ADDIS ABABA UNIVERSITY
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REACHABILITY AND CONTROLLABLITY OF LINEAR TIME INVARIANT DYNAMICAL SYSTEM

A Thesis Submitted to the Department of Mathematics of Addis Ababa University in Partial Fulfillment of the Requirements of the Master of Science Degree in Mathematics

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We,the undersigned, hereby certify that we have read and examined this thesis, a thesis on **Reachability And Controllablity of linear time invariant dynamical system**, which is done by MESERET ABIY in partial fulfillment of the requarements for the degree of master of science and recommend to the school of graduate studies for acceptance of the thesis.

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Abstract

The linear time invariant discrete-continuous systems, studied in this work, contain the two coupled subsystems: subsystem with continuous-time dynamics and subsystem with discrete-time dynamics. Continuous dynamics is described by ordinary linear differential equations, and a discrete one is described by difference equations for system's state jumps in prescribed time moments. For this class of models the reachability and the controllability properties are investigated, concerned with the following questions: Is it possible to steer the system , by suitable choice of the input function, from one particular state to another particular state? How long does the transition take? Can we find a concrete formula for an input function that force the system to go from one particular state to another particular state? As well controls are found for such systems states transition. Those solutions were applied for dynamical systems control design.

Notations

$\mathbb{R}[s]^{p\times q}$ a polynomial matrix in the variable s with real coefficients.

$\mathbb{R}^{n_1\times n_2}$ set of real $n_1 \times n_2$ matrices

$\mathbb{Z}_+$ set of positive integers

$\mathbb{Z}$ set of integers

$\mathbb{Q}$ set of rational numbers

$\mathbb{R}$ set of real numbers

$\mathbb{C}$ set of complex numbers

$\mathbf{T}$ model of time set.

$\mathbf{W}$ denote the set of signal value set

$\mathbf{W}^{\mathbf{T}}$ set of functions $\mathbf{T} \rightarrow \mathbf{W}$

$\mathcal{B} \subseteq \mathbf{W}^{T}$ called behaviour

$\sum$ denote the dynamic system

deg degree of polynomial

det determinant

σ, σ_t shift-operator

$\mathcal{C}^k(\mathbb{R})$ set of k times continuously differentiable functions

im image of linear map

ker kernel of linear map

$\chi_{\mathbf{A}}$ characteristic polynomial of the square matrix $\mathbf{A}$

Chapter 1

Preliminaries

1.1 Quotient field of an integral domain

Definition 1.1.1. *By a ring we mean a nonempty set* $\mathbf{R}$ *with two binary operations* $+$ *and* $\cdot$ *called addition and ,multiplication (also called product), respectively, such that*

- $(\mathbf{R}.+)$ *is an additive abelian group.*
- $(\mathbf{R},\cdot)$ *is a multiplicative semigroup.*
- *Multiplication is distributive (on both sides) over addition;*

Definition 1.1.2. *A ring* $\mathbf{R}$ *whose nonzero elements form a group under multiplication is called a division ring. If in addition,* $\mathbf{R}$ *is commutative, then* $\mathbf{R}$ *is called a field.*

The rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$ are examples of fields.

Definition 1.1.3. *A ring* $\mathbf{R}$ *is called an integral domain if* $xy = 0, x \in \mathbf{R}, y \in \mathbf{R}.$ *implies* $x = 0$ *or* $y = 0$.

The ring of integers $\mathbb{Z}$ is an integral domain. Every field (therefore every division ring) is an integral domain.

Linearity A system is considered **linear** if it satisfies the conditions of Additivity and Homogeneity.
In short, a system is linear if the following is true: Take two arbitrary inputs, and produce two arbitrary outputs:

$$y_1 = f(x_1)$$

$$y_2 = f(x_2)$$

Now, a linear combination of the inputs should produce a linear combination of the outputs:

$$(\mathbf{A}x + \mathbf{B}y) = f(\mathbf{A}x) + f(\mathbf{B}y) = \mathbf{A}f(x) + \mathbf{B}f(y)$$

Definition 1.1.4. *A nonempty subset* $\mathbf{W}$ *of a vector space* $\mathbf{V}$ *is called a subspace of* $\mathbf{V}$ *if* $\mathbf{W}$ *is a vector space under the operations addition and scalar multiplication defined in* $\mathbf{V}$.

Theorem 1.1.1. *Let* $\mathbf{V}$ *be a vector space and* $\mathbf{S} = \{v_1, v_2, \ldots, v_n\}$ *be a basis of* $\mathbf{V}$. *Then every set of vectors in* $\mathbf{V}$ *containing more than* n *vectors in* $\mathbf{V}$ *is linearly dependent.*

Theorem 1.1.2. *If one subspace is contained in another,* $\mathbf{S} \subseteq \mathbf{T}$, *then* $dim(\mathbf{S}) \leq dim(\mathbf{T})$. *If* $\mathbf{S} \subseteq \mathbf{T}$ *and* $dim(\mathbf{S}) = dim(\mathbf{T})$, *then* $\mathbf{S} = \mathbf{T}$

Theorem 1.1.3 (Linearly independent set extends to a basis)**.** *Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.*

Proof. Suppose V is finite dimensional and $\{v_1, \ldots v_m\}$ is linearly independent in V. We want to extend $\{v_1, \ldots v_m\}$ to a basis of V. We do this through the multistep process described below. First we let $\{w_1, \ldots, w_n\}$ be any list of vectors in V that spans V.

- step 1: If w_1 is in the span of $(v_1, \ldots, v_m)$, let $B = (v_1, \ldots, v_m)$. if w_1 is not in the span of $(v_1, \ldots, v_m)$, let $B = (v_1, \ldots, v_m, w_1)$

- step j: If w_j is in the span of B, leave B unchanged. If w_j is not in the span of B, extend B by adjoining w_j to it
  After each step, B is still linearly independent because otherwise the linear dependence (recall that $(v_1, \ldots, v_m)$ is linearly independent and any w_j that is adjoined to B is not in the span of the previous vectors in B). After step n, the span of B includes all the $w_j s$. Thus the B obtained after step n spans V and hence is a basis of V.

□

1.2 Eigenvalues and eigenvectors

Suppose $\mathbf{A}$ is a complex $n \times n$ matrix. (The use of the word complex also includes real.)

Definition 1.2.1. *A complex number* λ *is an eigenvalue of* $\mathbf{A}$ *if there is a nonzero vector* v *such that*

$$\mathbf{A}v = \lambda v$$

The nonzero vector v *is called an eigenvector of* $\mathbf{A}$ *corresponding to* λ.

So, an eigenvector of $\mathbf{A}$ is a nonzero complex n-vector with the property that there is a complex number λ such that the above relationship holds or, equivalently,

$$(\lambda\mathbf{I} - \mathbf{A})v = 0$$

Since v is nonzero, $\lambda\mathbf{I} - \mathbf{A}$ must be singular; so,

$$det(\lambda\mathbf{I} - \mathbf{A}) = 0$$

- $det(\lambda\mathbf{I}-\mathbf{A})$ is a monic polynomial of degree n. We call this polynomial **characteristic polynomial** of $\mathbf{A}$, and denote it by $\chi_{\mathbf{A}}(\lambda)$. Thus

$$\chi_{\mathbf{A}}(\lambda) = det(\lambda\mathbf{I} - \mathbf{A})$$

$$\Rightarrow \quad \chi_{\mathbf{A}}(\lambda) = det(\lambda\mathbf{I} - \mathbf{A}) = a_0 + a_1\lambda + \ldots + \lambda^n$$

- We conclude that a complex number λ is an eigenvalue of $\mathbf{A}$ if and only if it is a root of the characteristic polynomial of $\mathbf{A}$. Hence, $\mathbf{A}$ has at least one eigenvalue and at most n distinct eigenvalues. Suppose $\mathbf{A}$ has l distinct eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_l$; since these are the distinct roots of the characteristic polynomial of $\mathbf{A}$, we must have

$$det(s\mathbf{I} - \mathbf{A}) = \prod_{i=1}^{l}(s - \lambda_i)^{m_i}$$

- The integer m_i is called the algebraic multiplicity of λ_i if$(s - \lambda_i)^{m_i} \mid \chi_{\mathbf{A}}(\lambda)$ but $(s - \lambda_i)^{m_i+1} \nmid \chi_{\mathbf{A}}(\lambda)$.

- It should be clear that if v_1 and v_2 are any two eigenvectors corresponding to the same eigenvalue λ and ξ_1 and ξ_2 are any two numbers then (provided it is nonzero) $\xi_1 v_1 + \xi_2 v_2$ is also an eigenvector for λ. The set of eigenvectors corresponding to λ along with the zero vector is called the eigenspace of $\mathbf{A}$ associated with λ. This is simply the null space of $\lambda\mathbf{I} - \mathbf{A}$

- The geometric multiplicity of λ is the nullity of $\lambda\mathbf{I} - \mathbf{A}$, that is, it is the maximum number of linearly independent eigenvectors associated with λ.

- It follows that $\mathbf{A}$ is invertible if and only if all its eigenvalues are nonzero.

1.3 Similar Matrices

A square matrix $\mathbf{A}$ is said to be similar to another square matrix $\mathbf{A}$ if there exists a nonsingular matrix $\mathbf{T}$ such that $\mathbf{A} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$. If $\mathbf{A}$ and $\mathbf{B}$ are similar , then

- $det(\mathbf{A}) = det(\mathbf{B})$

- $\chi_{\mathbf{A}} = \chi_{\mathbf{B}}$ and, hence, $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues with the same algebraic multiplicities.

1.4 Cayley-Hamilton Theorem

The following result is a fundamental result in linear algebra and is very useful in systems and control.

Theorem 1.4.1. *If $\chi_{\mathbf{A}}$ is the characteristic polynomial of a square matrix $\mathbf{A}$, then*

$$\chi_{\mathbf{A}}(\mathbf{A}) = 0$$

Suppose the characteristic polynomial of $\mathbf{A}$ is

$$det(s\mathbf{I} - \mathbf{A}) = a_0 + a_1 s + \ldots + a_{n-1}s^{n-1} + s^n$$

Then the Cayley Hamilton theorem states that

$$a_0\mathbf{I} + a_1\mathbf{A} + \ldots + a_{n-1}\mathbf{A}^{n-1} + \mathbf{A}^n = 0$$

hence,

$$\mathbf{A}^n = -a_0\mathbf{I} - a_1\mathbf{A} - \ldots - a_{n-1}\mathbf{A}^{n-1}$$

From this one can readily show that for any $m \geq n, \mathbf{A}^m$ can be expressed as a linear combination of $\mathbf{I}, \mathbf{A}, \ldots, \mathbf{A}^{n-1}$. Hence, any polynomial of $\mathbf{A}$ can be expressed as a linear combination of $\mathbf{I}, \mathbf{A}, \ldots, \mathbf{A}^{n-1}$.

1.5 The state transition matrix: $e^{\mathbf{A}t}$

Consider a complex valued function f of a complex variable λ. Suppose f is analytic in some open disk $\lambda \in \mathbb{C} : |\lambda| < \rho$ of radius $\rho > 0$. Then, provided $|\lambda| < \rho$, $f(\lambda)$ can be expressed as the sum of a convergent power series; specifically

$$f(\lambda) = \sum_{0}^{\infty} a_k \lambda^k = a_0 + a_1\lambda_1 + a_2\lambda_2 + \ldots$$

where

$$a_k = \frac{1}{k!} f^{(k)}(0) = \frac{1}{k!}\frac{d^k f}{d\lambda^k}(0)$$

Suppose $\mathbf{A}$ is a square complex matrix P and $|\lambda_i| < \rho$ for every eigenvalue λ_i of $\mathbf{A}$. Then, it can be shown that the power series $\sum_{k=0}^{n} a_k \mathbf{A}^k = a_0\mathbf{I} + a_1\mathbf{A} + a_2\mathbf{A}^2 + \ldots$

The matrix exponential: $e^{\mathbf{A}}$

Recall the exponential function:

$$f(\lambda) = e^{\lambda}$$

This function is analytic in the whole complex plane (that is $\rho = \infty$) and

$$\frac{d^k e^{\lambda}}{d\lambda^k} = e^{\lambda}$$

for $k = 1, 2, \ldots$; hence for any complex number λ,

$$e^{\lambda} = \sum_{k=0}^{\infty} \frac{1}{k!}\lambda^k = 1 + \lambda + \frac{1}{2}\lambda^2 + \frac{1}{3!}\lambda^3 + \ldots$$

Thus the exponential of any square matrix $\mathbf{A}$ is defined by:

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{A}^k = \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + \ldots$$

and this power series converges for every square matrix $\mathbf{A}$.

The state transition matrix: $e^{\mathbf{A}t}$

Using the above definitions and abusing notation ($e^{\mathbf{A}t}$ instead of $e^{t\mathbf{A}}$),

$$e^{\mathbf{A}t} := \sum_{k=0}^{\infty} \frac{1}{k!}(t\mathbf{A})^k = \mathbf{I} + t\mathbf{A} + \frac{1}{2!}(t\mathbf{A})^2 + \frac{1}{3!}(t\mathbf{A})^3 + \dots \tag{1.1}$$

Chapter 2

Introduction

2.1 What is a system?

The word **"system"** comes from the Greek word, which originally meant something like to stand/put/place together.

Definition 2.1.1. *A system is a collection of parts that interact with each other to function as a whole.*

We are interested in dynamical systems. The components of a dynamical system evolve in time . Mathematically speaking, they are functions of time , they will be called " **Signals"**

Definition 2.1.2. ***A dynamical system*** Σ *is determined by the following data:*

- *a set* $\mathbf{T}$*, called the* ***time set;***
- *a set* $\mathbf{W}$*, called the* ***signal value set;***
- *a set* $\mathcal{B} \subseteq \mathbf{W}^{\mathbf{T}}$*, called the* ***behavior****.*

The set $\mathbf{T}$ is our mathematical model of time. We will deal exclusively with the following cases:

- $\mathbf{T} = \mathbb{Z}$ or a subinterval, especially $\mathbb{N} := \{0, 1, 2, ...\}$ (discrete time);
- $\mathbf{T} = \mathbb{R}$ or a subinterval, especially $\mathbb{R}^{+} := [0, \infty)$ (continuous time).

A **signal** w is a function of time, taking its values in the signal value set $\mathbf{W}$. We write

$$w : \mathbf{T} \longrightarrow \mathbf{W}, t \mapsto w(t).$$

set $\mathbf{W}^{T}$ is the set of all functions from $\mathbf{T}$ to $\mathbf{W}$, therefore it is **the set of all signals.** Typically, not all signals in $\mathbf{W}^{\mathbf{T}}$ can occur in our system (or at least, a system in which anything can happen would not be very interesting from the mathematical point of view). Usually, there will be a system **law** which is satisfied only by some signals.

The subset $\mathcal{B}$ of $\mathbf{W^T}$ formalizes this law which governs the system. The signals $w \in \mathcal{B}$ are precisely those which are compatible with the system law, that is, they may occur in our system. We also call them **admissible**, and we write

$$\mathcal{B} = \{w \in \mathbf{W^T} | w \text{ satisfies the system law}\}$$

Typically, a signal has several, say q, components coming from the same set $\mathbf{K}$ (usually, $\mathbf{K} = \mathbb{R}$). Then $\mathbf{W} = \mathbf{K}^q$, and a signal has the form

$$w : \mathbf{T} \longrightarrow \mathbf{K}^q, t \mapsto w(t) = \begin{bmatrix} w_1(t) \\ \vdots \\ w_q(t) \end{bmatrix}$$

In that case, we call w a **signal vector**, and $\mathbf{W^T}$ is the **set of all signal vectors.** Each component w_i of w is again called a **signal.**
This leads to a slight modefication to definition 2.1.3

Definition 2.1.3. *A dynamical system Σ is determined by the following data:*

- *a set $\mathbf{T}$, called the time set.*
- *a set $\mathcal{A}$, called the* ***signal set****;*
- *a positive integer q, called the* ***number of signals****;*
- *a set $\mathcal{B} \subseteq \mathcal{A}^q$, called the* ***behavior***

Define $\mathcal{A} := \mathbf{K}^T$, This will be called a signal set.
Finally, we arrive at the following definition.

Definition 2.1.4. *A dynamical system Σ is determined by the following data:*

- *a set $\mathcal{A}$, called the* ***signal set;***
- *a positive integer q, called the* ***number of signals;***
- *a set $\mathcal{B} \subseteq \mathcal{A}^q$, called the* ***behavior.***

Here, $\mathcal{A}$ and q define the setting/mathematical framework for our description of the system: $\mathcal{A}$ is the set of all signals, and $\mathcal{A}^q$ is the set of all signal vectors with q components. The signal vectors $w \in \mathcal{B}$ are precisely those which are compatible with the system law, that is,

$$\mathcal{B} = \{w \in \mathcal{A}^q | w\ satisfies\ the\ system\ law\}.$$

Remark 2.1. ***Prototypes of signal sets:***

- *the set of all functions from $\mathbb{N}$ to $\mathbb{R}$,*

$$\mathcal{A} = \mathbb{R}^{\mathbb{N}}$$

(such functions are usually called sequences);

- *the set of* **k** *times continuously differentiable functions from* $\mathbb{R}$ *to* $\mathbb{R}$ *(where* $0 \leq k \leq \infty$*)*

$$\mathcal{A} = \mathcal{C}^k(\mathbb{R});$$

- *the set of generalized functions or distributions*

$$\mathcal{A} = \mathcal{D}'(\mathbb{R}).$$

Example 2.1.1. *Consider the linear constant-coefficients differential equation* $\dot{x}(t) = ax(t) + b(t)$*, where* $t \in \mathbb{R}$ *and* $0 \neq a \in \mathbb{R}$*. show that, if* $b = h$*, the heaviside function, then there exists no classical solution, that is , no* $x \in \mathcal{C}^1(\mathbb{R})$ *that satisfies the equation* $\dot{x} = ax + b$*. However, there exist classical solutions for* $t > 0$ *and for* $t < 0$*, separately. Compute the general solutions in these intervals, and combine them into a global solution* $x \in \mathcal{C}^0(\mathbb{R})$*.*

Solution:-

Suppose that x *is a classical solution, that is, continuously differentiable. Then ,* $\dot{x}$ *is a continuous function, and so is* $\dot{x} - ax = h$ *because* h *is not continuous at zero.*

On $(-\infty, 0)$*, we have to solve* $\dot{x} = ax$*. The general solution is*

$$x(t) = e^{at}c_-$$

where c_- *is a constant. We have* $\lim_{t\to 0^-} x(t) = c^-$*.*

On $(0, \infty)$*, we have to solve* $\dot{x} = ax + 1$*. The general solution is*

$$x(t) = e^{at}c_+ + \frac{1}{a}(e^{at} - 1)$$

We have $\lim_{x\to 0^+} x(t) = c_+$ *Therefore, to make the global solution continuous, we choose* $c_+ = c_- =: c$ *and obtain*

$$x(t) = \begin{cases} e^{at}c & \text{if } t \leq 0 \\ e^{at}c + \frac{1}{a}(e^{at} - 1) & \text{if } t \geq 0 \end{cases}$$

Which we can write concisely as

$$x(t) = e^{at}c + \frac{1}{a}(e^{at} - 1)h(t).$$

Example 2.1.2. *Suppose we have two signals* x *and* b *which are linked via*

$$\dot{x}(t) = \mathbf{A}(t)x(t) + b(t)$$

where $\mathbf{A} : \mathbb{R} \to \mathbb{R}^{n\times n}$ *,* $t \mapsto \mathbf{A}(t)$ *is a smooth map. From the theory of ordinary differential equations, we know that if* $b : \mathbb{R} \to \mathbb{R}^n$ *,* $t \mapsto b(t)$ *is continuous, then the initial value problem (where* $t_0 \in \mathbb{R}$ *and* $x_0 \in \mathbb{R}^n$ *are arbitrary)*

$$\begin{aligned} \dot{x}(t) &= \mathbf{A}(t)x(t) + b(t) \\ x(t_0) &= x_0 \end{aligned}$$

has a unique solution $x : \mathbb{R} \to \mathbb{R}^n$, $t \mapsto x(t)$ *which is in* $(\mathcal{C}^1\mathbb{R})^n$. *Therefore, we may put* $\mathbf{T} = \mathbb{R}, \mathbf{W} = \mathbb{R}^{2n}$ *and*

$$\mathcal{B} = \left\{(x, b) \in \mathbf{W}^T \mid (x, b) \in \mathcal{C}^1(\mathbb{R})^n \times (\mathcal{C}^0\mathbb{R})^n \ \ \text{and} \ \ \dot{x}(t) = \mathbf{A}(t)x(t) + b(t)\right\}.$$

This behavior has the remarkable property

$$\forall b \in \mathcal{C}^0(\mathbb{R})^n \ \ \exists x \in \mathcal{C}^1(\mathbb{R})^n : (x, b) \in \mathcal{B}. \tag{2.1}$$

Later on, we will call signals b with this property ***inputs***.

