# Three-Level Laser Dynamics with Coherent and Squeezed Light

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of Doctor of Philosophy in Physics

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## DECLARATION

I hereby declare that this PhD dissertation is my original work and has not been presented for a degree in any other university, and that all sources of material used for the dissertation have been duly acknowledged.

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by

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# Abstract

We study the squeezing and statistical properties of the light produced by a three-level laser whose cavity contains a parametric amplifier and with the cavity mode driven by coherent light and coupled to a squeezed vacuum reservoir. We obtain stochastic differential equations associated with the normal ordering using the pertinent master equation. Making use of the solutions of the resulting differential equations, we calculate the quadrature variances and squeezing spectrum. We also determine the mean and variance of the photon number and the photon number distribution for the cavity mode employing the Q function. It is found that the parametric amplifier and squeezed vacuum reservoir increase the degree of squeezing, but the driving coherent light does not have any effect on the squeezing. Moreover, the mean photon number increases considerably due the driving coherent light, the parametric amplifier, and the squeezed vacuum reservoir.

# Contents

# 1

# Introduction

There has been a considerable interest in the analysis of the squeezing and statistical properties of the light generated by three-level lasers [1 - 22]. A three-level laser may be defined as a quantum optical system in which three-level atoms in a cascade configuration, initially prepared in a coherent superposition of the top and bottom levels, are injected into a cavity coupled to a vacuum reservoir via a single-port mirror. A three-level laser may have additional or modified features. One interesting additional feature of a three-level laser involves the coupling of the top and bottom levels of the atoms injected into the cavity by a strong coherent light. When a three-level atom in a cascade configuration makes a transition from the top to the bottom level via the intermediate level, two photons are generated. If the two photons have the same frequency, then the three-level atom is called degenerate three-level atom otherwise it is called nondegenerate.

Three-level lasers in which the crucial role is played by the coherent superposition of the top and bottom levels of the injected atoms have been studied by several authors [1 - 7]. These studies show that this quantum optical system can generate light in a squeezed state under certain conditions. Ansari [1] has calculated the quadrature variances of the cavity mode for a degenerate three-level laser employing the steady state solutions of the equations of evolution of the expectation values of the cavity mode variables. He has found that the cavity mode is in a squeezed state if the probability for the injected atoms to be in the bottom level is larger than the probability for the atoms to be in the top level. And almost perfect squeezing can be obtained for slightly more probability for the atoms to be in the bottom level and for large value of linear gain coefficient. In addition, with the aid of the Q function he has determined the photon number distribution. This result shows that the photon number distribution decreases with the number of photons. Recently, Tesfa [2] has studied the squeezing property of the cavity modes produced by a nondegenerate three-level laser applying the solutions of stochastic differential equations. He has found that the two-mode cavity radiation exhibits squeezing if the atoms are initially prepared with more atoms in the bottom level than in the upper level, and the degree of squeezing increases with the linear gain coefficient.

A three-level laser with the top and bottom levels of the atoms injected into the cavity coupled by a strong coherent light can also generate light in a squeezed state [8 - 14]. Ansari et al. [9] have considered a degenerate three-level laser in which the injected atoms are initially prepared in the top level and with the top and bottom levels of the atoms coupled by a strong coherent light. They have shown that for sufficiently strong coherent light the system behaves like a degenerate parametric oscillator.

Furthermore, it has been predicted theoretically and subsequently confirmed experimentally that a parametric oscillator produces light with a maximum intracavity squeezing of 50% below the coherent-state level [21 - 31]. Some authors have considered a three-level laser whose cavity contains a parametric amplifier [4, 10]. Fesseha has studied the squeezing and statistical properties of the light produced by a degenerate three-level laser whose cavity contains a degenerate parametric amplifier [4]. His study indicates that a more squeezed light could be generated by a combination of these two quantum optical systems. This study also shows that the presence of the parametric amplifier in the laser cavity increases significantly the mean photon number. On the other hand, Ambachew and Fesseha [10] have considered the same system with the injected atoms having equal probability to be in the upper and lower levels and with these two levels coupled by the pump mode emerging from the parametric amplifier. This study shows that the system generates light in a squeezed state with a maximum intracavity squeezing of 93% below the coherent-state level.

In this PhD dissertation we seek to investigate the squeezing and statistical properties of the light generated by a three-level laser whose cavity contains a parametric amplifier and with the cavity mode driven by coherent light and coupled to a squeezed vacuum reservoir. We consider a degenerate as well as a nondegenerate three-level laser in which the pump mode emerging from the parametric amplifier does not couple the top and bottom levels of the injected atoms. This could be realized by putting on the right-side of the nonlinear crystal a screen which absorbs the pump mode.

We carry out our analysis applying the solutions of the pertinent stochastic differential equations associated with the normal ordering. These differential equations are obtained applying the master equation derived in the linear and adiabatic approximation schemes following the approach described in Ref.[21]. We calculate, employing the solutions of the differential equations and the correlation properties of the noise forces, the quadrature variances and squeezing spectrum. Moreover, we determine the mean and variance of the photon number as well as the photon number distribution using the Q function. The Q function is obtained with the aid of the antinormally ordered characteristics function defined in the Hiesenberg picture. In addition, we obtain the mean and the normally-ordered variance of the photon count of the output mode for the degenerate system under consideration. Finally, we calculate the mean and the normally-ordered variances of the photon number sum and difference of the cavity modes as well as the photon count sum and difference of the output modes for the nondegenerate case.

# 2

# Degenerate Three-Level Laser

We consider a degenerate three-level laser into which three-level atoms in a cascade configuration and initially prepared in a coherent superposition of the top and bottom levels are injected at a constant rate and removed from the laser cavity after sometime. We discuss the case in which the cavity contains a parametric amplifier and the cavity mode is driven by coherent light and coupled to a squeezed vacuum reservoir as shown in Fig. 2.1. We carry out our analysis for the case in which the pump mode emerging from the parametric amplifier does not couple the top and bottom levels of the three-level atoms.

In this chapter we first seek to obtain the master equation and stochastic differential equations for the cavity mode variables. Applying the solutions of the differential equations, we calculate the quadrature variances and the squeezing spectrum. In addition, we obtain the Q function employing the antinormally ordered characteristic function defined in the Heisenbeg picture. We calculate applying the Q function the mean photon number, the variance of the photon number, and the photon number distribution for the cavity mode. Finally, we calculate the mean and the normally-ordered photon count variance of the output mode.

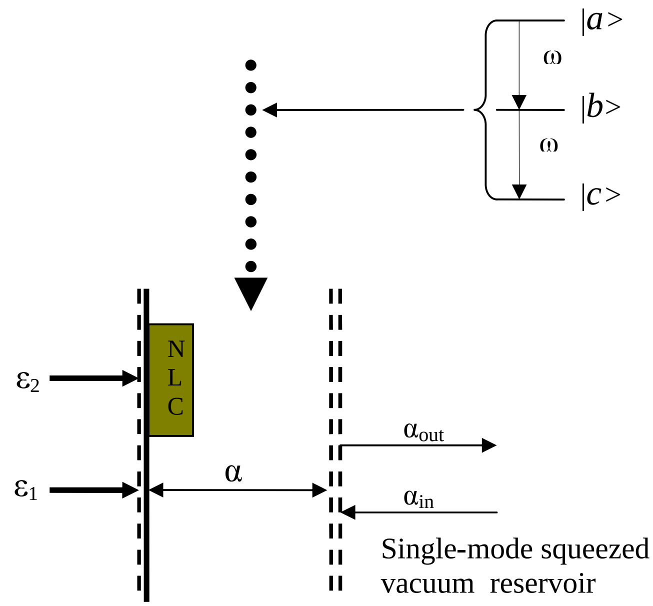


*Fig. 2.1:* Schematic representation of a degenerate three-level laser with a parametric amplifier, a driving coherent light, and a squeezing vacuum reservoir.

## 2.1 Stochastic Differential Equations

### 2.1.1 Master equation

In this section we wish to obtain the equation of evolution of the density operator for a degenerate three-level laser whose cavity contains a parametric amplifier and with the cavity mode driven by coherent light and coupled to a single-mode squeezed vacuum reservoir. We first drive the equation of evolution of the density operator for the three-level laser applying the linear and the adiabatic approximation schemes [4, 21]. Then after obtaining the properties of the reservoir submode operators, we drive the time evolution of the reduced density operator for a cavity mode coupled to a single-mode squeezed vacuum reservoir [22]. Finally, with the help of the two resulting equations, we write the master equation for the system under consideration.

**Degenerate Three-Level Laser**

Three-level atoms in a cascade configuration are injected at a constant rate $r_a$ and removed from the cavity after a certain time $\tau$. We denote the top, intermediate, and bottom levels of a three-level atom by $|a\rangle$, $|b\rangle$, and $|c\rangle$ as shown in Fig. 2.1. We assume the cavity mode to be at resonance with the two transitions $|a\rangle \rightarrow |b\rangle$ and $|b\rangle \rightarrow |c\rangle$, and with direct transition between levels $|a\rangle$ and $|c\rangle$ to be dipole forbidden.

The interaction of a degenerate three-level atom with a cavity mode can be described in the interaction picture by the Hamiltonian

$$\hat{H} = ig\left((|a\rangle\langle b| + |b\rangle\langle c|)\hat{a} - \hat{a}^\dagger(|b\rangle\langle a| + |c\rangle\langle b|)\right), \tag{2.1}$$

where $g$ is the coupling constant and $\hat{a}$ is the annihilation operator for the cavity mode. Considering the atom to be initially in the state

$$|\psi_A(0)\rangle = C_a|a\rangle + C_c|c\rangle, \tag{2.2}$$

the density operator for a single atom can be written as

$$\hat{\rho}_A(0) = \rho_{aa}^{(0)}|a\rangle\langle a| + \rho_{ac}^{(0)}|a\rangle\langle c| + \rho_{ca}^{(0)}|c\rangle\langle a| + \rho_{cc}^{(0)}|c\rangle\langle c|, \tag{2.3}$$

where

$$\rho_{aa}^{(0)} = |C_a|^2, \qquad \rho_{cc}^{(0)} = |C_c|^2 \tag{2.4}$$

are the probabilities for the atom to be in the upper and the bottom levels at the initial time and

$$\rho_{ac}^{(0)} = C_a C_c^* \tag{2.5}$$

is the atomic coherence at the initial time.

Suppose $\hat{\rho}_{AR}(t, t_j)$ is the density operator for a single atom plus the cavity mode at time $t$, with the atom injected at time $t_j$ such that $t - \tau \leq t_j \leq t$. Then the density operator for all atoms in the cavity plus the cavity mode at time $t$ can be written as

$$\hat{\rho}_{AR}(t) = r_a \sum \hat{\rho}_{AR}(t, t_j)\Delta t_j, \tag{2.6}$$

where $r_a$ represents the rate at which the atoms are injected into the cavity. Now converting the summation into integration in the limit $\Delta t_j \rightarrow 0$, we have at time $t$

$$\hat{\rho}_{AR}(t) = r_a \int_{t-\tau}^{t} \hat{\rho}_{AR}(t, t')dt' \tag{2.7}$$

and on differentiating with respect to $t$, there follows

$$\frac{d\hat{\rho}_{AR}(t)}{dt} = r_a\left(\hat{\rho}_{AR}(t,t) - \hat{\rho}_{AR}(t,t-\tau)\right) + r_a\int_{t-\tau}^{t}\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t')dt'. \tag{2.8}$$

We observe that $\hat{\rho}_{AR}(t,t)$ is the density operator for the cavity mode plus an atom injected at time $t$ and $\hat{\rho}_{AR}(t,t-\tau)$ represents the density operator for an atom plus the cavity mode at time $t$ with the atom being removed from the cavity at this time. Therefore, these density operators can be decoupled, so that

$$\hat{\rho}_{AR}(t,t) = \hat{\rho}_{A}(0)\hat{\rho}(t), \tag{2.9}$$

$$\hat{\rho}_{AR}(t,t-\tau) = \hat{\rho}_{A}(t-\tau)\hat{\rho}(t), \tag{2.10}$$

with $\hat{\rho}(t)$ being the density operator for the cavity mode alone. In view of Eqs. (2.9) and (2.10), Eq. (2.8) can be written as

$$\frac{d\hat{\rho}_{AR}(t)}{dt} = r_a\left(\hat{\rho}_{A}(0) - \hat{\rho}_{A}(t-\tau)\right)\hat{\rho}(t) + r_a\int_{t-\tau}^{t}\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t')dt'. \tag{2.11}$$

In the absence of damping of the cavity mode by a vacuum reservoir, the density operator $\hat{\rho}_{AR}(t,t')$ evolves in time according to

$$\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t') = -i\left[\hat{H},\hat{\rho}_{AR}(t,t')\right], \tag{2.12}$$

so that using Eq. (2.12) and taking into account Eq. (2.7), one can put Eq. (2.11) in the form

$$\frac{d\hat{\rho}_{AR}(t)}{dt} = r_a(\hat{\rho}_{A}(0) - \hat{\rho}_{A}(t-\tau))\hat{\rho}(t) - i\left[\hat{H},\hat{\rho}_{AR}(t)\right]. \tag{2.13}$$

Furthermore, tracing over the atomic variables, we have

$$\frac{d\hat{\rho}(t)}{dt} = -iTr_A\left[\hat{H},\hat{\rho}_{AR}(t)\right], \tag{2.14}$$

in which we have used the fact

$$Tr_A\left(\hat{\rho}_{A}(0)\right) = Tr_A\left(\hat{\rho}_{A}(t-\tau)\right) = 1. \tag{2.15}$$

Employing Eq. (2.1), one can put Eq. (2.14) in the form

$$\frac{d\hat{\rho}(t)}{dt} = g\left(\hat{a}\hat{\rho}_{ba} + \hat{a}\hat{\rho}_{cb} - \hat{\rho}_{ba}\hat{a} - \hat{\rho}_{cb}\hat{a} - \hat{a}^{\dagger}\hat{\rho}_{ab} - \hat{a}^{\dagger}\hat{\rho}_{bc} + \hat{\rho}_{ab}\hat{a}^{\dagger} + \hat{\rho}_{bc}\hat{a}^{\dagger}\right). \tag{2.16}$$

in which the matrix element $\hat{\rho}_{\alpha\beta}$ is defined by

$$\hat{\rho}_{\alpha\beta} = \langle\alpha|\hat{\rho}_{AR}(t)|\beta\rangle, \tag{2.17}$$

with $\alpha, \beta = a, b, c$.

We next proceed to determine the matrix elements $\hat{\rho}_{\alpha\beta}$. We see from Eq. (2.13) that

$$\frac{d\hat{\rho}_{\alpha\beta}}{dt} = r_a \left\{\langle\alpha|\hat{\rho}_A(0)|\beta\rangle - \langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle\right\}\hat{\rho}(t) - i\langle\alpha|[\hat{H}, \hat{\rho}_{AR}(t)]|\beta\rangle - \gamma\hat{\rho}_{\alpha\beta}, \tag{2.18}$$

where the last term is included to account for the decay of the atoms due to spontaneous emission. Here $\gamma$, considered to be the same for all the three levels, is the atomic decay rate. We assume that the atoms are removed after they have decayed to a level other than levels $|b\rangle$ and $|c\rangle$. We then see that

$$\langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle = 0 \tag{2.19}$$

and hence on account of (2.1) and (2.3), Eq (2.18) can be written as

$$\begin{aligned}\frac{d\hat{\rho}_{\alpha\beta}}{dt} =& r_a \left(\rho_{aa}^{(0)}\delta_{\alpha a}\delta_{a\beta} + \rho_{ac}^{(0)}\delta_{\alpha a}\delta_{c\beta} + \rho_{ca}^{(0)}\delta_{\alpha c}\delta_{a\beta} + \rho_{cc}^{(0)}\delta_{\alpha c}\delta_{c\beta}\right)\hat{\rho} \\ &+ g\left(\hat{a}\hat{\rho}_{b\beta}\delta_{\alpha a} + \hat{a}\hat{\rho}_{c\beta}\delta_{\alpha b} - \hat{a}^\dagger\hat{\rho}_{a\beta}\delta_{\alpha b} - \hat{a}^\dagger\hat{\rho}_{b\beta}\delta_{\alpha c}\right. \\ &\left. - \hat{\rho}_{\alpha a}\hat{a}a\delta_{b\beta} - \hat{\rho}_{\alpha b}\hat{a}\delta_{c\beta} + \hat{\rho}_{\alpha b}\hat{a}^\dagger\delta_{a\beta} + \hat{\rho}_{\alpha c}\hat{a}^\dagger\delta_{b\beta}\right) - \gamma\hat{\rho}_{\alpha\beta},\end{aligned} \tag{2.20}$$

from which follows

$$\frac{d\hat{\rho}_{ab}}{dt} = g\left(\hat{a}\hat{\rho}_{bb} - \hat{\rho}_{aa}\hat{a} + \hat{\rho}_{ac}\hat{a}^\dagger\right) - \gamma\hat{\rho}_{ab}, \tag{2.21}$$

$$\frac{d\hat{\rho}_{bc}}{dt} = g\left(\hat{a}\hat{\rho}_{cc} - \hat{\rho}_{bb}\hat{a} - \hat{a}^\dagger\hat{\rho}_{ac}\right) - \gamma\hat{\rho}_{bc}, \tag{2.22}$$

$$\frac{d\hat{\rho}_{aa}}{dt} = r_a\rho_{aa}^{(0)}\hat{\rho} + g\left(\hat{a}\hat{\rho}_{ba} + \hat{\rho}_{ab}\hat{a}^\dagger\right) - \gamma\hat{\rho}_{aa}, \tag{2.23}$$

$$\frac{d\hat{\rho}_{cc}}{dt} = r_a\rho_{cc}^{(0)}\hat{\rho} - g\left(\hat{a}^\dagger\hat{\rho}_{bc} - \hat{\rho}_{cb}\hat{a}\right) - \gamma\hat{\rho}_{cc}, \tag{2.24}$$

$$\frac{d\hat{\rho}_{ac}}{dt} = r_a\rho_{ac}^{(0)}\hat{\rho} + g\left(\hat{a}\hat{\rho}_{bc} - \hat{\rho}_{ab}\hat{a}\right) - \gamma\hat{\rho}_{ac}, \tag{2.25}$$

$$\frac{d\hat{\rho}_{bb}}{dt} = g\left(\hat{a}\hat{\rho}_{cb} - \hat{a}^{\dagger}\hat{\rho}_{ab} - \hat{\rho}_{ba}\hat{a} + \hat{\rho}_{bc}\hat{a}^{\dagger}\right) - \gamma\hat{\rho}_{bb}. \tag{2.26}$$

We confine ourselves to linear analysis and this can be achieved by dropping the $g$ terms in Eqs. (2.23), (2.24), (2.25), and (2.26). In the good cavity limit $\gamma \gg \kappa$, the cavity mode variables change slowly compared to the atomic variables and the atomic variables will reach steady state in relatively short time, so that we can set the time derivative of the atomic variables zero. This procedure is referred to as the adiabatic approximation scheme. Hence dropping the $g$ terms in Eqs. (2.23), (2.24), (2.25), (2.26) and applying the adiabatic approximation scheme, we get

$$\hat{\rho}_{aa} = \frac{r_a\rho_{aa}^{(0)}}{\gamma}\hat{\rho}, \tag{2.27}$$

$$\hat{\rho}_{cc} = \frac{r_a\rho_{cc}^{(0)}}{\gamma}\hat{\rho}, \tag{2.28}$$

$$\hat{\rho}_{ac} = \frac{r_a\rho_{ac}^{(0)}}{\gamma}\hat{\rho}, \tag{2.29}$$

$$\hat{\rho}_{bb} = 0. \tag{2.30}$$

In view of the above results, Eqs. (2.21) and (2.22) can be put in the form

$$\frac{d\hat{\rho}_{ab}}{dt} = \frac{gr_a}{\gamma}\left(\rho_{ac}^{(0)}\hat{\rho}\hat{a}^{\dagger} - \rho_{aa}^{(0)}\hat{\rho}\hat{a}\right) - \gamma\hat{\rho}_{ab}, \tag{2.31}$$

$$\frac{d\hat{\rho}_{bc}}{dt} = \frac{gr_a}{\gamma}\left(\rho_{cc}^{(0)}\hat{a}\hat{\rho} - \rho_{ac}^{(0)}\hat{a}^{\dagger}\hat{\rho}\right) - \gamma\hat{\rho}_{bc}. \tag{2.32}$$

Using once more the adiabatic approximation scheme, we see that

$$\hat{\rho}_{ab} = \frac{gr_a}{\gamma^2}\left(\rho_{ac}^{(0)}\hat{\rho}\hat{a}^{\dagger} - \rho_{aa}^{(0)}\hat{\rho}\hat{a}\right), \tag{2.33}$$

$$\hat{\rho}_{bc} = \frac{gr_a}{\gamma^2}\left(\rho_{cc}^{(0)}\hat{a}\hat{\rho} - \rho_{ac}^{(0)}\hat{a}^{\dagger}\hat{\rho}\right). \tag{2.34}$$

Finally on account of (2.33) and (2.34), Eq. (2.16) takes the form

$$\frac{d\hat{\rho}(t)}{dt} = \frac{A\rho_{aa}^{(0)}}{2}\left(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\right) + \frac{A\rho_{cc}^{(0)}}{2}\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right)$$
$$- \frac{A\rho_{ac}^{(0)}}{2}\left(2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{a}^{\dagger 2}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger 2}\right) - \frac{A\rho_{ca}^{(0)}}{2}\left(2\hat{a}\hat{\rho}\hat{a} - \hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^2\right), \tag{2.35}$$

where

$$A = \frac{2g^2 r_a}{\gamma^2} \tag{2.36}$$

is linear gain coefficient.

**Single-Mode Squeezed Vacuum Reservoir**

We assume that a reservoir is composed of a large number of submodes. We also assume that the reservoir submodes are independent and the reservoir radiation is incident on the system from one direction. The density operator for a single-mode squeezed vacuum reservoir can be expressed as

$$\hat{\rho}(r) = \prod_i \hat{S}_i(r)|0_i\rangle\langle_i 0|\hat{S}_i^\dagger(r), \tag{2.37}$$

where

$$\hat{S}_i(r) = e^{\frac{1}{2}r(\hat{b}_i^{\dagger 2} - \hat{b}_i^2)} \tag{2.38}$$

is the unitary squeeze operator with the squeeze parameter $r$ taken for convenience to be real and positive and $\hat{b}_i$ represents the annihilation operators for the reservoir submode. Applying Eq. (2.37), we can write

$$\langle\hat{b}_k\rangle = \prod_i \langle_i 0|\hat{b}_k(r)|0_i\rangle, \tag{2.39}$$

in which

$$\hat{b}_k(r) = \hat{S}_i^\dagger(r)\hat{b}_k\hat{S}_i(r). \tag{2.40}$$

We note that the derivative of Eq. (2.40) with respect to $r$ is

$$\frac{d}{dr}\hat{b}_k(r) = \frac{1}{2}[\hat{b}_k(r), \hat{b}_i^{\dagger 2} - \hat{b}_i^2] = \hat{b}_i^\dagger(r)\delta_{ik} \tag{2.41}$$

and hence

$$\frac{d}{dr}\hat{b}_k^\dagger(r) = \hat{b}_i(r)\delta_{ik}. \tag{2.42}$$

In order to decouple these equations, we differentiate Eq. (2.41) once more with respect to $r$. Thus we get

$$\frac{d^2}{dr^2}\hat{b}_k(r) = \hat{b}_k(r). \tag{2.43}$$

The solution of this equation can be put in the form

$$\hat{b}_k(r) = Ae^r + Be^{-r}. \tag{2.44}$$

Applying the condition $r = 0$, we see that

$$\hat{b}_k(r)\Big|_{r=0} = A + B = \hat{b}_i\delta_{ik}, \tag{2.45}$$

$$\frac{d}{dr}\hat{b}_k(r)\Big|_{r=0} = A - B = \hat{b}_i^\dagger\delta_{ik}. \tag{2.46}$$

It then follows that

$$A = \frac{1}{2}(\hat{b}_i + \hat{b}_i^\dagger)\delta_{ik}, \tag{2.47}$$

$$B = \frac{1}{2}(\hat{b}_i - \hat{b}_i^\dagger)\delta_{ik}, \tag{2.48}$$

so that substitution of Eqs. (2.47) and (2.48) into Eq. (2.44) yields

$$\hat{b}_k(r) = (\hat{b}_i\cosh r + \hat{b}_i^\dagger\sinh r)\delta_{ik}. \tag{2.49}$$

Using Eq. (2.49) in Eq. (2.39), we have

$$\langle\hat{b}_k\rangle = 0. \tag{2.50}$$

Furthermore, using Eq. (2.37) and the identity operator $\hat{I} = \hat{S}_i(r)\hat{S}_i^\dagger(r)$, we can write

$$\langle\hat{b}_k^\dagger\hat{b}_{k'}\rangle = \prod_i\langle_i 0|\hat{b}_k^\dagger(r)\hat{b}_{k'}(r)|0_i\rangle, \tag{2.51}$$

where $\hat{b}_k(r)$ is defined by (2.40). In view of Eq. (2.49), we see that

$$\langle\hat{b}_k^\dagger\hat{b}_{k'}\rangle = \prod_i\langle_i 0|(\hat{b}_i^\dagger\hat{b}_i\cosh^2 r + (\hat{b}_i^\dagger\hat{b}_i^\dagger + \hat{b}_i\hat{b}_i)\sinh r\cosh r + \hat{b}_i\hat{b}_i^\dagger\sinh^2 r)|0_i\rangle\delta_{ik}\delta_{ik'}, \tag{2.52}$$

from which follows

$$\langle\hat{b}_k^\dagger\hat{b}_{k'}\rangle = \langle_{k'}0|(\hat{b}_{k'}^\dagger\hat{b}_{k'}\cosh^2 r + (\hat{b}_{k'}^\dagger\hat{b}_{k'}^\dagger + \hat{b}_{k'}\hat{b}_{k'})\sinh r\cosh r + \hat{b}_{k'}\hat{b}_{k'}^\dagger\sinh^2 r)|0_{k'}\rangle\delta_{k'k}. \tag{2.53}$$

Applying the commutation relation $[\hat{b}_{k'}, \hat{b}^{\dagger}_{k'}] = 1$, we find

$$\langle \hat{b}^{\dagger}_{k} \hat{b}_{k'} \rangle = N\delta_{k'k}, \tag{2.54}$$

where

$$N = \sinh^2 r. \tag{2.55}$$

It can be also established in a similar manner that

$$\langle \hat{b}_{k} \hat{b}^{\dagger}_{k'} \rangle = (N+1)\delta_{k'k}, \tag{2.56}$$

$$\langle \hat{b}_{k} \hat{b}_{k'} \rangle = M\delta_{k'k}, \tag{2.57}$$

in which

$$M = \sinh r \cosh r. \tag{2.58}$$

Assuming that the wave number $k$ varies vary little around the central wave number of the reservoir submode $k_0$ , we can write $k \approx 2k_0 - k$. In view of this, Eq. (2.57) can be rewritten as

$$\langle \hat{b}_{k} \hat{b}_{k'} \rangle = M\delta_{k',2k_0-k}. \tag{2.59}$$

We next seek to drive the equation of evolution of the density operator for a cavity mode coupled to a single-mode squeezed vacuum reservoir. Let $\hat{\chi}(t)$ be the total density operator of the cavity mode plus the reservoir in the interaction picture. Then the equation of evolution of the density operator is

$$\frac{d}{dt}\hat{\chi}(t) = -i[\hat{H}_{SR}(t), \hat{\chi}(t)], \tag{2.60}$$

where $\hat{H}_{SR}(t)$ describes the interaction between the cavity mode and the reservoir. The density operator of the system is defined by

$$\hat{\rho}(t) = Tr_R(\hat{\chi}(t)), \tag{2.61}$$

in which $Tr_R$ indicates the trace over the reservoir variables only. A formal solution of Eq. (2.60) can be written as

$$\hat{\chi}(t) = \hat{\chi}(0) - i\int dt'[\hat{H}_{SR}(t'), \hat{\chi}(t')]. \tag{2.62}$$

On substituting Eq. (2.62) into Eq. (2.60), we have

$$\frac{d}{dt}\hat{\rho}(t) = -\, iTr_R[\hat{H}_{SR}(t), \hat{\chi}(0)] - \int dt' Tr_R[\hat{H}_{SR}(t), [\hat{H}_{SR}(t'), \hat{\chi}(t')]]. \tag{2.63}$$

Applying the Born approximation,

$$\hat{\chi}(t') = \hat{\rho}(t')R, \tag{2.64}$$

we see that

$$\frac{d}{dt}\hat{\rho}(t) = -\, i[\langle\hat{H}_{SR}(t)\rangle_R, \hat{\rho}(0)] - \int dt' Tr_R[\hat{H}_{SR}(t), [\hat{H}_{SR}(t'), \hat{\rho}(t')R]]. \tag{2.65}$$

The interaction between a cavity mode and a single-mode squeezed vacuum reservoir can be described by

$$\hat{H}_{SR}(t) = i\sum_k \lambda_k(\hat{a}^\dagger\hat{b}_k e^{i(\omega_0-\omega_k)t} - \hat{a}\hat{b}_k^\dagger e^{-i(\omega_0-\omega_k)t}), \tag{2.66}$$

where $\hat{a}$ is the annihilation operator for the cavity mode, $\hat{b}_k$ is the annihilation operator for a reservoir submode, and $\lambda_k$ is the coupling constant. In view of Eq. (2.50), we see that

$$\langle\hat{H}_{SR}(t)\rangle_R = 0. \tag{2.67}$$

Therefore, on account of this result, Eq. (2.65) reduces to

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) = &-\int_0^t dt' Tr_R(\hat{H}_{SR}(t)\hat{H}_{SR}(t')R)\hat{\rho}(t') + \int_0^t dt' Tr_R(\hat{H}_{SR}(t)\hat{\rho}(t')R\hat{H}_{SR}(t')) \\ &+ \int_0^t dt' Tr_R(\hat{H}_{SR}(t')\hat{\rho}(t')R\hat{H}_{SR}(t)) - \int_0^t dt' \hat{\rho}(t') Tr_R(R\hat{H}_{SR}(t')\hat{H}_{SR}(t)).\end{aligned} \tag{2.68}$$

Now employing the Hamiltonian (2.66) and the fact that the cavity mode and reservoir submode operators commute, we can write

$$Tr_R(\hat{H}_{SR}(t)\hat{H}_{SR}(t')R) = I_1\hat{a}^{\dagger 2} + I_2\hat{a}^\dagger\hat{a} + I_3\hat{a}\hat{a}^\dagger + I_4\hat{a}^2, \tag{2.69}$$

where

$$I_1 = -\sum_{jk} \lambda_j\lambda_k\langle\hat{b}_j\hat{b}_k\rangle e^{i(\omega_0-\omega_j)t + i(\omega_0-\omega_k)t'}, \tag{2.70}$$

$$I_2 = \sum_{jk} \lambda_j \lambda_k \langle \hat{b}_j \hat{b}_k^\dagger \rangle e^{i(\omega_0 - \omega_j)t - i(\omega_0 - \omega_k)t'}, \tag{2.71}$$

$$I_3 = \sum_{jk} \lambda_j \lambda_k \langle \hat{b}_j^\dagger \hat{b}_k \rangle e^{-i(\omega_0 - \omega_j)t + i(\omega_0 - \omega_k)t'}, \tag{2.72}$$

$$I_4 = -\sum_{jk} \lambda_j \lambda_k \langle \hat{b}_j^\dagger \hat{b}_k^\dagger \rangle e^{-i(\omega_0 - \omega_j)t - i(\omega_0 - \omega_k)t'}. \tag{2.73}$$

On account of Eq. (2.59), we see that

$$I_1 = -M \sum_{k} \lambda_{2k_0 - k} \lambda_k e^{i(\omega_0 - \omega_k)(t - t')}. \tag{2.74}$$

We assume the reservoir submode frequencies to be closely spaced. Then changing the summation into integration, we have

$$I_1 = -M \int_0^\infty d\omega g(\omega) \lambda(2\omega_0 - \omega) \lambda(\omega) e^{-i(\omega_0 - \omega)(t - t')}, \tag{2.75}$$

where $g(\omega)$ is the density of the reservoir submodes. We expect $\omega$ to vary very little around $\omega_0$. In view of this, we can replace $g(\omega)$ and $\lambda(2\omega_0 - \omega)\lambda(\omega)$ by $g(\omega_0)$ and $\lambda^2(\omega_0)$. Hence

$$I_1 = -M g(\omega_0) \lambda^2(\omega_0) \int_0^\infty d\omega e^{-i(\omega_0 - \omega)(t - t')}. \tag{2.76}$$

Upon setting $\omega' = \omega - \omega_0$ and extending the lower limit of the integration to $-\infty$, we see that

$$I_1 = -M g(\omega_0) \lambda^2(\omega_0) \int_{-\infty}^\infty d\omega' e^{i(t - t')\omega'}. \tag{2.77}$$

It then follows that

$$I_1 = -\kappa M \delta(t - t'), \tag{2.78}$$

where

$$\kappa = 2\pi g(\omega_0) \lambda^2(\omega_0) \tag{2.79}$$

is defined to be the cavity damping constant. Following a similar procedure, one can readily establish that

$$I_2 = \kappa(N+1)\delta(t-t'), \tag{2.80}$$

$$I_3 = \kappa N\delta(t-t'), \tag{2.81}$$

$$I_4 = -\kappa M\delta(t-t'). \tag{2.82}$$

Now combination of Eqs (2.69), (2.78), (2.80), (2.81), and (2.82) leads to

$$Tr_R\left(\hat{H}_{SR}(t)\hat{H}_{SR}(t')R\right) = \kappa\left((N+1)\hat{a}^\dagger\hat{a} + N\hat{a}\hat{a}^\dagger - M\hat{a}^{\dagger 2} - M\hat{a}^2\right)\delta(t-t'). \tag{2.83}$$

We also note that

$$Tr_R\left(\hat{H}_{SR}(t')\hat{H}_{SR}(t)R\right) == \kappa\left((N+1)\hat{a}^\dagger\hat{a} + N\hat{a}\hat{a}^\dagger - M\hat{a}^{\dagger 2} - M\hat{a}^2\right)\delta(t-t'). \tag{2.84}$$

Moreover, employing Eq. (2.66), one can write

$$Tr_R\left(\hat{H}_{SR}(t)\hat{\rho}(t')R\hat{H}_{SR}(t')\right) = I_1\hat{a}^\dagger\hat{\rho}(t')\hat{a}^\dagger + I_2\hat{a}\hat{\rho}(t')\hat{a}^\dagger + I_3\hat{a}^\dagger\hat{\rho}(t')\hat{a} + I_4\hat{a}\hat{\rho}(t')\hat{a}, \tag{2.85}$$

so that applying the results described by Eqs. (2.78), (2.80), (2.81), and (2.82), we have

$$\begin{aligned} Tr_R\left(\hat{H}_{SR}(t)\hat{\rho}(t')R\hat{H}_{SR}(t')\right) =& \kappa\left((N+1)\hat{a}\hat{\rho}(t')\hat{a}^\dagger + N\hat{a}^\dagger\hat{\rho}(t')\hat{a}\right. \\ & \left.-M\hat{a}^\dagger\hat{\rho}(t')\hat{a}^\dagger - M\hat{a}\hat{\rho}(t')\hat{a}\right)\delta(t-t'). \end{aligned} \tag{2.86}$$

It is easy to see that

$$\begin{aligned} Tr_R\left(\hat{H}_{SR}(t')\hat{\rho}(t')R\hat{H}_{SR}(t)\right) =& \kappa\left((N+1)\hat{a}\hat{\rho}(t')\hat{a}^\dagger + N\hat{a}^\dagger\hat{\rho}(t')\hat{a}\right. \\ & \left.-M\hat{a}^\dagger\hat{\rho}(t')\hat{a}^\dagger - M\hat{a}\hat{\rho}(t')\hat{a}\right)\delta(t-t'). \end{aligned} \tag{2.87}$$

On substituting (2.83), (2.84), (2.86), and (2.87) into Eq. (2.68) and carrying out the integration over t', we find

$$\begin{aligned} \frac{d}{dt}\hat{\rho}(t) =& \frac{\kappa}{2}(N+1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{\kappa}{2}N(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) \\ & - \frac{\kappa}{2}M(2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{a}^{\dagger 2}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger 2}) - \frac{\kappa}{2}M(2\hat{a}\hat{\rho}\hat{a} - \hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^2). \end{aligned} \tag{2.88}$$

This represents the equation of evolution of the reduced density operator for a cavity mode coupled to a single-mode squeezed vacuum reservoir. The effects of the reservoir are incorporated by the parameters N and M.

