Euler's Method and Error Analysis

By

Tadfie Denekew Amogne

Advisor: Dr. Addisalem Abathun

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Tadfie Denekew Amogne

Addis Ababa , Ethiopia.

Declaration

I, Tadfie Denekew, with student identification number $GSK/0132/06$, hereby declare that this project is my own work and that it has not been previously submitted for assessment or completion of any post graduate qualification to another university or for another qualification.

________________________________ Date __________

Tadfie Denekew

Certificate

I hereby certify that I have read this project prepared by Tadfie Denekew under my supervision and recommended that , it should be accepted as fulfilling the project requirement.

______________________________ Date __________

Addisalem Abathun

Abstract

In this paper we introduce the notion of Euler's method and error analysis. And we obtain various conditions for stability analysis and it has been shown that the Euler's method is convergent and consistent. The method is clarified with the help of numerical illustration and graphical analysis.

keywords: Euler's method , Error Analysis

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Chapter 1

Introduction

Differential equations are used to model problems that involve the change of some variable with respect to another. These problems require the solution to an initial-value problem that is, the solution to a differential equation that satisfied a given initial condition.([1])

In many real-life situations, the differential equation that models the problem is too complicated to solve exactly, and one of two approaches is taken to approximate mate the solution. The first approach is to simplify the differential equation to one that can be solved exactly, and then use the solution of the simplified equation to approximate the solution to the original equation. The other approach, the one we examine in this paper, involves finding methods for directly approximating the solution of the original problem. This is the approach commonly taken since more accurate results and realistic error information can be obtained.

The methods we consider in this paper is the Euler's method. It is a numerical technique to solve ordinary differential equations of the form

$$dy/dx = f(x, y), \qquad y(0) = y_0$$

This work also includes error analysis of this method.

1.1 Preliminaries

1.1.1 Ordinary Differential Equations

It will be useful to review some elementary definitions and concepts from the theory of differential equations. An equation involving a relation between the values of an unknown function and one or more of its derivatives is called **a differential equation** . We shall always assume that the equation can be solved explicitly for the derivative of highest order. An ordinary differential equation of order n will then have the form

$$y^{(n)}(x) = f(x, y(x), y^{'}(x), ..., y^{(n-1)}(x)) \qquad (1.1)$$

By a solution of the above equation we mean a function $\phi(x)$ which is n times continuously differentiable on a prescribed interval and which satisfies the given equation; that is $\phi(x)$ must satisfy $\phi^{(n)}(x) = f(x, \phi(x), \phi^{'}(x), ..., \phi^{(n-1)}(x))$ The general solution of (1.1) will normally contain n arbitrary constants,and hence there exists an n-parameter family of solutions. If $y(x_0), y^{'}(x_0).....y^{n-1}(x_0)$ are prescribed at one point $x = x_0$, we have an initial-value problem. A simple example of a first-order equation is $y^{'} = y$. Its general solution is $y(x) = Ce^x$, where C is an arbitrary constant. If the initial condition $y(x_0) = y_0$ is prescribed, the solution can be written $y(x) = y_0e^{x-x_0}$.Differential equations are further classified as linear and nonlinear. An equation is said to be linear if the function f in (1.1) involves y and its derivatives linearly. Linear differential equations possess the important property that if $y_1(x), y_2(x).....y_m(x)$ are any solutions of (1.1), then so is $C_1y_1(x)+C_2y_2(x)+$

$... + C_m y_m(x)$ for arbitrary constants C_i. A simple second-order equation is $y^{"} = y$. It is easily verified that e^x and e^{-x} are solutions of this equation, and hence by linearity the following sum is also a solution:

$$y(x) = C_1 e^x + C_2 e^{-x}$$

Two solutions y_1, y_2 of a second-order linear differential equation are said to be linearly independent if the Wronskian of the solution does not vanish, the Wronskian being defined by

$$W(y_1, y_2) = y_l y_2 - y_2 y_l.$$

The concept of linear independence can be extended to the solutions of equations of higher order. If $y_1(x), y_2(x)...., y_m(x)$ are n linearly independent solutions of a homogeneous differential equation of order n, then

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + ... + C_m y_m(x)$$

is called the general solution.

Among linear equations, those with constant coefficients are particularly useful since they lend themselves to a simple treatment. We write the n^{th}-order linear differential equation with constant coefficients in the form

$$L_y = y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_0 y^{(0)} = 0 \qquad (1.2)$$

where the a_i are assumed to be real. If we seek solutions of (1.2) in the form $e^{\beta(x)}$, then direct substitution shows that β must satisfy the polynomial equation

$$\beta^n + a_{n-1}\beta^{n-1} + ... + a_0 \qquad (1.3)$$

This is called the characteristic equation of the n^{th}-order differential equation (1.2). If the equation (1.3) has n distinct roots β_i $i = 1, ..., n$,then it can be shown that

$y(x) = C_1 e^{\beta_1}(x) + C_2 e^{\beta_2(x)} + ... + C_n e^{\beta_n(x)}$ where the C_i are arbitrary constants, is the general solution of (1.2). If $\beta_1 = \alpha + i\beta$ is a complex root of (1.3), so is its conjugate, $\beta_1 = \alpha - i\beta$. Corresponding to such a pair of conjugate-complex roots are two solutions $y_1 = e^{\alpha x}\cos\beta x$ and $y_2 = e^{\alpha x}\sin\beta x$, which are linearly independent solutions.When (1.3) has multiple roots, special techniques are available for obtaining linearly independent solutions. In particular, if β_1 is a double root of (1.3), then $y_1 = e^{\beta x}$ and $y_2 = xe^{\beta x}$ are linearly independent solutions of (1.2)([9]).

1.1.2 Taylor Series Method

Definition 1.1.1. *If a function has derivatives of all orders at $x = c$ then the series*

$$\sum_{n=0} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f^{'}(c)(x-c) + ... + \frac{f^{(n)}(c)}{n!}(x-c)^n + ...$$

is called the ***Taylor series*** *for $f(x)$ at c Moreover, if $c = 0$,then the series is the* ***Maclaurin series*** *for f.([2])*

Taylor series method is the fundamental numerical method for the solution of the initial value problem

$$y^{'} = f(x,y), \quad y(x_0) = y_0$$

Expanding $y(x)$ in Taylor series about any point x_i ,we obtain
$y(x_{i+1}) = y(x_i) + \frac{f(x_i,y_i)h}{1!} + \frac{f^{'}(x_i,y_i)h^2}{2!} + ...$
where h is the step size, i.e $h = x_i - x_{i-1}$

Theorem 1.1.2. *If $f \in C^n[a,b]$ and if $f^{(n+1)}$ exists on (a,b), then for any points c and x in [a, b] there exists a point ξ between c and x such that*