In order to avoid having to specify how many times a function is differentiable, it is convenient to work with distributions. We put $\mathcal{A} = \mathcal{D}'(\mathbb{R}), q = 2n$, *and*

$$\mathcal{B} = \{(x, b) \in \mathcal{A}^q \mid \dot{x} = \mathbf{A}(t)x(t) + b(t)\}.$$

This is more general because we may now have discontinuous b, such as, e.g., the Heaviside function. Again, we have (compare with (2.1))

$$\forall b \in \mathcal{A}^n \ \exists x \in \mathcal{A}^n : (x, b) \in \mathcal{B}.$$

Take the scalar example.

$$\mathbf{A}(t)x(t) + b(t).$$

The classical solutions are

$$x(t) = e^t x_0 + \int_0^t e^{t\tau} b(\tau) d\tau$$

where $x_0 \in \mathbb{R}$ *is arbitrary. If b is the Heaviside function*

$$b(t) = \begin{cases} 0 & \text{if} \ \ t < 0 \\ 1 & \text{if} \ \ t \geq \end{cases}$$

then we obtain

$$x(t) = e^t x_0 + \begin{cases} 0 & \text{if} \ \ t < 0 \\ e^t - 1 & \text{if} \ \ t \geq \end{cases}$$

which is $\mathcal{C}^0$, *but not* $\mathcal{C}^1$, *and hence not a classical solution.*

2.2 System properties: LTID systems

For the definition of linearity, we need to fix an underlying number field $\mathbf{K}$. We will focus on $\mathbf{K} = \mathbb{R}$, and therefore we give the definition for that case only.

Definition 2.2.1. *A dynamical system* $\Sigma = (\mathcal{A}, q, \mathcal{B})$ *is called linear if* $\mathcal{A}$ *is a real vector space, and* $\mathcal{B}$ *is a subspace of* $\mathcal{A}^q$

The first requirement means that linear combinations of signals are again signals,

$$a_1, a_2 \in \mathcal{A}, \lambda_1, \lambda_2 \in \mathbb{R} \Rightarrow \lambda_1 a_1 + \lambda_2 a_2 \in \mathcal{A}$$

and the second requirement means that linear combinations of admissible signal vectors are again admissible signal vectors,

$$w_1, w_2 \in \mathcal{B}, \lambda_1, \lambda_2 \in \mathbb{R} \Rightarrow \lambda_1 w_1 + \lambda_2 w_2 \in \mathcal{B}.$$

We call this condition the **superposition principle.**

Remark 2.2. *All signal spaces in Remark 2.1 are real vector spaces.*
Any $\mathcal{A}$ of the form $\mathcal{A} = \mathbb{R}^T$ is a real vector space. For $a_1, a_2 \in \mathcal{A}, \lambda_1, \lambda_2 \in \mathbb{R}$, we have

$$\lambda_1 a_1 + \lambda_2 a_2 : \mathbf{T} \longrightarrow \mathbb{R}, t \mapsto \lambda_1 a_1(t) + \lambda_2 a_2(t).$$

More generally: If $\mathbf{W}$ is a real vector space, then so is $\mathbf{W}^{\mathbf{T}}$.

Definition 2.2.2. *A dynamical system $\Sigma = (\mathbf{T}, \mathbf{W}, \mathcal{B})$ is called **linear** if $\mathbf{W}$ is a real vector space and $\mathcal{B}$ is a subspace of $\mathbf{W}^{\mathbf{T}}$.*

Definition 2.2.3. *Let $\mathbf{T}$ be such that*

$$t_1, t_2 \in \mathbf{T} \Rightarrow t_1 + t_2 \in \mathbf{T}. \tag{2.2}$$

For $\tau \in \mathbf{T}$, we define the shift operator σ_τ by

$$\sigma_\tau : \mathbf{W}^{\mathbf{T}} \longrightarrow \mathbf{W}^{\mathbf{T}}, w \longmapsto \sigma_\tau w$$

where

$$(\sigma_\tau w)(t) = w(t + \tau).$$

A dynamical system $\Sigma = (\mathbf{T}, \mathbf{W}, \mathcal{B})$ is called shift-invariant (or: time-invariant) if for all $\tau \in \mathbf{T}$

$$w \in \mathcal{B} \;\Rightarrow\; \sigma_\tau w \in \mathcal{B}.$$

Definition 2.2.4. *A dynamical system $\Sigma = (\mathbf{T}, \mathbf{W}, \mathcal{B})$ is called a **differential (difference)** system if its time set is continuous (discrete) and its system law is given by **differential (difference) equations**.*

This is the class of systems we will mainly study: **linear, time-invariant (LTI) differential (difference) systems**. The system laws will have the following form:

Differential systems: Systems of linear differential equations with constant coefficients. These can be put in the form

$$(\mathbf{R}_d \frac{d^d}{dt^d} + ... + \mathbf{R}_1 \frac{d}{dt} + \mathbf{R}_0)w = 0 \tag{2.3}$$

where $\mathbf{R}_i \in \mathbb{R}^{p\times q}$ are real matrices. We define

$$\mathbf{R} := \mathbf{R}_d s^d + \ldots + \mathbf{R}_1 s + \mathbf{R}_0.$$

Then $\mathbf{R}$ is a polynomial $p \times q$ matrix, and we may rewrite 2.3 in the concise form

$$\mathbf{R}(\frac{d}{dt})w = 0$$

Remark 2.3. *We have $\mathbb{R}[s]^{p\times q} = \mathbb{R}^{p\times q}[s]$. To see this Consider the 2×3 matrix given below;*

$$\mathbf{R} = \begin{bmatrix} s^3 & -2+s & 3 \\ -1+s^2 & 1+s+s^2 & s \end{bmatrix} \in \mathbb{R}[s]^{2\times 3}$$

Then $\mathbf{R}$ can also be written as :

$$\begin{aligned} \mathbf{R} &= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{=:\mathbf{R}_3} s^3 + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}}_{=:\mathbf{R}_2} s^2 + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{=:\mathbf{R}_1} s + \underbrace{\begin{bmatrix} 0 & -2 & 3 \\ -1 & 1 & 0 \end{bmatrix}}_{=:\mathbf{R}_0} \\ &= \mathbf{R_3} s^3 + \mathbf{R_2} s^2 + \mathbf{R_1} s + \mathbf{R_0} \in \mathbb{R}^{2\times 3}[s] \end{aligned}$$

thus,

$$\mathbf{R}(\frac{d}{dt})w = 0 \Leftrightarrow (\mathbf{R}_3 \frac{d^3}{dt^3} + \mathbf{R}_2 \frac{d^2}{dt^2} + \mathbf{R}_1 \frac{d}{dt} + \mathbf{R}_0)w = 0$$

The multivariable differential equation $\mathbf{R}(\frac{d}{dt})w = 0$ is :

$$\frac{d^3}{dt^3} w_1 - 2w_2 + \frac{d}{dt} w_2 + 3w_3 = 0$$

$$-w_1 + \frac{d^2}{dt^2} w_1 + w_2 + \frac{d}{dt} w_2 + \frac{d^2}{dt^2} w_2 + \frac{d}{dt} w_3 = 0$$

Difference systems: Systems of linear difference equations with constant coefficients. These can be put in the form

$$\mathbf{R}_d w(t+d) + ... + \mathbf{R}_1 w(t+1) + \mathbf{R}_0 w(t) = 0 \; for \; all \; t \in T \tag{2.4}$$

where $\mathbf{R}_i \in \mathbb{R}^{p\times q}$ are real matrices. Using the shift operator σ_τ , we may write

$$(\mathbf{R}_d \sigma_d + ... + \mathbf{R}_1 \sigma_1 + \mathbf{R}_0)w = 0.$$

We define $\sigma := \sigma_1, then\ \sigma_k = \sigma^k(k-$fold application of $\sigma)$. If we put again $\mathbf{R} := \mathbf{R}_d s^d + \ldots + \mathbf{R}_1 s + \mathbf{R}_0$ then we can write (2.4) in the concise form

$$\mathbf{R}(\sigma)w = 0.$$

Example 2.2.1. *Genetics/Gender-linked genes: An allele (a certain form of a gene) is located on the X-chromosome. Females have two X-chromosomes, males have one X- and one Y-chromosome. Let $p^f(i)$ be the frequency of the allele in the female gene pool of the $i-th$ generation and let $p^m(i)$ be the same for the male gene pool. Since a son inherits his X-chromosome from the mother,*

$$p^m(i+1) = p^f(i)$$

and since a daughter receives one X-chromosome from the father and one from the mother,

$$p^f(i+1) = \frac{2}{1}(p^f(i) + p^m(i)).$$

The time set is discrete. Both $\mathbf{T} = \mathbb{Z}$ and $\mathbf{T} = \mathbb{N}$ are suitable choices. We have two signals p^m and p^f , taking their values in $\mathbb{R}$, hence $\mathbf{W} = \mathbb{R}^2$ and

$$\mathcal{B} = \left\{ \begin{bmatrix} p^m \\ p^f \end{bmatrix} \in \mathbf{W}^T \mid \begin{bmatrix} p^m(i+1) \\ p^f(i+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} p^m(i) \\ p^f(i) \end{bmatrix} \quad for\ all \quad i \in \mathbf{T} \right\}.$$

In the following, let $\mathbf{R}$ be a $p \times q$ polynomial matrix in the variable s, with real coefficients. We write

$$\mathbf{R} \in \mathbb{R}[s]^{p \times q}.$$

From now on, we will restrict our discussion to the following standard models:

- The continuous standard model is

  $$\mathcal{B} = \{w \in \mathcal{A}^q \mid \mathbf{R}(\frac{d}{dt})w = 0\}$$

  . where $\mathcal{A} = \mathcal{D}'(\mathbf{T})$ *with* $\mathbf{T} = \mathbf{R}$ *or* $\mathbf{T} = \mathbf{R}_+$.

- The discrete standard model is

  $$\mathcal{B} = \{w \in \mathcal{A}^q \mid \mathbf{R}(\sigma)w = 0\}$$

  where $\mathcal{A} = \mathbf{R}^T$ *with* $\mathbf{T} = \mathbb{N}$ *or* $\mathbf{T} = \mathbb{Z}$.

2.3 Representations

The polynomial matrix $\mathbf{R}$ in the above two models is called a representation of $\mathcal{B}$. Note that once $\mathcal{A}$ and q are fixed, the behavior $\mathcal{B}$ is uniquely determined by $\mathbf{R}$. Conversely, there are many polynomial matrices which represent the same behavior. To see this we introduce the concept of unimodular matrix.

Definition 2.3.1. *A square polynomial matrix* $\mathbf{U}$ *is called unimodular if its determinant is a non-zero constant, that is,* $det(\mathbf{U}) \in \mathbb{R}\backslash\{0\}$. *This is equivalent to the existence of a polynomial matrix* $\mathbf{V}$ *such that*

$$\mathbf{UV} = \mathbf{VU} = \mathbf{I}.$$

Clearly, $\mathbf{V}$ is the inverse of $\mathbf{U}$, which exists because $\mathbf{U}$ is non-singular, i.e., $det(\mathbf{U}) \neq 0$. Unimodularity is much stronger than non-singularity, the crucial point is that $\mathbf{U}$ possesses a polynomial (rather than a rational) inverse.

We observe that pre-multiplication by a unimodular matrix U does not change the behavior represented by $\mathbf{R}$. More precisely, $\mathbf{R}$ and $\hat{\mathbf{R}} = \mathbf{UR}$ represent the same behavior, because

$$\begin{aligned} \mathbf{R}(\frac{d}{dt})w = 0 \quad &\Longrightarrow \quad \mathbf{U}(\frac{d}{dt})\mathbf{R}(\frac{d}{dt})w = \hat{\mathbf{R}}(\frac{d}{dt})w = 0 \\ \hat{\mathbf{R}}(\frac{d}{dt})w = 0 \quad &\Longrightarrow \quad \mathbf{V}(\frac{d}{dt})\hat{\mathbf{R}}(\frac{d}{dt})w = \mathbf{R}(\frac{d}{dt})w = 0 \end{aligned}$$

where $\mathbf{V}$ is the polynomial inverse of $\mathbf{U}$. The same holds if we replace $\frac{d}{dt}$ by σ

Definition 2.3.2. *A polynomial matrix* $\mathbf{R}$ *is called a minimal representation of* $\mathcal{B}$ *if there exists no polynomial matrix which represents the same behavior and has a smaller number of rows.*

In order to determine a minimal representation of $\mathcal{B}$ we need the following result, called the Smith form of a polynomial matrix.

Theorem 2.3.1. *(Smith form) For every polynomial matrix* $\mathbf{R} \in \mathbb{R}[s]^{p\times q}$ *there exist unimodular matrices* $\mathbf{U} \in \mathbb{R}[s]^{p\times p}$ *and* $\mathbf{V} \in \mathbb{R}[s]^{q\times q}$ *such that*

$$\mathbf{URV} = \begin{bmatrix} \mathbf{D} & 0 \\ 0 & 0 \end{bmatrix}$$

where $\mathbf{D} \in \mathbb{R}[s]^{r\times r}$ *is a non-singular diagonal matrix*

$$\mathbf{D} = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_r \end{bmatrix}$$

with $d_1|d_2|...|d_r$. *This notation means that for* $i = 1, \ldots, r-1$, *the polynomial* d_i *divides* d_{i+1}, *that is,*$d_{i+1} = d_i e_i$ *for some polynomial* e_i. *Clearly, the integer* r *is precisely the rank of the matrix* $\mathbf{R}$ *(over the quotient field* $\mathbf{R}(s)$ *of* $\mathbb{R}[s]$*).*

Corollary 2.1. *For every polynomial matrix* $\mathbf{R} \in \mathbb{R}[s]^{p\times q}$ *there exists a unimodular matrix* $\mathbf{U} \in \mathbb{R}[s]^{p\times p}$ *such that*

$$\mathbf{UR} = \begin{bmatrix} \mathbf{R}_1 \\ 0 \end{bmatrix}$$

where $\mathbf{R}_1 \in \mathbb{R}[s]^{r\times q}$ *has full row rank, that is, rank*$(\mathbf{R}_1) = r$.

Proof. Let $r = rank(\mathbf{R})$. Then , by theorem 2.3.1, there exist unimodular matrices $\mathbf{U} \in \mathbb{R}^{p\times p}$ and $\mathbf{V} \in \mathbb{R}^{q\times q}$ such that

$$\mathbf{URV} = \begin{bmatrix} \mathbf{D} & 0 \\ 0 & 0 \end{bmatrix}$$

where $\mathbf{D} = diag(d_1, d_2, \dots, d_r)$
Then,

$$\begin{aligned} \mathbf{UR} &= \begin{bmatrix} \mathbf{D} & 0 \\ 0 & 0 \end{bmatrix} \mathbf{V}^{-1} = \begin{bmatrix} \mathbf{D} & 0 \\ 0 & 0 \end{bmatrix} \mathbf{W}, \quad where\ \mathbf{W} = \mathbf{V}^{-1} \\ &= \begin{bmatrix} \mathbf{D} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{RW}_1 \\ 0 \end{bmatrix} \end{aligned}$$

Since $\mathbf{D}$ is non-singular and $\mathbf{W}_1$ has full row rank (because $\mathbf{W}$ has), $\mathbf{DW}_1$ has full row rank. Set

$$\mathbf{R}_1 := \mathbf{DW}_1.$$

□

By the observations above, $\mathbf{R}$ and $\mathbf{R}_1$ represent the same behavior. Thus we have the following result.

Lemma 2.3.1. *Any* $\mathcal{B}$ *possesses a representation matrix with full row rank. If* $\mathbf{R}$ *is a minimal representation of* $\mathcal{B}$*, then* $\mathbf{R}$ *has full row rank.*

Proof. The first statement follows directly from our considerations above. If $\mathbf{R} \in \mathbb{R}[s]^{p\times q}$ is a representation with $rank(\mathbf{R}) < p$, then there exists a representation $\mathbf{R}_1$ rows with less than p , according to Corollary 2.1 (noting that $rank(\mathbf{R}) = rank(\mathbf{R}_1)$), and thus $\mathbf{R}$ cannot be minimal. □

2.4 The fundamental principle

There are three version of the fundamental principle. Namely the scalar version , the square matrix version, and non-square matrix version. We will state the without proof.

Theorem 2.4.1 (Fundamental principle, square matrix version)**.** *Let* $\mathbf{P} \in \mathbb{R}[s]^{p\times p}$ *be a non-singular polynomial matrix. Then*

$$\mathbf{P}(\frac{d}{dt})y = v \quad or \quad \mathbf{P}(\sigma)y = v$$

possesses a solution $y \in \mathcal{A}^p$ for every choice of the right hand side signal $v \in \mathcal{A}^p$.

Theorem 2.4.2 (Fundamental principle, non-square matrix version). *Let $\mathbf{R} \in \mathbb{R}[s]^{p\times q}$ be a full row rank polynomial matrix, that is, $rank(\mathbf{R}) = p$. Then*

$$\mathbf{P}(\frac{d}{dt})w = v \quad or \quad \mathbf{P}(\sigma)w = v$$

possesses a solution $w \in \mathcal{A}^q$ for every choice of the right hand side signal $v \in \mathcal{A}^p$.

2.5 Elimination of latent variables

The most important consequence of the fundamental principel is given below. Often, the components of a signal vector can be divided into two classes: a set of components we are truly interested in (called **manifest variables**), and a set of components that were introduced as auxiliary variables during the modelling process (called **latent variables**). When we model the system the latent variables will be introduce but Finally these latent variables may give us wrong information about the system. Therefore we develop a mechanism to remove latent variables and use only manifest variables.

Consider the system with latent variable

$$\mathcal{B} = \left\{ \begin{bmatrix} w \\ l \end{bmatrix} \in \mathcal{A}^{q+r} | \mathbf{R}(\frac{d}{dt})w = \mathbf{M}(\frac{d}{dt})l \right\}$$

where $\mathbf{R} \in \mathbb{R}[s]^{p\times q}$ and $\mathbf{M} \in \mathbb{R}[s]^{p\times r}$ (we restrict to continuous time; the discrete case is analogous). Lets say that the components of $\mathbf{w}$ are the manifest variables, and l represents the latent variables. Then we are actually interested in the projection of $\mathcal{B}$ onto the first q variables, that is, in the system with latent variables given by

$$\mathcal{B}_l = \left\{ w \in \mathcal{A}^q | \exists l \in \mathcal{A}_r : \mathbf{R}(\frac{d}{dt})w = \mathbf{M}(\frac{d}{dt})l \right\}$$

In other words, we do not care about the precise form of the latent variables l, only about their existence. The question arises whether we can write $\mathcal{B}_l$ as a standard model, that is, whether we can find a polynomial matrix $\hat{\mathbf{R}}$ such that

$$\mathcal{B}_l = \hat{\mathcal{B}} = \left\{ w \in \mathcal{A}^q \mid \hat{\mathbf{R}}(\frac{d}{dt})w = 0 \right\}$$

The answer is yes, and moreover, there is an easy way to obtain the desired matrix $\hat{\mathbf{R}}$ from the given matrices $\mathbf{R}$ and $\mathbf{M}$.

We do this by computing, first, the left kernel of $\mathbf{M}$, over the polynomial ring . That is we solve, the linear system $\xi\mathbf{M} = 0$ where $\xi \in \mathbb{R}[s]^{p\times p}$
Using the Smith form, one can show that there exist $p - rank(\mathbf{M})$ linearly independent

solutions that span this kernel. Thus, let $\xi_1, \ldots, \xi_p - rank(\mathbf{M})$ be a generating system for the left kernel of M . Collecting these row vectors in a matrix X, we have constructed a matrix $\mathbf{X}$ which satisfies the following three conditions

1. $\mathbf{XM} = 0$;

2. any polynomial row vector ξ with $\xi\mathbf{M} = 0$ can be written as a polynomial linear combination of the rows of $\mathbf{X}$, that is,$\xi = \eta\mathbf{X}$ for some polynomial row vector η;

3. $\mathbf{X}$ has full row rank.