**Degenerate Three-Level Laser with Coherent and Squeezed Light**

With the driving coherent light and the pump mode treated classically, the Hamiltonian describing the interaction of the driving coherent light with the cavity mode and the parametric interaction is expressible as

$$\hat{H} = i\varepsilon_1(\hat{a}^\dagger - \hat{a}) + \frac{1}{2}i\varepsilon_2(\hat{a}^{\dagger 2} - \hat{a}^2), \tag{2.89}$$

in which $\varepsilon_1$ and $\varepsilon_2$, consider to be real and constant, are proportional to the amplitude of the driving coherent light and the pump mode, respectively. The equation of evolution of the density operator associated with this Hamiltonian has the form

$$\frac{d\hat{\rho}(t)}{dt} = \varepsilon_1\left(\hat{\rho}\hat{a} - \hat{a}\hat{\rho} + \hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger\right) + \frac{1}{2}\varepsilon_2\left(\hat{\rho}\hat{a}^2 - \hat{a}^2\hat{\rho} + \hat{a}^{\dagger 2}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger 2}\right). \tag{2.90}$$

Now taking into account Eqs. (2.35), (2.88), and (2.90), the master equation for the cavity mode produced by a degenerate three-level laser whose cavity contains a parametric amplifier and whose cavity mode is driven by coherent light and coupled to a squeezed vacuum reservoir can be written as

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} =& \varepsilon_1\left(\hat{\rho}\hat{a} - \hat{a}\hat{\rho} + \hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger\right) + \frac{1}{2}\varepsilon_2\left(\hat{\rho}\hat{a}^2 - \hat{a}^2\hat{\rho} + \hat{a}^{\dagger 2}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger 2}\right)\\ &+ \frac{1}{2}(A\rho_{aa}^{(0)} + \kappa N)\left(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\right)\\ &+ \frac{1}{2}(A\rho_{cc}^{(0)} + \kappa(N+1))\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right)\\ &+ \frac{1}{2}(A\rho_{ac}^{(0)} + \kappa M)\left(\hat{a}^{\dagger 2}\hat{\rho} + \hat{\rho}\hat{a}^{\dagger 2} - 2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\right)\\ &+ \frac{1}{2}(A\rho_{ca}^{(0)} + \kappa M)\left(\hat{a}^2\hat{\rho} + \hat{\rho}\hat{a}^2 - 2\hat{a}\hat{\rho}\hat{a}\right).\end{aligned} \tag{2.91}$$

## 2.1.2 Stochastic differential equations

We next seek to obtain, using the master equation, stochastic differential equations associated with the normal ordering. To this end, employing the relation

$$\frac{d}{dt}\langle\hat{a}\rangle = Tr\left(\frac{d\hat{\rho}(t)}{dt}\hat{a}\right) \tag{2.92}$$

along with Eq. (2.91), we see that

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}\rangle =&\varepsilon_1 Tr\left(\hat{a}\hat{\rho}\hat{a}-\hat{a}^2\hat{\rho}+\hat{a}\hat{a}^\dagger\hat{\rho}-\hat{a}\hat{\rho}\hat{a}^\dagger\right)+\frac{1}{2}\varepsilon_2 Tr\left(\hat{a}\hat{\rho}\hat{a}^2-\hat{a}^3\hat{\rho}+\hat{a}\hat{a}^{\dagger 2}\hat{\rho}-\hat{a}\hat{\rho}\hat{a}^{\dagger 2}\right)\\ &+\frac{1}{2}(A\rho_{aa}^{(0)}+\kappa N)Tr\left(2\hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}-\hat{a}^2\hat{a}^\dagger\hat{\rho}-\hat{a}\hat{\rho}\hat{a}\hat{a}^\dagger\right)\\ &+\frac{1}{2}(A\rho_{cc}^{(0)}+\kappa(N+1))Tr\left(2\hat{a}^2\hat{\rho}\hat{a}^\dagger-\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho}-\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\right)\\ &+\frac{1}{2}(A\rho_{ac}^{(0)}+\kappa M)Tr\left(\hat{a}\hat{a}^{\dagger 2}\hat{\rho}+\hat{a}\hat{\rho}\hat{a}^{\dagger 2}-2\hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\right)\\ &+\frac{1}{2}(A\rho_{ca}^{(0)}+\kappa M)Tr\left(\hat{a}^3\hat{\rho}-\hat{a}\hat{\rho}\hat{a}^2-2\hat{a}^2\hat{\rho}\hat{a}\right).\end{aligned} \tag{2.93}$$

Applying the cyclic property of the trace operation and the commutation relations

$$[\hat{a},\hat{a}^\dagger]=1 \tag{2.94}$$

and

$$[\hat{a},f(\hat{a}^\dagger,\hat{a})]=\frac{d}{d\hat{a}^\dagger}f(\hat{a}^\dagger,\hat{a}), \tag{2.95}$$

we get

$$\frac{d}{dt}\langle\hat{a}\rangle=-\frac{1}{2}\mu\langle\hat{a}\rangle+\varepsilon_2\langle\hat{a}^\dagger\rangle+\varepsilon_1, \tag{2.96}$$

where

$$\mu=A(\rho_{cc}^{(0)}-\rho_{aa}^{(0)})+\kappa. \tag{2.97}$$

Following the same procedure, it can also be easily verified that

$$\frac{d}{dt}\langle\hat{a}^2\rangle=-\mu\langle\hat{a}^2\rangle+2\varepsilon_2\langle\hat{a}^\dagger\hat{a}\rangle+2\varepsilon_1\langle\hat{a}\rangle+A\rho_{ac}+\kappa M+\varepsilon_2. \tag{2.98}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle=-\mu\langle\hat{a}^\dagger\hat{a}\rangle+\varepsilon_2\left(\langle\hat{a}^{\dagger 2}\rangle+\langle\hat{a}^2\rangle\right)+\varepsilon_1\left(\langle\hat{a}^\dagger\rangle+\langle\hat{a}\rangle\right)+A\rho_{aa}+\kappa N. \tag{2.99}$$

We note that Eqs. (2.96), (2.98), and (2.99) are in the normal order. The c-number equations corresponding to Eqs. (2.96), (2.98), and (2.99) are

$$\frac{d}{dt}\langle\alpha\rangle=-\frac{1}{2}\mu\langle\alpha\rangle+\varepsilon_2\langle\alpha^*\rangle+\varepsilon_1, \tag{2.100}$$

$$\frac{d}{dt}\langle\alpha^2\rangle=-\mu\langle\alpha^2\rangle+2\varepsilon_2\langle\alpha^*\alpha\rangle+2\varepsilon_1\langle\alpha\rangle+A\rho_{ac}+\kappa M+\varepsilon_2, \tag{2.101}$$

$$\frac{d}{dt}\langle\alpha^*\alpha\rangle = -\mu\langle\alpha^*\alpha\rangle + \varepsilon_2\left(\langle\alpha^{*2}\rangle + \langle\alpha^2\rangle\right) + \varepsilon_1\left(\langle\alpha^*\rangle + \langle\alpha\rangle\right) + A\rho_{aa} + \kappa N. \tag{2.102}$$

On the basis of Eq. (2.100), one can write [4, 21]

$$\frac{d}{dt}\alpha(t) = -\frac{1}{2}\mu\alpha(t) + \varepsilon_2\alpha^*(t) + \varepsilon_1 + f(t), \tag{2.103}$$

where $f(t)$ is a noise force. We now proceed to determine the properties of the noise force. We note that Eq. (2.100) and the expectation value of Eq. (2.103) will have identical form if

$$\langle f(t)\rangle = 0. \tag{2.104}$$

Applying the relation $\frac{d}{dt}\langle\alpha^2\rangle = 2\langle\alpha\frac{d}{dt}\alpha\rangle$ along with Eq. (2.103), we see that

$$\frac{d}{dt}\langle\alpha^2\rangle = -\mu\langle\alpha^2\rangle + 2\varepsilon_2\langle\alpha^*\alpha\rangle + 2\varepsilon_1\langle\alpha\rangle + 2\langle\alpha(t)f(t)\rangle, \tag{2.105}$$

so that comparison of Eqs. (2.101) and (2.105) shows that

$$\langle\alpha(t)f(t)\rangle = \frac{1}{2}(A\rho_{ac}^{(0)} + \kappa M + \varepsilon_2). \tag{2.106}$$

Employing the formal solution of Eq. (2.103)

$$\alpha(t) = \alpha(0)e^{-\mu t/2} + \int_0^t e^{-\mu(t-t')/2}\left[\varepsilon_2\alpha^*(t') + f(t') + \varepsilon_1\right]dt', \tag{2.107}$$

we see that

$$\begin{aligned}&\langle\alpha(0)f(t)\rangle e^{-\mu t/2} + \int_0^t e^{-\mu(t-t')/2}\left[\varepsilon_2\langle\alpha^*(t')f(t)\rangle + \langle f(t')f(t)\rangle + \varepsilon_1\langle f(t)\rangle\right]dt'\\ &= \frac{1}{2}(A\rho_{ac}^{(0)} + \kappa M + \varepsilon_2).\end{aligned} \tag{2.108}$$

Taking into account Eq. (2.104) and the fact that a noise force at a certain instant does not affect the cavity mode variables at earlier time, we have

$$\int_0^t e^{-\mu(t-t')/2}\langle f(t')f(t)\rangle dt' = \frac{1}{2}(A\rho_{ac}^{(0)} + \kappa M + \varepsilon_2). \tag{2.109}$$

In view of the property of the Dirac delta function

$$\int_0^t f(t')\delta(t-t')dt' = \frac{1}{2}f(t), \tag{2.110}$$

one can put Eq. (2.109) in the form

$$\int_0^t e^{-\mu(t-t')/2}\langle f(t')f(t)\rangle dt' = \int_0^t e^{-\mu(t-t')/2}(A\rho_{ac}^{(0)} + \kappa M + \varepsilon_2)\delta(t-t')dt'. \tag{2.111}$$

It then follows that

$$\langle f(t')f(t)\rangle = (A\rho_{ac}^{(0)} + \kappa M + \varepsilon_2)\delta(t-t'). \tag{2.112}$$

Furthermore, applying the relation $\frac{d}{dt}\langle\alpha^*\alpha\rangle = \langle\alpha\frac{d}{dt}\alpha^*\rangle + \langle\alpha^*\frac{d}{dt}\alpha\rangle$ along with Eq. (2.103) and its complex conjugate, we have

$$\begin{aligned}\frac{d}{dt}\langle\alpha^*\alpha\rangle =& -\mu\langle\alpha^*\alpha\rangle + \varepsilon_2(\langle\alpha^{*2}\rangle + \langle\alpha^2\rangle) + \varepsilon_1(\langle\alpha^*\rangle + \langle\alpha\rangle)\\ &+ \langle\alpha^*(t)f(t)\rangle + \langle f^*(t)\alpha(t)\rangle.\end{aligned} \tag{2.113}$$

Comparison of this equation with Eq. (2.102) indicates that

$$\langle\alpha^*(t)f(t)\rangle + \langle f^*(t)\alpha(t)\rangle = A\rho_{aa}^{(0)} + \kappa N. \tag{2.114}$$

Now using Eq. (2.107) and its complex conjugate, we get

$$\begin{aligned}&(\langle\alpha^*(0)f(t)\rangle + \langle\alpha(0)f^*(t)\rangle)e^{-\mu t/2} + \int_0^t e^{-\mu(t-t')/2}\Big[\varepsilon_2(\langle\alpha(t')f(t)\rangle + \langle f^*(t)\alpha(t')\rangle)\\ &+ \varepsilon_1(\langle f(t)\rangle + \langle f^*(t)\rangle) + \langle f^*(t')f(t)\rangle + \langle f^*(t)f(t')\rangle\Big]dt' = A\rho_{aa}^{(0)} + \kappa N.\end{aligned} \tag{2.115}$$

Taking into account Eq. (2.104) and the fact that a noise force at a certain instant does not affect the cavity mode variables at earlier time, we see that

$$\int_0^t e^{-\mu(t-t')/2}\left(\langle f^*(t')f(t)\rangle + \langle f^*(t)f(t')\rangle\right)dt' = A\rho_{aa}^{(0)} + \kappa N. \tag{2.116}$$

In view of Eq. (2.110), we can write

$$\int_0^t e^{-\mu(t-t')/2}\left(\langle f^*(t')f(t)\rangle + \langle f^*(t)f(t')\rangle\right)dt' = 2\int_0^t e^{-\mu(t-t')/2}\left(A\rho_{aa}^{(0)} + \kappa N\right)\delta(t-t')dt' \tag{2.117}$$

and assuming that $\langle f^*(t')f(t)\rangle = \langle f^*(t)f(t')\rangle$, we have

$$\int_0^t e^{-\mu(t-t')/2}\langle f^*(t)f(t')\rangle dt' = \int_0^t e^{-\mu(t-t')/2}\left(A\rho_{aa}^{(0)} + \kappa N\right)\delta(t-t')dt'. \tag{2.118}$$

It then follows that

$$\langle f^*(t')f(t)\rangle = (A\rho_{aa}^{(0)} + \kappa N)\delta(t-t'). \tag{2.119}$$

The results described by Eqs. (2.104), (2.112), and (2.119) represent the correlation properties of the noise forces associated with the normal ordering.

We next proceed to obtain the solution of Eq. (2.103). To this end, introducing new variables defined by

$$\alpha_{\pm}(t) = \alpha^*(t) \pm \alpha(t) \tag{2.120}$$

and applying Eq. (2.103) along with its complex conjugate, we find

$$\frac{d}{dt}\alpha_{\pm} = -\frac{1}{2}\lambda_{\mp}\alpha_{\pm} + f^*(t) \pm f(t) + \varepsilon_{\pm}, \tag{2.121}$$

where

$$\lambda_{\mp} = \mu \mp 2\varepsilon_2, \tag{2.122}$$

$$\varepsilon_{\pm} = \varepsilon_1 \pm \varepsilon_1. \tag{2.123}$$

The solution of Eq. (2.121) can be written in the form

$$\alpha_{\pm}(t) = \alpha_{\pm}(0)e^{-\lambda_{\mp}t/2} + \int_0^t e^{-\lambda_{\mp}(t-t')/2}\left[f^*(t') \pm f(t') + \varepsilon_{\pm}\right]dt', \tag{2.124}$$

so that with the aid of Eqs. (2.120) and (2.124), we finally obtain

$$\alpha(t) = A_+(t)\alpha(0) + A_-(t)\alpha^*(0) + F(t) + \frac{2\varepsilon_1}{\lambda_-}\left(1 - e^{-\lambda_- t/2}\right), \tag{2.125}$$

where

$$A_{\pm}(t-t') = \frac{1}{2}\left(e^{-\lambda_-(t-t')/2} \pm e^{-\lambda_+(t-t')/2}\right), \tag{2.126}$$

$$F(t) = \int_0^t [A_+(t-t')f(t') + A_-(t-t')f^*(t')]dt'. \tag{2.127}$$

## 2.2 Quadrature Squeezing

In this section we seek to calculate the quadrature variances of the cavity and the output modes as well as the squeezing spectrum of the output mode produced by a degenerate three-level laser whose cavity contains a parametric amplifier and whose cavity mode is driven by coherent light and coupled to a squeezed vacuum reservoir, using the solutions of the stochastic differential equations and the correlation properties of the noise forces.

### 2.2.1 Quadrature variance of the cavity mode

The variance of the quadratures represented by the operators

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a}, \tag{2.128}$$

and

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}), \tag{2.129}$$

can be expressed as

$$\Delta a_\pm^2 = 1 + \langle : \hat{a}_\pm, \hat{a}_\pm : \rangle, \tag{2.130}$$

in which :: stands for the normal order. We note that the c-number equation corresponding to Eq. (2.130) is

$$\Delta a_\pm^2 = 1 \pm \langle \alpha_\pm, \alpha_\pm \rangle, \tag{2.131}$$

where $\alpha_\pm$ is defined by Eq. (2.120). Using Eq. (2.124), one can write

$$\langle \alpha_\pm \rangle = \langle \alpha_\pm(0) \rangle e^{-\lambda_\mp t/2} + \int_0^t e^{-\lambda_\mp (t-t')/2} [\langle f^*(t') \rangle \pm \langle f(t') \rangle + \varepsilon_\pm] dt'. \tag{2.132}$$

Taking into account Eq. (2.104) and assuming the cavity mode to be initially a vacuum state, we have

$$\langle \alpha_\pm \rangle = \frac{2\varepsilon_\pm}{\lambda_\mp} \left(1 - e^{-\lambda_\mp t/2}\right). \tag{2.133}$$

Furthermore, employing Eq. (2.124) along with Eq. (2.104) and the fact that a noise force at a certain instant does not affect the cavity mode variables at earlier time, one can write

$$\begin{aligned} \langle \alpha_\pm^2 \rangle = \int_0^t dt'' dt' e^{-\lambda_\mp (2t - t' - t'')/2} \, [\langle f^*(t') f^*(t'') \rangle + \langle f(t') f(t'') \rangle \\ \pm \langle f^*(t') f(t'') \rangle \pm \langle f^*(t'') f(t') \rangle + \varepsilon_\pm^2 ] \, . \end{aligned} \tag{2.134}$$

Applying Eqs. (2.112) and (2.119) and carrying out the integration, we obtain

$$\langle \alpha_\pm^2 \rangle = \frac{A[\rho_{ac}^{(0)} + \rho_{ac}^{*(0)} \pm 2\rho_{aa}^{(0)}] + 2\varepsilon_2 \pm 2\kappa(N \pm M)}{\lambda_\mp} (1 - e^{-\lambda_\mp t}) + \frac{4\varepsilon_\pm^2}{\lambda_\mp^2} (1 - e^{-\lambda_\mp t/2})^2. \tag{2.135}$$

Hence using Eqs. (2.133) and (2.135), we finally find

$$\langle \alpha_\pm, \alpha_\pm \rangle = \frac{A[\rho_{ac}^{(0)} + \rho_{ac}^{*(0)} \pm 2\rho_{aa}^{(0)}] + 2\varepsilon_2 \pm 2\kappa(N \pm M)}{\lambda_\mp} \left(1 - e^{-\lambda_\mp t}\right). \tag{2.136}$$

It proves to be more convenient to introduce a new parameter defined by

$$\rho_{aa}^{(0)} = \frac{1-\eta}{2}, \tag{2.137}$$

so that in view of the fact that

$$\rho_{aa}^{(0)} + \rho_{cc}^{(0)} = 1 \tag{2.138}$$

and

$$|\rho_{ac}^{(0)}|^2 = \rho_{aa}^{(0)}\rho_{cc}^{(0)}, \tag{2.139}$$

one easily gets

$$\rho_{cc}^{(0)} = \frac{1+\eta}{2}, \tag{2.140}$$

$$|\rho_{ac}^{(0)}| = \frac{\sqrt{1-\eta^2}}{2}, \tag{2.141}$$

and

$$\rho_{cc}^{(0)} - \rho_{aa}^{(0)} = \eta. \tag{2.142}$$

On account of Eqs. (2.142) and (2.97), we see that

$$\mu = A\eta + \kappa, \tag{2.143}$$

so that Eq. (2.122) can be rewritten as

$$\lambda_\pm = A\eta + \kappa \pm 2\varepsilon_2. \tag{2.144}$$

Upon setting

$$\rho_{ac}^{(0)} = |\rho_{ac}^{(0)}|e^{i\theta}, \tag{2.145}$$

we have

$$\rho_{ac}^{(0)} + \rho_{ac}^{*(0)} = \sqrt{1-\eta^2}cos\theta. \tag{2.146}$$

Now employing Eqs. (2.137), (2.144), and (2.146), we put Eq. (2.136) in the form

$$\langle\alpha_\pm,\alpha_\pm\rangle = \frac{A[\sqrt{1-\eta^2}cos\theta \pm (1-\eta)] + 2\varepsilon_2 \pm 2\kappa(N \pm M)}{A\eta + \kappa \mp 2\varepsilon_2}\left(1 - e^{-(A\eta+\kappa\mp2\varepsilon_2)t}\right), \tag{2.147}$$

so that Eq. (2.131) becomes

$$\Delta a_\pm^2 = 1 + \frac{A[1-\eta \pm \sqrt{1-\eta^2}cos\theta] \pm 2\varepsilon_2 + 2\kappa(N \pm M)}{A\eta + \kappa \mp 2\varepsilon_2}\left(1 - e^{-(A\eta+\kappa\mp2\varepsilon_2)t}\right). \tag{2.148}$$

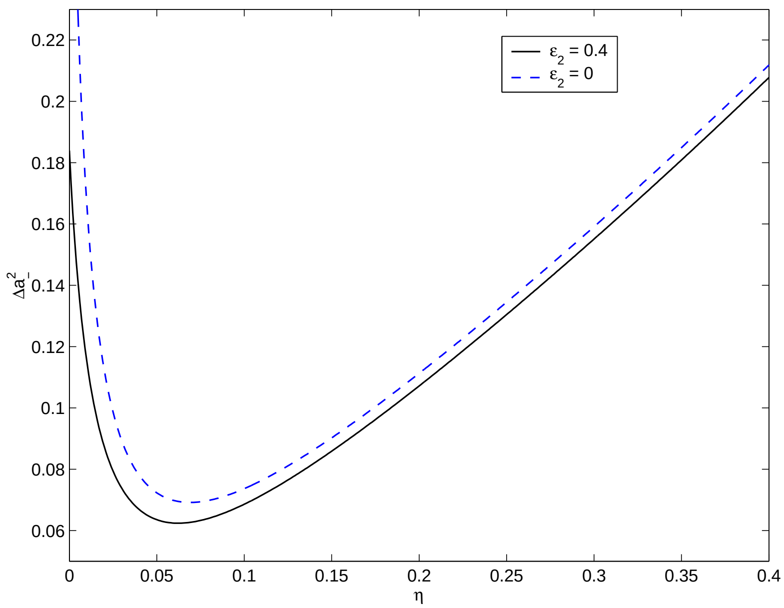


*Fig. 2.2:* Plots of the quadrature variance [Eq.(2.150)] versus $\eta$ for $A = 100$, $\kappa = 0.8$, $r = 0.5$, $\theta = 0$, and for $\varepsilon_2 = 0$ and 0.4.

This represents the quadrature variances of the cavity mode for the system under consideration. At steady state, we find

$$\Delta a_{\pm}^2 = \frac{A[1 \pm \sqrt{1-\eta^2}cos\theta] + \kappa[1 + 2(N \pm M)]}{A\eta + \kappa \mp 2\varepsilon_2}. \tag{2.149}$$

and in view of Eqs. (2.55) and (2.58), we see that

$$\Delta a_{\pm}^2 = \frac{A[1 \pm \sqrt{1-\eta^2}cos\theta] + e^{\pm 2r}\kappa}{A\eta + \kappa \mp 2\varepsilon_2}. \tag{2.150}$$

Since the parameter $\varepsilon_1$ does not appear in Eq. (2.150), we note that the driving coherent light does not have any effect on the quadrature variances.

Because Eq. (2.121) does not have a well-behaved solution for $A\eta + \kappa < 2\varepsilon_2$, we interpret $A\eta + \kappa = 2\varepsilon_2$ as the threshold condition. We then see that Eq. (2.150) takes at threshold the form

$$\Delta a_{+}^2 \to \infty, \tag{2.151}$$

$$\Delta a_{-}^2 = \frac{1}{2}\left[\frac{A[1 - \sqrt{1-\eta^2}cos\theta] + e^{-2r}\kappa}{A\eta + \kappa}\right]. \tag{2.152}$$

The term in the square brackets in Eq. (2.152) represents the quadrature variance for

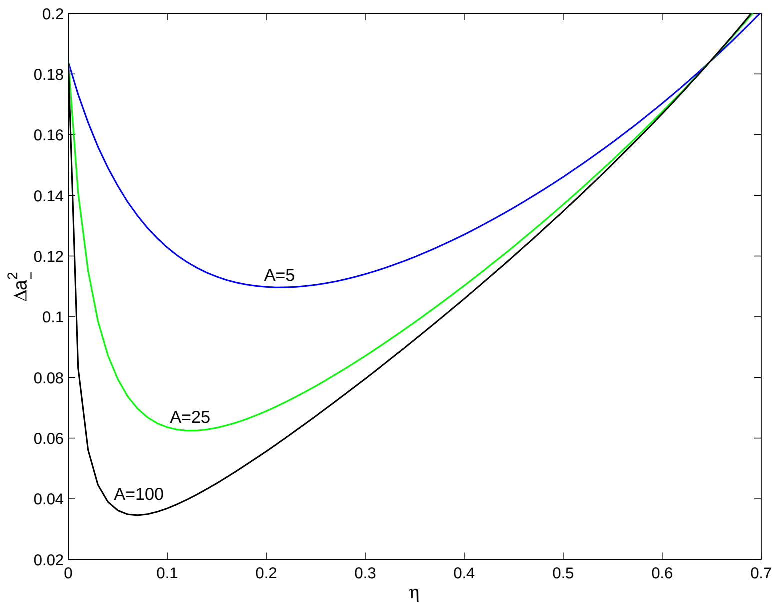


*Fig. 2.3:* Plots of the quadrature variance [Eq.(2.152)] versus $\eta$ for $\kappa = 0.8$, $\theta = 0$, $r = 0.5$, and for different values of the linear gain coefficient.

a degenerate three-level laser whose cavity mode is coupled to a squeezed vacuum reservoir [6]. We see from this equation that the variance of the minus quadrature at steady state and at threshold for the system under consideration is one-half of the quadrature variance for a degenerate three-level laser with the cavity mode coupled to a squeezed vacuum reservoir. Thus we observe that the effect of the parametric amplifier is to reduce the quadrature variance at steady state and at threshold by a factor of one-half. In addition, in Fig 2.2 we plot the quadrature variance [Eq.(2.150)] versus $\eta$ for $A = 100$, $\kappa = 0.8$, $r = 0.5$, $\theta = 0$, and for $\varepsilon_2 = 0.4$ (solid curve) and $\varepsilon_2 = 0$ (dashed curve) to see the effect of the parametric amplifier on the squeezing of the cavity mode. Hence we observe that the effect of the parametric amplifier is to increase the degree of squeezing.

Fig 2.3 represents the quadrature variance [Eq. (2.152)] versus $\eta$ for different values of $A$. This figure indicates that the degree of squeezing increases with the linear gain coefficient for small values of $\eta$ and almost perfect squeezing can be obtained for large values of the linear gain coefficient. Moreover, the minimum value of the quadrature variance described by Eq. (2.152) for $A = 100$, $\kappa = 0.8$, $\theta = 0$, and $r = 0.5$, is found to be $\Delta a_-^2 = 0.0346$ and occurs at $\eta = 0.070$. This result implies that the maximum intracavity

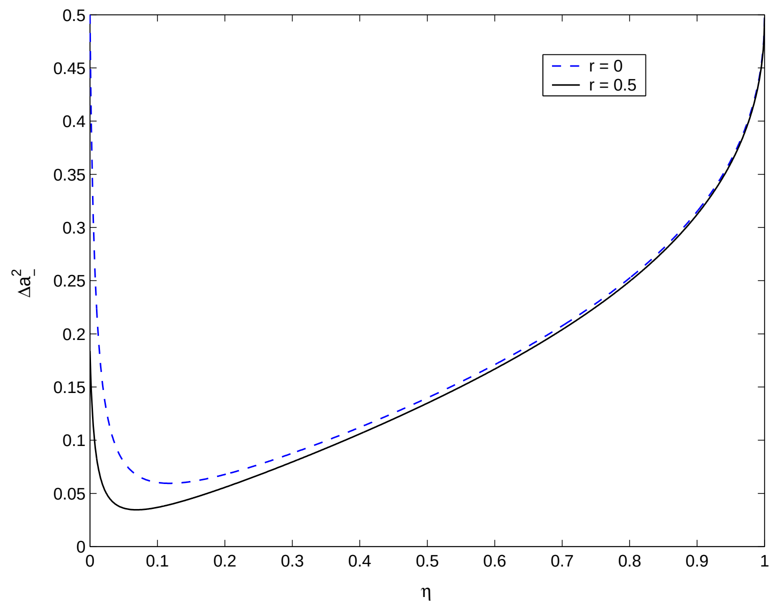


*Fig. 2.4:* Plots of the quadrature variances [Eq.(2.150)] (solid curve) and [Eq.(2.153)] (dashed curve) versus $\eta$ for $A = 100$, $\kappa = 0.8$, $\theta = 0$, $r = 0.5$, and at threshold.

squeezing for the above values is 96.5% below the coherent-state level.

We next proceed to consider some special cases. We first consider the case in which the cavity mode is coupled to a vacuum reservoir. Hence upon setting $r = 0$ in Eq. (2.150), we find

$$\Delta a_{\pm}^2 = \frac{A[1 \pm \sqrt{1-\eta^2} cos\theta] + \kappa}{A\eta + \kappa \mp 2\varepsilon_2}. \tag{2.153}$$

This represents the quadrature variance for the cavity mode of a degenerate three-level laser whose cavity contains a parametric amplifier and whose cavity mode is coupled to a vacuum reservoir. The result described by Eq (2.153) is exactly the same as the one obtained by Fesseha [4]. Fig 2.4 represents the quadrature variances [Eq. (2.150)] versus $\eta$ (solid curve) and [Eq. (2.153)] versus $\eta$ (dashed curve) for $A = 100$, $\kappa = 0.8$, $\theta = 0$, $r = 0.5$ and at threshold. Hence we see from the figure that the degree of squeezing increases with the squeeze parameter $r$.

Furthermore, we consider the case in which the nonlinear crystal is removed from the cavity and the cavity mode is coupled to a vacuum reservoir. Hence upon setting

$\varepsilon_2 = r = 0$ in Eq. (2.150), we get

$$\Delta a_{\pm}^2 = \frac{A[1 \pm \sqrt{1-\eta^2} cos\theta] + \kappa}{A\eta + \kappa}. \tag{2.154}$$

This represents the quadrature variance for a degenerate three-level laser [21].

### 2.2.2 Quadrature variance of the output mode

We next proceed to calculate the variance of the output quadratures. The variance of the output quadratures represented by the operators

$$\hat{a}_{+}^{out} = \hat{a}_{out}^{\dagger} + \hat{a}_{out}, \tag{2.155}$$

$$\hat{a}_{-}^{out} = i(\hat{a}_{out}^{\dagger} - \hat{a}_{out}), \tag{2.156}$$

can be expressed as

$$\Delta a_{\pm out}^2 = 1 + \langle : \hat{a}_{\pm}^{out}, \hat{a}_{\pm}^{out} : \rangle, \tag{2.157}$$

so that the corresponding c-number equation is

$$\Delta a_{\pm out}^2 = 1 \pm \langle \alpha_{\pm}^{out}, \alpha_{\pm}^{out} \rangle. \tag{2.158}$$

Using the input-output relation

$$\alpha_{\pm}^{out} = \sqrt{\kappa}\alpha_{\pm} - \alpha_{\pm}^{in}, \tag{2.159}$$

the output quadrature variances can be written as

$$\Delta a_{\pm out}^2 = \kappa \Delta a_{\pm}^2 + 1 - \kappa \mp 2\sqrt{\kappa}\langle \alpha_{\pm}, \alpha_{\pm}^{in} \rangle \pm \langle \alpha_{\pm}^{in}, \alpha_{\pm}^{in} \rangle. \tag{2.160}$$

We now proceed to evaluate $\langle \alpha_{\pm}, \alpha_{\pm}^{in} \rangle$. One can express the noise force $f(t)$ as a sum of the noise forces associated with the cavity mode and the reservoir modes as

$$f(t) = f_c(t) + f_r(t), \tag{2.161}$$

where we assume that $f_c(t)$ and $f_r(t)$ are not correlated. In view of Eqs. (2.104), (2.112), and (2.119), the correlation properties of these noise forces can be written as

$$\langle f_c(t) \rangle = \langle f_r(t) \rangle = 0, \tag{2.162}$$

$$\langle f_r^*(t) f_r(t')\rangle = \kappa N \delta(t-t'), \tag{2.163}$$

$$\langle f_r(t) f_r(t')\rangle = \kappa M \delta(t-t'), \tag{2.164}$$

$$\langle f_c^*(t) f_c(t')\rangle = A\rho_{aa}^{(0)} \delta(t-t'), \tag{2.165}$$

$$\langle f_c(t) f_c(t')\rangle = (A\rho_{ac}^{(0)} + \varepsilon_2) \delta(t-t'). \tag{2.166}$$

Using the definition

$$\alpha_\pm^{in}(t) = \frac{1}{\sqrt{\kappa}} \left(f_r^*(t) \pm f_r(t)\right) \tag{2.167}$$

and taking into account Eq. (2.162), we see that

$$\langle \alpha_\pm(t), \alpha_\pm^{in}(t)\rangle = \langle \alpha_\pm(t) \alpha_\pm^{in}(t)\rangle. \tag{2.168}$$

Now in view of (2.161), Eq. (2.124) can be rewritten as

$$\alpha_\pm(t) = \alpha_\pm(0) e^{-\lambda_\mp t/2} + \int_0^t e^{-\lambda_\mp (t-t')/2} \left[f_c^*(t') \pm f_c(t') + f_r^*(t') \pm f_r(t') + \varepsilon_\pm\right] dt', \tag{2.169}$$

so that multiplying by $\alpha_\pm^{in}(t)$ and then taking into account Eq. (2.167) along with (2.162), we find

$$\langle \alpha_\pm(t), \alpha_\pm^{in}(t)\rangle = \frac{1}{\sqrt{\kappa}} \int_0^t e^{-\lambda_\mp (t-t')/2} [\langle f_r^*(t') f_r^*(t)\rangle + \langle f_r(t') f_r(t)\rangle \\ \pm \langle f_r^*(t') f_r(t)\rangle \pm \langle f_r(t') f_r^*(t)\rangle] dt'. \tag{2.170}$$

Applying Eqs. (2.163) and (2.164) in Eq. (2.170) and carrying out the integration, we get

$$\langle \alpha_\pm(t), \alpha_\pm^{in}(t)\rangle = \sqrt{\kappa}(M \pm N). \tag{2.171}$$

We also note that for a squeezed vacuum

$$\langle \alpha_\pm^{in}, \alpha_\pm^{in}\rangle = \pm 2(N \pm M). \tag{2.172}$$

Hence substitution of Eqs. (2.171) and (2.172) into Eq. (2.160) leads to

$$\Delta a_{\pm out}^2(t) = \kappa \Delta a_\pm^2(t) + (1-\kappa)(1 + 2N \pm 2M). \tag{2.173}$$

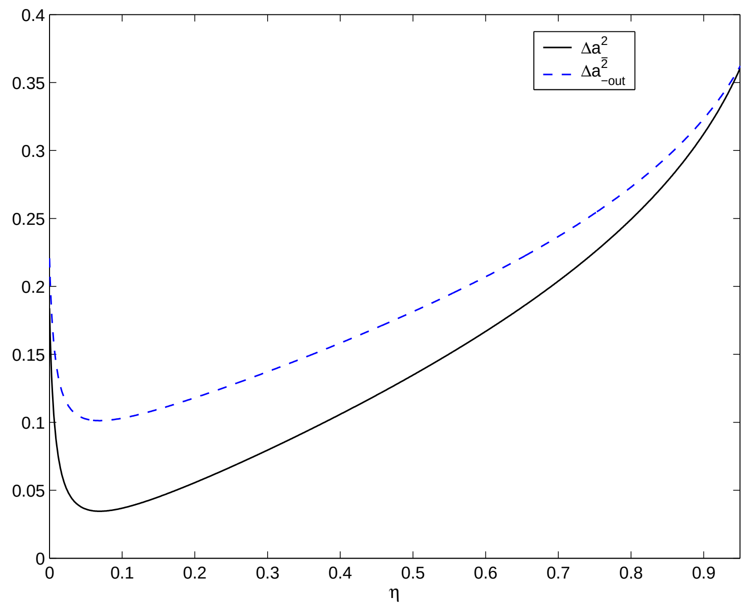


*Fig. 2.5:* Plots of the quadrature variances [Eq.(2.152)] (solid curve) and [Eq.(2.176)] (dashed curve) versus $\eta$ for $A = 100$, $\kappa = 0.8$, $r = 0.5$, and $\theta = 0$.

The first and the second terms on the right side of Eq. (2.173) represent the quadrature variances for the transmitted cavity mode and the reflected input mode, respectively. On account of Eq. (2.148), we can put Eq. (2.173) in the form

$$\Delta a^2_{\pm out}(t) = \kappa + \kappa \frac{A[1-\eta \pm \sqrt{1-\eta^2} cos\theta] \pm 2\varepsilon_2 + 2\kappa(N \pm M)]}{A\eta + \kappa \mp 2\varepsilon_2} \left(1 - e^{-(A\eta+\kappa\mp 2\varepsilon_2)t}\right) + (1-\kappa)(1+2N \pm 2M). \tag{2.174}$$

At steady state and at threshold, the output quadrature variances take the form

$$\Delta a^2_{+out} \to \infty, \tag{2.175}$$

$$\Delta a^2_{-out} = \kappa \frac{A[1-\sqrt{1-\eta^2} cos\theta] + \kappa e^{-2r}}{2(A\eta + \kappa)} + (1-\kappa)e^{-2r}. \tag{2.176}$$

The plots in Fig 2.5 represent the quadrature variance for the cavity mode [Eq.(2.152)] versus $\eta$ (solid curve) and the quadrature variance for the output mode [Eq.(2.176)] versus $\eta$ (dashed curve) for $A = 100$ $\kappa = 0.8$, $r = 0.5$, and $\theta = 0$. This figure shows that the squeezing of the cavity mode is greater than that of the output mode. Moreover, the minimum values of the quadrature variances for the cavity and output modes described by Eqs. (2.152) and (2.176) are found for $A = 100$, $\kappa = 0.8$, $\theta = 0$, and $r = 0.5$, to be $\Delta a_-^2 = 0.0.346$ and $\Delta a_{-out}^2 = 0.1012$, respectively and occur at $\eta = 0.070$. These results imply that the maximum squeezing for the above values is 96.5% for the cavity mode and 89.8% for the output mode below the coherent-state level.