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(x-c)^k + E_n(x)$$

where ,

$$E_n(x) = \frac{1}{n+1!} f^{(n+1)}(\xi)(x-c)^{n+1}$$

The special case $n = 1$ of Taylor's Theorem is often used in mathematical arguments. It is known as the **Mean-Value theorem**. It states that if f is in C[a, b] and if $f^{'}$ exists on (a, b), then for x and c in [a, b]

$$f(x) = f(c) + f^{'}(\xi)(x-c)$$

where ξ is between c and x. Taking $x = b$ and $c = a$, and rearranging, we have the important equation $f(b) - f(a) = f^{'}(\xi)(b-a)$ where $a \leq \xi \leq b$. A special case of the Mean-Value Theorem is **Rolle's Theorem**. It states that if f is continuous on [a.b], if $f^{'}$ exists on (a,b), and if $f(a) = f(b) = 0$, then $f^{'}(\xi) = 0$ for some ξ in (a,b). This is an immediate consequence of the preceding equation. (Actually, in a formal development, Rolle's Theorem is proved first, and from it, Taylor's Theorem is derived) .In both Rolle's Theorem and the Mean-Value Theorem, there may be more than one point η in the interval [a, b] that satisfies the given equation.

1.1.3 Types Of Errors

The numerical optimists ask how accurate the computed results; the numerical pessimists asks how much error has been introduced.The two questions, are of course, one and the same.Only rarely will the given data be exact,since it often originates in measurement process. So their is probably error in the input information.

If the number x^* is an approximation to the exact answer x ,then we call the difference $x - x^*$ the error in x^*; thus
$Exact = approximation + error$

Round -off Error

The error that is produced when a calculator or computer is used to perform real-number calculations is called **round-off error**. It occurs because the arithmetic performed in a machine involves numbers with only a finite number of digits, with the result that calculations are performed with only approximate representations of the actual numbers. In a typical computer, only a relatively small subset of the real number system is used for the representation of all the real numbers.

Truncation Error

Truncation error is made when a partial sum is used to represent an infinite series. If we consider the Taylor series
$y_{i+1} = y_i + y_i^{'}h + \frac{y_i^{''}h^2}{2!} + ... + \frac{y_i^{(n)}h^{(n)}}{n!} + ...$
is exact when all terms are kept. If we truncate this series the remaining terms are the approximation of the exact series and the terms which are truncated is error terms. **Local truncation error** is the error generated in every steps and the error from all the steps is termed as **Global error**.

Inherent Error

The inherent error is that quantity which is already present in the statement of the problem before its solution. The inherent error arises either due to the simplified assumptions in the mathematical formulation of the problem or due to the errors in the physical measurements of the parameters of the problem. Inherent error can be minimized by obtaining better data, by using high precision computing aids and by correcting obvious errors in the data.

Absolute Error

Absolute error is the numerical difference between the true value of a quantity and its approximate value. Thus if x^* is the approximate value of quantity x, then $|x - x^*|$ is called the absolute error and denoted by E_a. Therefore $E_a = |x - x^*|$. The unit of exact or unit of approximate values expresses the absolute error.

Relative Error

The relative error E_r defined by $E_r = |\frac{x-x^*}{x}| = \frac{E_a}{True \quad value}$, where x^* is the approximate value of quantity x. The relative error is independent of units.

Percentage Error

The percentage error in $x^{'}$ which is the approximate value of x is given by $E_p = 100 \times E_r = 100 \times |\frac{x-x^{'}}{x}|$.The percentage error is also independent of units.

Example 1.1.3. *Find the absolute, percentage and relative errors if x is rounded-off to three decimal digits. Given $x = 0.005998$.*

Sol: *If x is rounded-off to three decimal places we get $x = 0.006$. Therefore*

$Error = True\ \ Value - Approximate\ \ Value$

$Error = 0.005998 - 0.006 = -0.000002$

$Absolute\ \ Value\, of\, error = E_a = |Error| = 0.000002$

$Relative\ \ Error = E_r = \dfrac{E_a}{True\ \ value} = \dfrac{E_a}{0.005998} = \dfrac{0.000002}{0.005998} = 0.0033344$

$Percentage\ \ Error = E_p = E_r \times 100 = 0.33344$

1.1.4 Finite Difference

Finite difference have had a strong appeal to mathematicians for centuries. Issac Newton was an especially a heavy user , and much of the subject originated with him. Given a discrete function , that is, a finite set of arguments x_k each having a mate y_k , supposing the arguments equally spaced , so that $x_k - x_{k-1} = h$, the differences of y_k values are denoted
$\triangle y_k = y_{k+1} - y_k$
are called first differences.The difference of these first differences are denoted by
$\triangle^2 y_k = \triangle(\triangle y_k) = \triangle y_{k+1} - \triangle y_k = y_{k+2} - 2y_{k+1} + y_k$ are called second differences. In general ,

$$\triangle^n y_k = \triangle^{n-1} y_{k+1} - \triangle^{n-1} y_k$$

defines the n^{th} differences. Suppose f is a differentiable real-valued function on R. Let $x \in R$ and $h > 0$ .Then we have the following three popular difference approximations:

$$\begin{aligned} f'(x) &\approx \frac{f(x+h)-f(x)}{h} \\ &\approx \frac{f(x)-f(x-h)}{h} \\ &\approx \frac{f(x+h)-f(x-h)}{2h} \end{aligned}$$

These differences are called a **forward difference ,a backward difference** and a **centered difference**, respectively. Supposing f has a second derivative, it is easy to verify that the approximation errors for the forward and backward differences are both $O(h)$. If the third derivative of f exists,then the approximation error for the centered difference is $O(h^2)$. We see that if the function is smooth, the centered difference is a more accurate approximation to the derivative([**??**])

.

1.1.5 Numerical Solution of Initial-Value Problems

Before describing the methods for approximating the solution to our basic problem, we consider some situations that ensure the solution will exist. In fact, since we will not be solving the given problem, only an approximation to the problem,we need to know when problems that are close to the given problem have solutions that accurately approximate the solution to the given problem. This property of an initial-value problem is called well-posed, and these are the problems for which numerical methods are appropriate.

Definition 1.1.4 (Well-Posed Condition)**.** *Suppose that f and f_y, its first partial derivative with respect to y, are continuous for x in [a, b] and for all y. Then the initial-value problem*

$y' = f(x,y)$, for $a \leq x \leq b$, with $y(a) = \alpha$, has a unique solution $y(x)$ in $a \leq x \leq b$, and the problem is well-posed.

Example 1.1.5. *Consider the initial-value problem $y' = 1 + x \sin xy$, for $0 \leq x \leq 2$, with $y(0) = 0$.*
Since the functions $f(x,y) = 1 + x \sin xy$ and $f_y(x,y) = x^2 \cos xy$ are both continuous for $0 \leq x \leq 2$, and for all y, a unique solution exists to this well-posed initial-value problem.

