lemma 1 (} b_1 \text{).} \\ &= 0, \text{ since } x - x = 0 \end{aligned}$$

$$\text{implies } y \wedge w = y - (y - w) = 0$$

$$\text{which implies } w - (w - y) = 0$$

Hence $w - (w - y) = w \wedge y = x \wedge (z - y) \wedge y = 0$, since $w = x \wedge (z - y)$ which implies $w - (w - y) = 0$.

$$\begin{aligned}
\text{Thus } w - (x - y) &= (w - 0) - (x - y), \text{ since } w \leq x. \\
&= [w - [w - (w - y)]] - (x - y), \text{ since } [w - (w - y)] = 0 \\
&= (w - y) - (x - y), \text{ since } [w - [w - (w - y)]] = (w - y) \\
&= [[x - (x - w)] - y] - (x - y), \text{ since } w = x \wedge w = x - (x - w). \\
&= [(x - y) - (x - w)] - (x - y), \text{ by } S_3. \\
&= [(x - y) - (x - y)] - (x - w), \text{ by } S_3. \\
&= 0 - (x - w), \text{ since } (x - y) - (x - y) = 0 \\
&= 0, \text{ by } b_2.
\end{aligned}$$

which implies $w - (x - y) = 0$, and it gives $x \wedge (z - y) - (x - y) = 0$ which implies

$$x \wedge (z - y) \leq (x - y) \quad (2.14)$$

Therefore, from equation (13) and (14) we get,
 $x, y \leq z$ implies $x - y = x \wedge (z - y)$.

h. $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$ To prove this we need to apply the concept of *theorem(3)* and *Lemma(2)* above.

$$\text{i. } (x \wedge y) - (x \wedge z) = [x - (x - y)] - [x - (x - z)]$$

$$= [y - (y - x)] - [z - (z - x)], \text{ by } S_2.$$

$$\leq (y - z), \text{ since } [y - (y - x)] \leq y \text{ and } [z - (z - x)] \leq z$$

which implies $(x \wedge y) - (x \wedge z) \leq (y - z)$.

$$\text{ii. } (x \wedge y) - (x \wedge z) \leq x \wedge y$$

$$= x - (x - y) = y - (y - x), \text{ by } S_2$$

$$\leq x.$$

Thus from i and ii we get, $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.

□

Chapter 3

Ideal of Subtraction Algebra

Just as normal subgroups played a crucial role in the theory of groups, so ideals play an analogous role in the study of rings. It also plays an essential part to examine the definition and elementary properties of subtraction algebra. Ideals are defined on a subtraction algebras X and have their own character which makes them different from subtraction algebra. We also put some topics which have a related concept with ideal of subtraction algebra. This helps the reader to understand the topics with their detail proof and explanation. In addition to this some examples are provided to give more hint to the reader. After this review we pass to the definition and discus of properties of ideal of subtraction algebra.

3.1 Definition and Some properties Ideal of Subtraction Algebra

In this section we will see the definition of ideals of subtraction algebra and give an example of it.

Definiton 7. *A nonempty subset A of a subtraction algebra X is called an ideal of X , if it satisfies the following conditions:*

1. $0 \in A$
2. For all $x \in X$ and for all $y \in A$, $x - y \in A \Rightarrow x \in A$.

Example 8. *Let $X = \{0, a, b, c, d\}$ in which " $-$ " defined by the table below:*

-	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	0
b	b	b	0	b	0
c	c	c	c	0	c
d	d	d	d	d	0

Then $(X, -)$ is a subtraction algebra. Therefore, $A = \{0, c\}$ is an ideal of a subtraction algebra of X .

Proof. Using definition of ideal subtraction algebra, for all $x \in X$ and for all $y \in A$, $x - y \in A \Rightarrow x \in A$.

1. Let $x = 0, y = 0$. Then $x - y = 0 - 0 = 0 \in A \Rightarrow x \in A$.
2. Let $x = c, y = 0$. Then $x - y = c - 0 = c \in A \Rightarrow x \in A$.
3. Let $x = 0, y = c$. Then $x - y = 0 - c = 0 \in A \Rightarrow x \in A$.
4. Let $x = c, y = c$. Then $x - y = c - c = 0 \in A \Rightarrow x \in A$.

From these, we conclude that $A = \{0, c\}$ is an ideal of a subtraction algebra X . □

Lemma 3. *An ideal of a subtraction algebra X has the following property. For all $x \in X$ and for all $y \in A$, $x \leq y$ implies $x \in A$.*

Proof. i. By definition of ideal $0 \in A$.

- ii. Let $x \in X$ and $y \in A$ such that $x \leq y$ if and only if $x - y = 0 \in A$, since A is an ideal of X . This implies $x - y \in A$. Thus $x \in A$. □

Remark 2. 1. *If A is an ideal of X , then it is not empty, so there exist at least one element $a \in A$ such that $a - a = 0$ is also in A .*

2. *If A and B are two ideals of a subtraction algebra X , then $A \cap B$ is also an ideal of X .*

Proof. i. Since A and B are ideals of X ,

$0 \in A$ and $0 \in B$ implies $0 \in A \cap B$ implies $A \cap B \neq \emptyset$.

- ii. For all $x \in X$ and $y \in A \cap B$, $x - y \in A \cap B$

$\Rightarrow x - y \in A$ and $x - y \in B$

$\Rightarrow x \in A$, since A is an ideal of X and $x \in B$, since B is an ideal of X

$\Rightarrow x \in A \cap B$

Hence $A \cap B$ is an ideal of X . □

3. If A and B are two ideals of a subtraction algebra X , then $A \cup B$ is may not be an ideal of X .
4. If A and B are both an ideal of a subtraction algebra of X , then $A \cap B = A \wedge B$.

Proof. (\Rightarrow) If $x \in A \wedge B$, then $x = a \wedge b$, for some $a \in A$ and $b \in B$. Then

$$\begin{aligned} a \wedge b &= a - (a - b), \text{ by theorem 3(i)} \\ &= b - (b - a), \text{ by } S_2 \\ &\leq a, \text{ by Lemma2(a) .} \end{aligned}$$

$\Rightarrow a \wedge b \leq a$ and $a \in A$.

$\Rightarrow a \wedge b \in A$. Similarly: $a \wedge b = a - (a - b) \leq b$, by Lemma2(a).

$\Rightarrow a \wedge b \leq b$ and $b \in B$.

$\Rightarrow a \wedge b \in A \in B$. Thus $a \wedge b \in A \cap B$.