### 2.2.3 Squeezing spectrum of the output mode

The squeezing spectrum of the output mode can be expressed in terms of c-number variables associated with the normal ordering as [26]

$$S_\pm^{out}(\omega) = 1 \pm 2Re \int_0^\infty d\tau e^{i(\omega-\omega_0)\tau} \langle \alpha_\pm^{out}(t), \alpha_\pm^{out}(t+\tau)\rangle_{ss}, \tag{2.177}$$

where the subscript "ss" stands for steady state. Employing Eq. (2.159), we can write

$$\begin{aligned}\langle \alpha_\pm^{out}(t), \alpha_\pm^{out}(t+\tau)\rangle_{ss} =& \kappa\langle \alpha_\pm(t), \alpha_\pm(t+\tau)\rangle_{ss} + \langle \alpha_\pm^{in}(t), \alpha_\pm^{in}(t+\tau)\rangle_{ss} \\ &- \sqrt{\kappa}\langle \alpha_\pm^{in}(t), \alpha_\pm(t+\tau)\rangle_{ss} - \sqrt{\kappa}\langle \alpha_\pm(t), \alpha_\pm^{in}(t+\tau)\rangle_{ss}.\end{aligned} \tag{2.178}$$

On account of $\langle \alpha_\pm^{in}(t)\rangle = \langle \alpha_\pm^{in}(t+\tau)\rangle = 0$ and the fact that a noise force at a certain instant does not affect the cavity mode variables at earlier time, we see that

$$\begin{aligned}\langle \alpha_\pm^{out}(t), \alpha_\pm^{out}(t+\tau)\rangle_{ss} =& \kappa\langle \alpha_\pm(t)\alpha_\pm(t+\tau)\rangle_{ss} + \langle \alpha_\pm^{in}(t)\alpha_\pm^{in}(t+\tau)\rangle_{ss} \\ &- \sqrt{\kappa}\langle \alpha_\pm^{in}(t)\alpha_\pm(t+\tau)\rangle_{ss} - \kappa\langle \alpha_\pm(t)\rangle_{ss}\langle \alpha_\pm(t+\tau)\rangle_{ss}.\end{aligned} \tag{2.179}$$

We next proceed to obtain the explicit forms of the two-time correlation functions involved in Eq. (2.179). The solution of Eq. (2.121) can also be put in the form

$$\begin{aligned}\alpha_\pm(t+\tau) = \alpha_\pm(t)e^{-\lambda_\mp\tau/2} + \int_0^\tau & e^{-\lambda_\mp(\tau-\tau')/2}\left[f_c^*(t+\tau') \pm f_c(t+\tau')\right. \\ &\left. + f_r^*(t+\tau') \pm f_r(t+\tau') + \varepsilon_\pm\right] d\tau',\end{aligned} \tag{2.180}$$

so that on multiplying by $\alpha_\pm(t)$ and then taking into account the fact that a noise force at certain instant does not affect the cavity mode variables at earlier time, we have

$$\langle \alpha_\pm(t)\alpha_\pm(t+\tau)\rangle = \langle \alpha_\pm^2(t)\rangle e^{-\lambda_\mp\tau/2} + \varepsilon_\pm\langle \alpha_\pm(t)\rangle \int_0^\tau e^{-\lambda_\mp(\tau-\tau')/2} d\tau'. \tag{2.181}$$

It then follows

$$\langle \alpha_\pm(t)\alpha_\pm(t+\tau)\rangle = \langle \alpha_\pm^2(t)\rangle e^{-\lambda_\mp\tau/2} + \frac{2\varepsilon_\pm\langle \alpha_\pm(t)\rangle}{\lambda_\mp}(1 - e^{-\lambda_\mp\tau/2}). \tag{2.182}$$

Substituting Eqs. (2.133) and (2.135) into Eq. (2.182), we find at steady state

$$\langle\alpha_\pm(t)\alpha_\pm(t+\tau)\rangle_{ss} = \frac{A[\rho_{ac}^{(0)} + \rho_{ac}^{*(0)} \pm 2\rho_{aa}^{(0)}] + 2[\varepsilon_2 + \kappa(M \pm N)]}{\lambda_\mp} e^{-\lambda_\mp\tau/2} + \frac{4\varepsilon_\pm^2}{\lambda_\mp^2}. \quad (2.183)$$

With the aid of Eq. (2.167) along with (2.163) and (2.164), one can easily establish that

$$\langle\alpha_\pm^{in}(t+\tau)\alpha_\pm^{in}(t)\rangle_{ss} = \pm 2(N \pm M)\delta(\tau). \quad (2.184)$$

Furthermore, on multiplying Eq. (2.180) by $\alpha_\pm^{in}(t)$ and taking into account Eq. (2.167), we see that

$$\begin{aligned}\langle\alpha_\pm^{in}(t)\alpha_\pm(t+\tau)\rangle = {} & \frac{1}{\sqrt{\kappa}}\langle(f_r^*(t) \pm f_r(t))\alpha_\pm(t)\rangle e^{-\lambda_\mp\tau/2} \\ & + \frac{1}{\sqrt{\kappa}}\int_0^\tau e^{-\lambda_\mp(\tau-\tau')/2}[\langle f_r^*(t)f_r^*(t+\tau')\rangle + \langle f_r(t)f_r(t+\tau')\rangle \\ & \qquad \pm \langle f_r(t)f_r^*(t+\tau')\rangle \pm \langle f_r(t+\tau')f_r^*(t)\rangle \\ & \qquad + \varepsilon_\pm(\langle f_r^*(t)\rangle \pm \langle f_r(t)\rangle)]d\tau', \end{aligned} \quad (2.185)$$

so that on applying Eqs. (2.162), (2.163), (2.164), and carrying out the integration, there follows

$$\langle\alpha_\pm^{in}(t)\alpha_\pm(t+\tau)\rangle = \frac{1}{\sqrt{\kappa}}\langle(f_r^*(t) \pm f_r(t))\alpha_\pm(t)\rangle e^{-\lambda_\mp\tau/2} \pm \sqrt{\kappa}(N \pm M)e^{-\lambda_\mp\tau/2}. \quad (2.186)$$

In addition, multiplying Eq. (2.169) by $(f_r^*(t) \pm f_r(t))$ and then taking into account Eqs. (2.163) and (2.164), we have

$$\langle(f_r^*(t) \pm f_r(t))\alpha_\pm(t)\rangle = \pm 2\kappa(N \pm M)\int_0^t e^{-\lambda_\mp(t-t')/2}\delta(t-t')dt'. \quad (2.187)$$

Hence on carrying out the integration, we get

$$\langle(f_r^*(t) \pm f_r(t))\alpha_\pm(t)\rangle = \pm\kappa(N \pm M). \quad (2.188)$$

Now on substituting Eq. (2.188) into (2.186), we find at steady state

$$\langle\alpha_\pm^{in}(t)\alpha_\pm(t+\tau)\rangle_{ss} = \pm 2\sqrt{\kappa}(N \pm M)e^{-\lambda_\mp\tau/2}. \quad (2.189)$$

In view of Eq. (2.133), we have

$$\langle\alpha_\pm(t)\rangle_{ss} = \langle\alpha_\pm(t+\tau)\rangle_{ss} = \frac{2\sqrt{\kappa}\varepsilon_\pm}{\lambda_\mp}. \quad (2.190)$$

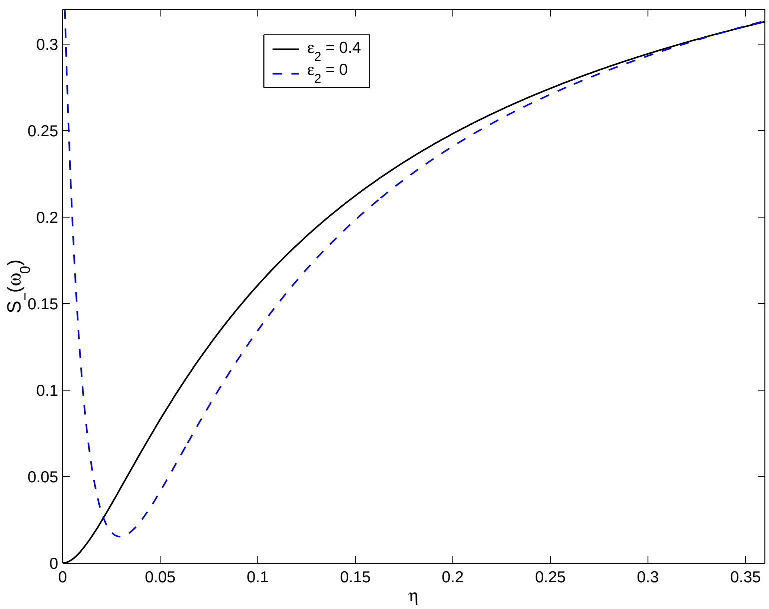


*Fig. 2.6:* Plots of the squeezing spectrum [Eq.(2.195)] versus $\eta$ for $\kappa = 0.8$, $\theta = 0$, $r = 0.5$, $A = 25$ and for $\varepsilon_2 = 0$ and 0.4.

Thus combination of Eqs. (2.179), (2.183), (2.184), (2.189), and (2.190) leads to

$$\langle \alpha_{\pm}^{out}(t), \alpha_{\pm}^{out}(t+\tau)\rangle_{ss} = \frac{A\kappa[\rho_{ac}^{(0)} + \rho_{ac}^{*(0)} \pm 2\rho_{aa}^{(0)}] + 2\kappa[\varepsilon_2 + (\kappa - \lambda_{\mp})(M \pm N)]}{\lambda_{\mp}} e^{-\lambda_{\mp}\tau/2} \pm 2(N \pm M)\delta(\tau). \quad (2.191)$$

Now using Eq. (2.191) in Eq. (2.177) and then carrying out the integration, we obtain

$$S_{\pm}^{out}(\omega) = 1 \pm 4\kappa \frac{A[\rho_{ac}^{(0)} + \rho_{ac}^{*(0)} \pm 2\rho_{aa}^{(0)}] + 2[\varepsilon_2 + (\kappa - \lambda_{\mp})(M \pm N)]}{4(\omega - \omega_0)^2 + \lambda_{\mp}^2} + 2(N \pm M). \quad (2.192)$$

This expression can be rewritten in terms of the parameter $\eta$ and $r$ as

$$S_{\pm}^{out}(\omega) = 4\kappa \frac{A[1 \pm \sqrt{1-\eta^2}\ cos\theta] - [A\eta \mp 2\varepsilon_2]e^{\pm 2r}}{4(\omega - \omega_0)^2 + (A\eta + \kappa \mp 2\varepsilon_2)^2} + e^{\pm 2r}. \quad (2.193)$$

At threshold and for $\omega = \omega_0$, we have

$$S_{+}^{out}(\omega_0) \to \infty, \quad (2.194)$$

$$S_{-}^{out}(\omega_0) = \frac{\kappa A[1 - \sqrt{1-\eta^2}\ cos\theta] + A^2\eta^2 e^{-2r}}{(A\eta + \kappa)^2}. \quad (2.195)$$

Fig 2.6 represents the squeezing spectrum of the output mode [Eq.(2.195)] versus $\eta$ for $A = 25$, $\kappa = 0.8$, $\theta = 0$, $r = 0.5$, and for $\varepsilon_2 = 0$ (dashed curve) and $\varepsilon_2 = 0.4$ (solid curve).

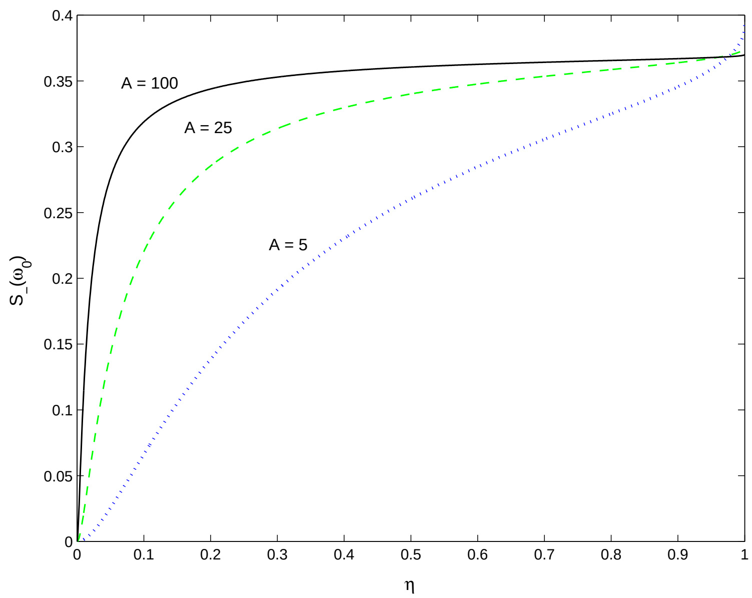


*Fig. 2.7:* Plots of the squeezing spectrum [Eq.(2.195)] versus $\eta$ for $\kappa = 0.8$, $\theta = 0$, $r = 0.5$, and for different values of A.

We observe from this figure that the effect of the parametric amplifier is to increase the degree of squeezing for $0 < \eta < 0.02$ and to decrease for $0.02 < \eta < 0.32$. Fig 2.7 represents the squeezing spectrum of the output mode [Eq. (2.195)] versus $\eta$ for $\kappa = 0.8$, $\theta = 0$, $r = 0.5$, and for different values of the linear gain coefficient. We see from this figure that there is perfect squeezing for $\eta = 0$. In fact a closer inspection of Eq. (2.195) shows that there is perfect squeezing for $\eta = 0$, $\omega = \omega_0$, and for any values of $A$, $\kappa$, and $r$. Furthermore, we plot [Eq.(2.195)] versus $\eta$ for $A = 100$, $\kappa = 0.8$, $\theta = 0$, and for $r = 0$ (dashed curve) and $r = 0.5$ (solid curve). We observe from Fig 2.8 that the effect of the squeezed vacuum reservoir is to increase the degree of squeezing.

## 2.3 Photon Statistics

The statistical properties of a light mode are described in terms of the mean and variance of the photon number as well as the photon number distribution. In this section we first obtain, using the antinormally ordered characteristic function defined in the Heisenberg picture, the Q function for the cavity mode. Then applying the resulting Q function, we calculate the mean and variance of the photon number as well as the photon number

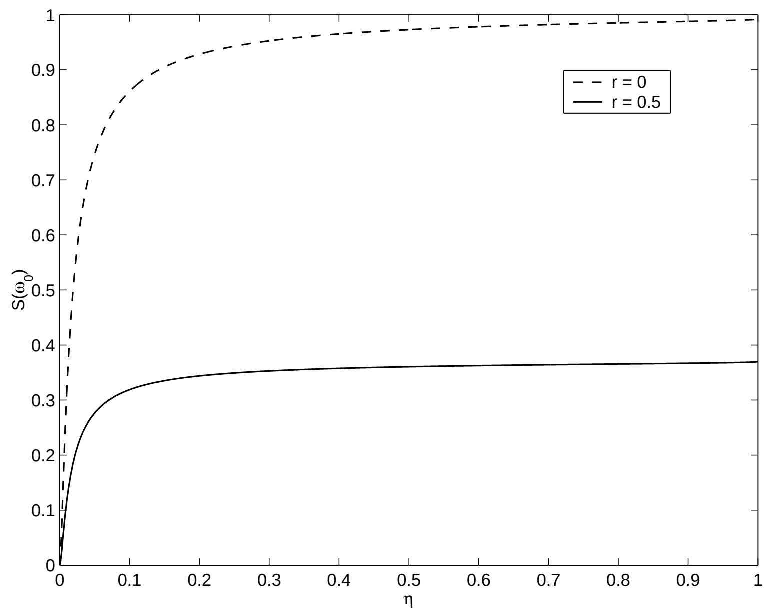


*Fig. 2.8:* Plots of the squeezing spectrum [Eq.(2.195)] versus $\eta$ for $A = 100$, $\kappa = 0.8$, $\theta = 0$, and for $r = 0$ and 0.5.

distributions for the cavity mode produced by a degenerate three-level laser whose cavity contains a parametric amplifier and whose cavity mode is driven by coherent light and coupled to a squeezing vacuum reservoir. Finally, we calculate the mean and the normally-ordered variance of the photon count.

### 2.3.1 Photon number statistics of the cavity mode

#### The Q function

We now proceed to obtain the Q function for the cavity mode. The Q function for a single-mode light can be expressed in terms of the antinormally ordered characteristic function as

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi} \int \frac{d^2 z}{\pi} \Phi_A(z^*, z, t) e^{z^*\alpha - z\alpha^*}, \tag{2.196}$$

where $\Phi_A(z^*, z, t)$ is defined in the Heisenberg picture by

$$\Phi_A(z^*, z, t) = Tr\left(\rho(0) e^{-z^*\hat{a}(t)} e^{z\hat{a}^\dagger(t)}\right). \tag{2.197}$$

Applying the relation

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{B}} e^{\hat{A}} e^{[\hat{A}, \hat{B}]}, \tag{2.198}$$

which holds for $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$, Eq. (2.197) can be rewritten as

$$\Phi_A(z^*, z, t) = e^{-z^*z} Tr\left(\rho(0) e^{z\hat{a}^\dagger(t)} e^{-z^*\hat{a}(t)}\right). \tag{2.199}$$

We note that Eq. (2.199) can be expressed in terms of c-number variables associated with the normal ordering as

$$\Phi_A(z^*, z, t) = e^{-z^*z} \langle e^{z\alpha^*(t) - z^*\alpha(t)} \rangle. \tag{2.200}$$

Using Eq. (2.290), we can write

$$\Phi_A(z^*, z, t) = exp\Big[ - z^*z + \frac{2\varepsilon_1}{\lambda_-}(1 - e^{-\lambda_- t/2})(z - z^*)\Big] \langle e^{z\alpha'^*(t) - z^*\alpha'(t)} \rangle, \tag{2.201}$$

where

$$\alpha'(t) = A_+(t)\alpha(0) + A_-(t)\alpha^*(0) + F(t). \tag{2.202}$$

With the aid of Eqs. (2.202), (2.122), and (2.126), it can be easily established that

$$\frac{d}{dt}\langle \alpha'(t) \rangle = -\frac{1}{2}[\mu \langle \alpha'(t) \rangle - 2\varepsilon \langle \alpha'^*(t) \rangle], \tag{2.203}$$

which is a linear differential equation for $\alpha'(t)$. On the other hand, in view of Eqs. (2.104) and (2.127), we see that

$$\langle F(t) \rangle = 0. \tag{2.204}$$

Assuming that the cavity mode is initially in a vacuum state and taking into account Eqs. (2.202) and (2.204), we have

$$\langle \alpha'(t) \rangle = 0. \tag{2.205}$$

Thus we observe that $\alpha'(t)$ is a Gaussian variable with zero mean. On account of this, Eq. (2.201) can be put in the form [21, 32]

$$\Phi_A(z^*, z, t) = exp\Big[ - z^*z + \frac{2\varepsilon_1}{\lambda_-}(1 - e^{-\lambda_- t/2})(z - z^*)\Big] e^{\langle \frac{1}{2}(z\alpha'^*(t) - z^*\alpha'(t))^2 \rangle}. \tag{2.206}$$

or

$$\Phi_A(z^*, z, t) = exp\Big[ - z^*z(1 + \langle \alpha'^*(t)\alpha'(t) \rangle) + \frac{1}{2}(z^2\langle \alpha'^{*2}(t) \rangle + z^{*2}\langle \alpha'^2(t) \rangle) + \frac{2\varepsilon_1}{\lambda_-}(1 - e^{-\lambda_- t/2})(z - z^*)\Big]. \tag{2.207}$$

We now proceed to obtain the expectation values involved in Eq. (2.207). Taking into account Eq. (2.202) with the cavity mode being initially in vacuum state and the fact that a noise force at a given instant does not affect the cavity mode variable at earlier time, we find

$$\langle \alpha'^*(t)\alpha'(t)\rangle = \langle F^*(t)F(t)\rangle, \tag{2.208}$$

so that using Eq. (2.127), we have

$$\begin{aligned}\langle \alpha'^*(t)\alpha'(t)\rangle = \int_0^t dt'dt'' \Big[ & A_+(t-t')A_+(t-t'')\langle f^*(t')f(t'')\rangle \\ & + A_-(t-t')A_-(t-t'')\langle f^*(t'')f(t')\rangle \\ & + A_+(t-t')A_-(t-t'')\langle f^*(t')f^*(t'')\rangle \\ & + A_-(t-t')A_+(t-t'')\langle f(t')f(t'')\rangle \Big].\end{aligned} \tag{2.209}$$

With the aid of Eqs. (2.112) and (2.119), we get

$$\begin{aligned}\langle \alpha'^*(t)\alpha'(t)\rangle = \int_0^t dt' \Big[ & (A_+^2(t-t') + A_-^2(t-t'))[A\rho_{aa}^{(0)} + \kappa N] \\ & + A_+(t-t')A_-(t-t')[A(\rho_{ac}^{(0)} + \rho_{ac}^{*(0)}) + 2\kappa M + 2\varepsilon_2] \Big]\end{aligned} \tag{2.210}$$

and employing Eq. (2.126), we have

$$\begin{aligned}\langle \alpha'^*(t)\alpha'(t)\rangle = \frac{1}{4}\int_0^t dt' \Big[ & (A[\rho_{ac}^{(0)} + \rho_{ac}^{*(0)} + 2\rho_{aa}^{(0)}] + 2[\varepsilon_2 + \kappa(M+N)])e^{-\lambda_-(t-t')} \\ & - (A[\rho_{ac}^{(0)} + \rho_{ac}^{*(0)} - 2\rho_{aa}^{(0)}] + 2[\varepsilon_2 + \kappa(M-N)])e^{-\lambda_+(t-t')} \Big].\end{aligned} \tag{2.211}$$

It then follows that

$$\begin{aligned}\langle \alpha'^*(t)\alpha'(t)\rangle = & \frac{1}{4\lambda_-}(A[\rho_{ac}^{(0)} + \rho_{ac}^{*(0)} + 2\rho_{aa}^{(0)}] + 2[\varepsilon_2 + \kappa(M+N)])(1 - e^{-\lambda_- t}) \\ & - \frac{1}{4\lambda_+}(A[\rho_{ac}^{(0)} + \rho_{ac}^{*(0)} - 2\rho_{aa}^{(0)}] + 2[\varepsilon_2 + \kappa(M-N)])(1 - e^{-\lambda_+ t}).\end{aligned} \tag{2.212}$$

Following a similar procedure, we easily find

$$\begin{aligned}\langle \alpha'^2(t)\rangle = & \frac{1}{4\lambda_-}(A[\rho_{ac}^{(0)} + \rho_{ac}^{*(0)} + 2\rho_{aa}^{(0)}] + 2[\varepsilon_2 + \kappa(M+N)])(1 - e^{-\lambda_- t}) \\ & + \frac{1}{4\lambda_+}(A[\rho_{ac}^{(0)} + \rho_{ac}^{*(0)} - 2\rho_{aa}^{(0)}] + 2[\varepsilon_2 + \kappa(M-N)])(1 - e^{-\lambda_+ t}) \\ & + \frac{A}{\lambda_- + \lambda_+}(\rho_{ac}^{(0)} - \rho_{ac}^{*(0)})(1 - e^{-(\lambda_- + \lambda_+)t/2}).\end{aligned} \tag{2.213}$$

On account of (2.212) and (2.213) along with the complex conjugate (2.213), the characteristic function described by Eq. (2.207) can be written as

$$\Phi_A(z^*, z, t) = e^{-az^*z + b^*z^2 + bz^{*2} + cz - cz^*}, \tag{2.214}$$

where the coefficients, expressed in terms of the parameter $\eta$, are

$$\begin{aligned} a = 1 &+ \frac{A[1-\eta+\sqrt{1-\eta^2}cos\ \theta] + 2\varepsilon_2 + 2\kappa(N+M)}{4(A\eta+\kappa-2\varepsilon_2)}(1-e^{-(A\eta+\kappa-2\varepsilon_2)t}) \\ &+ \frac{A[1-\eta-\sqrt{1-\eta^2}cos\ \theta] - 2\varepsilon_2 + 2\kappa(N-M)}{4(A\eta+\kappa+2\varepsilon_2)}(1-e^{-(A\eta+\kappa+2\varepsilon_2)t}), \end{aligned} \tag{2.215}$$

$$\begin{aligned} b =& \frac{A[1-\eta+\sqrt{1-\eta^2}cos\ \theta] + 2\varepsilon_2 + 2\kappa(N+M)}{8(A\eta+\kappa-2\varepsilon_2)}(1-e^{-(A\eta+\kappa-2\varepsilon_2)t}) \\ &- \frac{A[1-\eta-\sqrt{1-\eta^2}cos\ \theta] - 2\varepsilon_2 + 2\kappa(N-M)}{8(A\eta+\kappa+2\varepsilon_2)}(1-e^{-(A\eta+\kappa+2\varepsilon_2)t}) \\ &+ \frac{iA\sqrt{1-\eta^2}sin\ \theta}{4(A\eta+\kappa)}(1-e^{-(A\eta+\kappa)t}), \end{aligned} \tag{2.216}$$

$$c = \frac{2\varepsilon_1}{A\eta+\kappa-2\varepsilon_2}(1-e^{-(A\eta+\kappa-2\varepsilon_2)t/2}). \tag{2.217}$$

Hence applying (2.214) in Eq. (2.196) and carrying out the integration with the help of the relation

$$\int \frac{d^2\alpha}{\pi} e^{-a\alpha^*\alpha + b\alpha + c\alpha^* + B\alpha^2 + C\alpha^{*2}} = \frac{1}{\sqrt{a^2-4BC}} exp\left\{\frac{abc + Bc^2 + Cb^2}{a^2-4BC}\right\}, \quad a > 0, \tag{2.218}$$

the Q function for the cavity mode is found to be

$$Q(\alpha^*, \alpha, t) = \frac{\sqrt{u^2-4v^*v}}{\pi} exp\Big[-u|\alpha-c|^2 + v^*(\alpha-c)^2 + v(\alpha^*-c)^2\Big], \tag{2.219}$$

where

$$u = \frac{a}{a^2-4b^*b}, \tag{2.220}$$

$$v = \frac{b}{a^2-4b^*b}. \tag{2.221}$$

This represents the Q function for the cavity mode which is initially in a vacuum state.

### The mean and variance of the photon number

We next seek to calculate the mean and variance of the photon number for the cavity mode applying the Q function. We recall that the mean photon number can be expressed as

$$\overline{n} = \int d^2\alpha Q(\alpha^*, \alpha, t)(\alpha^*\alpha - 1). \tag{2.222}$$

Taking into account Eq. (2.219), the mean photon number can be put in the form

$$\overline{n} = \frac{\sqrt{u^2 - 4v^*v}}{e^{c^2(u-v^*-v)}}\left(\frac{\partial^2}{\partial q \partial q^*} - 1\right)\int \frac{d^2\alpha}{\pi} e^{-u\alpha^*\alpha + v^*\alpha^2 + v\alpha^{*2} + q\alpha^* + q^*\alpha}\bigg|_{q=c(u-2v)}. \tag{2.223}$$

Upon carrying out the integration with the help of Eq. (2.218), we obtain

$$\overline{n} = e^{-c^2(u-v^*-v)}\left(\frac{\partial^2}{\partial q \partial q^*} - 1\right)e^{\frac{uq^*q + v^*q^2 + vq^{*2}}{u^2 - 4v^*v}}\bigg|_{q=c(u-2v)}, \tag{2.224}$$

from which follows

$$\overline{n} = \frac{u}{u^2 - 4v^*v} + c^2 - 1. \tag{2.225}$$

Using Eqs. (2.220) and (2.221) along with (2.215) and (2.217), we can write

$$\begin{aligned}\overline{n} =& \frac{A[1 - \eta + \sqrt{1-\eta^2} cos\ \theta] + 2\varepsilon_2 + 2\kappa(N + M)}{4(A\eta + \kappa - 2\varepsilon_2)}(1 - e^{-(A\eta + \kappa - 2\varepsilon_2)t}) \\ &+ \frac{A[1 - \eta - \sqrt{1-\eta^2} cos\ \theta] - 2\varepsilon_2 + 2\kappa(N - M)}{4(A\eta + \kappa + 2\varepsilon_2)}(1 - e^{-(A\eta + \kappa + 2\varepsilon_2)t}) \\ &+ \frac{4\varepsilon_1^2}{(A\eta + \kappa - 2\varepsilon_2)^2}(1 - e^{-(A\eta + \kappa - 2\varepsilon_2)t/2})^2\end{aligned} \tag{2.226}$$

and at steady state this can be put in the form

$$\begin{aligned}\overline{n}_{ss} =& \frac{A[1 + \sqrt{1-\eta^2} cos\ \theta] + \kappa e^{2r}}{4(A\eta + \kappa - 2\varepsilon_2)} + \frac{A[1 - \sqrt{1-\eta^2} cos\ \theta] + \kappa e^{-2r}}{4(A\eta + \kappa + 2\varepsilon_2)} \\ &+ \frac{4\varepsilon_1^2}{(A\eta + \kappa - 2\varepsilon_2)^2} - \frac{1}{2}.\end{aligned} \tag{2.227}$$

This represents the mean photon number for the cavity mode produced by a three-level laser whose cavity contains a parametric amplifier and whose cavity mode is driven by coherent light and coupled to a squeezed vacuum reservoir. We can easily see from Eq. (2.227) that the coherent driving light enhances the mean photon number.

Fig 2.9 represents the mean photon number [Eq.(2.227)] versus $\eta$ for $A = 100$, $\kappa = 0.8$, $\varepsilon_1 = 0.2$, $r = 0.5$, $\theta = 0$, and for $\varepsilon_2 = 0$ (solid curve) and $\varepsilon_2 = 0.3$ (dashed curve). We

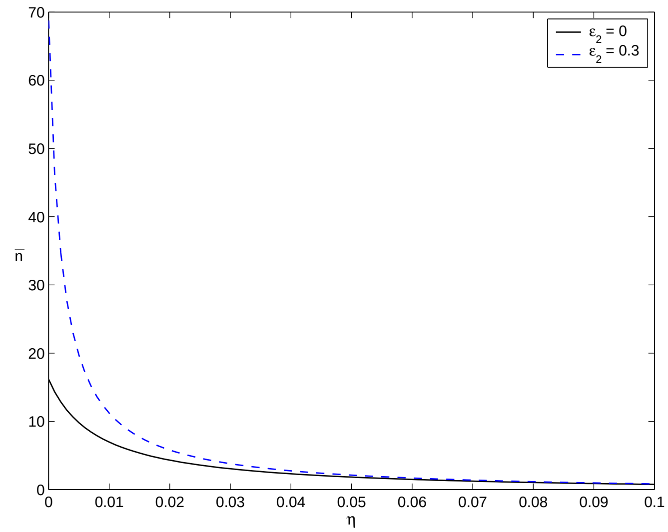


*Fig. 2.9:* Plots of the mean photon number [Eq.(2.227)] versus $\eta$ for $A = 100$, $\kappa = 0.8$, $\varepsilon_1 = 0.2$, $r = 0.5$, $\theta = 0$, and for different values of $\varepsilon_2$.

observe from this figure that the mean photon number deceases with $\eta$ and the parametric amplifier increases the mean photon number of the cavity mode.

We now proceed to consider some special cases. First we consider the case in which there is no parametric amplifier in the cavity. Thus upon setting $\varepsilon_2 = 0$ in Eq. (2.227), we get

$$\overline{n}_{ss} = \frac{A[1-\eta] + 2\kappa N}{2(A\eta + \kappa)} + \frac{4\varepsilon_1^2}{(A\eta + \kappa)^2}. \tag{2.228}$$

This is the mean photon number of a three-level laser whose cavity mode is driven by coherent light and coupled to a squeezed vacuum reservoir. The result described by Eq. (2.228) is the same as that obtained in [6].

We next consider the case in which the cavity mode is not driven by coherent light and coupled to a vacuum reservoir. Hence upon setting $\varepsilon_1 = 0$ and $r = 0$ in Eq. (2.227), we find

$$\overline{n}_{ss} = \frac{A[1 + \sqrt{1-\eta^2} cos\ \theta] + \kappa}{4(A\eta + \kappa - 2\varepsilon_2)} + \frac{A[1 - \sqrt{1-\eta^2} cos\ \theta] + \kappa}{4(A\eta + \kappa + 2\varepsilon_2)} - \frac{1}{2}. \tag{2.229}$$

This represents the mean photon number for a three-level laser with a parametric amplifier and with the cavity mode coupled to a vacuum reservoir. The result described by Eq. (2.229) is exactly the same as the one calculated by Fesseha [4]. Furthermore, setting

$\varepsilon_2 = 0$ in Eq. (2.229) leads to

$$\overline{n}_{ss} = \frac{A(1-\eta)}{2(A\eta + \kappa)}, \tag{2.230}$$

which is the mean photon number for a three-level laser.

We next proceed to calculate the variance of the photon number for the cavity mode. The variance of the photon number defined by

$$\Delta n^2 = \langle(\hat{a}^\dagger \hat{a})^2\rangle - \langle\hat{a}^\dagger \hat{a}\rangle^2 \tag{2.231}$$

can be expressed as

$$\Delta n^2 = \langle\hat{a}^2 \hat{a}^{\dagger 2}\rangle - \overline{n}^2 - 3\overline{n} - 2, \tag{2.232}$$

where $\overline{n} = \langle\hat{a}^\dagger \hat{a}\rangle$ is the mean photon number for the cavity mode. Using Eq. (2.219), we can write

$$\langle\hat{a}^2 \hat{a}^{\dagger 2}\rangle = \frac{\sqrt{u^2 - 4v^* v}}{e^{c^2(u - v^* - v)}} \frac{\partial^4}{\partial q^2 \partial q^{*2}} \int \frac{d^2\alpha}{\pi} e^{-u\alpha^*\alpha + v^*\alpha^2 + v\alpha^{*2} + q\alpha^* + q^*\alpha}\Big|_{q=c(u-2v)}, \tag{2.233}$$

so that carrying out the integration with the help of Eq. (2.218), we get

$$\langle\hat{a}^2 \hat{a}^{\dagger 2}\rangle = e^{-c^2(u - v^* - v)} \frac{\partial^4}{\partial q^2 \partial q^{*2}} exp\left[\frac{uq^*q + v^*q^2 + vq^{*2}}{u^2 - 4v^*v}\right]\Big|_{q=c(u-2v)}. \tag{2.234}$$

Then performing the differentiation, we find

$$\langle\hat{a}^2 \hat{a}^{\dagger 2}\rangle = \frac{2(u^2 + 2v^*v)}{(u^2 - 4v^*v)^2} + \frac{2c^2(2u + v^* + v)}{u^2 - 4v^*v} + c^4. \tag{2.235}$$

Combination of Eqs. (2.235), (2.232), (2.225), (2.220), and (2.221) results in

$$\Delta n^2 = a(a-1) + 4b^*b + 2c^2(a + b^* + b - 1/2). \tag{2.236}$$

With the aid of Eqs. (2.215), (2.216), and (2.217), the variance of the photon number for the cavity mode is found at steady state to be of the form

$$\begin{aligned}\Delta n_{ss}^2 =& 2\Big[\frac{A[1 + \sqrt{1-\eta^2}cos\ \theta] + \kappa e^{2r}}{4(A\eta + \kappa - 2\varepsilon_2)}\Big]^2 + 2\Big[\frac{A[1 - \sqrt{1-\eta^2}cos\ \theta] + \kappa e^{-2r}}{4(A\eta + \kappa + 2\varepsilon_2)}\Big]^2 \\ &+ 4\varepsilon_1^2 \frac{A[1 + \sqrt{1-\eta^2}cos\ \theta] + \kappa e^{2r}}{(A\eta + \kappa - 2\varepsilon_2)^3} + \frac{A^2(1-\eta^2)sin^2\ \theta}{4(A\eta + \kappa)^2} - \frac{1}{4}.\end{aligned} \tag{2.237}$$

Fig 2.10 represents the mean photon number [Eq.(2.227)] versus $\eta$ (solid curve) and the uncertainty in the photon number using [Eq.(2.237)] versus $\eta$ (dashed curve) for $A = 100$, $\kappa = 0.8$, $\varepsilon_1 = 0.2$, $\varepsilon_2 = 0.3$, $r = 0.5$, and $\theta = 0$. We observe from the figure that the uncertainty in the photon number is greater than the mean photon number.

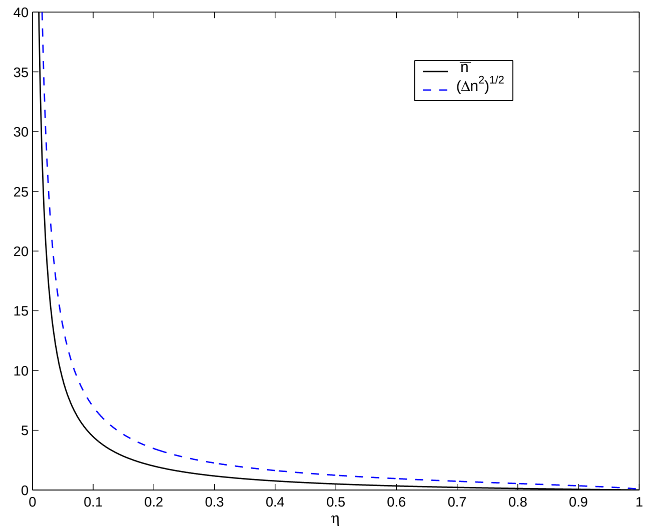


*Fig. 2.10:* Plots of the mean photon number [Eq.(2.227)] and the uncertainty in the photon number using [Eq.(2.237)] versus $\eta$ for $A = 100$, $\kappa = 0.8$, $\varepsilon_1 = 0.2$, $\varepsilon_2 = 0.3$, $r = 0.5$, and $\theta = 0$.

