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Game on Zero Sum

By

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Abstract

A game with two rational players in which the gain payoff for one is loss for the other is called two person zero-sum game. i.e the sum of payoffs for the two players are zero. Two person zero-sum game with finite sets of strategies are called matrix games. Rational players always seeks to maximize his payoff by choosing a best strategy. If the matrix of the game is payoff for Player 1, then player 1 at worst case guarantee himself to maximize the minimum loss of player 2. Similarly player 2 at worst case guarantee himself to minimize the maximum payoff player 1. Any mixed matrix game has optimal solution which is called saddle point in mixed strategies.

This project is focuses only on two person zero-sum game part of Game Theory with finite player strategies and present how to find the optimal value of the game or optimal solution strategies(saddle point) of the players. To find optimality solution method of primal(dual) linear programming problem and dominance strategy methods are used. The objective is to find the optimal strategies of the players in two person zero-sum game and optimal value of the game.

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Chapter 1

Introduction and Preliminary Concepts on Linear Programming

1.1 What is a Game?

Decision making is a rational process that results in selecting 'the best' of several alternatives, as judged by the individual or group of individuals who makes decision we call these decision makers, to realize the targeted objective. A game is type of decision making problem where the outcome depends on the action of more than one decision makers. It is a formal description of a strategic situation. The decision makers in game with finite or infinite strategies or alternatives are known as players. The decision makers in the game cooperate they do for the same objective or they do for not the same objective. In game there is a winner and lessor for players. A Game Theory is the study of what rational agent(player) do in such situation and is the study of decision-making where more than one participants(players) must make decision that influence the interests of the other agents.

A player is an agent who makes decisions in a game with the opponent or cooperater. A player is rational if he want to play in a manner which maximizes his own outcome payoff by assuming that the opponent or cooperater is acting in the best (for himself) possible way with out cheating. Strategy of player is the alternative from the given set of possible action to the player. A payoff is a number, also called utility, that reflects the desirability of an outcome to a player, for whatever reason. It is a measure of value, gain or loss of a player under a given alternatives(strategy). When the outcome is random, payoffs are usually weighted with their probabilities.

Game Theory applied in different area such as economics, politics, business, war and other areas can be analyzed by mathematical game theory. Mathematical games have strict rules and specify what is allowed and what is not. Games that can be analyzed mathematically have a fixed set of moves, usually all known in advance. Mathematical games may have many possible outcomes, each producing payoffs for the players. The payoffs may be measure in monetary, or it gives satisfaction for the players. Players play move to win the outcome of the game. The outcome of game is thrill and cannot predicted in advanced. Since its rules are fixed and known, this implies that a game must either contain some random elements or have more than one player. A game is always with decisions, if not boring for the mind. For example, most sport games have a result decisions, and can therefore at least partly be analyzed by game theory. Game in real-life cheating may possible. Cheating means not playing by the allowed and restricted rules. Game theory does not give acknowledge to the existence of cheating. We will see how to win the game without cheating with only two players opposite interest. It is assumed that the rationality the players is known by all players.

1.2 Linear Programming(LP)

Linear programming is deal with the optimization (minimization or maximization) of a linear function while satisfying a set of linear equality and/or inequality constraints or restrictions.
Linear programming(LP) has the following components.
Objective function: Mathematical expression of the objective of the LP.
Decision variables: Represents unknown quantity to be solved for.
Constraints: Mathematical relationship which is used to represent the restriction/limitation.
Parameters: Are numerical or fixed values that specify the impact that one unit of each decision variable will have on the objective and on any constraint.
Example

$$max\ z = x_1 + x_2 \longleftarrow objective\ function$$

$$\left.\begin{aligned} subject\ to\ 7x_1 + 3x_2 \quad &\leq 1 \\ 2x_1 + 5x_2 \quad &\leq 1 \end{aligned}\right\} \text{constraints}$$

$$x_1 \geq 0, x_2 \geq 0 \longleftarrow decision\ variables$$

For our purpose we only focus on maximization $\leq$ types of constraint linear programming(LP). To solving such problems there are different mechanisms but in this project we will see preliminaries of simplex and dual simplex methods.

1.2.1 Simplex Method

Solving problems involving more than two variables or problems involving a large number of constraints, it is better to use solution methods that are adaptable to computers, one such method is called the simplex method.

Maximizing linear programming problem in standard form is as follow

$$\begin{aligned} \max\ z = c_1x_1\ +\ c_2x_2\ +\ldots+\ c_nx_n \\ s.t\ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \\ \ldots \qquad\qquad \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \\ x_1 \geq 0, x_2 \geq 0, \ldots x_n \geq 0,\ b_i \geq 0. \end{aligned}$$

Note that $b = [b_1\ b_2 \ldots b_m]^T$ is right hand side non negative constant column vector of the constraint and $c = [c_1\ c_2 \ldots\ c_n]$ is the row vector coefficient of the objective function through out this project. We can rewrite this LP as below

$$\begin{aligned} \max cx^T \\ \text{s.t } Ax^T \leq b, \end{aligned}$$

Where $x = [x_1\ x_2 \ldots x_n]$ and $m \times n$ matrix A is the constant coefficient of the decision variable in the constraint.
After adding slack variables, to balance the right and left hand side of the constraints the corresponding system of constraint equations is

$$\begin{aligned} \max\ z = c_1x_1\ +\ c_2x_2\ +\ldots+\ c_nx_n + 0s_1 + 0s_2 + \ldots + 0s_m \\ s.t\ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + s_1 = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + s_0 = b_2 \\ \ldots \qquad\qquad \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n + s_m = b_m \end{aligned} \tag{1.1}$$

Where $x_1 \geq 0, x_2 \geq 0, \ldots x_n \geq 0, s_1 \geq 0, \ldots, s_m \geq 0,\ b_i \geq 0,\ i = 1, 2, \ldots, m.$
A basic solution of a linear programming problem in standard form is a solution $(x_1, x_2, \ldots, x_m, s_1, s_2, \ldots, s_m)$ of the constraint equations in which at most m variables are nonzero. The variables that are nonzero are called basic variables. The variables that are zero are called non-basic variables. A basic solution for which all variables are non-negative is called a basic feasible solution.
The Simplex Tableau: The simplex method is carried out by performing elementary row operations on a matrix that we call the simplex tableau. This

tableau consists of the augmented matrix corresponding to the constraint equations together with the coefficients of the objective function written in the form

$$-z + c_1x_1 + c_2x_2 + \ldots + c_nx_n = 0.$$

And put the coefficient of this linear equation in to the initial tableau as below. Note that the coefficient of z not need any elementary row operation it is simply to indicate the value of the objective function at the current tableau is the negative of the entry at the first row right corner of the tableau.

Initial tableau:

	x_1	x_2	$\ldots$	x_n	s_1	s_2	$\ldots$	s_m	b
$-z$	c_1	c_2	$\ldots$	c_n	0	0	$\ldots$	0	0
s_1	a_{11}	a_{12}	$\ldots$	a_{1n}	1	0	$\ldots$	0	b_1
s_2	a_{21}	a_{22}	$\ldots$	a_{2n}	0	1	0	$\ldots$	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s_m	a_{m1}	a_{m2}	$\ldots$	a_{mn}	0	0	$\ldots$	1	b_m

(1.2)

For the initial simplex tableau, the basic variables are $s_1, s_2, \ldots, s_m$ (since in the initial tableau the coefficient constraint matrix make identity matrix under these variables) and these are the current solution. The nonbasic variables (which have a value of zero) are $x_1, x_2, \ldots, x_n$.

If the current solution is not optimal, improving the current solution. To improve the current solution, we take a new basic variable into the solution from the non-basic variable. This variable is call the entering variable. This implies that one of the current basic variables must leave, otherwise we would have too many variables for a basic solution. We call this variable the departing variable. We choose the entering and departing variables as follows.

1. The entering variable corresponds to the greatest(the most positive) entry in the first(object) row of the tableau.

2. The departing variable corresponds to the smallest non-negative ratio of b_i/a_{ij}, in the column determined by the entering variable(j indicates the column of the entering variable).

3. The entry in the simplex tableau in the entering variable's column and the departing variable's row is called the pivot.

Finally, by performing elementary row operations on a matrix to the column that contains the pivot, we improved solution. (This process is called pivoting.) To solve a linear programming problem (1.1) in standard form, we use the following steps.

1. Create the initial simplex tableau as in (1.2).

2. Locate the most positive(greatest) entry in the first(objective) row. The column for this entry is called the entering column. (If ties occur, any of the tied entries can be used to determine the entering column).

3. Form the ratios of the entries in the b-column with their corresponding positive entries coefficient in the entering column j. The departing row corresponds to the smallest non-negative ratio, b_i/a_{ij}. (If all entries in the entering column j are non positive, then there is no entering variable at this column. For ties, choose either entry.) The entry in the departing row and the entering column is called the pivot.

4. Use elementary row operations so that the pivot is 1, and all other entries in the entering column are 0. This process is called pivoting.

5. If all entries in the first(objective) row are non-positive, this is the final tableau. If not, go back to Step 2.

6. If we obtain a final tableau, then the linear programming problem has a maximum solution, then the optimal value is the negative of the entry in the first(objective) row right corner of the tableau.

For solving minimization LP problem we use an equivalence relation with the LP in (1.1). i.e. $\min z = cx^T \ s.t \ Ax^T \leq b \ \Leftrightarrow -\max z = -cx^T \ s.t \ Ax^T \leq b$.

1.2.2 Primal and Dual LP Problem

Associated with each linear programming problem there is another linear programming problem called the dual. The dual problem is an LP defined directly and systematically from the primal (or original) LP model. The two problems are so closely related that the optimal solution of one problem automatically provides the optimal solution to the other. Let as denote primal LP by (P) and the dual by (D).

Suppose that the primal linear program is given by

$$\begin{aligned} &\max z = cx^T \\ (P) \quad &\text{s.t } Ax^T \leq b \\ &x \geq 0 \end{aligned} \tag{1.3}$$

Then the dual of (P) is given by

$$\begin{aligned} &\min w = yb \\ (D) \quad &\text{s.t } yA \geq c^T \\ &y \geq 0, \end{aligned} \tag{1.4}$$

where y is any row vector decision variable for the min LP(or D).
From (1.3) and (1.4) $cx^T \leq yAx^T \leq yb$, for any feasible solution x of (P) and y of (D), since $Ax^T \leq b$ and $yA \geq c^T$. This indicates that the objective function value of (D) for any feasible solution (D) is upper bound set of the objective function value of (P) for any feasible solution (P) and the objective function value of (P) for

any feasible solution (P) is lower bound set of the objective function value for any feasible solution (D). The problem (P) always depends on maximize the minimum value and (D) minimize the maximum value.

Theorem: Duality Theorem.

If both problems (P) and (D) have feasible solution then both of them have optimal solution x^*, y^*, respectively, and

$$cx^{*T} = y^*b$$

Proof. *Assume (P) and (D) have feasible solution set and denote by* $S = \{\ x|\ Ax^T \leq b\}$ *and* $\overline{S} = \{\ y|\ yA \geq c\}$ *respectivly. Since (D) has feasible solution* $\overline{S}$*, then the objective function of (P) has an upper bound set* $U = \{\ by|\ y \in \overline{S}\}$*. Any set that have an upper bound set and also have least upper bound. The least upper bound of the objective function value of (P) in* S *is the minimum of* U*, i.e.* $\min U = \min by$*, there exit* $u \in U$ *such that* $u = y^*b = \min U = \min yb$ *for some* $y^* \in \overline{S}$*. The least upper bound of the objective function (P) in* S *is* $u = y^*b$ *i.e.* $\max\ cx^T = y^*b$ *for all* $x \in S$*, this implies*

$$\max\ cx^T = y^*b = \min U = \min yb \tag{1.5}$$

for $x \in S$ *and* $y \in \overline{S}$*.*

Similarly, the objective function value of (P) in S *is the lower bound of the objective function value (D) in* $\overline{S}$*. Let* L *be the set lower bound of the objective function value (D) in* $\overline{S}$ *and given by* $L = \{\ cx^T|x \in S\}$*. A set that have lower bound set and also have greatest lower bound. So, the greatest lower bound of the objective function value (D) in* $\overline{S}$ *is* $\max L = \max cx^T$*, there exist* $l \in L$ *such that* $l = cx^{*T} = \max L = \max cx^T$ *for some* $x^* \in S$ *and for all* $x \in S$*, then greatest lower bound of the objective value (D) in* $\overline{S}$ *is*

$$maxL = cx^{*T} = \min yb, \tag{1.6}$$

for all $y \in \overline{S}$*. Therefore from (1.5) and (1.6) we have* $cx^{*T} = y^*b$ *and hold at* $x^* \in S$ *and* $y^* \in \overline{S}$ □

Optimal Dual Solution: The primal and dual solutions are so closely related that the optimal solution of either problem directly yields (with little additional computation) the optimal solution to the other. In the starting tableau, (1.2) the constraint coefficients under the starting basic variable form an identity matrix. With this arrangement, subsequent iterations of the simplex tableau generated by the elementary row operation will modify the elements of the identity matrix until we get optimal tableau to produce what is known as the inverse matrix and denote by B^{-1} (i.e. is the inverse of the original coefficient matrix of the basic variable as they appear in the base). Let y^* be the optimal solution of (D). The optimal solution y^* of (D) can be determined from the final optimal tableau of the primal (P) by

$$y^* = C_B B^{-1}, \tag{1.7}$$

where C_B is the original objective coefficients of optimal primal basic variables. The elements of the row must appear in the same order in which the basic variables are listed in the Basic column of the simplex tableau.