(\Leftarrow) If $x \in A \cap B$, then $x = x \wedge x \in A \wedge B$. i.e $x = x - (x - x)$

$\Rightarrow x = x \in A \wedge B$.

Therefore $A \cap B = A \wedge B$. □

5. Let $(X, -)$ be a subtraction algebra and let $x \in X$ and $y \in A$, where A is an ideal of X , then the following axioms are true.

a. $x = y - (y - x)$

b. $x \in A$.

Proof. Since for all $x, y \in X$, if $x \leq y$ if and only if $x - y = 0$, then

$$\begin{aligned} y - (y - x) &= x - (x - y), \text{ by } S_2 \\ &= x - 0, \text{ since } x - y = 0 \\ &= x. \end{aligned}$$

Hence $y - (y - x) = x$. But $y \in X$, then $x \in A$. □

Theorem 4. Let A be a non-empty subset of a subtraction algebra X . Then the set:

$K := \{x \in X : (\dots((x - a_1) - a_2) - \dots) - a_n = 0, \text{ for some } a_1, a_2, \dots, a_n \in A\}$,
for $n \in N$

is a minimal ideal of X containing A . Where N is denoted the set of natural number.

Proof. i. Since $0 \in X$, then $\{(\dots((0 - a_1) - a_2) - \dots) - a_n = 0 \in K$, for some $a_1, a_2, \dots, a_n \in A\}$, which implies $K \neq \emptyset$.

ii. Let $x \in X$ be such that $y \in K$ and $x - y \in K$. Then
 $(\dots((y - a_1) - a_2) - \dots) - a_n = 0$, for some $a_1, a_2, \dots, a_n \in A$, and
 $(\dots(((x - y) - b_1) - b_2) - \dots) - b_m = 0$ for some $b_1, b_2, \dots, b_m \in A$ and for $n, m \in \mathbb{N}$*

Now by applying S_3 in to (*), we get

$$\begin{aligned} & (\dots(((x - y) - b_1) - b_2) - \dots) - b_m = 0 \\ & ((\dots((x - b_1) - b_2) - \dots) - b_m) - y = 0, \text{ by } S_3 \\ & ((\dots((x - b_1) - b_2) - \dots) - b_m) \leq y, \text{ by poset property} \\ & (\dots((((\dots((x - b_1) - b_2) - \dots) - b_m) - a_1) - a_2) - \dots) - a_n \leq \\ & \quad (\dots((y - a_1) - a_2) - \dots) - a_n, \text{ by Lemma2(e)} \\ & (\dots((((\dots((x - b_1) - b_2) - \dots) - b_m) - a_1) - a_2) - \dots) - a_n = 0, \text{ by it self} \end{aligned}$$

It follows that $x \in K$, so that K is an ideal of X . Let G be an ideal of X such that $A \subseteq G$. We need to show that $K \subseteq G$

Let $x \in K$, by definition ideal there exist $a_1, a_2, \dots, a_n \in A$ such that

$$\begin{aligned} & (\dots((x - a_1) - a_2) - \dots) - a_n = 0 \in G \\ & (\dots((x - a_1) - a_2) - \dots) - a_{n-1} = 0 \in G \\ & (\dots((x - a_1) - a_2) - \dots) - a_{n-2} = 0 \in G \\ & \quad \dots \\ & \quad \dots \\ & \dots = 0 \in G \end{aligned}$$

Which implies $x \in G$. Hence $K \subseteq G$, is a minimal ideal of X containing A .

□

From the above theorem we have seen that the ideal K is generated by A and is denoted by $\langle A \rangle$.

Theorem 5. Let X be a subtraction algebra. For any $a, b \in X$ and $n \in \mathbb{N}$, the set

$$[a; b^n] := \{x \in X : (x - a) - b^n = 0, \text{ for } n \in \mathbb{N}\} \text{ is an ideal of } X.$$

Proof. i. Since $0 \in X$, then we have $(0 - a) - b^n = 0 \in [a; b^n]$, which implies $[a; b^n] \neq \emptyset$ implies $0 \in [a; b^n]$, thus $[a; b^n] \neq \emptyset$.

ii. Let $x \in X$ be such that $y \in [a; b^n]$ and $x - y \in [a; b^n]$. Then applying S_3 and b_2 , we have

$$\begin{aligned}
((x - y) - a) - b^n &= 0 \\
((x - a) - (y - a)) - b^n &= 0, \text{ by } b_5 \\
(((x - a) - (y - a)) - b) - b^{n-1} &= 0, \text{ since } b^n = b - b^{n-1} \\
(((x - a) - b) - ((y - a) - b)) - b^{n-1} &= 0, \text{ by } b_5 \\
((((x - a) - b) - ((y - a) - b)) - b) - b^{n-2} &= 0, \\
&\dots\dots\dots \\
((x - a) - b^n) - ((y - a) - b^n) &= 0 \\
((x - a) - b^n) - 0 &= 0, \text{ since } ((y - a) - b^n) = 0 \\
((x - a) - b^n) &= 0, \text{ by Lemma 1 (} b_2 \text{)}
\end{aligned}$$

$\Rightarrow x \in [a; b^n]$.

Therefore $[a; b^n]$ is an ideal of X .

□

In the next theorem, using the idea of the set $[a; b^n]$, we establish a condition for a subset of a subtraction algebra X to be an ideal of X .

Theorem 6. *Let A be a nonempty subset of a subtraction algebra X . Then A is an ideal of X if and only if $[a; b^n] \subseteq A$ for every $a, b \in A$ and $n \in N$.*

Proof. (\Rightarrow) Assume that A is an ideal of X and let $a, b \in A$ and $n \in N$. If $x \in [a; b^n]$, then $(x - a) - b^n = 0$. Since $a, b \in A$, it follows that $x \in A$, by ideal definition (2). Hence $[a; b^n] \subseteq A$

(\Leftarrow) Suppose that $[a; b^n] \subseteq A$, for every $a, b \in A$ and $n \in N$. Obviously $0 \in [a; b^n] \subseteq A$. Let $x, y \in X$ be such that $y \in A$ and $x - y \in A$. Then

$$\begin{aligned}
(x - (x - y)) - y^n &= ((x - (x - y)) - y) - y^{n-1} \\
&= ((x - y) - (x - y)) - y^{n-1}, \text{ by } S_3 \\
&= 0 - y^{n-1}, \text{ since } (x - y) - (x - y) = 0 \\
&= 0, \text{ by } b_2
\end{aligned}$$

Implies $(x - (x - y)) - y^n = 0$ and thus $x \in [x - y; y^n] \subseteq A$. Hence A is an ideal of X

□

Definiton 8. *A nonempty subset A of a subtraction algebra X is called subalgebra if for all $x, y \in A, x - y \in A$.*

Theorem 7. *Any ideal of a subtraction algebra X is a subalgebra.*

Proof. i. By definition of ideal $0 \in A$

ii. For all $a \in X$ and $b \in A$, if $a \leq b$, then $a - b = 0 \in A$
 $\Rightarrow a - b \in A$
 $\Rightarrow a \in A$.