### The photon number distribution

The photon number distribution for a single-mode light defined by

$$P(n,t) = \langle n|\hat{\rho}(\hat{a}^\dagger, \hat{a})|n\rangle, \tag{2.238}$$

is expressible in terms of the Q function as [28, 33]

$$P(n,t) = \frac{\pi}{n!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}Q(\alpha^*,\alpha,t)e^{\alpha^*\alpha}\Big|_{\alpha^*=\alpha=0}. \tag{2.239}$$

On account of Eq. (2.219), we see that

$$P(n,t) = \frac{\sqrt{u^2-4v^*v}}{n!e^{c^2(u-v^*-v)}}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}e^{(1-u)\alpha^*\alpha+v^*\alpha^2+v\alpha^{*2}+c(u-2v^*)\alpha+c(u-2v)\alpha^*}\Big|_{\alpha^*=\alpha=0}, \tag{2.240}$$

so that using the power series

$$e^{(1-u)\alpha^*\alpha} = \sum_k^\infty \frac{(1-u)^k}{k!}\alpha^{*k}\alpha^k, \tag{2.241}$$

$$e^{v^*\alpha^2} = \sum_l^\infty \frac{v^{*l}}{l!}\alpha^{2l}, \tag{2.242}$$

$$e^{v\alpha^{*2}} = \sum_m^\infty \frac{v^m}{m!}\alpha^{*2m}, \tag{2.243}$$

$$e^{c(u-2v^*)\alpha} = \sum_r^{\infty} \frac{[c(u-2v^*)]^r}{r!}\alpha^r, \tag{2.244}$$

$$e^{c(u-2v)\alpha^*} = \sum_r^{\infty} \frac{[c(u-2v)]^s}{s!}\alpha^{*s}, \tag{2.245}$$

we have

$$P(n,t) = \frac{\sqrt{u^2-4v^*v}}{n!e^{c^2(u-v^*-v)}} \sum_{k,l,m,r,s} \frac{(1-u)^k}{k!}\frac{v^{*l}}{l!}\frac{v^m}{m!}\frac{[c(u-2v^*)]^r}{r!}\frac{[c(u-2v)]^s}{s!} \frac{\partial^n}{\partial\alpha^n}\alpha^{k+2l+r}\frac{\partial^n}{\partial\alpha^{*n}}\alpha^{*k+2m+s}\bigg|_{\alpha^*=\alpha=0}. \tag{2.246}$$

Upon carrying out the differentiation with the help of the relation

$$\frac{\partial^m}{\partial x^m}x^n\bigg|_{x=0} = \frac{n!}{(n-m)!}\delta_{mn}, \tag{2.247}$$

we get

$$P(n,t) = \frac{\sqrt{u^2-4v^*v}}{n!e^{c^2(u-v^*-v)}} \sum_{k,l,m,r,s} \frac{(1-u)^k}{k!}\frac{v^{*l}}{l!}\frac{v^m}{m!}\frac{[c(u-2v^*)]^r}{r!}\frac{[c(u-2v)]^s}{s!} \frac{(k+2l+r)!}{(k+2l+r-n)!}\frac{(k+2m+s)!}{(k+2m+s-n)!} \delta_{k+2l+r,n}\delta_{k+2m+s,n}. \tag{2.248}$$

Hence applying the property of the Kronecker delta, we have

$$r = n-k-2l \tag{2.249}$$

and

$$s = n-k-2m. \tag{2.250}$$

Now in view of these results, the photon number distribution takes the form

$$P(n,t) = \frac{\sqrt{u^2-4v^*v}}{e^{c^2(u-v^*-v)}} \sum_{k=0}^{n}\sum_{l,m=0}^{[n-k]} n!\frac{(1-u)^k}{k!}\frac{v^{*l}}{l!}\frac{v^m}{m!}\frac{[c(u-2v^*)]^{n-k-2l}}{(n-k-2l)!}\frac{[c(u-2v)]^{n-k-2m}}{(n-k-2m)!}, \tag{2.251}$$

where $[n-k] = (n-k)/2$ for even $n-k$ and $[n-k] = (n-k-1)/2$ for odd $n-k$. Fig 2.11 represents the photon number distribution [Eq.(2.251)] versus $n$ for $\kappa = 0.8$, $\varepsilon_1 = 0.2$,

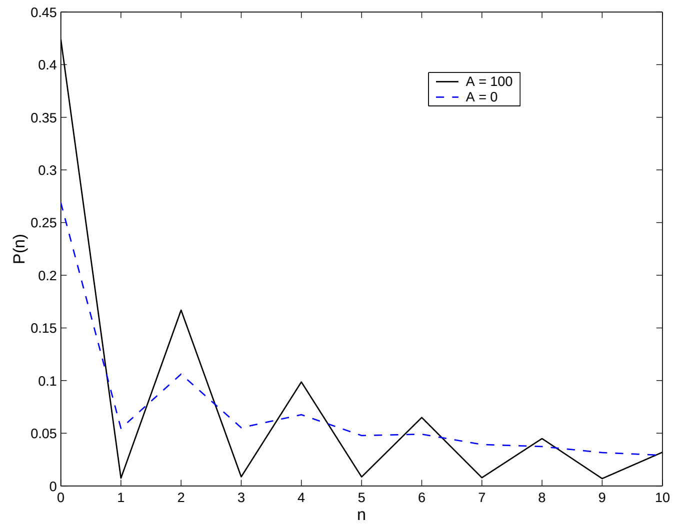


*Fig. 2.11:* Plots of the photon number distribution [Eq.(2.251)] versus n for $\kappa = 0.8$, $\varepsilon_1 = 0.2$, $\varepsilon_2 = 0.3$, $\eta = 0.1$, $r = 0.5$, and $\theta = 0$ and for $A = 0$ and 100.

$\varepsilon_2 = 0.3$, $r = 0.5$, $\eta = 0.1$, $\theta = 0$, and for $A = 100$ (solid curve) and $A = 0$ (dashed curve). The figure indicates that the probability for observing n photons in the cavity decreases as n increases. we also see from the same figure that the probability for observing even number of photons is greater than that for observing odd number of photons. Moreover, we see from the same figure that there is more probability for observing even number of photons for $A \neq 0$ than for $A = 0$.

### 2.3.2 Mean and variance of the photon count

In this section we wish to calculate the mean and the normally-ordered variance of the photon count for the output mode employing a moment generating function. The moment generating function is defined by [34]

$$M(\lambda) = \sum_{m=0}^{\infty} P(m)(1-\lambda)^m, \tag{2.252}$$

where $P(m)$ is the photon count distribution. We note that the photon count distribution can be expressed as [26, 34]

$$P(m) = \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!} P_{out}(n) \nu^m (1-\nu)^{n-m}, \tag{2.253}$$

in which $P_{out}(n)$ is the photon number distribution for the output mode and $\nu$ is the probability for detecting a single photon. On substituting Eq. (2.253) into Eq. (2.252), we have

$$M(\lambda) = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!} P_{out}(n) \nu^m (1-\nu)^{n-m} (1-\lambda)^m. \tag{2.254}$$

Inverting the order of the summations, we see that

$$M(\lambda) = \sum_{n=0}^{\infty} P_{out}(n) \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} (1-\nu)^{n-m} [\nu(1-\lambda)]^m, \tag{2.255}$$

so that applying the binomial theorem, the moment generating function can be expressed in terms of the photon number distribution for the output mode as

$$M(\lambda) = \sum_{n=0}^{\infty} P_{out}(n)(1-\nu\lambda)^n. \tag{2.256}$$

Multiplying Eq. (2.252) from the left by $(\lambda - 1)\frac{d}{d\lambda}$ and then setting $\lambda = 0$, we get

$$\overline{m} = (\lambda - 1)\frac{d}{d\lambda} M(\lambda)\Big|_{\lambda=0}, \tag{2.257}$$

where

$$\overline{m} = \sum_{m=0}^{\infty} mP(m) \tag{2.258}$$

is the mean photon count. Using Eq. (2.256) in Eq. (2.257), we have

$$\overline{m} = \sum_{n=0}^{\infty} P_{out}(n)(\lambda - 1)\frac{d}{d\lambda}(1-\nu\lambda)^n\Big|_{\lambda=0} \tag{2.259}$$

and on carrying out the differentiation, we get

$$\overline{m} = \nu \overline{n}_{out}, \tag{2.260}$$

where $\overline{n}_{out}$ is the mean photon number of the output mode.

Using the input-output relation

$$\alpha_{out} = \sqrt{\kappa}\alpha - \alpha_{in} \tag{2.261}$$

and

$$\overline{n}_{out} = \langle \alpha^*_{out} \alpha_{out} \rangle, \tag{2.262}$$

the mean photon number of the output mode can be expressed as

$$\overline{n}_{out} = \kappa\overline{n} + \overline{n}_{in} - \sqrt{\kappa}\langle\alpha^*(t)\alpha_{in}(t)\rangle - \sqrt{\kappa}\langle\alpha_{in}^*(t)\alpha(t)\rangle, \tag{2.263}$$

where

$$\overline{n} = \langle\alpha^*(t)\alpha(t)\rangle \tag{2.264}$$

is the mean photon number of the cavity mode and

$$\overline{n}_{in} = \langle\alpha_{in}^*(t)\alpha_{in}(t)\rangle = N \tag{2.265}$$

is the mean photon number of the input mode.

We now proceed to find the explicit forms of $\langle\alpha^*(t)\alpha_{in}(t)\rangle$. To this end, we rewrite Eq. (2.290) as

$$\alpha(t) = B(t) + E_c(t) + E_r(t), \tag{2.266}$$

where

$$B(t) = A_+(t)\alpha(0) + A_-(t)\alpha^*(0) + \frac{2\varepsilon_1}{\lambda_-}\left(1 - e^{-\lambda_- t/2}\right), \tag{2.267}$$

$$E_c(t) = \int_0^t [A_+(t-t')f_c(t') + A_-(t-t')f_c^*(t')]dt', \tag{2.268}$$

$$E_r(t) = \int_0^t [A_+(t-t')f_r(t') + A_-(t-t')f_r^*(t')]dt'. \tag{2.269}$$

On account of Eq. (2.266), we see that

$$\langle\alpha^*(t)\alpha_{in}(t)\rangle = \langle B^*(t)\alpha_{in}(t)\rangle + \langle E_c^*(t)\alpha_{in}(t)\rangle + \langle E_r^*(t)\alpha_{in}(t)\rangle. \tag{2.270}$$

With the aid of Eqs. (2.104), (2.267), and the fact that a noise force at a certain time does not affect the cavity mode variables at earlier time, we easily get

$$\langle B^*(t)\alpha_{in}(t)\rangle = 0. \tag{2.271}$$

Furthermore, assuming that the noise forces associated with the cavity and reservoir modes are not correlated, we have

$$\langle E_c^*(t)\alpha_{in}(t)\rangle = 0, \tag{2.272}$$

In view of (2.269), (2.271), (2.272), and the relation

$$\alpha_{in}(t) = \frac{1}{\sqrt{\kappa}} f_r(t), \tag{2.273}$$

Eq. (2.270) takes the form

$$\langle \alpha^*(t)\alpha_{in}(t)\rangle = \frac{1}{\sqrt{\kappa}} \int_0^t [A_+(t-t')\langle f_r^*(t')f_r(t)\rangle + A_-(t-t')\langle f_r(t')f_r(t)\rangle]dt'. \tag{2.274}$$

so that applying Eqs. (2.126), (2.163), and (2.164), we find

$$\langle \alpha^*(t)\alpha_{in}(t)\rangle = \sqrt{\kappa}N/2. \tag{2.275}$$

Therefore, using Eq. (2.275) and its complex conjugate in Eq. (2.263), we get

$$\overline{n}_{out} = \kappa\overline{n} + (1-\kappa)N. \tag{2.276}$$

The first and the second terms on the right side of Eq. (2.275) represent the mean number of the transmitted cavity photons and the mean number of reflected input photons. Taking into account Eq. (2.276), the mean photon count for the output mode takes the form

$$\overline{m} = \nu\kappa\overline{n} + \nu(1-\kappa)N. \tag{2.277}$$

We next seek to calculate the normally-ordered photon count variance for the output mode. The normally-ordered variance of the photon count can be expressed as

$$: \Delta m^2 := \Delta m^2 - \overline{m}, \tag{2.278}$$

where

$$\Delta m^2 = \overline{m^2} - \overline{m}^2 \tag{2.279}$$

is the variance of the photon count. Using Eq. (2.252), one can easily verify that

$$\overline{m^2} = [(\lambda-1)\frac{d}{d\lambda}]^2 M(\lambda)\bigg|_{\lambda=0}. \tag{2.280}$$

On account of Eq. (2.256), we see that

$$\overline{m^2} = \sum_{n=0}^{\infty} P_{out}(n)[(\lambda-1)\frac{d}{d\lambda}]^2(1-\nu\lambda)^n\bigg|_{\lambda=0}. \tag{2.281}$$

Carrying out the differentiation and then applying the condition $\lambda = 0$, we get

$$\overline{m^2} = \nu^2 \langle \hat{n}_{out}^2 \rangle + \nu(1-\nu)\overline{n}_{out}. \tag{2.282}$$

Hence substituting (2.260) and (2.282) into Eq. (2.279), we get

$$\Delta m^2 = \nu^2 \Delta n_{out}^2 + \nu(1-\nu)\overline{n}_{out}, \tag{2.283}$$

where

$$\Delta n_{out}^2 = \langle \hat{n}_{out}^2 \rangle - \overline{n}_{out}^2 \tag{2.284}$$

is the photon number variance of the output mode. Now applying Eqs. (2.260) and (2.283) in Eq. (2.278), the normally-ordered photon count variance of the output mode can be put in the form

$$: \Delta m^2 : = \nu^2 : \Delta n_{out}^2 :, \tag{2.285}$$

in which

$$: \Delta n_{out}^2 : = \langle \alpha_{out}^{*2} \alpha_{out}^2 \rangle - \overline{n}_{out}^2 \tag{2.286}$$

is the normally-ordered photon number variance of the output mode.

Using the input-output relation, we can write

$$\begin{aligned}\langle \alpha_{out}^{*2}\alpha_{out}^2 \rangle &= \kappa^2 \langle |\alpha^2|^2 \rangle + \langle |\alpha_{in}^2|^2 \rangle + \kappa(4\langle |\alpha\alpha_{in}|^2 \rangle + \langle \alpha^{*2}\alpha_{in}^2 \rangle + \langle \alpha_{in}^{*2}\alpha^2 \rangle) \\ &\quad - 2\kappa\sqrt{\kappa}(\langle \alpha^{*2}\alpha\alpha_{in} \rangle + \langle \alpha^*\alpha_{in}^*\alpha^2 \rangle) - 2\sqrt{\kappa}(\langle \alpha_{in}^{*2}\alpha\alpha_{in} \rangle + \langle \alpha^*\alpha_{in}^*\alpha_{in}^2 \rangle).\end{aligned} \tag{2.287}$$

We next proceed to evaluate the expectation values involved in Eq. (2.287). To this end, we rewrite Eq. (2.290) as

$$\alpha = \alpha' + \varepsilon, \tag{2.288}$$

where

$$\alpha' = A_+(t)\alpha(0) + A_-(t)\alpha^*(0) + F(t), \tag{2.289}$$

$$\varepsilon = \frac{2\varepsilon_1}{\lambda_-}\left(1 - e^{-\lambda_- t/2}\right). \tag{2.290}$$

On account of Eq. (2.288) and its complex conjugate, we see that

$$\langle\alpha^*\alpha\alpha_{in}^*\alpha_{in}\rangle = \langle\alpha'^*\alpha'\alpha_{in}^*\alpha_{in}\rangle + \varepsilon^2\langle\alpha_{in}^*\alpha_{in}\rangle + \varepsilon\langle\alpha'^*\alpha_{in}^*\alpha_{in}\rangle + \varepsilon\langle\alpha_{in}^*\alpha'\alpha_{in}\rangle. \tag{2.291}$$

We recall that $\alpha'$ and $\alpha_{in}$ are Gaussian variables with zero mean. Hence Eq. (2.291) can be put in the form

$$\langle\alpha^*\alpha\alpha_{in}^*\alpha_{in}\rangle = \langle\alpha'^*\alpha'\rangle\langle\alpha_{in}^*\alpha_{in}\rangle + \langle\alpha'^*\alpha_{in}^*\rangle\langle\alpha'\alpha_{in}\rangle + \langle\alpha'^*\alpha_{in}\rangle\langle\alpha_{in}^*\alpha'\rangle + \varepsilon^2\langle\alpha_{in}^*\alpha_{in}\rangle. \tag{2.292}$$

With the aid of Eq. (2.288) and the assumption that the cavity mode is initially in a vacuum state, it can be easily verified that

$$\langle\alpha'\alpha_{in}\rangle = \sqrt{\kappa}M/2 \tag{2.293}$$

and

$$\langle\alpha'^*\alpha_{in}\rangle = \sqrt{\kappa}N/2. \tag{2.294}$$

We note that for a squeezed vacuum reservoir

$$\langle\alpha_{in}^*\alpha_{in}\rangle = N \tag{2.295}$$

and

$$\langle\alpha_{in}\alpha_{in}\rangle = M. \tag{2.296}$$

Now combination of Eqs. (2.292), (2.293), (2.294), and (2.295) results in

$$\langle\alpha^*\alpha\alpha_{in}^*\alpha_{in}\rangle = \kappa N^2/4 + \kappa M^2/4 + \overline{n}N, \tag{2.297}$$

where

$$\overline{n} = \langle\alpha'^*\alpha'\rangle + \varepsilon^2. \tag{2.298}$$

Furthermore, taking into account Eq. (2.288) and the fact that $\alpha'$ and $\alpha_{in}$ are Gaussian variables with zero mean, we obtain

$$\langle\alpha^{*2}\alpha_{in}^2\rangle = \langle\alpha'^{*2}\rangle\langle\alpha_{in}^2\rangle + 2\langle\alpha'^*\alpha_{in}\rangle^2 + \varepsilon^2\langle\alpha_{in}^2\rangle, \tag{2.299}$$

so that applying Eqs. (2.294) and (2.296), we get

$$\langle \alpha^{*2}\alpha_{in}^2\rangle = M'M + \kappa N^2/2, \tag{2.300}$$

in which

$$M' = b + \varepsilon^2, \tag{2.301}$$

with $b = \langle \alpha'^{*2}(t)\rangle$ given by Eq. (2.216). Following the same procedure, we find

$$\langle \alpha^{*2}\alpha\alpha_{in}\rangle = \sqrt{\kappa}(M'M/2 + \overline{n}N), \tag{2.302}$$

$$\langle \alpha_{in}^{*2}\alpha\alpha_{in}\rangle = \sqrt{\kappa}(M^2/2 + N^2), \tag{2.303}$$

$$\langle \alpha_{in}^{*2}\alpha_{in}^2\rangle = M^2 + 2N^2. \tag{2.304}$$

Substitution of Eqs. (2.297), (2.300), (2.302), (2.303), and (2.304) along with their complex conjugate into Eq. (2.287) yields

$$\langle \alpha_{out}^{*2}\alpha_{out}^2\rangle = \kappa^2\langle \alpha^{*2}\alpha^2\rangle + (1-\kappa)^2(M^2 + 2N^2) + 2\kappa(1-\kappa)(M'M + 2\overline{n}N). \tag{2.305}$$

Now taking into account of (2.305) and (2.276), we can put Eq. (2.286) in the form

$$:\Delta n_{out}^2: = \kappa^2 :\Delta n^2: + (1-\kappa)^2 :\Delta n_{in}^2: + 2\kappa(1-\kappa)(M'M + \overline{n}N). \tag{2.306}$$

where

$$:\Delta n^2: = \langle \alpha^{*2}\alpha^2\rangle - \overline{n}^2 \tag{2.307}$$

is the normally-ordered photon number variance of the cavity mode and

$$:\Delta n_{in}^2: = N^2 + M^2 \tag{2.308}$$

is the normally-ordered photon number variance of the input mode. Therefore, with Eq. (2.306) introduced into Eq. (2.285), there emerges

$$:\Delta m^2: = \nu^2\kappa^2 :\Delta n^2: + (1-\kappa)^2\nu^2 :\Delta n_{in}^2: + 2\kappa(1-\kappa)\nu^2(M'M + \overline{n}N). \tag{2.309}$$

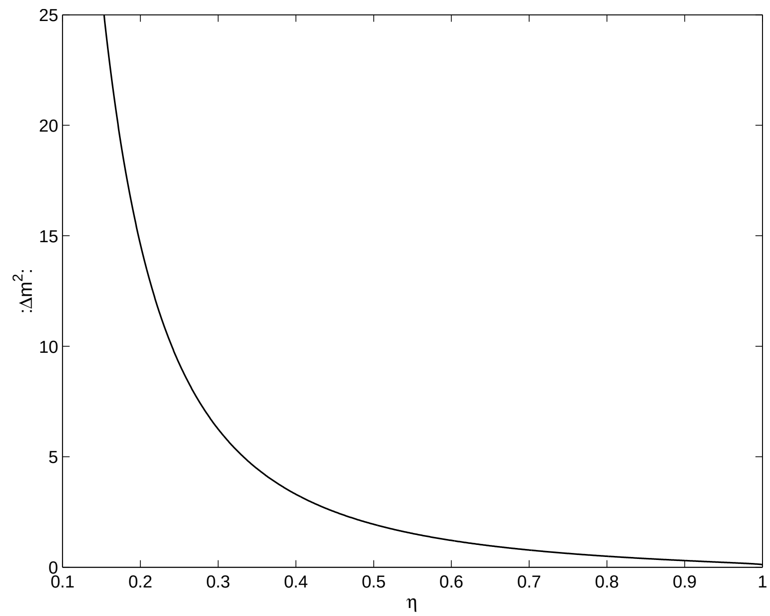


*Fig. 2.12:* A plot of the normally-ordered photon count variance [Eq. (2.309)] versus $\eta$ for $A = 100$, $\kappa = 0.8$, $\varepsilon_1 = 0.2$, $\varepsilon_2 = 0.3$, $r = 0.5$, $\nu = 0.6$, and $\theta = 0$.

Fig 2.12 represents the normally-ordered photon count variance [Eq. (2.309)] versus $\eta$ for $A = 100$, $\kappa = 0.8$, $\varepsilon_1 = 0.2$, $\varepsilon_2 = 0.3$, $\nu = 0.6$, $r = 0.5$, and $\theta = 0$. We observe from the figure that the photon count statistics is super-Poissonian.

# 3

# Nondegenerate Three-Level Laser

In this chapter we seek to study the squeezing and statistical properties of the light produced by a nondegenerate three-level laser whose cavity contains a parametric amplifier and in which the cavity modes are driven by coherent light and coupled to a two-mode squeezed vacuum reservoir. Three-level atoms initially prepared in a coherent superposition of the top and bottom levels are injected into the cavity at a constant rate and removed from the cavity after sometime.

We first obtain the master equation and stochastic differential equations for the cavity mode variables associated with the normal ordering. Using the solutions of the resulting differential equations and the correlation properties of the noise forces, we calculate the quadrature variances and the squeezing spectrum. In addition, we determine the mean and variances of the photon number sum and difference as well as the photon number distribution for the cavity modes employing the Q function. The Q function is obtained with the aid of the antinormally ordered characteristic function defined in the Heisenbeg picture. Finally, we calculate the mean and the normally-ordered variances of the photon count sum and difference for the output modes.

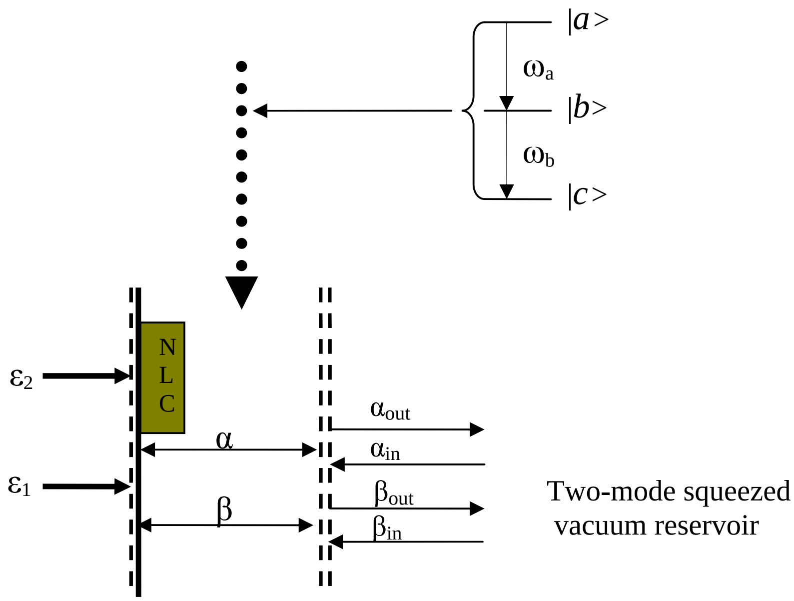


*Fig. 3.1:* Schematic representation of a nondegenerate three-level laser with a parametric amplifier, a driving coherent light and a two-mode squeezing vacuum reservoir.

# 3.1 Stochastic Differential Equations

## 3.1.1 Master equation

Here we want to drive the master equation for a nondegenerate three-level laser whose cavity contains a parametric amplifier and with the cavity modes driven by a two-mode coherent light and coupled to a two-mode squeezed vacuum reservoir applying the same approximations that we have used to drive the master equation for the degenerate case considered in chapter two.

### Nondegenerate Three-Level Laser

We represent the top, intermediate, and bottom levels of a three-level atom in a cascade configuration by $|a\rangle$, $|b\rangle$, and $|c\rangle$, respectively, as shown in Fig. 3.1. In addition, we assume the two modes $a$ and $b$ to be at resonance with the two transitions $|a\rangle \rightarrow |b\rangle$ and $|b\rangle \rightarrow |c\rangle$, respectively and direct transition between level $|a\rangle$ and level $|c\rangle$ to be dipole forbidden. The interaction of a nondegenerate three-level atom with the cavity modes can be described by the Hamiltonian

$$\hat{H} = ig\left(|a\rangle\langle b|\hat{a} - \hat{a}^{\dagger}|b\rangle\langle a| + |b\rangle\langle c|\hat{b} - \hat{b}^{\dagger}|c\rangle\langle b|\right), \tag{3.1}$$

where $g$ is the coupling constant, $\hat{a}$ and $\hat{b}$ are the annihilation operators for the cavity modes. We consider a three-level atom initially in the state

$$|\psi_A(0)\rangle = C_a|a\rangle + C_c|c\rangle. \tag{3.2}$$

The density operator for a single atom can then be written as

$$\hat{\rho}_A(0) = \rho_{aa}^{(0)}|a\rangle\langle a| + \rho_{ac}^{(0)}|a\rangle\langle c| + \rho_{ca}^{(0)}|c\rangle\langle a| + \rho_{cc}^{(0)}|c\rangle\langle c|, \tag{3.3}$$

where

$$\rho_{aa}^{(0)} = |C_a|^2 \quad and \quad \rho_{cc}^{(0)} = |C_c|^2 \tag{3.4}$$

are the probability for the atom to be in the upper and the lower levels at the initial time,

$$\rho_{ac}^{(0)} = C_aC_c^* \quad and \quad \rho_{ca}^{(0)} = C_cC_a^* \tag{3.5}$$

represent the atomic coherence at the initial time. We note that

$$|\rho_{ac}^{(0)}|^2 = \rho_{aa}^{(0)}\rho_{cc}^{(0)}. \tag{3.6}$$

With the aid of Eqs. (2.14) and (3.1), the equation of evolution of the density operator for the cavity modes can be put in the form

$$\frac{d\hat{\rho}(t)}{dt} = g\left(\hat{a}\hat{\rho}_{ba} - \hat{\rho}_{ba}\hat{a} - \hat{a}^{\dagger}\hat{\rho}_{ab} + \hat{\rho}_{ab}\hat{a}^{\dagger} + \hat{b}\hat{\rho}_{cb} - \hat{\rho}_{cb}\hat{b} - \hat{b}^{\dagger}\hat{\rho}_{bc} + \hat{\rho}_{bc}\hat{b}^{\dagger}\right), \tag{3.7}$$

where

$$\hat{\rho}_{\alpha\beta} = \langle\alpha|\hat{\rho}_{AR}(t)|\beta\rangle, \tag{3.8}$$

with $\alpha, \beta = a, b, c$. We next proceed to determine the matrix elements $\hat{\rho}_{\alpha\beta}$ involved in Eq. (3.8). Taking into account Eqs. (2.18), (2.19), (3.1), and (3.3), we can write

$$\begin{aligned}\frac{d\hat{\rho}_{\alpha\beta}}{dt} =& r_a(\rho_{aa}^{(0)}\delta_{\alpha a}\delta_{a\beta} + \rho_{ac}^{(0)}\delta_{\alpha a}\delta_{c\beta} + \rho_{ca}^{(0)}\delta_{\alpha c}\delta_{a\beta} + \rho_{cc}^{(0)}\delta_{\alpha c}\delta_{c\beta})\hat{\rho}\\ &+ g\Big(\hat{a}\hat{\rho}_{b\beta}\delta_{\alpha a} - \hat{a}^{\dagger}\hat{\rho}_{a\beta}\delta_{\alpha b} + \hat{b}\hat{\rho}_{c\beta}\delta_{\alpha b} - \hat{b}^{\dagger}\hat{\rho}_{b\beta}\delta_{\alpha c}\\ &\quad - \hat{\rho}_{\alpha a}\hat{a}\delta_{b\beta} + \hat{\rho}_{\alpha b}\hat{a}^{\dagger}\delta_{a\beta} - \hat{\rho}_{\alpha b}\hat{b}\delta_{c\beta} + \hat{\rho}_{\alpha c}\hat{b}^{\dagger}\delta_{b\beta}\Big) - \gamma\hat{\rho}_{\alpha\beta},\end{aligned} \tag{3.9}$$

where $\gamma$, considered to be the same for all the three levels, is the atomic decay rate. Using Eq. (3.9), we obtain

$$\frac{d\hat{\rho}_{ab}}{dt} = g\left(\hat{a}\hat{\rho}_{bb} - \hat{\rho}_{aa}\hat{a} + \hat{\rho}_{ac}\hat{b}^{\dagger}\right) - \gamma\hat{\rho}_{ab}, \tag{3.10}$$

$$\frac{d\hat{\rho}_{bc}}{dt} = g\left(\hat{b}\hat{\rho}_{cc} - \hat{\rho}_{bb}\hat{b} - \hat{a}^{\dagger}\hat{\rho}_{ac}\right) - \gamma\hat{\rho}_{bc}, \tag{3.11}$$

$$\frac{d\hat{\rho}_{aa}}{dt} = r_a\rho_{aa}^{(0)}\hat{\rho} + g\left(\hat{a}\hat{\rho}_{ba} + \hat{\rho}_{ab}\hat{a}^{\dagger}\right) - \gamma\hat{\rho}_{aa}, \tag{3.12}$$

$$\frac{d\hat{\rho}_{cc}}{dt} = r_a\rho_{cc}^{(0)}\hat{\rho} - g\left(\hat{b}^{\dagger}\hat{\rho}_{bc} + \hat{\rho}_{cb}\hat{b}\right) - \gamma\hat{\rho}_{cc}, \tag{3.13}$$

$$\frac{d\hat{\rho}_{ac}}{dt} = r_a\rho_{ac}^{(0)}\hat{\rho} + g\left(\hat{a}\hat{\rho}_{bc} - \hat{\rho}_{ab}\hat{b}\right) - \gamma\hat{\rho}_{ac}, \tag{3.14}$$

$$\frac{d\hat{\rho}_{bb}}{dt} = -g\left(\hat{a}^{\dagger}\hat{\rho}_{ab} - \hat{b}\hat{\rho}_{cb} + \hat{\rho}_{ba}\hat{a} - \hat{\rho}_{bc}\hat{b}^{\dagger}\right) - \gamma\hat{\rho}_{bb}. \tag{3.15}$$

Dropping the $g$ terms in Eqs. (3.12), (3.13), (3.14), (3.15) and then applying the adiabatic approximation scheme, we get

$$\hat{\rho}_{aa} = \frac{r_a\rho_{aa}^{(0)}}{\gamma}\hat{\rho}, \tag{3.16}$$

$$\hat{\rho}_{cc} = \frac{r_a\rho_{cc}^{(0)}}{\gamma}\hat{\rho}, \tag{3.17}$$

$$\hat{\rho}_{ac} = \frac{r_a\rho_{ac}^{(0)}}{\gamma}\hat{\rho}, \tag{3.18}$$

$$\hat{\rho}_{bb} = 0. \tag{3.19}$$

Substituting the above results into Eqs. (3.10) and (3.11), we have

$$\frac{d\hat{\rho}_{ab}}{dt} = \frac{gr_a}{\gamma}\left(\rho_{ac}^{(0)}\hat{\rho}\hat{b}^{\dagger} - \rho_{aa}^{(0)}\hat{\rho}\hat{a}\right) - \gamma\hat{\rho}_{ab}, \tag{3.20}$$

$$\frac{d\hat{\rho}_{bc}}{dt} = \frac{gr_a}{\gamma}\left(\hat{b}\rho_{cc}^{(0)}\hat{\rho} - \hat{a}^{\dagger}\rho_{ac}^{(0)}\hat{\rho}\right) - \gamma\hat{\rho}_{bc}, \tag{3.21}$$

so that employing once more the adiabatic approximation scheme, we get

$$\hat{\rho}_{ab} = \frac{gr_a}{\gamma^2}\left(\rho_{ac}^{(0)}\hat{\rho}\hat{b}^\dagger - \rho_{aa}^{(0)}\hat{\rho}\hat{a}\right), \tag{3.22}$$

$$\hat{\rho}_{bc} = \frac{gr_a}{\gamma^2}\left(\rho_{cc}^{(0)}\hat{b}\hat{\rho} - \rho_{ac}^{(0)}\hat{a}^\dagger\hat{\rho}\right). \tag{3.23}$$

Finally on account of Eqs. (3.22) and (3.23), the equation of evolution of the density operator for the cavity modes given by Eq. (3.7) takes the form

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} &= \frac{A\rho_{aa}^{(0)}}{2}\left(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\right) + \frac{A\rho_{cc}^{(0)}}{2}\left(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\hat{\rho}\right)\\ &- \frac{A\rho_{ac}^{(0)}}{2}\left(2\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{a}^\dagger\right) - \frac{A\rho_{ca}^{(0)}}{2}\left(2\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b} - \hat{a}\hat{b}\hat{\rho}\right).\end{aligned} \tag{3.24}$$

where

$$A = \frac{2g^2 r_a}{\gamma^2} \tag{3.25}$$

is linear gain coefficient.

### Two-Mode Squeezed Vacuum Reservoir

The density operator for a two-mode squeezed vacuum reservoir can be written as

$$\hat{\rho}(r) = \prod_i \hat{S}_i(r)|0_i 0_i\rangle\langle 0_i 0_i|\hat{S}_i^\dagger(r) \tag{3.26}$$

where

$$\hat{S}_i(r) = e^{r(\hat{c}_i^\dagger\hat{d}_i^\dagger - \hat{c}_i\hat{d}_i)} \tag{3.27}$$

is the squeeze operator with the squeeze parameter $r$ taken for convenience to be real and positive, and $\hat{c}_i$ and $\hat{d}_i$ represent the annihilation operators for the reservoir submodes. With the aid of Eq. (3.26) and the identity operator $\hat{I} = \hat{S}_i(r)\hat{S}_i^\dagger(r)$ , we can write as

$$\langle\hat{c}_j\hat{d}_k\rangle = \prod_i \langle 0_i 0_i|\hat{c}_j(r)\hat{d}_k(k)|0_i 0_i\rangle, \tag{3.28}$$

in which

$$\hat{c}_j(r) = \hat{S}_i^\dagger(r)\hat{c}_j\hat{S}_i(r) \tag{3.29}$$

and

$$\hat{d}_k(r) = \hat{S}_i^\dagger(r)\hat{d}_k\hat{S}_i(r). \tag{3.30}$$

Differentiating Eq. (3.29) and the adjoint of Eq. (3.30) with respect to $r$, we obtain

$$\frac{d}{dr}\hat{c}_j(r) = \hat{d}_i^\dagger(r)\delta_{ij} \tag{3.31}$$

and

$$\frac{d}{dr}\hat{d}_k^\dagger(r) = \hat{c}_i(r)\delta_{ik}. \tag{3.32}$$

In order to decouple these equations, we differentiate Eq. (3.31) once more with respect to $r$. We then get

$$\frac{d^2}{dr^2}\hat{c}_j(r) = \hat{c}_j(r). \tag{3.33}$$

The solution of this equation can be put in the form

$$\hat{c}_j(r) = Ae^r + Be^{-r}. \tag{3.34}$$

Applying the condition $r = 0$, we see that

$$\hat{c}_j(r)\Big|_{r=0} = A + B = \hat{c}_i\delta_{ij}, \tag{3.35}$$

$$\frac{d}{dr}\hat{c}_j(r)\Big|_{r=0} = A - B = \hat{d}_i^\dagger\delta_{ij}. \tag{3.36}$$

It then follows that

$$A = \frac{1}{2}(\hat{c}_i + \hat{d}_i^\dagger)\delta_{ij}, \tag{3.37}$$

$$B = \frac{1}{2}(\hat{c}_i - \hat{d}_i^\dagger)\delta_{ij}, \tag{3.38}$$

so that on account of these, Eq. (3.34) takes the form

$$\hat{c}_j(r) = (\hat{c}_i \cosh r + \hat{d}_i^\dagger \sinh r)\delta_{ij}. \tag{3.39}$$

Following a similar procedure, we can easily verify that

$$\hat{d}_k(r) = (\hat{d}_i \cosh r + \hat{c}_i^\dagger \sinh r)\delta_{ik}. \tag{3.40}$$

In view of Eqs. (3.39) and (3.40), we have

$$\langle\hat{c}_j\hat{d}_k\rangle = \prod_i \langle 0_i 0_i|\hat{c}_i\hat{d}_i \cosh^2 r + (\hat{d}_i^\dagger\hat{d}_i + \hat{c}_i\hat{c}_i^\dagger)\cosh r \sinh r + \hat{d}_i^\dagger\hat{c}_i^\dagger \sinh^2 r|0_i 0_i\rangle\delta_{ij}\delta_{ik}. \tag{3.41}$$

Applying the relation $\hat{c}_i\hat{c}_i^\dagger = 1 + \hat{c}_i^\dagger\hat{c}_i$, we find

$$\langle\hat{c}_j\hat{d}_k\rangle = M\delta_{jk}, \tag{3.42}$$

where $M = \sinh r \cosh r$. We assume that $k$ is of the order of the central wave number $k_0$, so that we can replace $k$ by $2k_0 - k$. In view of this, Eq. (3.42) can be rewritten as

$$\langle\hat{c}_j\hat{d}_k\rangle = M\delta_{j,2k_0-k}. \tag{3.43}$$

One can also easily establish in a similar manner that

$$\langle\hat{c}_k\rangle = \langle\hat{d}_k\rangle = \langle\hat{c}_j\hat{c}_k\rangle = \langle\hat{d}_j\hat{d}_k\rangle = \langle\hat{c}_j\hat{d}_k^\dagger\rangle = \langle\hat{c}_j^\dagger\hat{c}_k^\dagger\rangle = \langle\hat{d}_j^\dagger\hat{d}_k^\dagger\rangle = \langle\hat{c}_j^\dagger\hat{d}_k\rangle = 0, \tag{3.44}$$

$$\langle\hat{c}_j^\dagger\hat{d}_k^\dagger\rangle = M\delta_{j,2k_0-k}, \tag{3.45}$$

$$\langle\hat{c}_j^\dagger\hat{c}_k\rangle = \langle\hat{d}_j^\dagger\hat{d}_k\rangle = N\delta_{jk}, \tag{3.46}$$

$$\langle\hat{c}_j\hat{c}_k^\dagger\rangle = \langle\hat{d}_j\hat{d}_k^\dagger\rangle = (N+1)\delta_{jk}, \tag{3.47}$$

in which

$$N = \sinh^2 r. \tag{3.48}$$

We now proceed to drive the equation of evolution of the density operator for cavity modes coupled to a two-mode squeezed vacuum reservoir. In general, the reduced density operator for cavity modes coupled to a reservoir can be expressed in Born approximation as

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) =& -i[\langle\hat{H}_{SR}(t)\rangle, \hat{\rho}(0)] - \int dt' Tr_R(\hat{H}_{SR}(t)\hat{H}_{SR}(t')\hat{\rho}(t')R) \\ &+ \int dt' Tr_R(\hat{H}_{SR}(t)\hat{\rho}(t')R\hat{H}_{SR}(t')) + \int dt' Tr_R(\hat{H}_{SR}(t')\hat{\rho}(t')R\hat{H}_{SR}(t)) \\ &- \int dt' Tr_R(\hat{\rho}(t')R\hat{H}_{SR}(t')\hat{H}_{SR}(t)).\end{aligned} \tag{3.49}$$