Dual simplex: In primal simplex start with $b \geq 0$ and preserve this primal feasibility but in some case this may not hold. Primal feasibility fail means that there $b_i < 0$, $i = 1, 2, \ldots, m$. In this case we use dual simplex start with $c_j \leq 0$ and preserve this dual feasibility and work to ward making $b_i \geq 0$. If $b_i \geq$ for all $i = 1, 2, \ldots, m$, then the current tableau is optimal.

	x_1	x_2	$\ldots$	x_n	s_1	s_2	$\ldots$	s_m	b
$-z$	c'_1	c'_2	$\ldots$	c'_n	c'_{n+1}	$\ldots$	$\ldots$	c'_{n+m}	z'
x_{B1}	a'_{11}	a'_{12}	$\ldots$	a'_{1n}	a'_{1n+1}	$\ldots$	$\ldots$	a'_{1n+m}	$\bar{b}_1$
x_{B2}	a'_{21}	a'_{22}	$\ldots$	a'_{2n}	a'_{2n+1}	$\ldots$	$\ldots$	a'_{2n+m}	$\bar{b}_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{Bm}	a'_{m1}	a'_{m2}	$\ldots$	a'_{mn}	a'_{mn+1}	$\ldots$	$\ldots$	a'_{mn+m}	$\bar{b}_m$

(1.8)

Consider the tableau in (1.8) representing a basic solution of (1.1) the maximization problem at some iteration. Suppose that the tableau is dual feasible (that is, $c'_j \leq 0$ for a maximization problem), for all $j = 1, 2, \ldots, n$. If the tableau is also primal feasible (that is, all $\bar{b}_i \geq 0$), then we have an optimal solution.

Dual simplex algorithm for maximization:

Initial step:

check dual feasibility i.e. $c'_j \leq 0$, for all $j = 1, 2, \ldots, m$

min step:

1. If $\bar{b}_i \geq 0$, for all $i = 1, 2, \ldots, m$ Stop; the current solution is optimal. Otherwise, select a pivot row r with $\bar{b}_r < 0$; say, $\bar{b}_r = \text{minimum}\{\bar{b}_i\}$
2. If $a'_{rj} \geq 0$ for all j, stop; the dual is unbounded and the primal is infeasible. Otherwise, select the pivot column k by the following minimum ratio test:

$$\frac{c'_k}{a'_{rk}} = \text{minimum}_j\{\frac{c'_j}{a'_{rj}},\ a'_{rj} < 0\}$$

3. Pivot at a_{rk} and return to Step 1.

We will use (P) and (D) LP in the next chapter to find optimal solution in Game Theory what we called optimal strategies of of players in the max min and min max problem.

Chapter 2

Normal Form Games

2.1 Two-person Zero-sum Game in Normal Form

Definition 1. *The system*

$$\Gamma = (X, Y, K), \tag{2.1}$$

where X and Y are nonempty finite sets, and the function $K : X \times Y \to \mathbb{R}$ is bounded.

The elements $x \in X$ and $y \in Y$ are called the strategies of players 1 and 2(possible set of actions of a player 1 or 2), respectively, in the game Γ. The strategies x and y are the alternative action taken by player 1 and 2 respectively to make a decision. The elements of the Cartesian product $X \times Y$ (i.e. the pairs of strategies (x, y), where $x \in X$ and $y \in Y$) are called situations, and the function K is the payoff of Player 1. It measures the gain and loss of player 1 at each situation. Player 1's payoff in situation (x, y) is set equal to $K(x, y)$ and player 2's payoff is $-K(x, y)$. Therefore the function K also called payoff function of the game Γ. The value $K(x, y)$ at situation (x, y) indicates the gain or loss of player 1, i.e. if the value is positive indicate gain and loss if negative. The gain $K(x, y)$ at situation (x, y) for player 1 is loss for player 2 and vice versa, then the game Γ is called two-person zero-sum game, and the function K is called the payoff function for the game Γ. In two person zero sum game the sum of the payoffs to payer 1 and player 2 is zero. In order to specify the game Γ, it is necessary to define the sets of strategies X, Y for players 1 and 2, and the payoff function K given on the set of all situations $X \times Y$. The game Γ is interpreted as follows. Players simultaneously and independently choose strategies $x \in X$, $y \in Y$. There after Player 1 receives the payoff equal to $K(x, y)$ and Player 2 receives the payoff equal to $-K(x, y)$. The value of the function $K(x, y)$ can not be the same at all situation (x, y), since game is need decision based on the payoff outcome of the function, so no need a decision for such the same value at all.

A game in normal form, is representation of a game with table(matrix) in which players independently and simultaneously choose their strategies. The resulting payoffs are presented in a matrix for each strategy combination.

The restriction of a function is a new function obtained from the original function by choosing a smaller domain sub set of the original function.

Definition 2. *The game $\Gamma' = (X', Y', K')$ is called a sub-game of the game $\Gamma = (X, Y, K)$ if $X' \subseteq X$, $Y' \subseteq Y$, and the function $K' : X' \times Y' \to \mathbb{R}$ is a restriction of function K on $X' \times Y'$.*

Definition 3. *Two-person zero-sum games in which both players have finite sets of strategies are called matrix games and denoted by Γ_A, where A is the payoff matrix for player 1.*

Definition 4. *A pure strategy is a strategy that provides a complete definition of how a player will play a game.*

We focus only on two-person zero-sum matrix games in which the pure strategy sets of the players' are finite.

Suppose that Player 1 in matrix game (2.1) has a total of m strategies. Let us order the strategy set X of the first player and correspond to the row of matrix i.e. set up a one-to-one correspondence between the set $M = \{1, 2, \ldots, m\}$ and X . The set M is called pure strategies of player 1. Similarly, if Player 2 has n strategies, it is possible to set up a one-to-one correspondence between the sets $N = \{1, 2, \ldots, n\}$ and Y. The set N is called pure strategies of player 2. The game Γ is then fully defined by specifying the matrix $A = (a_{ij})$, where $a_{ij} = K(x_i, y_i), (i, j) \in M \times N$, $(x_i, y_j,) \in X \times Y, i \in M, j \in N$. In this case the game Γ is realized as follows. Player 1 chooses row $i \in M$ and Player 2 (simultaneously and independently from Player 1) chooses column $j \in N$. Thereafter Player 1 receives the payoff a_{ij} and Player 2 receives the payoff $-a_{ij}$. If the payoff is equal to a negative number, then we are dealing with the actual loss of Player 1.

Example 1. *Suppose player 1 in war zone has m regiments and his enemy(player 2) has n regiments. Player 2 is defending two posts. The post will be taken by player 1 if when attacking the post he is more powerful in strength on this post. The opposing parties are two separate regiments between the two posts. Define the payoff to the Player 1 at each post. If player 1 has more regiments than the enemy(Player 2) at the post, then his payoff at this post is equal to the number of the enemy's regiments plus one (the occupation of the post is equivalent to capturing of one regiment). If Player 2 has more regiments than Player 1 at the post, Player 1 loses his regiments at the post plus one (for the lost of the post). If each side has the same number of regiments at the post, it is a draw and each side gets zero. The total payoff to Player 1 is the sum of the payoffs at the two posts.*

The game is zero-sum. We shall describe strategies of the players. Suppose that $x_0, x_1, \ldots, x_m$, *and* $y_0, y_2, \ldots, y_n$ *be strategies of player 1 and player 2 respectively. Assume* $m > n$. *And let as place the regiments in the two post for Player 1 strategies:* $x_0 = (m, 0)$ *- to place all of the regiments at the first post and no at the second post;* $x_1 = (m-1, 1)$ *- to place* $(m-1)$ *regiments at the first post and one at the second;* $x_2 = (m-2, 2), \ldots,$ $x_{m-1} = (1, m-1)$, $x_m = (0, m)$. *And for (Player 2) has the following strategies:* $y_0 = (n, 0)$, $y_1 = (n-1, 1), \ldots$, $y_n = (0, n)$. *Suppose that the Player 1 chooses strategy* x_0 *and Player 2 chooses strategy* y_0*.Compute the payoff* a_{00} *of Player 1 in this situation.Since* $m > n$, *Player 1 wins at the first post. His payoff is* $n+1$ *(one for holding the post). At the second post it is draw. Therefore* $a_{00} = n + 1$*.Compute* a_{01}*.Since* $m > n - 1$, *then in the first post Player 1's payoff is* $n - 1 + 1 = n$. *Player 2 wins at the second post. Therefore the loss of Player 1 at this post is one. Thus,* $a_{01} = n - 1$. *Similarly, we obtain* $a_{0j} = n - j + 1 - 1 = n - j$, $1 \leq j \leq n$. *Further, if* $m - 1 > n$ *then* $a_{10} = n+1+1 = n+2$, $a_{11} = n-1+1 = n$, $\ldots$, $a_{1j} = n-j+1-1-1 = n-j-l$, $2 < j < n$*.In a general case (for any* m *and* n*) the elements* a_{ij}, $i = 0, 1, \ldots, m$, $j = 0, \ldots, n$, *of the payoff matrix are computed as follow:*

$$a_{ij} = \begin{cases} n+2, & if \quad m-i > n-j, i > i, \\ n-j+1, & if \quad m-i > n-j, i = j, \\ n-j-i, & if \quad m-i > n-j, i < j, \\ -m+i+j, & if \quad m-i < n-j, i > j, \\ j+1, & if \quad m-i = n-j, i > j, \\ -m-2, & if \quad m-i < n-j, i < j, \\ -i-1, & if \quad m-i = n-j, i < j, \\ -m+i-1, & if \quad m-i < n-j, i = j, \\ 0, & if \quad m-i = n-j, i = j. \end{cases}$$

Thus, with $m = 4$, $n = 3$, *considering all possible situations, we obtain the payoff matrix* A *of this game:*

$$A = \begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_0 & y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \end{array} \tag{2.2}$$

Note that the negative numbers in the payoff matrix indicates the actual loss of player 1 and 0 *indicates no gain and no loss.*

Example 2. *Suppose that Player 1 is searching for a mobile object (Player 2) for the purpose of detecting it. Player 2's objective is the opposite one (i.e. he seeks to avoid being detected). Suppose there are three detecting places and the velocities of player 1 at each place is denoted by* α_1, α_2 *and* α_3 *and of player 2 at each place* β_1, β_2, β_3 *. Player 1 can move at velocities of* $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$, *respectively to the three places. The range of the detecting device used by Player 1, depending on the velocities of the players is determined by the matrix*

$$D = \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{array} \begin{array}{c} \begin{array}{ccc} \beta_1 & \beta_2 & \beta_3 \end{array} \\ \begin{bmatrix} 4 & 5 & 6 \\ 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix} \end{array}$$

Strategies of the players are the velocities, and Player 1's payoff in the situation (α_i, β_j) *is assumed to be the search efficiency* $a_{ij} = \alpha_i \delta_{ij}$, *where* $i = 1, 2, 3$, $j = 1, 2, 3$ *and* δ_{ij} *is an element of the matrix* D*. Then the problem of selecting velocities in a noisy search can be represented by the game with matrix*

$$A = \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{array} \begin{array}{c} \begin{array}{ccc} \beta_1 & \beta_2 & \beta_3 \end{array} \\ \begin{bmatrix} 4 & 5 & 6 \\ 6 & 8 & 10 \\ 3 & 6 & 9 \end{bmatrix} \end{array} \qquad (2.3)$$

2.2 Maximin and Minimax Strategies

Consider a two-person zero-game $\Gamma = (X, Y, K)$ as in 2.1. In this game each of the players wants to maximize his payoff by choosing a proper strategy. But for Player 1 the payoff is determined by the function $K(x, y)$, and for Player 2 it is determined by $(-K(x, y))$, i.e. the players' interest are directly opposite, since the gain for the one is loss for the other. Note that the payoff of Player 1 (or 2) (the payoff function) is determined on the set of situations $(x, y) \in X \times Y$. Each situation, and hence the player's payoff do not depend only on his own choice, but also on what strategy will be chosen by his opponent whose objective is directly opposite, since player 1 wants to maximize the minimum gain and player 2 want to minimize the maximum loss this is the objective of the players. Therefore, seeking to obtain the maximum possible payoff, each player must take into account the opponent's behavior.

