DYNAMICS OF DEGENERATE THREE-LEVEL LASER WITH

SPONTANEOUS EMISSION AND NOISELESS

VACUUM RESERVOIR

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This is to certify that the thesis prepared by **Getachew Asmelash**, entitled: **Dynamics of Degenerate Three-Level Laser with Spontaneous Emission and Noiseless Vacuum Reservoir** and submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy in Physics **(Quantum Optics)** complies with the regulations of the University and meets the accepted standards with respect to originality and quality. Signed by the Examining Committee

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Abstract

In this dissertation we have studied the quantum properties of the cavity light produced by a coherently pumped three-level laser in an open cavity and coupled to a vacuum reservoir via a single-port mirror. We have carried out our analysis by putting the noise operators associated with the vacuum reservoir in normal order and by taking into consideration the interaction of the three-level atoms with the vacuum reservoir outside the cavity. Employing the pertinent master equation for a coherently pumped degenerate three-level atom, we have obtained coupled equations of evolution for the expectation values of the atomic operators. Then applying the large-time approximation scheme to the quantum Langevin equations for the cavity mode operators, we have managed to decouple the equations of evolution for the expectation values of the atomic operators. Applying the solutions of the pertinent equations of evolution, we have calculated the local and global mean photon number, the local and global variance of the photon number, and the local and global quadrature squeezing of the degenerate three-level laser.

We have established that a large part of the total mean photon number and the total variance of the photon number are confined in relatively small frequency intervals. Moreover, we have observed that the cavity light is in a squeezed state and the squeezing occurs in the minus qudrature. In addition, we have found that the maximum global quadrature squeezing of the cavity light is 43.42% (and occurs at $\Omega = 0.1717$ for $\gamma = 0$). We have also seen that the cavity light produced by the laser operating under the conditions $\Omega \gg \gamma_c$ and $\Omega \gg \gamma$ is in a chaotic state. In addition, we have shown that the presence of the spontaneous emission process leads to a decrease in the mean photon number, the variance of the photon number, and the maximum quadrature squeezing. Furthermore, we have established that the

maximum local quadrature squeezing of the cavity light is 78.48% (and occurs at $\lambda = 0.0606$ for $\gamma = 0$).

We have also analyzed the squeezing and statistical properties of a pair of superposed degenerate three-level laser light beams. We have found that the global mean photon number of the superposed two-mode laser light beams is the sum of the global mean photon numbers of the constituent two-mode laser light beams. Furthermore, we have noted that unlike the mean photon number, the variance of the photon number for the superposed two-mode laser light beams is not the sum of the photon-number variances of the constituent two-mode laser light beams. However, it turns out to be four times that of a two-mode laser light beam, for the case in which the separate two-mode laser light beams are identical. Finally, we have found that the superposition of the two-mode laser light beams does not affect the local and global quadrature squeezing, but it increases the global (local) mean photon number and the global (local) variance of the photon number. Thus, we note that the superposition of the two-mode laser light beams leads to a more bright squeezed light.

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Dedication

To my parents: Asmelash Gebru and Abrehet Hishe.

For their love and support,

and for passing on to me their thirst for knowledge.

“In physics, you don’t have to go around making trouble for yourself - nature does it for you.”

Frank Wilczek

“It is often stated that of all the theories proposed in this century, the silliest is quantum theory. In fact, some say that the only thing that quantum theory has going for it is that it is unquestionably correct.”

Michio Kaku, author of **Physics of the Impossible.**

Contents

1

Introduction

Over the years, a considerable attention has been paid to squeezed states of light. Squeezing is one of the nonclassical features of light that has been predicted theoretically [1-11] and subsequently experimentally observed [12,13]. In a squeezed state the quantum noise in one quadrature is below the vacuum level at the expense of enhanced fluctuations in the conjugate quadrature, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation [1,3,12,14]. Because of the quantum noise reduction achievable below the vacuum level, squeezed light has potential applications in the detection of week signals and in low-noise communications [1,2,4,12,14]. Squeezed light can be generated by various quantum optical processes such as subharmonic generations [1-4,12,14,15,16], resonance fluorescence [1,17,18], second harmonic generation [1,18,19], and four-wave mixing [5,10,20]. Recently, it has been predicted theoretically that a three-level laser under certain conditions can generate squeezed light [1,2,4,21-31].