Hence A is an ideal of X . Now from (ii) we see that $a - b \in A$, for all $a, b \in A$.

Thus is A a subalgebra.

□

Chapter 4

Quotient Subtraction Algebra

Certain techniques are fundamental to the study of algebra. One such technique is the construction of the quotient set of an algebraic object by means of an equivalence relation on the underlying set. This quotient construction can be applied to numerous algebraic structures including groups, rings, boolean algebras and vector spaces. In this chapter we introduce the basic concept and properties of quotient subtraction algebra.

Before we will examine the meaning and examples of quotient of subtraction algebra, we will discuss about equivalent relation

4.1 Equivalence Relation

A relation R from a set X to a set A is a subset of $X \times A$. We say that a is related to b under R if the pair (a, b) belongs to the subset, and we write this as aRb . Then in this sub topic we will see a relation based on subtraction algebra.

Definiton 9. *Let X be a subtraction algebra and A be an ideal of X . For any $x, y \in X$, we define a relation as the following:*

$$x \sim y \Leftrightarrow x - y \in A \text{ and } y - x \in A.$$

Theorem 8. *Let $(X, -)$ be a subtraction algebra and A be an ideal of X . The relation " \sim " define on X , for $x, y \in X$, by:*

$$x \sim y \Leftrightarrow x - y \in A \text{ and } y - x \in A$$

is an equivalence relation.

Proof. i. Reflexive: For all $x \in X$ and A is an ideal of X , we have $x - x = 0 \in A$
 $\Rightarrow x - x \in A \Leftrightarrow x \sim x$.
Hence \sim is a Reflexive.

ii. Symmetry: For all $x, y \in X, x \sim y \Leftrightarrow x - y \in A, y - x \in A$. We want to show that $y \sim x$.

$$\begin{aligned} \Rightarrow x \sim y &\Leftrightarrow x - y \in A, y - x \in A. \\ &\Leftrightarrow y - x \in A, x - y \in A. \\ x \sim y &\Leftrightarrow y \sim x. \end{aligned}$$

Hence \sim is a symmetry.

iii. Transitive: Suppose, $x \sim y$ and $y \sim z$, then $x \sim z$, for all $x, y, z \in X$. That is $x \sim y \Leftrightarrow x - y \in A, y - x \in A$ and $y \sim z \Leftrightarrow y - z \in A, z - y \in A$, then we want to show that $x \sim z$. By definition (9) above, let $x - y \in A$ be given, then by Lemma 2(e), we have

$$\begin{aligned} (x - z) - (y - z) &\leq (x - y) \in A \\ (x - z) - (y - z) &\in A, \text{ by least upper bound property} \\ (x - z) &\in A, \text{ since } A \text{ is an ideals of } X. \end{aligned}$$

Similarly from

$$\begin{aligned} (z - y) &\in A, \text{ we get} \\ (z - x) - (y - x) &\leq (z - y) \in A, \text{ by Lemma 2(e)} \\ (z - x) - (y - x) &\in A, \text{ by least upper bound property} \\ (z - x) &\in A, \text{ since } A \text{ is an ideals of } X. \end{aligned}$$

Which implies,

$$(x - z) \in A \text{ and } (z - x) \in A \Leftrightarrow x \sim z$$

Hence " \sim " is a transitive.

Therefore, " \sim " is equivalence relation on X .

□

The set of equivalence class containing 'x' is denoted by

$$\bar{x} = \{y \in X : y \sim x\}$$

Proposition 1. Let $(X, -)$ be a subtraction algebra and A be an ideals of X . Let " \sim " be a relation which defined on X by:

$$x \sim y \Leftrightarrow x - y \in A \text{ and } y - x \in A,$$

for all $x, y, z \in X$. Then:

- i. $0 \in \bar{x} \Leftrightarrow x \in A$
- ii. $(x - y) - y \in \overline{(x - y)}$
- iii. If $y \in \bar{x}$, then $x - (x - y) \in \bar{x}$.
- iv. $(x - y) - (y - x) \in \bar{x}$.
- v. $x - (x - (x - y)) \in \overline{x - y}$.
- vi. If $y \in \bar{x}$, then $(x - z) - (y - z) \in \overline{x - y}$.

Proof. i. Let $0 \in \bar{x} \Leftrightarrow 0 \sim x$

$$\begin{aligned} &\Leftrightarrow 0 - x \in A \text{ and } x - 0 \in A \\ &\Leftrightarrow x \in A. \\ 0 \in \bar{x} &\Leftrightarrow x \in A. \end{aligned}$$

ii. $(x - y) - y \in \overline{x - y}$.

To prove this we need to check the following two conditions:

1. $[(x - y) - y] - (x - y) \in A$
2. $(x - y) - [(x - y) - y] \in A,$

$$\begin{aligned} 1. \quad &[(x - y) - y] - (x - y) = [(x - y) - (x - y)] - y, \text{ by } S_3 \\ &= 0 - y, \text{ since } (x - y) - (x - y) = 0 \\ &= 0, \text{ by Lemma1 (} b_2 \text{)} \\ &\Rightarrow [(x - y) - y] - (x - y) = 0 \in A \\ &\Rightarrow [(x - y) - y] - (x - y) \in A \\ 2. \quad &(x - y) - [(x - y) - y] = (x - y) - (x - y), \text{ by theorem 1(i)} \\ &= 0, \text{ since } (x - y) - (x - y) = 0 \\ &\Rightarrow (x - y) - [(x - y) - y] = 0 \in A, \text{ since } A \text{ is an ideal of } X \\ &\Rightarrow (x - y) - [(x - y) - y] \in A \end{aligned}$$

Thus from (1) and (2) we have

$$[(x - y) - y] - (x - y) \in A$$

and

$$\begin{aligned} (x - y) - [(x - y) - y] &\in A \\ \Leftrightarrow (x - y) - y &\sim (x - y) \end{aligned}$$

Therefore

$$(x - y) - y \in \overline{x - y}$$

iii. If $y \in \bar{x}$, then $x - (x - y) \in \bar{x}$.