The game in example 1 (2.2) provides a good example of the foregoing. If Player 1 wants to gain maximum payoff, he must choose the strategy x_0(or x_4) row in order to get the maximum entry value 4. In this case, if Player 2 chooses y_0 for strategy x_4 player 1, or y_3 for strategy x_0 player 1 , then the first player receives the payoff 0, i.e. he loses 4 units. Similar reasonings are applicable to Player 2.

In the theory of games it is supposed that the behavior of both players is rational, i.e. they wish to maximize the minimum gain and minimize the maximum loss, assuming that the opponent is acting in the best (for himself) possible way. What maximal payoff can Player 1 guarantee himself? Suppose player 1 chooses strategy x. Then, at worst case he will win $\min_y K(x, y)$. Therefore, Player 1 can always guarantee himself the payoff $\max_x \min_y K(x, y)$. This is the objective of player 1. If the max and min are not reached, Player 1 can guarantee himself obtaining the payoff arbitrarily close to the quantity

$$\underline{v} = \sup_{x \in X} \inf_{y \in Y} K(x, y). \tag{2.4}$$

The principle of constructing such objective strategy x of based on the maximization of the minimal payoff is called the max min principle, and the strategy x selected by this principle is called the max min strategy of Player 1. As in example 1 (2.2) the guarantee strategy for player 1 is x_3 since the $\max_i \min_j a_{ij} = 1$ is exist at strategy x_4.

For Player 2 it is possible to provide similar reasonings. Suppose he chooses strategy y. Then, at worst case, he will lose $\max_x K(x, y)$. Therefore, the second player can always guarantee himself the payoff $\min_y \max_x K(x, y)$ and is the objective of player 2. If the min and max is not reachable, Player 2 can guarantee himself obtaining the payoff arbitrarily close to the quantity

$$\overline{v} = \inf_{y \in Y} \sup_{x \in X} K(x, y). \tag{2.5}$$

The principle of constructing a strategy y, based on the minimization of maximum losses, is called the *minmax* principle, and the strategy y selected for this principle is called the *minimax* strategy of Player 2. In example 1 (2.2) the guarantee strategy for player 2 are exist y_1 and y_2 and the $\min_j \max_i a_{ij} = 3$.

Consider the $(m \times n)$ matrix game Γ_A. Then the extrema in (2.4) and (2.5) are reached and $\underline{v}$ and $\overline{v}$ of the game are, respectively equal to

$$\underline{v} = \max_{1 \le i \le m} \min_{1 \le j \le n} a_{ij} \tag{2.6}$$

$$\overline{v} = \min_{1 \le j \le n} \max_{1 \le i \le m} a_{ij} \tag{2.7}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The max min principle:

$$\max_i \min_j a_{ij} = \max\{min\{a_{1j}\}_{j=1}^n, \min\{a_{2j}\}_{j=1}^n, \dots, \min\{a_{mj}\}_{j=1}^n\}$$
$$= \{\max_i\{\min_j a_{ij}\}_{j=1}^n\}_{i=1}^m$$

And the min max principle:

$$\min_j \max_i a_{ij} = \min\{\max\{a_{i1}\}_{i=1}^m, \max\{a_{i2}\}_{i=1}^m, \dots, \max\{a_{in}\}_{i=1}^m\}$$
$$= \{\min_j\{\max_i a_{ij}\}_{i=1}^m\}_{j=1}^n$$

The following theorem holds for any game $\Gamma = (X, Y, K)$

Theorem 1. *The lower value of the game is always leas than the upper value, i.e.*

$$\underline{v} \leq \overline{v} \tag{2.8}$$

or

$$\sup_{x\in X} \inf_{y\in Y} K(x, y) \leq \inf_{y\in Y} \sup_{x\in X} K(x, y) \tag{2.9}$$

Proof. *Let $x \in X$ be an arbitrary strategy of Player 1. Then we have*

$$K(x, y) \leq \sup_{x\in X} K(x, y)$$

Hence we get

$$\inf_{y\in Y} K(x, y) \leq \inf_{y\in Y} \sup_{x\in X} K(x, y).$$

Note that we have a constant on the right-hand side of the latter inequality, and the value $x \in X$ has been chosen arbitrarily. Therefore, the following inequality holds

$$\sup_{x\in X} \inf_{y\in Y} K(x, y) \leq \inf_{y\in Y} \sup_{x\in X} K(x, y).$$

□

The values $\underline{v}$ and $\overline{v}$ are called the lower and upper value of the game respectively.As in Example 1 in (2.2) $\max_i \min_j a_{ij} = 1 < \min_j \max_i a_{ij} = 3$.

If $\underline{v} = \max_x \inf_y K(x, y) = \overline{v} = \min_y \sup_x K(x, y)$, then the max min and min max principles are called optimal principle. And the strategies x and y are called the optimal strategies of player 1 and 2 in the game respectively.

Thus, the game Γ_A in Example 2 with the matrix (2.3), the lower value (max min) $\underline{v}$ and the max min strategy of the first player are $\underline{v} = 6$, α_2, respectively, and the upper value (min max) $\overline{v}$ and the min max strategy of the second player are $\overline{v} = 6$, β_1, respectively. Then $\underline{v} = \overline{v} = 6$, this implies max min and min max are optimal principle of the game and the strategies α_2, β_1 are the optimal strategies of player 1 and 2 respectively.

2.3 Saddle points

Consider the optimal behavior of players in a two-person zero-sum game. In the game $\Gamma = (X, Y, K)$ it is natural to consider as optimal a situation $(x^*, y^*) \in X \times Y$ the deviation from which there is no advantage for both players. Such a point (x^*, y^*) is called the equilibrium point and the optimality principle based on constructing an equilibrium point is called the equilibrium principle. For two-person zero-sum games, as will be shown later, the equilibrium principle is equivalent to the principles of min max and max min the discussed above in section (2.2) . This, of course, requires the existence of an equilibrium.

Definition 5. *In the two-person zero-sum game* $\Gamma = (X, Y, K)$ *the point* (x^*, y^*) *is called an equilibrium point, or a saddle point, if*

$$K(x, y^*) \leq K(x^*, y^*), \tag{2.10}$$

$$K(x^*, y^*) \leq K(x^*, y) \tag{2.11}$$

for all $x \in X$ *and* $y \in Y$*.*

The set of all equilibrium points in the game Γ will be denoted as

$$Z(\Gamma), (x^*, y^*) \in Z(\Gamma) \subseteq X \times Y.$$

In the matrix game Γ the equilibrium points are the saddle points of the payoff matrix A, i.e. the points (i^*, j^*) for which for all $i \in M$ and $j \in N$ the inequalities (2.10) and (2.11) are satisfied

$$a_{ij^*} \leq a_{i^*j^*} \leq a_{i^*j}.$$

The element of the matrix $a_{i^*j^*}$ at the saddle point is simultaneously the minimum of its row and the maximum of its column. For example, in the game with the matrix $\begin{bmatrix} 1 & 0 & 4 \\ 5 & 3 & 8 \\ 6 & 0 & 1 \end{bmatrix}$ the point $(2, 2)$ is a saddle point (equilibrium).

Theorem 2. *Let* (x_1^*, y_1^*), (y_2^*, x_2^*) *be two arbitrary saddle points in the two person zero-sum game* Γ*. Then;*

$$1. K(x_1^*, y_1^*) = K(x_2^*, y_2^*);$$

$$2. (x_1^*, y_2^*) \in Z(\Gamma), (x_2^*, y_1^*) \in Z(\Gamma).$$

Proof. *1. From the definition of a saddle point for all $x \in X$ and $y \in Y$ we have*

$$K(x, y_1^*) \leq K(x_1^*, y_1^*) \leq K(x_1^*, y) \tag{2.12}$$

$$K(x, y_2^*) \leq K(x_2^*, y_2^*) \leq K(x_2^*, y) \tag{2.13}$$

We substitute x_2^* into the left-hand side of the inequality (2.12), y_2^* into the right-hand side, x_1^* into the left-hand side of the inequality (2.13) and y_1^* into the right-hand side. Then we get

$$K(x_2^*, y_1^*) \leq K(x_1^*, y_1^*) \leq K(x_1^*, y_2^*) \leq K(x_2^*, y_2^*) \leq K(x_2^*, y_1^*)$$

From this it follows that:

$$K(x_1^*, y_1^*) = K(x_2^*, y_2^*) = K(x_1^*, y_2^*) = K(x_2^*, y_1^*) \tag{2.14}$$

2. To show the second, consider the point (x_1^*, y_2^*). From (2.12)-(2.14), we have

$$K(x, y_2^*) \leq K(x_2^*, y_2^*) = K(x_1^*, y_1^*) = K(x_1^*, y_2^*) \leq K(x_1^*, y) \tag{2.15}$$

for all $x \in X, y \in Y$. And also for the point x_2^*, y_1^* can be proved in the same manner as the point (x_1^*, y_2^*).

□

From the theorem it follows that the payoff function takes the same values at all saddle points.

In game theory optimal strategy is a strategy that satisfies the equilibrium principle and/or the optimal principle.

Definition 6. *Let (x^*, y^*) be a saddle point in the game Γ. Then the number*

$$v = K(x^*, y^*) \tag{2.16}$$

is called the value of the game Γ.

From Theorem 2 by the second assertion suggests, in particular the following fact. Let $X^* \subset X$, $Y^* \subset Y$ and

$$X^* = \{x^* \in X | \exists y^* \in Y, (x^*, y^*) \in Z(\Gamma)\},$$

$$Y^* = \{y^* \in Y | \exists x^* \in X, (x^*, y^*) \in Z(\Gamma)\}$$

then

$$Z(\Gamma) = X^* \times Y^* \tag{2.17}$$

Definition 7. *The set X^* and Y^* is called the set of optimal strategies of Player 1 and 2 respectively in the game Γ, and their elements-optimal strategies of the player 1 and 2. We say these optimal strategies because for each player any change in the chosen strategies does not improve the payoff to either player i.e. satisfies the optimal principle.*

Optimality of the players' behavior remains unaffected if the strategy sets in the game remain the same and the payoff function is multiplied by a positive constant, or a constant number is added thereto.

Theorem 3. *For the existence of the saddle point in the game $\Gamma = (X, Y, K)$, it is necessary and sufficient that the quantities*

$$\min_y \sup_x K(x,y), \max_x \inf_y K(x,y) \tag{2.18}$$

exist and the following equality holds:

$$\underline{v} = \max_x \inf_y K(x,y) = \min_y \sup_x K(x,y) = \overline{v}. \tag{2.19}$$

Proof. *Necessity: Let $x^*, y^* \in Z(\Gamma)$. Then for all $x \in X$ and $y \in Y$ the inequality holds*

$$K(x,y^*) \le K(x^*,y^*) \le K(x^*,y)$$

and hence

$$\sup_x K(x,y^*) \le K(x^*,y^*) \tag{2.20}$$

$$\inf_y \sup_x K(x,y) \le \sup_x K(x,y^*) \tag{2.21}$$

comparing (2.20)and (2.21) we get

$$\inf_y \sup_x K(x,y) \le \sup_x K(x,y^*) \le K(x^*,y^*)$$

In the similar way we get the inequality

$$K(x^*,y^*) \le \inf_y K(x^*,y) \le \sup_x \inf_y K(x,y).$$

The inverse inequality holds in (2.9). We get

$$\sup_x \inf_y K(x,y) = \inf_y \sup_x K(x,y),$$

and finally we get

$$\min_y \sup_x K(x,y) = \sup_x K(x,y^*) = K(x^*,y^*)$$

$$\max_x \inf_y K(x,y) = \inf_y K(x^*,y) = K(x^*,y^*)$$

i.e. the exterior extrema of the min sup and max inf are reached at the points y^* and x^* respectively.