1.1.6 Single Step And Multi Step Methods

The methods for the solution of the initial value problem

$$y' = f(x,y), \qquad y(x_0) = y_0 \qquad 1.1.6.1$$

can be classified mainly into two types. They are (i) single step methods, and (ii) multi step methods. **Single step methods**: In single step methods, for solving IVP (1.1.6.1) are those methods in which the solution y_{j+1} at the $(j+1)^{th}$ grid point involves only one previous grid point where the solution is already known. Accordingly, a general one step method may be written as

$$y_{j+1} = y_j + h\phi(x_j, y_j, h)$$

The increment function ϕ depends on solution y_j at previous grid point x_j and step size h. If y_{j+1} can be determined simply by evaluating right hand side then the method is explicit method.
It may be reasonable to develop methods that use more information about the solution (functional values and derivatives) at previously known values while computing solution at the next grid point. Such methods using information at more than one previous grid points are known as **multi-step methods** and are expected to give better results than one step methods. To

determine solution y_{j+1} , a multi-step method or k-step method uses values of $y(x)$ and $f(x, y(x))$ at k previous grid points $x_{j-k},\;\; k = 0, 1, 2, \ldots k-1, .$ y_j is called the initial point while y_{j-k} are starting points. The starting points are computed using some suitable one step method. Thus multi-step methods are not self starting method ([**??**]).

1.1.7 Stability of Numerical Methods

In any initial value problem, we require solution for $x > x_0$ and usually up to a point $x = b$. The step length h for application of any numerical method for the initial value problem must be properly chosen. The computations contain mainly two types of errors: truncation error and round-off error. Truncation error is in the hand of the user. It can be controlled by choosing higher order methods. Round-off errors are not in the hands of the user. They can grow and finally destroy the true solution. In such case, we say that the method is *numerically unstable.*This happens when the step length is chosen larger than the allowed limiting value. All explicit methods have restrictions on the step length that can be used. Many implicit methods have no restriction on the step length that can be used. Such methods are called *unconditionally stable methods.*

The behavior of the solution of the given initial value problem is studied by considering the linearized form of the differential equation $y^{'} = f(x, y)$. The linearized form of the initial value problem is given by $y^{'} = \lambda y, \lambda < 0, y(x_0) = y_0$. The single step methods are applied to this differential equation to obtain the difference equation $y_{i+1} = E(\lambda h)y_i$, where $E(\lambda h)$ is called the *amplification factor.*If $|E(\lambda h)| < 1$, then all the errors (round-off and other errors) decay and the method gives convergent solutions. We say that the method is *stable*. This condition gives a bound on the step length h that can be used in the computations.

Chapter 2

Euler's Method Of Solving Differential Equations

The most popular numerical methods for solving the differential equation of the form

$$y^{'} = f(x, y), \qquad y(x_0) = y_0 \qquad (2.1)$$

Are called finite difference methods.Approximate values are obtained for the solution at a set of grid points

$$x_o < x_1 < x_2 < ... < x_n < ...$$

And the approximate value at each x_i, is obtained by using some of the values in previous steps. We begin with a simple but computationally inefficient method attributed to Leonhard Euler. The analysis of it has many of the features of the analyses of the more efficient finite difference methods, but without their additional complexity. As before, $y(x)$ will denote the true solution to (2.1):

$$y'(x) = f(x, y(x)) \qquad (2.2)$$

The approximate solution will be denoted by $y(x_j)$, and the values $y(x_0), y(x_1), ..., y(x_n), ...$ will often be denoted by

$$y_0, y_1, \dots, y_n, \cdots$$

An equal grid size $h > 0$ will be used to define the node points,

$$x_j = x_0 + jh \qquad j = 0, 1, ...$$

When we are comparing numerical solutions for various values of h, we will also use the notation $y_h(x)$ to refer to $y(x)$ with step size h. The problem (2.1) will be solved on a fixed finite interval, which will always be denoted by $[x_0, b]$.

2.1 Derivation of Euler's Method

Euler's method is defined by

$$y(x_{n+1}) = y(x_n) + hf(x_n, y(x_n), \quad y(x_0) = y_0, \quad n = 0, 1, 2, ... \qquad (2.3).$$

Different viewpoints of it are given.

1. *A geometric viewpoint*

   At $x = 0$,we are given the value of $y_0 = y(x_0)$. Let us call $x = 0$ as x_0. Now since we know the slope of y with respect to x , that is, $f(x, y)$, then at $x = x_0$, the slope is $f(x_0, y_0)$. Both x_0 and y_0 are known from the initial condition $y(x_0) = y_0$.

   From the graph let y_1 denotes $y(x_1)$ and y_0 denotes $y(x_0)$. So the slope

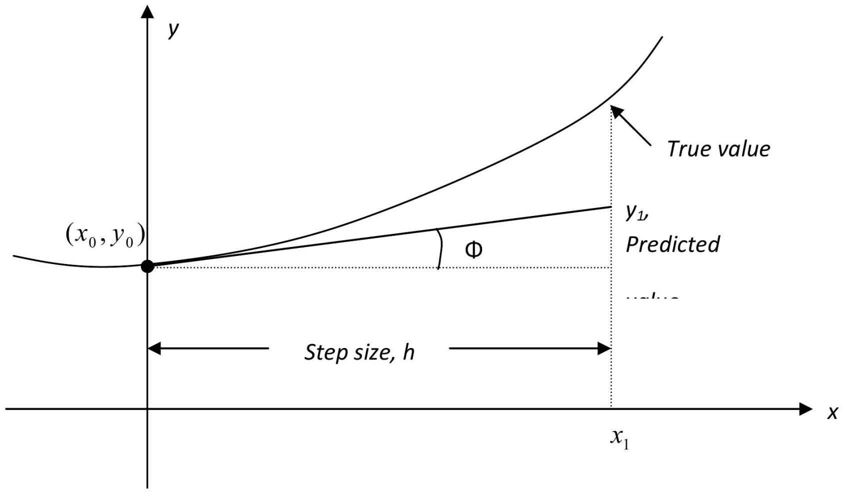


Figure 2.1: graphical representation of Euler's method

at $x = x_0$ as shown in Figure 2.1 is

$$Slope = \frac{Rise}{Run}$$
$$f(x_0, y(x_0)) = \frac{y(x_1) - y(x_0)}{x_1 - x_0}$$

From here

$$y(x_1) = y(x_0) + f(x_0, y(x_0))(x_1 - x_0)$$

calling $x_1 - x_0$ step size h , we get

$$y(x_1) = y(x_0) + f(x_0, y(x_0))h$$

One can use the value of $y(x_i)$ (an approximate value of y at $x = x_i$) to calculate $y(x_2)$, that would be the predicted value at x_2 , given by

$$y(x_2) = y(x_1) + hf(x_1, y_1)$$

$$x_2 = x_1 + h$$

Based on the above equations , if we know the value of $y = y_n$ at x_n , then

$$y(x_{n+1}) = y(x_n) + f(x_n, y_n)h$$

This formula is known as Euler's Method.

