ANALYSIS OF HEAT EQUATION IN $\mathbb{R}^n$

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Declaration

The undersigned here by certify that they have read and recommended to the department of mathematics for acceptance of this thesis entitled **"Analysis Of Heat Equation In** $\mathbb{R}^n$**"** by **Mehari Temesgen ID N$\underline{o}$ GSE/9145/09** in partial fulfillment of the requirements for degree of Master of Science in Differential Equation.

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Abstract

This paper is concerned with **Analysis of heat equation in $\mathbb{R}^n$.** Analysis of heat equation in $\mathbb{R}^n$ regarded as Fourier Transformation which has different properties and Inverse of Fourier Transformation that is important to solve heat equation in $'n'$ spaces. using separation of variables for solving with boundary conditions like Dirichlet, Neumann and periodic conditions. By using these condition we solve heat equations in $\mathbb{R}^n$ is solved.

Naotation

∇: The gradient operator.

$\Re$: The set of real numbers.

R^n: n dimensional Euclidian space.

M: The region in $\mathbb{R}^n$.

C: The specific heat capacity of the region in $\mathbb{R}^n$

∂M: The boundary of M.

$||g||_{L^2}^2$: The norm of g define on Ω .

$\langle .,.\rangle$: The inner product related to in L^2.

$B(x_0, r)$: Ball of the radius r about x_0 in $\mathbb{R}^n$.

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Chapter 1

INTRODUCTION

1.1 Definition of heat equation

Definition 1.1.1. *Heat equation is a very useful partial differential equation that describes the distribution of heat in a given region.*
The general formula of heat equation is:

$$u_t = ku_{xx}$$

For a function u(x, y, z), which are x, y, z and the time t are variables of the function.
The heat equation is:

$$u_t = k\left(u_{xx} + u_{yy} + u_{zz}\right)$$

Or

$$u_t = k\nabla^2 u$$

where k is a constant.

The heat equation is plays an important role in its the scientific fields of studies. The diffusion equation is the general version of the heat equation, it increases in connection with the study of chemical diffusion and other related processes. The heat equation indicets that if a hot body is placed in a container of cold water, then the temperature of the body will decreases and eventually the temperature in container will be equalized.
The heat equation is derived from Fourier's law and conservation of energy. By Fourier's law the flow rate of heat energy in a surface is proportional to the negative temperature gradient over the surface.

$$q = -k\nabla u$$

where, k is the constant conductivity of the heat and u is the temperature. In 1-D the gradient is the part of ODE and the Fourier's law is $q = -ku_x$

1.2 Diffusion

Heat Equation that indicates the derivatives of the equation given by:

$$u_t - ku_{xx} = 0, \quad k > 0 \tag{1.1}$$

We call it this the diffusion equation. Assume that a fluid in which a dye is being that diffused through the liquid. The dye will move from higher concentration to a lower concentration.
Let $u(x,t)$ be the proportion of the dye at position x in the pipe at time t. The total amount of the dye in the pipe from a to b at time t is given by:

$$N(t) = \int_a^b u(x,t)dx$$

Therefore,

$$N^{'}(t) = \int_a^b u(x,t)dt$$

where, $k > 0$ is a constant value. That is the flow rate is proportional to the concentration. Therefore,

$$\int_a^b u_t(x,t)dt = ku_{xx}(b,t)$$

Differentiating with respect to b, we have

$$u_t(b,t) = ku_{tt}(b,t)$$

or equivalently we have:

$$u_t(b,t) - ku_{xx}(b,t) = 0$$

We call it this the Diffusion Equation.

Chapter 2

The Heat Flow

Let us see the other derivation of the study of heat flow through in the region. Let M be a region in $\mathbb{R}^n$. Let

$$X = [x_1, x_2, x_3, ..., x_n]^t$$

be a vector in $\mathbb{R}^n$.

Let $u(x,t)$ be the temperature at point x, time t and G(t) be the total amount of heat contained in M. Let C be the specific heat capacity of the material and $'\rho'$ be its density. Then

$$G(t) = \int_M c\rho u(x,t)dx$$

So, the change in heat is given by:

$$G^{'}(t) = \frac{d}{dt}\left(\int_M c\rho u(x,t)dx\right) = c\rho\int_M \frac{d}{dt}u(x,t)dx = \int_M c\rho u_t(x,t)dt$$

Fourier's law saying that heat flow from hot regoin to cold regions at a rate $k > 0$ is proportional to the given temperature gradient. There is one way the heat will leave in the region M through the boundary. That is

$$G^{'}(t) = \int_{\partial M} k\nabla ujds$$

where, ∂M is the boundary of M, 'j' is the outward unit normal vector to ∂M and ds is the surface measure over ∂M. Therefore, we have

$$\int_M c\rho u_t dx = \int_{\partial M} k\nabla ujds$$

remained that for a vector field F, then the Divergent theorem gives

$$\int_{\partial M} Fjds = \int_M \nabla F dx$$

Now, we have

$$\int_M c\rho u_t(x,t)dx = \int_M \nabla(k\nabla u)dx$$

This guide us to the partial differential equation.

$$c\rho u_t = \nabla(k\nabla u)dx$$

If c, ρ and k are constant, we are lead the heat equation $u_t = k\nabla$ u where k, c and $\rho > 0$ and

$$\triangle u = \sum_{i=1}^{n} u_i u_i$$

2.1 Heat Equation on an interval in $\mathbb{R}^n$

2.1.1 The Separation of Variables

Assume that IVP on an interval I in $\mathbb{R}^n$.

$$\begin{cases} u_t \ = \ ku_{xx}, \quad x \in I, \, t > 0 \\ u(x,0) = \phi(x) \end{cases} \tag{2.1}$$

The most common boundary conditions are the following

(i) Drichilet: $u(0,t) = 0 = u(l,t)$

(ii) Neumann: $u_x(0,t) = 0 = u_x(l,t)$

Now, let us present the techniques of separation of variables. This technique to connect something looking for a solution of a particular form. Now, we look for the solution in the form of:

$$u(x,t) = X(x)T(t)$$

For the functions X, T to be determined. Our supposition that we can looking for a solution of (1.1) of in the form of substituting this function $u = XT$ into the heat equation, we have the equation in the form

$XT' - kX''T = 0$ dividing the equation by kXT. we have

$\frac{T'}{kT} = \frac{X'}{X} = -\lambda$

for some constant λ. There exist a solution

$u(x,t) = X(x)T(t)$ is solution of the heat equation, then T and X must satisfy the equation

$\frac{T'}{kT} = -\lambda$ and $\frac{X''}{X} = -\lambda$ for some constant λ. We need the function X to satisfy the boundary conditions. That is we need to find the functions X and scalars λ such that

$$\begin{cases} X''(x) = \lambda X(x) \quad x \in I \\ x \text{ satisfy the } BCS \end{cases}$$

This problem is called an eigenvalue problem. Particularly, a constant λ which is satisfies (2.1) for some function, not identically zero is called an eigenvalue of $-\lambda^2$ for the given boundary conditions associated with eigenvalue λ. In order to find the solution of (1.1) given by in the form of:

$$u(x,t) = X(x)T(t)$$

The first aim is to find all the solution of the eigenvalue problem of (2.1).

Example 2.1.1 (Drichilet Boundary conditions)**.** *Find all solutions to the eigenvalue problem.*

$$\begin{cases} X'' + \lambda x = 0, \quad 0 < x < l \\ X(0) = 0 = X(l) \end{cases} \tag{2.2}$$

Solution:

Now, let us see some cases to show the eigenvalue problem.

<u>**Case 1**</u>

If $\lambda = \alpha^2 > 0$

The eigenvalue problem (2.2) becomes:

$$\begin{cases} X'' + \alpha^2 x = 0, \quad 0 < x < l \\ X(0) = 0 = X(l) \end{cases} \tag{2.3}$$

The characteristics equation of the given equation we have

$\lambda^2 + \alpha^2 = 0.$

therefore, the solution of this ODE is given by:

$X(x) = C_1 \cos(\alpha x) + C_2 \sin(\alpha x)$

From the boundary conditions, we have that:

$X(0) = 0 \Rightarrow c_1 = 0$

$X(l) = 0 \Rightarrow \sin(\alpha l) = 0$

$\Rightarrow \alpha = \frac{n\pi}{l}$, where n = 1,2,3,...

Therefore, we have a sequence of positive eigenvalues,

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2$$

the corresponding eigenfunction is given by:

$$X_n(x) = C_2 \sin\left(\frac{n\pi}{l}\right)$$

Case 2

If $\lambda = 0$

then the eigenvalue problem of (2.2) becomes:

$$\begin{cases} X'' = 0, \quad 0 < x < l \\ X(0) = 0 = X(l) \end{cases}$$

The characteristics equation becomes $\lambda^2 = 0$.

therefore, the solution of this ODE is given by:

$$X(x) = C_1 + C_2$$

The boundary conditions are gives us:

$$\begin{cases} X(0) = 0 \Rightarrow C_1 = 0, \\ X(l) = 0 \Rightarrow C_2 = 0 \end{cases}$$

Therefore, the solution of the eigenvalue problem $\lambda = 0$ is $X(x) = 0$. By definition, the zero function is not an eigenfunction. Therefore, $\lambda = 0$ is not an eigenvalue.

