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Ms-Project On

Capacitated Transportation Problems

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Abstract

This paper presents one way of getting a maximum or minimum cost about given Capacitated transportation problem for certain company. The understanding the transportation cost problem of company with a bounded rim condition will tell us the capacitated transportation

In Mathematics Optimization is intended to optimize mathematical problems in which one is transportation problem. Transportation problem involve optimization of a linear function called the objective function, subject to linear constraints, which may be either equalities, or inequalities in the unknowns. My aim is to minimize cost, with given demand and supply constraints of certain company in which the capacities of each demand location is fixed with in some range of numbers.

Keywords : Variables, Constraints, Feasible Solution, Optimality test.

Contents

0.1 Introduction

Capacitated transportation problem is special case of Transportation problem. Transportation problem is a subclasses of linear optimization problem in which the objective is to transport various quantities of a single homogenous, commodity that are initially stored at various origins to different destinations in such a way that the transportation cost is minimum and to achieve maxi-mum profit. Such transportation problems with out flow restriction are called **Un-capacitated** transportation problems

The **capacitated** transportation problem is the case in which the capacity of supply places are restricted to deliver only certain amount of quantities. The capacities that flow across a route is bounded. This is the major target of this paper, that is formulating and finding the solution for such kind of transportation problem(capacitated transportation problem). The basic goal of capacitated transportation problem is applied when only certain amount of demand/supply is possible to move. This mean that when there is only restricted production capacity or when there is only fixed customer are available. It can be used extensively in telecommunication networks, production distribution systems when there is a limited capacity of resources such as vehicles, equipment capacity etc.These are bounded variable transportation problems.

The non-linear Transportation problem considers the total non-linear cost of product produced in each of the origins. There is a wide scope extended transportation problem from which one is capacitated transportation problem with bounds on rim conditions.

Chapter 1

Capacitated Transportation Problem(CTP)

1.1 Introduction

Among extension of transportation problems few of them are The generalized Transportation Problem

The capacitated Transportation Problem.

The Transportation Problem with mix constraints.

The Transportation Problem with quantity discounts.

The fixed charge Transportation Problem.

The single source Transportation Problem.

The time minimizing (Bottleneck) Transportation Problem.

The general transportation problem: is used in a production system having machine loading systems. There are many situations where mix constraints are used in Transportation problem. For example; Production planning for a production period during which an upward trends of demand is expected for certain products.

The Single source Transportation model: is developed to be used in situation when there is only one shipping source for all destinations. This model is used in assignment of jobs to computers and machine loading.

The time minimizing Transportation problem: is used in situations when shelf life of item is low, it is also used in situations where timely delivery is highly essential (Such as perishable goods, Military equipment). The emphasis in this model is to minimize the maximum transpiration time from point of origin to point of consumption. To reduce time of delivery Aircrafts are also used.

Capacitated-Transportation Problem: is used in situations where shipping capacity is finite for all or some routes like number of trucks and aircrafts available is limited.

When we say that the TP is capacitated the flow along that way is fixed with in some range of numbers. So every assignment of flow amount depend up on the restricted ranges. The following figure indicates how flows are dependent of ranged capacities.

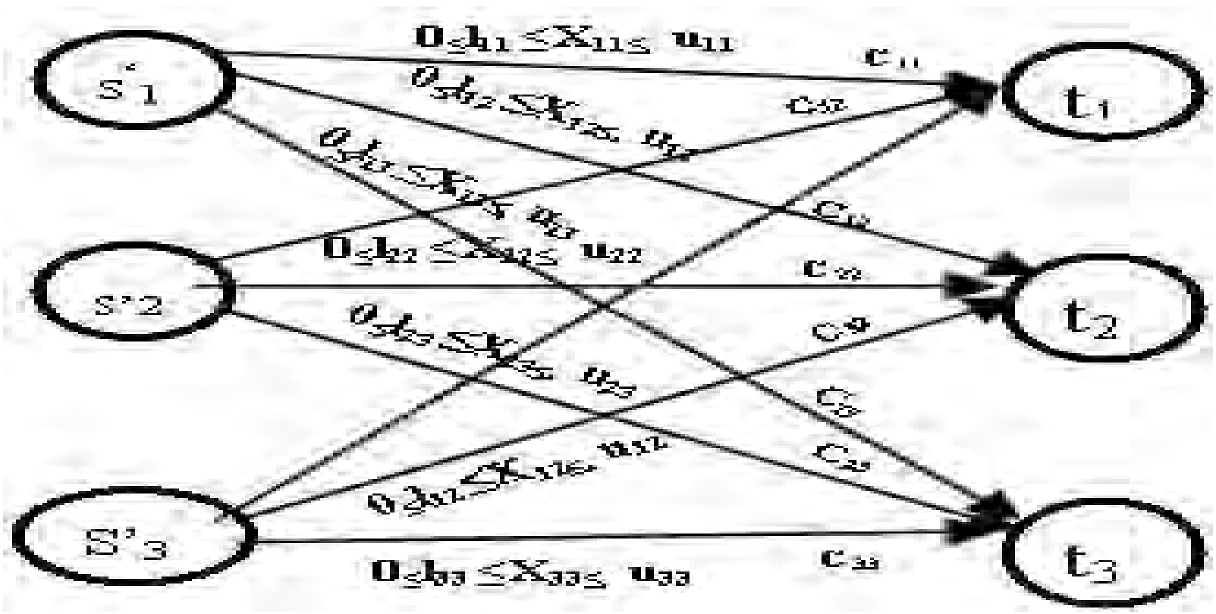


Figure 1.1: Cost and Capacity Flow.

1.2 Mathematical Formulation Of CTP

Consider that a company is going to transport a_i amount of goods from some supply places $A_i's$ where $i = i : m$ to a given demand places $B_j's$ which need b_j amount of goods, where $j = 1 : n$ and C_{ij} cost of transporting from i supply place to j demand place, as to that of regular transportation problem and in addition every X_{ij} are bounded in some restricted numbers($l_{ij} \leqslant U_{ij}$)

.

$$min \sum_{j=1}^{n} \sum_{i=1}^{m} C_{ij} X_{ij}$$

for some arbitrary integer m and n

$$\sum_{j}^{n} X_{ij} = a_i$$

$$\sum_{i}^{m} X_{ij} = b_j$$

$$l_{ij} \leqslant X_{ij} \leqslant U_{ij} \quad Bounded \quad rim \quad condition$$

$$b_j, \quad and \quad , a_i \geqslant 0$$

But sometimes the bounded rim condition will have minimum value zero or greater than zero, say

$$0 \leqslant l_{ij} \leqslant X_{ij} \leqslant U_{ij}$$

when this condition happens we can adjust the equation by subtracting l_{ij} from $l_{ij}s, U_{ij}s$. Which implies recent X'_{ij} is replaced by

$X'_{ij} = X_{ij} - l_{ij}$

and the recent U'_{ij} is replaced by

$U'_{ij} = U_{ij} - l_{ij}$.

The supply and Demands are replaced by considering the locations as a node(after or Before one another)

$a'_i = a_i + \sum_{j \in NA(i)} l_{ij} - \sum_{k \in NB(i)} l_{ki}$

$b'_j = b_j + \sum_{k \in NA(j)} l_{jj} - \sum_{k \in NB(j)} l_{ij}$

Then the restricted condition can be re-written as:

$$min \sum_{j=1}^{n} \sum_{i=1}^{m} C_{ij} X_{ij}$$

for some arbitrary integer m and n

$$\sum_{j}^{n} X'_{ij} = a'_i$$

$$\sum_{i}^{m} X'_{ij} = b'_j$$

$$l'_{ij} \leqslant X'_{ij} \leqslant U'_{ij} \quad Bounded \quad rim \quad condition$$

$$b'_j, \quad and \quad , a'_i \geqslant 0$$

From this we can find the difference between Transportation problem and Capacitated transportation problems is the bounded rim condition found during CTP.