2. *Taylor Series*

   Expand

   $y(x_{n+1}) = y(x_n) + hy^{'}(x_n) + \frac{h^2}{2}y^{''}(\xi_n)$ where $x_n \leq \xi_n \leq x_{n+1}$

   By dropping the error term ,we obtain the Euler's method (2.3).The term

$$T(x_n) = \frac{h^2}{2}y^{''}(\xi_n)$$

   is called the truncation error or discretization error at x_{n+1}.

3. *Numerical Differentiation*

   From the definition of a derivative ,

$$\frac{y(x_{n+1}) - y(x_n)}{h} = y^{'}(x_n) = f(x_n, y(x_n))$$

   From this equation , we get

$$y(x_{n+1}) = y(x_n) + hf(x_n, y(x_n))$$

Example 2.1.1. *Use Euler's method on the interval* $[0, 1]$ *to find the solution to* $\frac{dy}{dx} = y$ *when* $x = 1$. *(use four steps,i.e,*$n = 4$*)*

Sol: *From the given example step size* $h = \frac{a-b}{n} = \frac{1-0}{4} = 0.25$.

We wanted to find the value of y *at* $x = 1$ *, provided that* $y(0) = 1$.

From Euler's method we have;

$$y(x_1) = y(x_0) + h(f(x_0, y(x_0))$$

$$y(x_1) = y(0) + 0.25(y_0)$$

$$y(x_1) = 1 + 0.25(1)$$

$$y(x_1) = y(0.25) = 1.25$$

$$y(x_2) = y(0.5) = y(x_1) + h(f(x_1, y(x_1))$$

$$y(x_2) = y(0.5) = 1.25 + 0.25y(x_1) = 1.25 + 0.25(1.25)$$

$$y(x_2) = y(0.5) = 1.5625$$

$$y(x_3) = y(0.75) = y(x_2) + h(f(x_2, y(x_2))$$

$$y(x_3) = y(0.75) = 1.5625 + 0.25(y(x_2)) = 1.5625 + 0.25(1.5625)$$

$$y(x_3) = y(0.75) = 1.931$$

$$y(x_4) = y(1) = y(x_3) + h(f(x_3, y(x_3))$$

$$y(x_4) = y(1) = 1.931 + 0.25(y(x_3)) = 1.931 + 0.25(1.931)$$

$$y(x_4) = y(1) =\approx 2.4414$$

But from the given equation ,by using the method of separation of variables, the exact value of $y(x_4)$ *is*

$$y(x_4) = e^1 = e \approx 2.71828...$$

2.2 Modified Euler's Method

Here we wish to modify Euler's method to get a better approximation . We know that Euler's method states that

$$y(x_{n+1}) = y(x_n) + hf(x_n, y(x_n)).$$

Suppose $f(x_0, y(x_0))$ is the slope of a tangent line through $(x_0, y(x_0))$ and $f(x_1, y(x_1))$ is the slope of a tangent line through $(x_1, y(x_1))$ .Thus the slope of another line between the two lines is the average of the two slopes.i.e, the new slope is given by the formula

$$slope = \frac{f(x_0, y(x_0)) + f(x_1, y(x_1))}{2}$$

And from the Euler's formula,we have

$$y(x_1) = y(x_0) + hf(x_0, y(x_0))$$

By replacing the new slope in the place of the slope at $(x_0, y(x_0)) = f(x_0, y(x_0))$. So we get

$$y^m(x_1) = y(x_0) + h(\frac{f(x_0, y(x_0)) + f(x_1, y(x_1))}{2})$$
$$y^m(x_1) = y(x_0) + \frac{h}{2}(f(x_0, y(x_0)) + f(x_1, y(x_1)),$$

where y^m is the modified value of y at x_1 and with similar prcedure at x_2

$$y^m(x_2) = y(x_1) + \frac{h}{2}(f(x_1, y(x_1)) + f(x_2, y(x_2))$$

at x_{n+1} we will get

$$y^m(x_{n+1}) = y(x_n) + \frac{h}{2}(f(x_n, y(x_n)) + f(x_n, y(x_n)).$$

This method is called the *modified Euler's method*([**??**]) .

1^{st} **Remark** :In the modified formula $y(x_n)$ is obtained by applying the concept of Euler's method.That is

$$y(x_n) = y(x_{n-1}) + hf(x_{n-1}, y(x_{n-1}))$$

2^{nd} **Remark** :We obtain the iteration formula

$$y^{n+1}(x_1) = y(x_0) + \frac{h}{2}(f(x_0, y(x_0)) + f(x_1, y^n(x_1)) \qquad n = 0, 1, 2, ..$$

where, $y^n(x_1)$ is the n^{th} approximation to $y(x_1)$. The above iteration formula can be started by $y^1(x_1)$ from Euler's method.

$$y^0(x_1)) = y(x_0) + h(x_0, y(x_0))$$

Example 2.2.1. *Solve* $\frac{dy}{dx} = f(x, y) = x + y + 1$ *for* $x = [0, 2]$ *with initial conditions* $y(0) = 0$ *using the Euler's method: when*

I .h=1

II .h=0.2

And compare the result graphically with the exact solution $y = 2e^x - x - 2$.

sol*. First consider* $h = 1$

$$y(x_1) = y(x_0) + hf(x_0, y(x_0)$$

$$y(x_1) = y(x_0) + h(x_0 + y(x_0) + 1)$$

$$y(1) = y(0) + 1(0 + 0 + 1)$$

$$y(x_1) = 0 + 1 = 1$$

$$y(x_2) = y(x_1) + hf(x_1, y(x_1))$$

$$y(x_2) = y(x_1) + h(x_1 + y(x_1) + 1)$$

$$y(x_2) = y(2) = 1 + 1(1 + 1 + 1) = 4$$

Hence $y(x_2) = y(2) = 4$.