A three-level laser is a quantum optical device in which light is generated by three-level atoms in a cavity usually coupled to a vacuum reservoir via a single-port mirror. When a three-level atom in a cascade configuration makes a transition from the top to the bottom level via the intermediate level, two photons are generated. If the photons have the same frequency, then the three-level atom is called degenerate otherwise it is called nondegenerate. In one model of a three-level laser, three-level atoms initially prepared in a coherent superposition of the top and bottom levels are injected into a cavity and then removed from the cavity after they have decayed due to spontaneous emission [21-27]. In another model of a three-level laser, the top and bottom levels of the three-level atoms injected into a cavity are coupled by coherent light [28-31]. It is found that a three-level laser in either model generates squeezed light under certain conditions. The coupling of the top and bottom levels is responsible for the squeezing of the generated light. It appears to be quite difficult to prepare the atoms in a coherent superposition of the top and bottom levels before they are injected into the laser cavity [1]. In addition, it should certainly be hard to find out that the atoms have decayed spontaneously before they are removed from the cavity [33].

In order to avoid the aforementioned problems, Fesseha [33] has studied the squeezing and the statistical properties of the light produced by a three-level laser with the atoms in a closed cavity and pumped by electron bombardment. He has shown that the maximum quadrature squeezing of the light generated by the laser, operating far below threshold, is 50% below the vacuum-state level. Alternatively, the three-level atoms available in a closed cavity and pumped by coherent light can also generate squeezed light under certain conditions, with the maximum quadrature squeezing being 43% below the vacuum-state level [1]. It appears to be practically more convienent to pump the atoms by coherent light than electrobombardment [1].

In this dissertation, we seek to study the quantum properties of the light produced by a three-level laser pumped by coherent light coupled to a vacuum reservoir via a single-port mirror. We carry out our analysis by putting the noise operators associated with the vacuum reservoir in the normal order and by taking into consideration the interaction of the three-level atoms with the vacuum reservoir outside the cavity. We first obtain the quantum Langevin equations for the cavity mode operators and the pertinent master equation for a coherently pumped degenerate three-level atom. We next determine, employing the master equation, equations of evolution for the expectation values of the atomic operators.

Applying the solutions of the pertinent equations of evolution, we calculate the local and global mean photon number, the local and global variance of the photon number, and the local and global quadrature squeezing of the two-mode cavity light. Moreover, using the input-output relation and by considering the cavity to be coupled to a two-mode vacuum reservoir with the input mode operator put in normal order, we obtain the local and global mean photon number, the local and global variance of the photon number, and the local and global quadrature squeezing of the two-mode output laser light beam.

We aslo wish to analyze the squeezing and statistical properties of a pair of superposed two-mode laser light beams. To this end, we first determine the Q function for a single two-mode laser light beam. We then obtain the density operator for the superposed two-mode laser light beams. Employing the resulting density operator, we calculate the local and global mean photon number, the local and global variance of the photon number, and the local and global quadrature squeezing. Finally, we determine the local and global mean photon number, the local and global variance of the photon number, and the local and global quadrature squeezing of the superposed two-mode output laser light beams.

2

Operator Dynamics

In this chapter we wish to obtain the equations of evolution of the atomic and cavity mode operators for a degenerate three-level laser with an open cavity. Here we consider the case in which N three-level atoms available in an open cavity are pumped by coherent light. We assume that the laser cavity is coupled to a two-mode vacuum reservoir via a single-port mirror as depicted in Fig. 2.1. To this end, we first seek to obtain the quantum Langevin equations for the cavity mode operators and the pertinent master equation for a coherently pumped degenerate three-level atom. In addition, employing the master equation, we determine equations of evolution for the expectation values of the atomic operators. Finally, we find the solutions of the equations of evolution for the expectation values of the atomic operators and the cavity mode operators, respectively.