Suppose $y \in \bar{x}$. That is $x \sim y \Leftrightarrow x - y \in A$ and $y - x \in A$, then we want to show that: $x - (x - y) \in \bar{x}$. To prove this we need to check the following axioms,

- a. $x - (x - (x - y)) \in A$
- b. $(x - (x - y)) - x \in A$

which implies,

$$\begin{aligned} \text{a. } [x - (x - (x - y))] &= (x - y), \text{ by Lemma 1 (} b_4 \text{)} \\ &= (x - y) \in A \\ (x - (x - (x - y))) &\in A, \\ \text{b. } (x - (x - y)) - x &= (x - x) - (x - y), \text{ by } S_3 \\ &= 0 - (x - y), \text{ since } x - x = 0 \\ &= 0, \text{ by Lemma 1 (} b_2 \text{)} \\ x - (x - (x - y)) &= 0 \in A, \text{ since } A \text{ is an ideal of } X \\ x - (x - (x - y)) &\in A \end{aligned}$$

Thus from a and b we have

$[x - (x - (x - y))] \in A$ and $[x - (x - (x - y))] \in A$ if and only if $x - (x - y) \sim x$

Therefore, $[x - (x - y)] \in \bar{x}$.

iv. $(x - y) - (y - x) \in \bar{x}$. Then we need to check the following two conditions:

- 1. $[(x - y) - (y - x)] - x \in A$
- 2. $x - [(x - y) - (y - x)] \in A$. Which implies

$$\begin{aligned}
1. \quad & [(x - y) - (y - x)] - x = [(x - (y - x)) - y] - x, \text{ by } S_3 \\
& = (x - y) - x, \text{ by } S_1 \\
& = (x - x) - y, \text{ by } S_3 \\
& = 0 - y, \text{ since } (x - x = 0). \\
& = 0, \text{ by Lemma 1 (} b_2 \text{)} \\
\Rightarrow & [(x - y) - (y - x)] - x = 0 \in A, \text{ since } A \text{ is an ideal of } X \\
\Rightarrow & [(x - y) - (y - x)] - x \in A, \\
2. \quad & x - [(x - y) - (y - x)] = x - [(x - (y - x)) - y] \\
& = [x - (x - y)], \text{ Since } [x - (y - x)] = x \\
& \leq x - x, \text{ since } (x - y) \leq x \\
& \leq 0, \text{ since } x - x = 0 \\
\Rightarrow & x - [(x - y) - (y - x)] = 0 \in A, \text{ since } A \text{ is an ideal of } X . \\
\Rightarrow & x - [(x - y) - (y - x)] \in A.
\end{aligned}$$

Thus from (1) and (2) we have

$$[(x - y) - (y - x)] - x \in A$$

and

$$x - [(x - y) - (y - x)] \in A$$

if and only if

$$(x - y) - (y - x) \sim x$$

Hence

$$(x - y) - (y - x) \in \bar{x}$$

v. $x - (x - (x - y)) \in \overline{x - y}$.

To prove this we need to check the following axioms:

- a. $x - (x - (x - y)) - (x - y) \in A$.
- b. $(x - y) - [x - (x - (x - y))] \in A$. Then

$$\begin{aligned}
a. \quad & x - (x - (x - y)) - (x - y) = (x - y) - (x - y), \text{ by Lemma 1 (} b_4 \text{)} \\
& = 0, \text{ since } (x - y) - (x - y) = 0 \\
\Rightarrow & x - (x - (x - y)) - (x - y) = 0 \in A, \text{ Since } A \text{ is an ideal of } X .
\end{aligned}$$

$\Rightarrow x - (x - (x - y)) - (x - y) \in A$ and

$$\begin{aligned} \text{b. } (x - y) - [x - (x - (x - y))] &= (x - y) - (x - y), \text{ by Lemma 1 (b}_4\text{)} \\ (x - y) - [x - (x - (x - y))] &= 0, \text{ since } (x - y) - (x - y) = 0 \\ (x - y) - [x - (x - (x - y))] &= 0 \in A, \text{ Since } A \text{ is an ideal of } X. \\ (x - y) - [x - (x - (x - y))] &\in A, \end{aligned}$$

Now from (a) and (b) above we get

$$x - (x - (x - y)) - (x - y) \in A$$

and

$$(x - y) - [x - (x - (x - y))] \in A$$

if and only if

$$x - (x - (x - y)) \sim (x - y)$$

Hence we obtain

$$x - (x - (x - y)) \in \overline{x - y}$$

vi. If $y \in \bar{x}$, then $(x - z) - (y - z) \in \overline{x - y}$.

Suppose $y \in \bar{x}$. That is $x \sim y \Leftrightarrow x - y \in A$ and $y - x \in A$, then we want to show that: $(x - z) - (y - z) \in \overline{x - y}$.

To prove this we need to check the following axioms.

1. $[(x - z) - (y - z)] - (x - y) \in A$ And
2. $(x - y) - [(x - z) - (y - z)] \in A$

$$\begin{aligned} 1. [(x - z) - (y - z)] - (x - y) &= [(x - z) - (x - y)] - (y - z), \text{ by } S_3 \\ &= [(x - (x - y)) - z] - (y - z), \text{ by } S_3 \\ &= [(y - (y - x)) - z] - (y - z), \text{ by } S_2 \\ &= [(y - z) - (y - x)] - (y - z), \text{ by } S_3 \\ &= [(y - z) - (y - z)] - (y - x), \text{ by } S_3 \\ &= 0 - (y - x) \\ &= 0, \text{ by Lemma 1 (b}_2\text{)}. \\ &= 0 \in A \end{aligned}$$

$\Rightarrow [(x - z) - (y - z)] - (x - y) = 0 \in A$, since A is an ideal of X .

$\Rightarrow [(x - z) - (y - z)] - (x - y) \in A$. And

$$\begin{aligned} 2. (x - y) - [(x - z) - (y - z)] &= (x - y) - [(x - y) - z], \text{ by Lemma 1 (b}_5\text{)} \\ &= z - (z - (x - y)), \text{ by Lemma 1(b}_4\text{)} \\ &\leq (x - y), \text{ by Lemma 2(a)} \\ &\leq (x - y), \text{ and } (x - y) \in A. \end{aligned}$$

$\Rightarrow (x - y) - [(x - z) - (y - z)] \in A$.