When $h = 0.2$*;*

$$y(x_1) = y(x_0) + hf(x_0, y(x_0))$$

$$y(x_1) = y(x_0) + h(x_0 + y(x_0) + 1)$$

$$y(0.2) = y(0) + 0.2(0 + 0 + 1) = 0.2$$

$$y(x_2) = y(x_1) + hf(x_1, y(x_1))$$

$$y(x_2) = y(x_1) + h(x_1 + y(x_1) + 1)$$

$$y(0.4) = 0.2 + 0.2(0.2 + 0.2 + 1) = 0.48$$

$$y(x_3) = y(x_2) + hf(x_2, y(x_2))$$

$$y(0.6) = 0.48 + 0.2(0.4 + 0.48 + 1) = 0.856$$

$$y(x_4) = y(x_3) + hf(x_3, y(x_3))$$

$$y(0.8) = 0.856 + 0.2(0.6 + 0.856 + 1) = 1.3472$$

$$y(x_5) = y(x_4) + hf(x_4, y(x_4))$$

$$y(1) = 1.3472 + 0.2(0.8 + 1.3472 + 1) = 1.97664$$

Continue with similar procedure,and finally we will get,

$$y(2) \approx 8.289$$

Which is a better approximation to the exact solution of $y = 2e^x - x - 2 \approx 10.778$

Remark: From the example above and the graph as the step size(h)is getting very small, we will have a better approximation.

Example 2.2.2. *Use modified Euler method to solve* $\dfrac{dy}{dx} = f(x, y) = x+y+1$ *for* $x = [0, 2]$ *with the initial condition* $y(x_0) = 0$*. For* $h = 1$ *and compare* *the results graphically with Euler method solution?*

sol. *Let's find the solution using modified Euler method:*

$$y^m(x_1) = y(x_0) + \frac{h}{2}(f(x_0, y(x_0)) + f(x_1, y(x_1)), where \;\; y(x_1) = y(x_0) + hf(x_0 + y(x_0))$$

$$y^m(x_1) = y(0) + \frac{h}{2}(f(x_0, y(x_0)) + f(x_1, y(x_1))$$

$$y^m(x_1) = 0 + \frac{1}{2}((0 + 0 + 1) + (1 + y(x_1) + 1). But \;\; y(x_1) = 0 + 1(0 + 0 + 1) = 1$$

$$y^m(x_1) = 0 + 0.5((0 + 0 + 1) + (1 + 1 + 1) = 2$$

$$y^m(x_2) = y(x_1) + \frac{h}{2}(f(x_1, y(x_1)) + f(x_2, y(x_2)). Where \;\; y(x_2) = y(x_1) + hf(x_1 + y(x_1))$$

$$y^m(x_2) = 1 + \frac{1}{2}((1 + 1 + 1) + (2 + y(x_2) + 1). But \;\; y(x_2) = 1 + 1(1 + 1 + 1) = 4$$

$$y^m(x_2) = 1 + 0.5((1 + 1 + 1) + (2 + 4 + 1) = 6.$$

And for Euler method:

$$y(x_1) = x_0 + hf(x_0 + y(x_0))$$

$$y(x_1) = 0 + 1(x_0 + y(x_0) + 1)$$

$$y(x_1) = 0 + 1(0 + 0 + 1) = 1$$

$$y(x_2) = x_1 + hf(x_1 + y(x_1))$$

$$y(x_2) = 1 + 1(x_1 + y(x_1) + 1)$$

$$y(x_2) = 1 + 1(1 + 1 + 1) = 4.$$

The given example and the graph above reveals the modified Euler's method is a better approximation to the exact solution as compared to the Euler's method .

Example 2.2.3. *Give* $\frac{dy}{dx} = x + y$ *with initial conditions* $y(0) = 1$ *then find* $y(0.05)$ *and*$y(0.1)$ *and correct to 6 decimal places where* $h = 0.05$.

sol. *Using Euler's method , we obtain*

$$y^0(x_1) = y(x_1) = y(x_0) + hf(x_0, y(x_0) = 1 + 0.05(0 + 1) = 1.05$$

We improve $y(x_1)$ *by using Euler's modified method*

$$\begin{aligned}
y^1(x_1) &= y(x_0) + \frac{h}{2}\left[f(x_0, y(x_0)) + f(x_1, y^0(x_1)\right] \\
&= 1 + \frac{0.05}{2}\left[(0 + 1) + (0.05 + 1.05)\right] \\
&= 1.0525 \\
y^2(x_1) &= y(x_0) + \frac{h}{2}\left[f(x_0, y(x_0)) + f(x_1, y^1(x_1)\right] \\
&= 1 + \frac{0.05}{2}\left[(0 + 1) + (0.05 + 1.0525)\right] \\
&= 1.0525625 \\
y^3(x_1) &= y(x_0) + \frac{h}{2}\left[f(x_0, y(x_0)) + f(x_1, y^2(x_1)\right] \\
&= 1 + \frac{0.05}{2}\left[(0 + 1) + (0.05 + 1.0525625)\right] \\
&= 1.052564 \\
y^4(x_1) &= y(x_0) + \frac{h}{2}\left[f(x_0, y(x_0)) + f(x_1, y^3(x_1)\right] \\
&= 1 + \frac{0.05}{2}\left[(0 + 1) + (0.05 + 1.052564)\right] \\
&= 1.0525641
\end{aligned}$$

Since $y^3(x_1) = y^4(x_1) = 1.052564$ *correct to 6 decimal places. Hence we take* $y^1(x_1) = y_1 = 1.052564$