Lemma 2.5.1. *Let X_1, X_2 be two matrices with the three properties from above. Then we must have $X_1 = \mathbf{U}X_2$ for some unimodular matrix $\mathbf{U}$.*

Proof. Condition 1 implies

$$X_1\mathbf{M} = 0 \quad and \quad X_2\mathbf{M} = 0.$$

Thus by condition 2, there exist polynomial matrices $\mathbf{U}$ and $\mathbf{V}$ such that

$$X_1 = \mathbf{U}X_2 \quad and \quad X_2 = \mathbf{V}X_1.$$

because each row of X_1 can be written as a polynomial linear combination of the rows of X_2 and vice versa.

$$\Rightarrow X_1 = \mathbf{UV}_1X_2 \quad and \quad X_2 = \mathbf{VU}X_2.$$

Then

$$\Rightarrow (\mathbf{I} - \mathbf{UV})X_1 = 0 \;\; and \quad (\mathbf{I} - \mathbf{VU})X_2 = 0.$$

Finally, condition 3 implies that $\mathbf{UV} = \mathbf{VU} = \mathbf{I}$ that is, $\mathbf{U}$ has a polynomial inverse, namely $\mathbf{V}$, and thus $\mathbf{U}$ is unimodular matrices.

□

Theorem 2.5.1. *Let $\mathbf{R}$, $\mathbf{M}$ be given polynomial matrices, with the same number of rows. Let $\mathbf{X}$ be as described above, and define $\hat{\mathbf{R}} := \mathbf{XR}$. Then*

$$\exists l \in \mathcal{A} : \mathbf{R}(\frac{d}{dt})w = \mathbf{M}(\frac{d}{dt})l \Leftrightarrow \hat{\mathbf{R}}(\frac{d}{dt})w = 0$$

Proof. Since $\mathbf{M} \in \mathbb{R}[s]^{p\times r}$, by Corollary 2.1, there exists a unimodular matrix $\mathbf{V}$ such that

$$\mathbf{VM} = \begin{bmatrix} \mathbf{M}_1 \\ 0 \end{bmatrix}, \; \mathbf{M}_1 \text{ has full } n \text{ row rank.}$$

Thus,

$$\begin{aligned} \exists l \in \mathcal{A}^r : \mathbf{R}w = \mathbf{M}l \;\; &\Leftrightarrow \;\; \exists l \in \mathcal{A}^r : \mathbf{VR}w = \mathbf{VM}l \\ &\Leftrightarrow \;\; \exists l \in \mathcal{A}^r : \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{bmatrix} w = \begin{bmatrix} \mathbf{M}_1 \\ 0 \end{bmatrix} l \\ &\Rightarrow \;\; \mathbf{R}_2 w = 0. \end{aligned}$$

To prove the converse, Suppose $\mathbf{R}_2 w = 0$. Since $\mathbf{M}_1$ has full row rank and $w \in \mathcal{A}^2$, by the fundamental principle for non-square matrix, $\exists l$:

$$\mathbf{M}_1 l = \underbrace{\mathbf{R}_1 w}_{=:\mathbf{V}}$$

Thus, $\exists l \in \mathcal{A}^r : \mathbf{M}_1 l = \mathbf{R}_1 w$

$$\Rightarrow \quad \exists l \in \mathcal{A}^r : \mathbf{R}_1 l = \mathbf{M}_1 w \;\&\; \mathbf{R}_2 w = 0$$

$$\Rightarrow \quad \exists l \in \mathcal{A}^r : \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{bmatrix} w = \begin{bmatrix} \mathbf{M}_1 \\ 0 \end{bmatrix} l$$

$$\Rightarrow \quad \exists l \in \mathcal{A}^r : \mathbf{R} w = \mathbf{M} l$$

Hence, $\exists l \in \mathcal{A}^r : \mathbf{R}w = \mathbf{M}l \Leftrightarrow \mathbf{R}_2 w = 0$.
To see the relation between $\hat{\mathbf{R}}$ and $\mathbf{R}_2$, let $\hat{X} = \begin{bmatrix} 0 & \mathbf{I} \end{bmatrix} \mathbf{V}$. Then, $\hat{X}$ satisfies all the three properties listed above. Then, $\exists \mathbf{U}$ unimodular such that

$$X = \mathbf{U}\hat{X}$$

Therefore, $\hat{\mathbf{R}} = X\mathbf{R} = \mathbf{U}\hat{X}\mathbf{R} = \mathbf{U}\begin{bmatrix} 0 & \mathbf{I} \end{bmatrix}\mathbf{V}\mathbf{R} = \mathbf{U}\begin{bmatrix} 0 & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{bmatrix} = \mathbf{U}\mathbf{R}_2$ and hence $\hat{\mathbf{R}}$ represent the same behavior. Thus,

$$\hat{\mathbf{R}}(\frac{d}{dt})w = 0 \quad \Leftrightarrow \quad \mathbf{R}_2(\frac{d}{dt})w = 0.$$

$$i.e, \qquad \exists l \in \mathcal{A}^r : \mathbf{R}w = \mathbf{M}l \quad \Leftrightarrow \quad \hat{\mathbf{R}}w = 0.$$

□

Example 2.5.1. *Consider* $\mathcal{B}_l = \{(\mathbf{I}, u)^T \in \mathcal{A}^2 \mid \exists (u_1, u_2, u_3)^T \in \mathcal{A}^3 : u_1 + U_2 + u_3 + u = 0, u_1 = r\mathbf{I}, c\dot{u}_2 = \mathbf{I} = u_3, l\dot{\mathbf{I}} = u_3\}$ *where,* r, l, c *are positive real numbers. Rewrite* $\mathcal{B}_l$ *as a standard model, i.e, eliminate the latent variables* u_i*.*
Solution :- *The latent variable system is given by*

$$\begin{bmatrix} 0 & -1 \\ r & 0 \\ 1 & 0 \\ l\frac{d}{dt} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ u \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & c\frac{d}{dt} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Thus, we need to compute the left kernel of

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & cs & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}[s]^{4\times 3}$$

This kernel is generated by $x = [-cs \quad cs \quad 1 \quad cs]$. *We obtain*

$$\hat{\mathbf{R}} = x\mathbf{R} = [-cs \quad cs \quad 1 \quad cs] \begin{bmatrix} 0 & -1 \\ r & 0 \\ 1 & 0 \\ ls & 0 \end{bmatrix} = \begin{bmatrix} rcs + 1 + cls^2 & cs \end{bmatrix}$$

Thus we have $\mathcal{B}_l = \left\{ (\mathbf{I}, u) \in \mathcal{A}^2 \mid (cr\frac{d}{dt} + 1 + cl\frac{d^2}{dt^2})\mathbf{I} + c\frac{d}{dt}u = 0 \right\}$. *In other words, the system law relating the manifest variable* $\mathbf{I}, u$ *is*

$$cr\dot{\mathbf{I}} + \mathbf{I} + cl\ddot{\mathbf{I}} + c\dot{u} = 0.$$

2.6 Input output Representation of LTID system

Definition 2.6.1. *Let the signal vector* w *be partitioned as*

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

where $w_i \in \mathcal{A}^{q_i}$, where q_1 and q_2 are positive integers with $q_1 + q_2 = q$. The subvector w_1 is called a vector of free variables (or: inputs) of $\mathcal{B}$ if it is unconstrained by the system law, that is,

$$\forall w_1 \in \mathcal{A}^{q_1} \exists w_2 \in \mathcal{A}^{q_2} : \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathcal{B}.$$

We can find Free variables from the system as follows:
We have,

$$\mathbf{R}w = 0, \mathbf{R} \in \mathbb{R}[s]^{p \times q}$$

The behavior is

$$\mathcal{B} = \{w \in \mathcal{A}^q \mid \mathbf{R}w = 0\} \quad \longrightarrow \quad \textit{Kernel Representations of the system}$$

We call $\mathbf{R}$ representation of the system. since any system has a minimal representation (i.e) a representation with full row runk, with out loss of generality we may assume that $\mathbf{R} \in \mathbb{R}[s]^{p \times q}$ has full row rank. That is, $rank(\mathbf{R}) = p \leq q$.
We can always permute the columns of R (this just corresponds to renumbering our signal components) such that

$$\mathbf{R} = [-\mathbf{Q} \quad \mathbf{P}]$$

$$where, \mathbf{P} \in \mathbb{R}[s]^{p \times p} \; and, \; \mathbf{Q} \in \mathbb{R}[s]^{p \times (q-p)}$$

Moreover $\mathbf{P}$ is invertible since it is $p \times p$ non singular matrix. Writing

$$w = \begin{bmatrix} u \\ y \end{bmatrix}$$

$$\mathbf{R}w = 0 \;\Leftrightarrow\; \begin{bmatrix} -\mathbf{Q} & \mathbf{P} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}$$
$$\Leftrightarrow\; \mathbf{P}y = \mathbf{Q}u$$

Therefore, our system law $\mathbf{R}(\frac{d}{dt})w = 0$ takes the form

$$\mathbf{P}(\frac{d}{dt})y = \mathbf{Q}(\frac{d}{dt})u \tag{2.5}$$

Definition 2.6.2. *A system law in the form of equation (2.5), where* $\mathbf{P}$ *is square and non-singular, is called an* ***input-output representation*** *of* $\mathcal{B}$*. One calls* $p = rank(\mathbf{R})$ *the number of outputs (or output-dimension), and* $m = q - p$ *the number of inputs (or input-dimension). The signal subvector* u *is called input, and the signal subvector* y *is called output.*

Example 2.6.1. *Determine all input output structure of*

$$\mathcal{B} = \left\{ w \in \mathcal{D}'(\mathbb{R})^3 | \begin{pmatrix} \frac{d}{dt} + 1 & \frac{d^2}{dt^2} - 1 & 1 \\ \frac{d}{dt} & \frac{d^2}{dt^2} - \frac{d}{dt} & \frac{d}{dt} \end{pmatrix} w = 0 \right\}$$

Compute the transfer function for the partition forming an input-output structure
For a representation $\mathbf{R} = \begin{bmatrix} -\mathbf{Q} & \mathbf{P} \end{bmatrix}$ *wit input output structure* $w^T = (u^T, y^T)$*, the rational matrix* $G = \mathbf{P}^{-1}\mathbf{Q}$ *is called transfer function.*
solution *Let*

$$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

Since $\mathbf{R}(s) = \begin{bmatrix} s+1 & s^2-1 & 1 \\ s & s^2-1 & s \end{bmatrix} \in \mathbb{R}[s]^{2\times 3}$ *has full row rank , we have three possiblities for the choice of the input* $u \in \mathcal{D}'(\mathbb{R})$

1. $y = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \; u = w_3$

   This can not be an input output stracture since,

$$\begin{vmatrix} \frac{d}{dt} + 1 & \frac{d^2}{dt^2} - 1 \\ \frac{d}{dt} & \frac{d^2}{dt^2} - \frac{d}{dt} \end{vmatrix} = 0$$

2. $y = \begin{bmatrix} w_1 \\ w_3 \end{bmatrix}, \; u = w_2$

   Is an input output structure since,

$$det(\mathbf{R}(s)) = \begin{vmatrix} s+1 & 1 \\ s & s \end{vmatrix} = s^2 + s - s = s^2 \neq 0$$

The transfer function is

$$\begin{aligned}\mathbf{H}(s) &= \begin{pmatrix} s+1 & 1 \\ s & s \end{pmatrix}^{-1} \begin{pmatrix} s^2-1 \\ s^2-s \end{pmatrix} \\ &= \frac{1}{s^2}\begin{pmatrix} s & -1 \\ -s & s+1 \end{pmatrix}\begin{pmatrix} s^2-1 \\ s^2-s \end{pmatrix} \\ &= \begin{pmatrix} s-1 \\ 0 \end{pmatrix}\end{aligned}$$

3. $y = \begin{bmatrix} w_2 \\ w_3 \end{bmatrix}, \; u = w_1$

Is an input output structure since,

$$det(\mathbf{R}(s)) = \begin{vmatrix} s-1 & 1 \\ s^2-s & s \end{vmatrix} = s^3 - s - s^2 + s = s^3 - s^2 \neq 0$$

The transfer function is

$$\mathbf{H}(s) = \begin{pmatrix} s+1 & 1 \\ s & s \end{pmatrix}^{-1} \begin{pmatrix} s^2-1 \\ s^2-s \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

2.7 Reduction to first order

In one of the previous section we have seen how to eliminate latent variables. We now see how one can introduce latent variables. A polynomial matrix $\mathbf{R} \in \mathbb{R}[s]^{p\times q}$ can be written in the form

$$\mathbf{R} = \mathbf{R}_d s^d + \ldots + \mathbf{R}_1 s + \mathbf{R}_0$$

where $\mathbf{R}_i \in \mathbb{R}^{p\times q}$. We may assume that $\mathbf{R}_d$ is not the zero matrix. Our system law takes the form

$$(\mathbf{R}_d(\frac{d^d}{dt^d}) + \ldots + \mathbf{R}_1(\frac{d}{dt}) + \mathbf{R}_0)w = 0$$

or

$$(\mathbf{R}_d(\sigma^d) + \ldots + \mathbf{R}_1(\sigma) + \mathbf{R}_0)w = 0$$

See (2.3) and (2.4). If we put

$$\xi = \begin{bmatrix} w \\ (\frac{d}{dt})w \\ \vdots \\ (\frac{d^{d-1}}{dt^{d-1}})w \end{bmatrix} \quad or \quad \begin{bmatrix} w \\ (\sigma)w \\ \vdots \\ (\sigma^{d-1})w \end{bmatrix}$$

in the continuous or discrete case, respectively, then we can rewrite the system law as

$$\mathbf{K}\dot{\xi} = \mathbf{L}\xi \quad or \quad \mathbf{K}\sigma\xi = \mathbf{L}\xi$$

where

$$\mathbf{K} = \begin{bmatrix} \mathbf{I}_q & & & \\ & \ddots & & \\ & & \mathbf{I}_q & \\ & & & \mathbf{R}_d \end{bmatrix} \quad \textit{and} \quad \mathbf{L} = \begin{bmatrix} 0 & \mathbf{I}_q & & \\ 0 & & \ddots & \\ \vdots & & & \mathbf{I}_q \\ -\mathbf{R}_0 & -\mathbf{R}_1 & \dots & -\mathbf{R}_{d-1} \end{bmatrix}$$

Putting $n = dq$ and $k = (d-1)q + p$, we have $\mathbf{K}, \mathbf{L} \in \mathbb{R}^{k \times n}$ and

$$\mathbf{R}(\frac{d}{dt})w = 0 \quad \Leftrightarrow \quad \exists \xi \in \mathcal{A}^n : \begin{cases} \mathbf{K}\dot{\xi} = \mathbf{L}\xi \\ w = [\mathbf{I}_q, 0, \dots, 0,]\xi \end{cases}$$

and similarly in the discrete case. This shows that reduction to first order is nothing but a special way of introducing latent variables.

2.8 State space representation of LTID system

2.8.1 State Space

In a state space system, the internal state of the system is explicitly accounted for by an equation known as the state equation. The system output is given in terms of a combination of the current system state, and the current system input, through the output equation. These two equations form a system of equations known collectively as state- space equations. The state-space is the vector space that consists of all the possible internal states of the system.

State

A state of a system is the current value of internal elements of the system, that change separately (but not completely unrelated) to the output of the system. In essence, the state of a system is an explicit account of the values of the internal system components.

State Variables

When modeling a system using a state-space equation, we first need to define three vectors:

Input variables

A SISO (Single Input Single Output) system will only have a single input value, but a MIMO system may have multiple inputs. We need to define all the inputs to the system, and we need to arrange them into a vector.

Output Variables

This is the system output value, and in the case of MIMO systems, we may have several. Output variables should be independent of one another, and only dependent on a linear combination of the input vector and the state vector.

State variable

The state variables represent values from inside the system, that can change over time. In an electric circuit, for instance, the node voltages or the mesh currents can be state variables. In a mechanical system, the forces applied by springs, gravity, and dashpots can be state variables.

We denote the input variables with u, the output variables with y, and the state variables with x. In essence, we have the following relationship:

$$y = f(x, u)$$

where $f(x, u)$ is our system. Also, the state variables can change with respect to the current state and the system input:

$$\dot{x} = g(x, u)$$

where $\dot{x}$ is the rate of change of the state variables.

2.8.2 State-Space Equations

We have already seen the kernel and input output representation of a given system. However we have seen how one can obtain the input output representation from the kernel representation. We now introduce how a state space representation is obtained from the input output representation.

Lemma 2.8.1. *Let* $\mathbf{R} \in \mathbb{R}[s]^{p\times q}$. *There exists matrices* $\mathbf{K} \in \mathbb{R}^{n\times n}, \mathbf{L} \in \mathbb{R}^{n\times q}, \mathbf{M} \in \mathbb{R}^{p\times n}, \mathbf{N} \in \mathbb{R}^{p\times q}$ *such that*

$$\mathbf{R}w = 0 \quad \Leftrightarrow \quad \exists x \in \mathbf{R}^n : \begin{array}{l} \dot{x} = \mathbf{K}x + \mathbf{L}w \\ 0 = \mathbf{M}x + \mathbf{N}w \end{array}$$

Proof. Let $\mathbf{R} := \mathbf{R}_d s^d + \ldots + \mathbf{R}_1 d + \mathbf{R}_0$

$$\mathbf{K} = \begin{bmatrix} 0 & \ldots & \ldots & 0 \\ \mathbf{I}_p & \ddots & & \vdots \\ & \ddots & \ddots & \vdots \\ & & \mathbf{I}_p & 0 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} \mathbf{R}_0 \\ \vdots \\ \mathbf{R}_{d-1} \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 0 & \ldots & 0 & \mathbf{I}_p \end{bmatrix}, \mathbf{N} = \mathbf{R}_d$$

Then, consider

$$\dot{x} = \mathbf{K}x + \mathbf{L}w$$

$$0 = \mathbf{M}x + \mathbf{N}w$$

$$\Rightarrow \quad \begin{bmatrix} \frac{d}{dt} - \mathbf{K} \\ -\mathbf{M} \end{bmatrix} x = \begin{bmatrix} \mathbf{L} \\ \mathbf{N} \end{bmatrix} w$$

If we multiply by the unimodular matrix

$$\mathbf{U} = \begin{bmatrix} \mathbf{I}_p & s\mathbf{I}_p & \dots & s^d\mathbf{I}_p \\ & \ddots & \ddots & \vdots \\ & & \ddots & s\mathbf{I}_p \\ & & & \mathbf{I}_p \end{bmatrix}$$

We set

$$\begin{bmatrix} 0 \\ -\mathbf{I}_n \end{bmatrix} x = \begin{bmatrix} \mathbf{R} \\ * \end{bmatrix} (\frac{d}{dt})w$$

Hence, $\exists x$ such that $(*) \Leftrightarrow \mathbf{R}w = 0$.

□

Now consider the input-output representation

$$\mathbf{P}y = \mathbf{Q}u$$

Then,

$$\underbrace{\begin{bmatrix} -\mathbf{Q} & \mathbf{P} \end{bmatrix}}_{\mathbf{R}} \underbrace{\begin{bmatrix} u \\ y \end{bmatrix}}_{w} = 0$$

$$\begin{aligned} \therefore \quad \mathbf{P}y = \mathbf{Q}u \quad \Leftrightarrow \quad \exists : \dot{x} &= \mathbf{K}x + \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_2 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \\ 0 &= \mathbf{M}x + \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \end{aligned}$$

An interesting case is when $\mathbf{N}_2$ is invertible , we set

$$y = (\underbrace{-\mathbf{N}_2^{-1}\mathbf{M}}_{=:\mathbf{C}})x + (\underbrace{-\mathbf{N}_2^{-1}\mathbf{N}_1}_{=:\mathbf{D}})u$$

Upon substitution, we set

$$\begin{aligned} \dot{x} &= \mathbf{K}x + \mathbf{L}_1 u + \mathbf{L}_1(-\mathbf{N}_2^{-1}\mathbf{M})x + \mathbf{L}_2(-\mathbf{N}_2^{-1}\mathbf{N}_1)u \\ &= (\underbrace{\mathbf{K} - \mathbf{N}_2^{-1}\mathbf{M}}_{=:\mathbf{A}})x + (\underbrace{\mathbf{L}_1 - \mathbf{L}_2\mathbf{N}_2^{-1}\mathbf{N}_1}_{=:\mathbf{B}})u \end{aligned}$$

$$\therefore \qquad \dot{x} = \mathbf{A}(t)x + \mathbf{B}u$$

$$y(t) = \mathbf{C}x + \mathbf{D}u$$

$$\textit{Hence,} \qquad \mathbf{P}y = \mathbf{Q}u \quad \Leftrightarrow \quad \exists x \in \mathcal{A}^n : \quad \begin{aligned} \dot{x} &= \mathbf{A}(t)x + \mathbf{B}u \\ y(t) &= \mathbf{C}x + \mathbf{D}u. \end{aligned}$$

State Equation

$$\dot{x} = \mathbf{A}x(t) + \mathbf{B}u(t)$$

Output Equation

$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

This equations called **State Space Representation**.