1.3 Tabular expression

Tables of transportation problems are like other tables containing rows and columns. The columns contains the demand places(D_{ij}) and each raws indicate the source or supply positions(A_{ij}). Each cells of TP contain the cost(C_{ij}) assigned for flows (X_{ij}). In Case Of capacitated transportation problem the flows are not fixed to specific number instead flow is possible with in a range of numbers, that is what I try to put with in the following table as follows.

Source	To D_1	To D_2	To D_3	To D_4	Amount this sites can supply
From A_1	$l_{11} \leqslant X_{11} \leqslant u_{11}/C_{11}$	$l_{12} \leqslant X_{12} \leqslant u_{12}/ C_{12}$	$l_{13} \leqslant X_{13} \leqslant u_{13}/ C_{13}$	$l_{14} \leqslant X_{14} \leqslant u_{14}/ C_{14}$	a_1
From A_2	$l_{21} \leqslant X_{21} \leqslant u_{21}/C_{21}$	$l_{22} \leqslant X_{22} \leqslant u_{22}/ C_{22}$	$l_{23} \leqslant X_{23} \leqslant u_{23}/ C_{23}$	$l_{24} \leqslant X_{24} \leqslant u_{24}/ C_{24}$	a_2
From A_3	$l_{31} \leqslant X_{31} \leqslant u_{31}/C_{31}$	$l_{32} \leqslant X_{32} \leqslant u_{32}/ C_{32}$	$l_{33} \leqslant X_{33} \leqslant u_{33}/ C_{33}$	$l_{34} \leqslant X_{34} \leqslant u_{34}/ C_{34}$	a_3
Requirement	b_1	b_2	b_3	b_4	Total amount

1.4 Solution Method of Capacitated Transportation Problems

In This paper we need to solve CTP by using Minimum cost flow method(MCFP). Before proceeding to MCFP let define and discuss about the followings

- **Networks**

  A network is characterized by a collection of nodes and directed edges, called a directed graph. Each edge points from one node to another. We will denote an edge pointing from a node i to a node j by (i, j)

- **Graph terminology**

  A graph $G = (N, A)$ consists of a set of nodes, N, and a set of arcs, A. In an undirected graph the arcs are unordered pairs of nodes $i, j \in A, i, j \in N$. In a directed graph (also called a network) the arcs are ordered pairs of nodes (i, j). In our solution we will consider ordered list of nodes $i_1, i_2, ..., i_m$it such that, in an undirected graph, $(i_k, i_{k+1}) \in A$, or, in a directed graph, that either $(i_k, i_k + 1) \in A$ or $(i_{k+1}, i_k) \in A$, for $k = 1, ..., t - 1$. A walk is a path is $i_1, i_2, ..., i_k$ are distinct, and a cycle if $i_1, i_2, ..., i_{k-1}$ are distinct and $i_1 = i_k$. A graph is connected if there is a path connecting every pair of nodes.

- **Augmenting arc:** An arc A with flow $X_A^* > 0$ is an augmenting arc with respect to flow x if $0 \leqslant x_{ij} + X_A^* \leqslant u_{ij}$ for each arc $(i.j)$ i.e. adding flow in cycle does not exceed flow bounds.

- The cost of an **augmenting path** is the sum of the costs of its edges.

- A network is **acyclic** if it contains no **cycles**.

  A network is a **tree** if it is connected and acyclic.

  A network (N', A') is a **subnetwork** of (N, A) if N' subset of N and A'subset of A. A subnetwork (N', A') is a **spanning tree** if it is a tree and $N' = N$.

- An augmenting cycle W is a directed cycle, whose edges all have positive capacity. The cost of W is the sum of the cost of directed edges in W.

- If we have a negative cost cycle, then we can augment the cycle to our flow to get another flow of same value but smaller cost.

- Residual network, is that gives information about how we can push or extract some flow into or from the network. When constructing a residual graph, put $-C_{ij}$ on the reverse edges.

1.5 Minimum Cost Flow Problem

Consider a directed graph with a set N of nodes and a set$A$ of edges. In a min-cost-flow problem, each edge $(i, j) \in A$ is associated with a cost c_{ij} and a capacity constraint U_{ij} . There is one decision variable X_{ij} per edge $(i, j) \in A$. Each X_{ij} is represents a flow of objects from i to j. The cost of a flow X_{ij} is $C_{ij}x_{ij}$. Each node $j \in N/s, d$ satisfies a flow constraint:where s Source and d is for destination. The Minimum Cost Flow (MCF) Problem is to send flow from a set of supply nodes, through the arcs of a network, to a set of demand nodes, at minimum total cost, and without violating the lower and upper bounds on flows through the arcs. The MCF framework is particularly broad, and may be used to model a number of more specialised network problems, including Assignment, Transportation and Transshipment problems, the Shortest Path Problem, and the Maximum Flow problem.

1.5.1 Formulation and Specialisations

Let $G = (N.A)$ be a directed network consisting of a finite set of nodes, $N = 1, 2, ...m$ and a set of directed arcs, $A = 1, 2, ...n$ linking pairs of nodes in N. We associate with every arc of $(i; j) \in A$, a flow x_{ij} , a cost per unit flow C_{ij} , a lower bound on the flow l_{ij} and a capacity U_{ij} . To each node $i \in N$ we assign an integer number b_i representing the available supply of, or demand for flow at that node. If $b_i > 0$ then node i is a supply node, if $b_i < 0$ then node i is a demand node, and otherwise, where $b_i = 0$, node i is referred to as a transshipment node. Total supply must equal total demand. The Minimum Cost Flow (MCF) Problem is to send the required flows from the supply nodes to the demand nodes as in the next page(i.e. satisfying the demand constraints (1.2)), at minimum cost. The flow bound constraints,as indicated in the next page (1.3), must be satisfied. The demand constraints are also known as mass balance or flow balance constraints.

The formulation of the problem as a Linear Programming (LP) problem is as follows:
The minimum-cost flow problem is to find a vector such that an optimal solution of the liner program

$$\sum_{(i,j)\in A} C_{ij}X_{ij} \tag{1.1}$$

Subjected to

$$\sum_{j\in NA(i)} X_{ij} - \sum_{k\in NB(i)} X_{ki} = b_i \qquad for\ (i,j) \in N \tag{1.2}$$

$$0 \leqslant l_{ij} \leqslant x_{ij} \leqslant u_{ij}, \qquad for\ all\ (i,j) \in A \tag{1.3}$$

The equations (1.2) are the so-called flow conservation constraints and the inequalities (1.3) are the flow bounds on x. A flow x is called a feasible flow, if it satisfies the flow conservation constraints and the flow bounds.

Theorem 1.5.1. ***(The equivalence theorem)***
Every instance of the transportation problem can be reduced to a minimum cost flow problem.

Proof. Consider an arbitrary instance of the transporting problem (TrP) as given above:

$$\min \sum_{(i,j)\in A} C_{ij}X_{ij}$$

Subject to

$$\sum_{j=1} X_{ij} \leqslant a_i$$

$$\sum_{i=1} X_{ij} \leqslant b_j$$

$$X_{ij} \geqslant 0.$$

Design a network General Transportation problem(GTrP) as follows. It consists of a source s, m nodes $(1, ..., m)$ corresponding to the depots, n nodes corresponding to the customers, and a sink t. GTrP has the following edges:

- an edge from s to every depot i with capacity a_i and cost 0.
- an edge from every depot i to every customer j with capacity X_{ij} and cost C_{ij}, and
- an edge from every customer j to the sink t with capacity b_j and cost 0.