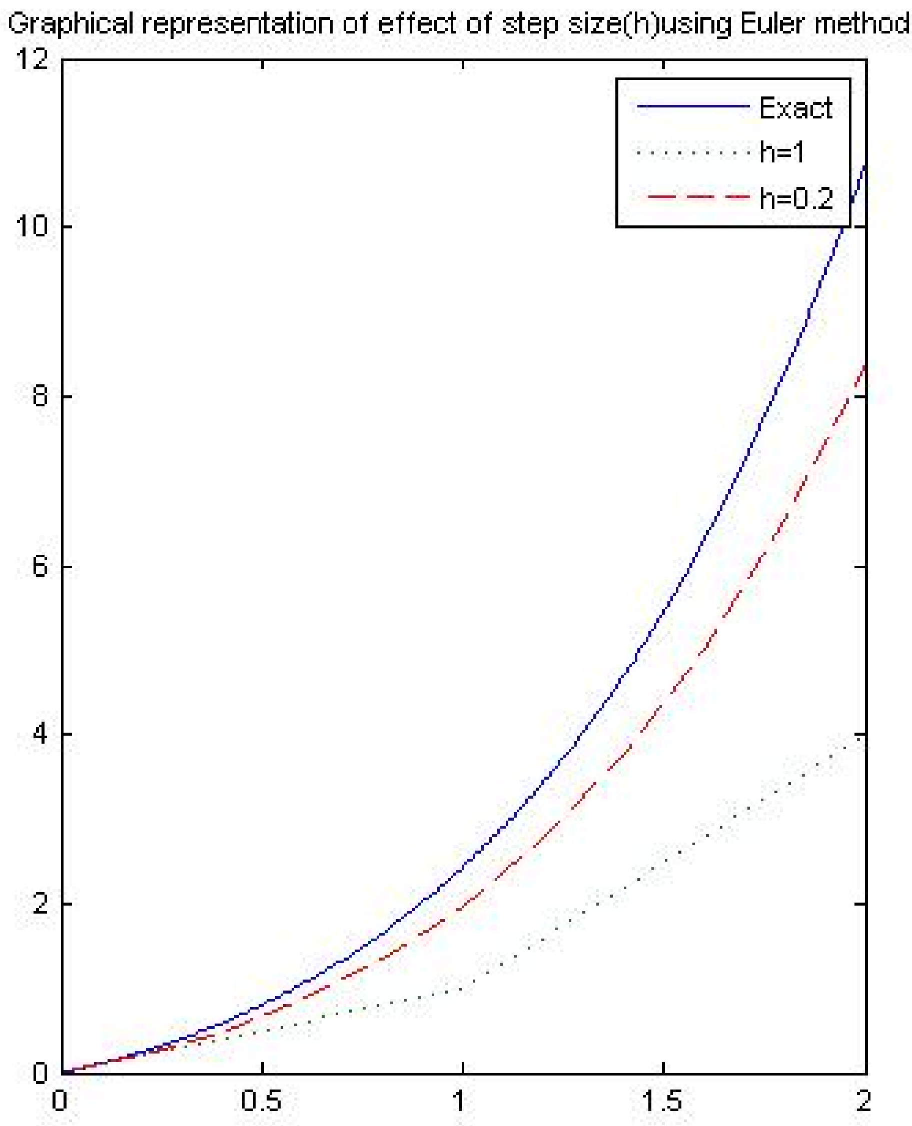
Again,using Euler's method, we obtain

$y^0(x_2) = y(x_1)+hf(x_0, y(x_1) = 1.052564+0.05(1.052564+0.05) = 1.1076922$

We improve $y(x_2)$ *by using Euler's modified method*

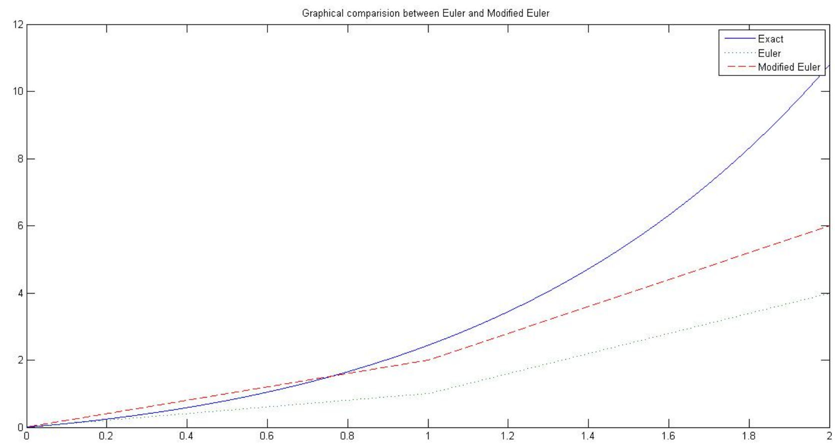
$$
\begin{aligned}
y^1(x_2) &= y(x_1) + \frac{h}{2}\left[f(x_1, y(x_1)) + f(x_2, y^0(x_1)\right] \\
&= 1.052564 + \frac{0.05}{2}\left[(1.052564 + 0.05) + (1.1076922 + 0.1)\right] \\
&= 1.1120511 \\
y^2(x_2) &= y(x_1) + \frac{h}{2}\left[f(x_1, y(x_1)) + f(x_2, y^1(x_2)\right] \\
&= 1.052564 + \frac{0.05}{2}\left[(1.052564 + 0.05) + (1.1120511 + 0.1)\right] \\
&= 1.1104294 \\
y^3(x_2) &= y(x_1) + \frac{h}{2}\left[f(x_1, y(x_1)) + f(x_2, y^2(x_2)\right] \\
&= 1.052564 + \frac{0.05}{2}\left[(1.052564 + 0.05) + (1.11042940.1)\right] \\
&= 1.1103888 \\
y^4(x_2) &= y(x_1) + \frac{h}{2}\left[f(x_1, y(x_1)) + f(x_2, y^3(x_2)\right] \\
&= 1.052564 + \frac{0.05}{2}\left[(1.052564 + 0.05) + (1.1103888 + .1)\right] \\
&= 1.1103878 \\
y^5(x_2) &= y(x_1) + \frac{h}{2}\left[f(x_1, y(x_1)) + f(x_2, y^4(x_2)\right] \\
&= 1.052564 + \frac{0.05}{2}\left[(1.052564 + 0.05) + (1.1103878 + .1)\right] \\
&= 1.1103878
\end{aligned}
$$

Since $y^4(x_2) = y^5(x_2) = 1.1103878$ *correct to 7 decimal places. Hence we take* $y^m(x_2) = y_2 = 1.1103878$. *Therefore*
$y^m(x_2) = y(0.1) = 1.110388$ *correct to 6 decimal places.*



0.5cm0.5cm

Figure 2.2:


Graphical comparision between Euler and Modified Euler
Exact
Euler
Modified Euler
0
2
4
6
8
10
12
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

22

Figure 2.3:

Chapter 3

Error Analysis

3.1 Order Of Euler's Method

Definition 3.1.1. *The difference method*

$$w_0 = \alpha$$

$$w_{i+1} = w_i + h\phi(x_i, w_i) \quad for\ each\ i = 1, 2, 3, ..., N$$

has local truncation error

$$\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\phi(x_i, y_i))}{h} = \frac{y_{i+1} - y_i}{h} - \phi(x_i, y_i).$$

For Euler's method , the local truncation error at i^{th} step for the problem

$$\frac{dy}{dx} = f(x, y), \quad a \le x \le b, \quad y(a) = \alpha$$

is

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - f(x_i, y_i) \quad for\ each\ i = 0, 1, 2, ..N.$$

This error is a local error because it measures the accuracy of the method at specified step , assuming that the method was exact at the previous step.as such , it depends on the differential equation, the step size,and the particular step at the approximation.

In the introduction part we see that Euler's method has an error

$$\tau_{i+1}(h) = \frac{h}{2}y^{''}(\xi_i) \qquad for\ \ some\ \ \xi_i\ \ in\ \ (x_i, x_{i+1})$$

When $y^{''}(x)$ is known to be bounded by a constant M in $[a, b]$, this implies

$$|\tau_{i+1}(h)| \leq \frac{h}{2}M \quad ,$$

so the local truncation error in Euler's method is $O(h)$.