2.8.3 Solution Of State Space Equations

continuous-time Equation

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t) \tag{2.6}$$

$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t) \tag{2.7}$$

Where $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ are respectively, $n \times n$, $n \times p$, $q \times n$ and $q \times p$ constant matrices.
To find the solution we need the property

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$$

to develop the solution. Premultipying $e^{-\mathbf{A}t}$ on both side of (2.6) yields

$$e^{-\mathbf{A}t}\dot{x}(t) - e^{-\mathbf{A}t}\mathbf{A}x(t) = e^{-\mathbf{A}t}\mathbf{B}u(t)$$

which implies

$$\frac{d}{dt}(e^{-\mathbf{A}t}x(t)) = e^{-\mathbf{A}t}\mathbf{B}u(t)$$

Its integration from 0 to t yields

$$e^{-\mathbf{A}\tau}x(\tau)|_{\tau=0}^{t} = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}u(\tau)d_\tau$$

Thus we have

$$e^{-\mathbf{A}t} - e^0 x(0) = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}u(\tau)d_\tau \tag{2.8}$$

Because the inverse of $e^{-\mathbf{A}t}$ is $e^{\mathbf{A}t}$ and $e^0 = \mathbf{I}$ (2.8) implies

$$x(t) = e^{\mathbf{A}t}x(0) + \int_0^t e^{-\mathbf{A}t}\mathbf{B}u(\tau)d\tau \tag{2.9}$$

is the solution of (2.6).
It is instructive to verify that (2.9) is the solution of (2.6). to verify this, we must show that (2.9) satisfies (2.6) and the intial condition $x(t) = x(0)$ at $t = 0$,
(2.9) reduces to

$$x(0) = e^{\mathbf{A}\cdot 0}x(0) = e^0 x(0) = Ix(0) = x(0)$$

Thus (2.9) satisfies the intial condition. We need the equation

$$\frac{\partial}{\partial t}\int_{t_0}^{t} f(t,\tau))d\tau = \int_{t_0}^{t}(\frac{\partial}{\partial t}f(t,\tau))d\tau + f(t,\tau)|_{\tau=t} \tag{2.10}$$

To sho that (2.9) satisfies (2.6) differentiating (2.9) and using (2.10). We obtain

$$\begin{aligned}
\dot{x}(t) &= \frac{d}{dt}[e^{\mathbf{A}t}x(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau] \\
&= \mathbf{A}e^{\mathbf{A}t}x(0) + \int_0^t \mathbf{A}e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau] + e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)|_{\tau=t} \\
&= \mathbf{A}(e^{\mathbf{A}t}x(0) + \int_0^t \mathbf{A}e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau) + e^{\mathbf{A}\cdot 0}\mathbf{B}u(t)
\end{aligned}$$

Which becomes, after substituting (2.9),

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t)$$

Thus (2.9) meets (2.6) and the intial condtion $x(0)$ and is the solution of (2.6).

$$y(t) = \mathbf{C}e^{\mathbf{A}t}x(0) + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau + \mathbf{D}u(t) \tag{2.11}$$

Solution Of Discrete-Time Equation

Consider the discrete-time state-space equation

$$x(k+1) = \mathbf{A}x(k) + \mathbf{B}u(k)$$

$$y(k) = \mathbf{C}x(k) + \mathbf{D}u(k)$$

$$\begin{aligned}
x(1) &= \mathbf{A}x(0) + \mathbf{B}u(0) \\
x(2) &= \mathbf{A}x(1) + \mathbf{B}u(1) \\
&= \mathbf{A}^2x(0) + \mathbf{A}\mathbf{B}u(0) + \mathbf{B}u(1) \\
x(3) &= \mathbf{A}x(2) + \mathbf{B}u(2) \\
&= \mathbf{A}^3x(0) + \mathbf{A}^2\mathbf{B}u(0) + \mathbf{A}\mathbf{B}u(1) + \mathbf{B}u(2) \\
&\vdots
\end{aligned}$$

proceeding forward, we can readily obtain for $k \geq 0$,

$$x(k) = \mathbf{A}^k x(0) + \sum_{m=0}^{k-1}\mathbf{A}^{k-1-m}\mathbf{B}u(m)$$

$$y(k) = \mathbf{C}\mathbf{A}^k x(0) + \sum_{m=0}^{k-1}\mathbf{A}^{k-1-m}\mathbf{B}u(m) + \mathbf{D}u(k)$$

2.9 Stability

2.9.1 Stability of Input output Representation

Definition 2.9.1. *Let $\mathcal{B}$ be represented by $\mathbf{P}(\frac{d}{dt})y = \mathbf{Q}(\frac{d}{dt})u$ or $\mathbf{P}(\sigma)y = \mathbf{Q}(\sigma)u$, respectively. The input-output representation is called stable if any two outputs y_1, y_2 belonging to the same input u satisfy*

$$||y_1(t) - y_2(t)|| \leq M$$

for all $t \in T_+$ for some constant M which is independent of t (but may depend on the specific choice of y_1, y_2). It is called asymptotically stable if we have additionally

$$\lim_{t\to\infty} ||y_1(t) - y_2(t)|| = 0.$$

Remark 2.4. *For a square non-singular matrix $\mathbf{p}$, $\mathbf{p}y = 0$ is any stable if the zeros $det(\mathbf{P})$ have a negative real part.*

2.9.2 Stability of state space representations

Consider the state space equations

$$\dot{x} = \mathbf{A}x + \mathbf{B}u \qquad or \qquad \sigma x = \mathbf{A}x + \mathbf{B}u$$

$$y = \mathbf{C}x + \mathbf{D}u \qquad\qquad y = \mathbf{C}x + \mathbf{D}u$$

We call them **stable** if two states x_1, x_2 belonging to the same input u satisfy

$$||x_1(t) - x_2(t)|| \leq M \qquad for\ all \quad t \in T_+$$

for some constant M . Then

$$||y_1(t) - y_2(t)|| = ||\mathbf{C}(x_1(t) - x_2(t))|| \leq M_1 \quad for\ all \quad t \in T_+.$$

We call the state space equations **asymptotically stable** if additionally

$$\lim_{t\to\infty} ||(x_1(t) - x_2(t))|| = 0$$

Then

$$\lim_{t\to\infty} ||(y_1(t) - y_2(t))|| = 0$$

Note that here, $||.||$ is used to denote both a norm on $\mathbb{R}^n$ and a norm on $\mathbb{R}^p$. One may think of the respective Euclidean norms, for instance.

Let $\mathbf{A}$ be a square real matrix.
The autonomous system represented by $\dot{x} = \mathbf{A}x + \mathbf{B}u$ *or* $\sigma x = \mathbf{A}x + \mathbf{B}u$ respectively, is

continuous-time asymptotically stable if and only if the eigenvalues of $\mathbf{A}$ have a negative real part;
continuous-time stable if and only if the eigenvalues of $\mathbf{A}$ have a non-positive real part and moreover, each eigenvalue λ with $Re(\lambda) = 0$ is semi-simple i.e, the algebraic multiplicity of λ equals its geometric multiplicity.
discrete-time asymptotically stable if and only if the eigenvalues of $\mathbf{A}$ have modulus less than one;
discrete-time stable if and only if the eigenvalues of $\mathbf{A}$ have modulus less than or equal to one, and moreover, each eigenvalue λ with $|\lambda| = 1$ is semi-simple.

Example 2.9.1. *Show that* $\dot{x} = \mathbf{A}x + \mathbf{B}u$ *where*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is not an asymptotically stable state space system, but the associated input output system is asymptotically stable.
Solution:-
The eigenvalues of $\mathbf{A}$ *are the zeros of* $\lambda^2 - \lambda - 2$, *i.e* $\lambda_1 = 2$ *and* $\lambda_2 = -1$

Due to the eigenvalue $\lambda_1 = 2$, *the state space system is not asymptotically stable.*
If we only look the transition from u *to* y *(elimination of the latent variable* x*, here, we do this by straightforward manipulation of the system equations):*

$$y = \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 - 2x_2$$

$$\Longrightarrow \dot{y} = \dot{x_1} - 2\dot{x_2}$$

$$\text{But} \quad \dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

Plugging in $\dot{y}$ *then we get*

$$\begin{aligned} \dot{y} = x_1 + 2x_2 + u - 2x_1 &= u + 2x_2 - x_1 \\ &= u - x_1 + 2x_2 \\ &= u(x_1 - 2x_2) = u - y \end{aligned}$$

$$\begin{aligned} &\Rightarrow \dot{y} + y = u \\ &\Rightarrow (s+1)y = u \\ &\Rightarrow p(s)y = u \\ &\Rightarrow det(p(s)) = s + 1 \\ &\Rightarrow -1 \quad \textit{is the only zero of} \;\; det(p(s)). \end{aligned}$$

Thus, the input-output system is asymptotically stable since real part -1 *is is negative.*

Chapter 3

Reachability and controllability

3.1 Basic notions of state space systems

Consider the state space equations

$$\dot{x} = \mathbf{A}x + \mathbf{B}u \qquad or \qquad \sigma x = \mathbf{A}x + \mathbf{B}u$$

$$y = \mathbf{C}x + \mathbf{D}u \qquad\qquad y = \mathbf{C}x + \mathbf{D}u$$

Where

$$\mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{B} \in \mathbb{R}^{n\times m}, \mathbf{C} \in \mathbb{R}^{p\times n}, and \;\; \mathbf{D} \in \mathbb{R}^{p\times m}.$$

Let $\mathcal{U}$ denote the space of **admissible input functions**(until now, we had $\mathcal{U} = \mathcal{D}'(\mathbf{T})^m$ in the continuous and $\mathcal{U} = (\mathbb{R}^{\mathbf{T}})^m$ in the discrete case, where $\mathbf{T} = \mathbb{R}, \mathbb{R}_+$ and $\mathbf{T} = \mathbb{N}, \mathbb{Z}$ are our usual time sets). In this project, we restrict to piecewise continuous input functions u in the continuous case, then the state function x is piecewise $\mathcal{C}^1$ (in particular, it is a classical function). One calls $\mathbf{X} = \mathbb{R}^n$ the **state space of these systems**. Its elements are called **states**. The state transition map

$$\varphi : \{(t,t_0) \in \mathbf{T}^2 \mid t \geq t_0\} \times \mathbf{X} \times \mathcal{U} \to \mathbf{X}, (t,t_0,x_0,u) \mapsto \varphi(t,t_0,x_0,u)$$

yields the state at time t if the state at time t_0 was x_0, and the input function was u. More concretely, we have

$$\varphi(t,t_0,x_0,u) = e^{\mathbf{A}(t-t_0)}x_0 + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau$$

and

$$\varphi(t,t_0,x_0,u) = \mathbf{A}^{t-t_0}x_0 + \sum_{i=t_0}^{t-1} \mathbf{A}^{t-i-1}\mathbf{B}u(i).$$

3.1.1 properties of state transition map

1. **The state transition maps are consistent**
   At one time we have only one state, i.e.,

$$\varphi(t,t,x,u) = x \qquad \text{for all} \quad t \in \mathbf{T}, x \in \mathbf{X}, u \in \mathcal{U} \tag{3.1}$$

2. **They are strictly causal**
   That is, if $u_1(t) = u_2(t)$ for all $t_0 \le t < t_1$ (note the strict inequality!), then

$$\varphi(t_1,t_0,x,u_1) = \varphi(t_1,t_0,x,u_2) \qquad\qquad \text{for all} \;\; x \in \mathbf{X}. \tag{3.2}$$

3. **The semigroup property**

$$\varphi(t_2,t_1,\varphi(t_1,t_0,x,u),u) = \varphi(t_2,t_0,x,u) \tag{3.3}$$

   As a consequence, we get the following concatenation property: Let $t_1 \in \mathbf{T}$, and let $u_1, u_2 \in \mathcal{U}$ be two admissible inputs. Define their concatenation at time t_1 by

$$u(t) = \begin{cases} u_1(t) & \text{if } t < t_1, \\ u_2(t) & \text{if } t_1 \le t \end{cases}$$

   which is again an admissible input. Then,

$$\varphi(t_2,t_0,x,u) = \varphi(t_2,t_1,\varphi(t_1,t_0,x,u_1),u_2)$$

   for all $x \in \mathbf{X}$, and all $t_0 \le t_1 \le t_2 \in \mathbf{T}$.

4. **Linearity property**

$$\varphi(t,t_0,\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 u_1 + \lambda_2 u_2) = \lambda_1 \varphi(t,t_0,x_1,u_1) + \lambda_2 \varphi(t,t_0,x_2,u_2) \tag{3.4}$$

   for all $t \le t_0 \in \mathbf{T}, \lambda_1, \lambda_2 \in \mathbb{R}, x_1, x_2 \in \mathbf{X}, u_1, u_2 \in \mathcal{U}$. As a consequence,

$$\varphi(t,t_0,0,0) = 0$$

   for all $t \ge t_0 \in \mathbf{T}$. One says that the zero state is an **equilibrium** of the system when the zero input function is applied.

5. **Time – invariance**
   We have

$$\varphi(t,t_0,x_0,u) = \varphi(t-\tau, t_0-\tau, x_0, \sigma_\tau u) \tag{3.5}$$

   for all $x_0 \in \mathbf{X}, u \in \mathcal{U}$, and all $t \ge t_0 \in \mathbf{T}$ and all $\tau \in \mathbf{T}$ with the property that $t-\tau, t_0-\tau \in \mathbf{T}$. Here, σ_τ is the shift operator defined by $\sigma_\tau u(s) = u(s+\tau)$ for all s. In particular,

$$\varphi(t,t_0,x_0,u) = \varphi(t-t_0, 0, x_0, \sigma_{t_0} u).$$

Remark 3.1. *If* $\mathbf{A}, \mathbf{B}, \mathbf{C}$, *and* $\mathbf{D}$ *depend on* t, *we will lose this time-invariance property.*

Reachability and controllability are concerned with the following questions: Is it possible to steer the system, by a suitable choice of the input function, from one particular state $\mathbf{x_0}$ (which is determined by the past of the system and may be thought of as unwanted) to another particular state $\mathbf{x_1}$ (which is prescribed by us and thus desired)? How long does the transition from $\mathbf{x_0}$ to $\mathbf{x_1}$ take? Can we find a concrete formula for an input function that forces the system to go from $\mathbf{x_0}$ to $\mathbf{x_1}$? To formulate and answer these questions, some new concepts need to be introduced.

Definition 3.1.1. *Let* $t_0 \in \mathbf{T}$ *be fixed. One says that the state* $x_1 \in \mathbf{X}$

- ***can be reached from*** $x_0 \in \mathbf{X}$ *in time* $\tau \in \mathbf{T}(\tau \geq 0)$ *if there exists* $u \in \mathcal{U}$ *such that*

$$\varphi(t_0 + \tau, t_0, x_0, u) = x_1$$

  Equivalently, we say that x_0 ***can be controlled to*** x_1 ***in time*** τ

- ***can be reached from*** $x_0 \in X$ *if this holds for at least one* $\tau \geq 0$. *Equivalently,* x_0 ***can be controlled to*** x_1.

we say that the system is:-

- ***completely reachable from*** $x_0 \in \mathbf{X}$ *if any* $x_1 \in \mathbf{X}$ *is reachable from* x_0.
- ***completely controllable to*** $x_1 \in X$ *if any* $x_0 \in X$ *can be controlled to* x_1.
- ***completely reachable (controllable)*** *if* x_1 *can be reached from* x_0 *(or:* x_0 *can be controlled to* x_1*) for all* x_0, $x_1 \in \mathbf{X}$.

The choice of the starting time t_0 plays no role since we are dealing with time invariant systems: We have

$$\varphi(t_0 + \tau, t_0, x_0, u) = \varphi(\tau, 0, x_0, \sigma_{t_0} u)$$

Therefore, if there exists a $t_0 \in \mathbf{T}$ such that x_1 is reachable from x_0 (when starting at time t_0) then this is true for any other starting time, e.g., for $t_0 = 0$. Thus we can often choose $t_0 = 0$ for simplicity.

Let $\tau \in \mathbf{T}, \tau \geq 0$, and $x_0, x_1 \in \mathbf{X}$. Define the following sets;

- $\mathcal{R}(\tau, x_0) :=\{ x \in \mathbf{X} : x$ is reachable from x_0 *in time* $\tau\}$
- $\mathcal{C}(\tau,x_1) :=\{x \in \mathbf{X} : x$ is controllable to x_1 in the time $\tau\}$

Moreover, we let $\mathcal{R}(\tau) := \mathcal{R}(\tau, 0)$ denote the set of states that are reachable from $x_0 = 0$ in time τ , and $\mathcal{C}(\tau) := \mathcal{C}(\tau, 0)$ is the set of states that are controllable to $x_1 = 0$ in time τ. Finally,

$$\mathcal{R} := \bigcup_{\tau \geq 0} \mathcal{R}(\tau) \qquad \textit{and} \qquad \mathcal{C} := \bigcup_{\tau \geq 0} \mathcal{C}(\tau)$$

are the set of states that are reachable from zero, and the set of states that are controllable to zero, respectively. The system is completely reachable from zero if and only if $\mathcal{R} = \mathbf{X}$, and it is completely controllable to zero if and only if $\mathcal{C} = \mathbf{X}$.

Theorem 3.1.1. *Let $s, t \in \mathbf{T}, 0 \leq s \leq t$. We have*

1. $\mathcal{R}(s) \subseteq \mathcal{R}(t)$ *and* $\mathcal{C}(s) \subseteq \mathcal{C}(t)$*;*

2. $\mathcal{R}(t), \mathcal{C}(t), \mathcal{R}, \mathcal{C}$ *are subspaces of* $\mathbf{X} = \mathbb{R}^n$

3. *There exists* $\tau^* \in \mathbf{T}, \tau^* \geq 0$ *such that*

$$\mathcal{R} = \mathcal{R}(\tau) \quad \textit{and} \quad \mathcal{C} = \mathcal{C}(\tau) \quad \textit{for all} \quad \tau \geq \tau^*.$$

Proof. :

1. Let $x \in \mathcal{R}(s)$. Then there exists an input function u such that

$$\varphi(t, t-s, 0, u) = x.$$

because the starting time doesn't affect. On the other hand ,

$$\varphi(t-s, 0, 0, 0) = 0.$$

because 0 is equilibrium when 0 imput is applied.
Now, let $\tilde{u}$ be the input function defined by

$$\tilde{u}(\tau) = \begin{cases} 0 & \text{if } \tau < t-s, \\ u(\tau) & \text{if } \tau \geq t-s \end{cases}$$

Then $\tilde{u}$ is an admissible input and we have

$$\varphi(t, 0, 0, \tilde{u}) = \varphi(t, t-s, \varphi(t-s, 0, 0, 0), u) = \varphi(t, t-s, 0, u) = x.$$

which shows that $x \in \mathcal{R}(t)$. The statement for $\mathcal{C}$ is analogous.

2. We show that $\mathcal{R}(t)$ is a vector space, the statement for $\mathcal{R}, \mathcal{C}(t), \mathcal{C}$ is analogous. We have $0 \in \mathcal{R}(t)$, because

$$\varphi(t, 0, 0, 0) = 0$$

.