2.1 The Master equation

A light mode confined in a cavity usually formed by two mirrors is called a cavity mode. A commonly used cavity has a single-port mirror. One side of such a cavity is a mirror through which light can enter or leave the cavity, with the other side being a mirror through which light may enter but can not leave the cavity [1]. We consider here the case in which N degenerate three-level atoms in cascade configuration are available in an open cavity. The top and bottom levels of the three-level atoms are coupled by coherent light. When a three-level atom decays from the top level to the bottom levels via the middle level, two photons of the same frequency are emitted.

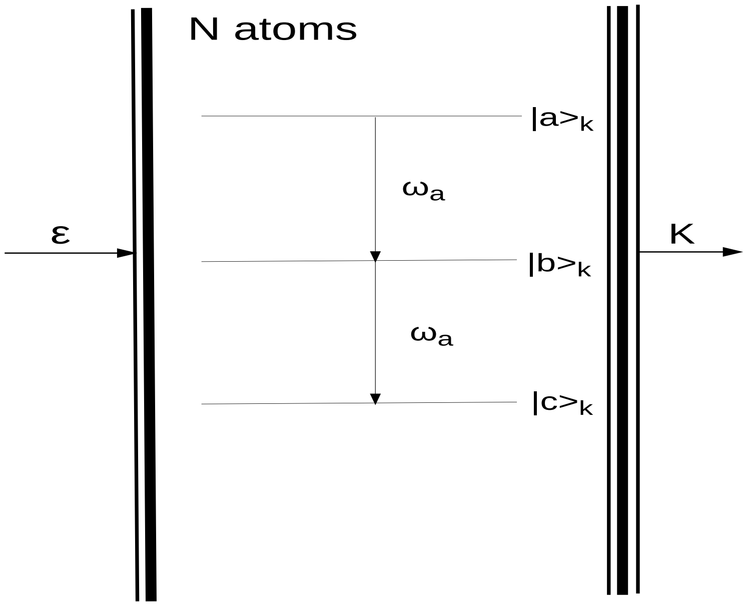

Figure 2.1: Schematic representation of coherently pumped three-level laser coupled to a vacuum reservoir

We denote the top, middle, and bottom levels of these atoms by $| \, a\rangle_k$, $| \, b\rangle_k$, and $| \, c\rangle_k$, respectively. We wish to represent the light emitted from the top level by $\hat{a}_1$ and the light emitted from the middle by $\hat{a}_2$. In addition, we assume that the two cavity modes a_1 and a_2 to be at resonance with the two transitions $| \, a\rangle_k \rightarrow | \, b\rangle_k$ and $| \, b\rangle_k \rightarrow | \, c\rangle_k$, with direct transitions between levels $| \, a\rangle_k$ and $| \, c\rangle_k$ to be dipole forbidden. Moreover, we assume that levels $| \, a\rangle_k$ and $| \, b\rangle_k$ have the same spontaneous emission decay constant γ.

Now we seek to obtain the equation of evolution of the reduced density operator for a degenerate three-level atom. The interaction of one of the three-level atoms with light modes a_1 and a_2 can be described at resonance by the Hamiltonian

$$\hat{H} = ig\left[\hat{\sigma}_1^{\dagger k}\hat{a}_1 - \hat{a}_1^{\dagger}\hat{\sigma}_1^k + \hat{\sigma}_2^{\dagger k}\hat{a}_2 - \hat{a}_2^{\dagger}\hat{\sigma}_2^k\right], \tag{2.1}$$

where

$$\hat{\sigma}_1^k = \mid b\rangle_{kk}\langle a \mid \tag{2.2}$$

and

$$\hat{\sigma}_2^k = \mid c\rangle_{kk}\langle b \mid \tag{2.3}$$

are lowering atomic operators, $\hat{a}_1$ and $\hat{a}_2$ are the annihilation operators for light modes a_1 and a_2, and g is the coupling constant between the atom and light mode a_1 or light mode a_2. And the interaction of the three-level atom with the driving coherent light can be described at resonance by the Hamiltonian