We seek to obtain the equation of evolution of the reduced density operator for cavity modes coupled to a two-mode squeezed vacuum. The Hamiltonian describing the interaction of the cavity modes with the two-mode squeezed vacuum reservoir can be written as

$$\hat{H}_{SR}(t) = i\sum_{k}\lambda_k(\hat{a}^\dagger\hat{c}_k e^{i(\omega_a-\omega_k)t} - \hat{a}\hat{c}_k^\dagger e^{-i(\omega_a-\omega_k)t} + \hat{b}^\dagger\hat{d}_k e^{i(\omega_b-\omega_k)t} - \hat{b}\hat{d}_k^\dagger e^{-i(\omega_b-\omega_k)t}), \quad (3.50)$$

where $\hat{c}_k$ and $\hat{d}_k$ are the annihilation operators for reservoir submodes and $\lambda_k$ is the coupling constant. Using (3.50), one can write Eq. (3.49) as

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) = \int dt'\Big(& I_1(\hat{a}^{\dagger 2}\hat{\rho}(t') + \hat{\rho}(t')\hat{a}^{\dagger 2} - 2\hat{a}^\dagger\hat{\rho}(t')\hat{a}^\dagger) + I_2(\hat{a}^\dagger\hat{a}\hat{\rho}(t') + \hat{\rho}(t')\hat{a}^\dagger\hat{a} - 2\hat{a}\hat{\rho}(t')\hat{a}^\dagger)\\
&+ I_3(\hat{a}\hat{a}^\dagger\hat{\rho}(t') + \hat{\rho}(t')\hat{a}\hat{a}^\dagger - 2\hat{a}^\dagger\hat{\rho}(t')\hat{a}) + I_4(\hat{a}^2\hat{\rho}(t') + \hat{\rho}(t')\hat{a}^2 - 2\hat{a}\hat{\rho}(t')\hat{a})\\
&+ I_5(\hat{b}^{\dagger 2}\hat{\rho}(t') + \hat{\rho}(t')\hat{b}^{\dagger 2} - 2\hat{b}^\dagger\hat{\rho}(t')\hat{b}^\dagger) + I_6(\hat{b}^\dagger\hat{b}\hat{\rho}(t') + \hat{\rho}(t')\hat{b}^\dagger\hat{b} - 2\hat{b}\hat{\rho}(t')\hat{b}^\dagger)\\
&+ I_7(\hat{b}\hat{b}^\dagger\hat{\rho}(t') + \hat{\rho}(t')\hat{b}\hat{b}^\dagger - 2\hat{b}^\dagger\hat{\rho}(t')\hat{b}) + I_8(\hat{b}^2\hat{\rho}(t') + \hat{\rho}(t')\hat{b}^2 - 2\hat{b}\hat{\rho}(t')\hat{b})\\
&+ 2I_9(\hat{a}^\dagger\hat{b}^\dagger\hat{\rho}(t') + \hat{\rho}(t')\hat{a}^\dagger\hat{b}^\dagger - \hat{b}^\dagger\hat{\rho}(t')\hat{a}^\dagger - \hat{a}^\dagger\rho(t')\hat{b}^\dagger)\\
&+ 2I_{10}(\hat{a}^\dagger\hat{b}\hat{\rho}(t') + \hat{\rho}(t')\hat{a}^\dagger\hat{b} - \hat{b}\hat{\rho}(t')\hat{a}^\dagger - \hat{a}^\dagger\hat{\rho}(t')\hat{b})\\
&+ 2I_{11}(\hat{a}\hat{b}^\dagger\hat{\rho}(t') + \hat{\rho}(t')\hat{a}\hat{b}^\dagger - \hat{b}^\dagger\hat{\rho}(t')\hat{a} - \hat{a}\hat{\rho}(t')\hat{b}^\dagger)\\
&+ 2I_{12}(\hat{a}\hat{b}\hat{\rho}(t') + \hat{\rho}(t')\hat{a}\hat{b} - \hat{b}\hat{\rho}(t')\hat{a} - \hat{a}\hat{\rho}(t')\hat{b})\Big), \qquad (3.51)\end{aligned}$$

where

$$I_1 = \sum_{kk'}\lambda_k\lambda_{k'}\langle\hat{c}_k\hat{c}_{k'}\rangle e^{i(\omega_a-\omega_k)t+i(\omega_a-\omega_{k'})t'}, \quad (3.52)$$

$$I_2 = -\sum_{kk'}\lambda_k\lambda_{k'}\langle\hat{c}_k\hat{c}_{k'}^\dagger\rangle e^{i(\omega_a-\omega_k)t-i(\omega_a-\omega_{k'})t'}, \quad (3.53)$$

$$I_3 = -\sum_{kk'}\lambda_k\lambda_{k'}\langle\hat{c}_k^\dagger\hat{c}_{k'}\rangle e^{-i(\omega_a-\omega_k)t+i(\omega_a-\omega_{k'})t'}, \quad (3.54)$$

$$I_4 = \sum_{kk'}\lambda_k\lambda_{k'}\langle\hat{c}_k^\dagger\hat{c}_{k'}^\dagger\rangle e^{-i(\omega_a-\omega_k)t-i(\omega_a-\omega_{k'})t'}, \quad (3.55)$$

$$I_5 = \sum_{kk'}\lambda_k\lambda_{k'}\langle\hat{d}_k\hat{d}_{k'}\rangle e^{i(\omega_b-\omega_k)t+i(\omega_b-\omega_{k'})t'}, \quad (3.56)$$

$$I_6 = -\sum_{kk'}\lambda_k\lambda_{k'}\langle\hat{d}_k\hat{d}_{k'}^\dagger\rangle e^{i(\omega_b-\omega_k)t-i(\omega_b-\omega_{k'})t'}, \quad (3.57)$$

$$I_7 = -\sum_{kk'} \lambda_k \lambda_{k'} \langle \hat{d}_k^\dagger \hat{d}_{k'} \rangle e^{-i(\omega_b-\omega_k)t+i(\omega_b-\omega_{k'})t'}, \tag{3.58}$$

$$I_8 = \sum_{kk'} \lambda_k \lambda_{k'} \langle \hat{d}_k^\dagger \hat{d}_{k'}^\dagger \rangle e^{-i(\omega_b-\omega_k)t-i(\omega_b-\omega_{k'})t'}, \tag{3.59}$$

$$I_9 = \sum_{kk'} \lambda_k \lambda_{k'} \langle \hat{c}_k \hat{d}_{k'} \rangle e^{i(\omega_a-\omega_k)t+i(\omega_b-\omega_{k'})t'}, \tag{3.60}$$

$$I_{10} = -\sum_{kk'} \lambda_k \lambda_{k'} \langle \hat{c}_k \hat{d}_{k'}^\dagger \rangle e^{i(\omega_a-\omega_k)t-i(\omega_b-\omega_{k'})t'}, \tag{3.61}$$

$$I_{11} = -\sum_{kk'} \lambda_k \lambda_{k'} \langle \hat{c}_k^\dagger \hat{d}_{k'} \rangle e^{-i(\omega_a-\omega_k)t+i(\omega_b-\omega_{k'})t'}, \tag{3.62}$$

$$I_{12} = \sum_{kk'} \lambda_k \lambda_{k'} \langle \hat{c}_k^\dagger \hat{d}_{k'}^\dagger \rangle e^{i(\omega_b-\omega_k)t+i(\omega_a-\omega_{k'})t'}. \tag{3.63}$$

In view of Eq. (3.44), we easily see that

$$I_1 = I_4 = I_5 = I_8 = I_{10} = I_{11} = 0. \tag{3.64}$$

On account of Eq. (3.47), we have

$$I_2 = -(N+1) \sum_k \lambda_k^2 e^{i(\omega_a-\omega_k)(t-t')}. \tag{3.65}$$

Now replacing $\omega_a$ by the average value $\omega_0 = \frac{\omega_a+\omega_b}{2}$, and assuming the reservoir submode frequencies to be closely spaced then the summation can be converted into integration. We then write

$$I_2 = -\,(N+1) \int_0^\infty d\omega g(\omega) \lambda^2(\omega) \lambda(\omega) e^{i(\omega_0-\omega)(t-t')}, \tag{3.66}$$

where $g(\omega)$ is the density of the reservoir submodes. We expect $\omega$ to vary very little around $\omega_0$. In view of this, we can replace $g(\omega)$ and $\lambda(\omega)$ by $g(\omega_0)$ and $\lambda(\omega_0)$ and extend the lower limit of the integration to $-\infty$, so that

$$I_2 = -\,(N+1) g(\omega_0) \lambda^2(\omega_0) \int_{-\infty}^\infty d\omega e^{i(\omega_0-\omega)(t-t')}. \tag{3.67}$$

Moreover, upon setting $\omega' = \omega - \omega_0$, we see that

$$I_2 = -(N+1)g(\omega_0)\lambda^2(\omega_0)\int_{-\infty}^{\infty} d\omega' e^{i(t-t')\omega'}, \tag{3.68}$$

from which follows

$$I_2 = -\kappa(N+1)\delta(t-t'), \tag{3.69}$$

where

$$\kappa = 2\pi g(\omega_0)\lambda^2(\omega_0) \tag{3.70}$$

is defined to be the cavity damping constant. Following a similar procedure, we can also easily obtain

$$I_3 = I_7 = -\kappa N\delta(t-t'), \tag{3.71}$$

$$I_6 = -\kappa(N+1)\delta(t-t'), \tag{3.72}$$

$$I_9 = I_{12} = \kappa M\delta(t-t'). \tag{3.73}$$

Upon substituting Eqs. (3.64), (3.69), (3.71), (3.72), and (3.73) into Eq. (3.51), we have

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) = \kappa\int dt'\Big(&(N+1)(2\hat{a}\hat{\rho}(t')\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho}(t') - \hat{\rho}(t')\hat{a}^\dagger\hat{a})\\ &+ N(2\hat{a}^\dagger\hat{\rho}(t')\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho}(t') - \hat{\rho}(t')\hat{a}\hat{a}^\dagger)\\ &+ (N+1)(2\hat{b}\hat{\rho}(t')\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho}(t') - \hat{\rho}(t')\hat{b}^\dagger\hat{b})\\ &+ N(2\hat{b}^\dagger\hat{\rho}(t')\hat{b} - \hat{b}\hat{b}^\dagger\hat{\rho}(t') - \hat{\rho}(t')\hat{b}\hat{b}^\dagger)\\ &- 2M(\hat{b}^\dagger\hat{\rho}(t')\hat{a}^\dagger + \hat{a}^\dagger\rho(t')\hat{b}^\dagger - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}(t') - \hat{\rho}(t')\hat{a}^\dagger\hat{b}^\dagger)\\ &- 2M(\hat{b}\hat{\rho}(t')\hat{a} + \hat{a}\hat{\rho}(t')\hat{b} - \hat{a}\hat{b}\hat{\rho}(t') - \hat{\rho}(t')\hat{a}\hat{b})\Big)\delta(t-t'),\end{aligned} \tag{3.74}$$

so that carrying out the integration, we get

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) =& \frac{\kappa}{2}(N+1)\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a} + 2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}\right)\\ &+ \frac{\kappa}{2}N\left(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger + 2\hat{b}^\dagger\hat{\rho}\hat{b} - \hat{b}\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^\dagger\right)\\ &- \kappa M\left(\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger + \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger + \hat{b}\hat{\rho}\hat{a} + \hat{a}\hat{\rho}\hat{b} - \hat{b}\hat{a}\hat{\rho} - \hat{\rho}\hat{b}\hat{a}\right).\end{aligned} \tag{3.75}$$

This represents the equation of evolution of the reduced density operator for cavity modes coupled to a two-mode squeezed vacuum reservoir.

**Nondegenerate Three-Level Laser with Coherent and Squeezed Light**

The interaction of the driving light modes, treated classically, and cavity modes is described by the Hamiltonian

$$\hat{H} = i\varepsilon_1(\hat{a}^\dagger - \hat{a} + \hat{b}^\dagger - \hat{b}), \tag{3.76}$$

where $\varepsilon_1$ is proportional to the amplitude of the driving light modes. In addition, the Hamiltonian describing the parametric interaction, with the pump mode treated classically, can be written as

$$\hat{H} = i\varepsilon_2(\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b}), \tag{3.77}$$

in which $\varepsilon_2$ is proportional to the amplitude of the pump mode. The equation of evolution of the density operator associated with these Hamiltonians has the form

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} &= \varepsilon_1\left(\hat{\rho}\hat{a} - \hat{a}\hat{\rho} + \hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger + \hat{\rho}\hat{b} - \hat{b}\hat{\rho} + \hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}^\dagger\right)\\ &+ \frac{1}{2}\varepsilon_2\left(\hat{\rho}\hat{a}\hat{b} - \hat{a}\hat{b}\hat{\rho} + \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\right).\end{aligned} \tag{3.78}$$

Taking into account Eqs. (3.24), (3.75), and (3.78), the master equation for the cavity modes of a nondegenerate three-level laser whose cavity contains a nondegenerate parametric amplifier and whose cavity modes are driven by a two-mode coherent light and coupled to a two-mode squeezed vacuum reservoir can be written as

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} &= \varepsilon_1\left(\hat{\rho}\hat{a} - \hat{a}\hat{\rho} + \hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger + \hat{\rho}\hat{b} - \hat{b}\hat{\rho} + \hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}^\dagger\right) + \varepsilon_2\left(\hat{\rho}\hat{a}\hat{b} - \hat{a}\hat{b}\hat{\rho} + \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\right)\\ &+ \frac{1}{2}\left[(A\rho_{aa}^{(0)} + \kappa N)\left(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\right) + (A\rho_{cc}^{(0)} + \kappa(N+1))\left(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\hat{\rho}\right)\right]\\ &- \frac{1}{2}\left[(A\rho_{ac}^{(0)} + \kappa M)\left(2\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{a}^\dagger\right) + (A\rho_{ca}^{(0)} + \kappa M)\left(2\hat{b}\hat{\rho}\hat{a} - \hat{a}\hat{b}\hat{\rho} - \hat{\rho}\hat{a}\hat{b}\right)\right]\\ &+ \frac{1}{2}\kappa\left[(N+1)\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right) + N\left(2\hat{b}^\dagger\hat{\rho}\hat{b} - \hat{b}\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^\dagger\right)\right]\\ &- \frac{1}{2}\kappa M\left(2\hat{b}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger + 2\hat{a}\hat{\rho}\hat{b} - \hat{b}\hat{a}\hat{\rho} - \hat{\rho}\hat{b}\hat{a}\right).\end{aligned} \tag{3.79}$$

### 3.1.2 Stochastic differential equations

We now proceed to obtain stochastic differential equations associated with the normal ordering. Employing Eq. (3.79), we see that

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}\rangle =& \varepsilon_1 Tr\left(\hat{\rho}\hat{a}\hat{a} - \hat{a}\hat{\rho}\hat{a} + \hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a} + \hat{\rho}\hat{b}\hat{a} - \hat{b}\hat{\rho}\hat{a} + \hat{b}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{a}\right)\\ &+ \varepsilon_2 Tr\left(\hat{\rho}\hat{a}\hat{b}\hat{a} - \hat{a}\hat{b}\hat{\rho}\hat{a} + \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}\right)\\ &+ \frac{1}{2}(A\rho_{aa}^{(0)} + \kappa N)Tr\left(2\hat{a}^\dagger\hat{\rho}\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}\right)\\ &+ \frac{1}{2}(A\rho_{cc}^{(0)} + \kappa(N+1))Tr\left(2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a} - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a}\right)\\ &- \frac{1}{2}(A\rho_{ac}^{(0)} + \kappa M)Tr\left(2\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger\hat{a} - \hat{b}^\dagger\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{a}^\dagger\hat{a}\right)\\ &- \frac{1}{2}(A\rho_{ca}^{(0)} + \kappa M)Tr\left(2\hat{b}\hat{\rho}\hat{a}\hat{a} - \hat{a}\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b}\hat{a}\right)\\ &+ \frac{\kappa}{2}\left[(N+1)Tr\left(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}\right) + NTr\left(2\hat{b}^\dagger\hat{\rho}\hat{b}\hat{a} - \hat{b}\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{b}\hat{b}^\dagger\hat{a}\right)\right]\\ &- \frac{\kappa}{2}MTr\left(2\hat{b}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a} + 2\hat{a}\hat{\rho}\hat{b}\hat{a} - \hat{b}\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}\hat{a}\hat{a}\right).\end{aligned} \quad (3.80)$$

Applying the cyclic property of the trace operation and the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1, \quad (3.81)$$

we get

$$\frac{d}{dt}\langle\hat{a}\rangle = -\frac{1}{2}\mu_a\langle\hat{a}\rangle + \frac{1}{2}\nu_-\langle\hat{b}^\dagger\rangle + \varepsilon_1, \quad (3.82)$$

where

$$\mu_a = \kappa - A\rho_{aa}^{(0)}, \quad (3.83)$$

$$\nu_- = 2\varepsilon_2 - A\rho_{ac}^{(0)}. \quad (3.84)$$

Following the same procedure, it can also be easily verified that

$$\frac{d}{dt}\langle\hat{b}\rangle = -\frac{1}{2}\mu_c\langle\hat{b}\rangle + \frac{1}{2}\nu_+\langle\hat{a}^\dagger\rangle + \varepsilon_1, \quad (3.85)$$

$$\frac{d}{dt}\langle\hat{a}^2\rangle = -\mu_a\langle\hat{a}^2\rangle + \nu_-\langle\hat{b}^\dagger\hat{a}\rangle + 2\varepsilon_1\langle\hat{a}\rangle, \quad (3.86)$$

$$\frac{d}{dt}\langle\hat{b}^2\rangle = -\,\mu_c\langle\hat{b}^2\rangle + \nu_+\langle\hat{a}^\dagger\hat{b}\rangle + 2\varepsilon_1\langle\hat{b}\rangle, \tag{3.87}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle = -\,\mu_a\langle\hat{a}^\dagger\hat{a}\rangle + \frac{1}{2}\nu_-\langle\hat{a}^\dagger\hat{b}^\dagger\rangle + \frac{1}{2}\nu_-^*\langle\hat{a}\hat{b}\rangle + \varepsilon_1(\langle\hat{a}^\dagger\rangle + \langle\hat{a}\rangle) + A\rho_{aa}^{(0)} + \kappa N, \tag{3.88}$$

$$\frac{d}{dt}\langle\hat{b}^\dagger\hat{b}\rangle = -\,\mu_c\langle\hat{b}^\dagger\hat{b}\rangle + \frac{1}{2}\nu_+\langle\hat{b}^\dagger\hat{a}^\dagger\rangle + \frac{1}{2}\nu_+^*\langle\hat{a}\hat{b}\rangle + \varepsilon_1(\langle\hat{b}^\dagger\rangle + \langle\hat{b}\rangle) + \kappa N, \tag{3.89}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{b}\rangle = -\,\frac{1}{2}(\mu_a + \mu_c)\langle\hat{a}^\dagger\hat{b}\rangle + \frac{1}{2}\nu_+\langle\hat{a}^{\dagger 2}\rangle + \frac{1}{2}\nu_-^*\langle\hat{b}^2\rangle + \varepsilon_1(\langle\hat{a}^\dagger\rangle + \langle\hat{b}\rangle), \tag{3.90}$$

$$\frac{d}{dt}\langle\hat{a}\hat{b}\rangle = -\,\frac{1}{2}(\mu_a + \mu_c)\langle\hat{a}\hat{b}\rangle + \frac{1}{2}\nu_+\langle\hat{a}^\dagger\hat{a}\rangle + \frac{1}{2}\nu_-\langle\hat{b}^\dagger\hat{b}\rangle + \varepsilon_1(\langle\hat{a}\rangle + \langle\hat{b}\rangle) + \frac{1}{2}(\nu_+ + 2\kappa M), \tag{3.91}$$

in which

$$\mu_c = \kappa + A\rho_{cc}^{(0)}, \tag{3.92}$$

$$\nu_+ = 2\varepsilon_2 + A\rho_{ac}^{(0)}. \tag{3.93}$$

We note that the c-number equations corresponding to Eqs. (3.82), (3.85), (3.86), (3.87), (3.88), (3.89), (3.90), and (3.91), which are in the normal order, are

$$\frac{d}{dt}\langle\alpha\rangle = -\,\frac{1}{2}\mu_a\langle\alpha\rangle + \frac{1}{2}\nu_-\langle\beta^*\rangle + \varepsilon_1, \tag{3.94}$$

$$\frac{d}{dt}\langle\beta\rangle = -\,\frac{1}{2}\mu_c\langle\beta\rangle + \frac{1}{2}\nu_+\langle\alpha^*\rangle + \varepsilon_1, \tag{3.95}$$

$$\frac{d}{dt}\langle\alpha^2\rangle = -\,\mu_a\langle\alpha^2\rangle + \nu_-\langle\beta^*\alpha\rangle + 2\varepsilon_1\langle\alpha\rangle, \tag{3.96}$$

$$\frac{d}{dt}\langle\beta^2\rangle = -\,\mu_c\langle\beta^2\rangle + \nu_+\langle\alpha^*\beta\rangle + 2\varepsilon_1\langle\beta\rangle, \tag{3.97}$$

$$\frac{d}{dt}\langle\alpha^*\alpha\rangle = -\,\mu_a\langle\alpha^*\alpha\rangle + \frac{1}{2}\nu_-\langle\alpha^*\beta^*\rangle + \frac{1}{2}\nu_-^*\langle\alpha\beta\rangle + \varepsilon_1(\langle\alpha^*\rangle + \langle\alpha\rangle) + A\rho_{aa}^{(0)} + \kappa N, \tag{3.98}$$

$$\frac{d}{dt}\langle\beta^*\beta\rangle = -\,\mu_c\langle\beta^*\beta\rangle + \frac{1}{2}\nu_+\langle\beta^*\alpha^*\rangle + \frac{1}{2}\nu_+^*\langle\alpha\beta\rangle + \varepsilon_1(\langle\beta^*\rangle + \langle\beta\rangle) + \kappa N, \tag{3.99}$$

$$\frac{d}{dt}\langle\alpha^*\beta\rangle = -\frac{1}{2}(\mu_a+\mu_c)\langle\alpha^*\beta\rangle + \frac{1}{2}\nu_+\langle\alpha^{*2}\rangle + \frac{1}{2}\nu_-^*\langle\beta^2\rangle + \varepsilon_1(\langle\alpha^*\rangle + \langle\beta\rangle), \tag{3.100}$$

$$\frac{d}{dt}\langle\alpha\beta\rangle = -\frac{1}{2}(\mu_a+\mu_c)\langle\alpha\beta\rangle + \frac{1}{2}\nu_+\langle\alpha^*\alpha\rangle + \frac{1}{2}\nu_-\langle\beta^*\beta\rangle + \varepsilon_1(\langle\alpha\rangle + \langle\beta\rangle) + \frac{1}{2}(\nu_+ + 2\kappa M). \tag{3.101}$$

On the basis of Eqs. (3.94) and (3.95), we can write [21]

$$\frac{d}{dt}\alpha(t) = -\frac{1}{2}\mu_a\alpha(t) + \frac{1}{2}\nu_-\beta^*(t) + \varepsilon_1 + f_\alpha(t), \tag{3.102}$$

$$\frac{d}{dt}\beta^*(t) = -\frac{1}{2}\mu_c\beta^*(t) + \frac{1}{2}\nu_+^*\alpha(t) + \varepsilon_1 + f_\beta^*(t), \tag{3.103}$$

where $f_\alpha(t)$ and $f_\beta(t)$ are noise forces. The formal solutions of these equations can be put in the form

$$\alpha(t) = \alpha(0)e^{-\mu_a t/2} + \int_0^t dt' e^{-\mu_a(t-t')/2}\left[\frac{1}{2}\nu_-\beta^*(t') + f_\alpha(t') + \varepsilon_1\right], \tag{3.104}$$

$$\beta^*(t) = \beta^*(0)e^{-\mu_c t/2} + \int_0^t dt' e^{-\mu_c(t-t')/2}\left[\frac{1}{2}\nu_+^*\alpha(t') + f_\beta^*(t') + \varepsilon_1\right]. \tag{3.105}$$

We now proceed to determine the properties of the noise forces. We note that Eq. (3.94) and the expectation value of Eq. (3.102) as well as Eq. (3.95) and the expectation value of Eq. (3.103) will have the same form provided that

$$\langle f_\alpha(t)\rangle = \langle f_\beta(t)\rangle = 0. \tag{3.106}$$

Applying the relation $\frac{d}{dt}\langle\alpha^2\rangle = 2\langle\alpha\frac{d}{dt}\alpha\rangle$ along with Eq. (3.304), we see that

$$\frac{d}{dt}\langle\alpha^2\rangle = -\mu_a\langle\alpha^2\rangle + \nu_-\langle\beta^*\alpha\rangle + 2\varepsilon_1\langle\alpha\rangle + 2\langle\alpha(t)f_\alpha(t)\rangle. \tag{3.107}$$

Comparison of this equation with (3.96) leads to

$$\langle\alpha(t)f_\alpha(t)\rangle = 0. \tag{3.108}$$

On account of Eq. (3.104) along with (3.108), we see that

$$\langle\alpha(0)f_\alpha(t)\rangle e^{-\mu_a t/2} + \int_0^t e^{-\mu_a(t-t')/2}\left[\frac{1}{2}\nu_-\langle\beta^*(t')f_\alpha(t)\rangle + \langle f_\alpha(t')f_\alpha(t)\rangle + \varepsilon_1\langle f_\alpha(t)\rangle\right]dt' = 0, \tag{3.109}$$

so that taking into account Eq. (3.106) and the fact that a noise force at a certain instant does not affect the cavity mode variables at earlier time, we have

$$\langle f_\alpha(t')f_\alpha(t)\rangle = 0. \tag{3.110}$$

Similarly, we can easily establish that

$$\langle f_\beta(t')f_\beta(t)\rangle = \langle f_\alpha^*(t')f_\beta(t)\rangle = 0. \tag{3.111}$$

Furthermore, using Eq. (3.102) and its complex conjugate, we have

$$\begin{aligned}\frac{d}{dt}\langle\alpha^*\alpha\rangle =& -\mu_a\langle\alpha^*\alpha\rangle + \frac{1}{2}\nu_-\langle\alpha^*\beta^*\rangle + \frac{1}{2}\nu_-^*\langle\alpha\beta\rangle + \varepsilon_1(\langle\alpha^*\rangle + \langle\alpha\rangle)\\ &+ \langle\alpha^*(t)f_\alpha(t)\rangle + \langle f_\alpha^*(t)\alpha(t)\rangle.\end{aligned} \tag{3.112}$$

Comparison of this equation with Eq. (3.98) shows that

$$\langle\alpha^*(t)f_\alpha(t)\rangle + \langle f_\alpha^*(t)\alpha(t)\rangle = A\rho_{aa}^{(0)} + \kappa N. \tag{3.113}$$

Now taking into account (3.104), (3.105), and the complex conjugate of (3.104), we find

$$\int_0^t e^{-\mu_a(t-t')/2}\left(\langle f_\alpha^*(t')f_\alpha(t)\rangle + \langle f_\alpha^*(t)f_\alpha(t')\rangle\right)dt' = A\rho_{aa}^{(0)} + \kappa N, \tag{3.114}$$

so that assuming $\langle f_\alpha^*(t')f_\alpha(t)\rangle = \langle f_\alpha^*(t)f_\alpha(t')\rangle$, we have

$$\int_0^t e^{-\mu_a(t-t')/2}\langle f_\alpha^*(t)f_\alpha(t')\rangle dt' = \frac{1}{2}\left(A\rho_{aa}^{(0)} + \kappa N\right) \tag{3.115}$$

and in view of Eq. (2.110), this can be rewritten as

$$\int_0^t e^{-\mu_a(t-t')/2}\langle f_\alpha^*(t)f_\alpha(t')\rangle dt' = \int_0^t e^{-\mu_a(t-t')/2}\left(A\rho_{aa}^{(0)} + \kappa N\right)\delta(t-t')dt'. \tag{3.116}$$

It then follows that

$$\langle f_\alpha^*(t')f_\alpha(t)\rangle = (A\rho_{aa}^{(0)} + \kappa N)\delta(t-t'). \tag{3.117}$$

It can also be established in a similar fashion that

$$\langle f_\beta^*(t')f_\beta(t)\rangle = \kappa N\delta(t-t'), \tag{3.118}$$

$$\langle f_\alpha(t')f_\beta(t)\rangle = \frac{1}{2}(\nu_+ + 2\kappa M)\delta(t-t'). \tag{3.119}$$

The results described by Eqs. (3.106), (3.110), (3.111), (3.117), (3.118), and (3.119) represent the correlation properties of the noise forces $f_\alpha(t)$ and $f_\beta(t)$ associated with the normal ordering.

We next proceed to obtain the solutions of the coupled differential equations Eqs. (3.102) and (3.103) following the procedure described in Ref. [22]. To this end, we rewrite these equations in matrix form as

$$\frac{d}{dt}Y(t) = -\frac{1}{2}MY(t) + F(t), \tag{3.120}$$

where

$$Y(t) = \begin{pmatrix} \alpha(t) \\ \beta^*(t) \end{pmatrix}, \tag{3.121}$$

$$M = \begin{pmatrix} \mu_a & -\nu_- \\ -\nu_+^* & \mu_c \end{pmatrix}, \tag{3.122}$$

$$F(t) = \begin{pmatrix} f_\alpha(t) + \varepsilon_1 \\ f_\beta^*(t) + \varepsilon_1 \end{pmatrix}. \tag{3.123}$$

Introducing a matrix defined by

$$V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}, \tag{3.124}$$

with $V_1 = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix}$ and $V_2 = \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix}$ being the eigenvectors of the matrix $M$, Eq. (3.120) can be written as

$$\frac{d}{dt}Y(t) = -\frac{1}{2}VV^{-1}MVV^{-1}Y(t) + F(t). \tag{3.125}$$

Multiplying both sides from the left by $V^{-1}$, we see that

$$\frac{d}{dt}(V^{-1}Y(t)) = -\frac{1}{2}R(V^{-1}Y(t)) + V^{-1}F(t), \tag{3.126}$$

where

$$R = V^{-1}MV = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}, \tag{3.127}$$

in which $\lambda_+$ and $\lambda_-$ are the eigenvalues of the matrix $M$. We note that Eq. (3.126) has a well defined solution for $\lambda_+ > 0$ and $\lambda_- > 0$. The solution of this equation can be written as

$$Y(t+\tau) = Ve^{-\frac{1}{2}R(\tau)}V^{-1}Y(t) + \int_0^{\tau} Ve^{-\frac{1}{2}R(\tau-\tau')}V^{-1}F(t+\tau')d\tau'. \tag{3.128}$$

We next proceed to find the eigenvalues and eigenvectors of the matrix $M$. Applying the eigenvalue equation

$$MV_i = \lambda V_i \tag{3.129}$$

along with Eq (3.122), we find the characteristic equation

$$\lambda^2 - (\mu_a + \mu_c)\lambda + (\mu_a\mu_c - \nu_+^*\nu_-) = 0. \tag{3.130}$$

The roots of this quadratic equation are found to be

$$\lambda_\pm = \frac{1}{2}\left\{(\mu_a + \mu_c) \pm \sqrt{(\mu_a - \mu_c)^2 + 4\nu_+^*\nu_-}\right\}. \tag{3.131}$$

Taking into account Eqs. (3.83), (3.92), and the relation

$$\rho_{aa}^{(0)} + \rho_{cc}^{(0)} = 1, \tag{3.132}$$

Eq. (3.131) can be rewritten as

$$\lambda_\pm = \frac{1}{2}\left(2\kappa + A\eta \pm \lambda\right), \tag{3.133}$$

where

$$\eta = \rho_{cc}^{(0)} - \rho_{aa}^{(0)}, \tag{3.134}$$

$$\lambda = \sqrt{A^2 + 4\nu_+^*\nu_-}. \tag{3.135}$$

With the aid of Eqs. (3.83), (3.92), (3.122), (3.132), and (3.133), we have

$$(A + \lambda)v_{11} + 2\nu_- v_{21} = 0, \tag{3.136}$$

and taking into account the normalization condition:

$$v_{11}^2 + v_{21}^2 = 1, \tag{3.137}$$

we get

$$v_{11} = \frac{2\nu_-}{\sqrt{(A+\lambda)^2 + 4\nu_-^2}},$$
$$v_{21} = -\frac{A+\lambda}{\sqrt{(A+\lambda)^2 + 4\nu_-^2}}. \tag{3.138}$$

Similarly we can also easily show that the elements of the eigenvector corresponding to $\lambda_-$ to be

$$v_{12} = -\frac{2\nu_-}{\sqrt{(A-\lambda)^2 + 4\nu_-^2}},$$
$$v_{22} = \frac{A-\lambda}{\sqrt{(A-\lambda)^2 + 4\nu_-^2}}. \tag{3.139}$$

Now substitution of Eqs. (3.138) and (3.139) into Eq. (3.124) yeids

$$V = \begin{pmatrix} \frac{2\nu_-}{\sqrt{A_+^2+4\nu_-^2}} & -\frac{2\nu_-}{\sqrt{A_-^2+4\nu_-^2}} \\ -\frac{A_+}{\sqrt{A_+^2+4\nu_-^2}} & \frac{A_-}{\sqrt{A_-^2+4\nu_-^2}} \end{pmatrix}, \tag{3.140}$$

in which

$$A_\pm = A \pm \lambda. \tag{3.141}$$

And the inverse of the matrix $V$ is found to be

$$V^{-1} = -\frac{1}{4\nu_-\lambda} \begin{pmatrix} A_-\sqrt{A_+^2 + 4\nu_-^2} & 2\nu_-\sqrt{A_+^2 + 4\nu_-^2} \\ A_+\sqrt{A_-^2 + 4\nu_-^2} & 2\nu_-\sqrt{A_-^2 + 4\nu_-^2} \end{pmatrix}. \tag{3.142}$$

Since Eq. (3.127) describes a diagonal matrix, we observe that

$$e^{-\frac{1}{2}R\tau} = \begin{pmatrix} e^{-\frac{1}{2}\lambda_+\tau} & 0 \\ 0 & e^{-\frac{1}{2}\lambda_-\tau} \end{pmatrix}, \tag{3.143}$$

$$e^{-\frac{1}{2}R(\tau-\tau')} = \begin{pmatrix} e^{-\frac{1}{2}\lambda_+(\tau-\tau')} & 0 \\ 0 & e^{-\frac{1}{2}\lambda_-(\tau-\tau')} \end{pmatrix}, \tag{3.144}$$

from which follows

$$Ve^{-\frac{1}{2}R\tau}V^{-1} = \begin{pmatrix} p_1(\tau) & q_1(\tau) \\ q_2(\tau) & p_2(\tau) \end{pmatrix} \tag{3.145}$$

and

$$Ve^{-\frac{1}{2}R(\tau-\tau')}V^{-1} = \begin{pmatrix} p_1(\tau-\tau') & q_1(\tau-\tau') \\ q_2(\tau-\tau') & p_2(t-t') \end{pmatrix}, \tag{3.146}$$

where

$$p_1(\tau) = \frac{A_+}{2\lambda} e^{-\frac{1}{2}\lambda_- \tau} - \frac{A_-}{2\lambda} e^{-\frac{1}{2}\lambda_+ \tau}, \tag{3.147}$$

$$p_2(\tau) = \frac{A_+}{2\lambda} e^{-\frac{1}{2}\lambda_+ \tau} - \frac{A_-}{2\lambda} e^{-\frac{1}{2}\lambda_- \tau}, \tag{3.148}$$

$$q_1(\tau) = \frac{2\nu_-}{2\lambda} e^{-\frac{1}{2}\lambda_- \tau} - \frac{2\nu_-}{2\lambda} e^{-\frac{1}{2}\lambda_+ \tau}, \tag{3.149}$$

$$q_2(\tau) = \frac{2\nu_+^*}{2\lambda} e^{-\frac{1}{2}\lambda_- \tau} - \frac{2\nu_+^*}{2\lambda} e^{-\frac{1}{2}\lambda_+ \tau}, \tag{3.150}$$

$$p_1(\tau - \tau') = \frac{A_+}{2\lambda} e^{-\frac{1}{2}\lambda_- (\tau - \tau')} - \frac{A_-}{2\lambda} e^{-\frac{1}{2}\lambda_+ (\tau - \tau')}, \tag{3.151}$$

$$p_2(\tau - \tau') = \frac{A_+}{2\lambda} e^{-\frac{1}{2}\lambda_+ (\tau - \tau')} - \frac{A_-}{2\lambda} e^{-\frac{1}{2}\lambda_- (\tau - \tau')}, \tag{3.152}$$

$$q_1(\tau - \tau') = \frac{2\nu_-}{2\lambda} e^{-\frac{1}{2}\lambda_- (\tau - \tau')} - \frac{2\nu_-}{2\lambda} e^{-\frac{1}{2}\lambda_+ (\tau - \tau')}, \tag{3.153}$$

$$q_2(\tau - \tau') = \frac{2\nu_+^*}{2\lambda} e^{-\frac{1}{2}\lambda_- (\tau - \tau')} - \frac{2\nu_+^*}{2\lambda} e^{-\frac{1}{2}\lambda_+ (\tau - \tau')}. \tag{3.154}$$

With the aid of Eqs. (3.121), (3.122), (3.123), (3.128), (3.145), and (3.146), we finally obtain

$$\alpha(t + \tau) = p_1(\tau)\alpha(t) + q_1(\tau)\beta^*(t) + G_1(t + \tau) + \varepsilon_{11}(\tau), \tag{3.155}$$

$$\beta^*(t + \tau) = p_2(\tau)\beta^*(t) + q_2(\tau)\alpha(t) + G_2(t + \tau) + \varepsilon_{12}(\tau), \tag{3.156}$$

where

$$G_1(t + \tau) = \int_0^\tau \left[p_1(\tau - \tau') f_\alpha(\tau' + t) + q_1(\tau - \tau') f_\beta^*(\tau' + t)\right] d\tau', \tag{3.157}$$

$$G_2(t + \tau) = \int_0^\tau \left[p_2(\tau - \tau') f_\beta^*(\tau' + t) + q_2(\tau - \tau') f_\alpha(\tau' + t)\right] d\tau', \tag{3.158}$$

$$\varepsilon_{11}(\tau) = \frac{\varepsilon_1}{\lambda}\left[\frac{A_+ + 2\nu_-}{\lambda_-}(1 - e^{-\frac{1}{2}\lambda_-\tau}) - \frac{A_- + 2\nu_-}{\lambda_+}(1 - e^{-\frac{1}{2}\lambda_+\tau})\right], \tag{3.159}$$

$$\varepsilon_{12}(\tau) = \frac{\varepsilon_1}{\lambda}\left[\frac{A_+ - 2\nu_+^*}{\lambda_+}(1 - e^{-\frac{1}{2}\lambda_+\tau}) - \frac{A_- - 2\nu_+^*}{\lambda_-}(1 - e^{-\frac{1}{2}\lambda_-\tau})\right]. \tag{3.160}$$

Furthermore, upon setting $t = 0$ and $\tau = t$, the cavity mode variables $\alpha(t)$ and $\beta(t)$ take the form

$$\alpha(t) = p_1(t)\alpha(0) + q_1(t)\beta^*(0) + G_1(t) + \varepsilon_{11}(t), \tag{3.161}$$

$$\beta^*(t) = p_2(t)\beta^*(0) + q_2(t)\alpha(0) + G_2(t) + \varepsilon_{12}(t). \tag{3.162}$$

## 3.2 Quadrature Squeezing

In this section we seek to calculate the quadrature variances of the cavity and the output modes as well as the squeezing spectrum of the output modes, employing the solutions of the stochastic differential equations and the correlation properties of the noise forces.