Remark: A method with truncation error $O(h^p)$ is called an order p method

3.2 Stability of Euler's Method

We analyze the stability condition of forward euler method and backward euler method by using the so called test equation.

The test equation reads

$$y^{'} = \lambda y$$

$$y(0) = \alpha$$

Where λ is a complex number. We also $\alpha \neq 0$, other wise we get the trivial solution. The forward euler method reads

$$y(x_{i+1}) = y(x_i) + hf(x_i, y(x_i)).$$

Substituting ODE to the above formula, we get

$$y(x_{i+1}) = y(x_i) + h\lambda y(x_i)$$
$$= (1 + h\lambda)y(x_i).$$

Therefore , by induction we have

$$(1 + h\lambda)^i y(x_i)\alpha.$$

The exact solution of the test equation is

$$y(x) = \alpha e^{\lambda x}$$

(To see this, we take derivative with respect to x on both sides of the above equation, we have $y^{'} = \lambda\alpha e^{\lambda x} = y\lambda$ which is the same as the ODE.)If we restrict λ such that its real part is negative,i.e. $Re(\lambda) < 0$,then the exact solution eventually decays to 0 as x goes to infinity.)

Numerically, this corresponds to the case that $y(x_i)$ goes to 0 when i goes to infinite in Equation $(1 + h\lambda)^i y(x_i)\alpha$. To satisfy this, we have to require.

$$|1 + h\lambda| < 1.$$

This equation is the stability condition for euler method([**??**]). If we restrict that λ is a real number (which is often the case in practice),then we can simplify the stability condition. From inequality $|1 + h\lambda| < 1$, we obtain

$$-1 < 1 + h\lambda < 1$$
$$-2 < h\lambda < 0$$
$$0 < h < \frac{-2}{\lambda}$$

h is the step size, so it is always positive. Then the stability condition is simplify to

$$0 < \frac{-2}{\lambda}$$

If λ is complex($= a + bi$), then it is stable if

$$|1 + h\lambda| < 1$$

$$|1 + h(a + bi)| < 1$$

by using the property $|a + bi| = (a^2 + b^2)$, we have

$$(1 + ha)^2 + (bh)^2| < 1$$

i.e.

$$(x + 1)^2 + y^2 < 1 \qquad where \quad x = ha \;\; y = bh$$

i.e., it lies with in the unit circle center at $(-1, 0)$

If λ is pure imaginary,i.e. $\lambda = bi$, then $|1 + hbi| = (1 + (bh)^2) > 1$ so the method is not stable.

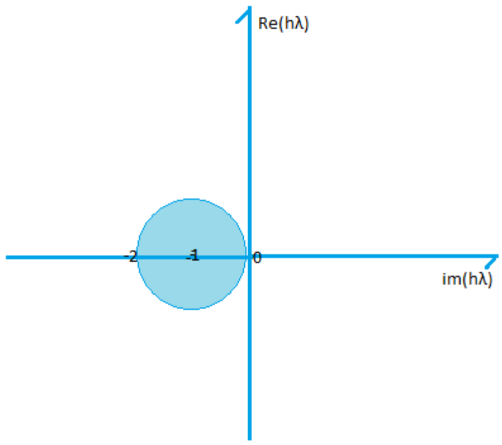


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Figure 3.1: Stability region (shadowed circle) for Euler method

Example 3.2.1. *Suppose the initial-value problem*

$y^{'} = -50y \quad ,0 \le x_i \le 1, \quad ,y(0) = 1$

is approximated by the Euler's method so that

$$y(x_{i+1}) = y(x_i) + hf(x_i + y(x_i))$$

then examine the effect of h on the stability of the Euler's method and give graphical explanation when:

I .n=20

II .n=25

III .n=30

Sol: The exact solution of the euler method is

$$y(x) = e^{-50x}$$

Now we generate the numerical solution with $n = 20$ in this case $h = \frac{1-0}{20} = 0.05$

$$y(x_0) = y(0) = 1$$
$$y(x_1) = y(x_0) + hf(x_0 + y(x_0)) = 1 + 0.05(-50)(1) = -1.5$$
$$y(x_2) = y(x_1) + hf(x_1 + y(x_1)) = -1.5 + 0.05(-50)(-1.5) = 2.25$$
$$and \quad so \quad on...$$

Here errors are growing and the method is **unstable.** Take $n = 25$, then $h = \frac{1-0}{25} = 0.04$

$$y(x_0) = y(0) = 1$$
$$y(x_1) = y(x_0) + hf(x_0 + y(x_0)) = 1 + 0.04(-50)(1) = -1$$
$$y(x_2) = y(x_1) + hf(x_1 + y(x_1)) = -1 + 0.04(-50)(-1) = 1$$
$$and \quad so \quad on.$$

Here the solution oscillates and the error grows slowly. The method is **unstable.**

Take $n = 30$, then $h = \frac{1-0}{30} = \frac{1}{30}$

$$y(x_0) = y(0) = 1$$
$$y(x_1) = y(x_0) + hf(x_0 + y(x_0)) = 1 + \frac{1}{30}(-50)(1) = \frac{-2}{3}$$
$$y(x_2) = y(x_1) + hf(x_1 + y(x_1)) = \frac{-2}{3} + \frac{1}{30}(-50)(\frac{-2}{3}) = \frac{4}{9}$$
$$and \quad so \quad on.$$

Here the solution oscillates and the error grows slowly. The method is in some sense **stable.**