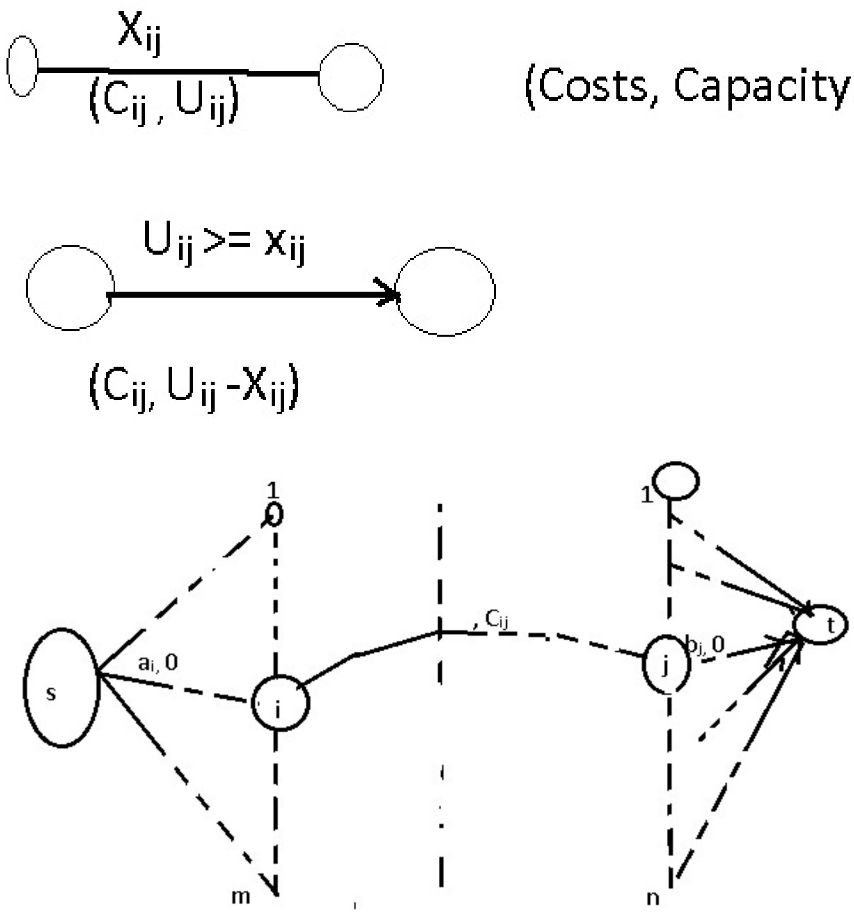


Figure 1.2: Cost and Capacity Flow.

the Forward arcs : Possible flow increase, $Capacity \ = \ U_{ij} - X_{ij}$, $Costs \ = \ C_{ij}$

the Backward Arcs : Possible flow decrease, $Capacity \ = \ X_{ij}$, $Cost \ = \ -C_{ij}$

The Transportation Problem is based on a bi-partite graph, where N_1 is the set of source nodes, and N_2 is the set of sink or destination nodes, such that $N = N_1 \bigcup N_2$, and the set of arcs is defined by $A = (i.j)|i \in N_1, j \in N_2$

The objective is to find the least cost shipping plan from the sources of supply (N_1) to the destinations (N_2), where X_{ij} is the number of units shipped from source i to destination j, and c_{ij} is the cost of shipping one unit from i to j. The supply and demands at sources and destinations respectively are denoted bya_i, b_j. There are no upper bounds on flows. (A problem with no upper flow bounds is said to be un-capacitated) and with upper bound is Capacitated in this paper we use only the capacitated one.

In LP form, the problem is stated:

$$\sum_{i,j\in A} C_{ij}X_{ij}$$

Subjected to

$$\sum_{j:(i,j)\in A} X_{ij} = b_i for i \in N_1$$

$$\sum_{i:(i,j)\in A} X_{ij} = -b_i for j \in N_2$$

$$0 \leqslant l_{ij} \leqslant x_{ij} \leqslant u_{ij}$$

As a network formulation this can be written as:

$$\sum_{i,j\in A} C_{ij}X_{ij}$$

Subjected to

$$\sum_{j\in NA(i)} X_{ij} - \sum_{k\in NB(i)} X_{ki} \;=\; b_i for(i.j)j \in N$$

$$0 \leqslant l_{ij} \leqslant x_{ij} \leqslant u_{ij}$$

□

This shows, The minimum cost flow problem is equivalent to the capacitated transportation problem.
The result implies that efficient algorithms to solve the minimum cost flow problem can be converted into efficient algorithms to solve the transportation problem.

1.5.2 Feasible flow problem(FFP)

FFP ensure that the sum of outflow minus the sum of inflow equals the supply at that source with the restriction constraints.

$$\sum_{j\in NA(i)} X_{ij} - \sum_{k\in NB(i)} X_{ki} = b_i \quad for\ each\ i \in N$$

$$0 \leqslant X_{ij} \leqslant U_{ij} \quad for\ each\ (i,j) \in A.$$

we solve the FFP by solving the maximum flow problem on the extended $s-t$ network. Sometimes given problems like transportation problems will not be given as $s-t$ network.But we can change the given transportation problem to $s - t$ network format by introducing source node and sink node in such a way we saw above and by introducing super source s and super sink t. For each supply node i (with $b_i > 0$), add an arc (s, i) with capacity $U_{si} = b_i$, and for each demand node j (with $b_j < 0$) add an arc (j, t) with capacity $U_{jt} = |b_{jt}|$

Theorem 1.5.2. *Let x be a maximum flow in the extended $s - t$ network with the maximum value b. Then, x is feasible flow in the network of the FFP if and only if*

$$X_{si} = b_i \quad for\ each\ (s, i) \in A.$$

Proof. obviously the flow is feasible when the total capacity along that edge moved.
And also if all available flow along that edge moved it implies no more flow to move along that edge, that mean the flow is feasible. □

1.5.3 Optimality Condition

Theorem 1.5.3. ***(Negative Cost Cycle Optimality Condition)*** *A flow x (of value v) is a minimum cost flow if and only if the residual graph $G(x)$ has no negative cost directed cycle.*

Proof.

- ($\Rightarrow$) Proof by transposition. Suppose there is a negative cost cycle, then we can add that cycle to our flow to get smaller cost flow of the same value. Then the original flow is not minimum cost flow.

- ($\Leftarrow$) Proof by transposition. Suppose x is not minimum-cost flow. Then, there is a cheaper flow g of value v. The flow $g - x$ can be decomposed into a union of augmenting cycles. One of the cycles has a negative cost, since $g - x$ has negative cost.

□

This gives a way for checking whether a flow has minimum cost or not.

Reduced cost

Let us take minimum cost flow algorithms, we measure the cost of an arc related to some cost associated with the nodes. These costs are typical intermediate data of the algorithm. Assume that we associate to each node N a potential node π. We then define the reduced cost with respect to π as C^{π} defined by:

$$C^{\pi}ij = C_{ij} + \pi_j - \pi_i$$

if $(i, j) \in A$)

In those algorithms, the reduced cost replaces the cost, especially in the residual network. Thus, it is crucial to understand the relationship between C and C^{π}, with respect to