$$\hat{H}' = \frac{i\Omega}{2}\left[\hat{\sigma}_3^{\dagger k} - \hat{\sigma}_3^k\right], \tag{2.4}$$

in which

$$\hat{\sigma}_3^k = \mid c\rangle_{kk}\langle a \mid \tag{2.5}$$

and

$$\Omega = 2g'\varepsilon. \tag{2.6}$$

Here ε is the amplitude of the driving coherent light and g' is the coupling constant between the atom and coherent light. Thus upon combining Eqs. (2.1) and (2.4), the interaction of the three-level atom with the driving coherent light and cavity light modes a_1 and a_2 is described at resonance by the Hamiltonian

$$\hat{H}_{sys} = ig\left[\hat{\sigma}_1^{\dagger k}\hat{a}_1 - \hat{a}_1^\dagger\hat{\sigma}_1^k + \hat{\sigma}_2^{\dagger k}\hat{a}_2 - \hat{a}_2^\dagger\hat{\sigma}_2^k\right] + \frac{i\Omega}{2}\left[\hat{\sigma}_3^{\dagger k} - \hat{\sigma}_3^k\right]. \tag{2.7}$$

On the other hand, the degenerate three-level atoms available in an open cavity are coupled to the two-mode vacuum reservoir. The master equation for the three-level atom interacting with a two-mode vacuum reservoir has the form [1,4]

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) = -i\left[\hat{H}_{sys}, \hat{\rho}(t)\right] &+ \frac{\gamma}{2}\left[2\hat{\sigma}_1^k\hat{\rho}\hat{\sigma}_1^{\dagger k} - \hat{\sigma}_1^{\dagger k}\hat{\sigma}_1^k\hat{\rho} - \hat{\rho}\hat{\sigma}_1^{\dagger k}\hat{\sigma}_1^k\right] \\ &+ \frac{\gamma}{2}\left[2\hat{\sigma}_2^k\hat{\rho}\hat{\sigma}_2^{\dagger k} - \hat{\sigma}_2^{\dagger k}\hat{\sigma}_2^k\hat{\rho} - \hat{\rho}\hat{\sigma}_2^{\dagger k}\hat{\sigma}_2^k\right],\end{aligned} \tag{2.8}$$

where γ is the spontaneous emission decay constant. Now with the aid of Eq. (2.7), one can put Eq. (2.8) in the form

$$\frac{d}{dt}\hat{\rho}(t) = g\Big[\hat{\sigma}_1^{\dagger k}\hat{a}_1\hat{\rho} - \hat{a}_1^{\dagger}\hat{\sigma}_1^{k}\hat{\rho} - \hat{\rho}\hat{\sigma}_1^{\dagger k}\hat{a}_1 + \hat{\rho}\hat{a}_1^{\dagger}\hat{\sigma}_1^{k} + \hat{\sigma}_2^{\dagger k}\hat{a}_2\hat{\rho} - \hat{a}_2^{\dagger}\hat{\sigma}_2^{k}\hat{\rho} - \hat{\rho}\hat{\sigma}_2^{\dagger k}\hat{a}_2 + \hat{\rho}\hat{a}_2^{\dagger}\hat{\sigma}_2^{k}\Big]$$
$$+\frac{\gamma}{2}\left[2\hat{\sigma}_1^{k}\hat{\rho}\hat{\sigma}_1^{\dagger k} - \hat{\sigma}_1^{\dagger k}\hat{\sigma}_1^{k}\hat{\rho} - \hat{\rho}\hat{\sigma}_2^{\dagger k}\hat{\sigma}_1^{k}\right] + \frac{\gamma}{2}\left[2\hat{\sigma}_2^{k}\hat{\rho}\hat{\sigma}_2^{\dagger k} - \hat{\sigma}_2^{\dagger k}\hat{\sigma}_2^{k}\hat{\rho} - \hat{\rho}\hat{\sigma}_2^{\dagger k}\hat{\sigma}_2^{k}\right]$$
$$+\frac{\Omega}{2}\left[\hat{\sigma}_3^{\dagger k}\hat{\rho} - \hat{\sigma}_3^{k}\hat{\rho} + \hat{\rho}\hat{\sigma}_3^{k} - \hat{\rho}\hat{\sigma}_3^{\dagger k}\right] \tag{2.9}$$

This represents the master equation for a coherently pumped degenerate three-level atom.