Sufficiency: *Suppose there exist the* $\min_y \sup_x$ *and* $\max_x \inf_y$

$$\max_x \inf_y K(x,y) = \inf_y K(x^*,y);$$

$$\min_y \sup_x K(x,y) = \sup_x K(x,y^*) \tag{2.22}$$

and the equality (2.19) holds. We need to show that (x^*,y^*) is a saddle point.

$$K(x^*,y^*) \geq \inf_y K(x^*,y) = \max_x \inf_y K(x,y) \tag{2.23}$$

$$K(x^*,y^*) \leq \sup_x K(x,y^*) = \min_y \sup_x K(x,y) \tag{2.24}$$

By (2.19) the min sup *is equal to the* max inf*, and from (2.23), (2.24) it follows that the* min sup *is also equal to the $K(x^*,y^*)$, i.e. the inequalities in (2.23) and (2.24) are satisfied as equalities. Now we have*

$$K(x^*,y^*) = \inf_y K(x^*,y) \leq K(x^*,y)$$

$$K(x^*,y^*) = \sup_x K(x,y^*) \geq K(x,y^*)$$

for all $x \times y \in X \times Y$, this implies $(x^*,y^*) \in Z(\Gamma)$

□

The proof shows that the common value of the min sup and max inf is equal to $K(x^*,y^*) = v$, the value of the game, and any min sup and max inf strategies y^* and x^* are optimal in terms of the theorem, i.e. the point (x^*,y^*) is a saddle point. In game the a pair of strategies player 1 and 2 that satisfy the $\max\min = \min\max$ is called optimal strategies.

From the proof of this theorem the following corollary

Corollary 1. *If the* min sup *and* max inf *in (2.18) exist and are reached on $\overline{y}$ and $\overline{x}$, respectively, then*

$$\max_x \inf_y K(x,y) = K(\overline{x},\overline{y}) = \min_y \sup_x K(x,y) \tag{2.25}$$

The games, in which saddle points exist and is said to be strictly determined. Therefore, this theorem establishes a criterion for strict determination of the game and can be restated as follows. For a game to be strictly determined it is necessary and sufficient that the min sup and max inf in (2.18) exist and the equality (2.19) is satisfied.

Theorem 4. *Let* $\Gamma = (X, Y, K)$ *and* $\Gamma' = (X, Y, K')$ *be two zero-sum games and*

$$K' = \beta K + \alpha, \;\; \beta > 0, \;\; \alpha = constant, \;\; \beta = constant \tag{2.26}$$

Then

$$Z(\Gamma') = Z(\Gamma), \;\; v_{\Gamma'} = \beta v_{\Gamma} + \alpha. \tag{2.27}$$

Proof. *Let* (x^*, y^*) *be a saddle point in the game* Γ. *Then we have*

$$K'(x^*, y^*) = \beta K(x^*, y^*) + \alpha \leq \beta K(x^*, y) + \alpha = K'(x^*, y)$$

$$K'(x^*, y^*) = \beta K(x^*, y^*) + \alpha \geq \beta K(x, y^*) + \alpha = K'(x, y^*)$$

for all $x \in X$ *and* $y \in Y$
this implies

$$K'(x, y^*) \leq K'(x^*, y^*) \leq K'(x, y^*)$$

for all $x \times y \in X \times Y$,
Therefore $(x^*, y^*) \in Z(\Gamma')$
Conversely let $(x^*, y^*) \in Z(\Gamma')$. *Then*

$$K(x^*, y^*) = \frac{1}{\beta} K'(x^*, y^*) - \frac{\alpha}{\beta}$$

and, by similar reasoning, we have that $(x^*, y^*) \in Z(\Gamma)$. *Therefore* $Z(\Gamma) = Z(\Gamma')$.
And

$$K'(x^*, y^*) = \beta K(x^*, y^*) + \alpha = \beta v_{\Gamma} + \alpha = v_{\Gamma'}$$

□

Corollary 2. *For the* $m \times n$ *matrix game to be strictly determined it is necessary and sufficient that the following equalities hold*

$$\min_{1 \leq j \leq n} \max_{1 \leq i \leq m} a_{ij} = \max_{1 \leq i \leq m} \min_{1 \leq j \leq n} a_{ij} \tag{2.28}$$

For example, in the game with the matrix $\begin{bmatrix} 1 & 4 & 1 \\ 2 & 2 & 4 \\ 0 & \text{-}1 & 7 \end{bmatrix}$ the point $(2, 1)$ is a saddle point. In this case

$$\max_i \min_j a_{ij} = \min_j \max_i a_{ij} = 2.$$

On the other hand, the game with the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ does not have a saddle point, since

$$\min_j \max_i a_{ij} = 1 > \max_i \min_j a_{ij} = 0.$$

so, the game is not strictly determined, this implies that the game have not optimal strategies and no value of the game at these strategies. we will see in the next section to find equilibrium point in the non strictly determined case.

Note that the game in Example 1 (2.2) is non strictly determined, but the game in example 2 is strictly determined and the value of the game is 6 with optimal strategy (α_2, β_1).

2.4 Mixed Strategies of a Game

Consider the matrix game Γ_A. If the game has a saddle point, then the min max is equal to the max min; and each of the players can, by the definition of the saddle point, inform the opponent of his optimal (max min, min max) strategy and hence no player can receive extra benefits. Now assume that the game Γ_A has no saddle point. Then by Theorem 1 and Theorem 4

$$\min_j \max_i a_{ij} - \max_i \min_j a_{ij} > 0. \tag{2.29}$$

In this case the max min and min max strategies are not optimal and the value of the game v_A is occur in some where between max min and min max, i.e.

$$\max\min < v_A < \min\max$$

$\begin{bmatrix} 7 & 3 \\ 2 & 5 \end{bmatrix}$ from this matrix game the $\min_j \max_i a_{ij} = 5$, $\max_i \min_j a_{ij} = 3$, that is the saddle point does not exist. Denote by i^* the max min strategy of Player 1 ($i^* = 1$), and by j^* the min max strategy of Player 2 ($j^* = 2$). Suppose Player 2 adopts strategy $j^* = 2$ and Player 1 chooses strategy $i = 2$. Then the latter receives the payoff 5, i.e. 2 units greater than the max min. If, however, Player 2 guesses the choice by Player 1, he alters his strategy to $j = 1$ and then Player 1 receives a payoff of 2 units only, i.e. 1 unit less than in the case of the max min. Similar reasonings apply to the second player.

Definition 8. *The random variable whose values are strategies of a player is called a mixed strategy of the player.*

For the matrix game Γ_A, a mixed strategy of Player 1 is a random variable whose values are the row numbers $i \in M, M = \{1, 2, \ldots, m\}$. A similar definition applies to Player 2's mixed strategy whose values are the column numbers $j \in N$ of the matrix A. Since the random variable is characterized by its distribution, the mixed strategy will be identified in what follows with the probability distribution over the set of pure strategies. Thus, *Player* 1's mixed strategy x in the game is the m-dimensional vector,

$$x = (\xi_1, \xi_2, \ldots, \xi_m), \xi_i \geq 0, \ \sum_{i=1}^{m} \xi_i = 1, \ i = 1, 2, \ldots, m. \tag{2.30}$$

Similarly, *player* 2's mixed strategy y is n-dimensional vector,

$$y = (\gamma_1, \gamma_2, \ldots, \gamma_n), \gamma_i \geq 0, \ \sum_{j=1}^{n} \gamma_j = 1, \ j = 1, 2, \ldots, n \tag{2.31}$$

In this case, $\xi_i \geq 0$ and $\gamma_j \geq 0$ are the probabilities of choosing the pure strategies $i \in M$ and $j \in N$, respectively, when the players use mixed strategies x and y. Denote by set of vectors X and Y the sets of mixed strategies for the first and second players, respectively.

Definition 9. *Let $x = (\xi_1, \xi_2, \ldots, \xi_m) \in X$ be a mixed strategy of Player 1. The set of indices*

$$M_x = \{i | i \in M, \xi_i \neq 0\} \tag{2.32}$$

where $M = \{1, 2, \ldots, m\}$, is called the spectrum of strategy x.

Similarly, for the mixed strategy $y = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ of Player 2 the spectrum N_y is determined as follows:

$$N_y = \{j | j \in N, \gamma_j \neq 0\} \tag{2.33}$$

where $N = \{1, 2, \ldots, n\}$

For any mixed strategy x the spectrum $M_x \neq \emptyset$, since the vector x has non-negative components with the sum equal to 1.

Any pure strategy $i \in M$ can expressed as a mixed strategy $u_i = (\xi_1, \xi_2, \ldots, \xi_m) \in X$, where $\xi_i = 1, \xi_j = 0, j \neq i, i = 1, 2, \ldots, m$. Such a strategy prescribes a selection of the i^{th} row of the matrix A with probability 1. Similarly for any pure strategy $j \in N$ of player 2 can be expressed by $v_j = (\gamma_1, \gamma_2, \ldots, \gamma_n), \gamma_j = 1, \gamma_i = 0, i \neq j, j = 1, 2, \ldots, n$

Definition 10. *The pair (x, y) of mixed strategies in the matrix game Γ_A is called the situation in mixed strategies.*

If we mixed the game Γ_A with respect the mixed strategies $x = (\xi_1, \xi_2, \ldots, \xi_m)$ of player 1, we have this row matrix $xA = \left[\sum_{i=1}^{m} a_{i1}\xi_i \ \sum_{i=1}^{m} a_{i2}\xi_i \ldots \sum_{i=1}^{m} a_{in}\xi_i\right]$, is the moderate payoff value for player 1 between $\min_i a_{ij}$ and $\max_i a_{ij}$, where

$j = 1, 2, \ldots, n$ and also mix this with respect mixed strategy $y = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ of player 2 and we have get

$$K(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \xi_i \gamma_j = xAy^T \tag{2.34}$$

is called the payoff function of Player 1 at the point (x, y) in mixed strategies $x = (\xi_1, \xi_2, \ldots, \xi_m) \in X$, $y = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in Y$ for the $(m \times n)$ matrix game Γ_A. Mean that the game is mixed with the probabilities of the pure strategy. The function $K(x, y)$ is continuous in $x \in X$ and $y \in Y$. When one player uses pure strategy $i(or\ j)$ and the other player uses mixed strategy x or y respectively then the payoffs $K(i, y), K(x, j)$ are computed as follow:

$$K(i, y) = K(u_i, y) = \sum_{j=1}^{n} a_{ij} \gamma_j = a_i y^T, i = 1, 2, \ldots, m$$

$$K(x, j) = K(x, v_j) = \sum_{i=1}^{m} \xi_i a_{ij} = x a^j, j = 1, 2, \ldots, n$$

where a_i, a^j are respectively the i^{th} row and the j^{th} column of the $(m \times n)$ matrix A.

The game $\overline{\Gamma}_A = (X, Y, K)$, where X and Y are the sets of mixed strategies in the game $\Gamma_A = (M, N, A)$ and K is the payoff function in mixed strategies, is called a mixed extension of the game Γ_A. The game Γ_A is a subgame for $\overline{\Gamma}_A$, i.e. $\Gamma_A \subseteq \overline{\Gamma}_A$

Definition 11. *The point (x^*, y^*) in the game $\overline{\Gamma}_A$ is saddle point and the number $v = K(x^*, x^*)$ is the value of the game if for all $x \in X$ and $y \in Y$*

$$K(x, y^*) \leq K(x^*, y^*) \leq K(x^*, y) \tag{2.35}$$

The strategies (x^*, y^*) appearing in the saddle point are called optimal strategies. Moreover, by Theorem 4, the strategies x^* and y^* are respectively the max min and min max strategies, since the exterior extrema in (2.18) are reachable (the function K(x,y) is continuous on the compact sets X and Y).

Theorem 5. *Let Γ_A and $\Gamma_{A'}$ be two $m \times n$ matrix games, where $A' = \alpha A + B$, $\alpha > 0$, $\alpha = const$, and B is the matrix with the same elements β, i.e. $\beta_{i,j} = \beta$ for all i and j. Then $Z(\overline{\Gamma}_{A'}) = Z(\overline{\Gamma}_A)$ and $\overline{v}_{A'} = \alpha \overline{v}_A + \beta$, where $\overline{\Gamma}_{A'}$ and $\overline{\Gamma}_A$ are the mixed extensions of the games $\Gamma_{A'}$ and Γ_A, respectively, and $\overline{v}_{A'}$ and $\overline{v}_A$ are the values of the games $\overline{\Gamma}_{A'}$ and $\overline{\Gamma}_A$.*

Proof. *matrices A and A′ are of dimension* $m \times n$*; therefore the sets of mixed strategies in the games* $\Gamma_{A'}$ *and* Γ_A *coincide. We shall show that for any situation in mixed strategies* (x, y) *the following equality holds*

$$K'(x, y) = \alpha K(x, y) + \beta, \tag{2.36}$$

where K' *and* K *are Player 1's payoffs in the games* $\overline{\Gamma}_{A'}$ *and* $\overline{\Gamma}_A$*, respectively. Indeed, for all* $x \in X$ *and* $y \in Y$ *we have*

$$K'(x, y) = xA'y^T = \alpha(xAy^T) + xBy^T = \alpha K(x, y) + \beta.$$

From Theorem 3 it follows that $Z(\overline{\Gamma}_{A'}) = Z(\overline{\Gamma}_A)$, $\overline{v}_{A'} = \overline{v}_A + \beta$.