Thus from (1) and (2) we obtain

$$(x - y) - [(x - z) - (y - z)] \in A$$

and

$$\begin{aligned} [(x - z) - (y - z)] - (x - y) &\in A \\ \Leftrightarrow (x - z) - (y - z) &\sim (x - y) \end{aligned}$$

Therefore,

$$(x - z) - (y - z) \in \overline{x - y}$$

□

Lemma 4. *Let $(X, -)$ be a subtraction algebra and A be an ideal of X and for all $x, y \in X$. Then,*

$$x \sim y \Leftrightarrow x - y \in A \text{ and } y - x \in A$$

is a congruence relation on X .

Proof. Suppose $x \sim y$ and $u \sim v$, i.e $x \sim y \Leftrightarrow x - y \in A$, $y - x \in A$ and $u \sim v \Leftrightarrow u - v \in A$, $v - u \in A$, for all $x, y, u, v \in X$. Then we want to show that $x - u \sim y - v$. Using definition (9) above, let $x - y \in A$ be given, then by applying Lemma 2(e) and Lemma 3 we have,

$$\begin{aligned} (x - u) - (y - u) &\leq (x - y) \in A \\ (x - u) - (y - u) &\in A, \text{ by least upper bound property} \end{aligned}$$

Similarly by definition (9), let $y - x \in A$ be given, then using Lemma 2(e), we have

$$\begin{aligned} (y - u) - (x - u) &\leq (y - x) \in A \\ (y - u) - (x - u) &\in A, \text{ by least upper bound property} \end{aligned}$$

Hence $(x - u) - (y - u) \in A$, $(y - u) - (x - u) \in A \Leftrightarrow (x - u) \sim (y - u)$ which implies

$$(x - u) \sim (y - u) \tag{4.1}$$

And using the same Lemma 2(a) above we get,

$$y - (y - v) \leq v$$

and

$$y - (y - u) \leq u$$

then it becomes

$$(y - (y - v)) - u \leq (v - u) \in A$$

and

$$(y - (y - u)) - v \leq (u - v) \in A, \text{ by Lemma 2(e)}$$

$$(y - (y - v)) - u \in A$$

and

$$(y - (y - u)) - v \in A, \text{ by definition (9)}$$

$$(y - u) - (y - v) \in A$$

and

$$(y - v) - (y - u) \in A, \text{ by } S_3$$

From this we get

$$(y - u) - (y - v) \in A$$

and

$$(y - v) - (y - u) \in A$$

$$\Leftrightarrow (y - u) \sim (y - v)$$

which implies

$$(y - u) \sim (y - v) \tag{4.2}$$

Hence from (4.1) and (4.2) we have,

$$(x - u) \sim (y - u)$$

and

$$(y - u) \sim (y - v)$$

Then by transitivity property we have,

$$(x - u) \sim (y - v)$$

Therefore, " \sim " is a congruence relation on X . □

Corollary 1. *Let X be a subtraction algebra and A be an ideal of X . Let " \sim " be the relation which defined as*

$$x \sim y \Leftrightarrow x - y \in A \text{ and } y - x \in A.$$

Then

i. If $x \sim y$, then $x - z \sim y - z$,

ii. If $x \sim y$, then $z - x \sim z - y$.

Proof. i. Suppose $x \sim y$. i.e $x \sim y \Leftrightarrow x - y \in A$ and $y - x \in A$, for all $x, y, z \in X$. Then we want to show that $x - z \sim y - z$. Thus by definition (9) above, let $x - y \in A$ be given, then Lemma2(e) above we have,

$$\begin{aligned} (x - z) - (y - z) &\leq (x - y) \in A \\ (x - z) - (y - z) &\in A \end{aligned}$$

by least upper bound property

$$(x - z) - (y - z) \in A \quad (4.3)$$

And similarly By definition (9), let $y - x \in A$ be given, again applying Lemma2(e) we get,

$$\begin{aligned} (y - z) - (x - z) &\leq (y - x) \in A \\ (y - z) - (x - z) &\in A \end{aligned}$$

by least upper bound property

$$(y - z) - (x - z) \in A \quad (4.4)$$

Hence from (4.3) and (4.4) we have

$$(x - z) - (y - z) \in A$$

and

$$(y - z) - (x - z) \in A$$

if and only if

$$(x - z) \sim (y - z)$$

Therefore, if $x \sim y$, then $x - z \sim y - z$,

ii. Similarly suppose $x \sim y$ i.e $x \sim y \Leftrightarrow x - y \in A$ and $y - x \in A$, for all $x, y, z \in X$. We have to show that $z - x \sim z - y$.

To prove this we need to apply Lemma2(a) i.e

$$\begin{aligned} [z - (z - y)] &\leq y \text{ and } [z - (z - x)] \leq x \\ [z - (z - y)] - x &\leq (y - x) \end{aligned}$$

and

$$[z - (z - x)] - y \leq (x - y), \text{ by Lemma2(e)}$$

$$[z - (z - y)] - x \leq (y - x) \in A$$

and

$$[z - (z - x)] - y \leq (x - y) \in A, \text{ by definition(9) above}$$

$$[z - (z - y)] - x \in A$$

and

$$[z - (z - x)] - y \in A, \text{ by Lemma(4)}$$

$$(z - x) - (z - y) \in A$$

and

$$(z - y) - (z - x) \in A, \text{ by } S_3$$

$$(z - x) - (z - y) \in A$$

and

$$(z - y) - (z - x) \in A$$

If and only if

$$(z - x) \sim (z - y)$$

Therefore, if $x \sim y$, then $z - x \sim z - y$

□

Proposition 2. *Let X be a subtraction algebra and A be an ideal of X , then*

$$A = \bar{0}.$$

Proof. (\Rightarrow) If $x \in A$, then $x - 0 = x \in A$ and $0 - x = 0 \in A$, implies $x \sim 0$.

Hence $x \in \bar{0}$.

(\Leftarrow) let $x \in \bar{0}$, $\Leftrightarrow x \sim 0$

$$\Leftrightarrow x - 0 \in A \text{ and } 0 - x \in A,$$

$$\Leftrightarrow x \in A \text{ and } 0 \in A,$$

$$\Leftrightarrow x \in A$$

Hence $A = \bar{0}$.

□

4.2 Quotient Subtraction Algebras

In this section, we will see the definition and elementary properties of Quotient subtraction algebra.

Definiton 10. Let X be a subtraction algebra and A be an ideal of X . Let " \sim " be an equivalence relation which defined as

$$x \sim y \Leftrightarrow x - y \in A \text{ and } y - x \in A.$$

Then the set of all equivalence classes in X is denoted X/A and called the quotient set of X by A .