Let $x_1, x_2 \in \mathcal{R}(t)$, and $\lambda_1, \lambda_2 \in \mathcal{R}$. We need to show that $\lambda_1 x_1 + \lambda_2 x_2 \in \mathcal{R}(t)$. By assumption, there exist input functions u_1 and u_2 such that

$\varphi(t,0,0,u_1) = x_1$ and $\varphi(t,0,0,u_2) = x_2$

$$\begin{aligned}\lambda_1 x_1 + \lambda_2 x_2 &= \lambda_1\varphi(t,0,0,u_1) + \lambda_2\varphi(t,0,0,u_2)\\ &= \varphi(t,0,0,\lambda_1 u_1) + \varphi(t,0,0,\lambda_2 u_2)\\ &= \varphi(t,0,0,\lambda_1 u_1 + \lambda_2 u_2)\end{aligned}$$

$\Rightarrow$ $\lambda_1 u_1 + \lambda_2 u_2$ is an input function that steers the system from 0 to $\lambda_1 x_1 + \lambda_2 x_2$ in time t.
Thus, $\lambda_1 x_1 + \lambda_2 x_2 \in \mathcal{R}(t)$ and hence $\mathcal{R}(t)$ is a vector space over $\mathbb{R}$.

3. Consider a strictly increasing sequence

$$0 = \tau_0 < \tau_1 < \tau_2 < \dots$$

in $\mathbf{T}$ with $\lim_{i\to\infty}\tau_i = \infty$. By part 1 ,

$$\mathcal{R}(\tau_0) \subseteq \mathcal{R}(\tau_1) \subseteq \mathcal{R}(\tau_2) \subseteq \dots$$

By part 2, this is a sequence of subspaces of $\mathbf{X} = \mathbb{R}^n$, with

$$dim(\mathcal{R}(\tau_0)) \leq dim(\mathcal{R}(\tau_1)) \leq dim(\mathcal{R}(\tau_2)) \leq \dots \leq n$$

This is an increasing sequence of integers less than or equal to n. Such a sequence must become stationary, that is, there exists i_0 such that

$$dim(\mathcal{R}(\tau_i)) = dim(\mathcal{R}(\tau_{i_0})) \qquad \text{for all} \;\; i \geq i_0.$$

We use the following fact from linear algebra: If a vector space is contained in another vector space of the same finite dimension, then the two vector spaces must be the same. Thus

$$\mathcal{R}(\tau_i) = \mathcal{R}(\tau_{i_0}) \qquad for \;\; all \quad i \geq i_0.$$

For any $\tau \in \mathbf{T}$, $\tau \geq \tau_{i_0}$, there exists $j \geq i_0$ such that $\tau \leq \tau_j$. Then $\mathcal{R}(\tau_{i_0}) \subseteq R(\tau) \subseteq \mathcal{R}(\tau_j) = \mathcal{R}(\tau_{i_0})$. We conclude that

$$\mathcal{R}(\tau) = \mathcal{R}(\tau_{i_0}) \qquad for \;\; all \quad \tau \geq \tau_{i_0}.$$

and thus

$$\mathcal{R} = \bigcup_{\tau \geq 0} \mathcal{R}(\tau) = \bigcup_{\tau \geq \tau_{i_0}} \mathcal{R}(\tau) = \mathcal{R}(\tau_{i_0}).$$

Put $\tau^* := \tau_{i_0}$, then for $\tau \geq \tau^*$, we have

$$\mathcal{R} \supseteq \mathcal{R}(\tau) \supseteq \mathcal{R}(\tau^*) = \mathcal{R}(\tau_{i_0}) = \mathcal{R}$$

and thus

$$\mathcal{R}(\tau) = \mathcal{R} \qquad for \ \ all \quad \tau \geq \tau^*.$$

□

Corollary 3.1. *In discrete time,*

$$\mathcal{R}(n) = \mathcal{R} \quad \text{ and } \quad \mathcal{C}(n) = \mathcal{C}$$

where n is the dimension of the state space. In continuous time,

$$\mathcal{R}(\epsilon) = \mathcal{R} \quad \text{ and } \quad \mathcal{C}(\epsilon) = \mathcal{C}$$

for every $\epsilon > 0$.

Remark 3.2. *In a discrete system, if x can be reached from x_0 at all, then it can also be reached in time n, where n is the dimension of the state space. In a continuous system, if x can be reached from x_0 at all, then it can also be reached in an arbitrarily small time ϵ. This is counterintuitive at first sight: In a real world system, it certainly takes some time to change from one state to another. The reason is that we admit arbitrarily large input values here, i.e., we make the optimistic assumption that we can put as much energy as we like into the system. In a real world system, there are constraints which limit the size of the admissible inputs, and this has the consequence that the transition from one state to another cannot be done arbitrarily fast in practice.*

Proof. For discrete time, from the previous proof we know that

$$\mathcal{R}(0) \subseteq \mathcal{R}(1) \subseteq \mathcal{R}(2) \ldots \tag{3.6}$$

becomes stationary, that is, there exists i_0 such that $\mathcal{R}(i) = \mathcal{R}(i_0)$ for all $i \geq i_0$ and then $\mathcal{R}(i_0) = \mathcal{R}$. We have to show that this happens for some $i_0 \leq n$. Then we are finished, because $\mathcal{R} = \mathcal{R}(i_0) \subseteq \mathcal{R}(n) \subseteq \mathcal{R}$ yields the desired result. Considering the dimensions $d_i := dim\mathcal{R}(i) \leq n$, we have

$$0 = d_0 \leq d_1 \leq d_2 \leq \ldots \leq d_n \leq d_{n+1}.$$

These inequalities cannot all be strict, i.e., we must have $d_i = d_{i+1}$ and hence

$$\mathcal{R}(i) = \mathcal{R}(i+1) \tag{3.7}$$

for some $i \leq n$. The claim is that then we may put $i_0 = i$, that is, the first equality in (3.6) will already yield stationarity. Thus we have to show that (3.7) implies

$$\mathcal{R}(i) = \mathcal{R}(i+1) \qquad \text{for all} \quad k \geq 0.$$

We will proof this by using induction.

1. The statement is trivial for k = 0, i.e, $\mathcal{R}(i) = \mathcal{R}(i+1)$ by (3.7).
2. Let us assume that the statement is true for k. $\mathcal{R}(i) = \mathcal{R}(i+k)$
3. We need to show it for k + 1.
   The inclusion $\mathcal{R}(i) \subseteq \mathcal{R}(i+k+1)$ is clear because $i \leq i+k+1$ For the converse, let $x \in \mathcal{R}(i+k+1)$. This means that there exists an input function u such that

   $$\varphi(i+k+1, 0, 0, u) = x$$

   set

   $$x_1 := \varphi(i+k, 0, 0, u)$$

   Then

   $$\begin{aligned} x &= \varphi(i+k+1, i+k, \varphi(i+k, 0, 0, u), u) \\ &= \varphi(i+k+1, i+k, x_1, u) \\ &= \varphi(i+1, i, x_1, \sigma^k u) \end{aligned}$$

   and $x_1 \in \mathcal{R}(i+k)$, which equals $\mathcal{R}(i)$ by the inductive assumption. Thus there exists an input function u_1 with

   $$\varphi(i, 0, 0, u_1) = x_1$$

   Let $\tilde{u}$ be defined by

   $$\tilde{u}(\tau) = \begin{cases} u_1(\tau) & \text{if } \tau < i, \\ \sigma^k u(\tau) & \text{if } \tau \geq i. \end{cases}$$

Then

$$\varphi(i+1, 0, 0, \tilde{u}) = \varphi(i+1, i, \varphi(i, 0, 0, u_1), \sigma^k u) = \varphi(i+1, i, x_1, \sigma^k u) = x$$

which shows that $x \in \mathcal{R}(i+1)$. Finally, (3.7) implies that $x \in \mathcal{R}(i)$ as desired.
For continuous time, let $\epsilon > 0$ be given. Consider

$$\mathcal{R}(0) \subseteq \mathcal{R}(\frac{\epsilon}{n}) \subseteq \mathcal{R}(\frac{2\epsilon}{n}) \subseteq \dots$$

and apply the same argument as the discrete time case.

□

Corollary 3.2. *The following are equivalent: The system is*

1. *completely reachable (controllable);*

2. *completely reachable from zero, that is, $\mathcal{R} = \mathcal{X}$.*

Proof. (1) $\Rightarrow$ (2) It is clear that if the system is comletely reachable, then every states can be reached from zero. Therefore, the system is completely reachable from zero.
(2) $\Rightarrow$ (1)
Let $x_0, x_1 \in \mathbf{X}$ be given. We wish to show that x_1 can be reached from x_0.
In continuous time, pick any $\tau^* = \epsilon > 0$. In discrete time, choose $\tau^* = n$, where n is the dimension of $\mathbf{X}$. Define $x := x_1 - \varphi(\tau^*, 0, x_0, 0)$. By assumption, since the system is completely reachable from zero, we have, $x \in \mathcal{R} = \mathcal{R}(\tau^*)$, that is, there exists an input function $u \in \mathcal{U}$ with

$$x = \varphi(\tau^*, 0, 0, u).$$

This can be rewritten as

$$x_1 = \varphi(\tau^*, 0, x_0, 0) + \varphi(\tau^*, 0, 0, u) = \varphi(\tau^*, 0, x_0, u)$$

showing that x_1 can be reached from x_0 . □

Corollary 3.3. *Consider $\dot{x} = \mathbf{A}x + \mathbf{B}u$ or $\sigma x = \mathbf{A}x + \mathbf{B}u$. In the discrete case, assume that $\mathbf{A}$ is invertible. Then the following are equivalent: The system is*

1. *completely reachable (controllable);*

2. *completely reachable from zero, that is, $\mathcal{R} = \mathbf{X}$;*

3. *completely controllable to zero, that is, $\mathcal{C} = \mathbf{X}$.*

Proof. We only need to prove (3) $\Rightarrow$ (1) because from Corollary 3.2, we have completely reachable from zero implies completely controlable which inturn implies completely controllable to zero.
Let $x_0, x_1 \in \mathbf{X}$ be given. We wish to show that x_1 can be reached from x_0 .
In continuous time, pick $\epsilon > 0$ and define $x := x_0 - e^{\mathbf{A}\epsilon} x_1$. By assumption, $x \in \mathcal{C} = \mathcal{C}(\epsilon)$, that is, there exists an input function $u \in \mathcal{U}$ with

$$0 = \varphi(\epsilon, 0, x, u) = e^{\mathbf{A}\epsilon} x + \int_0^{\epsilon} e^{\mathbf{A}(\epsilon - \tau)} \mathbf{B} u(\tau) d\tau$$

Plugging in for x, this can be rewritten as

$$0 = e^{\mathbf{A}\epsilon}x_0 - x_1 + \int_0^\epsilon e^{-\mathbf{A}(\epsilon-\tau)}\mathbf{B}u(\tau)\mathrm{d}\tau$$

$$x_1 = e^{\mathbf{A}\epsilon}x_0 + \int_0^\epsilon e^{\mathbf{A}(\epsilon-\tau)}\mathbf{B}u(\tau)\mathrm{d}\tau = \varphi(\epsilon, 0, x_0, u)$$

showing that x_1 can be reached from x_0 . In discrete time, let $\tau^* = n$ and define $x := x_0 - \mathbf{A}^{-n}x_1$.

Since $x \in \mathcal{C} = \mathcal{C}(n)$, there exists $u \in \mathcal{U}$ such that

$$0 = \varphi(n, 0, x, u) = \mathbf{A}^n x + \sum_{i=0}^{n-1} \mathbf{A}^{n-i-1}\mathbf{B}u(i).$$

Upon substitution, we get

$$\begin{aligned} 0 &= \mathbf{A}^n(x_0 - \mathbf{A}^{-n}x_1) + \sum_{i=0}^{n-i-1} \mathbf{B}u(i) \\ &= \mathbf{A}^n x_0 - x_1 + \sum_{i=0}^{n-i-1} \mathbf{B}u(i) \\ \Rightarrow \quad x_1 &= \mathbf{A}^n x_0 + \sum_{i=0}^{n-i-1} \mathbf{B}u(i) \end{aligned}$$

Thus, x_1 is reachable from x_0.

□

Remark 3.3. *Here we have another difference between continuous and discrete systems. The reason is that* $e^{\mathbf{A}t}$ *is always an invertible matrix, whereas its discrete counterpart* $\mathbf{A}^t$ *is invertible if and only if* $\mathbf{A}$ *is. Therefore, we have to make this additional assumption in the discrete case. Without it, complete controllability to zero may be strictly weaker than complete controllability. Take for instance* $\sigma x = \mathbf{A}x + \mathbf{B}u$ *with*

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $x_0 = x(0) \in \mathbb{R}^2$ *be any given initial state. Put* $u(0) = -x_1(0)$*, then* $x_1(1) = x_1(0) + u(0) = 0$ *and* $x_2(1) = x_1(0) + u(0) = 0$*, that is,* $x(1) = 0$*. This shows that any* x_0 *can be controlled to zero (in time 1). Hence the system is completely controllable to zero, or* $\mathcal{C} = \mathcal{C}(1) = \mathbb{R}^2$ *. However, it is not completely controllable. If we start in* $x_0 = 0$*, we have*

$$\begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

which implies $x_1(1) = u$ and $x_2(1) = u$, i.e, $x_1(1) = x_2(1)$. If we continue this way, we can see that every state $x(t)$ will satisfy $x_1(t) = x_2(t)$.
$\Rightarrow \; dim\mathcal{R} = 1$
$\Rightarrow \; \mathcal{R} = \mathbb{R}^2$.
Thus, the system is not completely reachable from zero, and thus it is not completely reachable (controllable).

Example 3.1.1. *Given the discrete time control system*

$$x(t+1) = \begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix} x(t) + \begin{pmatrix} -1 \\ 3 \end{pmatrix} u(t).$$

Show that for the intial state $x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and any control $u(t)_{t\in\mathbb{N}}$ the state $x(t)$ remains in the space

$$span \begin{pmatrix} 1 \\ -3 \end{pmatrix} \subsetneqq \mathbb{R}^2.$$

Solution:- *The state $x(T)$ is determined by*

$$x(T) = \sum_{t=0}^{i-1} \mathbf{A}^{T-1-t}\mathbf{B}u(t)$$

Using the fact that

$$\mathbf{AB} = \begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -7 \\ 21 \end{pmatrix} = 7\mathbf{B},$$

we get

$$x(T) = \sum_{t=0}^{i-1} 7^{T-1-t}\mathbf{B}u(t) \in im\mathbf{B} = span \begin{pmatrix} -1 \\ 3 \end{pmatrix} \subsetneqq \mathbb{R}^2.$$

The next goal is to determine conditions for (complete) reachability/controllability in terms of the matrices $\mathbf{A}, \mathbf{B}$. For this, we define

Definition 3.1.2. *The* **Controllability Gramians** *(named after the Danish mathematician J. P. Gram, 1850 - 1916) of the system $\dot{x} = \mathbf{A}x + \mathbf{B}u$ is given*

$$\mathbf{W}(t) = \int_0^{\tau} e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} \;\; OR \;\; \mathbf{W}(t) = \sum_{i=0}^{t-1} \mathbf{A}^i\mathbf{B}\mathbf{B}^T(\mathbf{A}^T)^i \tag{3.8}$$

and the **Kalman reachability/controllability matrix** *(named after the Hungarian control scientist R. E. Kalman, 1930- 2016)of the system*

$\dot{x} = \mathbf{A}x + \mathbf{B}u$ *is given*

$$\mathbf{K} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & . & . & . & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \tag{3.9}$$

Remark 3.4. *The definitions* (3.8) *and* (3.9) *are motivated by the following observations: Consider* $\sigma x = \mathbf{A}x + \mathbf{B}u$. *We know that* $\mathcal{R} = \mathcal{R}(n)$. *Moreover,* $x \in \mathcal{R}$ *if and only if there exists* $u \in \mathcal{U}$, *that is, a sequence of input vectors* $u(0), \ldots, u(n-1) \in \mathbb{R}^m$ *such that*

$$\begin{aligned} x &= \sum_{i=0}^{n-1} \mathbf{A}^{n-1-i}\mathbf{B}u(i) = \mathbf{A}^{n-1}\mathbf{B}u(0) + \mathbf{A}^{n-2}\mathbf{B}u(1) + \ldots + \mathbf{AB}u(n-2) + \mathbf{B}u(n-1) \\ &= \begin{bmatrix} \mathbf{B} & \mathbf{AB} & . & . & . & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix} =: \mathbf{Kv} \end{aligned}$$

Therefore, x *is reachable from zero if and only if the equation* $x = \mathbf{Kv}$ *possesses a solution* $\mathbf{v} \in \mathbb{R}^{nm}$. *This is the case if and only if* $x \in im(\mathbf{K})$. *So we have* $\mathcal{R} = im(\mathbf{K})$. *Moreover, note that* $\mathbf{W}(n) = \mathbf{KK}^T$. *Since* $im(\mathbf{K}) = im(\mathbf{KK}^T)$ *holds for any real matrix* $\mathbf{K}$, *we also have* $\mathcal{R} = im(\mathbf{W}(n))$.

If $x \in \mathcal{R}$, *we therefore have* $x = \mathbf{W}(n)z = \mathbf{KK}^T z$ *for some* $z \in \mathbb{R}^n$. *Then* $\mathbf{v}^* := \mathbf{K}^T z$ *is a special solution of* $\mathbf{Kv} = x$. *This corresponds to*

$$\begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix} = \begin{bmatrix} \mathbf{B}^T \\ \mathbf{B}^T\mathbf{A}^T \\ \vdots \\ \mathbf{B}^T(\mathbf{A}^T)^{(n-1)} \end{bmatrix} z$$

In other words, $u(t) = \mathbf{B}^T(\mathbf{A}^T)^{n-t-1}z$ is a special input function that steers the system from state 0 to state x in time n. The analogous statement for continuous system is given in the next theorem.

Theorem 3.1.2. *Consider* $\dot{x} = \mathbf{A}x + \mathbf{B}u$ *or* $\sigma x = \mathbf{A}x + \mathbf{B}u$. *In continuous time, let* $\tau^* = \epsilon > 0$ *be arbitrary. In discrete time, let* $\tau^* = n$, *where* n *is the dimension of the state space. We have*

$$\mathcal{R} = \mathcal{R}(\tau^*) = im(\mathbf{W}(\tau^*)) = im(\mathbf{K}).$$

Therefore the followings are equivalent

1. $\dot{x} = \mathbf{A}x + \mathbf{B}u$ *or* $\sigma x = \mathbf{A}x + \mathbf{B}u$ *is reachable/controllable.*

2. $\mathbf{W}(\tau^*)$ *is non-singular.*

3. *K has full row rank.*

Moreover in that case, an input function which steers the system from state 0 to state x in time τ^* is given by

$$u(t) = \mathbf{B}^T e^{\mathbf{A}^T(\tau^*-t)}\mathbf{W}(\tau^*)^{-1}x \quad or \quad \mathbf{B}^T(\mathbf{A}^T)^{\tau^*-t-1}\mathbf{W}(\tau^*)^{-1}x$$

Note that this special input function is smooth in the continuous case (although only piecewise continuity has been required at the beginning).

Remark 3.5. *Before giving the proof of the theorem, we will have the following observations:*

1. *Since $\mathbf{W}(\tau^*)$ is always positive semi-definite due to its form, condition 2 from above is also equivalent to: $\mathbf{W}(\tau^*)$ is positive definite, i.e, $x^T\mathbf{W}(\tau^*)x > 0$ for all $0 \neq x \in \mathbb{R}^n$. Then, we write $\mathbf{W}(\tau^*) > 0$.*

2. *It is worth noting that we obtain the same, purely algebraic condition for reachability/controllability both in continuous and in discrete time, namely, $rank(\mathbf{K}) = n$, where n is the dimension of the state space. In that case, it is not ambiguous to simply say that the matrix pair $(\mathbf{A}, \mathbf{B})$ is reachable/controllable (this notion is independent of the time set).*

3. *The well-known Hamilton-Cayley theorem, i.e.,*

$$\chi_{\mathbf{A}}(\mathbf{A}) = \mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + \ldots + a_1\mathbf{A} + a_0 I = 0.$$

implies the $n-th$ power (and hence all higher powers) of an $n \times n$ matrix A is a linear combination of the first n powers of $\mathbf{A}$, that is, $\mathbf{A}^0 = \mathbf{I}, \mathbf{A}^1 = \mathbf{A}, \ldots, \mathbf{A}^{n-1}$. Therefore we have

$$\begin{aligned} im(\mathbf{K}) = \mathbf{K}\mathbb{R}^{nm} &= span\{\mathbf{A}^i b_j | i = 0, \ldots, n-1, j = 1, \ldots, m\} \\ &= span\{\mathbf{A}^i b_j | i \in \mathbb{N}, j = 1, \ldots, m\} \end{aligned}$$

where $b_j \in \mathbb{R}^n$ are the columns of $\mathbf{B}$, that is, $\mathbf{B} = [b_1, \ldots, b_m]$.