□

Example 3. *Let us verify that the strategies* $y^* = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $x^* = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ *are optimal and* $v = 0$ *is the value of the game* Γ_A *with the matrix*

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & 3 & -1 \end{bmatrix}$$

For simplicity we add 1 to all entry of matrix A and then multiply the result matrix by $\frac{1}{2}$ *we get the following matrix*

$$A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

i.e. $A' = \frac{1}{2}(A + B)$*, where* $B = \beta_{ij}$ *and* $\beta_{ij} = 1$*, for all* $i = 1, 2, 3$ *and* $j = 1, 2, 3$.

for any, $y = (\gamma_1, \gamma_2, \gamma_3) \in Y$*, we have* $K'(x^*, y) = x^*A'y^T = \frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2 + \frac{1}{2}\gamma_2 = \frac{1}{2}$*, and for all* $x = (\xi_1, \xi_2, \xi_3)$, $K'(x, y^*) = xA'y^{*T} = \frac{1}{2}\xi_1 + \frac{1}{2}\xi_2 + \frac{1}{2}\xi_3 = \frac{1}{2}$*, this implies* $K'(x, y^*) \leq K'(x^*, y^*) \leq K'(x^*, y)$ *for all* $(x, y) \in X \times Y$*, hence* $(x^*, y^*) \in Z(\overline{\Gamma}_{A'})$*. i.e.* (x^*, y^*) *is the saddle point of the game* $\overline{\Gamma}_{A'}$.

$K'(x^*, y^*) = x^*A'y^{*T} = \frac{1}{2}$*. Since* $A' = \frac{1}{2}(A + B)$*, then* $v_{A'} = \frac{1}{2}(v_A + 1)$, $v_A = 0$

In this example only see how to check the saddle point condition, but we will see later how to find these saddle points.

2.5 Existence of a solution of the matrix game in mixed strategies

Theorem 6. *Any matrix game has a saddle point in mixed strategies.*

Proof. ***case 1:*** *Let Γ_A be an arbitrary game with $(n \times m)$ a strictly positive matrix $A = (a_{ij})$ i.e. $a_{ij} > 0$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. We need to show the Theorem is true in this case.*

Player 1s' mixed optimal strategy $x = (\xi_1, \xi_2, \ldots, \xi_m)$ can be determined by solving the following max min *problem.*

$$\max_x \min_{v_j} \; xAv_j^T = \max_x \{\min_{1 \le j \le n} xa^j\}$$

$$= \max_{\xi_i}\{\min\{\sum_1^m a_{i1}\xi_i, \sum_1^m a_{i2}\xi_i, \ldots, \sum_1^m a_{in}\xi_i\}\}, \qquad (2.37)$$

where $v_j = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_j = 1$ for $k = j$, and $\lambda_k = 0$ for $k \neq j$ and a^j is column vectors of matrix A, $j, k = 1, 2, \ldots, n$, $\sum_1^m \xi_i = 1$.
Now let

$$v = \min\{\sum_{i=1}^m a_{i1}\xi_i, \sum_{i=1}^m a_{i2}\xi_i, \ldots, \sum_{i=1}^m a_{in}\xi_i\} > 0, \quad (\textit{since } A \textit{ isstrictly positive})$$

The equality in (2.37) can be modified by

$$\max \;\; z = v$$

$$s.t \;\; \sum_{i=1}^m a_{ij}\xi_i = xA \ge v$$

$$\xi_1 + \xi_2 + \ldots + \xi_m = 1.$$

$$\xi_1, \xi_2, \ldots, \xi_m \ge 0, j = 1, 2, \ldots, n$$

$$v > 0.$$

Dividing all the constraints both side by v, we get the following.

$$\max \;\; z = v$$

$$s.t \;\; a_{11}\frac{\xi_1}{v} + a_{21}\frac{\xi_2}{v} + \ldots + a_{m1}\frac{\xi_m}{v} \ge 1$$

$$a_{12}\frac{\xi_1}{v} + a_{22}\frac{\xi_2}{v} + \ldots + a_{m2}\frac{\xi_m}{v} \ge 1$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{1n}\frac{\xi_1}{v} + a_{2n}\frac{\xi_2}{v} + \ldots + a_{mn}\frac{\xi_m}{v} \ge 1$$

$$\frac{\xi_1}{v} + \frac{\xi_2}{v} + \ldots + \frac{\xi_m}{v} = \frac{1}{v}$$

$$\xi_i \geq 0,\ v > 0,\ i = 1, 2, \ldots, m.$$

$$\max\ z = v \Leftrightarrow \min\ z = \frac{1}{v}$$

Let $q_i = \frac{\xi_i}{v}$, *and* $\frac{\xi_i}{v}$ *substituting by* q_i *we get the following*

$$\max\ z = v \Leftrightarrow \min\ z = \frac{1}{v} = q_1 + q_2 + \ldots + q_m$$

$$s.t\ a_{11}q_1 + a_{21}q_2 + \ldots + a_{m1}q_m \geq 1$$

$$a_{12}q_1 + a_{22}q_2 + \ldots + a_{m2}q_2 \geq 1$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{1n}q_1 + a_{2n}q_2 + \ldots + a_{mn}q_m \geq 1$$

$$\xi_i \geq 0, v > 0, i = 1, 2, \ldots, m.$$

To add simplicity, the above linear programming problem we can rewrite as below

$$\min\ z = qu^T\ s.t\ qA \geq w^T,\ q \geq 0 \tag{2.38}$$

where, $q = (q_1, q_2, \ldots, q_m) \in \mathbb{R}^m$, $u = (1, 1, \ldots, 1) \in \mathbb{R}^m$, $w = (1, 1, \ldots, 1) \in \mathbb{R}^n$

Player 2s' mixed optimal strategy $y = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ *can be determined by solving the following* $\min\max$ *problem.*

$$\min_y \max_{u_i} u_i A y^T = \min_y \{\max\{\sum_{j=1}^{n} a_{1j}\gamma_j, \sum_{j=1}^{n} a_{2j}\gamma_j, \ldots, \sum_{j=1}^{n} a_{mj}\gamma_j\}\}, \tag{2.39}$$

where $\sum_{j=1}^{n} \gamma_j = 1$, $u_i = (\eta_1, \eta_2, \ldots, \eta_m)$, $\eta_k = 1$, *for* $k = i$,$\eta_i = 0$, *for* $i \neq k$, $i = 1, 2, \ldots, m$.

Let

$$v = \max\{\sum_{j=1}^{n} a_{1j}\gamma_j, \sum_{j=1}^{n} a_{2j}\gamma_j, \ldots, \sum_{j=1}^{n} a_{mj}\gamma_j\}$$

The $\min\max$ *problem in (2.39) can be modified by the following:*

$$\min\ z = v$$

$$s.t\ \sum_{j=1}^{n} a_{ij}\gamma_j = Ay^T \leq v, i = 1, 2, \ldots, m$$

$$\gamma_1 + \gamma_2 + \ldots + \gamma_j = 1$$

$$\gamma_j \geq 0,\ v > 0\ (since\ A\ is\ strictly\ positive\ matrix)$$

$\forall j = 1, 2, \ldots, n$

Dividing all the constraints both side by v,

$$\min \ z = v$$

$$s.t \ a_{11}\frac{\gamma_1}{v} + a_{12}\frac{\gamma_2}{v} + \ldots + a_{1n}\frac{\gamma_n}{v} \leq 1$$

$$a_{21}\frac{\gamma_1}{v} + a_{22}\frac{\gamma_2}{v} + \ldots + a_{2n}\frac{\gamma_n}{v} \leq 1$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1}\frac{\gamma_1}{v} + a_{m2}\frac{\gamma_2}{v} + \ldots + a_{mn}\frac{\gamma_n}{v} \leq 1$$

$$\frac{\gamma_1}{v} + \frac{\gamma_2}{v} + \ldots + \frac{\gamma_n}{v} = \frac{1}{v}$$

$$\gamma_j \geq 0, v > 0, j = 1, 2, \ldots, n.$$

$$\min \ z = v \Leftrightarrow \max \ z = \frac{1}{v}$$

Let $p_j = \frac{\gamma_j}{v}$ *and substituting* $\frac{\gamma_j}{v}$ *by* p_j *and we have the following*

$$\min \ z = v \Leftrightarrow \max \ z = \frac{1}{v} = p_1 + p_2 + \ldots + p_n$$

$$s.t \ a_{11}p_1 + a_{12}p_2 + \ldots + a_{1n}p_n \leq 1$$

$$a_{21}p_1 + a_{22}p_2 + \ldots + a_{2n}p_n \leq 1$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1}p_1 + a_{m2}p_2 + \ldots + a_{mn}p_n \leq 1$$

$$p_j \geq 0, v > 0, j = 1, 2, \ldots, n.$$

This linear programming can be rewrite as below

$$\max \ z = wp^T \ s.t \ Ap^T \leq u^T, p \geq 0 \tag{2.40}$$

where, $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$, $u = (1, 1, \ldots, 1) \in \mathbb{R}^m$, $w = (1, 1, \ldots, 1) \in \mathbb{R}^n$

The linear programming problem in (2.40) is dual linear programming to the linear programming problem in(2.38).

The linear programming problem in ((2.38)) has a feasible solution $x > 0$, *for which* $xA > w^T$, *since* A *is strictly positive matrix game. Similarly* $y = 0$ *is feasible solution of the linear programming (2.40). Therefore by the Duality Theorem both problems have optimal solution* $\overline{x}$, $\overline{y}$ *respectively, and*

$$\overline{x}u^T = w\overline{y}^T = \delta > 0 \tag{2.41}$$

$$(\overline{x}u^T)/\delta = (w\overline{y}^T)/\delta = 1$$

$\overline{x}/\delta > 0,\ \overline{y}/\delta > 0$ since $\overline{x}$ and $\overline{y}$ are the optimal solution of the linear programming problem (2.38) and (2.40) respectively. i.e. $\overline{x}/\delta$ and $\overline{y}/\delta$ are mixed strategies of player 1 and 2 in the game Γ_A.

$$1 = (w\overline{y}^T)/\delta \leq (\overline{x}A\overline{y}^T)/\delta \leq (\overline{x}u^T)/\delta = 1$$

this implies, $(\overline{x}A\overline{y}^T)/\delta = 1 \Leftrightarrow \overline{x}A\overline{y}^T/\delta^2 = 1/\delta$ (i.e. dividing both side by δ), implies $K(\overline{x}/\delta, \overline{y}/\delta) = 1/\delta$

Let $x \in X$ and $y \in Y$ be arbitrary mixed strategies for players 1 and 2. The following inequalities hold:

$$K(\overline{x}/\delta, y) = \overline{x}Ay^T/\delta \geq (wy^T)/\delta = 1/\delta$$

$$K(x, \overline{y}/\delta) = xA\overline{y}^T/\delta \leq xu^T/\delta = 1/\delta$$

Therefore $(\overline{x}/\delta, \overline{y}/\delta)$ is the saddle point and $1/\delta$ is the value of the game Γ_A with a strictly positive matrix.

case 2: *Now consider the game $\Gamma_{A'}$ with $(m \times n)$ an arbitrary matrix $A' = (a_{ij})$. Then there exists constant $\beta > 0$ such that the matrix $A = A' + B$ is strictly positive, where $B = (\beta_{ij})$ is an $(m \times n)$ matrix, $\beta_{ij} = \beta$, $i = 1, 2, \ldots, m$, $j = l, 2, \ldots, n$. In the game Γ_A there exists a saddle point (x^*, y^*) in mixed strategies, and the value of the game equals $v_A = 1/\delta$, where δ is determined as in case 1.*

From Theorem 5, it follows that $(x^*, y^*) \in Z(\overline{\Gamma}_{A'})$ is a saddle point in the game $\Gamma_{A'}$ in mixed strategies and the value of the game is equal to
$v_{A'} = v_A - \beta = 1/\delta - \beta$ □

Note that from the proof of the Theorem the solution of the any matrix game is correlate to a linear programming problem, and the solution algorithm for the game $\Gamma_{A'}$ is as follows.