Case 3

If $\lambda = -\gamma^2$, then the eigenvalue problem of (2.2) becomes:

$$\begin{cases} X'' - \gamma^2 x = 0, \quad 0 < x < l \\ X(0) = 0 = X(l) \end{cases}$$

Using the characteristics equation we have: $\lambda^2 - \gamma^2 = 0 \Rightarrow \lambda = \gamma$

and by using the matrix product we have:

$$\begin{bmatrix} 1 & 1 \\ e^{-\gamma x} & e^{\gamma x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, the boundary conditions gives us:

$$\begin{cases} X(0) = 0 \Rightarrow C_1 = 0, \\ X(0) = 0 = X(l) = 0 \Rightarrow C_2 = 0 \end{cases}$$

The general solution of the ODE is given by:

$$X(x) = C_1 \cosh(\gamma x) + C_2 \sinh(\gamma x)$$

therefore, the solution of (2.2) is given by $\lambda = (\frac{n\pi}{l})^2$ then the general solution of the equation is given by:

$$X_n(x) = C_n \sin\left(\frac{n\pi}{l}\right)^2$$

where, n = 1,2,3,...

Example 2.1.2 (Periodic Boundary Conditions)**.** *Find all solutions to the eigenvalue problem*

$$\begin{cases} X'' + \lambda x = 0, \quad -l < x < l \\ X(-l) = X(l) \\ X'(-l) = X'(l) \end{cases} \tag{2.4}$$

Solution:

We have the following cases to find the eigenvalue problem:

<u>**Case 1**</u>

If $\lambda = \alpha^2$, the eigenvalue problem of (2.4) becomes:

$$\begin{cases} X'' + \alpha^2 x = 0, \quad -l < x < l \\ X(-l) = X(l) \\ X'(-l) = X'(l) \end{cases}$$

So, the characteristics equation of the given equation becomes:

$\lambda^2 + \alpha^2 = 0$

therefore, the solution of the ODE is given by:

$$X(x) = C_1 \cos(\alpha x) + C_2 \sin(\alpha x)$$

The boundary conditions are given by:

$X(-l) = X(l) \Rightarrow C_2 \sin(\alpha l) = 0$

$C_2 = 0$ or $\alpha = \frac{n\pi}{l}$

The boundary conditions are ginven by:

$X'(-l) = X'(l) \Rightarrow C_1 \alpha l \sin(\alpha l) = 0 \Rightarrow C_1 = 0$ or $\alpha = \frac{n\pi}{l}$ Therefore, we have a sequence of positive eigenvalues $\lambda = (n\pi/l)^2$. Where, n = 1,2,3,... with corresponding eigenfunctions.

$$X_n(x) = C_n \left(\cos\left(\frac{n\pi}{l}x\right) + \sin\left(\frac{n\pi}{l}x\right)\right)$$

<u>Cases 2</u>

If $\lambda = 0$, then the eigenvalue problem of (2.4) becomes:

$$\begin{cases} X'' = 0, \quad -l < x < l \\ X(-l) = X(l) \\ X'(-l) = X'(l) \end{cases}$$

Now, the characteristics equation becomes: $\lambda^2 = 0$. Then the solution of the given ODE is becomes:

$$X(x) = C_1 x + C_2$$

The boundary conditions are given by:

$-C_1 l + C_2 = C_1 l + C_2, \Rightarrow C_1 = 0$, if $l \neq 0$

$X(-l) = X(l) \Rightarrow C_2 = 0$,

$X'(-l) = X'(l)$ that satisfied $C_2 = 0$. Therefore, $\lambda = 0$ is an eigenvalue with corresponding eigenfunction

$X_0(x) = C_0, \quad C_0 \neq 0,$

Case 3

If $\lambda = -\gamma^2$, then the eigenvalue problem of (2.4) becomes:

$$\begin{cases} X'' - \gamma^2 x = 0, \quad -l < x < l \\ X(-l) = X(l) \\ X'(-l) = X'(l) \end{cases}$$

Using the characteristics equation we have: $\lambda^2 - \gamma^2 = 0 \Rightarrow \lambda = \gamma$

and by using the matrix product we have:

$$\begin{bmatrix} 1 & 1 \\ e^{-\gamma x} & e^{\gamma x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, the solution of the ODE is given by:

$$X(x) = C_1 \cosh(\gamma x) + C_2 \sinh(\gamma x)$$

So, the boundary value conditions we have:

$X(-l) = X(l) \Rightarrow C_2 \sinh(\gamma l) = 0 \Rightarrow C_2 = 0,$

$X'(-l) = X'(l) \Rightarrow C_2 \gamma l \sinh(\gamma l) = 0 \Rightarrow C_2 = 0.$

Therefore, all the solutions of (2.4) are given by: $\lambda = (\frac{n\pi}{l})^2$.

$$X_n(x) = C_n \left(\cos\left(\frac{n\pi}{l} x\right) + \sin\left(\frac{n\pi}{l} x\right)\right)$$

$\lambda_0 = 0$, $X_0(x) = C_0$, choose $X_0(x) = 1 \quad$ where, n = 1,2,3,...

So far, we have done some examples on eigenvalue problems. We return to using the method of separation of variables to solve (2.1). We have that a function in the form of :

$$u(x,t) = X(x)T(t)$$

To be a solution of the heat equation on an interval $I \in \Re$ which satisfies the given boundary conditions. We need X to be a solution of the problem.

$$\begin{cases} X'' = -\lambda x, \quad x \in I \\ X \text{ satisfies certain } BCs. \end{cases}$$

Some scalar λ and T to be a solution of the ODE. We have solve the eigenvalue problem. We need to solve equation for T and for some scalar λ be the solution of the ODE, T is given by:

$$T(t) = ae^{-k\lambda t}$$

a is an arbitrary constant. Now, for every eigenfunction X_n with corresponding eigenvalue λ_n we have a solution of T_n such that the function is given by:

$$u_n(x,t) = X_n(x)T_n(t)$$

is the solution of the heat equation on the interval I that satisfies the boundary conditions. For the initial condition: $u(x,0) = \varphi(x)$. From u_n is the sequence of the solutions of the heat equation on an interval I that satisfy the boundary conditions, then any finite linear combination of these solutions will also gives as a solution.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

will be a solution of the heat equation on I that satisfies the boundary conditions assuming that each u_n is a solution. We have that an infinite series of the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

will be a solution of the heat equation under proper convergence by an assumption of in this series.

2.1.2 satisfying the initial conditions

Let X_n be sequence of eigenfunctions and λ_n is a sequence of eigenvalues. Then for every λ_n, we have a solution of T_n. So, we have

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} a_n X_n e^{-k\lambda_n t}$$

we need to choose a_n then the initial condition is satisfied. Therefore, we have that:

$$u(x,0) = \sum_{n=1}^{\infty} a_n X_n(x) \quad = \quad \varphi(x)$$

To find a_n that satisfies this conditions we have use the following orthogonality property of eigenfunctions. For two real valued functions h and g is defined on Ω.

$$< h, g >= \int_{\Omega} h(x)g(x)dx$$

h and g are defined on L^2. The norm of h on Ω is defined as:

$$||h|\,|^2_{L^2}(\Omega) =< h, g > \quad = \quad \int_{\Omega} |h(x)|^2 \, dx$$

We say functions h and g are orthogonal on $\Omega \subset \Re^n$ if

$$< h, g >= \int_\Omega h(x)g(x)dx = 0$$

When the boundary conditions are symmetric if

$$[h'(x)g(x) - h(x)g'(x)]_{x=a}^{x=b} = 0$$

for all functions h and g satisfying the initial boundary conditions.

Lemma 2.1.1. *Assume that the eigenvalue problem of (2.4) symmetric with the boundary conditions if X_n, X_m are two eigenfunctions of (2.4) with distinct eigenvalues. X_n and X_m are orthogonal.*

Proof. Let us take an interval $I = [c, d]$
$\lambda_n = \int_c^d X_n(x)X_m(x)dx$
$= \int_c^d X_n''(x)X_m(x)dx$
$= \int_c^d X_n'(x)X_m'(x)dx - X_n'(x)X_m(x)|_{x=c}^{x=d}$
$= -\int_c^d X_n(x)X_m''(x)dx + [X_nX_m']\,|_{x=c}^{x=d}$
$= -\lambda_m \int_c^d X_n(x)X_m(x)dx$
By using the reality the boundary conditions are symmetric. Therefore,

$$(\lambda_n - \lambda_m) \int_c^d X_n(x)X_m(x)dx$$

but $\lambda_n \neq \lambda_m$, because the eigenvalues are assumed to be distinct. Therefore,

$$\int_c^d X_n(x)X_m(x)dx = 0$$

To find the coefficient of a_n we can use this Lemma such that we have:

$$\sum_{n=1} a_nX_n(x) = \varphi(x)$$

Now, multiplying both sides of the equation by X_m for some fixed value m and integrating over I, then we have that:

$$a_m < X_m, X_m > \quad = \quad < X_m, \varphi >$$

therefore, we have:

$$a_m = \frac{< X_m, \varphi >}{< X_m, X_m >}$$

□

Example 2.1.3 (Dirichlet Boundary conditions)**.** *Assume that the Dirichlet boundary conditions on an interval [0,l], we discussed the above eigenvalues and eigenfunctions are given by:* $\lambda_n = (\frac{n\pi x}{l})^2$, *n = 1,2,3,...*

So, the solution of T_n is given by:

$$T_n(t) = a_n e^{-k\lambda_n t} = a_n e^{-k(\frac{n\pi}{l})^2 t}$$

Now, we have:

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) e^{-k(\frac{n\pi}{l})^2 t}$$

By using the reality of the Dirichlet boundary conditions are symmetric, then the coefficient of a_m is given by:

$$a_m = \frac{< X_m, \varphi >}{< X_m, X_m >} = \frac{\int_0^l \sin\left(\frac{m\pi}{l}\right) \varphi(x) dx}{\int_0^l \sin^2(\frac{m\pi}{l}) dx}$$

$$= \frac{2}{l} \int_0^l \sin\left(\frac{m\pi x}{l}\right) \varphi(x) dx$$

Therefore, the solution of (2.1) on the interval I = [0,l] with Dirichlet boundary conditions is given by:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{l}) e^{-k(\frac{n\pi x}{l})^2 t}$$

where, $a_n = \frac{2}{l}\int_0^l \sin(\frac{n\pi}{l}x)\varphi(x)dx$ n = 1,2,3,...