Theorem 9. Let X be a subtraction algebra and A be an ideal of X . Then X/A is a subtraction algebra with the operation " $-$ " given by

$$\bar{x} - \bar{y} = \overline{x - y} \text{ for all } x, y \in X.$$

Proof. Since " \sim " is a congruence relation on X , the operation " $-$ " is well defined. Then using S_1 , S_2 and S_3 we have to done in the following:

$$\begin{aligned} S_1. \quad \bar{x} - (\bar{y} - \bar{x}) &= \overline{x - (y - x)} \\ &= \overline{x - (y - x)} \\ &= \bar{x} \end{aligned}$$

$$\text{Hence } \bar{x} - (\bar{y} - \bar{x}) = \bar{x}$$

$$\begin{aligned} S_2. \quad \bar{x} - (\bar{x} - \bar{y}) &= \overline{x - (x - y)} \\ &= \overline{x - (x - y)} \\ &= \overline{y - (y - x)} \\ &= \overline{y - (y - x)} \\ &= \bar{y} - (\bar{y} - \bar{x}) \end{aligned}$$

$$\text{Hence } \bar{x} - (\bar{x} - \bar{y}) = \bar{y} - (\bar{y} - \bar{x})$$

$$\begin{aligned} S_3. \quad (\bar{x} - \bar{y}) - \bar{z} &= \overline{(x - y) - z} \\ &= \overline{(x - y) - z} \\ &= \overline{(x - z) - y} \\ &= \overline{(x - z) - y} \\ &= (\bar{x} - \bar{z}) - \bar{y}. \end{aligned}$$

$$\text{Hence } (\bar{x} - \bar{y}) - \bar{z} = (\bar{x} - \bar{z}) - \bar{y}.$$

Therefore X/A is a subtraction algebra. □

Theorem 10. *If A and B are any two ideals of a subtraction algebra of X and $A \subset B$, then the following conditions are satisfied.*

(a) *A is an ideal of subalgebra B .*

(b) *B/A is an ideal of quotient algebra of X/A .*

Proof. (a) To prove this we need to check the following conditions.

- i. Since A is an ideal of X , then $0 \in A$ implies $A \neq \emptyset$.
- ii. For all $x \in B$, and $y \in A$ such that $x - y \in A$, by definition of ideal. Since $B \subseteq X$ and A is an ideal of X . Then $x \in B$ implies $x \in X$.

Therefore $x - y \Rightarrow x \in A$. Hence A is an ideal of a subalgebra B .

(b) $B/A = \{\bar{x} : x \in B\} \subset \{\bar{x} : x \in X\} = X/A$.

1. Now from hypothesis, since B is an ideal of X we have $0 \in B$ and this implies $\bar{0} \in B/A$, which implies $B/A \neq \emptyset$
2. Let $\bar{x}, \bar{y} \in X/A$ be such that $\bar{y} \in B/A$ and $\bar{x} - \bar{y} \in B/A$. Then $y \in B$ and $x - y \in B$. Again from hypothesis, since B is an ideal of X , we get $x \in B$. Then $\bar{x} \in B/A$.
Hence B/A is an ideal of quotient algebra of X/A .

□

Theorem 11. *If B^* is an ideal of quotient subtraction algebra of X/A , then $B = \cup_{\bar{x} \in B^*} \bar{x}$ is an ideal of X and $A \subseteq B$.*

Proof. To prove this we need to check that,

- a. Since A is an ideal of X and $A = \bar{0} \in B^*$, we get $0 \in B$, by prop.(2)
Which implies $B \neq \emptyset$
- b. Again since A is an ideal of X and for any $x \in A, x \in \bar{0} \subset B$, then $A \subseteq B$, by prop.(2)
Let $x - y \in B$ be such that $y \in B$, and it becomes $\bar{x} - \bar{y} = \overline{x - y} \in B^*$ and $\bar{y} \in B^*$, by theorem (9). Which implies $y \in B$
Therefore, B is an ideal of X .

□

The set of all ideals of subtraction algebra of X is denoted by $A(X)$. Also we show that the set of all ideals which are containing A on X by $A(X, A)$.

Theorem 12. *Let A be an ideal of subtraction algebra of X . There exists a one-to-one and onto mapping $f : A(X, A) \rightarrow A(X/A)$ defined by, for any $B \in A(X, A)$, $f(B) = B/A$.*

Proof. Using the Theorem (11), f is onto. For any $I, J \in A(X, A)$, let $f(I) = f(J)$ but $I \neq J$. Then, at least for any $x \in J$, we have $x \notin I$. Since $f(I) = f(J)$, we get $\bar{x} \in f(I)$ and $\bar{x} \in f(J)$. Then there exists $z \in I$ such that $\bar{z} = \bar{x}$. Hence we obtain $x \sim z$, that is, $x - z \in A$. By symmetry, we have $z - x \in A$. Since $A \subseteq I$, we have $x - z \in I$ and also since $z \in I$, we obtain $x \in I$ by the definition of an ideal, which contradicts x is not in I . Therefore, f is one to one. \square

Chapter 5

Isomorphism Theorems of Subtraction Algebras

In almost all the algebraic structures in mathematics, there is a natural question of isomorphism. Given two algebraic structures of the same type, most of the time we want to determine if they have the same structure regardless of their elements. The most common examples are; isomorphisms of groups; isomorphisms of rings; isomorphisms of vector spaces and so on. In this part of our work we consider isomorphisms of our main algebraic structures; subtraction algebras.

5.1 Definitions and Some basic Properties of Subtraction Algebra Homomorphisms

In this section we will explain the definition and some properties of homomorphism of subtraction algebras.

Definiton 11. *Let $(X, -)$ and (Y, \ominus) be any two subtraction algebras. A mapping*

$$\phi : X \longrightarrow Y$$

that satisfies the condition

$$\phi(x - y) = \phi(x) \ominus \phi(y)$$

for all $x, y \in X$ is called a homomorphism of subtraction algebras.

Let $(X, -)$ and (Y, \ominus) be two subtraction algebras and

$$f : X \longrightarrow Y$$

be a homomorphism of subtraction algebras.

- (a) If the mapping f is an onto function, that is for each element in Y there exist a pre-image in X such that , $f(X) = Y$, where

$$f(X) = \{f(x) : x \in X\},$$

then f is called an epimorphism and Y is said to be homomorphic image of X .

- (b) If the mapping f is a one-to-one function, then f is called an monomorphism.
- (c) The mapping is called an isomorphism if it is both an onto and one-to-one.