1. By employing the matrix A', construct a strictly positive matrix $A = A' + B$, where $B = (\beta_{ij})$, $\beta_{ij} = \beta > 0$.

2. Solve the linear programming problems (2.38),(2.40). Find optimal solution vectors $\overline{x}$, $\overline{y}$ and optimal value. i.e. δ (as in Theorem 7).

3. Construct optimal strategies for the players 1 and 2, respectively,

$$x^* = \overline{x}/\delta,\ y^* = \overline{y}/\delta.$$

4. Compute the value of the game $\Gamma_{A'}$

$$v_{A'} = 1/\delta - \beta.$$

Example 4. *Consider the matrix game Γ_A determined with the matrix*

$$A = \begin{bmatrix} 7 & 3 \\ 2 & 5 \end{bmatrix}$$

Associated linear programming problems(LPP) are of the form:

$$\begin{array}{rl} \min\ w = x_1 + x_2 & \max\ z = y_1 + y_2 \\ 7x_1 + 2x_2 \geq 1 & 7y_1 + 3y_2 \leq 1 \\ 3x_1 + 5x_2 \geq 1 & 2y_1 + 5y_2 \leq 1 \\ x_1 \geq 0, x_2 \geq 0 & y_1 \geq 0, y_2 \geq 0 \end{array}$$

Note that, these problems may be written in the equivalent standard form with constraints in the form of equalities

$$\begin{array}{ll} \min\ w = x_1 + x_2 & \max\ z = y_1 + y_2 \\ s.t\ 7x_1 + 2x_2 - x_3 = 1 & s.t\ 7y_1 + 3y_2 + y_3 = 1 \\ 3x_1 + 5x_2 - x_4 = 1 & 2y_1 + 5y_2 + y_4 = 1 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0 & y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0 \end{array}$$

Solution:

Since the given LP problems are Primal and Dual each other finding solution for one of the LP problem by some modification gives the solution of the other. So, let as start to find the solution of the maximization LP problem by primal/dual simplex.

$$\max\ z = y_1 + y_2 + 0y_3 + 0y_4$$
$$\text{s.t}\ \ 7y_1 + 3y_2y_3 = 1$$
$$2y_1 + 5y_2 + y_4 = 1$$
$$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, y_3 \geq 0$$

Initial tableau:

	y_1	y_2	y_3	y_4	b
$-z$	1	1	0	0	0
y_3	7	3	1	0	1
y_4	2	5	0	1	1

In initial tableau the basic feasible solution is $(y_1, y_2, y_3, y_4) = (0, 0, 1, 1)$ with objective function value $z = 0$, but tableau is not optimal since there are entries in the first with positive value , so choose y_1 to enter the base column, since it has positive objective coefficient. And y_4 is the row with minimum positive ratio, so leave the base and replace by the new entering and pivoting the pivot entry 2.

tableau 2:

	y_1	y_2	y_3	y_4	b
$-z$	0	$-3/2$	0	$-1/2$	$-1/2$
y_3	0	$-29/2$	1	$-7/2$	$-5/2$
y_1	1	$5/2$	0	$1/2$	$1/2$

In tableau 2 failed the primal simplex feasibility, since $b'_1 = -5/2 < 0$, and then we do with dual simplex feasibility (i.e. $c_j \leq 0,$) for all $j = 1,2,3,4$, in this method we need make $b_1 \geq 0$ for all $i = 1,2$. The second row is the pivot row, since $b'_1 = -5/2 < 0$(the entry of the second row in b column is negative), and y_3 leave the base. Choose the pivot column by the minimum positive ratio test $\min\{c_j/a_{1j}\}$, by this test the enter variable is y_2 and pivoting the pivot entry. This we will do in tableau 3

tableau 3:

	y_1	y_2	y_3	y_4	b
$-z$	0	0	$-3/29$	$-4/29$	$-7/29$
y_2	0	1	$-2/29$	$7/29$	$5/29$
y_1	1	0	$5/29$	$-3/29$	$2/29$

Tableau 3 is optimal tableau since $b > 0$ and $c_j \leq 0$ for all $j = 1,2,3,4$.. we have the optimal solution $(\overline{y}_1, \overline{y}_2) = (2/29, 5/29)$ and the optimal value $\overline{z} = 7/29$. For minimization problem optimal solution can get from its dual maximization. As in (1.7) the inverse matrix in the final optimal tableau(tableau 3) is $B^{-1} = \begin{bmatrix} -2/29 & 7/29 \\ 5/29 & -3/29 \end{bmatrix}$. The optimal solution for min problem is given by as in (1.7)$\overline{x} = C_B B^{-1}$, where $C_B = [1\ 1]$ is the coefficient of the original max problem. So, $\overline{x} = C_B B^{-1} = (3/29, 4/29) = (\overline{x}_1, \overline{x}_2)$ and $\overline{w} = 7/29 = \overline{z}$. We need to find the value of the game for such LP problems. The optimal value of the game Γ_A is $v_A = 1/\overline{w} = 29/7$ and the optimal strategies are $x^* = \overline{x}/\overline{w} = (3/7, 4/7)$, $y^* = \overline{y}/\overline{w} = (2/7, 5/7)$

The linear programming problem that have optimal solution is equivalent to the matrix game Γ_A.

Consider the following primal and dual problems of linear programming

$$\begin{aligned} &\min\ xu^T \\ xA \geq& w^T, \\ x \geq& 0, \end{aligned} \tag{2.42}$$

$$\begin{aligned} &\max\ wy^T \\ Ay \leq& u^T, \\ y \geq& 0, \end{aligned} \tag{2.43}$$

where $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ (u and w as in Theorem 7) Let $\overline{X}$ and $\overline{Y}$ be the sets of optimal solutions of the problems (2.42) and (2.43), respectively. Denote $(1/\delta)\overline{X} = \{\overline{x}/\delta |\ \overline{x} \in \overline{X}\}$, $(1/\delta)\overline{Y} = \{\overline{y}/\delta |\ \overline{y} \in Y\}$, $\delta > 0$.

Theorem 7. *Let Γ_A be the $m \times n$ game with the positive matrix A (all elements are positive) and let there be given two dual problems of linear programming (2.42) and (2.43). Then the following conditions hold.*

1. Both linear programming problems have a solution set $\overline{X} \neq \emptyset$ *and* $\overline{Y} \neq \emptyset$*, in which case*

$$\delta = \min_x xu^T = \max_y wy^T$$

2. The value v_A *of the game* Γ_A *is*

$$v_A = 1/\delta,$$

and the strategies

$$x^* = \overline{x}/\delta, \; y^* = \overline{y}/\delta$$

are optimal, where $\overline{x} \in \overline{X}$ *is an optimal solution of the primal problem (2.42) and* $\overline{y} \in \overline{Y}$ *is the optimal solution of the dual problem (2.43).*

3. Any optimal strategies $x^* \in \overline{X}^*$ *and* $y^* \in \overline{Y}^*$ *of the players can be constructed as shown above, i.e.*

$$X^* = (1/\delta)\overline{X}, \; Y^* = (1/\delta)\overline{Y}$$

Proof. *The proof of (1),(2) and the inclusions* $(1/\delta)\overline{X} \subseteq X^*$, $(1/\delta)\overline{Y} \subseteq Y^*$ *immediately follows from proof of Theorem 7 (case 1 and 2). To show for the inverse inclusion. Let* $x^* \in X^*$ *and let as compute the value of objective function (2.42) at* $\overline{x} = \delta x^*$, $\overline{x}u^T = \delta x^* u^T = \delta$ *i.e.* $\overline{x} \in \overline{X}$ *is the optimal solution of (2.42), this implies* $X^* \subseteq (1/\delta)\overline{X}$*. In similar manner* $Y^* \subseteq (1/\delta)\overline{Y}$ □

2.6 Properties of optimal strategies and value of the game

Let $(x^*, y^*) \in X \times Y$ be saddle in mixed strategies for the game Γ_A. To test the point (x^*, y^*) is equilibrium point it is sufficient to check the conditions of saddle point in (2.35) only for $i \in M$ and $j \in N$ not for all $x \in X$ and $y \in Y$.

Theorem 8. *For the situation* (x^*, y^*) *to be an equilibrium (saddle point) in the game* Γ_A*, and the number* $v = K(x^*, y^*)$ *be the value, it is necessary and sufficient that the following inequalities hold for all* $i \in M$ *and* $j \in N$*:*

$$K(i, y^*) \leq K(x^*, y^*) \tag{2.44}$$

$$K(x^*, y^*) \leq K(x^*, j) \tag{2.45}$$

Proof. ***Necessity:*** *Let* (x^*, y^*) *be saddle in the* Γ_A*. The*

$$K(x, y^*) \leq K(x^*, y^*) \leq K(x^*, y)$$

for all $x \in X$ and $y \in Y$. In particular for $u_i \in X$ and $w_j \in Y$, we have this

$$K(i, y^*) = K(u_i, y^*) \leq K(x^*, y^*) \leq K(x^*, w_i) = K(x^*, j)$$

for $i \in M$ and $j \in N$

Sufficiency: *Let $(x^*, y^*) \in X \times Y$ and satisfies (2.44, 2.45) and also let $x = (\xi_1, \xi_2, \ldots, \xi_m) \in X$ and $y = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in Y$. Multiply (2.44) by ξ_i and (2.45) and summing, we have the following*

$$\sum_{i=1}^{m} \xi_i K(i, y^*) = K(x, y^*) \leq \sum_{i=1}^{m} \xi_i K(x^*, y^*) = K(x^*, y^*) \tag{2.46}$$

$$\sum_{j=1}^{n} \gamma_j K(x^*, y^*) = K(x^*, y^*) \leq \sum_{j=1}^{n} \gamma_j K(x^*, j) = K(x^*, y) \tag{2.47}$$

From (2.46) and (2.47), we have obtained for any arbitraries of strategies $x \in X$ and $y \in Y$ the saddle point (x^*, y^*) condition satisfies. □

Corollary 3. *Let (i^*, j^*) be a saddle point in the game Γ_A. Then the situation (i^*, j^*) is also a saddle point in the mixed extension game $\overline{\Gamma}_A$.*

Example 5. *Players 1 and 2 choose integers i and j from the set $1, 2, \ldots, n$. Player 1 wins the amount $|i - j|$. The game is zero-sum. The payoff matrix is square $(n \times n)$ matrix, where $a_{ij} = |i - j|$.*

Suppose the players select integers i and j between 1 and n, and Player 1 wins the amount $a_{ij} = |i - j|$, i.e. the distance between the numbers i and j.

Suppose the first player uses strategy $x^* = (1/2, 0, \ldots, 0, 1/2)$. Then $K(x^*, j) = 1/2|1 - j| + 1/2|n - j| = 1/2(j - 1) + 1/2(n - j) = (n - 1)/2$ for all $1 \leq j \leq n$.

a. Let $n = 2k + 1$ be odd. Then Player 2 has a pure strategy $j^* = (n + 1)/2 = k + 1$ such that $a_{ij^*} = |i - (n + 1)/2| = |i - k - 1| \leq k = (n - l)/2$ for all $i = 1, 2, \ldots, n$.

b. Let $n = 2k$ be even. Then Player 2 has a strategy $y^* = (0, 0, \ldots, 1/2, 1/2, 0, \ldots, 0)$, where $\gamma_k^* = 1/2$, $\gamma_{k+1}^* = 1/2$, $\gamma_k^* = 0$, $j \neq k + 1$, $j \neq k$, $j = 1, 2, \ldots, n$ and $K(i, y^*) = 1/2|i - k| + 1/2|i - k - 1| \leq 1/2k + 1/2(k - 1) = (n - 1)/2$ for all $1 \leq i \leq n$. From the theorem the value of the game is $v = (n - 1)/2$ with optimal strategy x^* for player 1 and for player 2 if n is odd j^*, and y^*, if n even. i.e. $(x^*, j^*) \in Z(\Gamma_A)$ if n odd, and $(x^*, y^*) \in Z(\Gamma_A)$ if n even.