Example 2.1.4 (Periodic Boundary conditions)**.** *Consider the Periodic boundary conditions on an interval [-l,l], we have discussed eigenvalues and eigenfunctions on the above examples. That is given by:*
$\lambda_n = (\frac{n\pi}{l})^2$

$$X_n(x) = \begin{cases} \cos(\frac{n\pi x}{l}), & n = 1,2,3,... \\ \sin(\frac{n\pi x}{l}) \\ \lambda_0 = 0, \quad X_0(x) = C_0, \quad C_0 \neq 0 \end{cases}$$

Therefore, the solution of T_n is given by:

$$T_n(t) = \begin{cases} a_n e^{-k\lambda_n t}, & n = 1,2,3,... \\ a_n e^{-k(\frac{n\pi}{l})^2 t} \\ a_0, \quad n = 0 \end{cases}$$

Where, $n \geq 0$.

$$u_n(x,t) = X_n(x)T_n(t)$$

this is a solution of the heat equation that satisfies the periodic boundary conditions. Which is defined by:

$$u(x,t) = \sum_{n=1}^{\infty}[X_n(x)T_n(t)] = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right)\right] e^{-k}(\frac{n\pi}{l})^2 t$$

Now, insert this equation into the initial condition, we have that:

$$u(x,0) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{l}x\right) + b_n \sin\left(\frac{n\pi}{l}x\right)\right] = \phi(x)$$

We have identified that eigenfunctions corresponding to distinct eigenvalues will be orthogonal, for a periodic boundary conditions. However, we have two an eigenfunctions for every positive eigenvalue. We have also $\cos(\frac{n\pi}{l}x)$ and $\sin\left(\frac{n\pi}{l}x\right)$ are eigenfunctions corresponding to the eigenvalue $\lambda_n = (\frac{n\pi}{l})^2$. We can calculate directly, we have:

$$\int_{-l}^{l} \cos\left(\frac{n\pi}{l}x\right) \sin\left(\frac{n\pi}{l}x\right) dx = 0$$

Therefore, it is an orthogonal. For every eigenvalue λ of (2.3) with multiply linearly independent eigenfunction, the eigenfunctions may always be chosen to be orthogonal. This process is known as the Gram-Schmidt orthogonalization method. Since all the eigenfunctions are mutually orthogonal we can calculate the coefficients a_n, b_n, so that in the initial condition is satisfied using the technique to described above cases and $< h, g >$ be under L^2 (i.e inner product) on [-l,l], we have that:

$$a_0 = \frac{< l, \phi >}{< l, l >} = \frac{1}{2l} \int_{-l}^{l} \phi(x)dx$$

$$a_n = \frac{< \cos\left(\frac{n\pi}{l}x\right), \phi >}{< \cos\left(\frac{\pi}{l}x\right), \cos\left(\frac{n\pi}{l}x\right) >} = \frac{1}{l} \int_{-l}^{l} \cos\left(\frac{n\pi}{l}x\right) \phi(x)dx$$

$$b_n = \frac{< \sin\left(\frac{n\pi}{l}x\right), \phi >}{< \sin\left(\frac{n\pi}{l}x\right), \sin\left(\frac{n\pi}{l}x\right) >} = \frac{1}{l} \int_{-l}^{l} \sin\left(\frac{n\pi}{l}x\right) \phi(x)dx$$

Therefore, the solution of (2.4) on the interval I = [-l,l] with periodic boundary conditions is given by:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{l}x\right) + b_n(\sin\left(\frac{n\pi}{l}x\right))\right] e^{-k(\frac{n\pi x}{l})^2}$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} \phi(x)dx$$

$$a_n = \frac{1}{2l}\int_{-l}^{l} \cos\left(\frac{n\pi}{l}x\right)\phi(x)dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} \sin\left(\frac{n\pi}{l}x\right)\phi(x)dx$$

Neumann Boundary Conditions

The Neumann problem on the half-line,

Now, the Neumann Boundary Condition we have,

$$\begin{cases} v_t - kv_{tt} = 0, \quad 0 < x < \infty, \quad 0 < t < \infty \\ v(x,0) = \phi(x), \quad x > 0 \\ v_x(0,t) = 0, \quad t > 0 \end{cases} \tag{2.5}$$

To find the solution of (2.5), we use a similar idea apply in the case of Dirichlet boundary problem. Since, if $\psi(x)$ is an even function, (which is $\psi(-x) = \psi(x)$), then its derivative function will be odd. Indeed, differentiating in the definition of the even function, we get $-\psi'(-x) = \psi'(x)$, which is the same as $\psi'(-x) = -\psi'(x)$. Hence, for an arbitrary even function $\psi'(x)$, $\psi'(0) = 0$. The resulting function is will produce solutions to the initial value probleme on the whole line that satisfies the Neumann condition of (2.5).

Define the function $\phi(x)$, then we have:

$$\phi_{even} = \begin{cases} \phi(x), \quad for \quad x > 0 \\ \phi(-x), \quad for \quad x < 0 \end{cases} \tag{2.6}$$

Assume that the initial value problem on the whole line, we have given that:

$$\begin{cases} u_t - ku_{tt} = 0, \quad 0 < x < \infty, \quad 0 < t < \infty \\ u(x,0) = \phi_{even}, \quad x > 0 \end{cases} \tag{2.7}$$

The solution $u(x,t)$ of the initial value problem of (2.7) will be even in x, since the difference $[u(-x,t) - u(x,t)]$ solves the heat equation and has zero initial condition, we use the solution the formula for the initial value problem on the whole line to write.

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\phi_{even}(y)dy, \quad t > 0$$

And take

$$z(x,t) = u(x,t)|_{x \geq 0}$$

equivalent to the case of the Dirichlet problem. This can indicate that $z(x,t)$ to solve the initial value problem of (2.6), and use this expression for the heat kernel as well as the definition of (2.7), to write the solution of the formula is given by:

$$z(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left[e^{\frac{-(x-y^2)}{4kt}} + e^{\frac{-(x+y^2)}{4kt}} \right] \phi(y)dy$$

Example 2.1.5. *Solve the following initial value problem.*

$$\begin{cases} v_t - kv_{tt} = 0, \\ v_t(0,t) = v_t(l,t) = 0, \\ v(x,0) = x \end{cases}$$

***Solution**:*

Applying the Fourier coefficients we have that:

$a_0 = \frac{1}{l}\int_0^l xdx = \frac{1}{l}\frac{x^2}{2}|_0^l = \frac{l}{2}$

$a_n = \frac{2}{l}\int_0^l xcos\left(\frac{n\pi x}{l}\right)dx$

$= \frac{2}{n\pi}xsin\left(\frac{n\pi x}{l}\right)|_0^l - \frac{2}{n\pi}\int_0^l sin\left(\frac{n\pi x}{l}\right)$

$= \frac{-2l}{n^2\pi^2}cos\left(\frac{n\pi x}{l}\right)|_0^l$

$= \frac{2l(1-cos(n\pi))}{n^2\pi^2}$, *n= 1, 2, ...*

Therefore, the solution of the equation is give by:

$$v(x,t) = \frac{l}{2} + \sum_{n=1}^{\infty} \frac{2l((1 - cos(n\pi)))}{n^2\pi^2} cos(\frac{n\pi x}{l}) e^{\frac{-kn^2\pi^2 t}{l^2}}$$

Neumann boundary value problem has an infinite series solutions, then we have the general solution:

$$v(x,t) = a_0 + \sum_{n=1}^{\infty} a_n cos\left(\frac{n\pi x}{l}\right) e^{\frac{-kn^2\pi^2 t}{l^2}}$$

Chapter 3

Fourier Series

Consider the Dirichlet boundary conditions, we can find the coefficients of a_n we have that:

$$\phi(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)$$

The Fourier coefficient is given by:

$$a_n = \frac{2}{l}\int_0^l \sin\left(\frac{n\pi x}{l}\right)\phi(x)dx$$

A function ϕ is defined on an interval (0,l), such that an infinite series is given by the function:

$$\phi \sim \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)$$

where the Fourier coefficient is:

$$a_n = \frac{2}{l}\int_0^l \sin\left(\frac{n\pi x}{l}\right)\phi(x)dx$$

this is called the Fourier sine series of ϕ. From the conditions we have that the equation,

$$\phi(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right] \tag{3.1}$$

In this condition, the coefficients must be given by:

$$\begin{cases} a_0 = \frac{1}{2l}\int_{-l}^{l} \phi(x)dx \\ a_n = \frac{1}{l}\int_{-l}^{l} \cos\left(\frac{n\pi x}{l}\right)\phi(x)dx \\ b_n = \frac{1}{l}\int_{-l}^{l} \sin\left(\frac{n\pi x}{l}\right)\phi(x)dx \end{cases} \tag{3.2}$$

The given function ϕ is defined on an interval (-l,l) the series function is given by:

$$\phi \sim a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

Where, a_n and b_n are defined in (3.1) is known as the Full Fourier Series of ϕ. Generally, for a sequence of eigenfunction X_n of (3.2) that satisfy certain boundary conditions defined on the general Fourier series of a function ϕ as:

$$\phi \sim \sum_{n=1}^{\infty} a_n X_n(x)$$

which implies that,

$$a_n = \frac{< X_n(x), \phi >}{< x_n, x_n >}$$

If ϕ could be represented in terms of this infinite series we need to find coefficients a_n this implies:

$$\phi(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)$$

The coefficient is given by:

$$a_n = \frac{2}{l}\int_0^l \sin\left(\frac{n\pi x}{l}\right)\phi(x)dx$$

Where, n = 1,2,,3,...