If there exists an isomorphism $f : X \longrightarrow Y$, then we call X to be isomorphic to Y , written $X \cong Y$. Obviously $X \cong Y$ implies $Y \cong X$ and $X \cong Y, Y \cong Z$ implies $X \cong Z$. In case if $X = Y$, then a homomorphism is called an endomorphism and an isomorphism from X into itself is called an automorphism.

Example 9. Let $(X, -)$ and (Y, \ominus) be two subtraction algebras.

- (a) The identity map $1 : X \longrightarrow X$ defined by $1(x) = x$ for all $x \in X$ is an isomorphism. Thus $X \cong X$.
- (b) The zero homomorphism $\mathbf{0} : X \longrightarrow Y$ defined by $\mathbf{0}(x) = 0'$ for all $x \in X$, where $0'$ is the zero element for Y .

We denote the set of all homomorphisms from X into Y by $Hom(X, Y)$ and for any $f \in Hom(X, Y)$ and any nonempty subset $B \subseteq Y$, the set

$$\{x \in X : f(x) \in B\}$$

is denoted by $f^{-1}(B)$, called the inverse image of B under f .

In particular, $f^{-1}(0')$ is called kernel of f . Note that

$$f^{-1}(0') = \{x \in X : f(x) = 0'\},$$

we will write $\ker f$ instead of $f^{-1}(0')$.

Theorem 13. Let A be an ideal of a subtraction algebra of X . The mapping

$$\varphi : X \longrightarrow X/A$$

which is given by $\varphi(x) = \bar{x}$ for all x in X is a homomorphism.

Proof. For any two elements $x, y \in X$,

$$\begin{aligned}\varphi(x - y) &= \overline{(x - y)} \\ &= \bar{x} - \bar{y} \\ &\quad \text{(by the definition of ' - ' on the set } X/A) \\ &= \varphi(x) - \varphi(y) \\ &\quad \text{(by the definition of } \varphi \text{ on the set } X/A).\end{aligned}$$

This implies $\varphi(x - y) = \varphi(x) - \varphi(y)$ and hence φ is a homomorphism. \square

Theorem 14. *If B is an ideal of quotient subtraction algebra of X/A , then $f^{-1}(B)$ is an ideal of X and $A \subseteq f^{-1}(B)$.*

Proof. A mapping $f : X \rightarrow X/A$, given by $f(x) = \bar{x}$, is a homomorphism. Then we need to check that the following axioms are true.

- i. Let $0 \in X$, $f(0) = \bar{0} \in B$, implies $0 \in f^{-1}(B)$, thus $f^{-1}(B) \neq \emptyset$.
- ii. For any $x \in X$ and $y \in f^{-1}(B)$ such that $x - y \in f^{-1}(B)$. Then $f(y) = \bar{y} \in B$ and $f(x - y) = \overline{x - y} = \bar{x} - \bar{y} \in B$. Since B is an ideal of X/A , we have $\bar{x} \in B$ or $x \in f^{-1}(B)$. Hence $f^{-1}(B)$ is an ideal of X . Then by Prop.(2) above, for all $x \in A$, $x \in \bar{0}$ in B , we get $0 \in f^{-1}(B)$. And this implies $x \in f^{-1}(B)$. And $A \subseteq f^{-1}(B)$.

\square

Let $(X, -)$ and (Y, \ominus) be two subtraction algebras. For any two elements $x, y \in X$ and $x \leq y$ if $f(x) \leq f(y)$, then f is called isotone. In the following theorem, we shall see that every homomorphism of between two subtraction algebras is an isotone.

Theorem 15. *Let $(X, -)$ and (Y, \ominus) be two subtraction algebras and*

$$f : X \longrightarrow Y$$

be a homomorphism. Then,

- (a) $f(0) = 0'$ and
- (b) f is an isotone.

Proof.

- (a) If 0 is the identity element in X and $0'$ is the identity element in Y , then we have

$$\begin{aligned}
 f(0) &= f(0 - 0) \text{ (since } (0 - 0 = 0 \text{ in } X)) \\
 &= f(0) \ominus f(0) \text{ (by the definition of } f) \\
 &= 0' \ominus 0'. \\
 &= 0'
 \end{aligned}$$

This implies $f(0) = 0'$.

- (b) If $x, y \in X$ and $x \leq y$, then $x - y = 0$. Now using (a) we have

$$\begin{aligned}
 f(x) - f(y) &= f(x - y), \text{ since } f \text{ is homomorphism} \\
 &= f(0), \text{ since } x - y = 0 \\
 &= 0', \text{ by (a) above}
 \end{aligned}$$

Thus $f(x) - f(y) = 0'$ which implies $f(x) \leq f(y)$. Hence f is an isotone.

□

Homomorphisms of subtraction algebras can be used to define some ideals of subtraction algebras that are related with the given homomorphisms. Some of these ideals are given in the following theorem

Theorem 16. *Let $(X, -)$ and (Y, \ominus) be two subtraction algebras, B be an ideal in Y and $f : X \rightarrow Y$ be a homomorphism. Then,*

- (a) *the set $f^{-1}(B)$ is an ideal of X .*
 (b) *$\text{Ker}f = \{a \in X : f(a) = 0\}$ is an ideal of X .*
 (c) *The set $\text{Im}f = \{b \in Y : b = f(a) \text{ for some } a \in X\}$ is called the image of f and it is an ideal of Y .*

Proof.

- (a) Since B is an ideal of Y and $f(0) = 0'$, we have $f(0) = 0' \in B$. This implies $0 \in f^{-1}(B)$.

Assume that $x - y \in f^{-1}(B)$ and $y \in f^{-1}(B)$. Then $f(x - y) \in B$ and $f(y) \in B$. This implies $f(x) \ominus f(y) \in B$ and $f(y) \in B$. Since B is an ideal of Y , $f(x) \in B$. Thus $x \in f^{-1}(B)$ and hence $f^{-1}(B)$ is an ideal of X .

(b) Since $\{0'\}$ is an ideal of Y and $\text{Ker}f = f^{-1}(\{0'\})$, by (a) above we have $\text{Ker}f$ is an ideal of X .

(c) We have $f(0) = 0'$, which implies $0' \in \text{Im}f$.

□

Example 10. If $(X, -)$ is a subtraction algebra and A is an ideal of X , then in Theorem 9 we have seen that X/A is a quotient subtraction algebra and in Theorem 13 we have seen that X/A is a homomorphic image of X , as the mapping

$$\varphi : X \longrightarrow X/A$$

given by

$$\varphi(x) = \bar{x} \text{ for all } x \in X$$

a homomorphism. Thus X/A is a homomorphic image of X .