Example 3.2.2. *Examine the stability of the above example when*

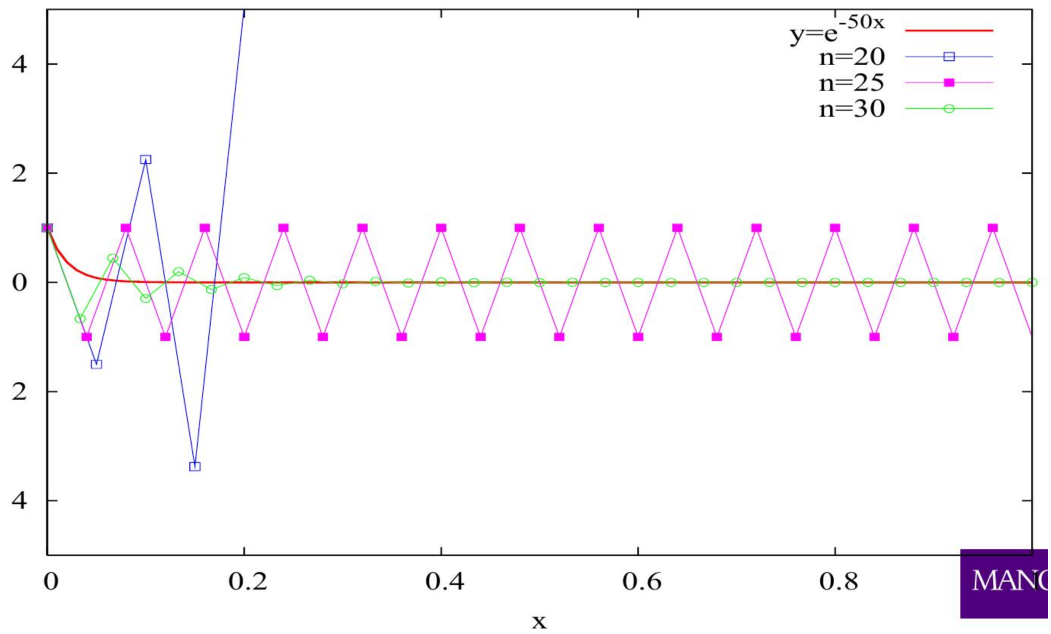

Figure 3.2: graphical comparison for h=20,h=25 and h=30

I .n=50

II .n=100

Sol: We know that the Euler solution is stable when $h < \frac{-2}{\lambda} = \frac{-2}{-50} = 0.04$. But the value of $h = \frac{1-0}{50} = 0.02$ so we have $h < \frac{-2}{\lambda}$ which implies the solution **stable** and similarly when $n = 100$ the solution is **stable**

Remark: The above examples indicates that a numerical method can not be applied as we like to a given initial value problem. The choice of the step length is very important and it is governed by the stability condition.

3.3 consistency

Definition 3.3.1. *A one step difference-equation method with truncation error $\tau_i h$ at the i^{th} step is said to be consistent with the difference equation*

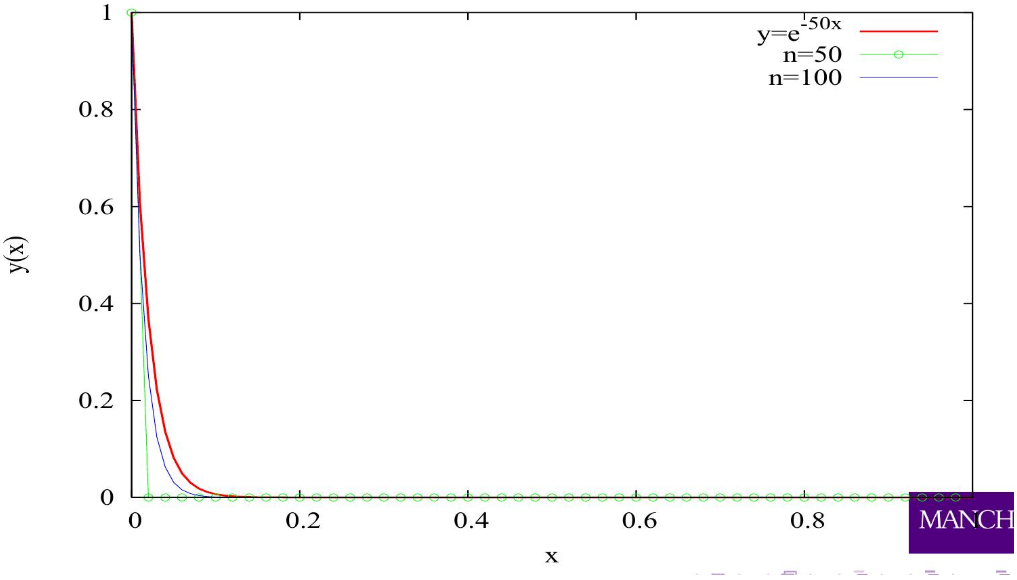

Figure 3.3: Stability graph for h=50 and h=100

it approximates if

$\lim_{h \to 0} \; max|\tau_i h| = 0 \qquad 0 \le i \le n$

Consider the ordinary differential equation given by

$$\frac{dy}{dx} = f(x, y), \quad a \le x \le b, \quad y(a) = \alpha.$$

From the discussion in (3.0.1) The local truncation error of Euler's method is given by

$$\tau_{i+1}(h) = \frac{h}{2} y^{''}(\xi_i).$$

If $y^{''}$ is bounded by the constant M on the interval $[a, b]$,then

$$|\tau_{i+1}(h)| \le \frac{h}{2} M$$

Now

$$\lim_{h\to 0} \; max|\tau_i h|$$
$$\leq \lim_{h\to 0} \; max(\frac{h}{2}M) = 0$$

this implies,

$$\lim_{h\to 0} \; max|\tau_i h| = 0 \qquad 0 \leq i \leq n$$

Hence by the definition of consistency Euler's method is consistent.

3.4 Convergence

Definition 3.4.1. *A one step difference-equation method is said to be convergent with the difference equation it approximates if*

$$\lim_{h\to 0} \; max|w_i - y(x_i)| = 0 \qquad 0 \leq i \leq n$$

where $y(x_i)$ denotes the exact value of the solution of the differential equation and w_i is the approximation obtained from the difference method at the i^{th} step.

Based on the above definition let's examine the convergence of the Euler method.

We know that $|w_1 - y(x_i)|$ is the truncation error of the Euler method which is $|\tau_{i+1}(h)| = \frac{h}{2}y^{''}(\xi_i)$.

Thus

$\lim\limits_{h\to 0} \; max|w_i - y(x_i)| \approx \lim\limits_{h\to 0} \; max|\tau_i h| = \lim\limits_{h\to 0} \dfrac{h}{2}y^{''}(\xi) = 0$. Which shows that the Euler method is convergent.

3.5 Conclusion

This paper examines one of the most popular numerical methods for solving the differential equation of the form

$$y^{'} = f(x, y), \qquad y(x_0) = y_0.$$

One of the finite difference methods to solve the above types of initial value problems is the Euler's method. The purpose of the paper to investigate the of derivation of Euler's method and its error analysis. Euler's method is defined by

$$y(x_{n+1}) = y(x_n) + hf(x_n, y(x_n)), \quad y(x_0) = y_0.$$

This method is simple but computationally inefficient when we modify this method we will get the modified Euler's method which is a better approximaton to the exact solution than the Euler's method. Through numerical examples and graphical method it has been shown that the modified Euler's method is a better aproximation to the exact solution as compared to the Euler's method.

This paper also dealt with the error analysis of the Euler's method. And we are able to show that the Eulers method is convergent as well as consistent.Further more we investigated the stability conditons of the Euler's method and it is found that the stability condition is

$$|1 + h\lambda| < 1.$$

If λ is real the stability condition is $-2 < h\lambda < 0$. And if λ is complex the stability region of the method lies with in the unit circle center at $(-1, 0)$ but if λ is pure imaginary the method is not stable. In general a numerical method can not be applied as we like to a given initial value problem. The choice of the step length is very important and it is governed by the stability condition.

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