Lemma 1.5.1. *minimum cost flow problem has the same optimal solutions if we replace the cost C by C^{π}. Furthermore, for any feasible flow X we have:*

$$\sum_{(i.j)\in A} C_{ij}X_{ij} - \sum_{(i.j)\in A} C^{\pi}_{ij}X_{ij} = \sum_{j\in N} \pi_j b_j$$

for b_j is demand at node i

Proof. First, we establish the relationship on the cost:

$$\sum_{(i.j)\in A} C^{\pi}_{ij}X_{ij} = \sum_{(i.j)\in A} C_{ij}X_{ij} + \sum_{(i.j)\in A} X_{ij}\pi_j - \sum_{(i.j)\in A} X_{ij}\pi_i$$

$$= \sum_{(i.j)\in A} C_{ij}X_{ij} + \sum_{j\in NA(i)} \pi_j \sum_{(i.k)\in N-j} X_{ik} - \sum_{i\in NB(j)} \pi_i \sum_{(i.k)\in N+j} X_{kj}$$

$$= \sum_{(i.j)\in A} C_{ij}X_{ij} + \sum_{j\in NA(i)} \pi_j(\sum_{(i.k)\in N-j} X_{ik} - \sum_{(i.k)\in N+j} X_{ik})$$

$$\sum_{(i.j)\in A} C_{ij}X_{ij} - \sum_{j\in N} \pi_j b_j$$

Thus the total decrease in the objective function does not depend on the flow. Therefore if X minimizes the objective function for C, it minimizes the objective function for C^{π}. □

Lemma 1.5.2. *Assume there is a source node $s \in N$ such that every node $n \in N$ is accessible from s. Let d be a potential function and $\pi = -d$. Then if there is no negative cost cycle in the network, the following two conditions are equivalent:*

(i) d_n is the shortest path distance from s to n with respect to C

(ii) $d_s = 0$ and $C^{\pi}_{ij} > 0$ for $(i.j) \in A$

Proof. We will implicitly use the fact that d is well-defined, that is, there is no negative cost cycle.

(i)$\Rightarrow$ (ii) Let $e = (i, n) \in A$. Now write the fact that d_n is the shortest path distance from s to n with respect to C: $d_n \leqslant d_i + C_e \Rightarrow C_e + d_i - d_n > 0$
$\Rightarrow C^{\pi}_{i,n} = C^{-d}_{e} > 0$

$\neg(i) \Rightarrow \neg(ii)$ There are two cases: either $d_s = 0$ and in this case d trivially cannot be the shortest path distance (since the shortest path distance from s to s must be 0. Either $\exists e_0$;such that $C^{\pi}_{e_0} < 0$. By writing $e_0 = (i_0; n_0)$ we get that: $C^{\pi}_{e_o} < 0 \Rightarrow C_{e_0} + d_{i_0} - d_{n_0} < 0$
$\Rightarrow d_{i_0} + C_{e_0} < d_{n_0}$ Which contradicts the fact that d is the shortest path distance function.

□

Theorem 1.5.4. ***(Reduced Cost Optimality Condition)*** *A flow x (of value N) is a minimum cost flow if and only if for some set of node labels π_i, $i \in N$, the reduced cost optimality condition holds in the residual graph $G(x)$:*

$$C^{\pi}_{ij} := C_{ij} - \pi_i + \pi_j \geqslant 0 \quad for\ all\ (i, j)$$

Proof.

- $\Rightarrow$Assume that $x*$ is an optimal solution. By the negative cycle optimality condition theorem (1.5.3), there is no negative cost cycle in $G(x*)$. Thus the shortest path distance d(.) in G(x*) is well-defined.Now apply lemma 1.5.2 to $G(x*)$ and we get that the potential $\pi = -d$ satisfy the reduced cost optimality condition.

- $\Leftarrow$ Assume that there exists a potential node π such that the reduced cost optimality condition is satisfied. We will show that any cycle in G(x*) has nonnegative cost. Then by the negative cycle optimality condition theorem (1.5.3), we will conclude that $x*$ is optimal. Let W be a directed cycle in $G(x*)$. we can link the cost of W in $G(x*)$ to its reduced cost and because all the reduced costs are nonnegative, the cycle must have a nonnegative cost:

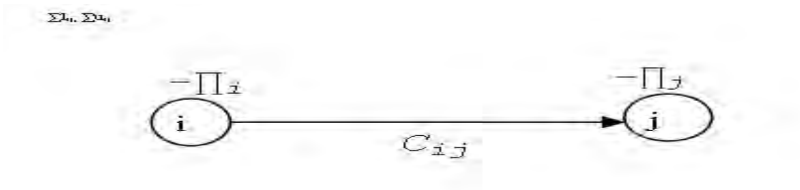
$$\sum_{(ij) \in W} C_{i,j} = \sum_{(i,j) \in W} C^{\pi}_{i,j} \geqslant 0$$

because $C^{\pi} \geqslant 0$ for all $(i, j) \in A$

□

- Then for a given set of node potentials π

$$-\pi_j \leqslant -\pi_i + C_{ij}$$



if and only if The reduced cost of an arc $(i, j)$is

$$C_{ij}^{\pi} = C_{ij} - \pi_i + \pi_j$$

Relating Optimal Flows to Optimal Node Potentials

(1) Given an optimal flow, find node potentials

Let $x*$ = optimal flow

$N(x*)$ = the residual graph w.r.t. $x*$

C_{ij} = the cost of $(i.j)$.

Let d(:) = the shortest path from node i to any node

$$d_j \leqslant d_i + c_{ij} \qquad for\ all\ (i, j) \in N(x*)$$

Let $\pi = -d$

(2) Given node potential, find an optimal flow. Let π be node potential, C_{ij}^{π} be reduce cost

Case 1 : $C_{ij}^{\pi} > 0$; $x*_{ij} = 0$ delete (i, j) from network.

Case 2 : $C_{ij}^{\pi} < 0$; $x*_{ij} = u_{ij}$ delete (i, j) from network and

Case 3 : $C_{ij}^{\pi} = 0$ we allow the flow on arc (i, j) to assume any value between 0 and u_{ij}

1.6 Optimality Test

Complementary Slackness Theorem

Theorem 1.6.1. ***(Complementary Slackness Theorem)*** *A feasible solution x' is an optimal solution of the Minimum cost flow problem if and only if for some set of node potential π, the reduced costs and the values satisfy the optimality conditions:*

(i) *If* $C^{\pi}_{ij} > 0$ *then* $x_{ij} = 0$.

(ii) *If* $C^{\pi}_{ij} = 0$ *then* $0 \leqslant x_{ij} \leqslant u_{ij}$

(iii) *If* $C^{\pi}_{ij} < 0$ *then* $x_{ij} = u_{ij}$.

Proof. For $(i, j) \in A$, set $C^{\pi}_{ij} = C_{ij} - \pi_i + \pi_j$.
Then, for every feasible flow x'.
$\sum_{(ij)\in A} C^{\pi}_{ij} x'_{ij} = \sum_{(ij)\in A} C_{ij} x'_{ij} - \sum_{i\in N} \pi_i (\sum_{j:(i,j)\in A} x'_{ij} - \sum_{j:(i,j)\in A} x'_{ji}) = \sum_{(ij)\in A} C^{\pi}_{ij} x'_{ij}$
$\sum_{(ij)\in A} C_{ij} x'_{ij} = \sum_{(ij)\in A} C^{\pi}_{ij} x'_{ij}$ and as of *lemma*1.5.1 above' □

(1) . if node potential π and flow vector x satisfy the reduced cost optimality conditions, then they must satisfy $i - iii$

(2) . if the flow vector x and node potential π satisfy the complementary slackness conditions, then they satisfy the reduced cost optimality conditions.

Now let use a basis structure of the network problem and define by partitioning the set of arc A, into three disjoint sets, as follows:
B: the set of basic arcs, which form a spanning tree of the network,
L: the set of non-basic arcs at their lower bounds,
U: the set of non-basic arcs at their upper bounds,
Then We suppose the network contains no negative cost augmenting cycles and start of finding the reduced costs for each arcs. Given a basis structure $(B; L; U)$ we compute node potentials as follows,

(1) Set $\pi_i(root) = 0$

(2) Find an arc (i, j) in B for which one node potential is known and the other not. If $\pi_(i)$ is known, then set

$$\pi_j = \pi_i - c_{ij}$$

If π_j is known, then set

$$\pi_i = \pi_j + c_{ij}$$

(3) Repeat (2) until all π are known.