4. *The result*

$$\mathcal{R} = im(\mathbf{K}) = im(\mathbf{B}) + \mathbf{A}im(\mathbf{B}) + \ldots + \mathbf{A}^{n-1}im(\mathbf{B})$$

can also be formulated as follows: $\mathcal{R}$ is the smallest $\mathbf{A}-invariant$ (that is, $\mathbf{A}\mathcal{R} \subseteq \mathcal{R}$) subspace of the state space $\mathbf{X}$ that contains $im(\mathbf{B})$.

Proof. : Due to Remark 3.4, we only need to proof for the continuous case: Then, $\tau^* = \epsilon > 0$ is arbitrary.

We start with showing that $\mathcal{R}(\epsilon) \subseteq im(\mathbf{K})$: If $x \in \mathcal{R}(\epsilon)$, then there exists $u \in \mathcal{U}$ such that

$$\begin{aligned} x = \varphi(\epsilon, 0, 0, u) &= \int_0^\epsilon e^{\mathbf{A}(\epsilon-\tau)}\mathbf{B}u(\tau)\mathrm{d}\tau \\ &= \int_\epsilon^0 -e^{\mathbf{A}\tilde{\tau}}\mathbf{B}u(\epsilon-\tilde{\tau})\mathrm{d}\tilde{\tau} \\ &= \int_0^\epsilon e^{\mathbf{A}\tilde{\tau}}\mathbf{B}u(\epsilon-\tilde{\tau})\mathrm{d}\tilde{\tau} = \int_0^\epsilon e^{\mathbf{A}\tau}\mathbf{B}u(\epsilon-\tau)\mathrm{d}\tau \end{aligned}$$

By equation (1.1) we have,

$$e^{\mathbf{A}\tau} = \sum_{i=0}^{\infty} \frac{\tau^i}{i!}\mathbf{A}^i$$

$$x = \int_0^\epsilon e^{\mathbf{A}\tau}\mathbf{B}u(\epsilon-\tau)\mathrm{d}\tau = \int_0^\epsilon \sum_{i=0}^{\infty} \frac{\tau^i}{i!}\mathbf{A}^i\mathbf{B}u(\epsilon-\tau)\mathrm{d}\tau \in im(\mathbf{K})$$

Therefore, $\mathcal{R}(\epsilon) \subseteq im(\mathbf{K})$

Secondly, we wish to show that $im(\mathbf{K}) = im(\mathbf{W}(\epsilon))$. From linear algebra, we know that it is equivalent to prove that $im(\mathbf{K})^\perp = im(\mathbf{W}(\epsilon))^\perp$
Let $x \in im(\mathbf{K})^\perp$, that is, $< x, y >= 0$ for all $y \in im(\mathbf{K})$. Then $x^T\mathbf{K}z = 0$ for all $z \in \mathbb{R}^{nm}$, which means that $x^T\mathbf{K} = 0$ and hence $x^T\mathbf{A}^i\mathbf{B} = 0$ for all i. Then also

$$x^T e^{\mathbf{A}t}\mathbf{B} = x^T \sum_{i=0}^{\infty} \frac{\tau^i}{i!}\mathbf{A}^i\mathbf{B} = 0 \quad \textit{and thus} \quad x\mathbf{W}(\epsilon) = 0.$$

This shows that $x \in im(\mathbf{W}(\epsilon))^\perp$.
Conversely, let $x \in im(\mathbf{W}(\epsilon))^\perp$, then

$$x^T\mathbf{W}(\epsilon) = \int_0^\epsilon x^T e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}\tau}\mathrm{d}\tau = 0.$$

Post-multiplying this by x, we obtain

$$\int_0^\epsilon ||\mathbf{B}^T e^{\mathbf{A}^T\tau}x||^2\mathrm{d}\tau$$

We conclude that the smooth function $f(\tau) = \mathbf{B}^T e^{\mathbf{A}^T\tau}x$ is the constant zero function because of the norm of a $\mathcal{C}^\infty$ - function is zero, then the function is zero. Then also all its derivatives are zero. Evaluating them at

$\tau = 0$, we obtain

$$\mathbf{B}^T x = 0, \quad \mathbf{B}^T \mathbf{A}^T x = 0, \quad \mathbf{B}^T (\mathbf{A}^T)^2 x = 0, \ldots$$

that is, $x^T \mathbf{K} = 0$ and hence $x \in im(\mathbf{K})^{\perp}$.

Finally, we need to show that $im(\mathbf{W}(\epsilon)) \subseteq \mathcal{R}(\epsilon)$. Let $x \in im(\mathbf{W}(\epsilon))$, then there exists $z \in \mathbb{R}^n$ such that

$$x = \mathbf{W}(\epsilon) z.$$

Set $u(t) = \mathbf{B}^T e^{\mathbf{A}^T(\epsilon - t)} z$. Then

$$\begin{aligned}
\varphi(\epsilon, 0, 0, u) &= \int_0^{\epsilon} e^{\mathbf{A}(\epsilon-\tau)} \mathbf{B} u(\tau) \mathrm{d}\tau \\
&= \int_0^{\epsilon} e^{\mathbf{A}(\epsilon-\tau)} \mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T(\epsilon-\tau)} z \mathrm{d}\tau \\
&= \int_0^{\epsilon} e^{\mathbf{A}(\tau)} \mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T(\tau)} z \mathrm{d}\tau = \mathbf{W}(\epsilon) z = x
\end{aligned}$$

which shows that $x \in \mathcal{R}(\epsilon)$. □

Example 3.1.2 (An electric water katlle). *Let x_1, x_2 be the temperature of the heater coil and of the water, respectively. The change of x_1 is proportional to the electrical power fed into the system minus the coil's heat loss to the water ; the electrical power is proportional to v^2; the heat loss is proportional to the temperature difference $x_1 - x_2$. The change of x_2 is proportional to the heat loss of the coil. This leads to the model*

$$\dot{x}_1(t) = a v(t)^2 - b(x_1(t) - x_2(t))$$

$$\dot{x}_2(t) = c(x_1(t) - x_2(t))$$

where $a, b, c \in \mathbb{R}$. setting $u := v^2$, this is a state space system. Discuss its stability and controllability in terms a, b, c.

Solution:- *We have*

$$\begin{pmatrix} \dot{x}_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{bmatrix} -b & b \\ c & -c \end{bmatrix}}_{=:\mathbf{A}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} \dot{x}_1 \\ x_2 \end{pmatrix}}_{=:\mathbf{B}} u$$

Thus, $spec(\mathbf{A}) = \{0, -b-c, \}$

Stability:-

$$\begin{aligned}
\textit{If } b+c &< 0, \textit{ the system is stable.}\\
\textit{If } b+c &= 0, \textit{ the eigenvalue 0 has algebraic multiplicity 2.}\\
c=-b \quad \therefore \quad \mathbf{A} &= \begin{bmatrix} -b & b \\ c & -c \end{bmatrix}\\
\textit{if } \ b &= 0, dim\ ker(\lambda\mathbf{I}-\mathbf{A})\\
&= dim\ ker(0\mathbf{I}-0)\\
&= dim\mathbb{R}^2 = 2\\
\textit{If } \ b &= 0, \textit{ the geometric multiplicity is also } 2, \textit{ and the system is stable.}\\
\textit{If } \ b &\neq 0, \textit{ the geometric multiplicity is } 1, \textit{ and thus the system is unstable.}\\
\textit{If } \ b+c &> 0, \textit{ the system is stable (but not asymptotically).}
\end{aligned}$$

Controllability:- *The Kalman matrix is*

$$\mathbf{K} = \begin{bmatrix} a & -ab \\ 0 & ac \end{bmatrix}$$

and $det(\mathbf{K}) = a^2c$. *Thus, the system is controllable if and only if* $a \neq 0$ *and* $c \neq 0$.

Example 3.1.3. *Let a scalar control system* $\dot{x} = x + u$ *be given. Compute a control* u *which steering the state from* $x(0) = 0$ *to* $x(T) = x_f$ *according to the formula in the proof above. What happens if* $T \to 0$?

solution:-

We have

$$\dot{x} = (1)x + (1)u \Rightarrow \ \mathbf{A} = (1), \quad \mathbf{B} = (1)$$

$$Therefore, \quad \mathbf{W}(T) = \int_0^T e^t 1 \cdot 1 \cdot e^t dt = \int_0^T e^{2t} dt = \frac{1}{2}(e^{2T}-1)$$

and thus

$$u(t) = 1 \cdot e^{1(T-t)} \cdot \frac{1}{2}(e^{2T}-1)x_f = \frac{2e^{T-t}}{e^{2T}-1} \cdot x_f = \frac{2e^T x_f}{e^t(e^{2T}-1)}$$

When, $\mathbf{T \to 0}$

One can see that the control function gets "bigger" if T *tends to zero. As* $T \to 0$, $e^{2T} - 1 \to 0$ *and thus* $u(t)$ *gets biger.*

Example 3.1.4. *Consider the state space system given by* $\dot{x}_1 = x_2, \dot{x}_2 = u$. *let* $x_0 = (0 \quad 0)^T$, $x_f = (1 \quad 3)^T$, *and* $t_f = 1$ *be given. Compute an input function* u *that steers the system from* $x(0) = x_0$ *to* $x(t_f) = x_f$.

Solution:- *The formula above is*

$$u(t) = \mathbf{B}^T e^{\mathbf{A}^T(1-t)} \mathbf{W}_1^{-1} x_f.$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus,

$$e^{\mathbf{A}t} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

$$\mathbf{W}_1 = \int_0^1 e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t} dt = \int_0^1 \begin{bmatrix} t^2 & t \\ t & 1 \end{bmatrix} dt = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}.$$

Plugging this in, we obtain $u(t) = 6t$.

Example 3.1.5. *Show that the minimal "energy" needed to steer a controllable system $\dot{x} = \mathbf{A}x + \mathbf{B}u$, from 0 to $\bar{x}$ in time $\epsilon > 0$ is given by*

$$\min\{\mathbf{E}(u)|u(\epsilon, 0, 0, u(.)) = \bar{x}\} = \bar{x}^T\mathbf{W}(\epsilon)^{-1}\bar{x},$$

where, $\mathbf{E}(u) = \int_0^t ||u(t)||^2 dt$ and $\mathbf{W}(.)$ denotes the controllability Gramian

$$\mathbf{W}(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} d\tau$$

We prove this result step by step by proving the following sequence of results:

a) *Show that $\dot{\mathbf{W}}(t) = \mathbf{A}\mathbf{W}(t) + \mathbf{W}(t)\mathbf{A}^T + \mathbf{B}\mathbf{B}^T$*

Proof. : We have by the fundamental of calculus, $\dot{\mathbf{W}}(t) = e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t}$ and

$$\begin{aligned} \mathbf{A}\mathbf{W}(t) + \mathbf{W}(t)\mathbf{A}^T &= \int_0^t \mathbf{A}e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} d\tau + \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau}\mathbf{A}^T d\tau \\ &= \int_0^t (\mathbf{A}e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} + e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau}\mathbf{A}^T) d\tau \\ &= \int_0^t \frac{d}{d\tau}(e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau}) \\ &= e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau}|_0^t \\ &= \underbrace{e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau}}_{\dot{\mathbf{W}}} - \mathbf{B}\mathbf{B}^T \end{aligned}$$

$$\Rightarrow \quad \dot{\mathbf{W}}(t) = \mathbf{A}\mathbf{W}(t) + \mathbf{W}(t)\mathbf{A}^T + \mathbf{B}\mathbf{B}^T$$

□

b) *Show that $\frac{d}{dt}(x(t)^{-1}) = -x(t)\dot{x}(t)x(t)^{-1}$ for every invertible matrix.*

Proof. We have $xx^{-1} = \mathbf{I}$ and hence, by product rule,

$$\dot{x}x^{-1} + x\frac{d}{dx}(x^{-1}) = 0$$

$$\Rightarrow \quad \frac{d}{dx}(x^{-1}) = -x^{-1}\dot{x}x^{-1}$$

□

c) *Consider $v(t) := x(t)^T\mathbf{W}(t)x(t)$, where $x(\cdot)$ is a solution of $\dot{x} = \mathbf{A}x + \mathbf{B}u$ and compute $\dot{v}(t)$.*

Proof. Omitting the argument t,

$$\begin{aligned}
\dot{v}(t) &= \dot{x}^T\mathbf{W}^{-1}x + x^T(\frac{d}{dt}\mathbf{W}^{-1})x + x^T\mathbf{W}^{-1}\dot{x} \\
&= (x^T\mathbf{A}^T + u^T\mathbf{B}^T)\mathbf{W}^{-1}x - x^T\mathbf{W}^{-1}\dot{\mathbf{W}}\mathbf{W}^{-1}x + x^T\mathbf{W}^{-1}(\mathbf{A}x + \mathbf{B}u) \\
&= x^T(\mathbf{A}^T)\mathbf{W}^{-1} - \mathbf{W}^{-1}\dot{\mathbf{W}}\mathbf{W}^{-1}\mathbf{A})x + u^T\mathbf{B}^T\mathbf{W}^{-1}x \\
&\quad -(\mathbf{W}^{-1}\mathbf{A} + \mathbf{A}^T\mathbf{W}^{-1} + \mathbf{W}^{-1}\mathbf{B}\mathbf{B}^T\mathbf{W}^{-1}) \\
&= -x^T\mathbf{W}^{-1}\mathbf{B}\mathbf{B}^T\mathbf{W}^{-1}x + u^T\mathbf{B}^T\mathbf{W}^{-1}x + x^T\mathbf{W}^{-1}\mathbf{B}u \\
&= -x^T\mathbf{W}^{-1}\mathbf{B}\mathbf{B}^T\mathbf{W}^{-1}x + 2u^T\mathbf{B}^T\mathbf{W}^{-1}x
\end{aligned}$$

where we have used that $\mathbf{W}$ and hence $\mathbf{W}^{-1}$ are symmetric. □

d) *Rewrite the result of* ***(c)*** *as (omitting the argument t)*

$$\dot{v} = -||\mathbf{B}^T\mathbf{W}^{-1}x||^2 + 2 < u, \mathbf{B}^T\mathbf{W}^{-1}x > .$$

Now use quadratic completion, integrate form 0 to ϵ, and conclude that for all u with $\varphi(\epsilon, 0, 0, u(\cdot)) = \bar{x}$, we must have

$$\bar{x}^T\mathbf{W}(\epsilon)^{-1}\bar{x} \leq \mathbf{E}(u).$$

Proof. The result of **(c)** can be written as

$$\dot{v} = -||\mathbf{B}^T\mathbf{W}^{-1}x||^2 + 2 < u, \mathbf{B}^T\mathbf{W}^{-1}x >$$

Using quadratic completion, we get

$$\begin{aligned}
\dot{v} &= -||\mathbf{B}^T\mathbf{W}^{-1}x||^2 + 2 < u, \mathbf{B}^T\mathbf{W}^{-1}x > -||u||^2 + ||u||^2 \quad \text{(adding and subtracting} ||u||^2) \\
&= -||u - \mathbf{B}^T\mathbf{W}^{-1}x||^2 + ||u||^2
\end{aligned}$$

Integrating from 0 to ϵ yields

$$v(\epsilon) - v(0) = -\int_0^\epsilon ||u(t) - \mathbf{B}^T\mathbf{W}(t)^{-1}x(t)||^2 dt + \int_0^\epsilon ||u(t)||^2 dt \leq \int_0^\epsilon ||u(t)||^2 dt$$

Plugging in $x(0) = 0$ and $x(\epsilon) = \bar{x}$, we have $v(0) = 0$, and hence

$$v(\epsilon) = \bar{x}^T\mathbf{W}(\epsilon)^{-1}\bar{x} \leq \mathbf{E}(u).$$

□

e) *Finally, show that equality is achived for*

$$u(t) = \mathbf{B}^T e^{\mathbf{A}^T(\epsilon - t)}\mathbf{W}(\epsilon)^{-1}\bar{x}.$$

Remark 3.6. :- *This explains the trade off between the speed and the energy consumption of control : the smaller ϵ is, the larger is $\bar{x}^T\mathbf{W}(\epsilon)^{-1}\bar{x}$.*

Proof. Plugging the given u into the solution formula with $x(0) = 0$, we get

$$x(\epsilon) = \int_0^{\epsilon} e^{\mathbf{A}^{(\epsilon-\tau)}}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T(\epsilon-\tau)} d\tau \mathbf{W}^{-1}\bar{x} = \bar{x}$$

plugging u into the solution formula with $x(0) = 0$, we get ;

$$x(\epsilon) = \int_0^{\epsilon} e^{\mathbf{A}(\epsilon-t)}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T(\epsilon-\tau)}\mathbf{W}^{-1}\bar{x}d\tau = \bar{x}$$

Showing that u steers the system to $x(\epsilon) = \bar{x}$ as desired. finally,

$$\mathbf{E}(u) = \int_0^{\epsilon} \bar{x}^T\mathbf{W}(t)e^{\mathbf{A}^{(\epsilon-t)}}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T(\epsilon-t)}\mathbf{W}^{-1}\bar{x}dt = \bar{x}^T\mathbf{W}(\epsilon)^{-1}\bar{x}.$$

$$\begin{aligned} As\ \ \epsilon \to 0 \quad , \quad & \mathbf{W}(\epsilon) \to 0 \\ \Rightarrow \quad & \mathbf{W}(\epsilon) \to +\infty \\ \Rightarrow \quad & \bar{x}^T\mathbf{W}(\epsilon)^{-1}\bar{x} \to +\infty \end{aligned}$$

i.e, the smaller ϵ is, the larger is $\bar{x}^T\mathbf{W}(\epsilon)^{-1}\bar{x}$. □

Remark 3.7.

Corollary 3.4. *Consider $\dot{x} = \mathbf{A}x + \mathbf{B}u$ or $\sigma x = \mathbf{A}x + \mathbf{B}u$. In the discrete case, assume that $\mathbf{A}$ is invertible. Then $\mathcal{C} = \mathcal{R}$.*

In general, we only have $\mathcal{R} \subseteq \mathcal{C}$, to see this consider the system $x(t+1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \end{pmatrix} u$. Then, $\mathcal{R} = \{0\}$ and $\mathcal{C} = \mathbb{R}^2$ because every state is controllable in 2 steps to zero because $\mathbf{A}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

Proof. Let τ^* be as usual. We have $x \in \mathcal{C} = \mathcal{C}(\tau^*)$ if and only if there exists $u \in \mathcal{U}$ such that

$$\varphi(\tau^*, 0, x, u) + \varphi(\tau^*, 0, x, 0) + \varphi(\tau^*, 0, 0, u) = 0,$$

that is, $\varphi(\tau^*, 0, x, 0) + \varphi(\tau^*, 0, 0, u) = 0$ and hence $\varphi(\tau^*, 0, x, 0) = \varphi(\tau^*, 0, 0, -u)$. Thus $x \in \mathcal{C}$ if and only if $\varphi(\tau^*, 0, x, 0) \in \mathcal{R}$. In continuous time, this means

$$\begin{aligned} \varphi(\tau^*, 0, x, 0) &= e^{\mathbf{A}\tau^*-0}x + \int_0^{\tau^*} e^{\mathbf{A}\tau^*-s}\mathbf{B}u(s)ds \\ &= e^{\mathbf{A}\tau^*}x \end{aligned}$$

$$x \in \mathcal{C} \Leftrightarrow e^{\mathbf{A}\tau^*}x \in \mathcal{R}. \tag{3.10}$$

Since $e^{\mathbf{A}\tau}$ is invertible, this shows that $e^{\mathbf{A}\tau}\mathcal{C} = \mathcal{R}$ and hence $dim(\mathcal{C}) = dim(\mathcal{R})$. In discrete time,

$$x \in \mathcal{C} \Leftrightarrow \mathbf{A}^{\tau}x \in \mathcal{R}. \tag{3.11}$$

If $\mathbf{A}$ is invertible, we can argue as in the continuous case to see that $\mathcal{R}$ and $\mathcal{C}$ have the same dimension. Thus it suffices to show that $\mathcal{R} \subseteq \mathcal{C}$. If $x \in \mathcal{R} = \mathcal{R}(\tau^*)$, then $\mathbf{A}^{\tau^*}x$ and $e^{\mathbf{A}\tau^*}x$ are also in $\mathcal{R}$ (this is due to the $\mathbf{A} - invariance\ of\ \mathcal{R}$). According to (3.10) and (3.11), this implies $x \in \mathcal{C}$. Therefore, $\mathcal{C} = \mathcal{R}$.