2.2 The Quantum Langevin equations

We recall that the laser cavity is coupled to a two-mode vacuum reservoir via a single-port mirror. In addition, we carry out our analysis by putting the noise operators associated with the vacuum reservoir in normal order. Thus the noise operators will not have any effect on the dynamics of the cavity mode operators [1,32]. In view of this, we can drop the noise operators and write the quantum Langevin equation for the operators $\hat{a}_1$ and $\hat{a}_2$ as

$$\frac{d}{dt}\hat{a}_1(t) = -i\left[\hat{a}_1, \hat{H}_{sys}\right] - \frac{k}{2}\hat{a}_1, \tag{2.10}$$

$$\frac{d}{dt}\hat{a}_2(t) = -i\left[\hat{a}_2, \hat{H}_{sys}\right] - \frac{k}{2}\hat{a}_2, \tag{2.11}$$

where k is the cavity damping constant. Then with the aid of Eq. (2.7), we easily find

$$\frac{d}{dt}\hat{a}_1 = -\frac{k}{2}\hat{a}_1 - g\hat{\sigma}_1^{k} \tag{2.12}$$

and

$$\frac{d}{dt}\hat{a}_2 = -\frac{k}{2}\hat{a}_2 - g\hat{\sigma}_2^{k}. \tag{2.13}$$

The procedure of normal ordering the noise opertators renders the vacuum reservoir to be a noiseless physical entity [33].

2.3 Expectation values of atomic operators

We next wish to obtain, using the pertinent master equation, the equations of evolution of the expectation values of atomic operators. Applying the relation

$$\frac{d}{dt}\langle \hat{A}\rangle = Tr\left(\frac{d\hat{\rho}(t)}{dt}\hat{A}\right) \tag{2.14}$$

along with Eq. (2.9), one can readily establish that

$$\frac{d}{dt}\langle\hat{\sigma}_1^k\rangle = g\Big[\langle\hat{\eta}_b^k\hat{a}_1\rangle - \langle\hat{\eta}_a^k\hat{a}_1\rangle + \langle\hat{a}_2^\dagger\hat{\sigma}_3^k\rangle\Big] + \frac{\Omega}{2}\langle\hat{\sigma}_2^{\dagger k}\rangle - \gamma\langle\hat{\sigma}_1^k\rangle, \tag{2.15}$$

$$\frac{d}{dt}\langle\hat{\sigma}_2^k\rangle = g\Big[\langle\hat{\eta}_c^k\hat{a}_2\rangle - \langle\hat{\eta}_b^k\hat{a}_2\rangle - \langle\hat{a}_1^\dagger\hat{\sigma}_3^k\rangle\Big] - \frac{\Omega}{2}\langle\hat{\sigma}_1^{\dagger k}\rangle - \frac{\gamma}{2}\langle\hat{\sigma}_2^k\rangle, \tag{2.16}$$

$$\frac{d}{dt}\langle\hat{\sigma}_3^k\rangle = g\Big[\langle\hat{\sigma}_2^k\hat{a}_1\rangle - \langle\hat{\sigma}_1^k\hat{a}_2\rangle\Big] + \frac{\Omega}{2}\Big[\langle\hat{\eta}_c^k\rangle - \langle\hat{\eta}_a^k\rangle\Big] - \frac{\gamma}{2}\langle\hat{\sigma}_3^k\rangle, \tag{2.17}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = g\Big[\langle\hat{\sigma}_1^{\dagger k}\hat{a}_1\rangle + \langle\hat{a}_1^\dagger\hat{\sigma}_1^k\rangle\Big] + \frac{\Omega}{2}\Big[\langle\hat{\sigma}_3^{\dagger k}\rangle + \langle\hat{\sigma}_3^k\rangle\Big] - \gamma\langle\hat{\eta}_a^k\rangle, \tag{2.18}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = g\Big[\langle\hat{\sigma}_2^{\dagger k}\hat{a}_2\rangle + \langle\hat{a}_2^\dagger\hat{\sigma}_2^k\rangle - \langle\hat{\sigma}_1^{\dagger k}\hat{a}_1\rangle - \langle\hat{a}_1^\dagger\hat{\sigma}_1^k\rangle\Big] + \gamma\Big[\langle\hat{\eta}_a^k\rangle - \langle\hat{\eta}_b^k\rangle\Big], \tag{2.19}$$