### 3.2.1 Quadrature variance of the cavity modes

Here we wish to calculate the quadrature variances for the cavity modes produced by the system under consideration. The quadrature operators for a two-mode light are defined by

$$\hat{c}_\pm = \sqrt{\pm 1}(\hat{c}^\dagger \pm \hat{c}), \tag{3.163}$$

where

$$\hat{c} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{b}). \tag{3.164}$$

Using Eq. (3.81), it can be readily verified that

$$[\hat{c}, \hat{c}^\dagger] = 1 \tag{3.165}$$

and

$$[\hat{c}_+, \hat{c}_-] = 2i. \tag{3.166}$$

The variances of the quadratures represented by the operators defined by (3.163) can be expressed in terms of c-number variables associated with the normal ordering as

$$\Delta c_{\pm}^2 = 1 \pm \langle \gamma_{\pm}(t), \gamma_{\pm}(t)\rangle, \tag{3.167}$$

where

$$\gamma_{\pm}(t) = \frac{1}{\sqrt{2}}(\alpha^*(t) + \beta^*(t) \pm \alpha(t) \pm \beta(t)). \tag{3.168}$$

On account of Eq. (3.168), we see that

$$\langle \gamma_{\pm}(t), \gamma_{\pm}(t)\rangle = \frac{1}{2}\bigg(\langle \alpha(t), \alpha(t)\rangle + \langle \beta^*(t), \beta^*(t)\rangle + 2\langle \alpha(t), \beta(t)\rangle \\ \pm \langle \alpha^*(t), \alpha(t)\rangle \pm \langle \beta^*(t), \beta(t)\rangle \pm 2\langle \beta^*(t), \alpha(t)\rangle\bigg) + c.c., \tag{3.169}$$

in which *c.c.* stands for complex conjugate. Using Eqs. (3.106), (3.110), (3.111), (3.157), (3.158), (3.161), (3.162), and assuming the cavity modes are initially in vacuum states along with the fact that a noise force at a certain time does not affect the cavity mode variables at earlier time, we easily find

$$\langle \alpha(t), \alpha(t)\rangle = \langle \beta(t), \beta(t)\rangle = \langle \beta^*(t), \alpha(t)\rangle = 0, \tag{3.170}$$

so that in view of these results, Eq. (3.169) reduces to

$$\langle \gamma_{\pm}(t), \gamma_{\pm}(t)\rangle = \langle \alpha(t), \beta(t)\rangle + \langle \alpha^*(t), \beta^*(t)\rangle \pm \langle \alpha^*(t), \alpha(t)\rangle \pm \langle \beta^*(t), \beta(t)\rangle. \tag{3.171}$$

Furthermore, taking into account Eq. (3.161) along with its complex conjugate, we get

$$\langle \alpha^*(t), \alpha(t)\rangle = \langle G_1^*(t) G_1(t)\rangle. \tag{3.172}$$

With the aid of Eqs. (3.157), (3.117), (3.118), and (3.119), we have

$$\langle \alpha^*(t), \alpha(t)\rangle = \int_0^t \bigg(|p_1(t-t')|^2 f_{\alpha^*\alpha} + p_1^*(t-t') q_1(t-t') f_{\alpha\beta}^* \\ + q_1^*(t-t') p_1(t-t') f_{\alpha\beta} + |q_1(t-t')|^2 \kappa N\bigg) dt'. \tag{3.173}$$

where

$$f_{\alpha^*\alpha} = A\rho_{aa}^{(0)} + \kappa N, \qquad f_{\alpha\beta} = (\nu_+ + 2\kappa M)/2. \tag{3.174}$$

Applying (3.151) and (3.153) in Eq. (3.173) and then carrying out the integration, we get

$$\begin{aligned}\langle\alpha^*(t),\alpha(t)\rangle =& \frac{A_+^*[A_+f_{\alpha^*\alpha}+2\nu_-f_{\alpha\beta}^*]+2\nu_-^*[A_+f_{\alpha\beta}+2\nu_-\kappa N]}{|2\lambda|^2(\lambda_-^*+\lambda_-)/2}(1-e^{-\frac{1}{2}(\lambda_-^*+\lambda_-)t})\\ &-\frac{A_+^*[A_-f_{\alpha^*\alpha}+2\nu_-f_{\alpha\beta}^*]+2\nu_-^*[A_-f_{\alpha\beta}+2\nu_-\kappa N]}{|2\lambda|^2(\lambda_-^*+\lambda_+)/2}(1-e^{-\frac{1}{2}(\lambda_-^*+\lambda_+)t})\\ &-\frac{A_-^*[A_+f_{\alpha^*\alpha}+2\nu_-f_{\alpha\beta}^*]+2\nu_-^*[A_+f_{\alpha\beta}+2\nu_-\kappa N]}{|2\lambda|^2(\lambda_+^*+\lambda_-)/2}(1-e^{-\frac{1}{2}(\lambda_+^*+\lambda_-)t})\\ &+\frac{A_-^*[A_-f_{\alpha^*\alpha}+2\nu_-f_{\alpha\beta}^*]+2\nu_-^*[A_-f_{\alpha\beta}+2\nu_-\kappa N]}{|2\lambda|^2(\lambda_+^*+\lambda_+)/2}(1-e^{-\frac{1}{2}(\lambda_+^*+\lambda_+)t}).\end{aligned} \tag{3.175}$$

Following a similar procedure, we also find

$$\begin{aligned}\langle\beta^*(t),\beta(t)\rangle =& \frac{A_+^*[A_+\kappa N-2\nu_+^*f_{\alpha\beta}]-2\nu_+[A_+f_{\alpha\beta}^*-2\nu_+^*f_{\alpha^*\alpha}]}{|2\lambda|^2(\lambda_+^*+\lambda_+)/2}(1-e^{-\frac{1}{2}(\lambda_+^*+\lambda_+t})\\ &-\frac{A_+^*[A_-\kappa N-2\nu_+^*f_{\alpha\beta}]-2\nu_+[A_-f_{\alpha\beta}^*-2\nu_+^*f_{\alpha^*\alpha}]}{|2\lambda|^2(\lambda_+^*+\lambda_-)/2}(1-e^{-\frac{1}{2}(\lambda_+^*+\lambda_-t})\\ &-\frac{A_-^*[A_+\kappa N-2\nu_+^*f_{\alpha\beta}]-2\nu_+[A_+f_{\alpha\beta}^*-2\nu_+^*f_{\alpha^*\alpha}]}{|2\lambda|^2(\lambda_-^*+\lambda_+)/2}(1-e^{-\frac{1}{2}(\lambda_-^*+\lambda_+)t})\\ &+\frac{A_-^*[A_-\kappa N-2\nu_+^*f_{\alpha\beta}]-2\nu_+[A_-f_{\alpha\beta}^*-2\nu_+^*f_{\alpha^*\alpha}]}{|2\lambda|^2(\lambda_-^*+\lambda_-)/2}(1-e^{-\frac{1}{2}(\lambda_-^*+\lambda_-)t})\end{aligned} \tag{3.176}$$

and

$$\begin{aligned}\langle\alpha(t),\beta(t)\rangle =& \frac{A_+^*[A_+f_{\alpha\beta}+2\nu_-\kappa N]-2\nu_+[A_+f_{\alpha^*\alpha}+2\nu_-f_{\alpha\beta}^*]}{|2\lambda|^2(\lambda_+^*+\lambda_-)/2}(1-e^{-\frac{1}{2}(\lambda_+^*+\lambda_-)t})\\ &-\frac{A_+^*[A_-f_{\alpha\beta}+2\nu_-\kappa N]-2\nu_+[A_-f_{\alpha^*\alpha}+2\nu_-f_{\alpha\beta}^*]}{|2\lambda|^2(\lambda_+^*+\lambda_+)/2}(1-e^{-\frac{1}{2}(\lambda_+^*+\lambda_+)t})\\ &-\frac{A_-^*[A_+f_{\alpha\beta}+2\nu_-\kappa N]-2\nu_+[A_+f_{\alpha^*\alpha}+2\nu_-f_{\alpha\beta}^*]}{|2\lambda|^2(\lambda_-^*+\lambda_-)/2}(1-e^{-\frac{1}{2}(\lambda_-^*+\lambda_-)t})\\ &+\frac{A_-^*[A_-f_{\alpha\beta}+2\nu_-\kappa N]-2\nu_+[A_-f_{\alpha^*\alpha}+2\nu_-f_{\alpha\beta}^*]}{|2\lambda|^2(\lambda_-^*+\lambda_+)/2}(1-e^{-\frac{1}{2}(\lambda_-^*+\lambda_+)t}).\end{aligned} \tag{3.177}$$

Now substitution of Eqs. (3.175), (3.176), (3.177), and the complex conjugate of

Eq. (3.177) into Eq. (3.171) leads to

$$\begin{aligned}\langle\gamma_\pm(t),\gamma_\pm(t)\rangle &= \pm\frac{1}{|2\lambda|^2}\bigg[\frac{(A_+\pm 2\nu_+^*)[(A_+^*\pm 2\nu_+)f_{\alpha^*,\alpha}\mp(A_-^*\mp 2\nu_-^*)f_{\alpha,\beta}]}{\lambda_-+\lambda_-^*}\\ &+\frac{(A_-\mp 2\nu_-)[(A_-^*\mp 2\nu_-^*)\kappa N\mp(A_+^*\pm 2\nu_+)f_{\alpha,\beta}^*]}{\lambda_-+\lambda_-^*}\bigg](1-e^{-\frac{1}{2}(\lambda_-+\lambda_-^*)t})\\ &\pm\frac{1}{|2\lambda|^2}\bigg[\frac{(A_-\pm 2\nu_+^*)[(A_-^*\pm 2\nu_+)f_{\alpha^*,\alpha}\mp(A_+^*\mp 2\nu_-^*)f_{\alpha,\beta}]}{\lambda_++\lambda_+^*}\\ &+\frac{(A_+\mp 2\nu_-)[(A_+^*\mp 2\nu_-^*)\kappa N\mp(A_-^*\pm 2\nu_+)f_{\alpha,\beta}^*]}{\lambda_++\lambda_+^*}\bigg](1-e^{-\frac{1}{2}(\lambda_++\lambda_+^*)t})\\ &\mp\frac{2}{|2\lambda|^2}\bigg[\frac{(A_-\pm 2\nu_+^*)[(A_+^*\pm 2\nu_+)f_{\alpha^*,\alpha}\mp(A_-^*\mp 2\nu_-^*)f_{\alpha,\beta}]}{\lambda_++\lambda_-^*}\\ &+\frac{(A_+\mp 2\nu_-)[(A_-^*\mp 2\nu_-^*)\kappa N\mp(A_+^*\pm 2\nu_+)f_{\alpha,\beta}^*]}{\lambda_++\lambda_-^*}\bigg](1-e^{-\frac{1}{2}(\lambda_++\lambda_-^*)t})\\ &+c.c. \end{aligned} \tag{3.178}$$

Hence on account of (3.178), Eq. (3.167) takes at steady state the form

$$\begin{aligned}\Delta c_\pm^2 &= 1+\frac{2}{|2\lambda|^2}\bigg[\frac{|A_+\pm 2\nu_+^*|^2}{\lambda_-+\lambda_-^*}+\frac{|A_-\pm 2\nu_+^*|^2}{\lambda_++\lambda_+^*}\\ &-\frac{(A_+^*\pm 2\nu_+)(A_-\pm 2\nu_+^*)}{\lambda_++\lambda_-^*}-\frac{(A_+\pm 2\nu_+^*)(A_-^*\pm 2\nu_+)}{\lambda_+^*+\lambda_-}\bigg]f_{\alpha^*\alpha}\\ &+\frac{2}{|2\lambda|^2}\bigg[\frac{|A_-\mp 2\nu_-|^2}{\lambda_-+\lambda_-^*}+\frac{|A_+\mp 2\nu_-|^2}{(\lambda_++\lambda_+^*)}\\ &-\frac{(A_+\mp 2\nu_-)(A_-^*\mp 2\nu_-^*)}{\lambda_++\lambda_-^*}-\frac{(A_+^*\mp 2\nu_-^*)(A_-\mp 2\nu_-)}{\lambda_+^*+\lambda_-}\bigg]\kappa N\\ &\mp\frac{2}{|2\lambda|^2}\bigg[\frac{(A_+\pm 2\nu_+^*)(A_-^*\mp 2\nu_-^*)}{\lambda_-+\lambda_-^*}+\frac{(A_+^*\mp 2\nu_-^*)(A_-\pm 2\nu_+^*)}{\lambda_++\lambda_+^*}\\ &-\frac{(A_-^*\mp 2\nu_-^*)(A_-\pm 2\nu_+^*)}{\lambda_++\lambda_-^*}-\frac{(A_+^*\mp 2\nu_-^*)(A_+\pm 2\nu_+^*)}{\lambda_+^*+\lambda_-}\bigg]f_{\alpha\beta}\\ &\mp\frac{2}{|2\lambda|^2}\bigg[\frac{(A_+^*\pm 2\nu_+)(A_-\mp 2\nu_-)}{\lambda_-+\lambda_-^*}+\frac{(A_+\mp 2\nu_-)(A_-^*\pm 2\nu_+)}{\lambda_++\lambda_+^*}\\ &-\frac{(A_+\mp 2\nu_-)(A_+^*\pm 2\nu_+)}{\lambda_++\lambda_-^*}-\frac{(A_-^*\pm 2\nu_+)(A_-\mp 2\nu_-)}{\lambda_+^*+\lambda_-}\bigg]f_{\alpha\beta}^*. \end{aligned} \tag{3.179}$$

This represents the quadrature variances of the cavity modes for a nondegenerate three-level laser whose cavity contains a parametric amplifier and whose cavity modes are driven by coherent light and coupled to a two-mode squeezed vacuum reservoir. In order to have a mathematically manageable analysis, we take $\rho_{ac}=\rho_{ca}$. Hence in view of this as well as Eqs. (3.6), (3.132), and (3.134), we can write Eqs. (3.84), (3.93), (3.133), (3.135), (3.141),

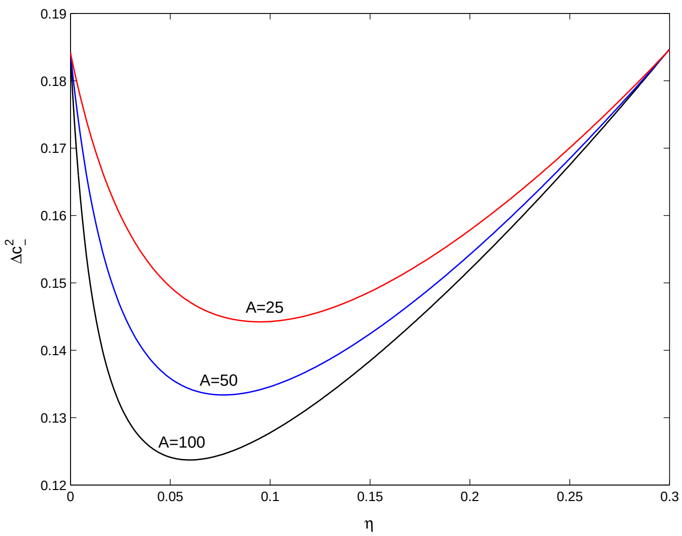


*Fig. 3.2:* Plots of the quadrature variance [Eq.(3.181)] versus $\eta$ for $\kappa = 0.8$, $\varepsilon_2 = 0.399$, $r = 0.5$, and for different values of the linear gain coefficient.

and (3.174) as

$$\begin{aligned}
&2\nu_\pm = 2\nu_\pm^* = 4\varepsilon_2 \pm A\sqrt{1-\eta^2}, \qquad \lambda = \lambda^* = \sqrt{A^2\eta^2 + 16\varepsilon_2^2},\\
&A_\pm = A_\pm^* = A \pm \sqrt{A^2\eta^2 + 16\varepsilon_2^2}, \quad \lambda_\pm = \lambda_\pm^* = \frac{1}{2}(2\kappa + A\eta \pm \sqrt{A^2\eta^2 + 16\varepsilon_2^2}),\\
&f_{\alpha^*\alpha} = (A(1-\eta) + 2\kappa N)/2, \qquad f_{\alpha\beta} = f_{\alpha\beta}^* = (4\varepsilon_2 + A\sqrt{1-\eta^2} + 4\kappa M)/4, \qquad (3.180)
\end{aligned}$$

so that with the aid Eqs. (3.179) and (3.180), we get

$$\begin{aligned}
\Delta c_\pm^2 = 1 &+ \frac{2\kappa A(1-\eta)(2\kappa + 2A\eta + A) + 16\varepsilon_2^2 A\eta - 4\kappa A^2\eta^2 N}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)}\\
&\pm \frac{2\kappa(4\varepsilon_2 + A\sqrt{1-\eta^2})(2\kappa + A\eta + A \pm 4\varepsilon_2)}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)}\\
&+ \frac{4\kappa[(2\kappa + A\eta)(2\kappa + A\eta \pm 4\varepsilon_2)(N \pm M) + A^2(1 \pm \sqrt{1-\eta^2})(N \mp M)]}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)}. \qquad (3.181)
\end{aligned}$$

Since the parameter $\varepsilon_1$ does not appear in this equation, the driving coherent light has no effect on the quadrature variances. Fig 3.2 represents the variances of the minus quadrature [Eq. (3.181)] versus $\eta$ for different values of $A$. This figure indicates that the degree of squeezing increases with the linear gain coefficient and almost perfect squeezing can be obtained for large values of the linear gain coefficient and for small values of $\eta$. Moreover, the minimum value of the quadrature variance described by Eq. (3.181) for $A = 100$, $\kappa = 0.8$, $\varepsilon_2 = 0.399$, and $r = 0.5$, is found to be $\Delta c_-^2 = 0.1237$ and occurs

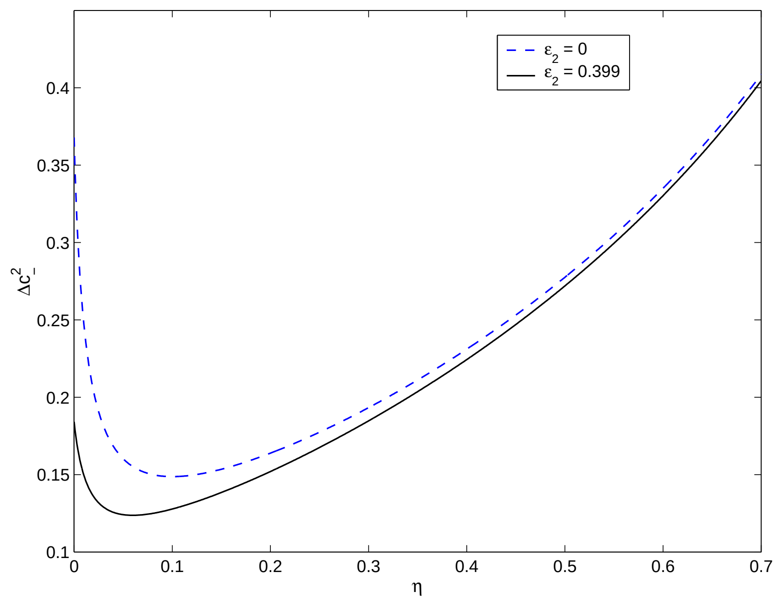


*Fig. 3.3:* Plots of the quadrature variances [Eq.(3.181)] (solid curve) and [Eq.(3.182)] (dashed curve) versus $\eta$ for $A = 100$, $\kappa = 0.8$, $r = 0.5$, and $\varepsilon_2 = 0.399$.

at $\eta = 0.06$. This result implies that the maximum intracavity squeezing for the above values is 87.6% below the coherent-state level.

We next consider some special cases. We first consider the case in which the parametric amplifier is removed from the cavity. Thus setting $\varepsilon_2 = 0$ in Eq. (3.181), we have

$$\begin{aligned}\Delta c_{\pm}^2 = 1 &+ \frac{A(1-\eta)(2\kappa + 2A\eta + A) - 2A^2\eta^2 N}{2(\kappa + A\eta)(2\kappa + A\eta)} \\ &\pm \frac{A\sqrt{1-\eta^2}(2\kappa + A\eta + A + 2A(N \mp M))}{2(\kappa + A\eta)(2\kappa + A\eta)} \\ &+ \frac{2[(2\kappa + A\eta)^2(N \pm M) + A^2(N \mp M)]}{2(\kappa + A\eta)(2\kappa + A\eta)}. \end{aligned} \tag{3.182}$$

This represents the quadrature variances of the cavity modes for a nondegenerate three-level laser coupled to a two-mode squeezed vacuum reservoir. The minimum value of the quadrature variance is found to be $\Delta c_{-}^2 = 0.1487$ and occurs at $\eta = 0.1$ for $A = 100$, $\kappa = 0.8$, $r = 0.5$, and $\varepsilon_2 = 0$. This result indicates that the maximum intracavity squeezing for the above values and in the absence of the parametric amplifier is 85.1% below the coherent-state level. Fig 3.3 is the plots of the variance of the minus quadrature [Eq. (3.181)] versus $\eta$ in the presence (solid curve) and [Eq. (3.182)] versus $\eta$ in the absence (dashed curve) of parametric amplifier in a nondegenerate three-level laser cavity. This

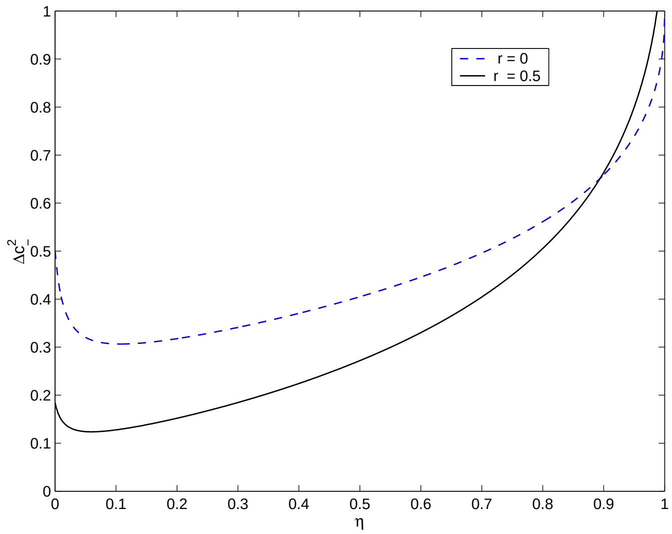


*Fig. 3.4:* Plots of the quadrature variances [Eq.(3.181)] (solid curve) and [Eq.(3.183)] (dashed curve) versus $\eta$ for $A = 100$, $\kappa = 0.8$, $\varepsilon_2 = 0.399$, and $r = 0.5$.

figure shows that the increase of the degree of squeezing due to the parametric amplifier is not significant.

Next upon setting $N = M = 0$ in Eq.(3.181), we have

$$\Delta c_{\pm}^2 = 1 + \frac{2\kappa A(1-\eta)(2\kappa + 2A\eta + A) + 16\varepsilon_2^2 A\eta}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)} \pm \frac{2\kappa(4\varepsilon_2 + A\sqrt{1-\eta^2})(2\kappa + A\eta + A \pm 4\varepsilon_2)}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)}. \tag{3.183}$$

This is the quadrature variances of the cavity modes for a nondegenerate three-level laser whose cavity contains a parametric amplifier and whose cavity modes are coupled to a two-mode vacuum reservoir. In Fig 3.4 we plot the variance of the minus quadrature [Eq. (3.181)] versus $\eta$ (solid curve) and [Eq. (3.183)] versus $\eta$ (dashed curve) for $A = 100$, $\kappa = 0.8$, $\varepsilon_2 = 0.399$, and $r = 0.5$. We see from this figure that the degree of squeezing increases with the squeeze parameter $r$ for $\eta$ between 0 and 0.9, and decreases for other values of $\eta$. Moreover, the maximum intracavity squeezing for $A = 100$, $\kappa = 0.8$, $\varepsilon_2 = 0.399$, and $r = 0$ is 69.34% and occurs at $\eta = 0.11$. Comparison of this result with the 87.6% squeezing that could be obtained in the presence of the squeezed vacuum reservoir shows that the squeezed vacuum reservoir has significant effect on the squeezing of the cavity modes.

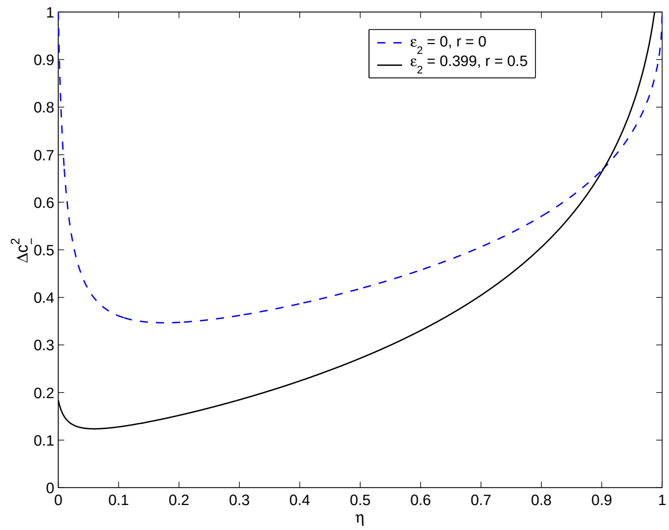


*Fig. 3.5:* Plots of the quadrature variances [Eq.(3.181)] (solid curve) and [Eq.(3.184)] (dashed curve) versus $\eta$ for $A = 100$, $\kappa = 0.8$, $\varepsilon_2 = 0.399$, and $r = 0.5$.

We now consider the case in which the nonlinear crystal is removed from the cavity and the cavity is coupled to a two-mode vacuum reservoir. Then upon setting $\varepsilon_2 = N = M = 0$ in Eq. (3.181), we get

$$\Delta c_{\pm}^2 = 1 + \frac{A(1-\eta)(2\kappa + 2A\eta + A) \pm A\sqrt{1-\eta^2}(2\kappa + A\eta + A)}{2(\kappa + A\eta)(2\kappa + A\eta)}. \tag{3.184}$$

This is the quadrature variances of the cavity modes for a nondegenerate three-level laser. The minimum value of the quadrature variance described by Eq. (3.184) for $A = 100$, $\kappa = 0.8$, is found to be $\Delta c_{-}^2 = 0.3467$ and occurs at $\eta = 0.16$. This result implies that the maximum intracavity squeezing for the above values is 65.3% below the coherent-state level. The plots in Fig. 3.5 represent the variances of the minus quadrature of the cavity modes for a nondegenerate three-level laser alone (dashed curve) and with parametric amplifier and squeezed vacuum reservoir (solid curve). This figure indicates that better squeezing can be obtained from a nondegenerate three-level laser with parametric amplifier and squeezed vacuum reservoir.

### 3.2.2 Quadrature variance of the output modes

The squeezing properties of the output modes are described by the quadrature operators

$$\hat{c}_{\pm}^{out} = \sqrt{\pm 1}(\hat{c}_{out}^{\dagger} \pm \hat{c}_{out}), \tag{3.185}$$

where

$$\hat{c}_{out} = \frac{1}{\sqrt{2}}\left(\hat{a}_{out} + \hat{b}_{out}\right). \tag{3.186}$$

The quadrature variances of the output modes

$$\Delta c^2_{\pm out} = 1 + \langle : \hat{c}^{out}_{\pm}, \hat{c}^{out}_{\pm} : \rangle \tag{3.187}$$

can be expressed in terms of c-number variables as

$$\Delta c^2_{\pm out} = 1 \pm \langle \gamma^{out}_{\pm}, \gamma^{out}_{\pm} \rangle. \tag{3.188}$$

Using the input-output relation

$$\gamma^{out}_{\pm} = \sqrt{\kappa}\gamma_{\pm} - \gamma^{in}_{\pm}, \tag{3.189}$$

Eq. (3.188) can be rewritten as

$$\Delta c^2_{\pm out} = \kappa \Delta c^2_{\pm} + 1 \pm \langle \gamma^{in}_{\pm}, \gamma^{in}_{\pm} \rangle - \kappa(1 \pm \frac{2}{\sqrt{\kappa}}\langle \gamma_{\pm}, \gamma^{in}_{\pm} \rangle), \tag{3.190}$$

where

$$\gamma^{in}_{\pm}(t) = \frac{1}{\sqrt{2\kappa}}(f^*_{\alpha r}(t) + f^*_{\beta r}(t) \pm f_{\alpha r}(t) \pm f_{\beta r}(t)). \tag{3.191}$$

We now proceed to evaluate $\langle \gamma_{\pm}, \gamma^{in}_{\pm} \rangle$. Taking into account Eqs. (3.168), (3.161), (3.162), (3.191) along with $\langle \gamma^{in}_{\pm} \rangle = 0$ and the fact that a noise force at a certain time does not affect the cavity mode variables at earlier time, we have

$$\langle \gamma_{\pm}, \gamma^{in}_{\pm} \rangle = \frac{1}{2\sqrt{\kappa}}\langle (G^*_1(t) + G_2(t) \pm G_1(t) \pm G^*_2(t))(f^*_{\alpha r}(t) + f^*_{\beta r}(t) \pm f_{\alpha r}(t) \pm f_{\beta r}(t)) \rangle. \tag{3.192}$$

We note that the noise forces can be expressed as a sum of the noise forces associated with the cavity modes and the reservoir modes as

$$f_{\alpha}(t) = f_{\alpha c}(t) + f_{\alpha r}(t), \tag{3.193}$$

$$f_{\beta}(t) = f_{\beta c}(t) + f_{\beta r}(t). \tag{3.194}$$

We assume that the noise forces associated with the cavity modes are not correlated with the noise forces associated with the reservoir modes. Furthermore, in view of Eqs. (3.106), (3.110), (3.111), (3.117), (3.118), and (3.119), the correlation properties of the noise forces associated with the cavity modes and the reservoir modes can be written as

$$\langle f_{\alpha c}(t)\rangle = \langle f_{\alpha r}(t)\rangle = \langle f_{\beta c}(t)\rangle = \langle f_{\beta r}(t)\rangle = 0, \tag{3.195}$$

$$\langle f_{\alpha c}(t) f_{\alpha c}(t')\rangle = \langle f^*_{\beta c}(t) f_{\beta c}(t')\rangle = \langle f_{\beta c}(t) f_{\beta c}(t')\rangle = \langle f^*_{\alpha c}(t) f_{\beta c}(t')\rangle = 0, \tag{3.196}$$

$$\langle f_{\alpha r}(t) f_{\alpha r}(t')\rangle = \langle f_{\beta r}(t) f_{\beta r}(t')\rangle = \langle f^*_{\alpha r}(t) f_{\beta r}(t')\rangle = 0, \tag{3.197}$$

$$\langle f^*_{\alpha r}(t) f_{\alpha r}(t')\rangle = \langle f^*_{\beta r}(t) f_{\beta r}(t')\rangle = \kappa N \delta(t-t'), \tag{3.198}$$

$$\langle f_{\alpha r}(t) f_{\beta r}(t')\rangle = \kappa M \delta(t-t'), \tag{3.199}$$

$$\langle f_{\alpha c}(t) f_{\beta c}(t')\rangle = \frac{1}{2}\nu_+ \delta(t-t'), \tag{3.200}$$

$$\langle f^*_{\alpha c}(t) f_{\alpha c}(t')\rangle = A\rho^{(0)}_{aa}\delta(t-t'). \tag{3.201}$$

Using Eqs. (3.193) and (3.194), we can rewrite Eqs. (3.157) and (3.158) as

$$G_1(t) = \int_0^t \left\{p_1(t-t')(f_{\alpha c}(t') + f_{\alpha r}(t')) + q_1(t-t')(f^*_{\beta c}(t') + f^*_{\beta r}(t'))\right\} dt', \tag{3.202}$$

$$G_2(t) = \int_0^t \left\{p_2(t-t')(f^*_{\beta c}(t') + f^*_{\beta r}(t')) + q_2(t-t')(f_{\alpha c}(t') + f_{\alpha r}(t'))\right\} dt', \tag{3.203}$$

so that with the aid of Eqs. (3.192), (3.196), (3.197), (3.198), (3.202), and (3.203), we find

$$\begin{aligned}\langle \gamma_\pm, \gamma^{in}_\pm\rangle = &\frac{\sqrt{\kappa}}{4}[p^*_1(t-t) + p_1(t-t) + p^*_2(t-t) + p_2(t-t)](M \pm N)\\ &\pm \frac{\sqrt{\kappa}}{4}[q^*_1(t-t) + q_1(t-t) + q^*_2(t-t) + q_2(t-t)](N \pm M).\end{aligned} \tag{3.204}$$

On account of Eqs. (3.151), (3.152), (3.153), and (3.154), we obtain

$$\langle \gamma_\pm(t), \gamma^{in}_\pm(t)\rangle = \sqrt{\kappa}(M \pm N). \tag{3.205}$$

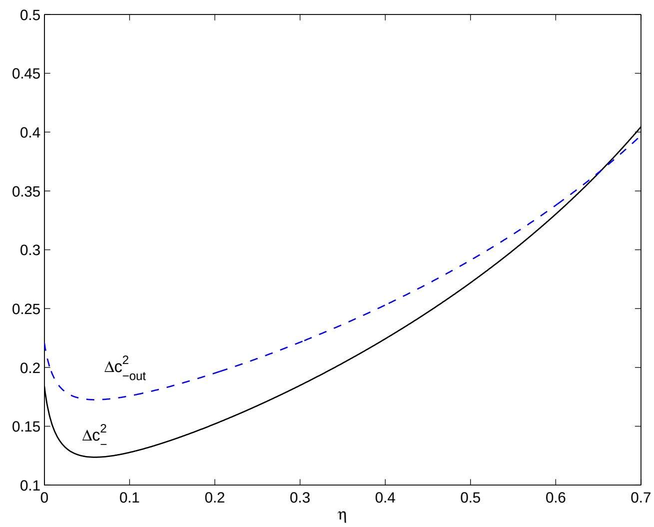


*Fig. 3.6:* Plots of the quadrature variances [Eqs .(3.181) (solid curve) and (3.207) (dashed curve)] versus $\eta$ for $A = 100$, $\kappa = 0.8$, $\varepsilon_2 = 0.399$, and $r = 0.5$.

We note that for a squeezed vacuum reservoir

$$\langle \gamma_{\pm}^{in}, \gamma_{\pm}^{in} \rangle = 2(M \pm N). \tag{3.206}$$

Hence on substituting Eqs. (3.205) and (3.206) into Eq. (3.190), we get

$$\Delta c_{\pm out}^2 = \kappa \Delta c_{\pm}^2 + (1 - \kappa)(1 + 2N \pm 2M). \tag{3.207}$$

The first and second terms on the right side of Eq. (3.207) represent the quadrature variances of the transmitted cavity modes and reflected input modes, respectively. We easily observe from Fig. 3.6 that the degree of squeezing of the output modes is less that of the cavity modes. For $A = 100$, $\varepsilon_2 = 0.399$, and $r = 0.5$, the cavity and output mode squeezing are found to be respectively 87.6% and 82.8% below the coherent-state level and occur at $\eta = 0.06$.