All π values will be found since there are $n-1$ arcs in B, and they form a spanning tree. Suppose there is some arc (l,k) in L or U for which $C_{ij}^{\pi} < 0$. We can add basic arcs to form a cycle W with arc (l,k) such that: the simplex algorithm for a Minimum cost flow problems moves from one feasible spanning tree structure (B,L,U) to another until optimality is achieved.
The simplex algorithm always maintains the condition that the reduced cost for basic variables is 0; i.e., $C_{ij}^{\pi} = 0$ for all $(i,j) \in B$, for certain node potential π Given the current spanning tree structure (B,L,U) , the method first determines the value of the node potential π that satisfies

$$c_{ij}^{\pi} = c_{ij} - \pi_i + \pi_j = 0 \quad for\ all\ (i,j) \in B$$

$$\pi_i - \pi_j = c_{i,j}, \quad for\ all\ (i,j) \in B$$

Since adding a constant on every node potential does not change the reduced cost of arcs, we start with setting any one node potential to 0, say $\pi_1 = 0$
In this case node 1 is called the root node. Thus if $G = (N,A)$ describes the graph of the problem, then A = $B \bigcup L \bigcup U$, and (B,L,U) describes a Basic feasible solution. In a problem in which all lower flow bounds are zero, a basic solution will be feasible if the flow, x_{ij} , in each arc (i,j) in A, is as follows:

$$X_{ij} = \left\{ \begin{array}{l} 0 : if \quad (i,j) \in L \\ U_{ij} \quad if \quad (i,j) \in U \\ 0 \leqslant x_{ij} \leqslant u_{ij} \quad Otherwise \end{array} \right\}$$

The simplex method is a one procedure that moves from one point to another extreme point having a better objective value. It also discovers whether the feasible region is empty and whether the optimal objective value is unbounded. In practice, the method only enumerates a small portion of the extreme points of the feasible region.
The key to the simplex method lies in recognizing the optimality of a given extreme point solution based on local considerations without having to (globally) enumerate all extreme points or basic feasible solutions.

The Simplex Method for Solving Capacitated Transportation Problem: The general steps in the application of the simplex method to a linear program are as follows:

Step1: . Find a starting basic feasible solution.

Step2: Let us now define For any basic arc (i, j), such that

$$\pi_j = \pi_i + C_{ij}$$

That is, we break even if we purchase a unit at market i, transport it along the arc (i, j), and sell it at market j. thus at this step compute

$$\pi_j - C_{ij} \text{ or } \pi_i + C_{ij}$$

for each nonbasic variable. Stop or select an entering column.

Step3: Determine an exiting column.

Step4: Obtain the new basic feasible solution and repeat Step 2.

We shall show how each of these steps can be carried out directly on the transportation tableau. Network simplex method iterates towards the optimal solution by exchanging basic with non-basic arcs and adjusting the flows accordingly. The algorithm terminates when a basic feasible solution is reached which satisfies the reduced cost optimality conditions. A feasible flow in a spanning tree structure (B,L,U) is optimal for the Minimum cost flow problem if for some node potential π, the reduced cost C_{ij}^{π} satisfy the following conditions:

$$\begin{aligned} c_{ij}^{\pi} &= 0, \qquad for\ all\ (i, j) \in B \\ c_{ij}^{\pi} &\geqslant 0, \qquad for\ all\ (i, j) \in L \\ c_{ij}^{\pi} &\leqslant 0, \qquad for\ all\ (i, j) \in U \end{aligned}$$

Step 0: Initialisation

a) First, all non-zero lower bounds are removed so that $l_{ij} = 0$ for all (i, j) in A.

b) Determine a node potential π

Step 1: Start

a) If the current flow in (B, L, U) satisfies the Optimality Test, STOP.

b) Otherwise, Go TO Step 2.

Step 2: Select an Entering Arc and Determine the Leaving Arc.

An arc satisfying the following condition are candidates for entering arc.

a) Any arc $(r, s) \in L$ such that $c^{\pi}_{rs} < 0$

b) Any arc $(r, s) \in U$ such that $c^{\pi}_{rs} > 0$ If $(r, s) \in L$ with $c^{\pi}_{rs} < 0$ is the entering arc. then it forms a cycle W and c^{π}_{rs} is the cost of the pivot cycle w.r.t B, oriented W such that (r, s) is forward in it.

If (r, s) in U with $c^{\pi}_{rs} > 0$ is the entering arc, then c^{π}_{rs} is the cost of the pivot cycle

w.r.t B, oriented W such that (r, s) is backward in it.

Step 3: If one of the arcs in B, say (p, q), becomes non-basic (out-of-tree) i.e., (p, q) moves either into L or U and we have a new (B, L, U) structure. Thus, update B, L, U, and the node potential π.

1.6.1 Updating Node Potential

Let (r, s) be an entering arc and (p, q) in B be a leaving arc. Removal of the arc (p, q) from B decomposes B into two sub trees T_1 and T_2 , where T_1 denotes the sub tree containing the root node s of T. The entering arc (r, s) connects T_1 with T_2 and forms new spanning tree T^*. Now to update the node potential π to a new potential π^*.

Since root node s in T_1, there is no change of π_i for all i in T_1. i.e., $\pi^*_i = \pi_i$ for all i in T_1. But, the node potential of vertices in T_2 will change by a constant δ i.e., $\pi^*_i = \pi_i + \delta$ for all $i \in T_2$. where $\delta = -c^{\pi}_{rs}$

Thus,

$$\pi^*_i = \left\{ \begin{array}{lll} \pi_i & for & r \in T_1 \\ \pi_i - c^{\pi}_{rs} & for & r \in T_2 \end{array} \right\}$$

1.6.2 Numerical Example

Consider the above example stated above in formulating initial basic feasible solution of capacitated transportation problem.Consider Company **R** needs to transport certain commodity from different centers(A_1**,** "A_2" **and** "A_3") 12, 24 and 33 quantities of the commodity. while "B_1", "B_2", "B_3" and "B_4" wants to receive 10, 18, 17 and 24 quantities respectively from this supply centers as indicated in the following cost table.

Source	B_1	B_2	B_3	B_4	Capacity
A_1	$C_{11} = 27$	$C_{12} = 69$	$C_{13} = 90$	$C_{14} = 20$	$a_1 = 12$
A_2	$C_{21} = 2$	$C_{22} = 42$	$C_{23} = 40$	$C_{21} = 80$	$a_2 = 24$
A_3	$C_{31} = 80$	$C_{32} = 12$	$C_{33} = 60$	$C_{34} = 24$	$a_3 = 33$
Demand	$b_1 = 10$	$b_2 = 18$	$b_3 = 17$	$b_4 = 24$	$\sum_i a_i = \sum_i b_i = 69$

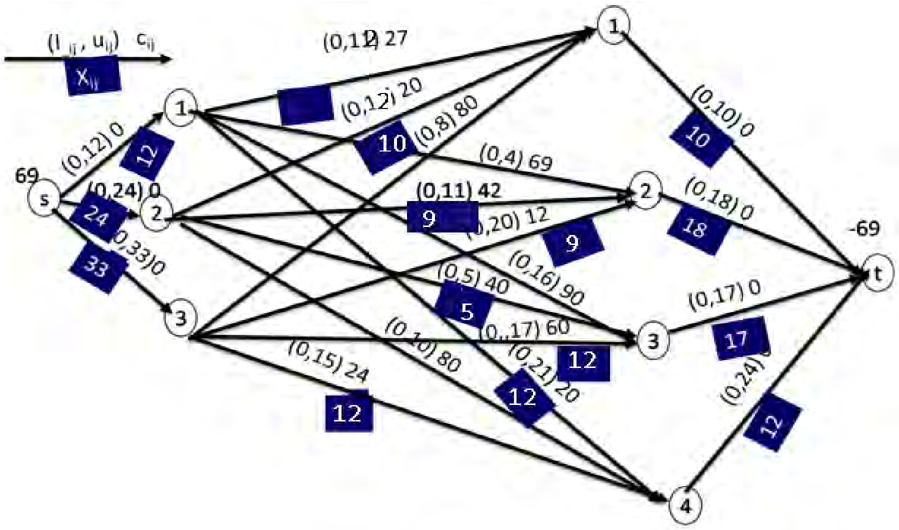
$0 \leqslant X_{11} \leqslant 20$, $0 \leqslant X_{12} \leqslant 4$, $0 \leqslant X_{13} \leqslant 16$, $0 \leqslant X_{14} \leqslant 21$