$$\frac{d}{dt}\langle\hat{\eta}_c^k\rangle = -g\Big[\langle\hat{\sigma}_2^{\dagger k}\hat{a}_2\rangle + \langle\hat{a}_2^\dagger\hat{\sigma}_2^k\rangle\Big] - \frac{\Omega}{2}\Big[\langle\hat{\sigma}_3^{\dagger k}\rangle + \langle\hat{\sigma}_3^k\rangle\Big] + \gamma\langle\hat{\eta}_b^k\rangle, \tag{2.20}$$

where

$$\hat{\sigma}_3^k = \mid c\rangle_{kk}\langle a \mid, \tag{2.21}$$

$$\hat{\eta}_a^k = \mid a\rangle_{kk}\langle a \mid, \tag{2.22}$$

$$\hat{\eta}_b^k = \mid b\rangle_{kk}\langle b \mid, \tag{2.23}$$

$$\hat{\eta}_c^k = \mid c\rangle_{kk}\langle c \mid . \tag{2.24}$$

We see that Eqs. (2.15)-(2.20) are nonlinear and coupled differential equations. Therefore, it is not possible to obtain the exact time-dependent solutions. We intend to overcome this problem by applying the large-time approximation [1,32]. Then using this approximation scheme, we get from Eqs. (2.12) and (2.13) the approximately valid relations

$$\hat{a}_1 = -\frac{2g}{k}\hat{\sigma}_1^k \tag{2.25}$$

and

$$\hat{a}_2 = -\frac{2g}{k}\hat{\sigma}_2^k. \tag{2.26}$$

Evidently, these turn out to be exact relations at steady state. Now introducing Eqs. (2.25) and (2.26) into Eqs. (2.15)-(2.20), we get

$$\frac{d}{dt}\langle\hat{\sigma}_1^k\rangle = -\Big[\gamma_c + \gamma\Big]\langle\hat{\sigma}_1^k\rangle + \frac{\Omega}{2}\langle\hat{\sigma}_2^{\dagger k}\rangle, \tag{2.27}$$

$$\frac{d}{dt}\langle\hat{\sigma}_2^k\rangle = -\frac{1}{2}\Big[\gamma_c + \gamma\Big]\langle\hat{\sigma}_2^k\rangle - \frac{\Omega}{2}\langle\hat{\sigma}_1^{\dagger k}\rangle, \tag{2.28}$$

$$\frac{d}{dt}\langle\hat{\sigma}_3^k\rangle = -\frac{1}{2}\Big[\gamma_c + \gamma\Big]\langle\hat{\sigma}_3^k\rangle + \frac{\Omega}{2}\Big[\langle\hat{\eta}_c^k\rangle - \langle\hat{\eta}_a^k\rangle\Big], \tag{2.29}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -\Big[\gamma_c + \gamma\Big]\langle\hat{\eta}_a^k\rangle + \frac{\Omega}{2}\Big[\langle\hat{\sigma}_3^{\dagger k}\rangle + \langle\hat{\sigma}_3^k\rangle\Big], \tag{2.30}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = -\Big[\gamma_c + \gamma\Big]\langle\hat{\eta}_b^k\rangle + \Big[\gamma_c + \gamma\Big]\langle\hat{\eta}_a^k\rangle, \tag{2.31}$$

$$\frac{d}{dt}\langle\hat{\eta}_c^k\rangle = \Big[\gamma_c + \gamma\Big]\langle\hat{\eta}_b^k\rangle - \frac{\Omega}{2}\Big[\langle\hat{\sigma}_3^{\dagger k}\rangle + \langle\hat{\sigma}_3^k\rangle\Big], \tag{2.32}$$