□

Example 3.1.6. *a) With the notion introduced above, show that*

$$\mathcal{R}(t, x) = \varphi(t, 0, x, 0) + \mathcal{R}(t), \text{ and}$$

$$\mathcal{C}(t, x) = \Phi(t)^{-1}(x + \mathcal{R}(t))$$

where $\Phi(t)x = \varphi(t, 0, x, 0)$.

b) Consider that in discrete time, $\mathcal{C} = (\mathbf{A}^n))^{-1}\mathcal{R} \supseteq \mathcal{R}$. Thus, a discrete state space system is completely controllable to zero if and only if $im(\mathbf{A}^n) \subseteq im(\mathbf{K})$, where $\mathbf{K}$ is the Kalman matrix.

Note: (.) denotes the inverse image in (a) and (b).

c) Consider $\mathbf{P}(\frac{d}{dt})y = u$, where $\mathbf{P}$ is a scalar monic polynomial of degree n. Transform this into a state space system of size n and show that the system is controllable.

Solution:-

a) We have

. $$x_1 \in \mathcal{R}(t, x) \Leftrightarrow \exists u : x_1 = \varphi(t, 0, x, u) = \varphi(t, 0, x, 0) + \underbrace{\varphi(t, 0, 0, u)}_{\mathcal{R}(t)}$$

$$\Leftrightarrow x_1 \in \varphi(t, 0, x, 0) + \mathcal{R}(t)$$

.

$$\begin{aligned} x_0 \in \mathcal{C}(t, x) \quad &\Leftrightarrow \quad \exists u : x = \varphi(t, 0, x_0, u) = \underbrace{\varphi(t, 0, x_0, 0)}_{\Phi(t)x_0} + \varphi(t, 0, 0, u) \\ &\Leftrightarrow \quad x \in \Phi(t)x_0 + \mathcal{R}(t) \\ &\Leftrightarrow \quad \Phi(t)x_0 \in x + \mathcal{R}(t) \\ &\Leftrightarrow \quad x_0 \in \Phi(t)^{-1}(x + \mathcal{R}(t)) \end{aligned}$$

b) We have

$$\mathcal{C}(t) = \Phi(t)^{-1}\mathcal{R}(t) = (\mathbf{A}^t)^{-1}\mathcal{R}(t)$$

and thus $\mathcal{C} = \mathcal{C}(n) = (\mathbf{A}^n)^{-1}\mathcal{R}(n) = (\mathbf{A}^n)^{-1}\mathcal{R}$.

Due to $\mathbf{A}$- invariance of $\mathcal{R}$, we have $\mathcal{R} \subseteq (\mathbf{A}^n)^{-1}\mathcal{R}$.

Thus,$\mathcal{C} = \mathbb{R}^n$ if and only if $\forall x \in \mathbb{R}^n : \mathbf{A}^n x \in \mathcal{R} = im(\mathbf{K})$, i.e, $im(\mathbf{K})$.

c) *Let* $\mathbf{P} = s^n + p_{n-1}s^{n-1} + \ldots + p_1 s + p_0$.
set $x = \begin{bmatrix} y & y^2 & \ldots & y^{(n-1)} \end{bmatrix}^T$
Then,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ -p_0 & -p_1 & \ldots & -p_{n-1} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

and hence the Kalman matrix has the shape

$$\mathbf{K} = \begin{bmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & * \end{bmatrix}$$

which shows that $\mathbf{K}$ *has rank* n *independently of the choice of the* p_i.

3.2 Controllable matrix pairs

Let

$$\mathbf{A} \in \mathbb{R}^{n\times n} \quad \textit{and} \quad \mathbf{B} \in \mathbb{R}^{n\times m}.$$

We say that the matrix pair $(\mathbf{A}, \mathbf{B})$ is controllable if the associated Kalman controllability matrix is

$$\mathbf{K} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & . & . & . & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

has full row rank, that is, $rank(\mathbf{K}) = n$.
If a state space system

$$\dot{x} = \mathbf{A}x + \mathbf{B}u \quad or \quad \sigma z = \mathbf{A}z + \mathbf{B}u$$

is subject to a coordinate transform x = T z, where $\mathbf{T} \in \mathbb{R}^{n\times n}$ is invertible, then we get

$$\dot{z} = \mathbf{T}^{-1}\mathbf{AT}z + \mathbf{T}^{-1}\mathbf{B}u \quad or \quad \sigma z = \mathbf{T}^{-1}\mathbf{AT}z + \mathbf{T}^{-1}\mathbf{B}u$$

We say that the matrix pair $(\mathbf{T}^{-1}\mathbf{AT}, \mathbf{T}^{-1}\mathbf{B})$ is similar to the matrix pair $(\mathbf{A}, \mathbf{B})$.
A coordinate transform should not change structural system properties such as stability and controllability. Indeed, similar matrices have the same eigenvalues, and the ranks of the Kalman controllability matrices of similar matrix pairs coincide.

The following result is limited to the single-input case, that is, m = 1. Then $\mathbf{B}$ is a single column vector. In that case, we simply write $\mathbf{b}$ instead of $\mathbf{B}$. The associated Kalman controllability matrix

$$\mathbf{K} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & . & . & . & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

If a system is completely controllable, there exist similarity transformations that convert it into special **standard forms**, or **canonical forms:**

- The controllability form
- The controller form

Theorem 3.2.1. *Let* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *and* $\mathbf{b} \in \mathbb{R}^n$ *, and let* $(\mathbf{A}, \mathbf{b})$ *be a controllable matrix pair. Then there exists an invertible matrix* $\mathbf{T} \in \mathbb{R}^{n\times n}$ *such that*

$$\tilde{\mathbf{A}} := \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} 0 & \dots & 0 & -a_0 \\ 1 & & & -a_1 \\ & \ddots & & \vdots \\ & & 1 & -a_n \end{bmatrix} \qquad \text{and} \qquad \tilde{\mathbf{b}} := \mathbf{T}^{-1}\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The numbers a_i *are precisely the coefficients of the characteristic polynomial, that is,*

$$\chi_{\mathbf{A}}(s) = \chi_{\mathbf{T}_{-1}\mathbf{A}\mathbf{T}}(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0.$$

This is called the controllability form of $(\mathbf{A}, \mathbf{b})$.
Moreover, there exists an invertible matrix $\mathbf{P} \in \mathbb{R}^{n\times n}$ *such that*

$$\tilde{\mathbf{A}} := \mathbf{P}\mathbf{A}\mathbf{P}^{-1} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & \dots & -a_n \end{bmatrix} \qquad \text{and} \qquad \tilde{\mathbf{b}} := \mathbf{P}\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

This is called the controller form of $(\mathbf{A}, \mathbf{b})$.

Proof. : **(The controllability form)** According to the Hamilton-Cayley theorem,

$$\mathbf{A}^n = -a_{n-1}\mathbf{A}^{n-1} - \ldots - a_1\mathbf{A} - a_0 I.$$

Thus we have

$$\mathbf{A}\mathbf{K} = \begin{bmatrix} \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & . & . & . & \mathbf{A}^n\mathbf{b} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & . & . & . & \mathbf{A}^{n-1}\mathbf{b} \end{bmatrix} \begin{bmatrix} 0 & \dots & 0 & -a_0 \\ 1 & \dots & 0 & -a_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & -a_n \end{bmatrix}$$

And

$$\mathbf{b} = \begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & . & . & . & \mathbf{A}^{n-1}\mathbf{b} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{K}\tilde{\mathbf{b}}$$

Since $(\mathbf{A}, \mathbf{b})$ is controllable, $\mathbf{K}$ is invertible

$$\mathbf{A}\mathbf{K} = \mathbf{K}\tilde{\mathbf{A}} \quad \Rightarrow \mathbf{K}^{-1}\mathbf{A}\mathbf{K} = \tilde{\mathbf{A}}$$

$$\mathbf{b} = \mathbf{K}\tilde{\mathbf{b}} \quad \Rightarrow \mathbf{K}^{-1}\mathbf{b} = \tilde{\mathbf{b}}$$

Thus we simply put $\mathbf{T} = \mathbf{K}$ □

Example 3.2.1. *Let $\dot{x} = \mathbf{A}x + bu$ be a controllable single-input system $(\mathbf{A} \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n)$ Let the characteristic polynomial of $\mathbf{A}$ be*

$$\chi_{\mathbf{A}} = det(s\mathbf{I} - \mathbf{A}) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0.$$

Recurssively, define the following vectors:

$$v^{(n)} := cb, \quad and$$

$$v^{(i)} := \mathbf{A}v^{i+1} + a_i b \ \ for \ \ i = n-1, \ldots, 0$$

a) Show that $\{v^{(1)}, \ldots, v^{(n)}\}$ is basis of $\mathbb{R}^n$ and $v^{(0)} = 0$.

b) Prove that the matrix $\mathbf{T} = \left[v^{(1)}, \ldots, v^{(n)}\right]$ that has the vectors $v^{(i)}$ as columns, is a transformation matrix that puts $(\mathbf{A}, b)$ in to controller form.

- *Apply thisto* $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Solution:-

a) We have

$$\begin{aligned} v^{(n)} &= b \\ v^{(n-1)} &= \mathbf{A}b + a_{n-1}b = (\mathbf{A} + a_{n-1}\mathbf{I})b \\ v^{(n-2)} &= \mathbf{A}v^{n-1} + a_{n-2}b = (\mathbf{A}^2 + a_{n-1}\mathbf{A} + a_{n-2}\mathbf{I})b \\ &\vdots \end{aligned}$$

Thus,

$$v^{(i)} = (\mathbf{A}^{n-i} + a_{n-1}\mathbf{A}^{n-i-1} + \ldots + a_i\mathbf{I})b = \sum_{j=0}^{n-i} a_{i+j}\mathbf{A}^j b,$$

where we set $a_n := 1$. Thus,

$$v^{(0)} = (\mathbf{A}^n + \ldots + a_0\mathbf{I})b$$

By the Hamilton-Cayley Theorem, we have $v^{(0)} = 0$. Then,

$$0 = \sum_{i=0}^{n} \lambda v^{(i)} = \sum_{i=1}^{n}\sum_{j=0}^{n-i} \lambda_{i+j}\mathbf{A}^j b = \sum_{j=0}^{n}(\sum_{i=1}^{n-j} \lambda_i a_{i+j})\mathbf{A}^j b$$

Since $(\mathbf{A}, b)$ is controllable, the vectors $b, \mathbf{A}b, \ldots, \mathbf{A}^{n-1}b$ are linearly independent Thus, we must have :

$$\sum_{i=1}^{n-j} \lambda_i a_{i+j} = 0$$

for all $j = 0, \dots, n-1$. *This implies (for*$j = n-1$*) that* $\lambda_1 a_n = 0$ *and thus* $\lambda_1 = 0$. *For* $j = n-2$, *we get* $\lambda_1 a_{n-1} + \lambda_2 a_n = 0$, *and thus* $\lambda_2 = 0$. *Proceeding like this , we see that all* $\lambda_i = 0$ *as desired.*

Hence, $\{v^{(1)}, \dots, v^{(n)}\}$ *is a basis of* $\mathbb{R}^n$.

b) *We have , for* $i = 1, \dots, n-1$

$$\mathbf{A}v^{(i+1)} - a_i b \quad \textit{and} \quad \mathbf{A}v^{(1)} = v^{(0)} - a_0 b = -a_0 b.$$

writing this column-wise into a matrix equation we obtain

$$\mathbf{A}\left[v^{(1)}, \dots, v^{(n)}\right] = \left[v^{(1)}, \dots, v^{(n)}\right] \underbrace{\begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix}}_{\mathbf{A}_{controller}}$$

and thus $\mathbf{AT} = \mathbf{TA}_{controller}$ *or* $\mathbf{T}^{-1}\mathbf{AT} = \mathbf{A}_{controller}$. *Finally,*

$$\left[v^{(1)} \dots v^{(n)}\right] \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = v^{(n)} = b$$

that is , $\mathbf{T}b_{controller} = b$ *or* $b_{controller} = \mathbf{T}^{-1}b$ *as desired.*

c) *The characteristic polynomial is* $\chi_{\mathbf{A}} = s^3 - s^2 - 3s - 1$. *Thus,*

$$v^{(3)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v^{(2)} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad v^{(3)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Thus, we set

$$\mathbf{T} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and we obtain

$$\mathbf{T}^{-1}\mathbf{AT} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 1 \end{bmatrix}, \quad \mathbf{T}^{-1}b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The above theorem can be extended for the case $m > 1$, and systems which are not necessarly controllable.

Theorem 3.2.2 (Kalman controllability decomposition). *Let* $\mathbf{A} \in \mathbb{R}^{n \times n}$ *and* $\mathbf{B} \in \mathbb{R}^{n \times m}$. *Let* $\mathbf{K}$ *be the associated Kalman controllability matrix and let*

$r := rank(\mathbf{K})$. *Then there exists an invertible matrix* $\mathbf{T} \in \mathbb{R}^{n\times n}$ *such that*

$$\tilde{\mathbf{A}} := \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ 0 & \mathbf{A}_3 \end{bmatrix} \qquad \text{and} \qquad \tilde{\mathbf{B}} := \begin{bmatrix} \mathbf{B}_1 \\ 0 \end{bmatrix}$$

where $\mathbf{A}_1 \in \mathbb{R}^{r\times r}, \mathbf{B}_1 \in \mathbb{R}^{r\times m}$ *is a controllable matrix pair.*

Proof. Since $\mathcal{R}$ is a space of reachable states from zero and $\mathcal{R} = im(\mathbf{K})$, we have $\mathcal{R}$ equals the span of columns of the Kalman matrix.
Thus, there exist vectors $v_1, ..., v_r \in \mathcal{R}$ that form a basis of $\mathcal{R}$ By theorem 1.1.3 Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space. Therefore, there exist $w_1, ..., w_{n-r} \in \mathbb{R}^n$ such that $v_1, ..., v_r, w_1, ..., w_{n-r}$ is a basis of $\mathbb{R}^n$. Then

$$\mathbf{T} := [\mathbf{V} \quad \mathbf{W}] := [v_1 \quad . \quad . \quad . \quad v_r \quad w_1 \quad . \quad . \quad .w_{n-r}] \in \mathbb{R}^{n\times n}$$

is an invertible matrix.
Due to the $\mathbf{A}-invariance\ \ of\ \ \mathcal{R}, the\ \ columns\ \ of\,\mathbf{AV}$ are again in $\mathcal{R}$. Thus they can be written as linear combinations of the vectors v_i , that is,

$$\begin{aligned}
\mathbf{AV} &= \mathbf{A}\begin{bmatrix} v_1 & v_2 & \dots & v_r \end{bmatrix} \\
&= \begin{bmatrix} a_{11}v_1 + a_{21}v_2 + \dots + a_{r1}v_r, & a_{12}v_1 + \dots + a_{r2}v_r, & \dots, & a_{1r}v_1 + \dots, +a_{rr}v_r \end{bmatrix} \\
&= \begin{bmatrix} v_1 & v_2 & \dots & v_r \end{bmatrix} \underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rr} \end{bmatrix}}_{=:\mathbf{A}_1} \\
&= \mathbf{V}\mathbb{A}_1
\end{aligned}$$

for some matrix $\mathbf{A}_1 \in \mathbb{R}^{r\times r}$. On the other hand, the columns of $\mathbf{AW}\ \ are\ \ in\ \ \mathbb{R}^n$ and thus they can be written as linear combinations of v_i, w_j , that is,

$$\mathbf{AW} = \mathbf{VA}_2 + \mathbf{WA}_3$$

for some matrices $\mathbf{A}_2, \mathbf{A}_3$ of appropriate sizes. Summing up, we have

$$\mathbf{AT} = [\mathbf{AV} \quad \mathbf{AW}] = [\mathbf{VA}_1 \quad \mathbf{VA}_2 + \mathbf{WA}_3] = [\mathbf{V} \quad \mathbf{W}]\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ 0 & \mathbf{A}_3 \end{bmatrix} = \mathbf{T}\tilde{\mathbf{A}}$$

$$\begin{aligned}
\text{since } \mathcal{R} = im(\mathbf{K}) &= im(\mathbf{B}) + \mathbf{A}im(\mathbf{B}) + \dots + \mathbf{A}^{n-1}im(\mathbf{B}) \\
&= (\mathbf{I} + \mathbf{A} + \dots + \mathbf{A}^{n-1})im(\mathbf{B}), \qquad \text{we have}
\end{aligned}$$

$$im(\mathbf{B}) \subseteq \mathcal{R}$$

Thus, the columns of $\mathbf{B}$ are linear combinations of the vectors v_i, that is,

$$\mathbf{B} = \mathbf{V}\mathbf{B}_1 = [\mathbf{V} \quad \mathbf{W}] \begin{bmatrix} \mathbf{B}_1 \\ 0 \end{bmatrix} = \mathbf{T}\tilde{\mathbf{B}}$$

some matrix $\mathbf{B}_1 \in \mathbb{R}^{r \times m}$. It remains to show that $(\mathbf{A}_1, \mathbf{B}_1)$ is a controllable matrix pair. The Kalman controllability matrix associated to $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})$ is

$$\tilde{\mathbf{K}} = \begin{bmatrix} \tilde{\mathbf{B}} & \tilde{\mathbf{A}}\tilde{\mathbf{B}} & \cdots & \tilde{\mathbf{A}}^{n-1}\tilde{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{A}_1\mathbf{B}_1 & \cdots & \mathbf{A}_1^{n-1}\mathbf{B}_1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

this has rank r, just like the Kalman controllability matrix of the original matrix pair. Therefore,

$$r = rank \begin{bmatrix} \mathbf{B}_1 & \mathbf{A}_1\mathbf{B}_1 & \cdots & \mathbf{A}_1^{n-1}\mathbf{B}_1 \end{bmatrix}.$$

Due to the Hamilton-Cayley theorem, this implies that

$$r = rank \begin{bmatrix} \mathbf{B}_1 & \mathbf{A}_1\mathbf{B}_1 & \cdots & \mathbf{A}_1^{r-1}\mathbf{B}_1 \end{bmatrix}.$$

which shows that $(\mathbf{A}_1, \mathbf{B}_1)$ is controllable. □

Remark 3.8. *Let $\chi_{\mathbf{A}}$ and $spec(\mathbf{A})$ denote the characterstic polynomial and the spectrum of $\mathbf{A}$, respectively.*
In Kalman controllability decomposition we clearly have

$$\chi_{\mathbf{A}} = \chi_{\mathbf{A}_1} \cdot \chi_{\mathbf{A}_3}$$

and thus

$$spec(\mathbf{A}) = spec\mathbf{A}_1 \cup spec\mathbf{A}_3.$$

We call $\chi_{\mathbf{A}_3}$ the uncontrollable part of the characterstic polynomial of $\mathbf{A}$ with respect to $\mathbf{B}$, and $\lambda \in spec(\mathbf{A}_3)$ an uncontrollable mode of $(\mathbf{A}, \mathbf{B})$. Of course, it has to be verified that these notions do not depend on the specific choice of the Kalman decomposition (which is non-unique, in general, since its construction involves several choices): In fact, the matrix $\mathbf{A}_1$ is the matrix representation of $\mathbf{A}|_{\mathcal{R}} : \mathcal{R} \to \mathcal{R}$ with respect to the chosen basis of $\mathcal{R}$. Thus, $\mathbf{A}_1$ depends on the choice of the basis, but $\chi_{\mathbf{A}_1}$ does not. Therefore, this holds also for $\chi_{\mathbf{A}_3} = \chi_{\mathbf{A}} \setminus \chi_{\mathbf{A}_1}$. The decomposition in the theorem is not unique but $\chi_{\mathbf{A}_3}$ is unique.

Example 3.2.2. *Consider the system $\dot{x} = \mathbf{A}x + \mathbf{B}u$ where, $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.*
The Kalman controllability matrix is

$$\mathbf{K} = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which has rankr $= 1$. We choose the vector $v_1 := \mathbf{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as a basis of $\mathcal{R} = im(\mathbf{K})$.

We can also choose $w_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ *so that* v_1, w_1 *form a basis of* $\mathbb{R}^2$*. Thus, we set*

$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then,

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \ \textit{and} \ \mathbf{T}^{-1}b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

observe that $\chi_{\mathbf{A}}$ *and hence* 0 *is uncontrollable mode of* $(\mathbf{A}, \mathbf{B})$.