$0 \leqslant X_{21} \leqslant 12$, $0 \leqslant X_{22} \leqslant 9$, $0 \leqslant X_{23} \leqslant 5$, $0 \leqslant X_{24} \leqslant 10$

$0 \leqslant X_{31} \leqslant 8$, $0 \leqslant X_{32} \leqslant 20$, $0 \leqslant X_{33} \leqslant 17$, $0 \leqslant X_{34} \leqslant 15$

This can be formulated as Minimum, flow problem. as we can see from the network graph given below.

After Removal of all $l'_{ij}s$ and $U'_{ij}s$ the basic feasible allocations are as indicated below



Allocations.png

Figure 1.3: Network graph with basic allocations as given.

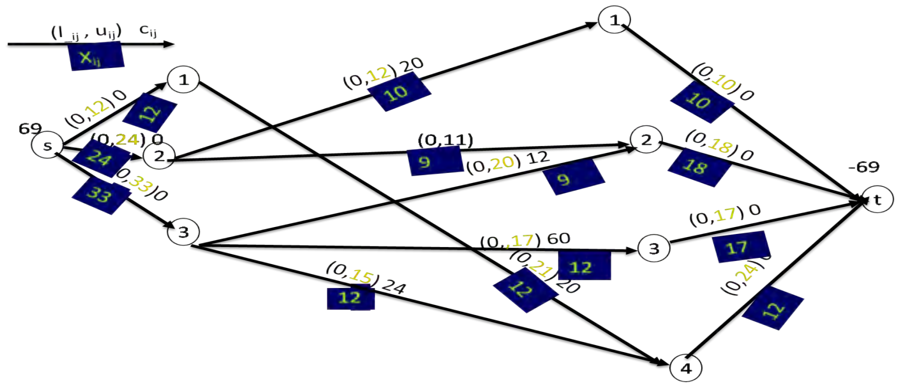


Figure 1.4: Basic feasible allocations.

$$B = (1,4), (2,1), (2,2), (3,2), (3,3), (3,4)$$

$$L = (1,1), (1,2), (1,3), (2,4), (3,1)$$

$$U = (2,3)$$

Now let us find each potential nods: Assigning the π_{a_1} as root node.

$\pi_{b_4} = -20$

$\pi_{a_3} = 4$

$\pi_{b3} = -56$

$\pi_{b_2} = -8$

$\pi_{a_2} = 34$ and $\pi_{b_1} = -33$

And then let find $C^{\pi_{ij}}s$ for set of Us and set Ls

$C^{\pi}_{11} = C_{11} + \pi_{b1} - \pi_{a1} = -6$ Violating arc

$C^{\pi}_{12} = C_{12} + \pi_{b2} - \pi_{a1} = 61$

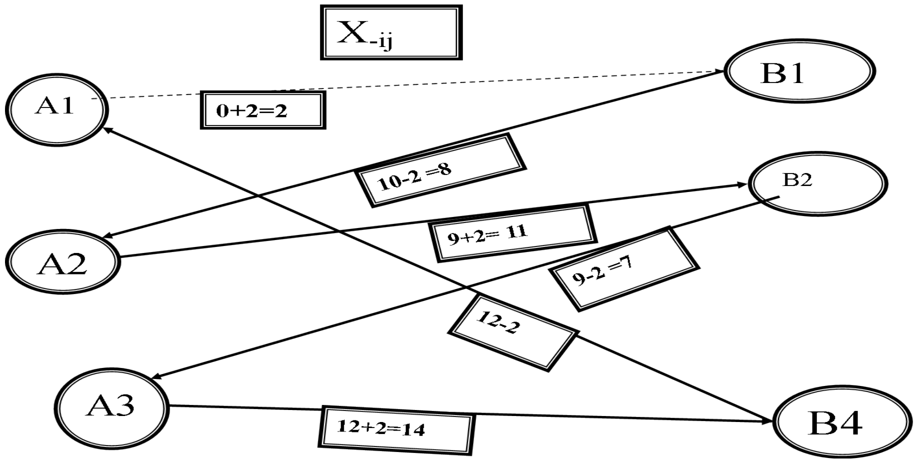
$C^{\pi}_{13} = C_{13} + \pi_{b3} - \pi_{a1} = 34$

$C^{\pi}_{23} = C_{23} + \pi_{b3} - \pi_{a2} = -50$

$C^{\pi}_{24} = C_{24} + \pi_{b4} - \pi_{a2} = 26$

$C^{\pi}_{31} = C_{31} + \pi_{b1} - \pi_{a3} = 148$

Now using the entering arc(1,1) and min(2, 9, 3,10,12) to add on the forward and subtract from the backward arcs.then the Cycle becomes



formed.png

Figure 1.5: Cycle.

From this, since arc (2,2) is get saturated, consider it is leaving arc. The new basic allocation is then

L= (1,2), (1,3), (2,4), (1,3), U= (2,2), (2,3) and B= (1,1), (1,4),(2,1), (3,2), (3,3), (3,4)

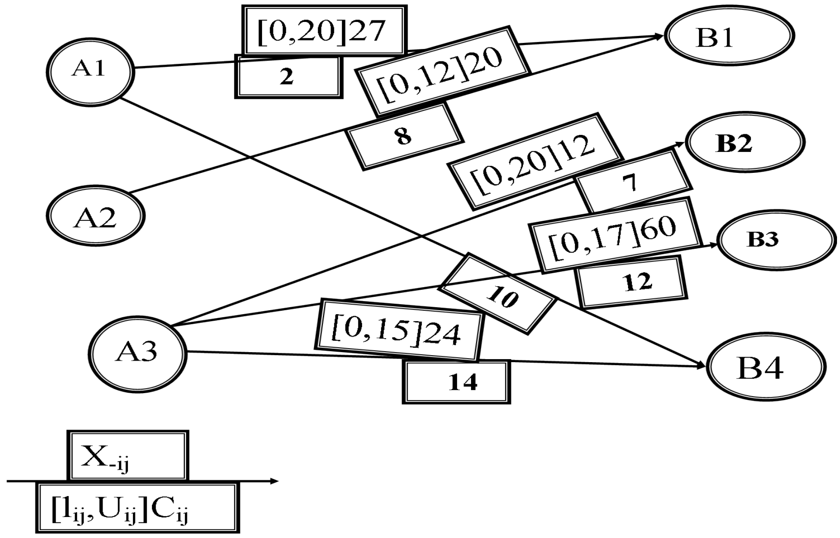


Figure 1.6: Re-Cycled.

The new potential nodes are then, $\pi_{b_4} = -20$, $\pi_{a_3} = 4$, $\pi_{b3} = -56$, $\pi_{a_2} = -7$, $\pi_{b_1} = -27$

Following find the updated cost residue,like, $C^{\pi}_{12} = 61$, $C^{\pi}_{13} = 34$, $C^{\pi}_{22} = 41$(Violating Arc), $C^{\pi}_{23} = -9$, $C^{\pi}_{31} = 109$, $C^{\pi}_{24} = 11$

10*20 + 9*42 + 9*12 + 12*60 + 12*20 + 24*12= 1,934 birr.