where

$$\gamma_c = \frac{4g^2}{k} \tag{2.33}$$

is the stimulated emission decay constant. We next sum Eqs. (2.27)-(2.32) over the N three-level atoms. We then obtain

$$\frac{d}{dt}\langle\hat{m}_1\rangle = -\Big[\gamma_c + \gamma\Big]\langle\hat{m}_1\rangle + \frac{\Omega}{2}\langle\hat{m}_2^\dagger\rangle, \tag{2.34}$$

$$\frac{d}{dt}\langle\hat{m}_2\rangle = -\frac{1}{2}\Big[\gamma_c + \gamma\Big]\langle\hat{m}_2\rangle - \frac{\Omega}{2}\langle\hat{m}_1^\dagger\rangle, \tag{2.35}$$

$$\frac{d}{dt}\langle\hat{m}_3\rangle = -\frac{1}{2}\Big[\gamma_c + \gamma\Big]\langle\hat{m}_3\rangle + \frac{\Omega}{2}\Big[\langle\hat{N}_c\rangle - \langle\hat{N}_a\rangle\Big], \tag{2.36}$$

$$\frac{d}{dt}\langle\hat{N}_a\rangle = -\Big[\gamma_c + \gamma\Big]\langle\hat{N}_a\rangle + \frac{\Omega}{2}\Big[\langle\hat{m}_3^\dagger\rangle + \langle\hat{m}_3\rangle\Big], \tag{2.37}$$

$$\frac{d}{dt}\langle\hat{N}_b\rangle = -\Big[\gamma_c + \gamma\Big]\langle\hat{N}_b\rangle + \Big[\gamma_c + \gamma\Big]\langle\hat{N}_a\rangle, \tag{2.38}$$

$$\frac{d}{dt}\langle\hat{N}_c\rangle = \Big[\gamma_c + \gamma\Big]\langle\hat{N}_b\rangle - \frac{\Omega}{2}\Big[\langle\hat{m}_3^\dagger\rangle + \langle\hat{m}_3\rangle\Big], \tag{2.39}$$

in which

$$\hat{m}_1 = \sum_{k=1}^{N} \hat{\sigma}_1^k, \tag{2.40}$$

$$\hat{m}_2 = \sum_{k=1}^{N} \hat{\sigma}_2^k, \tag{2.41}$$

$$\hat{m}_3 = \sum_{k=1}^{N} \hat{\sigma}_3^k, \tag{2.42}$$

$$\hat{N}_a = \sum_{k=1}^{N} \hat{\eta}_a^k, \tag{2.43}$$

$$\hat{N}_b = \sum_{k=1}^{N} \hat{\eta}_b^k, \tag{2.44}$$

$$\hat{N}_c = \sum_{k=1}^{N} \hat{\eta}_c^k, \tag{2.45}$$

with the operators $\hat{N}_a$, $\hat{N}_b$, and $\hat{N}_c$ representing the number of atoms in the top, middle, and bottom levels. In addition, employing the completeness relation

$$\hat{\eta}_a^k + \hat{\eta}_b^k + \hat{\eta}_c^k = \hat{I}, \tag{2.46}$$

we easily arrive at

$$\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle = N. \tag{2.47}$$

Furthermore, applying the definition given by Eq. (2.2) and setting for any k

$$\hat{\sigma}_1^k = \mid b\rangle\langle a \mid, \tag{2.48}$$

we have

$$\hat{m}_1 = N \mid b\rangle\langle a \mid . \tag{2.49}$$

Following the same procedure, one can easily find

$$\hat{m}_2 = N \mid c\rangle\langle b \mid, \tag{2.50}$$

$$\hat{m}_3 = N \mid c\rangle\langle a \mid, \tag{2.51}$$

$$\hat{N}_a = N \mid a\rangle\langle a \mid, \tag{2.52}$$

$$\hat{N}_b = N \mid b\rangle\langle b \mid, \tag{2.53}$$

$$\hat{N}_c = N \mid c\rangle\langle c \mid . \tag{2.54}$$

Moreover, using the definition [1]