### 3.2.3 Squeezing spectrum of the output modes

The squeezing spectrum of the output modes can be expressed in terms of c-number variables associated with the normal ordering as

$$S_{\pm}^{out}(\omega) = 1 \pm 2Re \int_0^{\infty} d\tau e^{i(\omega - \omega_0)\tau} \langle \gamma_{\pm}^{out}(t), \gamma_{\pm}^{out}(t + \tau) \rangle_{ss}, \tag{3.208}$$

where the subscript "ss" stands for steady state. Taking into account the input-output relation and the fact

$$\langle\gamma_{\pm}^{in}(t)\rangle_{ss} = \langle\gamma_{\pm}^{in}(t+\tau)\rangle_{ss} = \langle\gamma_{\pm}(t)\gamma_{\pm}^{in}(t+\tau)\rangle_{ss} = 0, \tag{3.209}$$

we can write

$$\begin{aligned}\langle\gamma_{\pm}^{out}(t), \gamma_{\pm}^{out}(t+\tau)\rangle_{ss} = \kappa\langle\gamma_{\pm}(t), \gamma_{\pm}(t+\tau)\rangle_{ss} + \langle\gamma_{\pm}^{in}(t)\gamma_{\pm}^{in}(t+\tau)\rangle_{ss} \\ - \sqrt{\kappa}\langle\gamma_{\pm}^{in}(t)\gamma_{\pm}(t+\tau)\rangle_{ss}.\end{aligned} \tag{3.210}$$

We next procedure to obtain the explicit forms of the two-time correlation functions involved in Eq. (3.210). To this end, with the aid of Eq. (3.168), we get

$$\begin{aligned}\langle\gamma_{\pm}(t), \gamma_{\pm}(t+\tau)\rangle_{ss} = \frac{1}{2}\Big[&\langle\alpha(t), \alpha(t+\tau)\rangle_{ss} + \langle\alpha^*(t), \beta^*(t+\tau)\rangle_{ss} + \langle\beta(t), \alpha(t+\tau)\rangle_{ss} \\ &+ \langle\beta^*(t), \beta^*(t+\tau)\rangle_{ss} \pm \langle\alpha^*(t), \alpha(t+\tau)\rangle_{ss} \pm \langle\beta^*(t), \alpha(t+\tau)\rangle_{ss} \\ &\pm \langle\alpha(t), \beta^*(t+\tau)\rangle_{ss} \pm \langle\beta(t), \beta^*(t+\tau)\rangle_{ss}\Big] + c.c.,\end{aligned} \tag{3.211}$$

On account of Eq. (3.155), we see that

$$\langle\alpha(t), \alpha(t+\tau)\rangle_{ss} = p_1(\tau)\langle\alpha(t), \alpha(t)\rangle_{ss} + q_1(\tau)\langle\alpha(t), \beta^*(t)\rangle_{ss} + \langle\alpha(t), G_1(t+\tau)\rangle_{ss}. \tag{3.212}$$

Applying Eq. (3.170) and the fact that a noise force at a certain time does not affect the cavity mode variables at earlier time, we easily obtain

$$\langle\alpha(t), \alpha(t+\tau)\rangle = 0, \tag{3.213}$$

Similarly, we can find

$$\langle\beta^*(t), \beta^*(t+\tau)\rangle = \langle\alpha(t), \beta^*(t+\tau)\rangle = \langle\beta^*(t), \alpha(t+\tau)\rangle = 0, \tag{3.214}$$

Furthermore, using Eq. (3.155), we have

$$\langle\beta(t), \alpha(t+\tau)\rangle_{ss} = p_1(\tau)\langle\beta(t), \alpha(t)\rangle_{ss} + q_1(\tau)\langle\beta(t), \beta^*(t)\rangle_{ss} + \langle\beta(t), G_1(t+\tau)\rangle_{ss}. \tag{3.215}$$

It then follows that

$$\langle \beta(t), \alpha(t+\tau)\rangle_{ss} = p_1(\tau)\langle \beta(t), \alpha(t)\rangle_{ss} + q_1(\tau)\langle \beta(t), \beta^*(t)\rangle_{ss}. \tag{3.216}$$

Following the same procedure, we can easily get

$$\langle \alpha^*(t), \beta^*(t+\tau)\rangle_{ss} = p_2(\tau)\langle \alpha^*(t), \beta^*(t)\rangle_{ss} + q_2(\tau)\langle \alpha^*(t), \alpha(t)\rangle_{ss}, \tag{3.217}$$

$$\langle \alpha^*(t), \alpha(t+\tau)\rangle_{ss} = p_1(\tau)\langle \alpha^*(t), \alpha(t)\rangle_{ss} + q_1(\tau)\langle \alpha^*(t), \beta^*(t)\rangle_{ss}, \tag{3.218}$$

$$\langle \beta(t), \beta^*(t+\tau)\rangle_{ss} = p_2(\tau)\langle \beta(t), \beta^*(t)\rangle_{ss} + q_2(\tau)\langle \beta(t), \alpha(t)\rangle_{ss}. \tag{3.219}$$

so that combination of Eqs. (3.213), (3.214), (3.216), (3.217), (3.218), and (3.219) yields

$$\begin{aligned}\langle \gamma_\pm(t), \gamma_\pm(t+\tau)\rangle_{ss} = &\frac{1}{2}(p_2(\tau) \pm q_1(\tau))(\langle \alpha^*(t), \beta^*(t)\rangle_{ss} \pm \langle \beta^*(t), \beta(t)\rangle_{ss}) \\ &+ \frac{1}{2}(p_1(\tau) \pm q_2(\tau))(\langle \alpha(t), \beta(t)\rangle_{ss} \pm \langle \alpha^*(t), \alpha(t)\rangle_{ss}) + c.c.\end{aligned} \tag{3.220}$$

Hence taking into account Eqs. (3.151), (3.152), (3.153), (3.154), and $\rho_{ac}^{(0)} = \rho_{ca}^{(0)}$, we find

$$\begin{aligned}\langle \gamma_\pm(t), \gamma_\pm(t+\tau)\rangle_{ss} = &\pm\frac{r_\pm(A_+ \pm 2\nu_+) - s_\pm(A_- \mp 2\nu_-)}{2\lambda}e^{-\frac{1}{2}\lambda_-\tau} \\ &\mp \frac{r_\pm(A_- \pm 2\nu_+) - s_\pm(A_+ \mp 2\nu_-)}{2\lambda}e^{-\frac{1}{2}\lambda_+\tau},\end{aligned} \tag{3.221}$$

where

$$r_\pm = \langle \alpha^*(t), \alpha(t)\rangle_{ss} \pm \langle \alpha(t), \beta(t)\rangle_{ss}, \tag{3.222}$$

$$s_\pm = \langle \beta^*(t), \beta(t)\rangle_{ss} \pm \langle \alpha(t), \beta(t)\rangle_{ss}. \tag{3.223}$$

With the aid of Eqs. (3.174), (3.175), (3.176), and (3.177), $r_\pm$ and $s_\pm$ can be written in terms of $\eta$ as

$$\begin{aligned}r_\pm = &\frac{(2\kappa A(1-\eta) + 16\varepsilon_2^2)[\kappa + A\eta] + \kappa[2\kappa + A\eta + A](A(1-\eta) \pm 4\varepsilon_2 \pm A\sqrt{1-\eta^2})}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)} \\ &+ \frac{2[4\kappa(\kappa + A\eta) + A^2 + (2\kappa + A\eta)(A \pm 4\varepsilon_2) - A\sqrt{1-\eta^2}(4\varepsilon_2 \mp A)]\kappa N}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)} \\ &+ \frac{2[2\kappa + A\eta + A](4\varepsilon_2 - A\sqrt{1-\eta^2} \pm [2\kappa + A\eta - A])\kappa M}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)},\end{aligned} \tag{3.224}$$

$$s_{\pm} = \frac{\kappa(4\varepsilon_2 + A\sqrt{1-\eta^2})(4\varepsilon_2 + A\sqrt{1-\eta^2} \pm [2\kappa + A\eta + A])}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)}$$
$$+ \frac{2(4\kappa(\kappa + A\eta) + A^2 - (2\kappa + A\eta)(A \mp 4\varepsilon_2) + A\sqrt{1-\eta^2}(4\varepsilon_2 \pm A))\kappa N}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)}$$
$$+ \frac{2[2\kappa + A\eta - A](4\varepsilon_2 + A\sqrt{1-\eta^2} \pm [2\kappa + A\eta + A])\kappa M}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)}. \tag{3.225}$$

Furthermore employing Eqs. (3.168), and (3.191), we have

$$\langle \gamma_{\pm}^{in}(t), \gamma_{\pm}(t+\tau)\rangle_{ss} = \frac{1}{2\sqrt{\kappa}}\Big(\langle f_{\alpha r}(t), \alpha(t+\tau)\rangle_{ss} + \langle f_{\alpha r}^*(t), \beta^*(t+\tau)\rangle_{ss} + \langle f_{\beta r}(t), \alpha(t+\tau)\rangle_{ss}$$
$$+ \langle f_{\beta r}^*(t), \beta^*(t+\tau)\rangle_{ss} \pm \langle f_{\alpha r}^*(t), \alpha(t+\tau)\rangle_{ss} \pm \langle f_{\beta r}^*(t), \alpha(t+\tau)\rangle_{ss}$$
$$\pm \langle f_{\alpha r}(t), \beta^*(t+\tau)\rangle_{ss} \pm \langle f_{\beta r}(t), \beta^*(t+\tau)\rangle_{ss}\Big) + c.c, \tag{3.226}$$

With the aid of Eqs. (3.155), (3.156), (3.195), and (3.196), we easily obtain

$$\langle f_{\alpha r}(t), \alpha(t+\tau)\rangle = \langle f_{\beta r}^*(t), \beta^*(t+\tau)\rangle = \langle f_{\alpha r}(t), \beta^*(t+\tau)\rangle = \langle f_{\beta r}^*(t), \alpha(t+\tau)\rangle = 0. \tag{3.227}$$

On account of these results, Eq. (3.226) reduces to

$$\langle \gamma_{\pm}^{in}(t), \gamma_{\pm}(t+\tau)\rangle_{ss} = \frac{1}{2\sqrt{\kappa}}\Big[\langle f_{\alpha r}^*(t), \beta^*(t+\tau)\rangle_{ss} + \langle f_{\beta r}(t), \alpha(t+\tau)\rangle_{ss}$$
$$\pm \langle f_{\alpha r}^*(t), \alpha(t+\tau)\rangle_{ss} \pm \langle f_{\beta r}(t), \beta^*(t+\tau)\rangle_{ss}\Big] + c.c. \tag{3.228}$$

We now proceed to obtain the explicit form of the two-time correlation function involved in Eq. (3.228). Using Eq. (3.156) and the fact that $\langle f_{\alpha r}^*(t)\rangle = 0$, we can write

$$\langle f_{\alpha r}^*(t), \beta^*(t+\tau)\rangle_{ss} = p_2(\tau)\langle f_{\alpha r}^*(t)G_2(t)\rangle_{ss} + q_2(\tau)\langle f_{\alpha r}^*(t)G_1(t)\rangle_{ss} + \langle f_{\alpha r}^*(t), G_2(t+\tau)\rangle_{ss}. \tag{3.229}$$

In view of Eq. (3.203), we see that

$$\langle f_{\alpha r}^*(t)G_2(t)\rangle_{ss} = \int_0^t [p_2(t-t')\langle f_{\alpha r}^*(t)f_{\beta r}^*(t)\rangle_{ss} + q_2(t-t')\langle f_{\alpha r}^*(t)f_{\alpha r}(t)\rangle_{ss}]dt'. \tag{3.230}$$

With the aid of Eqs. (3.152), (3.154), (3.198), and (3.199), we get

$$\langle f_{\alpha r}^*(t)G_2(t)\rangle_{ss} = \kappa M/2. \tag{3.231}$$

Similarly, we have

$$\langle f^*_{\alpha r}(t)G_1(t)\rangle_{ss} = \kappa N/2, \tag{3.232}$$

$$\langle f^*_{\alpha r}(t), G_2(t+\tau)\rangle_{ss} = \kappa[p_2(\tau)M + q_2(\tau)N]/2. \tag{3.233}$$

Employing Eqs. (3.231), (3.232), and (3.233) in Eq. (3.229), we find

$$\langle f^*_{\alpha r}(t), \beta^*(t+\tau)\rangle_{ss} = \kappa[p_2(\tau)M + q_2(\tau)N]. \tag{3.234}$$

Following the same procedure, we also obtain

$$\langle f_{\beta r}(t), \alpha(t+\tau)\rangle_{ss} = \kappa[p_1(\tau)M + q_1(\tau)N], \tag{3.235}$$

$$\langle f^*_{\alpha r}(t), \alpha(t+\tau)\rangle_{ss} = \kappa[p_1(\tau)N + q_1(\tau)M], \tag{3.236}$$

$$\langle f_{\beta r}(t), \beta^*(t+\tau)\rangle_{ss} = \kappa[p_2(\tau)N + q_2(\tau)M]. \tag{3.237}$$

Upon substituting Eqs, (3.234), (3.235), (3.236), and (3.237) into Eq. (3.228), we get

$$\langle \gamma^{in}_\pm(t), \gamma_\pm(t+\tau)\rangle_{ss} = \pm\frac{\sqrt{\kappa}}{2}\Big[p_2(\tau) \pm q_2(\tau) + p_1(\tau) \pm q_1(\tau)\Big](N \pm M) + c.c. \tag{3.238}$$

and then using Eqs. (3.151), (3.152), (3.153), and (3.154), and taking $\rho^{(0)}_{ac} = \rho^{(0)}_{ca}$, we arrive at

$$\begin{aligned}\langle \gamma^{in}_\pm(t)\gamma_\pm(t+\tau)\rangle_{ss} = &\pm \frac{\sqrt{\kappa}}{2}(N \pm M)\frac{(A_+ \pm 2\nu_+) - (A_- \mp 2\nu_-)}{2\lambda}e^{-\frac{1}{2}\lambda_-\tau}\\ &\mp \frac{\sqrt{\kappa}}{2}(N \pm M)\frac{(A_- \pm 2\nu_+) - (A_+ \mp 2\nu_-)}{2\lambda}e^{-\frac{1}{2}\lambda_+\tau}.\end{aligned} \tag{3.239}$$

We note that for a squeezed vacuum reservoir

$$\langle \gamma^{in}_\pm(t)\gamma^{in}_\pm(t+\tau)\rangle = \pm 2(N \pm M)\delta(\tau). \tag{3.240}$$

Now combination of Eqs. (3.228), (3.239), (3.240), and (3.210) results in

$$\begin{aligned}\langle \gamma^{out}_\pm(t), \gamma^{out}_\pm(t+\tau)\rangle_{ss} = &\pm 2(N \pm M)\delta(\tau)\\ &\pm \kappa\frac{(r_\pm - (N \pm M))}{2\lambda}[(A_+ \pm 2\nu_+)e^{-\frac{1}{2}\lambda_-\tau} - (A_- \pm 2\nu_+)e^{-\frac{1}{2}\lambda_+\tau}]\\ &\mp \kappa\frac{(s_\pm - (N \pm M))}{2\lambda}[(A_- \mp 2\nu_-)e^{-\frac{1}{2}\lambda_-\tau} - (A_+ \mp 2\nu_-)e^{-\frac{1}{2}\lambda_+\tau}],\end{aligned} \tag{3.241}$$

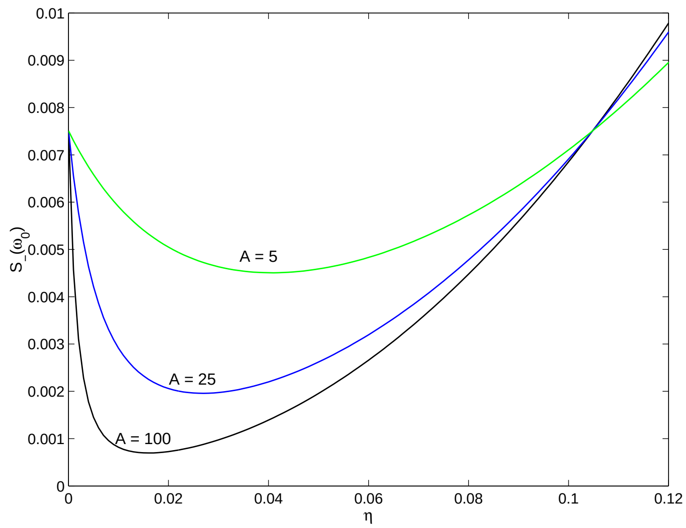


*Fig. 3.7:* Plots of the squeezing spectrum [Eq .(3.243)] versus $\eta$ for $\kappa = 0.8$, $\varepsilon_2 = 0.3$, $r = 0.5$, and different values of $A$.

so that applying this in Eq. (3.208) and carrying out the integration, we find

$$S_{\pm}^{out}(\omega) = 1 + 2(N \pm M) + 2\kappa \frac{r_{\pm} - (N \pm M)}{\lambda}\left[\frac{\lambda_{-}(A_{+} \pm 2\nu_{+})}{\lambda_{-}^{2} + 4(\omega - \omega_0)^2} - \frac{\lambda_{+}(A_{-} \pm 2\nu_{+})}{\lambda_{+}^{2} + 4(\omega - \omega_0)^2}\right] - 2\kappa \frac{s_{\pm} - (N \pm M)}{\lambda}\left[\frac{\lambda_{-}(A_{-} \mp 2\nu_{-})}{\lambda_{-}^{2} + 4(\omega - \omega_0)^2} - \frac{\lambda_{+}(A_{+} \mp 2\nu_{-})}{\lambda_{+}^{2} + 4(\omega - \omega_0)^2}\right]. \tag{3.242}$$

For $\omega = \omega_0$, Eq.(3.242) turns out to be

$$S_{\pm}^{out}(\omega_0) = 1 + 2(N \pm M)\left[1 - 2\kappa \frac{2\kappa + A\eta \pm 4\varepsilon_2}{\kappa(\kappa + A\eta) - 4\varepsilon_2^2}\right] + 2\kappa \left[\frac{(r_{\pm} - s_{\pm})(A \pm A\sqrt{1-\eta^2}) + (r_{\pm} + s_{\pm})(2\kappa + A\eta \pm 4\varepsilon_2)}{\kappa(\kappa + A\eta) - 4\varepsilon_2^2}\right]. \tag{3.243}$$

Fig 3.7 represents the plots of the squeezing spectrum [Eq .(3.243)] versus $\eta$ for $\kappa = 0.8$, $\varepsilon_2 = 0.3$, $r = 0.5$, and for $A = 5$, $A = 25$, $A = 100$. We see from the plots that the squeezing spectrum increases with linear gain coefficient for $\eta$ between 0 and 0.11, and almost perfect squeezing occurs for large values of linear gain coefficient and for small values of $\eta$. In Fig 3.8 we plot the squeezing spectrum [Eq .(3.243)] versus $\eta$ for $A = 100$, $\kappa = 0.8$, $r = 0.5$, and for $\varepsilon_2 = 0$ (dashed curve) and $\varepsilon_2 = 0.3$ (solid curve). We observe from these plots that the parametric amplifier increases the degree of squeezing.

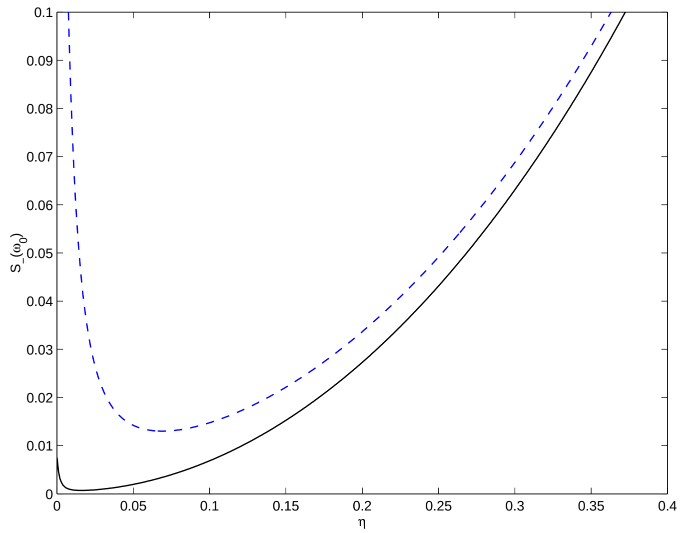


*Fig. 3.8:* Plots of the squeezing spectrum [Eq .(3.243)] versus $\eta$ for $A = 100$, $\kappa = 0.8$, $r = 0.5$, and for $\varepsilon_2 = 0$ (dashed curve) and $\varepsilon_2 = 0.3$ (solid curve).

## 3.3 Photon Statistics

In this section we study the statistical properties of the cavity and output modes produced by a nondegenerate three-level laser whose cavity contains a parametric amplifier and with the cavity modes driven by coherent light and coupled to a two-mode squeezed vacuum reservoir. We first obtain, using the antinormally ordered characteristic function defined in the Heisenberg picture, the Q function for the cavity modes. Then applying the resulting Q function, we calculate the mean and variances of the photon number sum and difference as well as the photon number distribution for the cavity modes. Finally, we calculate the mean and the normally-ordered variances of the photon count sum and difference.

### 3.3.1 Photon statistics of the cavity modes

#### The Q function

Here we wish to obtain the Q function for the cavity modes produced by the system under consideration. The Q function for a two-mode light can be expressed as

$$Q(\alpha, \beta, t) = \frac{1}{\pi^2} \int \frac{d^2 z}{\pi} \frac{d^2 w}{\pi}\ \Phi_A(z, w, t) e^{z^*\alpha - z\alpha^* + w^*\beta - w\beta^*}, \tag{3.244}$$

where

$$\Phi_A(z,w,t) = Tr\left(\rho(0)e^{-z^*\hat{a}(t)}e^{z\hat{a}^\dagger(t)}e^{-w^*\hat{b}(t)}e^{w\hat{b}^\dagger(t)}\right) \tag{3.245}$$

is the antinormally ordered characteristic function defined in the Heisenberg picture. Employing the Baker-Hausdorff identity, we can rewrite Eq. (3.245) in the normal order as

$$\Phi_A(z,w,t) = e^{-z^*z-w^*w}Tr\left(\rho(0)e^{z\hat{a}^\dagger(t)}e^{-z^*\hat{a}(t)}e^{w\hat{b}^\dagger(t)}e^{-w^*\hat{b}(t)}\right), \tag{3.246}$$

so that the corresponding c-number equation is

$$\Phi_A(z,w,t) = e^{-z^*z-w^*w}\left\langle e^{z\alpha^*(t)-z^*\alpha(t)+w\beta^*(t)-w^*\beta(t)}\right\rangle. \tag{3.247}$$

Now taking into account Eqs. (3.161) and (3.162) along with their complex conjugates, Eq. (3.247) can be put in the form

$$\Phi_A(z,w,t) = e^{-z^*z-w^*w+\varepsilon_{11}^*z-\varepsilon_{11}z^*+\varepsilon_{12}w-\varepsilon_{12}^*w^*}\left\langle e^{z\alpha'^*(t)-z^*\alpha'(t)+w\beta'^*(t)-w^*\beta'(t)}\right\rangle, \tag{3.248}$$

where

$$\alpha'(t) = p_1(t)\alpha(0) + q_1(t)\beta^*(0) + G_1(t), \tag{3.249}$$

$$\beta'(t) = p_2(t)\beta(0) + q_2(t)\alpha^*(0) + G_2^*(t). \tag{3.250}$$

With the aid of Eqs. (3.133), (3.151), (3.152), (3.153), (3.154), (3.157), and (3.158), it can be easily established that

$$\frac{d}{dt}\langle\alpha'(t)\rangle = -\frac{1}{2}\mu_a\langle\alpha'(t)\rangle + \frac{1}{2}\nu_-\langle\beta'^*(t)\rangle, \tag{3.251}$$

$$\frac{d}{dt}\langle\beta'(t)\rangle = -\frac{1}{2}\mu_c\langle\beta'(t)\rangle + \frac{1}{2}\nu_+\langle\alpha'^*(t)\rangle. \tag{3.252}$$

We see that Eqs. (3.251) and (3.252) are linear differential equations for $\alpha'(t)$ and $\beta'(t)$. On the other hand, taking into account Eqs. (3.157), (3.158), (3.106), and the assumption that the cavity modes are initially in a vacuum state, we have

$$\langle\alpha'(t)\rangle = \langle\beta'(t)\rangle = 0. \tag{3.253}$$

Thus we observe that $\alpha'(t)$ and $\beta'(t)$ are Gaussian variables with a vanishing mean. In view of this, Eq. (3.248) can be expressed as [32]

$$\Phi_A(z,w,t) = e^{-z^*z-w^*w+\varepsilon_{11}^*z-\varepsilon_{11}z^*+\varepsilon_{12}w-\varepsilon_{12}^*w^*} \times \exp\left[\left\langle \frac{1}{2}\Big(z\alpha'^*(t) - z^*\alpha'(t) + w\beta'^*(t) - w^*\beta'(t)\Big)^2\right\rangle\right] \tag{3.254}$$

or

$$\begin{aligned}\Phi_A(z,w,t) = \exp\Big[&-z^*z(1+\langle\alpha'^*(t)\alpha'(t)\rangle) + \frac{1}{2}(z^2\langle\alpha'^{*2}(t)\rangle + z^{*2}\langle\alpha'^2(t)\rangle)\\ &+ z^*(w^*\langle\alpha'(t)\beta'(t)\rangle - w\langle\alpha'(t)\beta'^*(t)\rangle - \varepsilon_{11})\\ &+ z(w\langle\alpha'^*(t)\beta'^*(t)\rangle - w^*\langle\alpha'^*(t)\beta'(t)\rangle + \varepsilon_{11}^*)\\ &- w^*w(1+\langle\beta'^*(t)\beta'(t)\rangle) + \frac{1}{2}(w^2\langle\beta'^{*2}(t)\rangle + w^{*2}\langle\beta'^2(t)\rangle)\\ &+ \varepsilon_{12}w - \varepsilon_{12}^*w^*\Big].\end{aligned} \tag{3.255}$$

Now on account of Eq. (3.249), we have

$$\begin{aligned}\langle\alpha'^2(t)\rangle = \langle(p_1(t)\alpha(0) + q_1(t)\beta^*(0))^2\rangle + 2\langle(p_1(t)\alpha(0) + q_1(t)\beta^*(0))G_1(t)\rangle\\ + \langle G_1(t)G_1(t)\rangle.\end{aligned} \tag{3.256}$$

With the aid of Eqs. (3.157), (3.110), and (3.111) along with the assumption that initially the cavity modes are in vacuum state and the fact that a noise force at a given instant does not affect the cavity mode variables at earlier time, we obtain

$$\langle\alpha'^2(t)\rangle = 0. \tag{3.257}$$

Similarly, we easily get

$$\langle\beta'^{*2}(t)\rangle = \langle\beta'^*(t)\alpha'(t)\rangle = 0, \tag{3.258}$$

$$\langle\alpha'(t)\beta'(t)\rangle = \langle G_2^*(t)G_1(t)\rangle, \tag{3.259}$$

$$\langle\alpha'^*(t)\alpha'(t)\rangle = \langle G_1^*(t)G_1(t)\rangle, \tag{3.260}$$

$$\langle\beta'^*(t)\beta'(t)\rangle =\langle G_2^*(t)G_2(t)\rangle. \tag{3.261}$$

Hence on account of Eqs. (3.257), (3.258), (3.259), (3.260), and (3.261), the characteristic function can be put in the form

$$\Phi_A(z,w,t) =e^{-a_\alpha z^*z+z^*(w^*b-\varepsilon_{11})+z(wb^*+\varepsilon_{11}^*)}e^{-a_\beta w^*w+\varepsilon_{12}w-\varepsilon_{12}^*w^*}, \tag{3.262}$$

where

$$a_\alpha =1+\langle G_1^*(t)G_1(t)\rangle, \tag{3.263}$$

$$a_\beta =1+\langle G_2^*(t)G_2(t)\rangle, \tag{3.264}$$

$$b =\langle G_2^*(t)G_1(t)\rangle. \tag{3.265}$$

With the aid of Eqs. (3.157), (3.158), (3.117), (3.118), (3.119), and (3.180), we can write (3.263), (3.264), and (3.265) as

$$\begin{aligned} a_\alpha = 1 &+ \frac{\kappa A(1-\eta)(4\kappa+3A\eta+A)+16\varepsilon_2^2(\kappa+A\eta)}{4[\kappa(\kappa+A\eta)-4\varepsilon_2^2](2\kappa+A\eta)} \\ &+ \frac{[2\kappa(2\kappa+2A\eta+A)+A^2(1+\eta)-4\varepsilon_2A\sqrt{1-\eta^2}]2\kappa N}{4[\kappa(\kappa+A\eta)-4\varepsilon_2^2](2\kappa+A\eta)} \\ &+ \frac{[4\varepsilon_2-A\sqrt{1-\eta^2}](2\kappa+A\eta+A)2\kappa M}{4[\kappa(\kappa+A\eta)-4\varepsilon_2^2](2\kappa+A\eta)}, \end{aligned} \tag{3.266}$$

$$\begin{aligned} a_\beta = 1 &+ \frac{\kappa(4\varepsilon_2+A\sqrt{1-\eta^2})^2+(4\varepsilon_2+A\sqrt{1-\eta^2})[2\kappa+A\eta-A]2\kappa M}{4[\kappa(\kappa+A\eta)-4\varepsilon_2^2](2\kappa+A\eta)} \\ &+ \frac{[2\kappa(2\kappa+2A\eta-A)+A^2(1-\eta)+4\varepsilon_2A\sqrt{1-\eta^2}]2\kappa N}{4[\kappa(\kappa+A\eta)-4\varepsilon_2^2](2\kappa+A\eta)}, \end{aligned} \tag{3.267}$$

and

$$\begin{aligned} b = &\frac{\kappa(4\varepsilon_2+A\sqrt{1-\eta^2})(2\kappa+A\eta+A)+[(2\kappa+A\eta)^2-A^2]2\kappa M}{4[\kappa(\kappa+A\eta)-4\varepsilon_2^2](2\kappa+A\eta)} \\ &+ \frac{[4\varepsilon_2(2\kappa+A\eta)+A^2\sqrt{1-\eta^2}]2\kappa N}{4[\kappa(\kappa+A\eta)-4\varepsilon_2^2](2\kappa+A\eta)}. \end{aligned} \tag{3.268}$$

Now using (3.262) in Eq (3.244), we have

$$\begin{aligned} Q(\alpha,\beta,t) = \frac{1}{\pi^2}\int\frac{d^2z}{\pi}\frac{d^2w}{\pi}\exp\Big\{ &-a_\alpha z^*z+z^*(\alpha-\varepsilon_{11}+w^*b)-z(\alpha^*-\varepsilon_{11}^*-wb^*)\Big\} \\ &\times\exp\Big\{-a_\beta w^*w+w^*(\beta-\varepsilon_{12}^*)-w(\beta^*-\varepsilon_{12})\Big\}, \end{aligned} \tag{3.269}$$

so that carrying out the integration with the help of Eq. (2.218), the Q function is found to be

$$Q(\alpha,\beta,t) = \frac{u_\alpha u_\beta - v^* v}{\pi^2}\exp\bigg[ - u_\beta \alpha^*\alpha + \alpha(p^* + v^*\beta) + \alpha^*(p + v\beta^*) - u_\alpha \beta^*\beta + \beta q^* + \beta^* q - \varepsilon_{11} p^* - \varepsilon_{12} q\bigg], \tag{3.270}$$

where

$$u_\alpha = \frac{a_\alpha}{a_\alpha a_\beta - b^* b}, \tag{3.271}$$

$$u_\beta = \frac{a_\beta}{a_\alpha a_\beta - b^* b}, \tag{3.272}$$

$$v = \frac{b}{a_\alpha a_\beta - b^* b}, \tag{3.273}$$

$$p = u_\beta \varepsilon_{11} - v\varepsilon_{12}, \tag{3.274}$$

$$q = u_\alpha \varepsilon_{12}^* - v\varepsilon_{11}^*. \tag{3.275}$$

#### Mean and variances of the photon number sum and difference

We next proceed to calculate the mean and variances of the photon number sum and difference of mode $a$ and mode $b$ applying the Q function. We define the operators representing the photon number sum and difference of mode $a$ and mode $b$ by

$$\hat{n}_\pm = \hat{a}^\dagger \hat{a} \pm \hat{b}^\dagger \hat{b}. \tag{3.276}$$

Then the mean of the photon number sum and difference can be written in terms of the Q function as

$$\overline{n}_\pm = \int d^2\alpha\ d^2\beta\ Q(\alpha,\beta,t)\left(\alpha^*\alpha \pm \beta^*\beta - 1 \mp 1\right). \tag{3.277}$$

On account of Eq. (3.270), we see that

$$\overline{n}_\pm = \frac{u_\alpha u_\beta - v^* v}{\pi^2} e^{-\varepsilon_{11} p^* - \varepsilon_{12} q} \int d^2\alpha\ d^2\beta\ \left(\alpha^*\alpha \pm \beta^*\beta - 1 \mp 1\right) \times e^{-u_\beta |\alpha|^2 + \alpha^*(p + v\beta^*) + \alpha(p^* + v^*\beta) - u_\alpha |\beta|^2 + \beta^* q + \beta q^*}. \tag{3.278}$$

This equation can be rewritten as

$$\overline{n}_{\pm} = \frac{u_\alpha u_\beta - v^* v}{\pi^2} e^{-\varepsilon_{11}p^* - \varepsilon_{12}q} \left( \frac{\partial^2}{\partial p^* \partial p} \pm \frac{\partial^2}{\partial q^* \partial q} - 1 \mp 1 \right) \times \int d^2\alpha \; d^2\beta \; e^{-u_\beta|\alpha|^2 + \alpha^*(p + v\beta^*) + \alpha(p^* + v^*\beta) - u_\alpha|\beta|^2 + \beta^* q + \beta q^*}. \tag{3.279}$$

Upon carrying out the integration with the help of Eq. (2.218), we obtain

$$\overline{n}_{\pm} = e^{-\varepsilon_{11}p^* - \varepsilon_{12}q} \left( \frac{\partial^2}{\partial p^* \partial p} \pm \frac{\partial^2}{\partial q^* \partial q} - 1 \mp 1 \right) \exp\left\{ \frac{u_\alpha p^* p + u_\beta q^* q + v^* pq + v p^* q^*}{u_\alpha u_\beta - v^* v} \right\}, \tag{3.280}$$

from which follows

$$\overline{n}_{\pm} = \overline{n}_a \pm \overline{n}_b, \tag{3.281}$$

where

$$\overline{n}_a = \frac{u_\alpha}{u_\alpha u_\beta - v^* v} + \varepsilon_{11}^* \varepsilon_{11} - 1 \tag{3.282}$$

and

$$\overline{n}_b = \frac{u_\beta}{u_\alpha u_\beta - v^* v} + \varepsilon_{12}^* \varepsilon_{12} - 1 \tag{3.283}$$

are the mean photon numbers of mode $a$ and mode $b$. With the aid of Eqs. (3.271), (3.272), (3.266), and (3.267), we can write

$$\begin{aligned} \overline{n}_a = {} & \frac{\kappa A(1-\eta)(4\kappa + 3A\eta + A) + 16\varepsilon_2^2(\kappa + A\eta)}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)} \\ & + \frac{[2\kappa(2\kappa + 2A\eta + A) + A^2(1+\eta) - 4\varepsilon_2 A\sqrt{1-\eta^2}]2\kappa N}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)} \\ & + \frac{[4\varepsilon_2 - A\sqrt{1-\eta^2}](2\kappa + A\eta + A)2\kappa M}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)} \\ & + \frac{\varepsilon_1^2(2\kappa + A\eta + A + 4\varepsilon_2 - A\sqrt{1-\eta^2})^2}{[\kappa(\kappa + A\eta) - 4\varepsilon_2^2]^2} \end{aligned} \tag{3.284}$$

and

$$\begin{aligned} \overline{n}_b = {} & \frac{\kappa(4\varepsilon_2 + A\sqrt{1-\eta^2})^2 + (4\varepsilon_2 + A\sqrt{1-\eta^2})[2\kappa + A\eta - A]2\kappa M}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)} \\ & + \frac{[2\kappa(2\kappa + 2A\eta - A) + A^2(1-\eta) + 4\varepsilon_2 A\sqrt{1-\eta^2}]2\kappa N}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)} \\ & + \frac{\varepsilon_1^2(2\kappa + A\eta - A + 4\varepsilon_2 + A\sqrt{1-\eta^2})^2}{[\kappa(\kappa + A\eta) - 4\varepsilon_2^2]^2}. \end{aligned} \tag{3.285}$$

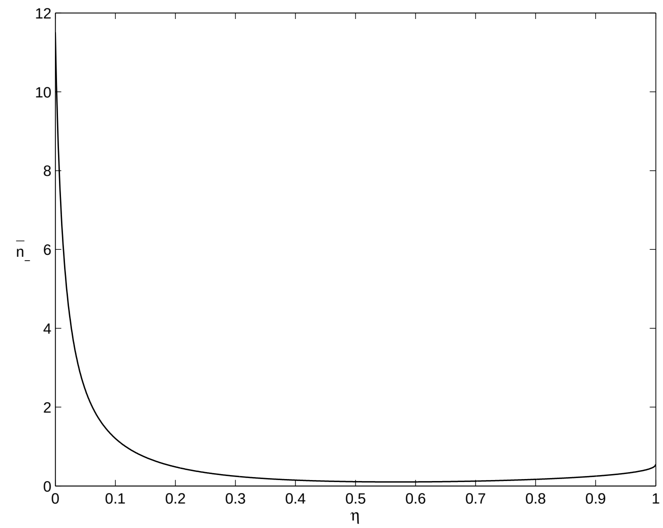


*Fig. 3.9:* A plot of the mean of the photon number difference [Eq .(3.286)] versus $\eta$ for $A = 100$, $\kappa = 0.8$, $\varepsilon_1 = 0.2$, $\varepsilon_2 = 0.399$, and $r = 0.5$.

We easily see from Eqs. (3.284) and (3.285) that the driving coherent light enhances the mean photon numbers of mode $a$ and mode $b$. On account of Eqs. (3.284) and (3.285), the mean of the photon number sum and difference can be written in the form

$$\overline{n}_{\pm} = \frac{2\kappa A(1-\eta)(2\kappa + A\eta) + 16\varepsilon_2^2 A\eta \pm \kappa 8\varepsilon_2 A\sqrt{1-\eta^2} + (1\pm 1)\kappa[A^2(1-\eta^2) + 16\varepsilon_2^2]}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)}$$
$$+ \frac{[(1\pm 1)[2\kappa(2\kappa + 2A\eta) + A^2] + (1\mp 1)A[2\kappa + A\eta - 4\varepsilon_2\sqrt{1-\eta^2}]]2\kappa N}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)}$$
$$+ \frac{[(1\pm 1)[4\varepsilon_2(2\kappa + A\eta) - A^2\sqrt{1-\eta^2}] + (1\mp 1)A[4\varepsilon_2 - (2\kappa + A\eta)\sqrt{1-\eta^2}]]2\kappa M}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)}$$
$$+ \frac{\varepsilon_1^2[(2\kappa + A\eta + A + 4\varepsilon_2 - A\sqrt{1-\eta^2})^2 \pm (2\kappa + A\eta - A + 4\varepsilon_2 + A\sqrt{1-\eta^2})^2]}{[\kappa(\kappa + A\eta) - 4\varepsilon_2^2]^2}. \tag{3.286}$$

Fig 3.9 represents a plot of the mean of the photon number difference [Eq. (3.286)] versus $\eta$ for $A = 100$, $\kappa = 0.8$, $\varepsilon_1 = 0.2$, $\varepsilon_2 = 0.399$, and $r = 0.5$. We see from this figure that the mean of the photon number difference is positive. This indicates that the mean photon number of mode $a$ is greater than that of mode $b$. Moreover, we observe in general that the mean of the photon number difference decreases as $\eta$ increases.