Which is an important form of the coefficients and

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) e^{-k(\frac{n\pi x}{l})^2 t}$$

It is useful to write the Full Fourier Series in complex form. The eigenfunctions are associated with the Full Fourier Series which are given by:

$$\left(\cos\left(\frac{n\pi x}{l}\right), \sin\left(\frac{n\pi x}{l}\right)\right)$$

for n = 1,2,3,...

By applying the de Morvre's formula we have:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This can be written us in the form of:

$$\cos\left(\frac{n\pi x}{l}\right) = \frac{e^{\frac{in\pi x}{l}} + e^{\frac{-in\pi x}{l}}}{2}$$

$$\sin\left(\frac{n\pi x}{l}\right) = \frac{e^{\frac{in\pi x}{l}} - e^{\frac{-in\pi x}{l}}}{2}$$

For any linear combination of eigenfunction is an eigenfunction. Therefore, we have also the following eigenfunctions.

$$\begin{cases} \cos\left(\frac{n\pi x}{l}\right) + i\sin\left(\frac{n\pi x}{l}\right) = e^{\frac{in\pi x}{l}} \\ \cos\left(\frac{n\pi x}{l}\right) - i\sin\left(\frac{n\pi x}{l}\right) = e^{\frac{-in\pi x}{l}} \end{cases}$$

Thus the eigenfunction associated with the Full Fourier Series can be written as: $e^{\frac{in\pi x}{l}}$.

n = ...,-2,-1,0,1,2,3,... Now, let us represent a given function ϕ as an infinite series expansion in terms of these eigenfunctions. Now, we need to find the coefficients C_n such that:

$$\phi(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$$

We are describing earlier the eigenfunctions corresponding to distinct eigenvalues will be orthogonal as periodic boundary conditions are symmetric. Therefore, we have

$$\int_{-l}^{l} e^{\frac{in\pi x}{l}} e^{\frac{im\pi x}{l}} dx = 0$$

For $m \neq n$, for eigenfunctions corresponding to the same eigenvalue we need to check the L^2 inner product. For the eigenvalue $\lambda_n = (\frac{n\pi}{l})^2$ we have the eigenfunctions, $e^{\frac{in\pi x}{l}}$ and $e^{\frac{-in\pi x}{l}}$. By direct calculation, we have that:

$$\int_{-l}^{l} e^{\frac{in\pi x}{l}} e^{\frac{-in\pi x}{l}} dx = 0$$

for $n \neq 0$

$$\int_{-l}^{l} e^{\frac{in\pi x}{l}} e^{\frac{-in\pi x}{l}} dx = 2l$$

Therefore, the coefficients C_n to be given by:

$$C_n = \frac{l}{2l} \int_{-l}^{l} e^{\frac{-in\pi x}{l}} dx$$

consequently, the complex form of the Full Fourier Series for a function ϕ is defined on an interval $(-l, l)$ is given by:

$$\phi \sim \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$$

Where, $C_n = \frac{l}{2l} \int_{-l}^{l} e^{\frac{-in\pi x}{l}} \phi(x) dx$

3.1 Fourier Transformations

Let us study the heat equation on the real number line. Think about the initial value problem

$$\begin{cases} u_t = ku_{tt}, \quad -\infty < x < \infty, \quad t > 0 \\ u(x,0) = \phi(x) \end{cases}$$

In the case of heat equation on an initial value problem, we found a solution u using Fourier Series. The case of the heat equation on the whole real line, the Fourier Series will be substituted by the Fourier Transformation. For a function ϕ defined on the interval [-l,l] we are defined its Full Fourier Series as

$$\phi \sim \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$$

Where, $C_n = \frac{l}{2l}\int_{-l}^{l} \phi(x) e^{\frac{-in\pi x}{l}} dx$

the constant C_n depend on ϕ. So C_n is the Fourier Transformation, then insert the coefficients C_n into an infinite series, we get that:

$$\phi \sim \left[\sum_{n=-\infty}^{\infty} \frac{l}{2l} \int_{-l}^{l} e^{\frac{-in\pi x}{l}} \phi(y) dy \right] e^{\frac{in\pi x}{l}}$$

Now, letting $k = \frac{n\pi}{l}$, then we can write this as in the form of:

$$\phi \sim \left[\sum_{n=-\infty}^{\infty} \frac{l}{2l} \int_{-l}^{l} \phi(y) e^{-i(y-x)k} dy \right] \frac{\pi}{l}$$

3.1.1 Definition of the Fourier Transformation

Definition 3.1.1. *Considеer $g \in L^1(\Re^n)$ if $\int_{\Re^n} |g(x)|dx < +\infty$*

For $g \in L^1(\Re^n)$, we have defined its Fourier Transformation at a point $\eta \in$

$\Re^n$ *as*

$$g(\eta) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\Re} e^{ix\eta} g(\eta) d\eta$$

The inverse of the Fourier Transformation at the point $\eta \in R^n$ is defined by:

$$g(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\Re} e^{ix\eta} g(\eta) d\eta$$

In the Fourier Transformation we can defined with a constant of $\frac{1}{(2\pi)^2}$ can be replaced by $\frac{1}{(2\pi)^{\frac{n}{2}}}$ and the inverse Fourier Transformation is defined with a constant 1 replaced by the constant $\frac{1}{(2\pi)^{\frac{n}{2}}}$.

Theorem 3.1.1 (Plancherel's Theorem)**.** *If* $u \in L^1(R^n)$*, then* $u, v \in L^2(R^n)$ *and*

$$||u||_{L^2(R^n)} = ||v||_{L^2(R^n)}$$

Proof.

$$g(\eta) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\Re^n} e^{-ix\eta} e^{-\epsilon|x|^2}$$

where, $\quad g(\eta) = \frac{1}{(2\epsilon)^{\frac{n}{2}}} e^{-(|\epsilon|)^2 4\epsilon}$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \left(\int_{-\infty}^{\infty} e^{-ix_1\eta_1} e^{-\epsilon x_1^2} dx \right) ... \int_{-\infty}^{\infty} e^{-ix_n\eta_n} e^{-\epsilon x_n^2} dx_n$$

For $x \in \Re$, and hence we defined that

$$\int_{-\infty}^{\infty} e^{-ix\eta} e^{-\epsilon x^2} dx$$

By using completing the square method gives us:

$$-\epsilon x^2 - ix\eta = -\epsilon \left[x^2 + \frac{i\eta x}{\epsilon} + \left(\frac{i\eta}{2\epsilon}\right)^2 - \left(\frac{i\eta}{2\epsilon}\right)^2 \right]$$

$$= -\epsilon \left[x + \left(\frac{i\eta}{2\epsilon}\right) \right]^2 + \epsilon \left(\frac{i\eta}{2\epsilon}\right)$$

Therefore, we have

$$\int_{-\infty}^{\infty} e^{-ix\eta} e^{-\epsilon x^2} dx = \int_{-\infty}^{\infty} e^{-\epsilon\left[x+(\frac{i\eta}{2\epsilon})\right]^2} e^{\frac{-\epsilon^2}{4\epsilon}} dx$$

Let $W = x + \left(\frac{i\eta}{2\epsilon}\right)$, then we can get

$$\int_{-\infty}^{\infty} e^{-\epsilon\left[x+(\frac{i\eta}{2\epsilon})\right]^2} e^{\frac{-\epsilon^2}{4\epsilon}} dx = e^{\frac{-\epsilon^2}{4\epsilon}} \int_{\delta} e^{-\epsilon W^2} dw$$

Where, δ is the line in the complex plane given by: $\delta = W \in Ç : y = x + \frac{i\eta}{2\epsilon}, x \in \Re$ with out loss of generality, we assume that $\eta > 0$. identically it works the analysis if $\eta < 0$ then we have that the value:

$$\int_{\delta} e^{-\epsilon w^2} dw = \lim_{\Re \to +\infty} \int_{\delta_{\Re}} e^{-\epsilon w^2} dw$$

Define $a_R^1, \quad a_R^2$ and a_R^3 for any $R > 0$.

$a_R^1 = \{x \in R : |x| \leq R\}$

$a_R^2 = \left\{w \in \ \mathcal{C} : w = x + iy, \quad x, \quad y \in \Re, x = \Re, \quad 0 \leq y \leq \frac{\eta}{2\epsilon}\right\}$

$a_R^3 = \left\{w \in \mathcal{C} : w = x + iy, \quad x, \quad y \in \Re, x = -\Re, \quad 0 \leq y \leq \frac{\eta}{2\epsilon}\right\}$

From complex analysis, we know that:

$$\int_{C} e^{-\epsilon w^2} dw = 0$$

where, C is a closed curve given by:

$C = \delta_R \cup a_R^1 \cup a_R^2 \cup a_R^3$. Transverse in the counter clock wise direction.