Another way to characterize the uncontrollable modes of a matrix pair is given in the next theorem:

Theorem 3.2.3. *Let* $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$, *and* $\lambda \in \mathbb{C}$ *. The following are equivalent:*

1. *λ is an uncontrollable mode of $(\mathbf{A}, \mathbf{B})$;*
2. *rank* $\begin{bmatrix} \lambda\mathbf{I} - \mathbf{A} & \mathbf{B} \end{bmatrix} \leq n$.

Proof. Let $\tilde{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ *and* $\tilde{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B}$. Since

$$[\lambda\mathbf{I} - \tilde{\mathbf{A}} \ \ \tilde{\mathbf{B}}] = \mathbf{T}^{-1}[\lambda\mathbf{I} - \mathbf{A} \ \ \mathbf{B}] \begin{bmatrix} \mathbf{T} & 0 \\ 0 & \mathbf{I} \end{bmatrix}$$

we have

$$rank \ \ [\lambda\mathbf{I} - \tilde{\mathbf{A}} \ \ \tilde{\mathbf{B}}] = rank \ \ [\lambda\mathbf{I} - \mathbf{A} \ \ \mathbf{B}]$$

Thus we may assume, without loss of generality, that a Kalman controllability decomposition has already been performed, i.e,

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ 0 & \mathbf{A}_3 \end{bmatrix} \ \textit{and} \ \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ 0 \end{bmatrix}$$

where $(\mathbf{A}_1, \mathbf{B}_1)$ is controllable. Then we have

$$\mathbf{H}(\lambda) := [\lambda\mathbf{I} - \mathbf{A} \ \ \mathbf{B}] = \begin{bmatrix} \lambda\mathbf{I} - \mathbf{A}_1 & -\mathbf{A}_2 & \mathbf{B}_1 \\ 0 & \lambda\mathbf{I} - \mathbf{A}_3 & 0 \end{bmatrix}$$

$(\Rightarrow)$

If λ is an uncontrollable mode, then it is an eigenvalue of $\mathbf{A}_3$. Thus it makes determinant of $\lambda\mathbf{I} - \mathbf{A}_3$ zero and thus $\lambda\mathbf{I} - \mathbf{A}_3$ is singular. Therefore, $\mathbf{H}(\lambda)$ can not have full row rank and hence $rank\,[\lambda\mathbf{I} - \mathbf{A} \quad \mathbf{B}] < n$. $(\Leftarrow)$ Conversely, Assume that the rank of $\mathbf{H}(\lambda)$ is not full. Then there exists a vector $\mathbf{x} \neq 0$ such that $\mathbf{x}\mathbf{H}(\lambda) = 0$, that is,

$$x_1(\lambda\mathbf{I} - \mathbf{A}_1) = 0$$

$$-x_1\mathbf{A}_2 + x_2(\lambda\mathbf{I} - \mathbf{A}_3) = 0$$

$$x_1\mathbf{B}_1 = 0$$

The first and third equations imply that

$$x_1\mathbf{B}_1 = 0, \quad x_1\mathbf{A}_1\mathbf{B}_1 = 0, \quad x_1\mathbf{A}_1^2\mathbf{B}_1 = 0, \quad ...$$

To see this, we multiply the 1^{st} equation by $\mathbf{B}_1$ from the right to gets

$$\underbrace{\lambda x_1\mathbf{B}_1}_{=:0} - x_1\mathbf{A}_1\mathbf{B}_1 = 0$$

$$\rightarrow \quad x_1\mathbf{A}_1\mathbf{B}_1 = 0.$$

Thus, $x_1\begin{bmatrix}\mathbf{B}_1 & \mathbf{A}_1\mathbf{B}_1 & \dots & \mathbf{A}_1^{r-1}\mathbf{B}_1\end{bmatrix} = 0$. The controllability of $(\mathbf{A}_1, \mathbf{B}_1)$ *yields that* $x_1 = 0$.*Then* $x_2 \neq 0$ and

$$x_2(\lambda\mathbf{I} - \mathbf{A}_3) = 0$$

which implies that λ *is an eigenvalue of* $\mathbf{A}_3$, that is, an uncontrollable mode of $(\mathbf{A}, \mathbf{B})$. □

Example 3.2.3. *Consider once more*

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then,

$$\mathbf{H}(\lambda) = \begin{bmatrix} \lambda - 1 & 0 & 1 \\ -1 & \lambda & 1 \end{bmatrix}$$

which has rank 2 *for all* $\lambda \neq 0$. *But,* $rank(\mathbf{H}(\mathbf{0})) = 1 < 2$, *showing that* 0 *is an uncontrollable mode of the system.*

As a consequence of the above theorem, we will have the following alternative criteria controllability of a system.

Corollary 3.5 (Hautus test for controllability)**.** *The following are equivalent:*

1. $(\mathbf{A}, \mathbf{B})$ *is controllable.*
2. *The matrix* $\mathbf{H}(\lambda) = [\lambda\mathbf{I} - \mathbf{A} \;\; \mathbf{B}]$ *has full row rank for all* $\lambda \in \mathbb{C}$.

We call the polynomial matrix $\mathbf{H} = [s\mathbf{I} - \mathbf{A} \quad \mathbf{B}] \in \mathbb{R}^{n\times(n+m)}$ *is called Hautus controllability matrix.*

3.3 Asymptotic controllability

Sometimes, it is not required that a system should go from one state to another in finite time τ . Instead, one is satisfied if this happens asymptotically as $\tau \to \infty$.

Definition 3.3.1. *We say that a state space system* $\dot{x} = \mathbf{A}x + \mathbf{B}u$ *or* $\sigma x = \mathbf{A}x + \mathbf{B}u$ *is asymptotically controllable (to zero) if for any* $x_0 \in \mathbf{X} = \mathbb{R}^n$ *, there exists an input function* $u \in \mathcal{U}$ *such that*

$$\lim_{t\to\infty} \varphi(t, 0, x_0, u) = 0.$$

Remark 3.9. *Controllability implies asymptotic controllability.*

Theorem 3.3.1. *A state space system is asymptotically controllable if and only if its uncontrollable modes* λ *are asymptotically stable, that is,* $Re(\lambda) < 0$ *in continuous time, and* $|\lambda| < 1$ *in discrete time.*

Let us convince ourselves at least of the simple direction of the proof. We may assume, without loss of generality, that

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ 0 & \mathbf{A}_3 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ 0 \end{bmatrix}$$

Then $x_2(t) = e^{A_3 t}x_2(0)$ or $x_2(t) = A_3^t x_2(0)$, respectively. If the system is asymptotically controllable, we must have $\lim_{t\to\infty} x_2(t) = 0$ for all $x_2(0)$, which implies that A_3 must be asymptotically stable.

Example 3.3.1. *Compute a Kalman controllability decomposition of*

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

What are the uncontrollable modes of $(\mathbf{A}, \mathbf{B})$? *Is the system asymptotically controllable? Consider both continuous and discrete time.*

solution :-

The corresponding Kalman matrix is

$$\mathbf{K} = \begin{pmatrix} 1 & 3 & 11 \\ 1 & 3 & 11 \\ 0 & 2 & 10 \end{pmatrix}$$

which has rank $r = 2$. *We may take the first* 2 *columns of* $\mathbf{K}$ *as a basis of* $\mathcal{R}$. *We need to find a third vector that completes the basis. A possible choice is* $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ *and so we put*

$$\mathbf{T} = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 3 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

We obtain

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} 0 & -4 & \frac{1}{2} \\ 1 & 5 & \frac{1}{2} \\ 0 & 0 & -1 \end{bmatrix} \quad \textit{and} \quad \mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The only uncontrollable mode is -1. Thus, the system is asymptotically controllable in continuous time, but not in discrete time.

3.4 Controllable behaviors

The Hautus test gives us an idea about how to generalize the notion of controllability from state space systems $\dot{x} = \mathbf{A}x + \mathbf{B}u$ to general systems $\mathbf{R}(\frac{d}{dt})w = 0$ where $\mathbf{R} \in \mathbb{R}[s]^{p\times q}$, and $w \in \mathcal{A}$. In a state space system

$$\mathbf{R} = [s\mathbf{I} - \mathbf{A} \quad -\mathbf{B}] \in \mathbb{R}[s]^{n(n+m)} \quad \textit{and} \quad w = \begin{bmatrix} x \\ u \end{bmatrix}$$

The polynomial matrix $\mathbf{R}$ is recognized as the Hautus controllability matrix (up to the sign of $\mathbf{B}$, which does not influence the rank).

In this section, we restrict to continuous systems, and we return to our original signal spaces, that is, $\mathcal{A} = \mathbf{D}'(\mathbf{T})$, where $\mathbf{T} = \mathbb{R}, \mathbb{R}_+$.

Definition 3.4.1. *Let $t_0 \in \mathbf{T}$ be fixed. Let $\mathbf{R} \in \mathbb{R}[s]^{p\times q}$ and*

$$\mathcal{B} = \{w \in \mathcal{A}^q | \mathbf{R}(\frac{d}{dt})w = 0\}$$

*where $\mathcal{A}$ is as described above. Let $w^{(1)}, w^{(2)} \in \mathcal{B}$ be two trajectories. We say that $w^{(2)}$ is reachable from $w^{(1)}$ in time $\tau \geq 0$ (or: $w^{(1)}$ is controllable to $w^{(2)}$ in time τ) if there exists a trajectory $w \in \mathcal{B}$ which coincides with $w^{(1)}$ on $(-\infty, t_0) \cap \mathbf{T}$ and with $w^{(2)}$ on $(t_0 + \tau, \infty) \cap \mathbf{T}$. If there is any such $\tau \geq 0$, then we say that $w^{(2)}$ is reachable from $w^{(1)}$ (or: $w^{(1)}$ is controllable to $w^{(2)}$). One says that $\mathcal{B}$ is **controllable** if any $w^{(1)} \in \mathcal{B}$ can be controlled to any $w^{(2)} \in \mathcal{B}$.*

To say that two distributions coincide on an open set $U \subseteq \mathbb{R}$ means that they assign the same value to each test function whose support lies in U . In the situation described above, one calls w a connecting trajectory for $w^{(1)}, w^{(2)}$. For classical functions, this means

$$w(t) = \begin{cases} w^{(1)} & \text{if } t < t_0 \\ w^{(2)} & \text{if } t > t_0 + \tau \end{cases}$$

The choice of the starting time t_0 makes no difference, because we consider only time-invariant systems. Similarly as with state space systems, the transition time τ can be made arbitrarily small, independently of the choice of the two trajectories to be connected.

Theorem 3.4.1 (Generalized Hautus test). *Let $\mathcal{A} = \mathcal{D}'(\mathbf{T})$ for $\mathbf{T} = \mathbb{R}$ or $\mathbb{R}_+$ and let $\mathbf{R} \in \mathbb{R}[s]^{p\times q}$. Without loss of generality, let $\mathbf{R}$ have full row rank. Then the behavior*

$$\mathcal{B} = \left\{ w \in \mathcal{A}^q | \mathbf{R}(\frac{d}{dt})w = 0 \right\}$$

is controllable if and only if $rank(\mathbf{R}(\lambda)) = p$ for all $\lambda \in \mathbb{C}$.

Proof. Let $\mathbf{R}$ have full row rank and let

$$\mathbf{URV} = \begin{bmatrix} \mathbf{D} & 0 \end{bmatrix}$$

be the Smith form of $\mathbf{R}$, with $\mathbf{D} = diag(d_1, \dots, d_p)$. Since $\mathbf{U}, \mathbf{V}$ are unimodular, we have, for all $\lambda \in \mathbb{C}$,

$$rank(\mathbf{R}(\lambda)) = rank(\mathbf{U}(\lambda)\mathbf{R}(\lambda)\mathbf{V}(\lambda)) = rank(\mathbf{D}(\lambda)).$$

Thus, $rank(\mathbf{R}(\lambda)) = p$ for all $\lambda \in \mathbb{C}$ if and only if $det(\mathbf{D}(\lambda)) \neq 0$ for all $\lambda \in \mathbb{C}$, that is, if no d_i has a zero in $\mathbb{C}$. This is true if and only if all d_i are constants. Since we may always assume that the d_i are monic polynomials (i.e., their leading coefficients are equal to one), we have

$$rank(\mathbf{R}(\lambda)) = p \quad \text{for all} \quad \lambda \in \mathbb{C} \quad \Leftrightarrow \quad \mathbf{D} = \mathbf{I}.$$

Consider as usual

$$\tilde{\mathcal{B}} = \left\{ \tilde{w} \in \mathcal{A}^q | \begin{bmatrix} \mathbf{D} & 0 \end{bmatrix} \tilde{w} = 0 \right\}$$

(w has no influence on the solution set but v has an influence.) which is related to $\mathbf{B}$ via the isomorphism $\tilde{\mathcal{B}} \to \mathcal{B}, \tilde{w} \mapsto w = \mathbf{V}(\frac{d}{dt})\tilde{w}$. because

$$\begin{aligned} \mathbf{R}w = 0 \quad &\Rightarrow \quad u\mathbf{R}w = 0 \\ &\Rightarrow \quad u\mathbf{R}vv^{-1}w = 0 \\ &\Rightarrow \quad [\mathbf{D} \quad 0]\,\tilde{w} = 0 \end{aligned}$$

$$\text{where} \quad \tilde{w} = v^{-1}w \quad \text{and thus} \quad w = v\tilde{w}.$$

Thus, $\tilde{\mathcal{B}}$ is controllable if and only if $\mathcal{B}$ is controllable. However, if $\mathbf{D} = \mathbf{I}$, then

$$\tilde{\mathcal{B}} = \{0\} \times \mathcal{A}^{q-p} \subseteq \mathcal{A}^q$$

which is clearly controllable. Conversely, suppose $\tilde{\mathcal{B}}$ is controllable, we need to show that $\mathbf{D} = \mathbf{I}$.

If $\mathbf{D} \neq \mathbf{I}$, there exists at least one d_i , say d_1 , which is not constant. Then the equation for the first component $\tilde{w}_1$ of $\tilde{w}$ reads

$$d_1(\frac{d}{dt})\tilde{w} = 0$$

which has precisely the solutions

$$\tilde{w}_1(t) = \sum_{\lambda} a_\lambda(t) e^{\lambda t} \tag{3.12}$$

where $\lambda \in \mathbb{C}$ are the zeros of d_1 (since d_1 is not a constant, there exists at least one such λ). Now consider $t_0 \in \mathbf{T}$ and two trajectories $\tilde{w}^{(1)}$, $\tilde{w}^{(2)}$ in $\tilde{\mathcal{B}}$, where the first component of $\tilde{w}^{(1)}$ is not identically zero on $(-\infty, t_0) \cap \mathbf{T}$, and $w^{(2)}$ is the zero trajectory. Then $\tilde{w}^{(2)}$ is not reachable from $\tilde{w}(1)$: A connecting trajectory $\tilde{w}$ would have to satisfy $\tilde{w}_1 = 0$ on $(t_0 + \tau, \infty)$, which implies that $\tilde{w}_1 = 0$ everywhere because (3.12) that $\tilde{w}_1$ must be an analytic function. Thus $\tilde{w}$ cannot coincide with $\tilde{w}^{(1)}$ on $(-\infty, t_0) \cap \mathbf{T}$. This shows that the system is not controllable, which is a contradiction to the assumption that $\tilde{\mathcal{B}}$ is controllable. Then, $\mathbf{D} = \mathbf{I}$ and hence, $rank(\mathbf{R}(\lambda)) = p \ \ \forall \lambda \in \mathbb{C}$.

□

For instance, a scalar input-output representation

$$y^{(n)} + a_{n1} y^{(n-1)} + \ldots + a_1 \dot{y} + a_0 y = u$$

is controllable, because putting

$$\mathbf{R} = \begin{bmatrix} -1 & \quad s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 \end{bmatrix} \quad \textit{and} \quad w = \begin{bmatrix} u \\ y \end{bmatrix}$$

we see that $rank(\mathbf{R}(\lambda)) = 1$ for all $\lambda \in \mathbb{C}$.

Polynomial matrices that represent controllable behaviors are characterized by the following theorem.

Theorem 3.4.2. *Let $\mathbf{R} \in \mathbb{R}[s]^{p \times q}$ be a polynomial matrix with full row rank. The following are equivalent:*

1. *$rank(\mathbf{R}(\lambda)) = p$ for all $\lambda \in \mathbb{C}$;*
2. *The Smith form of $\mathbf{R}$ is $[\, \mathbf{I} \quad 0 \,]$;*
3. *There exists a matrix $\mathbf{T} \in \mathbb{R}[s]^{(q-p) \times q}$ such that $\begin{bmatrix} \mathbf{R} \\ \mathbf{T} \end{bmatrix}$ is unimodular;*
4. *There exists a matrix $S \in \mathbb{R}[s]^{q \times p}$ such that $\mathbf{R}S = \mathbf{I}$;*
5. *If $\mathbf{R} = \mathbf{U}\mathbf{R}_1$ for some $\mathbf{U} \in \mathbb{R}[s]^{p \times p}, \mathbf{R}_1 \in \mathbb{R}[s]^{p \times q}$, then $\mathbf{U}$ must be unimodular.*

*If the equivalent conditions are satisfied, we say that $\mathbf{R}$ is **left prime** (or: **left irreducible**).*

Proof. : We have already seen in the previous proof that "1 ⇒ 2". The converse is obvious.

(2 $\Longrightarrow$ 3): Assume that

$$\mathbf{R} = \mathbf{U}\begin{bmatrix} \mathbf{I} & 0 \end{bmatrix}\mathbf{V} = \mathbf{U}\begin{bmatrix} \mathbf{I} & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \mathbf{U}V_1$$

where $\mathbf{U}$ and $\mathbf{V}$ are unimodular. Define $\mathbf{T} := V_2$, then

$$\begin{bmatrix} \mathbf{R} \\ \mathbf{T} \end{bmatrix} = \begin{bmatrix} \mathbf{U}V_1 \\ V_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{U} & 0 \\ 0 & \mathbf{I} \end{bmatrix}}_{unimodular} \underbrace{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}}_{unimodular}$$

which shows that the matrix is unimodular.

For 3 $\Longrightarrow$ 4,

Let $\mathbf{T}$ be a matrix according to assertion 3. Then there exist matrices S_1, S_2 such that

$$\begin{bmatrix} \mathbf{R} \\ \mathbf{T} \end{bmatrix}\begin{bmatrix} S_1 & S_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix}$$

In particular, $\mathbf{R}S_1 = \mathbf{I}$.

For 4 $\Longrightarrow$ 5,

Let S be such that $\mathbf{R}S = \mathbf{I}$ and $\mathbf{R} = \mathbf{U}R_1$. Then $\mathbf{U}R_1S = \mathbf{I}$, which shows that $\mathbf{U}$ is unimodular because $\Re S$ is a polynomial and inverse of $\mathbf{U}$.

Finally, we show 5 $\Longrightarrow$ 2 by negation.

Assume that the Smith form is $\begin{bmatrix} \mathbf{D} & 0 \end{bmatrix}$ with $\mathbf{D} = diag(d_1, \ldots, d_p)$ and at least one of the d_i is not a constant, say $d_1 \notin \mathbb{R}$. Then

$$\mathbf{R} = \mathbf{U}\begin{bmatrix} \mathbf{D} & 0 \end{bmatrix}\mathbf{V} = U_1R_1$$

with

$$U_1 = \begin{bmatrix} d_1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \quad and \quad R_1 = \begin{bmatrix} 1 & & & & \\ & d_2 & & & \\ & & \ddots & & 0 \\ & & & d_p & \end{bmatrix}\mathbf{V}$$

We have $det(U_1) = det(U)d_1$ and thus we have found a factorization $\mathbf{R} = U_1R_1$ in which U_1 is not unimodular. □

A polynomial matrix $\mathbf{R} \in \mathbb{R}[s]^{p\times q}$ with full row rank is left prime if and only if its $p \times p$ subdeterminants have no common zeros in $\mathbb{C}$ (since we have $rank(R(\lambda)) < p$ if and only if $\lambda \in \mathbb{C}$ is a common zero of all $p \times p$ minors). By the fundamental theorem of algebra, it is also equivalent to say that the $p \times p$ subdeterminants of $\mathbf{R}$ are devoid of common factors, i.e., they are coprime polynomials in $\mathbb{R}[s]$

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