We now proceed to consider some special cases. We first consider the case in which the parametric amplifier, the driving coherent light, and the squeezed vacuum reservoir are absent. Thus upon setting $\varepsilon_1 = \varepsilon_2 = r = 0$ in Eq. (3.286), we get

$$\overline{n}_{\pm} = \frac{2A(1-\eta)(2\kappa + A\eta) + (1\pm 1)A^2(1-\eta^2)}{4[(\kappa + A\eta)](2\kappa + A\eta)}. \qquad (3.287)$$

This is the mean of the photon number sum and difference for the cavity modes produced by a nondegenerate three-level laser coupled to a vacuum reservoir. We see from Eq. (3.287) that the mean of the photon number difference is positive. This also shows that the mean photon number of mode $a$ is greater than that of mode $b$. We next consider the case in which atoms are not injected into the cavity. Hence upon setting $A = 0$ in Eq. (3.286), we find

$$\overline{n}_{\pm} = (1\pm 1)\left[\frac{2\varepsilon_2^2 + \kappa^2 N + 2\varepsilon_2\kappa M}{\kappa^2 - 4\varepsilon_2^2} + \frac{4\kappa^2\varepsilon_1^2}{(\kappa^2 - 4\varepsilon_2^2)^2}\right]. \qquad (3.288)$$

This represents the mean of the photon number sum and difference of the cavity modes for a nondegenerate parametric oscillator driven by coherent light and coupled to a squeezed vacuum reservoir. We see from Eq. (3.288) that the mean of the photon number difference is zero. We observe from these two special cases that the mean photon number of mode $a$ is greater than that of mode $b$ due to the three-level laser. And the increase in the mean photon number of mode $a$ must be due to the decay of some atoms from the intermediate level to levels other than level $c$ spontaneously.

We next proceed to calculate the variances of the photon number sum and difference of mode $a$ and mode $b$. The variances of the photon number sum and difference defined by

$$\Delta n_{\pm}^2 = \langle(\hat{a}^\dagger\hat{a} \pm \hat{b}^\dagger\hat{b})^2\rangle - \langle\hat{a}^\dagger\hat{a} \pm \hat{b}^\dagger\hat{b}\rangle^2 \qquad (3.289)$$

can be expressed as

$$\Delta n_{\pm}^2 = \Delta n_a^2 + \Delta n_b^2 \pm 2n_{ab}, \qquad (3.290)$$

in which

$$\Delta n_a^2 = \langle(\hat{a}^\dagger\hat{a})^2\rangle - \overline{n}_a^2 \qquad (3.291)$$

is the photon number variance of mode $a$,

$$\Delta n_b^2 = \langle(\hat{b}^\dagger\hat{b})^2\rangle - \overline{n}_b^2 \qquad (3.292)$$

is the photon number variance of mode $b$, and

$$n_{ab} = \langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle - \overline{n}_a \overline{n}_b. \tag{3.293}$$

with $\overline{n}_a = \langle \hat{a}^\dagger \hat{a} \rangle$ and $\overline{n}_b = \langle \hat{b}^\dagger \hat{b} \rangle$. Using the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, we can write

$$\Delta n_a^2 = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - \overline{n}_a^2 - 3\overline{n}_a - 2. \tag{3.294}$$

The first term on the right side of Eq. (3.294) can be expressed in terms of the Q function as

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \int d\alpha^2 d\beta^2 Q(\alpha, \beta, t) \alpha^{*2} \alpha^2. \tag{3.295}$$

On account of Eq. (3.270), we have

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{u_\alpha u_\beta - v^* v}{\pi^2} e^{-\varepsilon_{11} p^* - \varepsilon_{12} q} \int d^2\alpha \; d^2\beta \; \alpha^2 \alpha^{*2} e^{-u_\beta |\alpha|^2 + \alpha^*(p + v\beta^*) + \alpha(p^* + v^*\beta) - u_\alpha |\beta|^2 + \beta^* q + \beta q^*} \tag{3.296}$$

or

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{u_\alpha u_\beta - v^* v}{\pi^2} e^{-\varepsilon_{11} p^* - \varepsilon_{12} q} \frac{\partial^4}{\partial p^2 \partial p^{*2}} \int d^2\alpha \; d^2\beta \; e^{-u_\beta |\alpha|^2 + \alpha^*(p + v\beta^*) + \alpha(p^* + v^*\beta) - u_\alpha |\beta|^2 + \beta^* q + \beta q^*}. \tag{3.297}$$

Hence carrying out the integration, we get

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = e^{-\varepsilon_{11} p^* - \varepsilon_{12} q} \frac{\partial^4}{\partial p^2 \partial p^{*2}} \exp \left\{ \frac{u_\alpha p^* p + u_\beta q^* q + v^* p q + v p^* q^*}{u_\alpha u_\beta - v^* v} \right\}. \tag{3.298}$$

Then performing the differentiation, we find

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{2u_\alpha^2}{(u_\alpha u_\beta - v^* v)^2} + \frac{4u_\alpha}{u_\alpha u_\beta - v^* v} \left| \frac{u_\alpha p + v q^*}{u_\alpha u_\beta - v^* v} \right|^2 + \left| \frac{(u_\alpha p + v q^*)^2}{(u_\alpha u_\beta - v^* v)^2} \right|^2. \tag{3.299}$$

With the aid of Eqs. (3.274), (3.275), and (3.282), we can write as

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = 2(\overline{n}_a + 1)^2 - |\varepsilon_{11}|^4. \tag{3.300}$$

Therefore, substitution of Eq. (3.300) into Eq. (3.294) yields

$$\Delta n_a^2 = \overline{n}_a^2 + \overline{n}_a - |\varepsilon_{11}|^4. \tag{3.301}$$

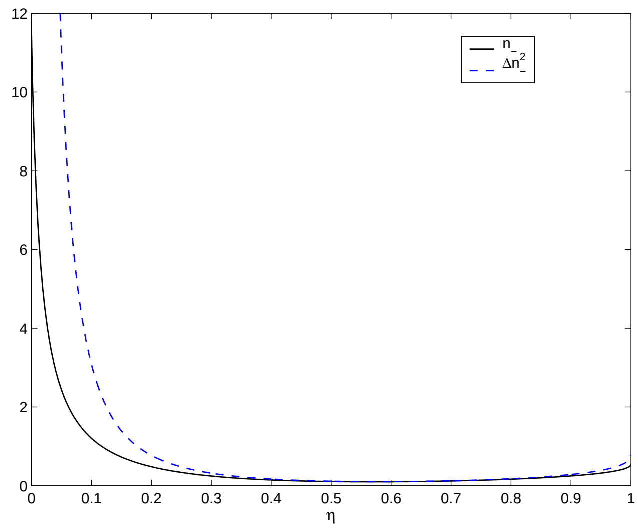


*Fig. 3.10:* Plots of the mean of the photon number difference [Eq .(3.286)] versus $\eta$ (solid curve) and the variance of the photon number difference [Eq .(3.304)] versus $\eta$ (dashed curve) for $A = 100$, $\kappa = 0.8$, $\varepsilon_2 = 0.399$, $\varepsilon_1 = 0.2$, and $r = 0.5$.

Following the same procedure, we easily obtain

$$\Delta n_b^2 = \overline{n}_b^2 + \overline{n}_b - |\varepsilon_{12}|^4 \tag{3.302}$$

and

$$n_{ab} = |b + \varepsilon_{11}\varepsilon_{12}^*|^2 - |\varepsilon_{11}|^2|\varepsilon_{12}^*|^2. \tag{3.303}$$

Hence combination of Eqs. (3.284), (3.295), (3.296), and (3.297) results in

$$\Delta n_\pm^2 = \overline{n}_a^2 + \overline{n}_a + \overline{n}_b^2 + \overline{n}_b \pm 2|b + \varepsilon_{11}\varepsilon_{12}^*|^2 - (|\varepsilon_{11}|^2 \pm |\varepsilon_{12}^*|^2)^2. \tag{3.304}$$

Fig 3.10 represents the mean and variance of the photon number difference versus $\eta$ for $A = 100$, $\kappa = 0.8$, $r = 0.5$, $\varepsilon_1 = 0.2$, and $\varepsilon_2 = 0.399$. We observe from the figure that the variance of the photon number difference is greater than the mean of the photon number difference.

### The photon number distribution

In this section we wish to obtain the joint probability for observing m photons of mode $a$ and n photons of mode $b$. The photon number distribution for a two-mode light can be expressed in terms of the Q function as [28, 33]

$$P(m,n,t) = \frac{\pi^2}{m!n!}\frac{\partial^{2m+2n}}{\partial\alpha^{*m}\partial\alpha^{m}\partial\beta^{*n}\partial\beta^{n}}Q(\alpha,\beta,t)e^{\alpha^*\alpha+\beta^*\beta}\bigg|_{\alpha=\beta=0}. \tag{3.305}$$

Employing Eq (3.270), we see that

$$\begin{aligned}P(m,n,t) = &\frac{u_\alpha u_\beta - v^*v}{m!n!}e^{-\varepsilon_{11}p^*-\varepsilon_{12}q}\\ &\frac{\partial^{2m+2n}}{\partial\alpha^{*m}\partial\alpha^{m}\partial\beta^{*n}\partial\beta^{n}}\exp\bigg\{(1-u_\beta)\alpha^*\alpha+(1-u_\alpha)\beta^*\beta+v\beta^*\alpha^*+v^*\beta\alpha\\ &+\alpha^*p+\alpha p^*+\beta^*q+\beta q^*\bigg\}\bigg|_{\alpha=\beta=0}.\end{aligned} \tag{3.306}$$

Expanding in power series, we have

$$\begin{aligned}P(m,n,t) = &\frac{u_\alpha u_\beta - v^*v}{m!n!}e^{-\varepsilon_{11}p^*-\varepsilon_{12}q}\sum_{k,l,i,j,r,s,t,w}\frac{(1-u_\beta)^k}{k!}\frac{(1-u_\alpha)^l}{l!}\frac{v^i}{i!}\frac{v^{*j}}{j!}\frac{p^r}{r!}\frac{p^{*s}}{s!}\frac{q^t}{t}\frac{q^{*w}}{w!}\\ &\frac{\partial^m\alpha^{*k+i+r}}{\partial\alpha^{*m}}\frac{\partial^m\alpha^{k+j+s}}{\partial\alpha^{m}}\frac{\partial^n\beta^{*l+i+t}}{\partial\beta^{*n}}\frac{\partial^n\beta^{l+j+w}}{\partial\beta^{n}}\bigg|_{\alpha=\beta=0}.\end{aligned} \tag{3.307}$$

It then follows that

$$\begin{aligned}P(m,n,t) = (u_\alpha u_\beta - v^*v)m!n!e^{-\varepsilon_{11}p^*-\varepsilon_{12}q}\sum_{k,l,i,j,r,s,t,w}\frac{(1-u_\beta)^k}{k!}\frac{(1-u_\alpha)^l}{l!}\frac{v^i}{i!}\frac{v^{*j}}{j!}\frac{p^r}{r!}\frac{p^{*s}}{s!}\frac{q^t}{t!}\frac{q^{*w}}{w!}\\ \delta(k+i+r,m)\delta(k+j+s,m)\delta(l+i+t,n)\delta(l+j+w,n).\end{aligned} \tag{3.308}$$

Hence applying the properties of the Kronecker delta, the joint probability for observing $m$ photons of mode $a$ and $n$ photons of mode $b$ can be put in the form

$$\begin{aligned}P(m,n,t) =&(u_\alpha u_\beta - v^*v)m!n!e^{-\varepsilon_{11}p^*-\varepsilon_{12}q}\sum_{k,l}^{m,n}\frac{(1-u_\beta)^k}{k!}\frac{(1-u_\alpha)^l}{l!}\\ &\times\bigg|\sum_{i}^{(m-k,n-l)_{min}}\frac{v^i}{i!}\frac{p^{m-k-i}}{(m-k-i)!}\frac{q^{n-l-i}}{(n-l-i)!}\bigg|^2.\end{aligned} \tag{3.309}$$

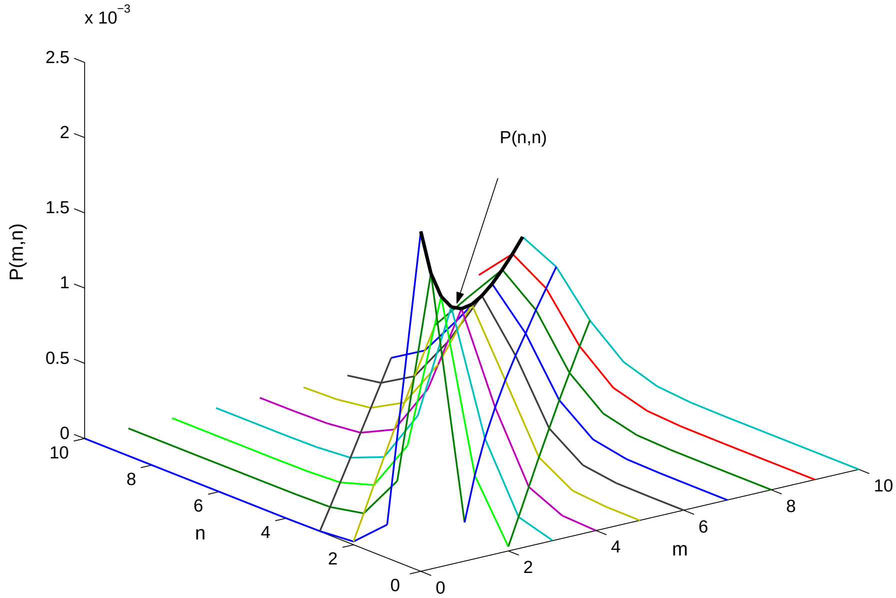


*Fig. 3.11:* A plot of the photon number distribution [Eq. (3.309)] versus m and n for $A = 100$, $\kappa = 0.8$, $\varepsilon_1 = 0.2$, $\varepsilon_2 = 0.339$, $\eta = 0.01$, and $r = 0.5$.

In Fig 3.11 we plot the probability to observe m photons of mode $a$ and n photons of mode $b$ versus m and n for $A = 100$, $\kappa = 0.8$, $\varepsilon_1 = 0.2$, $\varepsilon_2 = 0.339$, $\eta = 0.01$, and $r = 0.5$. The peak in the figure represents the joint probability to observe equal number of photons of mode $a$ and mode $b$. From this figure we see that the joint probability to observe $m$ photons of mode $a$ and $n$ photons of mode $b$ decreases as the difference between m and n increases.

Upon setting $m = n$ in Eq. (3.309), we find

$$P(n,n,t) = (u_\alpha u_\beta - v^* v)(n!)^2 e^{-\varepsilon_{11}p^* - \varepsilon_{12}q} \sum_{k,l}^{n} \frac{(1-u_\beta)^k}{k!} \frac{(1-u_\alpha)^l}{l!} \times \left| \sum_{i}^{(n-k,n-l)_{min}} \frac{v^i}{i!} \frac{p^{n-k-i}}{(n-k-i)!} \frac{q^{n-l-i}}{(n-l-i)!} \right|^2. \tag{3.310}$$

This represents the joint probability to observe equal number of photons of mode $a$ and mode $b$. Fig 3.12 is a plot of the probability to observe $n$ photons of mode $a$ and $n$ photons of mode $b$ versus $n$ for $A = 100$, $\kappa = 0.8$, $\varepsilon_1 = 0.2$, $\varepsilon_2 = 0.339$, $\eta = 0.01$, and $r = 0.5$. We observe from this figure that the joint probability to observe equal number of photons of mode $a$ and mode $b$ decreases as the number of photons increases.

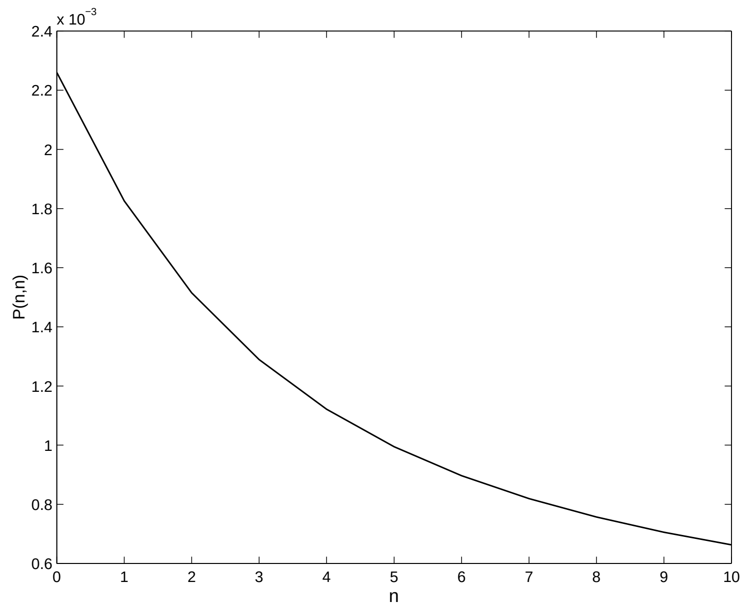


*Fig. 3.12:* A plot of the photon number distribution [Eq. (3.310)] versus n for $A = 100$, $\kappa = 0.8$, $\varepsilon_1 = 0.2$, $\varepsilon_2 = 0.339$, $\eta = 0.01$, and $r = 0.5$.

### 3.3.2 Mean and variances of the photon count sum and difference

In this section we wish to calculate the mean and the normally-ordered variances of the photon count sum and difference for the output modes. We define the moment generating function for a two-mode light by [34]

$$M(\lambda_1, \lambda_2) = \sum_{k,l=0}^{\infty} P(k,l)(1-\lambda_1)^k(1-\lambda_2)^l, \tag{3.311}$$

where $P(k,l)$ is the photon count distribution for a two-mode light. The mean of the photon count sum and difference, defined by

$$\overline{m}_{\pm} = \sum_{k,l=0}^{\infty} (k \pm l)P(k,l), \tag{3.312}$$

can be expressed in terms of the moment generating function as

$$\overline{m}_{\pm} = [(\lambda_1 - 1)\frac{d}{d\lambda_1} \pm (\lambda_2 - 1)\frac{d}{d\lambda_2}]M(\lambda_1, \lambda_2)\bigg|_{\lambda_1=\lambda_2=0}. \tag{3.313}$$

Furthermore, the photon count distribution can be expressed in terms of the photon number distribution for the output modes [26, 34] as

$$P(k,l) = \sum_{m=k,n=l}^{\infty} \frac{m!}{k!(m-k)!}\frac{n!}{l!(n-l)!}P_{out}(m,n)\nu^k(1-\nu)^{m-k}\nu^l(1-\nu)^{n-l}, \tag{3.314}$$

in which $\nu$, considered to be the same for the two modes, is the probability for detecting a single photon . Upon substituting Eq. (3.314) into Eq. (3.311), we get

$$M(\lambda_1,\lambda_2)=\sum_{m,n=0}^{\infty}P_{out}(m,n)\sum_{m=k}^{\infty}\frac{m!}{k!(m-k)!}[\nu(1-\lambda_1)]^k(1-\nu)^{m-k}$$
$$\times\sum_{n=l}^{\infty}\frac{n!}{l!(n-l)!}[\nu(1-\lambda_2)]^l(1-\nu)^{n-l}. \tag{3.315}$$

Inverting the order of the summations, we have

$$M(\lambda_1,\lambda_2)=\sum_{m,n=0}^{\infty}P_{out}(m,n)\sum_{k=0}^{m}\frac{m!}{k!(m-k)!}[\nu(1-\lambda_1)]^k(1-\nu)^{m-k}$$
$$\times\sum_{l=0}^{n}\frac{n!}{l!(n-l)!}[\nu(1-\lambda_2)]^l(1-\nu)^{n-l}, \tag{3.316}$$

so that applying the binomial theorem, the moment generating function can be written as

$$M(\lambda_1,\lambda_2)=\sum_{m,n=0}^{\infty}P_{out}(m,n)(1-\nu\lambda_1)^m(1-\nu\lambda_2)^n. \tag{3.317}$$

Using Eq. (3.317) in Eq. (3.313), we see that

$$\overline{m}_{\pm}=\sum_{m,n=0}^{\infty}P_{out}(m,n)[(\lambda_1-1)\frac{d}{d\lambda_1}\pm(\lambda_2-1)\frac{d}{d\lambda_2}](1-\nu\lambda_1)^m(1-\nu\lambda_2)^n\Big|_{\lambda_1=\lambda_2=0}. \tag{3.318}$$

Hence on carrying out the differentiation and then setting $\lambda_1=0$ and $\lambda_2=0$, we find

$$\overline{m}_{\pm}=\nu\overline{n}_{out\pm}, \tag{3.319}$$

where $\overline{n}_{out\pm}$ is the mean of the photon number sum and difference for the output modes. Moreover, in view of Eq. (2.275), we have

$$\overline{n}_{out\pm}=\kappa\overline{n}_{\pm}+(1-\kappa)(N\pm N). \tag{3.320}$$

Hence applying Eq. (3.320) in Eq. (3.319), the mean of the photon count sum and difference for the output modes can be put in the form

$$\overline{m}_{\pm}=\nu\kappa\overline{n}_{\pm}+\nu(1-\kappa)(N\pm N). \tag{3.321}$$

The first and second terms on the right side of this equation represent the mean of the photon count sum and difference for the transmitted cavity photons and the mean of the photon count sum and difference for the reflected input photons, respectively. We see from this equation that the mean of the photon count difference for the output modes is the mean of the photon count difference of the transmitted cavity photons only.

We next seek to calculate the normally-ordered variances of the photon count sum and difference for the output modes. We note that the variances of the photon count sum and difference is given by

$$\Delta m_{\pm}^2 = \overline{m^2}_{\pm} - \overline{m}_{\pm}^2. \tag{3.322}$$

With the aid of Eq. (3.311), we can easily verify that

$$\overline{m_{\pm}^2} =[(\lambda_1 - 1)\frac{d}{d\lambda_1} \pm (\lambda_2 - 1)\frac{d}{d\lambda_2}]^2 M(\lambda_1, \lambda_2)\Big|_{\lambda_1=\lambda_2=0}. \tag{3.323}$$

Using Eq. (3.317), we have

$$\overline{m_{\pm}^2} = \sum_{m,n=0}^{\infty} P_{out}(m,n)[(\lambda_1 - 1)\frac{d}{d\lambda_1} \pm (\lambda_2 - 1)\frac{d}{d\lambda_2}]^2 (1-\nu\lambda_1)^m (1-\nu\lambda_2)^n\Big|_{\lambda_1=\lambda_2=0}, \tag{3.324}$$

so that carrying out the differentiation and applying the condition $\lambda_1 = \lambda_2 = 0$, we get

$$\overline{m_{\pm}^2} = \nu^2 \langle \hat{n}_{out\pm}^2 \rangle + \nu(1-\nu)\overline{n}_{out+}. \tag{3.325}$$

Upon substituting (3.319) and (3.325) into Eq. (3.322), we find

$$\Delta m_{\pm}^2 = \nu^2 \Delta n_{out\pm}^2 + \nu(1-\nu)\overline{n}_{out+}. \tag{3.326}$$

where

$$\Delta n_{out\pm}^2 = \langle \hat{n}_{out\pm}^2 \rangle - \overline{n}_{out\pm}^2 \tag{3.327}$$

is the variances of the photon number sum and difference for the output modes. With the aid of the commutation relations $[\hat{a}_{out}, \hat{a}_{out}^\dagger] = [\hat{b}_{out}, \hat{b}_{out}^\dagger] = 1$, we can write Eq. (3.327) as

$$\Delta n_{out\pm}^2 =: \Delta n_{out\pm}^2 : + \overline{n}_{out+}, \tag{3.328}$$

where

$$: \Delta n^2_{out\pm} : = \langle (\alpha^*_{out}\alpha_{out} \pm \beta^*_{out}\beta_{out})^2 \rangle - \overline{n}^2_{out\pm}, \tag{3.329}$$

is the normally-ordered variances of the photon number sum and difference for the output modes. With the aid Eqs. (3.326) and (3.328), we can write the normally-ordered variances of the photon count sum and difference as

$$: \Delta m^2_{\pm} : = \nu^2 : \Delta n^2_{out\pm} : . \tag{3.330}$$

We next proceed to calculate the normally-ordered variances of the photon number sum and difference for the output modes. To this end, we note that

$$: \Delta n^2_{out\pm} : = (: \Delta n^2_{out} :)_a + (: \Delta n^2_{out} :)_b \pm 2(n_{out})_{ab} \tag{3.331}$$

where

$$(: \Delta n^2_{out} :)_a = \langle \alpha^{*2}_{out}\alpha^2_{out} \rangle - \langle \alpha^*_{out}\alpha_{out} \rangle^2 \tag{3.332}$$

$$(: \Delta n^2_{out} :)_b = \langle \beta^{*2}_{out}\beta^2_{out} \rangle - \langle \beta^*_{out}\beta_{out} \rangle^2 \tag{3.333}$$

$$(n_{out})_{ab} = \langle \alpha^*_{out}\alpha_{out}\beta^*_{out}\beta_{out} \rangle - \langle \alpha^*_{out}\alpha_{out} \rangle \langle \beta^*_{out}\beta_{out} \rangle. \tag{3.334}$$

Employing the input-output relation, we can write

$$\begin{aligned} \langle \alpha^{*2}_{out}\alpha^2_{out} \rangle = {} & \kappa^2 \langle \alpha^{*2}\alpha^2 \rangle /2 + \langle \alpha^{*2}_{in}\alpha^2_{in} \rangle /2 + 2\kappa \langle \alpha^*\alpha\alpha^*_{in}\alpha_{in} \rangle + \kappa \langle \alpha^{*2}_{in}\alpha^2 \rangle \\ & - 2\kappa\sqrt{\kappa} \langle \alpha^*\alpha^*_{in}\alpha^2 \rangle - 2\sqrt{\kappa} \langle \alpha^{*2}_{in}\alpha_{in}\alpha \rangle + c.c. \end{aligned} \tag{3.335}$$

We now proceed to evaluate the expectation values involved in Eq. (3.335). To this end, Eqs. (3.155) and (3.156) can be rewritten as

$$\alpha = \alpha' + \varepsilon_{11}, \tag{3.336}$$

$$\beta^* = \beta'^* + \varepsilon_{12}, \tag{3.337}$$

in which

$$\alpha' = p_1(t)\alpha(0) + q_1(t)\beta^*(0) + G_1(t), \tag{3.338}$$

$$\beta'^{*} = p_2(t)\beta^{*}(0) + q_2(t)\alpha(0) + G_2(t). \tag{3.339}$$

Taking into account Eq. (3.336), we see that

$$\langle \alpha^{*}\alpha\alpha^{*}_{in}\alpha_{in}\rangle = \langle \alpha'^{*}\alpha'\alpha^{*}_{in}\alpha_{in}\rangle + \varepsilon^{*}_{11}\varepsilon_{11}\langle \alpha^{*}_{in}\alpha_{in}\rangle + \varepsilon_{11}\langle \alpha'^{*}\alpha^{*}_{in}\alpha_{in}\rangle + \varepsilon^{*}_{11}\langle \alpha^{*}_{in}\alpha'\alpha_{in}\rangle. \tag{3.340}$$

We recall that $\alpha'$ and $\alpha_{in}$ are Gaussian variables with zero mean. Hence Eq. (3.340) can be put in the form

$$\langle \alpha^{*}\alpha\alpha^{*}_{in}\alpha_{in}\rangle = \langle \alpha'^{*}\alpha'\rangle\langle \alpha^{*}_{in}\alpha_{in}\rangle + \langle \alpha'^{*}\alpha^{*}_{in}\rangle\langle \alpha'\alpha_{in}\rangle + \langle \alpha'^{*}\alpha_{in}\rangle\langle \alpha'\alpha^{*}_{in}\rangle + \varepsilon^{*}_{11}\varepsilon_{11}\langle \alpha^{*}_{in}\alpha_{in}\rangle. \tag{3.341}$$

With the aid of Eqs. (3.338), (3.339), (3.202), (3.203), (3.197), (3.198), (3.199), and the relations

$$\alpha_{in} = \frac{1}{\sqrt{\kappa}} f_{\alpha r}, \qquad \beta_{in} = \frac{1}{\sqrt{\kappa}} f_{\beta r}, \tag{3.342}$$

we can easily verify that

$$\langle \alpha'\alpha_{in}\rangle = \langle \beta'\beta_{in}\rangle = \langle \alpha'^{*}\beta_{in}\rangle = \langle \beta'^{*}\alpha_{in}\rangle = 0, \tag{3.343}$$

$$\langle \alpha'^{*}\alpha_{in}\rangle = \langle \beta'^{*}\beta_{in}\rangle = \sqrt{\kappa}N/2, \tag{3.344}$$

$$\langle \alpha'\beta_{in}\rangle = \langle \beta'\alpha_{in}\rangle = \sqrt{\kappa}M/2. \tag{3.345}$$

Moreover, we note that for a two-mode squeezed vacuum reservoir

$$\langle \alpha^{2}_{in}\rangle = \langle \beta^{2}_{in}\rangle = \langle \alpha^{*}_{in}\beta_{in}\rangle = \langle \beta^{*}_{in}\alpha_{in}\rangle = 0, \tag{3.346}$$

$$\langle \alpha^{*}_{in}\alpha_{in}\rangle = \langle \beta^{*}_{in}\beta_{in}\rangle = N, \tag{3.347}$$

$$\langle \alpha_{in}\beta_{in}\rangle = \langle \beta_{in}\alpha_{in}\rangle = M. \tag{3.348}$$

On account of (3.344) and (3.347), Eq. (3.341) takes the form

$$\langle \alpha^{*}\alpha\alpha^{*}_{in}\alpha_{in}\rangle = \overline{n}_a N + \kappa N^2/4. \tag{3.349}$$

where

$$\overline{n}_a = \langle \alpha'^* \alpha' \rangle + \varepsilon_{11}^* \varepsilon_{11} \tag{3.350}$$

is the mean photon number of mode $a$.

Furthermore, taking into account Eq. (3.335) along with the fact that $\alpha'$ and $\alpha_{in}$ are Gaussian variables with zero mean, we can write

$$\langle \alpha_{in}^{*2} \alpha^2 \rangle = \langle \alpha_{in}^{*2} \rangle \langle \alpha'^2 \rangle + 2\langle \alpha_{in}^* \alpha' \rangle^2 + \varepsilon_{11}^2 \langle \alpha_{in}^{*2} \rangle. \tag{3.351}$$

In view of Eqs. (3.344) and (3.346), we have

$$\langle \alpha_{in}^{*2} \alpha^2 \rangle = \kappa N^2/2. \tag{3.352}$$

Following a similar procedure, we easily obtain

$$\langle \alpha^* \alpha_{in}^* \alpha^2 \rangle = \sqrt{\kappa}\overline{n}_a N \tag{3.353}$$

and

$$\langle \alpha^* \alpha_{in}^* \alpha_{in}^2 \rangle = \sqrt{\kappa} N^2. \tag{3.354}$$

Upon substituting Eqs. (3.349), (3.352), (3.353), and (3.354) into Eq. (3.335), we get

$$\langle \alpha_{out}^{*2} \alpha_{out}^2 \rangle = \kappa^2 \langle \alpha^{*2} \alpha^2 \rangle + \langle \alpha_{in}^{*2} \alpha_{in}^2 \rangle + 4\kappa(1-\kappa)\overline{n}_a N + 2(\kappa^2 - 2\kappa) N^2. \tag{3.355}$$

Moreover, on account of Eq. (2.275), we see that

$$\langle \alpha_{out}^* \alpha_{out} \rangle = \overline{n}_a + (1-\kappa) N, \tag{3.356}$$

so that in view of (3.355) and (3.356), one can express (3.332) as

$$(: \Delta n_{out}^2 :)_a = \kappa^2 : \Delta n_a^2 : + (1-\kappa)^2 N^2 + 2\kappa(1-\kappa)\overline{n}_a N, \tag{3.357}$$

where

$$: \Delta n_a^2 : = \langle \alpha^{*2} \alpha^2 \rangle - \overline{n}_a^2 \tag{3.358}$$

is the normally-ordered photon number variance for mode $a$. Following the same proce-

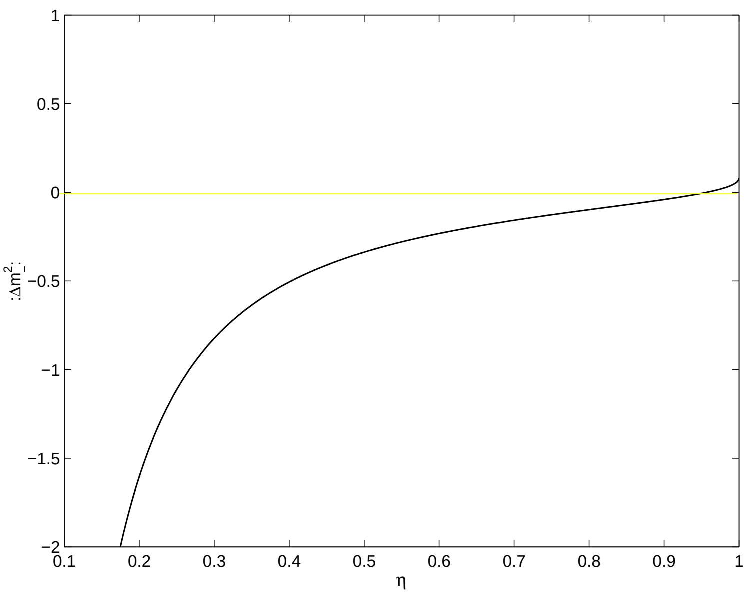


*Fig. 3.13:* A plot of the variance of the normally-ordered photon count difference [Eq. (3.367)] versus $\eta$ for $A = 100$, $\kappa = 0.8$, $\varepsilon_1 = 0.2$, $\varepsilon_2 = 0.399$, $r = 0.5$, and $\nu = 0.6$.

dure, we find

$$(:\Delta n_{out}^2:)_b = \kappa^2 : \Delta n_b^2 : +(1-\kappa)^2 N^2 + 2\kappa(1-\kappa)\overline{n}_b N, \tag{3.359}$$

where

$$:\Delta n_b^2: = \langle \beta^{*2}\beta^2\rangle - \overline{n}_b^2 \tag{3.360}$$

is the normally-ordered photon number variance for mode $b$ and

$$(n_{out}^2)_{ab} = \kappa^2 n_{ab} + (1-\kappa)^2 M^2 + 2\kappa(1-\kappa)M'M, \tag{3.361}$$

with

$$n_{ab} = \langle \alpha^*\alpha\beta^*\beta\rangle - \overline{n}_a\overline{n}_b \tag{3.362}$$

and

$$M' = \langle \alpha'\beta'\rangle + \varepsilon_{11}\varepsilon_{12}. \tag{3.363}$$

Therefore, on account of Eq. (3.357), (3.359), and (3.361), we can write Eq. (3.331) as

$$:\Delta n_{out\pm}^2: = \kappa^2 : \Delta n_{\pm}^2 : +(1-\kappa)^2 : \Delta n_{in\pm}^2 : + \, 2\kappa(1-\kappa)(\overline{n}_+ N \pm 2M'M), \tag{3.364}$$

where

$$:\Delta n_{\pm}^2 :=: \Delta n_a^2 : + : \Delta n_b^2 : \pm 2n_{ab} \tag{3.365}$$

and

$$:\Delta n_{in\pm}^2 := 2(N^2 \pm M^2) \tag{3.366}$$

are the normally-ordered variances of the photon number sum and difference for the cavity and the input modes. On substituting Eq. (3.364) into Eq. (3.330), the normally-ordered variances of the photon count sum and difference for the output mode is found to be

$$:\Delta m_{\pm}^2 : = \nu^2\kappa^2 : \Delta n_{\pm}^2 : + \nu^2(1-\kappa)^2 : \Delta n_{in\pm}^2 : + 2\kappa(1-\kappa)\nu^2(\overline{n}_+ N \pm 2M'M). \tag{3.367}$$

Fig 2.13 represents the variance of the normally-ordered photon count difference versus $\eta$ for $A = 100$, $\kappa = 0.8$, $\varepsilon_1 = 0.2$, $\varepsilon_2 = 0.399$, $r = 0.5$, and $\nu = 0.6$. We observe from the figure that the variance of the photon count difference is less than the mean of the photon count sum for $\eta$ between 0 and 0.95.

# 4

# Conclusion

In this dissertation we have studied the squeezing and statistical properties of the cavity and output modes produced by a degenerate as well as a nondegenerate three-level laser whose cavity contains a parametric amplifier and with the cavity modes driven by coherent light and coupled to a squeezed vacuum reservoir. We have obtained, using the master equation, stochastic differential equations associated with the normal ordering. Applying the solutions of the resulting differential equations, we have calculated the quadrature variances and squeezing spectrum. The light produced by the degenerate as well as the nondegenerate system is in a squeezed state. It is found that the parametric amplifier and the squeezed vacuum reservoir increase the degree of squeezing, but the driving coherent light does not have any effect on the squeezing. Moreover, the squeezing of the cavity modes is greater than that of the output modes for some values of $\eta$. We have also seen that the degree of squeezing increases with the linear gain coefficient for small values of $\eta$ and almost perfect squeezing can be obtained for large values of the linear gain coefficient. In addition, the squeezing spectrum of the output mode for the degenerate case shows that at threshold there is perfect squeezing for $\eta = 0$, $\omega = \omega_0$, and for any values of the linear gain coefficient, the cavity damping constant, and the squeeze parameter.

We have determined employing the Q function the mean photon number, the variance of the photon number, and the photon number distribution for the cavity modes. The mean photon number increases considerably due to the driving coherent light, squeezed vacuum reservoir, and parametric amplifier.

For the degenerate case, we have seen that the probability of observing even number of photons is greater than the probability of observing odd number of photons.

We have found that the mean photon number of mode $a$ is greater than that of mode $b$. This could be due to the spontaneous decay of some atoms from the intermediate level to levels other than the lower level. We have observed that the joint probability to observe $m$ photons of mode $a$ and $n$ photons of mode $b$ decreases as the difference between $m$ and $n$ increases. Finally, we have calculated the mean and normally-ordered variance of the photon count for the output mode. These result show that the normally-ordered variance of the photon count is greater than the mean of the photon count. For the nondegenerate case we have seen that the normally-ordered variance of the photon count difference is less than the mean of the photon count sum for $\eta$ between 0 and 0.95.

Since the effect of the parametric amplifier, the driving coherent light, and the squeezed vacuum reservoir on the three-level laser is to enhance both the degree of squeezing and the mean photon number, a bright and highly squeezed light can be produced by the quantum optical system considered in this dissertation.

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