We have

$$\int_{\delta_R} e^{-\epsilon w^2} dw = \int_{a_R} e^{-\epsilon w^2} dw$$

where, the integral on the right hand side is the line integral give by:

$a_R = a_R^1 \cup a_R^2 \cup a_R^3$. Transverse the direction shown. Hence,

$$\int_\delta e^{-\epsilon w^2} dw = \lim_{\Re \to +\infty} \int_{a_R} e^{-\epsilon w^2} dw$$

But, as $\Re \to +\infty$,

$$\int_{a_R^j} e^{-\epsilon w^2} \to 0$$

for $j = 2, 3$ and

$$\int_{a_R^1} e^{-\epsilon w^2} dw \to \int_{-\infty}^{\infty} e^{-\epsilon x^2} dx$$

Therefore, $\int_\delta e^{-\epsilon w^2} dw \to \int_{-\infty}^{\infty} e^{-\epsilon x^2} dx$

consequently

$$\int_{-\infty}^{\infty} e^{-\eta} i x^2 e^{-\epsilon x^2} dx$$

$$= e^{\frac{-\eta^2}{4\epsilon}} \int_\delta e^{-\epsilon w^2} dw$$

$$= e^{\frac{-\eta^2}{4\epsilon}} \int_{-\infty}^{\infty} e^{-\epsilon x^2} dx$$

$$= e^{\frac{-\eta^2}{4\epsilon}} \int_{-\infty}^{\infty} e^{-x^2} \frac{dx}{\sqrt{\epsilon}}$$

$$= \frac{e^{\frac{-\eta^2}{4\epsilon}}}{\sqrt{\epsilon}} \sqrt{\pi}$$

$\square$

Then we have,

$$g(\eta) = \frac{1}{(2\pi)^{\frac{n}{2}}} \left(\frac{e^{\frac{-\eta_1}{4\epsilon}}}{\sqrt{\epsilon}} \sqrt{\pi}\right) = \left(\frac{e^{\frac{-\eta_n^2}{4\epsilon}}}{\sqrt{\epsilon}} \sqrt{\pi}\right)$$

$$= \frac{1}{(2\eta)^{\frac{n}{2}}} e^{\frac{-|\eta|^2}{4\eta}}$$

Let us see the following claims

Claim 1

$$\mu(x) = (\nu * \omega)(x) = \int_{\Re} \nu(x-y)\omega(y)dy \in L^1(\Re) \tag{3.3}$$

where, $\nu, \omega \in L^1(\Re)$ (μ is the convolution of ν and ω). Then

$$\widehat{\mu}(\eta) = \widehat{\nu * \omega}(\eta) = (2\pi)^{\frac{n}{2}}\widehat{\nu}(\eta)\widehat{\omega}(\eta)$$

Proof. $\widehat{\mu}(\eta) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\Re^n} e^{-ix\eta}\mu(x)dx = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\Re^n} e^{-ix\eta}\int_{\Re}\nu(x-y)\omega(y)dydx$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\Re}\left[\int_{\Re^n} e^{-i(x-y)\eta}\nu(x-y)dx\right]e^{-iy\eta}\omega(y)dy$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\Re^n}(2\pi)^{\frac{n}{2}}\widehat{\nu(\eta)}e^{-iy\eta}\omega(y)dy$$

$$= \widehat{\nu}(\eta)\left[\int_{\Re^n}(2\pi)^{\frac{n}{2}}e^{-iy\eta}\omega(y)dy\right]$$

$$= \widehat{\nu}(\eta)\left[\int_{\Re^n}e^{-iy\eta}\omega(y)\right]dy$$

$$= \widehat{\nu}(\eta)(2\pi)^{\frac{n}{2}}\widehat{\omega}(\eta)$$

$$= (2\pi)^{\frac{n}{2}}\widehat{\nu}(\eta)\widehat{\omega}(\eta)$$

□

Next, we use the above claim (1) to prove Plancherel's theorem.

Proof. From the above theorem (3.1.1) By assumption $\nu, \omega, \mu \in L^1(\Re^n)\bigcap L^2(\Re^n)$ Let $\omega(x) = \overline{\nu}(-x)$ and $\mu(x) = (\nu * \omega)(x)$. We have

$$\widehat{\omega}(\eta) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\Re^n} e^{-ix\eta}\omega(x)dx$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\Re} e^{iy\eta}\overline{\nu}(-y)dy$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\Re} e^{iy\eta}\overline{\nu}(y)dy$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \overline{\int_{\Re^n} e^{-iy\eta}\nu(y)dy}$$

Therefore, using the above claim we have,

$$\widehat{\mu}(\eta) = (2\pi)^{\frac{n}{2}}\widehat{\nu}(\eta)\widehat{\omega}(\eta) = (2\pi)^{\frac{n}{2}}\widehat{\nu}(\eta)\overline{\widehat{\nu}}(\eta) = (2\pi)^{\frac{n}{2}}\,|\widehat{\nu}|^2$$

Now employing the fact, if g and h are in $L^1(\Re^n)$, then g, $\widehat{h}$ are in $L^\infty(\Re^n)$ and

$$\int_{\Re^n} g(x)\widehat{h}(x)dx = \int_{\Re^n} \widehat{g}(\eta)h(\eta)d\eta$$

Now, substitute directly and we have that:

$$\int_{\Re^n} g(x)\widehat{h}(x)dx = \int_{\Re^n} g(x)\left[\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\Re} e^{-ix\eta}h(\eta)d\eta\right]dx = \int_{\Re^n}\left[\frac{1}{(2\pi)\frac{n}{2}}\int_{\Re} e^{-ix\eta}g(x)dx\right]h(\eta)d\eta$$

$$= \int_{\Re^n} \widehat{g}(\eta)h(\eta)d\eta$$

Therefore, let $g(x) = e^{\epsilon|x|^2}$ and $h(x) = \mu(x)$ as defined the above. substituting g and h into (3.2) and applying the above claim to get the Fourier Transformation of g, we have.

$$\int_{\Re^n} e^{-\epsilon|x|^2}\widehat{\mu(\eta)}d\eta = \int_{\Re^n} \frac{1}{(2\pi)} e^{\frac{-|x|^2}{4\epsilon}}\mu(x)dx$$

Taking the limit of both sides as $\epsilon \to 0^+$,

$$\lim_{\epsilon\to 0^+}\int_{\Re^n} e^{-\epsilon|\eta|^2}\widehat{\mu}(\eta)d\eta = \int_{\Re_n}\widehat{\mu}(\eta)d\eta$$

□

Claim 2

$$\lim_{\epsilon\to 0^+}\frac{1}{(2\eta)^{\frac{n}{2}}}\int_{\Re^n} e^{\frac{-|x|^2}{4\eta}}\mu(x)dx = (2\pi)^{\frac{n}{2}}\mu(0)$$

Proof. We can prove the claim as follows:

$$\frac{1}{(4\pi\epsilon)^{\frac{n}{2}}}\int_{\Re^n} e^{\frac{-|x|^2}{4\epsilon}}\mu(x)dx \to \mu(0)$$

as $\epsilon \to 0^+$ Note that:

$$\frac{1}{(4\pi\epsilon)^{\frac{n}{2}}}\int_{\Re^n} e^{\frac{-|x|^2}{4\epsilon}}dx = 1$$

Now, we have

$$\int_{-\infty}^{\infty} e^{-\mu^2}d\mu = \sqrt{\pi}$$

Therefore,

$$\frac{1}{(4\pi\epsilon)^{\frac{n}{2}}}\int_{\Re^n} e^{\frac{-|x|^2}{4\epsilon}}\mu(x)dx - \mu(0) = \frac{1}{(4\pi\epsilon)^{\frac{n}{2}}}\int_{\Re^n} e^{\frac{-|x|^2}{4\epsilon}}[\mu(x)-\mu(0)]dx$$

Now, show that $\forall\delta > 0$. There exist $\bar{\epsilon} > 0$, such that $\left|\frac{1}{(4\pi\epsilon)^{\frac{n}{2}}}\int_{\Re^n} e^{\frac{-|x|^2}{4\epsilon}}\left[\mu(x)-\mu(0)\right]dx\right| < \delta$. For $0 < \epsilon < \bar{\epsilon}$, let $B(0,\delta)$ be the ball of radius δ about 0. Now, separate the integral into pieces as follows.

$\frac{1}{(4\pi\epsilon)^{\frac{n}{2}}}\int_{\Re^n} e^{\frac{-|x|^2}{4\epsilon}}\left[\mu(x)-\mu(0)\right]dx$

$$\leq \left|\frac{1}{(4\pi\epsilon)^{\frac{n}{2}}}\int_{B(0,\delta)} e^{\frac{-|x|^2}{4\epsilon}}\left[\mu(x)-\mu(0)\right]dx\right| + \left|\frac{1}{(4\pi\epsilon)^{\frac{n}{2}}}\int_{\Re^n - B(0,\delta)} e^{\frac{-|x|^2}{4\epsilon}}\left[\mu(x)-\mu(0)\right]dx\right|$$

$$= I + J$$

□

For term I: we have,

$$\left|\frac{1}{(4\pi\epsilon)^{\frac{n}{2}}}\int_{B(0,\delta)} e^{\frac{-|x|^2}{4\epsilon}}\left[\mu(x)-\mu(0)\right]dx\right|$$

$$\leq |\mu(x)-\mu(0)|_{L^\infty(B(0,\delta))}\frac{1}{(4\pi\epsilon)^{\frac{n}{2}}}\int_{\Re^n} e^{\frac{-|x|^2}{4\epsilon}}dx$$

$$< \frac{\delta}{2}$$

For δ sufficiently small, using the reality $\mu \in C(\Re^n)$ and (3.3). Taking δ fixed and small, we consider the term J,

$$\left| \frac{1}{(4\pi\epsilon)^{\frac{n}{2}}} \int_{\Re^n} e^{\frac{-|x|^2}{4\epsilon}} \left[\mu(x) - \mu(0)\right] dx \right|$$

$$\leq \frac{1}{(4\pi\epsilon)^{\frac{n}{2}}} \int_{\Re^n} e^{\frac{-|x|^2}{4\epsilon}} |\mu(x)|\, dx + \frac{1}{(4\pi\epsilon)^{\frac{n}{2}}} \int_{\Re^n - B(0,\delta)} e^{\frac{-|x|^2}{4\epsilon}} |\mu(0)|\, dx$$

$$\left| \frac{1}{(4\pi\epsilon)^{\frac{n}{2}}} e^{\frac{-|x|^2}{4\epsilon}} |_{L^\infty(\Re^n - B(0,\delta))} \right| \int_{\Re} |\mu(x)|\, dx + e^{\frac{-\delta^2}{8\epsilon}} |\mu(0)| \frac{2^{\frac{n}{2}}}{(8\pi\epsilon)^{\frac{n}{2}}} \int_{\Re^n - B(0,\delta)} e^{\frac{-|x|^2}{8\epsilon}} dx$$

$$\leq C \left| \frac{1}{(4\pi\epsilon)^{\frac{n}{2}}} e^{-\frac{|\delta|^2}{4\epsilon}} \right| + Ce^{-\frac{\delta^2}{8\epsilon}} \frac{1}{(8\pi\epsilon)^{\frac{n}{2}}} \int_{\Re^n} e^{\frac{-|x|^2}{8\epsilon}} dx \leq C \left| \frac{1}{(4\pi\epsilon)^{\frac{n}{2}}} e^{\frac{-|\delta|^2}{4\epsilon}} \right| + Ce^{\frac{-\delta^2}{8\epsilon}} < \frac{\delta}{2}.$$