Then the second allocation is optimal solution.

Chapter 2

UnFixed Supply and Demand Capacitated Transportation Problem

The Capacitated transportation problem we saw above is a type in which need and supply are fixed a_i and b_i respectively. Now in this section we will see a CTP in which need and capacities are also bounded in some range. i.e $a_i \leqslant \sum_{j=1}^{n} X_{ij} \leqslant A_i$ and $b_j \leqslant \sum_{i=1}^{m} X_{ij} \leqslant B_j$
This model is used in the situation at which the demand and supply units are not fixed to exactly fixed number. Now may case in this paper is formulating this case of CTP to minimum cost flow problem and finding optimal solution.

2.1 Problem Formulation of Bounded Rim CTP

Consider a capacitated transportation problem given by

$$\left\{ \begin{array}{l} minimize \;\; Z = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij} \\ \text{ Subjected to} \\ \\ a_i \leqslant \sum_{j=1}^{n} X_{ij} \leqslant A_i \quad \text{for all} \quad i = 1:m \\ \\ b_j \leqslant \sum_{i=1}^{m} X_{ij} \leqslant B_j \quad \text{for all} \quad \text{j=1:n} \end{array} \right\} = (P1).$$

$$l_{ij} \leqslant X_{ij} \leqslant U_{ij} \text{ for all i,j}$$

Term expressions

X_{ij} :number of units transported from origin i to destination j

C_{ij} :the cost of transporting a single unit form i origin to j destination

l_{ij} : the least amount ofX_{ij} to be transported to j destination from i source

U_{ij} :the maximum amount ofX_{ij} to be transported to j destination from i source

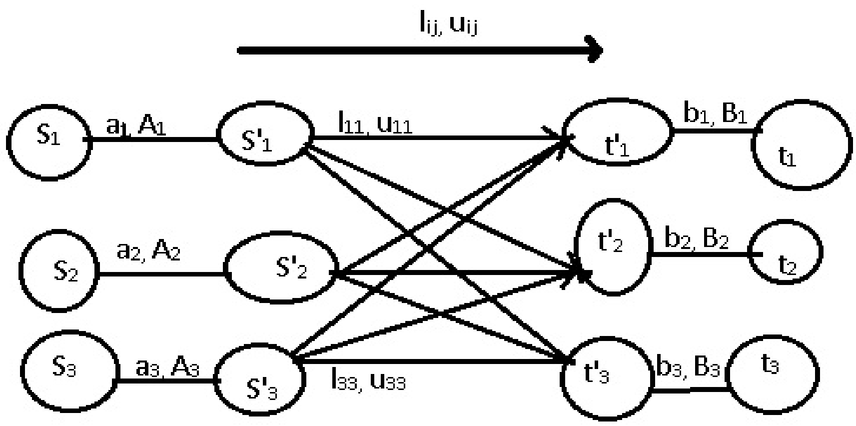
a_i, A_i :are bound of availability on i^{th} source

b_j, B_j :bound of demands onj^{th} destination

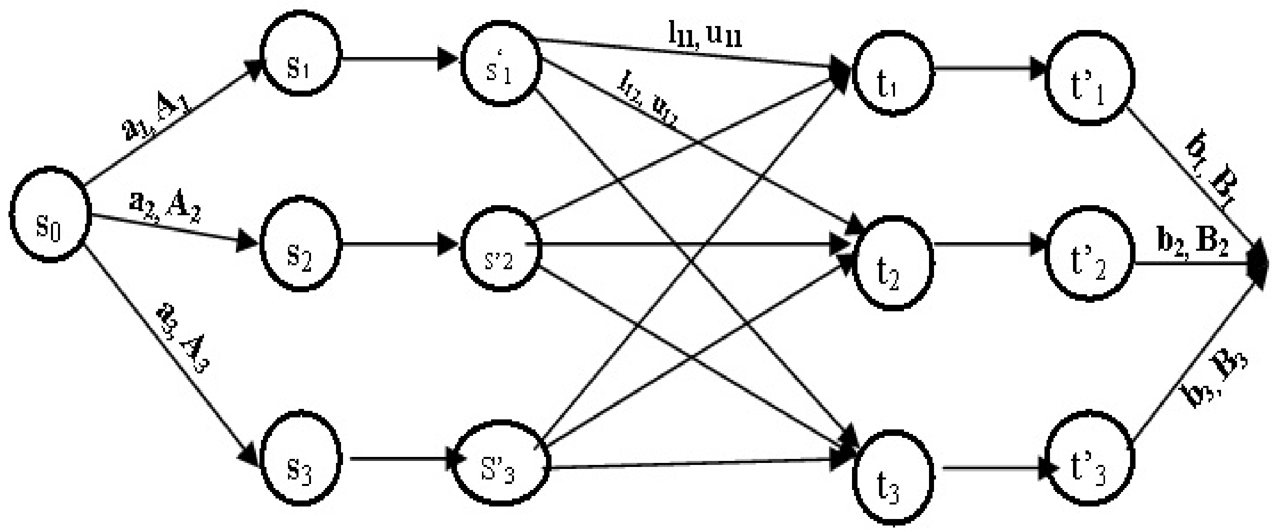
2.1.1 Formulating Bounded Rim Condition Of CTP As Network

To change this case of CTP to network flow form let first use as of done above. Then we need to define other node to consider the capacity and demand restriction. So, for supply node s_i define another node s_i' with arc (i, i') and define node t_j' for demand place t_j with arc (j,j')
The following graph will tell you graphically

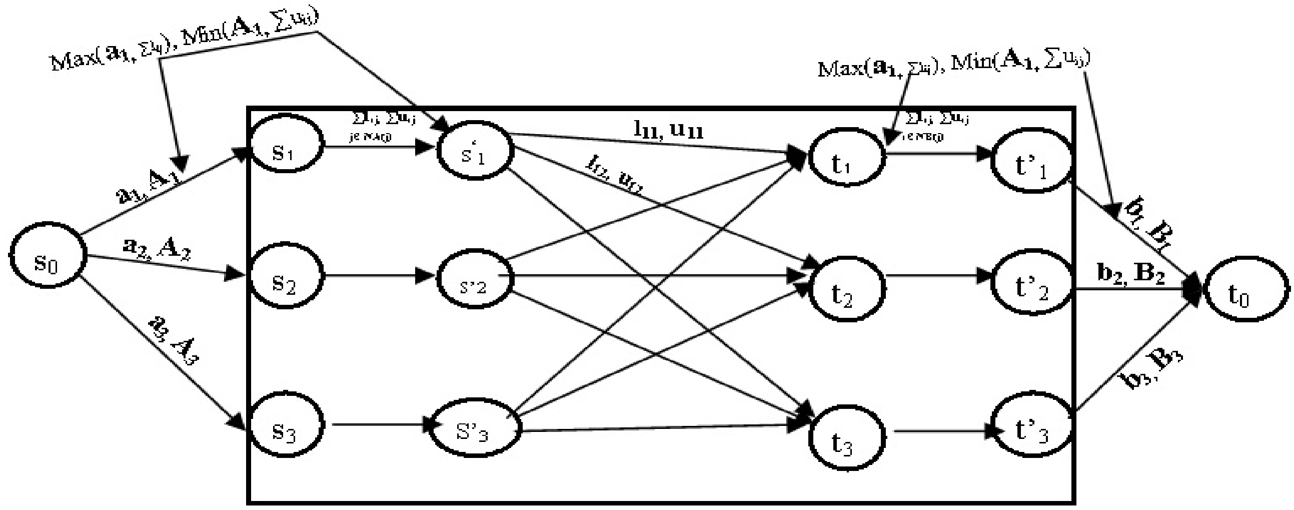
Since every transportation problem involves single sink and source node say t and s respectively



the following graph again tells how to assign the new nodes. These new nodes form an arc with those previous nods with weight of capacity given initially for supply nodes. The graph will tell you briefly. +6