Theorem 9. *Let Γ_A be an $(m \times n)$ game. For the situation in mixed strategies, let (x^*, y^*) be an equilibrium (saddle point) in the game $\overline{\Gamma}_A$, it is necessary and sufficient that the following equality holds*

$$\max_{1 \leq i \leq m} K(i, y^*) = \min_{1 \leq j \leq n} K(x^*, j) \tag{2.48}$$

Proof. ***Necessity:*** *If (x^*, y^*) is a saddle point,then in Theorem 9 by (2.44) and (2.45), we have*

$$K(i, y^*) \leq K(x^*, y^*) \leq K(x^*, j)$$

for all $i \in M$ and $j \in N$

$$K(i, x^*) \leq K(x^*, j)$$

for each i and j, i.e.

$$\max_{1 \leq i \leq m} K(i, y^*) \leq \min_{1 \leq j \leq n} K(x^*, j). \tag{2.49}$$

For the converse, we have

$$K(x^*, y^*) = \sum_{j=1}^{n} \gamma_j^* K(x^*, j) \geq \min_{1 \leq j \leq n} K(x^*, j) \tag{2.50}$$

and

$$K(x^*, y^*) = \sum_{i=1}^{m} \xi_i^* K(i, y^*) \leq \max_{1 \leq i \leq m} K(i, y^*) \tag{2.51}$$

from (2.50) and (2.51),we have the following

$$\min_{1 \leq j \leq n} K(x^*, j) \leq \max_{1 \leq i \leq m} K(i, y^*) \tag{2.52}$$

Therefore form (2.49) and (2.52), we get the following equality.

$$\max_{1 \leq i \leq m} K(i, y^*) = \min_{1 \leq j \leq n} K(x^*, j) = K(x^*, y^*). \tag{2.53}$$

Sufficiency:

Consider the point (x^*, y^*), and the equality in (2.48) holds. We need to show (x^*, y^*) is saddle point. From (2.53) we get the following result

$$K(i, y^*) \leq \max_{1 \leq i \leq m} K(i, y^*) = K(x^*, y^*) = \min_{1 \leq j \leq n} K(x^*, j) \leq K(x^*, j)$$

for each i and j. Therefore the point (x^*, y^*) is saddle point. □

Theorem 10. *The following relation holds for the matrix game Γ_A*

$$\max_x \min_j K(x, j) = v_A = \min_y \max_i K(i, y), \tag{2.54}$$

Proof. *To prove the theorem check the conditions*

$$i. \max_x \min_j K(x, j) \leq v_A \leq \min_y \max_i K(i, y),$$

$$ii. \max_x \min_j K(x,j) \geq v_A \geq \min_y \max_i K(i,y).$$

(i) Let (x^*, y^*) be saddle point. By saddle point condition we have

$$\min_j K(x,j) \leq \sum_{j=1}^{n} \gamma_j^* K(x,j) = K(x,y^*) \leq v_A$$

for any x. This implies

$$\max_x \min_j K(x,j) \leq v_A.$$

And

$$v_A \leq K(x^*, y) = \sum_{i}^{m} \xi^* K(i,y) \leq \max_i K(i,y)$$

for any y. This implies that

$$v_A \leq \min_y \max_i K(i,y),$$

hence

$$\max_x \min_j K(x,j) \leq v_A \leq \min_y \max_i K(i,y)$$

(ii) Let the situation (x^*, y^*) be saddle point of $\overline{\Gamma}_A$ game as in Theorem 10. By Theorem 9 in (2.44) and (2.45) we have $K(i,y^*) \leq K(x^*,y^*) = v_A \leq K(x^*,j)$ for any $i \times j \in M \times N$.

$$\max_i K(i,y^*) \leq K(x^*,y^*)$$

for any $i \in M$. From this we have the following

$$\min_y \max_i K(i,y) \leq \max_i K(i,y^*) \leq K(x^*,y^*) \tag{2.55}$$

for each $i \in M$ and $y \in Y$. And also

$$K(x^*,y^*) \leq \min_j K(x^*,j)$$

since $K(x^*,y^*) \leq K(x^*,j)$ for any $j \in N$ (by (2.45)in Theorem 9)

$$K(x^*,y^*) \leq \min_j K(x^*,j) \leq \max_x \min_j K(x,j) \tag{2.56}$$

And from (2.55) and (2.56) we get the following condition

$$\min_y \max_i K(i,y) \leq \max_i K(i,y^*) \leq K(x^*,y^*) \leq \min_j K(x^*,j) \leq \max_x \min_j K(x,j)$$

so, from (i) and (ii) we get the following result,

$$\min_y \max_i K(i,y) = \max_i K(i,y^*) = K(x^*,y^*) = v_A = \min_j K(x^*,j) = \max_x \min_j K(x,j)$$

□

Example 6. ***(Application of Theorem 11)***$(2 \times n)$ ***game:*** *We use in this example player 1 with 2 and player 2 with n strategies. The matrix have this form*

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \end{bmatrix}$$

Suppose Player 1 choose a mixed strategy $x = (\xi, 1-\xi)$ *and player 2 choose pure strategy* $j \in N$. *Then the payoff to player 1 at* (x, j) *is*

$$K(x, j) = \xi a_{1j} + (1-\xi)a_{2j}$$

The payoff function $K(x, j)$ *is linear function and geometrically a straight line graph with coordinate* (ξ, K).

$$H(\xi) = \min_j K(x, j) = \min\{K(x, 1), K(x, 2), \dots, K(x, n)\} =$$

$$\min\{(a_{11} - a_{21})\xi + a_{21}, (a_{12} - a_{22})\xi + a_{22}, \dots, (a_{1n} - a_{2n})\xi + a_{2n}\}$$

is the lower part of the straight lines $K(x, j)$ *graphs.The point* ξ^*, *at which the maximum of the function* $H(\xi^*)$ *is achieved with respect to* $\xi \in [0, 1]$, *yields the required optimal solution* $x^* = (\xi^*, 1 - \xi^*)$ *and the value of the game* $v_A = H(\xi^*)$ *For definiteness, we shall consider the game with the matrix*

$$A = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 2 & 1 & 4 & 0 \end{bmatrix}$$

For each $j = 1, 2, 3, 4$ $K(x, 1) = -\xi + 2$, $K(x, 2) = 2\xi + 1$,$K(i, 3) = -3\xi + 4$, $K(x, 4) = 4\xi$. *The lower intersection part of these lines is*

$$H(\xi) = \min\{K(x, 1) = -\xi+2,\ K(x, 2) = 2\xi+1,\ K(x, 3) = -3\xi+4,\ K(x, 4) = 4\xi\}$$

The $\max H(\xi^*)$ *of the function* $H(\xi)$ *is found as the intersection of* $K(x, 1) = -\xi+2$ *and the* $K(x, 4) = 4\xi$ *equations. Thus,* ξ^* *is a solution of the equation.*

$$K(x, 1) = -\xi + 2 = 4\xi = v_A$$

i.e. $\xi^* = 2/5$ *this implies* $x = (\xi^*, 1 - \xi^*) = (2/5, 3/5)$.

$$K(x^*, 1) = K(x^*, 4) = 8/5$$

The optimal strategy $y^* = (\gamma_1^*, \gamma_2^*, \gamma_3^*, \gamma_4^*)$ *for player 2 is found from below reasonings.*

$$K(x^*, y^*) = \gamma_1^* K(x^*, 1) + \gamma_2^* K(x^*, 2) + \gamma_3^* K(x^*, 3) + \gamma_4^* K(x^*, 4)$$

$\gamma_2^*=\gamma_3^*=0$, *since* $K(x^*, 2) = 9/5 > 8/5$ *and* $K(x^*, 3) = 14/5 > 8/5$.

From this we have $K(x^*, y^*) = \gamma_1^* K(x^*, 1) + \gamma_4^* K(x^*, 4)$ *and* γ_1^*,γ_4^* *can be found from* $\gamma_1^* + 4\gamma_4^* = 8/5$, $2\gamma_1^* = 8/5$.

$\gamma_1^* = 4/5$,$\gamma_4^* = 1/5$ *and the optimal strategy of Player 2 is* $y^* = (4/5, 0, 0, 1/5)$.

Theorem 11. *Let* $x^* = (\xi_1^*, \xi_2^*, \dots, \xi_m^*)$ *and* $y^* = (\gamma_1^*, \gamma_2^*, \dots, \gamma_n^*)$ *be optimal //strategies in the game* $\overline{\Gamma}_A$ *and* v_A *be the value of the game. Then for any* i*, for which* $K(i, y^*) < v_A$*, there must be* $\xi_i^* = 0$ *, and for any* j *such that* $v_A < K(x^*, j)$ *there must be* $\gamma_j^* = 0$*. Conversely, if* $\xi_i^* > 0$*, then* $K(i, y^*) = v_A$*, and if* $\gamma_j^* > 0$*, then* $K(x^*, j) = v_A$*.*

Proof. *Suppose that for some* $i_0 \in M$*,* $K(i_0, y^*) < v_A$ *and* $\xi_{i_0}^* \neq 0$*. Then we have*

$$K(i_0, y^*)\xi_{i_0}^* < v_A \xi_{i_0}^*.$$

For all $i \in M$*,* $K(i, y^*) \leq v_A$*, therefore*

$$K(i, y^*)\xi_i^* \leq v_A \xi_i^*.$$

Consequently, $K(x^*, y^*) < v_A$*, which contradicts to the fact that* v_A *is the value of the game. The second part of the Theorem can be proved in a similar manner.* □

Definition 12. *Player 1's (or 2's) pure strategy* $i \in M$*(or* $j \in N$*) is called an essential or active strategy if there exists the player's optimal strategy* $x^* = (\xi_1^*, \xi_2^*, \dots, \xi_m^*)$*(or* $y^* = (\gamma_1^*, \gamma_1^*, \gamma_2^*, \dots, \gamma_n^*)$*) for which* $\xi_i^* > 0$ *(or* $\gamma_j^* > 0$*) respectively*

From the definition, and from the latter theorem, it follows that for each essential strategy i of Player 1 and any optimal strategy $y^* \in Y^*$ of Player 2 in the game $\overline{\Gamma}_A$ the following equality holds:

$$K(i, y^*) = a_i y^* = v_A$$

similar equality holds for any essential strategy $j \in N$ of Player 2 and for the optimal strategy $x^* \in X^*$ of Player 1

$$K(x^*, j) = x^* a^j = v_A$$

If the equality $a_i y = v_A$ holds for the pure strategy $i \in M$ and mixed strategy $y \in Y$, then the strategy i is the best reply to the mixed strategy y in the game Γ_A. Thus, If the pure strategy of the player is essential, then it is the best reply to any optimal strategy of the opponent.

2.7 Dominance of strategies

The complexity of solving a matrix game increases as the dimensions of the matrix A increase. In some cases, however, the analysis of payoff matrices permits a conclusion that some pure strategies do not appear in the spectrum of optimal strategy. This can result in replacement of the original matrix by the payoff matrix of a smaller dimension.

Definition 13. *Strategy x' of Player 1 is said to dominate strategy x'' in the $(m \times n)$ game Γ_A if the following inequalities hold for all pure strategies $j \in 1, 2, \ldots, n$ of Player 2*

$$x'a^j \geq x''a^j \tag{2.57}$$

Similarly, strategy y' of Player 2 dominates his strategy y'' if for all pure strategies $i \in \{1, 2, \ldots, m\}$ of Player 1

$$a_i y'^T \leq a_i y''^T \tag{2.58}$$

If inequalities (2.57), (2.58) are satisfied as strict inequalities, then we are dealing with a strict dominance. A special case of the dominance of strategies is their equivalence.

Definition 14. *Strategies x' and x'' of Player 1 are equivalent in the game Γ_A if for all $j \in \{1, 2, \ldots, n\}$*

$$x'a^j = x''a^j.$$

We shall denote this fact by $x' \sim x''$.

For two equivalent strategies x' and x'' the following equality holds (for every $y \in Y$)

$$K(x', y) = K(x'', y)$$

Similarly, strategies y' and y'' of Player 2 are equivalent $(y' \sim y'')$ in the game Γ_A if for all $i \in \{1, 2, \ldots, m\}$

$$a_i y'^T = a_i y''^T.$$

Hence we have that for any mixed strategy $x \in X$ of Player 1 the following equality holds

$$K(x, y') = K(x, y'').$$

For pure strategies the above definitions are transformed as follows. If Player l's pure strategy i' dominates strategy i'' and Player 2's pure strategy j' dominates strategy j'' of the same player, then for all $i = 1, \ldots, m$; $j = 1, \ldots, n$ the following inequalities hold

$$a_{i'j} \geq a_{i''j}, \ a_{ij'} \leq a_{ij''}$$

This can be written in vector form as follows:

$$a_{i'} \geq a_{i''}, \ a^{j'} \leq a^{j''}$$

Equivalence of the pairs of strategies i', $i''(i' \sim i'')$ and j', $j''(j \sim j'')$ implies that the conditions $a_{i'} = a_{i''}$, $(a^{j'} = a^{j''})$ are satisfied.