For ϵ is sufficiently small, using the reality for a fixed $\delta \neq 0$.

$$\lim_{\epsilon \to 0^+} \frac{1}{(4\pi\epsilon)^{\frac{n}{2}}} e^{\frac{-|\delta|^2}{4\epsilon}} = 0$$

and

$$\lim_{\epsilon \to 0^+} e^{\frac{-\delta^2}{8\epsilon}} = 0$$

Therefore, $I + J < \delta$. For ϵ chosen sufficiently small and thus (3.4) combining to (3.3) and (3.4) with (3.2), we can conclude that

$\int_{\Re^n} \widehat{\mu}(\eta) d\eta = (2\pi)^{\frac{n}{2}} \mu(0)$

It follows that:

$\widehat{\mu}(\eta) d\eta = (2\pi)^{\frac{n}{2}} |\widehat{\nu}|^2$ and $\mu(0) = (\nu * \omega)(0)$

$$= \int_{\Re^n} \nu(x)\overline{\nu}(x) dx = (2\pi)^{\frac{n}{2}} \int_{\Re^2} |\widehat{\nu}|^2 d\eta$$

$$= (2\pi)^{\frac{n}{2}} \int_{\Re^n} |\nu|^2 dx$$

or equivalently $|\widehat{\nu}|_{L^2} = |\nu|_{L^2}$

3.1.2 Fourier Transformation on $L^2(R^n)$

For $v \in L^1(\Re)$, which means $\int_{\Re^n} |v(x)|\, dx < +\infty$. This indicates that the Fourier Transform is well-defined, since the integral converges.
Let $v \in L^2(\Re^n)$. Approximate v by a sequence of functions v_k such that $v_k \in L^2(\Re^n) \bigcap L^2(\Re^n)$ and $||v_k - v||\, L^2 \to 0$ as $k \to +\infty$. By the Plancherel's theorem, $||\widehat{v_k} - \widehat{v_j}||_{L^2} = \left|\left|\widehat{v_k - v_j}\right|\right|_{L^2} = ||v_k - v_j||_{L^2} \to 0$ as $k, j \to +\infty$ v_k is the Cauchy sequence in $L^2(\Re^n)$ and hence converges to some $u \in L^2(\Re^n)$. We can defined the Fourier Transform of v by the function g. That is $\widehat{v} = u$.

3.1.3 Inverse of the Fourier Transform

The inversion of the Fourier Transform, i.e finding $h(t)$ for a given $F(z)$, is sometimes possible using the inversion integral. However, in elementary cases we can use a table of standard Fourier Transform together, with the appropriate properties of the Fourier Transform.

Example 3.1.1. *Find the inverse Fourier Transform of*

$$F(z) = 20\frac{\sin(5z)}{5z}$$

Solution:

Apply the sine function, $h(t)$ is a function in the Fourier Transform. We have the standard form of:

$$F\left(k_c(t)\right) = 2c\frac{sinzc}{zc}$$

or

$$F^{-1}\left(2c\frac{sinzc}{zc}\right) = k_c(t)$$

Let $c = 5F^{-1}\left(10\frac{sin5z}{5z}\right) = k_5(t)$. Thus, by the linearity property
$h(t) = F^{-1}\left(20\frac{\sin 5z}{5z}\right) = 2k_5(t)$

Example 3.1.2. *Find the inverse of Fourier Transform of*

$$W(x) = 20\frac{\sin 5z}{5z}e^{-3iz}$$

solution:

Use the complex exponential factor in the Fourier Transform suggests the time shift property with the time $t_0 = +3$ (shift to the right). From example (3.3.1)
$F^{-1}\{20\frac{\sin 5z}{5z}\} = 2z_5(t)$
so

$$w(t) = F^{-1}\{20\frac{\sin 5z}{5z}e^{-3iz}\} = 2k_5(t-3)$$

Example 3.1.3. *Find the inverse Fourier Transform of*

$$Y(z) = \frac{2}{1+2(z-1)i}$$

solution:

put $z = (z-1)$. Hence, we assume that:
$w(z) = \frac{2}{1+2iz}$
considere the relevant standard form given by:

$$F\left(e^{-\beta t}u(t)\right) = \frac{1}{\beta + iz}$$

or equivalently

$$F^{-1}\left(\frac{1}{\beta + iz}\right) = e^{-\beta t}u(t)$$

And hence, we can write as $w(z) = \frac{1}{\frac{1}{2}+iz}$,
$w(t) = e^{\frac{-1}{2}t}u(t)$. Then, by the frequency shifting property with $z_0 = 1$

$$y(t) = F^{-1}\left(\frac{2}{1+2(z-1)i}\right) = e^{-\frac{1}{2}t}e^{it}u(t)$$

Chapter 4

Solving Heat Equation in$\mathbb{R}^n$

4.1 To solve the IVP for the Heat Equation in $\mathbb{R}^n$ Using Fourier Transform

Assume that the initial value problen for the heat equation on $\mathbb{R}$.

$$\begin{cases} u_t = ku_{xx}, \quad x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = \phi(x) \end{cases}$$

Applying the Fourier Transform to the heat equation, we have,

$$w_t(\eta,t) = kw_{tt}(\eta,t) \Rightarrow w_t = k(i\eta)^2 w = -k\eta^2 w$$

Now, solve this ordinary differential equation and applying the initial conditions:

$$u(x,0) = \phi(t)$$

We have $w(\eta,t) = \phi(\eta)e^{-k\eta^2 t}$. Then we have:

$$u(x,t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_R e^{ix\eta} w(\eta,t) d\eta = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\eta} \psi(\eta) e^{-k\eta^2 t} d\eta$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\eta} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iy\eta} \phi(y) dy \right] e^{-k\eta^2 t} d\eta$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(y) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(y-x)\eta} e^{-k\eta^2 t} d\eta \right] dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(y) g(y-x) dy$$

Where, $w(\eta) = e^{-k\eta^2 t}$ this implies that $g(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{4kt}}$

Therefore,

$g(y-x) = \frac{1}{\sqrt{2kt}} e^{\frac{-(x-y)^2}{4kt}}$

The solution of the Initial Value Problem for the heat equation on $\mathbb{R}$ is

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} \phi(y) e^{\frac{-(x-y)^2}{4kt}} dy$$

for $x \in \mathbb{R}^n$, $t > 0$.

We can use the same analysis to solve the problem in higher dimension. Consider the initial value problem for the heat equation in $\mathbb{R}^n$,

$$\begin{cases} u_t = k\triangle u, \quad x \in \mathbb{R}^n, \quad t > 0 \\ u(x,0) = \phi(t) \end{cases} \tag{4.1}$$

Using the Fourier Transform as in one dimension case, we can arrive at the solution formula.

$$u(x,t) = \frac{1}{\sqrt{(4k\pi t)^n}} \int_{\mathbb{R}^n} \phi(y) e^{-|x-y|^2} dy \quad for \quad t > 0 \tag{4.2}$$

Theorem 4.1.1. *Assume* $\phi \in C(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$ *and defined by* (4.2) *then*

1. $u \in L^\infty(\mathbb{R}^n \times (0,\infty))$

2. $u_t - k\triangle u \quad \forall_x \in \mathbb{R}^n, \quad t > 0$

3. $\lim_{(x,t)\to(x_0,0)} u(x,t) \quad = \quad \phi(x_0),\ x_0,\ x \in \mathbb{R}^n, \quad t > 0$

Proof. 1. Hence, the function $\frac{1}{l^{\frac{n}{2}}}e^{\frac{-|x|^2}{4kt}}$ is infinitely differentiable with uniformly bounded derivatives of all order on $\mathbb{R}^n \times (\delta, \infty)\,, \quad \delta > 0$

2. By direct calculation, we have seen that the function is

$u(x,t) = \frac{1}{(4k\pi t)^{\frac{n}{2}}} e^{\frac{-|x-y|^2}{4kt}}, \quad x, y \in \mathbb{R}^n, \quad \forall_t > 0.$

This function is infinitely differentiable. Hence,

$u_t(x,t) = k\triangle u(x,t) = \int_{\mathbb{R}^n} [(u_t - k\triangle u)(x-y,t)]\, \phi(y)dy = 0$

3. In this case we have $x_0 \in \mathbb{R}^n$ and $\epsilon < 0$ and we wants to show there exists a $\delta > 0$ such that

$|u(x,t) - \phi(x_0)| < \delta$ for $|(x,t) - (x_0, 0)| < \delta$.