To start allocations at each arc we first need to define the minimum and maximum flow at each arc. But as we see on the graph there an arcs with single node which doesn't has other direction to loose sink unit. Thus the minimum of both arcs is the maximum of the minimums and the maximum of both arcs is the minimum of maximums. See this graph. To find the basic



feasible allocations for this network flow problem first give it all the minimum cost for each arc, Then go to the minimum cost arc and assign as it will be. At last test the optimality using Network simplex algorithm seen above if the elements in B are n+m-1 arcs unless the current allocation is optimal

Numerical Example

Let three demand places need to transport their product three different demand places.Having the following restrictions on the quantities of product. The costs, demand and supply restriction are indicated in the table

Source	B_1	B_2	B_3	Capacity($a_i \leqslant A_i$)
A_1	5	9	9	$3 \leqslant \sum_{j=1}^{3} X_{1j} \leqslant 30$
A_2	4	6	2	$10 \leqslant \sum_{j=1}^{3} X_{2j} \leqslant 40$
A_3	2	1	1	$10 \leqslant \sum_{j=1}^{3} X_{3j} \leqslant 50$
Demand	$5 \leqslant \sum_{i=1}^{3} X_{i1} \leqslant 30$	$5 \leqslant \sum_{i=1}^{3} X_{i2} \leqslant 20$	$5 \leqslant sum_{i=1}^{3} X_{i3} \leqslant 30$	

With the following flow restrictions

$1 \leqslant x_{11} \leqslant 10$, $2 \leqslant x_{12} \leqslant 10$, $0 \leqslant x_{13} \leqslant 5$,$0 \leqslant x_{21} \leqslant 15$, $3 \leqslant x_{22} \leqslant 15$, $1 \leqslant x_{23} \leqslant 20$, $0 \leqslant x_{31} \leqslant 20$, $0 \leqslant x_{32} \leqslant 13$, $0 \leqslant x_{33} \leqslant 25$

Solution Method

- step1 draw graph as formulated above, then assign minimum value across each arc

- step2 if the sum of $\sum_{j \in NA(i)} l_{ij}$ < maximum of minimum value of arcs before i, then assign the difference to the most minimum (ij) putting the restriction under consideration

- step3 Check wether all minimum value conditions are satisfied, if not($\sum_{i \in NB(i)} l_{ij}$ <) the minimum value of arcs leaving from node j, assign to minimum cost to that node the amount remain to satisfy the minimum requirement from any node pass over the minimum requirement. Then the following graph is according to the given example.

- step4 assign the remaining values to the most minimum transportation cost, till it becomes saturated or block flow, from this example $C_{ij} = 1$ at arc (3,3) minimum of 0 and maximum of x+6(the minimum value assigned before)=30-7(already assigned value on arc(2,3))unit can pass across this arc thus assign the 17+6=23unit to this arc and subtract 23 unit from the maximum capacity of past node and the next node.

Continuing in such a way I get the following allocated graph

Thus the basic feasible allocation arcs are given as

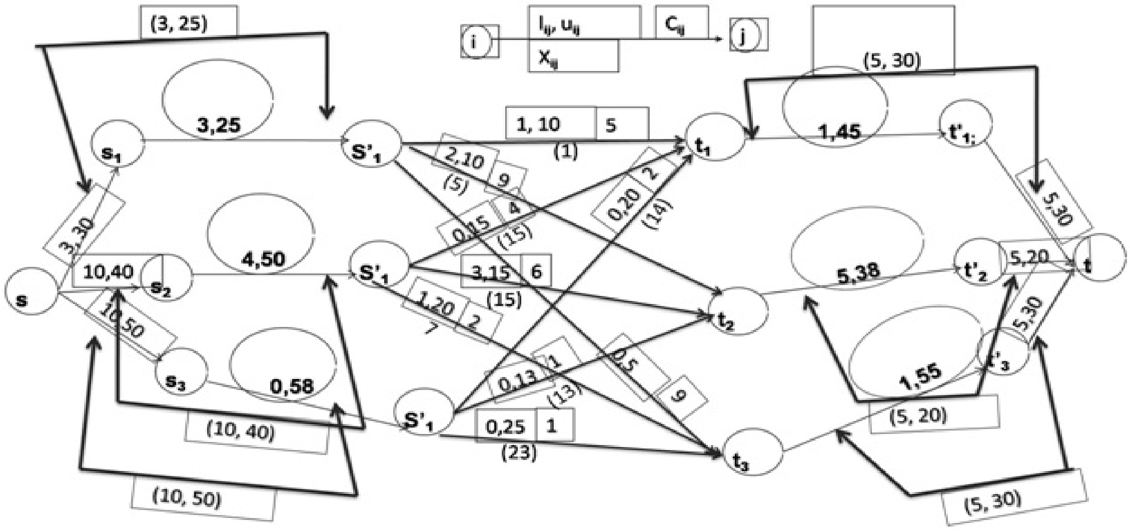


Figure 2.1: Allocated graph.

Then by using the above indicated Network simplex algorithm we can test the optimality of

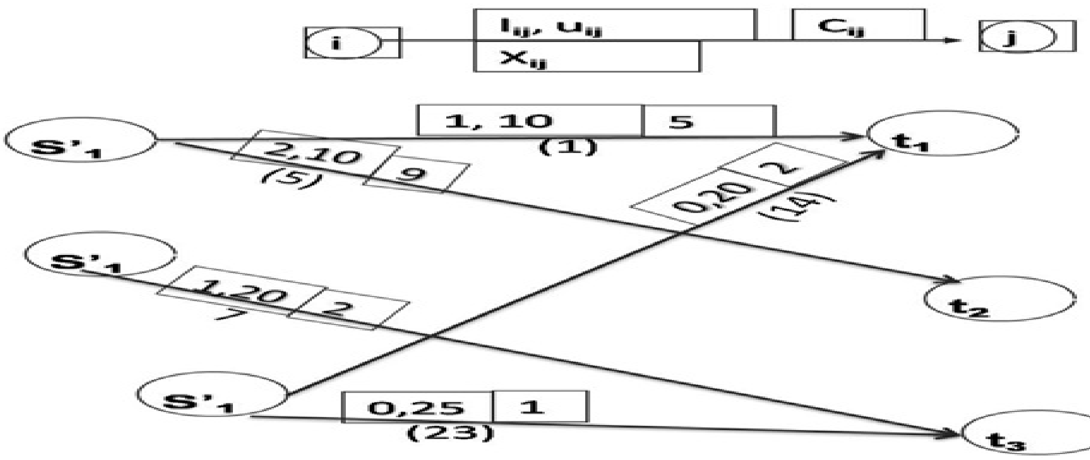


Figure 2.2: Basic feasible allocation arcs.

the CTP.

Chapter 3

Summary

- Transportation Problem(TP), Linear programming problems involve the optimization of a linear function, called the objective function, subject to linear constraints

- **capacitated** transportation problem is a transportation problem of the case in which the capacity of supply and demand places are restricted to deliver and receive only certain amount of quantities.

- To find solution of Capacitated transportation problem firstly find the feasible basic solution of the given problem, then having that as allocations find final solution using, Network simplex method as a minimum cost flow problem

- The Minimum Cost Flow (MCF) Problem is to send flow from a set of supply nodes, through the arcs of a network, to a set of demand nodes, at minimum total cost, and without violating the lower and upper bounds on flows through the arcs.

- **Transportation problem:** can be formulated as a minimum cost flow problem
  The following are basically used in this paper.

- Define Tp,CTP, Mathematically and graphically

- Finding initial feasible solution of CTP

- Finding finial or optimal solution of CTP
- Testing optimality of the optimal solution
- consider the unfixed case of CTP

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