5.2 Isomorphism Theorems on Subtraction Algebras

In this section we consider the isomorphism theorems on Subtraction Algebras. We start with the first isomorphism theorem.

Theorem 17. Let $(X, -)$ and (Y, \ominus) be two subtraction algebras and $f : X \longrightarrow Y$ be an epimorphism. Then $X/\text{Ker}f \cong Y$.

Proof. Since $\text{Ker}f$ is an ideal of X , $X/\text{Ker}f$ is a subtraction algebra. Define $g : X/\text{Ker}f \longrightarrow Y$ by $g(\bar{x}) = f(x)$.

(a) If $\bar{x} = \bar{y}$, then $x - y, y - x \in \text{Ker}f$. This implies

$$f(x - y) = f(x) \ominus f(y) = 0' \text{ and } f(y - x) = f(y) \ominus f(x) = 0'.$$

Then using Lemma 1, we have $f(y) = f(x)$ which implies $g(\bar{x}) = g(\bar{y})$. Hence g is well defined.

(b) If $\bar{x}, \bar{y} \in X/\text{Ker}f$, then

$$\begin{aligned} g(\bar{x} - \bar{y}) &= g(\overline{x - y}) \\ &= f(x - y) \\ &= f(x) \ominus f(y) \\ &= g(\bar{x}) - g(\bar{y}). \end{aligned}$$

Hence g is homomorphism

- (c) For $\bar{x}, \bar{y} \in X/\text{Ker}f$, if $\bar{x} \neq \bar{y}$, then $x - y \notin \text{Ker}f$ or $y - x \notin \text{Ker}f$. Suppose $x - y \notin \text{Ker}f$. Then $f(x) \ominus f(y) = f(x - y) \neq 0'$. Therefore $f(x) \neq f(y)$ and hence $g(\bar{x}) \neq g(\bar{y})$. Thus, g is one-to-one.
- (d) For any $y \in Y$, there exists $x \in X$ such that $y = f(x)$ as f is onto. Thus

$$g(\bar{x}) = f(x) = y,$$

which means that g is onto.

Therefore, g is an isomorphism and then $X/\text{Ker}f \cong Y$.

□

Theorem 18. Let $(X, -)$ and (Y, \ominus) be two subtraction algebras and $f : X \longrightarrow Y$ be an epimorphism. If B is an ideal of Y , then $X/A \cong Y/B$, where $A = f^{-1}(B)$,

Proof. The mapping $g : Y \longrightarrow Y/B$ given by $g(y) = \bar{y}$ is epimorphism. Then the composition mapping

$$g \circ f : X \longrightarrow Y/B$$

is an epimorphism. Now let us prove that

$$\text{Ker}(g \circ f) = f^{-1}(B).$$

For any $x \in X$, we have

$$(g \circ f)(x) = g(f(x)) = \overline{f(x)}.$$

Suppose $y \in f^{-1}(B)$. Then $f(y) \in B$ and hence $\overline{f(y)} = B$. That is

$$(g \circ f)(y) = B.$$

This implies $y \in \text{Ker}(g \circ f)$ and then

$$f^{-1}(B) \subseteq \text{Ker}(g \circ f).$$

Let $x \in \text{Ker}(g \circ f)$. That is, $(g \circ f)(x) = B$. Therefore, we have $\overline{f(x)} = B$, and so $f(x) \in B$. That is $x \in f^{-1}(B)$.

Thus $\text{Ker}(g \circ f) = f^{-1}(B)$.

Therefore by Theorem(17), we have $X/A \cong Y/B$. □

Lemma 5. Let $(X, -)$ and (Y, \ominus) be two subtraction algebras, $f : X \longrightarrow Y$ be an epimorphism and A be an ideal of X , then $f^{-1}(f(A)) = A$.

Proof. Since A is an ideal in X , it is obvious that $A \subseteq f^{-1}(f(A))$.
 Suppose $x \in f^{-1}(f(A))$, then $x \in f(A)$. There is exists $y \in A$ such that $f(x) = f(y)$, so

$$f(x - y) = f(x) \ominus f(y) = 0'.$$

Thus $x - y \in \text{Ker} f \subseteq A$. Since $x - y \in A$ and $y \in A$, we have $x \in A$.
 Therefore $f^{-1}(f(A)) \subseteq A$. \square

Theorem 19. *Let A and B be two ideals of a subtraction algebra X , $A \subseteq B$ and*

$$f : X \longrightarrow X/A$$

and

$$g : X/A \longrightarrow (X/A)/(B/A)$$

be epimorphisms. Then $X/B \cong (X/A)/(B/A)$.

Proof. Let $h = g \circ f$. Then $h : X \longrightarrow (X/A)/(B/A)$ is an epimorphism.
 Hence

$$X/\text{Ker} h \cong (X/A)/(B/A).$$

Then have prove that $\text{Ker} h = B$. Since $\text{Ker} h = \{x \in X : h(x) = B/A\}$, by theorem 10,19 and Lemma 5 we have

$$\text{Ker} h = h^{-1}(h(B)) = B.$$

\square

Conclusion

This Project work is conducted on quotient subtraction algebra. The key elements to proceed this project are subtraction algebra and ideal of subtraction algebra. With out these topics we can't talk about quotient of subtraction algebras. To have a good knowledge on quotient of subtraction algebra, firstly we know-it-all the meaning of subtraction algebras and ideals of subtraction algebras, unless and otherwise it is difficult to speak about quotient of subtraction algebra. In general I got a good image on the proposed topic to conduct another project around the topic.

Bibliography

- [1] Y. Çeven and S. Küçükkoç, *Quotient subtraction algebras*, International Mathematical Forum, 6(2011), 1241-1247..
- [2] J. C. Abbott, *Sets, Lattices and Boolean Algebras*, Allyn and Bacon, Boston, 1969.
- [3] Y. B. Jun, H. S. Kim and E. H. Roh, *Ideal theory of subtraction algebras*, Sci. Math. Jpn. Online e-2004(2004), 397-402
- [4] Mehment Aliöztürk and Hasret Yazarli, *Homomorphism theorems in subtraction algebras*, Tamkang Journal of Mathematics Volume 44, Number 4, 365-375, Winter 2013
- [5] Rabah Kellil, *External approach of ideals in subtraction algebras*, Asian Journal of Current Engineering and Maths 2: 2 March April (2013) 115 - 117.
- [6] William J. Gilbert and W.Keith Nicholson, *Modern Algebra with Application*, 2th edition, 2004