We have,

$$\left| \frac{1}{(4k\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{\frac{-|x-y|^2}{4kt}} \phi(y) - \phi(x_0) \right| < \epsilon$$

$|u(x,t) - \phi(x_0)| < \delta$. Where δ is chosen sufficiently small, then we have in particular applying:

$$\left| \frac{1}{(4\pi kt)^{\frac{n}{2}}} \int_{\mathbb{R}} e^{\frac{-|x-y|^2}{4kt}} \left[\phi(y) - \phi(x_0)\right] dy \right| = \left| \frac{1}{(4\pi kt)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-|x-y|^2} \left[\phi(y) - \phi(x_0)\right] \right| dy \tag{4.3}$$

Let $B(x_0, r)$ be a ball of radius r about x_0 we look at the integral in (4.3) over $B(x_0, r)$. using the reality ϕ is continuous, for δ chosen sufficiently small. From the choice of δ the integral in (4.3) something which completes $B(x_0, r)$. For $y \in \mathbb{R}^n - B(x_0, \delta), \quad |x_0 - x| < \frac{\delta}{2}$. We have,

$$|y - x_0| \leq |y - x| + |x_0 - x| < |y - x| + \frac{\delta}{2} < |y - x| + \frac{1}{2}\,|y - x_0|$$

Therefore, on a single items of the given integral we have that:

$|y - x_0| \leq |y - x| + |x_0 - x| < |y - x|$, the integral is bounded as

$$\left| \frac{1}{(4kt\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n - B(x_0, r)} e^{-|x-y|^{\frac{2}{4kt}}} \left[\phi(x) - \phi(y)\right] dy \right| \leq \frac{C}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n - B(x_0, r)} e^{\frac{-|x_0 - y|^2}{16kt}} dy$$

So, changing of the variables we have
$\frac{C}{t^{\frac{1}{2}}}\int_{\mathbb{R}^n-B(x_0,r)} e^{\frac{-(x_0-y)^2}{16kt}}dy = \frac{C}{t^{\frac{1}{2}}}\int_{\gamma}^{\infty} e^{\frac{-y^2}{16kt}}dy = C\int_{\gamma\sqrt{t}}^{\infty} e^{-z^2}dz < \frac{\in}{2}$
Hence,

$$\lim_{(x,t)\to(x_0,0)} u(x,t) = \phi(x_0)$$

□

4.2 The Fundamental Solution

Definition 4.2.1. *The delta distribution δ is a distribution of a mathematical object having a sufficient function of $\phi(x)$ as follows:*

$$\langle \delta, \phi \rangle = \phi(0)$$

The notation $\langle ., . \rangle$ is meant to remind us the L^2 inner product

$$\langle g, h \rangle = \int_{R^n} g(x)h(x)dx$$

think about the solution of (4.1) for the initial value problem of heat equation in $\mathbb{R}^n$. Define the function

$$\begin{cases} G(x,t) = \frac{1}{(4\pi kt)^{\frac{n}{2}}} e^{\frac{-|x|^2}{4kt}}, & t > 0 \\ 0, \quad t < 0 \end{cases} \tag{4.4}$$

Where, G is the solution of the heat equation for $t > 0$ and in addition, we can write the solution of (4.1) as

$$u(x,t) = [G(t) * \phi(x)] = \int_{\mathbb{R}^n} G((x-y),t)\phi(y)dy$$

$G(x,t)$ is the fundamental solution of the heat equation. G itself is a solution of the heat equation. This implies:

$$G_t - k\triangle G = 0, \quad x \in \mathbb{R}^n, \quad t > 0$$

We know that for $x \neq 0$, However, for

$$x = 0, \quad \lim_{t \to 0^+} G(x,t) = \infty$$

In addition, use the equation we see that:

$$\int_{\mathbb{R}^n} G(x,t)dx = 1, \quad for \quad t > 0$$

Therefore,

$$\lim_{t \to 0^+} \int_{\mathbb{R}^n} G(x,t)dx = \infty$$

for $x \in R^n$

Theorem 4.2.1. *If $\phi(x)$ is a bounded and continuous function then,*

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} \phi(y)dy \tag{4.5}$$

That satisfies the heat equation $u_t - ku_{tt} = 0$ for $t > 0$ and $\forall_x \in \mathbb{R}$ more over,

$$\lim_{t \to 0^+} u(x,t) = \phi(x), \quad \forall_x \in \mathbb{R}$$

The standard change of variables in the integral we have letting $z(x,y) = \frac{x-y}{\sqrt{4kt}}$, then $dz = \frac{-dy}{\sqrt{4kt}}$ then equation (4.5) be comes

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-^2} \phi(x - \sqrt{4kt}z)dz \tag{4.6}$$

For a constant value of x, if put $t = 0$ in this last integral, we could then factor ϕ out of the integral and write

$$\lim_{t \to 0^+} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} \phi\left(x - \sqrt{4ktz}\right) dz$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-z^2} \lim_{t\to 0^+} \phi\left(x-\sqrt{4kt}z\right) dz$$

$$= \frac{1}{\sqrt{\pi}}\phi(x)\int_{-\infty}^{\infty} e^{-z^2} dz$$

$$= \phi(x)$$

Example 4.2.1. *Let*

$$G(x,t) = \begin{cases} 1, & \text{if } x < t < x+1 \\ 0, & \text{otherwise} \end{cases}$$

and hence, consider

$$\lim_{x\to\infty}\int_0^{\infty} G(x,t)dy = 1, \quad x < t < x+1$$

and

$$\int_0^{\infty} \lim_{x\to\infty} G(x,t)dy = 1, \quad x < t < x+1$$

$$\Rightarrow \int_0^{\infty} G(x,t)dy = 1$$

For any positive value of x with a fixed positive value of x, but for any fixed value of t we have $G(x,t) = 0$, for $x > t$ and hence,

$$\lim_{x\to\infty} G(x,t) = 0, \quad \forall t$$

and the second limit is

$$\int_0^{\infty} \lim_{x\to\infty} G(x,t)dy = \int_0^{\infty} 0dy = 0$$

our aim is to show that

$$\lim_{t\to 0^+} u(x,t) = \phi(x)$$

In the other way we have that:

$$\lim_{t\to 0^+} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} \phi(x - \sqrt{4ktz}) dz = \phi(x)$$

We can break the integral into two parts, based on the following observations. If t is very small and z is not too large. We will have that $\phi(x - \sqrt{4ktz})$ is very closed to $\phi(x)$. If z is large, then e^{-z^2} is very small, so the value of the part of the integral where z is large should be close to zero. First, restate what are trying to prove:

$$\lim_{t\to 0^+} (u(x,t) - \phi(x)) = 0$$

This can be written as:

$$\begin{aligned} u(x,t) - \phi(x) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} \phi(x - \sqrt{4kt}z) dz - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} \phi(y) dz \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} \left[\phi(x - \sqrt{4kt})z - \phi(x)z \right] \end{aligned}$$

Since we are going to use

$$\lim_{t\to 0^+} F(t) = L$$

Which means, for any out put tolerance $\epsilon > 0$, we can choose an input tolerance $\delta > 0$ and then guarantee that whenever the input is with in δ of ϕ (i.e if $|t| < \delta$). Then, the out put $F(t)$ with in ϵ of L

$$|F(t) - L| < \epsilon$$

Limit as $t \to \infty$ are a little differently. When we say that, $\int_0^\infty e^{-z^2} dz$ converges. We mean that for any out put tolerance ϵ, there is a number D. Hence,
$\int_D^\infty e^{-z^2} dz < \epsilon$. which implies that the integral of e^{-z^2} from 0 to D is with in

the tolerance ϵ of the volume of the improper integral. The main aim is to show that given $\epsilon > 0$ we can find a $\delta > 0$ so that, if $0 < t < \delta$, then,

$$\left| \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} (\phi(x - \sqrt{4kt}z) - \phi(x)) dz \right| < \epsilon$$

Given that ϕ is a bounded function and that the improper integral $\int_{-\infty}^{\infty} e^{-z^2} dz$ from converges from the boundedness of $|\phi(x)| < B$, $x \in \mathbb{R}$. Then triangle inequalities gives $\left|\phi(x - \sqrt{4kt}z) - \phi(x)\right| < 2B$,

Hence, we can estimate that:

$$\left| \frac{1}{\sqrt{\pi}} \int_{D}^{\infty} e^{-z^2} (\phi(x - \sqrt{4kt}z) - \phi(x)) dz \right|$$

$$\leq \frac{1}{\sqrt{\pi}} \int_{D}^{\infty} e^{-z^2} (\phi(x - \sqrt{4kt}z) - \phi(x)) dz$$

$$\leq \frac{2B}{\sqrt{\pi}} \int_{D}^{\infty} e^{-z^2} dz$$

and

$$\left| \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-D} e^{-z^2} (\phi(x - \sqrt{4kt}z) - \phi(x)) dz \right|$$

$$\leq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-D} e^{-z^2} \left| \phi\left(x - \sqrt{4kt}z\right) - \phi(x) \right| dz$$

$$\leq \frac{2B}{\sqrt{\pi}} \int_{-\infty}^{-D} e^{-z^2} dz$$

Because, the integral of e^{-z^2} from $-\infty$ to ∞ converges, we can choose D large enough so that:

$$\frac{2B}{\sqrt{\pi}} \int_{D}^{\infty} e^{-z^2} dz + \frac{2B}{\sqrt{\pi}} \int_{-\infty}^{-D} e^{-z^2} dz < \frac{\epsilon}{2}$$

This implies that:

$$\left| \frac{1}{\sqrt{\pi}} \int_{|z|>D} e^{-z^2} (\phi(x - \sqrt{4kt}z) - \phi(x)) dz \right| \leq \frac{\epsilon}{2}$$

In this case we can choose D in the region of the boundary, we show that:

$$\left| \frac{1}{\sqrt{\pi}} \int_{|z|>D} e^{-z^2} (\phi(x - \sqrt{4kt}z) - \phi(x)) dz \right| \leq \frac{\epsilon}{2} \tag{4.7}$$

Because, ϕ is continuous at x, we can chosen a $\delta_1 > 0$.