Definition 15. *The strategy x''(or y'') of Player 1(or 2) is dominated if there exists a strategy $x \neq x''$(or $y' \neq y''$) of this player which dominates x'', (y''); otherwise strategy x''(or y'') is not dominated strategy.*

Similarly, strategy x''(or y'') of Player 1(or 2) is strictly dominated if there exists a strategy x'(or y') of this player which strictly dominates x''(or y''), i.e. for all $j = 1, 2, \ldots n$(or $i = 1, 2, \ldots, m$) the following inequalities hold

$$x'a^j > x''a^j, a_i y'^T < a_i y''^T;$$

otherwise strategy x''(or y'') of Player 1,(or 2) is not strictly dominated.

Theorem 12. *If, in the game $\overline{\Gamma}_A$, strategy x' of one of the players dominates an optimal strategy x^*, then strategy x' is also optimal.*

Proof. *Let x' and x^* be strategies of Player 1. Then, by dominance,*

$$x'a^j \geq x^*a^j$$

for all $j = 1, 2, \ldots, n$. Hence, using the optimality of strategy x^* (see Theorem 11), we get

$$v_A = \min_j x^*a^j \geq \min_j x'a^j \geq \min_j x^*a^j = v_A$$

for all $j = l, 2, \ldots, n$.

□

Theorem 13. *If, in the game $\overline{\Gamma}_A$, strategy x^* of one of the players is optimal, then strategy x^* is not strictly dominated.*

Proof. *For definiteness, let x^* be an optimal strategy of Player 1. Assume that x^* is strictly dominated, i.e. there exist such strategy $x' \in X$ that*

$$x'a^j > x^*a^j, j = 1, 2, \ldots, n.$$

Hence

$$\min_j x'a^j > \min_j x^*a^j.$$

However, by the optimality of $x^* \in X$, the equality $\min_j x^*a^j = v_A$ is satisfied. Therefore, the strict inequality

$$\max_x \min_j xa^j > v_A$$

holds and this contradicts to the fact that v_A is the value of the game (Theorem 11). The contradiction proves the theorem. □

Generally the reverse of this theorem is may not true. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ in this matrix game the first and second strategies of player 1 are no strictly dominated, and also they are not optimal.

If $x = (\xi_1, \xi_2, \ldots, \xi_m) \in X$ and $1 \leq i \leq m+1$, then the extension of strategy x at the i^{th} place is the vector $\overline{x} = (\xi_1, \xi_2, \ldots, \xi_{i-1}, 0, \xi_i, \ldots, \xi_m) \in \mathbb{R}^{m+1}$. Thus the extension of vector $(1/3, 2/3, 1/3)$ at the 2^{nd} place is the vector $(1/3, 0, 2/3, 1/3)$; the extension at the 4^{th} place is the vector $(1/3, 2/3, 1/3, 0)$; the extension at the 1^{st} place is the vector $(0, 1/3, 2/3, 1/3)$.

Theorem 14. *Let Γ_A be an $(m \times n)$ game. We assume that the i^{th} row of matrix A is dominated (i.e. Player 1's pure strategy i is dominated) and let $\Gamma_{A'}$ be the game with the matrix A' obtained from A by deleting the i^{th} row. Then the following assertions hold.*

1. $v_A = v_{A'}$.

2. Any optimal strategy y^* of Player 2 in the game $\Gamma_{A'}$ is also optimal in the game. Γ_A

3. If x^* is an arbitrary optimal strategy of Player 1 in the game $\Gamma_{A'}$ and $\overline{x}_i^*$ is the extension of strategy x^* at the i^{th} place, then $\overline{x}_i^*$ is an optimal strategy of that player in the game Γ_A.

4. If the i^{th} row of the matrix A is strictly dominated, then an arbitrary optimal strategy $\overline{x}^*$ of Player 1 in the game Γ_A can be obtained from an optimal strategy x^* in the game $\Gamma_{A'}$ by the extension at the i^{th} place.

Proof. *We assume, without loss of generality, that the last m^{th} row is dominated. Let $x = (\xi_1, \xi_2, \ldots, \xi_m)$ be a mixed strategy which dominates the row m. If $\xi_m = 0$, then from the dominance condition for all $j = 1, 2, \ldots, n$ we get*

$$\sum_{i=1}^{m} \xi_i a_{ij} = \sum_{i=1}^{m-1} \xi_i a_{ij} \geq a_{mj}$$

$$\sum_{i=1}^{m-1} \xi_i = 1, \ \xi_i \geq 0, \ i = 1, 2, \ldots, m. \tag{2.59}$$

Otherwise $(\xi_m > 0)$, consider the vector $x' = (\xi'_1, \xi'_2, \ldots, \xi'_m)$, where

$$\xi'_i = \begin{cases} \xi_i/(1-\xi_m), & if \quad i \neq m, \\ 0, & if \quad i = m. \end{cases} \tag{2.60}$$

Components of the vector x are non-negative, $(\xi'_i \geq 0, i = 1, 2, \ldots, m)$ and $\sum_{i=1}^{m} \xi'_i =$ 1. On the other hand, for all $j = 1, 2, \ldots, n$ we have

$$\frac{1}{1-\xi_m} \sum_{i=1}^{m} \xi_i a_{ij} \geq a_{mj} \sum_{i=1}^{m} \xi_i$$

or

$$\frac{1}{1-\xi_m}\sum_{i=1}^{m-1}\xi_i a_{ij} \geq a_{mj}\frac{i}{1-\xi_m}\sum_{i=1}^{m-1}\xi_i$$

Considering (2.60) we get

$$\sum_{i=1}^{m-1}\xi'_i a_{ij} \geq a_{mj}\sum_{i=1}^{m-1}\xi'_i = a_{mj},\ j=1,2,\ldots,n$$

$$\sum_{i=1}^{m-1}\xi'_i = 1,\ \xi'_i \geq 0,\ i=1,2,\ldots,m-1. \tag{2.61}$$

Thus, from the dominance of the m^{th} row it always follows that it does not exceed a linear combination of the remaining $m-1$ rows.

Let $(x^*, y^*) \in Z(\Gamma_{A'})$ be a saddle point in the game $\Gamma_{A'}$, $x^* = (\xi_1^*, \xi_2^*, \ldots, \xi_{m-1}^*)$, $y^* = (\gamma_1^*, \gamma_2^*, \ldots, \gamma_n^*)$. To prove 1,2,3 of the theorem, it suffices to show that $K(x_m^*, y^*) = v_{A'}$, and

$$\sum_{j=1}^{n} a_{ij}\gamma_j^* \leq v_A \leq \sum_{i=1}^{m-1} a_{ij}\xi_i^* + 0.a_{mj} \tag{2.62}$$

for all $i = 1,2,\ldots,m, j = l,2,\ldots,n$.

The first equality is straightforward, and the optimality of strategies (x^*, y^*) in the game $\Gamma_{A'}$ implies that the following inequalities are satisfied

$$\sum_{j=1}^{n} a_{ij}\gamma_j^* \leq v_{A'} \leq \sum_{i=1}^{m-1} a_{ij}\xi_i^*,\ i=1,2,\ldots,m-1,\ j=1,2,\ldots,n. \tag{2.63}$$

The first of the inequalities (2.62) is evident from (2.63). We shall prove the first inequality. To do this, it suffices to show that

$$\sum_{j=1}^{n} a_{mj}\gamma_j^* \leq v_{A'}$$

From inequalities (2.60), (2.61) we obtain

$$\sum_{j=1}^{n} a_{mj}\gamma_j^* \leq \sum_{j=1}^{n}\sum_{i=1}^{m-1} a_{ij}\xi_i^*\gamma_j^* \leq \sum_{i=1}^{m-1} v_{A'}\xi'_i = v_{A'}$$

which proves the first part of the theorem.

To prove the second part of the theorem (4), it suffices to note that in the case of strict dominance of the m^{th} row the inequalities (2.62), (2.63) are satisfied as strict inequalities for all $j = l, 2, \ldots, n$; hence

$$\sum_{j=1}^{n} a_{mj}\gamma_j^* < \sum_{j=1}^{n}\sum_{i=1}^{m-1} a_{ij}\xi_i'\gamma_j^* \leq v_{A'}.$$

From Theorem 12, we then have that the m^{th} component of any optimal strategy of Player 1 in the game Γ_A is zero. □

Theorem 15. *Let Γ_A be an $(m \times n)$ game. Assume that the j^{th} column of the matrix A is dominated and $\Gamma_{A'}$ is the game having the matrix A' obtained from A by deleting the j^{th} column. Then the following are true.*

1. $v_A = v_{A'}$.

2. Any optimal strategy x^* of Player 1 in the game $\Gamma_{A'}$ is also optimal in the game Γ_A

3. If y^* is an arbitrary optimal strategy of Player 2 in the game $\Gamma_{A'}$ and $\overline{y}^*$ is the extension of strategy y^* at the j^{th} place, then $\overline{y}^*$ is an optimal strategy of Player 2 in the game Γ_A.

4. Further, if the j^{th} column of the matrix A is strictly dominated, then an arbitrary optimal strategy $\overline{y}^*$ of Player 2 in the game Γ_A can be obtained from an optimal strategy y^* in the game $\Gamma_{A'}$ by extension at the j^{th} place.

The prove of this Theorem is in similar manner as Theorem 14. To summarize Theorem 14 and 15, if the matrix row (or column) is not greater (or not smaller) than a linear combination of the remaining rows (or columns) of the matrix, then to find solution of the game, this row(or column) can be deleted. In this case, an extension of optimal strategy in the reduced matrix game yields an optimal solution of the original game. Note that coefficients of the linear combination of row(or column) is in $[0, 1]$ and their sum is 1.

Example 7. *Let us consider the game with the matrix*

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 2 & 3 & 1 & 3 \\ 3 & 1 & 2 & 0 \\ 0 & 3 & 0 & 6 \end{bmatrix}$$

Since the 3^{rd} row a_3 dominates the 1^{st} row ($a_3 \geq a_1$), then, by deleting the 1^{st} row, we obtain

$$A_1 = \begin{bmatrix} 2 & 3 & 1 & 3 \\ 3 & 1 & 2 & 0 \\ 0 & 3 & 0 & 6 \end{bmatrix}$$

In this matrix the 3^{rd} *column* a^3 *dominates the* 1^{st} *column* a^1. *Hence we get*

$$A_2 = \begin{bmatrix} 3 & 1 & 3 \\ 1 & 2 & 0 \\ 3 & 0 & 6 \end{bmatrix}$$

In A_2 *matrix no row (or column) is dominated by the other row (or column). At the same time, the* 1^{st} *column* a^1 *of* A_2 *is dominated by the linear combination of columns* a^2 *and* a^3, *i.e.* $a^1 \geq 1/2a^2 + 1/2a^3$, *since* $3 > 1/2 + 1/2.3$, $1 = 1/2.2 + 1/2.0$, $3 = 0.1/2 + 1/2.6$. *By eliminating the* 1^{st} *column, we obtain*

$$A_3 = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 6 \end{bmatrix}$$

In this matrix the 1^{st} *row is equal to the linear combination of the second and third rows with a mixed strategy* $x = (0, 1/2, 1/2)$, *since* $1 = 1/22 + 01/2$, $3 = 0 - 1/2 + 6 - 1/2$. *Thus, by eliminating the* 1^{st} *row, we obtain the matrix*

$$A_4 = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$$

The players' optimal strategies x^* *and* y^* *in the game with this matrix are* $x^* = y^* = (3/4, 1/4)$, *in which case the game value* v *is 3/2. The* A_4 *matrix* A_4 *is obtained by deleting the first two rows and columns; hence the players' optimal strategies in the original game are extensions of these strategies at the 1st and 2nd places, i.e.* $\overline{x}^* = \overline{y}^* = (0, 0, 3/4, 1/4)$.

Conclusion

A game with two rational players in which the gain payoff for one is loss for the other is called two person zero-sum game. i.e the sum of payoffs for the two players are zero. Two person zero-sum game with finite sets of strategies are called matrix games. Rational players always seeks to maximize his payoff by choosing a best strategy. If the matrix of the game is payoff for Player 1, then player 1 at worst case guarantee himself to maximize the minimum loss of player 2. Similarly player 2 at worst case guarantee himself to minimize the maximum payoff player 1. Any mixed matrix game has optimal solution which is called saddle point in mixed strategies. A rational player does not choose dominated strategy.

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