So that, if $|x - t| < \delta_1$, then $|\phi(x) - \phi(t)| < \frac{1}{2}\epsilon$. For the integral in (4.7), this means that for t to be small enough therefore, $\left|\sqrt{4kt}z\right| < \delta_1$ for all p ranging from $-D$ to D. To do this, choose $\delta < \frac{\delta_1^2}{4kD^2}$. This we will have, $\forall_t < \delta$ and $-D < z < D$

$$\left|\sqrt{4kt}z\right| < \left|\sqrt{4k\delta}z\right| < \frac{\sqrt{4k\delta_1^2}}{4kD^2}$$

Where $D = \delta_1$. So that,

$$\left|\phi(x\sqrt{4kt}z) - \phi(x)\right| < \frac{\epsilon}{2}$$

$\forall_z$ between $-D$ and D. Now, we prove estimate (4.7) above. If $t < \delta$ with δ given as above, then

$$\begin{aligned}
&\left| \frac{1}{\pi} \int_{-D}^{D} e^{-z^2} (\phi(x - \sqrt{4kt}z) - \phi(x)) dz \right| \\
&\leq \frac{1}{\pi} \int_{-D}^{D} e^{-z^2} \left| (\phi(x - \sqrt{4kt}z) - \phi(x)) \right| dz \\
&< \frac{1}{\pi} \int_{-D}^{D} e^{-z^2} \frac{\epsilon}{2} dz \\
&< \frac{\epsilon}{2} \frac{1}{\pi} \int_{-D}^{D} e^{-z^2} dz \\
&= \frac{\epsilon}{2}
\end{aligned}$$

To get estimate (4.6) and (4.7) implies that if $0 < t < \delta$, then

$$|u(x,t) - \delta(x)| = \left| \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-z^2} (\phi(x - \sqrt{4kt}z) - \phi(x)) dz \right|$$

$$\leq \left|\frac{1}{\pi}\int_{|p|>D} e^{-z^2}(\phi(x-\sqrt{4kt}z)-\phi(x))dz\right| + \left|\frac{1}{\pi}\int_{-D}^{D} e^{-z^2}(\phi(x-\sqrt{4kt}z)-\phi(x))dz\right|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Which is just what we need to conclude that

$$\lim_{t\to 0^+} u(x,t) = \phi(x)$$

Chapter 5

Summery

Heat equation is a very useful partial differential equation that describes the distribution of heat in a given region.
The general formula of heat equation is:

$$u_t = ku_{xx}$$

For a function u(x, y, z), which are x, y, z and the time t are variables of the function.
The heat equation is:

$$u_t = k\left(u_{xx} + u_{yy} + u_{zz}\right)$$

Or

$$u_t = k\nabla^2 u$$

where k is a constant.
Heat Equation that indicates the derivatives of the equation given by:

$$u_t - ku_{xx} = 0, \quad k > 0$$

We call it this the diffusion equation.

Fourier's law saying that heat flow from hot regoin to cold regions at a rate $k > 0$ is proportional to the given temperature gradient. There is one way the heat will leave in the region M through the boundary. That is

$$G'(t) = \int_{\partial M} k \nabla u j ds$$

where, ∂M is the boundary of M, 'j' is the outward unit normal vector to ∂M and ds is the surface measure over ∂M. Therefore, we have

$$\int_M c\rho u_t dx = \int_{\partial M} k \nabla u j ds$$

remained that for a vector field F, then the Divergent theorem gives

$$\int_{\partial M} F j ds = \int_M \nabla F dx$$

Now, we have

$$\int_M c\rho u_t(x,t) dx = \int_M \nabla(k \nabla u) dx$$

This guide us to the partial differential equation.

$$c\rho u_t = \nabla(k \nabla u) dx$$

If c, ρ and k are constant, we are lead the heat equation $u_t = k\nabla$ u and

$$\triangle u = \sum_{i=1}^{n} u_i u_i$$

The most common boundary conditions are the following

(i) Drichilet: $u(0,t) = 0 = u(l,t)$

(ii) Neumann: $u_x(0,t) = 0 = u_x(l,t)$

Now, let us present the techniques of separation of variables. This technique to connect something looking for a solution of a particular form. Now, we look for the solution in the form of:

$$u(x,t) = X(x)T(t)$$

For the functions X, T to be determined. Our supposition that we can looking for a solution of (1.1) of in the form of substituting this function $u = XT$ into the heat equation, we have the equation in the form
$XT' - kX''T = 0$ dividing the equation by kXT. we have
$\frac{T'}{kT} = \frac{X'}{X} = -\lambda$
for some constant λ. There exist a solution
$u(x,t) = X(x)T(t)$ is solution of the heat equation, then T and X must satisfy the equation
$\frac{T'}{kT} = -\lambda$ and $\frac{X''}{X} = -\lambda$ for some constant λ.

Fourier Transformations

Consider the Dirichlet boundary conditions, we can find the coefficients of a_n we have that:

$$\phi(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)$$

The Fourier coefficient is given by:

$$a_n = \frac{2}{l}\int_0^l \sin\left(\frac{n\pi x}{l}\right)\phi(x)dx$$

A function ϕ is defined on an interval (0,l), such that an infinite series is given by the function:

$$\phi \sim \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)$$

where the Fourier coefficient is:

$$a_n = \frac{2}{l}\int_0^l \sin\left(\frac{n\pi x}{l}\right)\phi(x)dx$$

this is called the Fourier sine series of ϕ.

Let us study the heat equation on the real number line. Think about the initial value problem

$$\begin{cases} u_t = ku_{tt}, \quad -\infty < x < \infty, \quad t > 0 \\ u(x,0) = \phi(x) \end{cases}$$

In the case of heat equation on an initial value problem, we found a solution u using Fourier Series. The case of the heat equation on the whole real line, the Fourier Series will be substituted by the Fourier Transformation. For a function ϕ defined on the interval [-l,l] we are defined its Full Fourier Series as

$$\phi \sim \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$$

Where, $C_n = \frac{l}{2l} \int_{-l}^{l} \phi(x) e^{\frac{-in\pi x}{l}} dx$

the constant C_n depend on ϕ. So C_n is the Fourier Transformation, then insert the coefficients C_n into an infinite series, we get that:

$$\phi \sim \left[\sum_{n=-\infty}^{\infty} \frac{l}{2l} \int_{-l}^{l} e^{\frac{-in\pi x}{l}} \phi(y) dy \right] e^{\frac{in\pi x}{l}}$$

Now, letting $k = \frac{n\pi}{l}$, then we can write this as in the form of:

$$\phi \sim \left[\sum_{n=-\infty}^{\infty} \frac{l}{2l} \int_{-l}^{l} \phi(y) e^{-i(y-x)k} dy \right] \frac{\pi}{l}$$

Definition of the Fourier Transformation

Consideer $g \in L^1(\Re^n)$ if $\int_{\Re^n} |g(x)| dx < +\infty$

For $g \in L^1(\Re^n)$, we have defined its Fourier Transformation at a point $\eta \in \Re^n$ as

$$g(\eta) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\Re} e^{ix\eta} g(\eta) d\eta$$

Definition of Fundamental Solution

The delta distribution δ is a distribution of a mathematical object having a sufficient function of $\phi(x)$ as follows:

$$\langle \delta, \phi \rangle = \phi(0)$$

The notation $\langle ., . \rangle$ is meant to remind us the L^2 inner product

$$\langle g, h \rangle = \int_{R^n} g(x)h(x)dx$$

think about the solution of (4.1) for the initial value problem of heat equation in $\mathbb{R}^n$. Define the function

$$\begin{cases} G(x,t) = \frac{1}{(4\pi kt)^{\frac{n}{2}}} e^{\frac{-|x|^2}{4kt}}, \quad t > 0 \\ 0, \quad t < 0 \end{cases}$$

Where, G is the solution of the heat equation for $t > 0$ and in addition, we can write the solution of (4.1) as

$$u(x,t) = [G(t) * \phi(x)] = \int_{\mathbb{R}^n} G((x-y),t)\phi(y)dy$$

$G(x,t)$ is the fundamental solution of the heat equation. G itself is a solution of the heat equation. This implies:

$$G_t - k\triangle G = 0, \quad x \in \mathbb{R}^n, \quad t > 0$$

We know that for $x \neq 0$, However, for

$$x = 0, \quad \lim_{t \to 0^+} G(x,t) = \infty$$

To get estimate (4.6) and (4.7) implies that if $0 < t < \delta$, then

$$|u(x,t) - \delta(x)| = \left| \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-z^2} (\phi(x - \sqrt{4kt}z) - \phi(x))dz \right|$$

$$\leq \left|\frac{1}{\pi}\int_{|p|>D} e^{-z^2}(\phi(x-\sqrt{4kt}z)-\phi(x))dz\right| + \left|\frac{1}{\pi}\int_{-D}^{D} e^{-z^2}(\phi(x-\sqrt{4kt}z)-\phi(x))dz\right|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Which is just what we need to conclude that

$$\lim_{t\to 0^+} u(x,t) = \phi(x)$$

Chapter 6

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