$$\hat{m}_a = \hat{m}_1 + \hat{m}_2 \tag{2.55}$$

and taking into account Eqs. (2.49)-(2.54), it can be readily established that

$$\hat{m}_a^\dagger \hat{m}_a = N(\hat{N}_a + \hat{N}_b), \tag{2.56}$$

$$\hat{m}_a \hat{m}_a^\dagger = N(\hat{N}_b + \hat{N}_c), \tag{2.57}$$

$$\hat{m}_a^2 = N\hat{m}_3. \tag{2.58}$$

Upon Adding Eqs. (2.12) and (2.13), we have

$$\frac{d}{dt}\hat{a} = -\frac{k}{2}\hat{a} - g\left(\hat{\sigma}_1^k + \hat{\sigma}_2^k\right), \tag{2.59}$$

where

$$\hat{a} = \hat{a}_1 + \hat{a}_2. \tag{2.60}$$

In the presence of N three-level atoms, we rewrite Eq. (2.59) as [1]

$$\frac{d}{dt}\hat{a} = -\frac{k}{2}\hat{a} + \lambda\hat{m}_a, \tag{2.61}$$

in which λ is a constant whose value remains to be fixed. The steady-state solution of Eq. (2.59) is

$$\hat{a} = -\frac{2g}{k}\left(\hat{\sigma}_a^k + \hat{\sigma}_b^k\right). \tag{2.62}$$

Taking into account of (2.62) and its complex conjugate, the commutation relation for the cavity mode operator is found to be

$$\left[\hat{a}, \hat{a}^\dagger\right]_k = \frac{\gamma_c}{k}\left(\hat{\eta}_c^k - \hat{\eta}_a^k\right), \tag{2.63}$$

and on summing over all atoms, we have

$$\left[\hat{a}, \hat{a}^{\dagger}\right] = \frac{\gamma_c}{k}\left(\hat{N}_c - \hat{N}_a\right), \tag{2.64}$$

where

$$\left[\hat{a}, \hat{a}^{\dagger}\right] = \sum_{k=1}^{N}\left[\hat{a}, \hat{a}^{\dagger}\right]_k, \tag{2.65}$$

stands for the commutator of $(\hat{a}, \hat{a}^{\dagger})$ when the cavity light mode a is interacting with all the N three-level atoms. On the other hand, using the steady-state solution of Eq. (2.61), one can verify

$$\left[\hat{a}, \hat{a}^{\dagger}\right] = N\left[\frac{2\lambda}{k}\right]^2\left(\hat{N}_c - \hat{N}_a\right). \tag{2.66}$$

Comparison of Eqs. (2.64) and (2.66) shows that

$$\lambda = \pm\frac{g}{\sqrt{N}}. \tag{2.67}$$

With the help of this result, Eq. (2.61) takes the form [1]

$$\frac{d}{dt}\hat{a} = -\frac{k}{2}\hat{a} + \frac{g}{\sqrt{N}}\hat{m}_a. \tag{2.68}$$

We note that the expectation value of the solution of Eq. (2.68) can be expressed as

$$\langle\hat{a}(t)\rangle = \langle\hat{a}(0)\rangle e^{-kt/2} + \frac{g}{\sqrt{N}}e^{-kt/2}\int_0^t e^{kt'/2}\langle\hat{m}_a(t')\rangle dt'. \tag{2.69}$$

We next obtain the expectation value of the solution of $\hat{m}_a(t)$ that appear in Eq. (2.69). Thus applying the large-time approximation scheme to Eq. (2.35), we get

$$\langle\hat{m}_2(t)\rangle = -\left\{\frac{\Omega}{\gamma_c + \gamma}\right\}\langle\hat{m}_1^{\dagger}(t)\rangle, \tag{2.70}$$

and the adjoint of this expression is

$$\langle\hat{m}_2^{\dagger}(t)\rangle = -\left\{\frac{\Omega}{\gamma_c + \gamma}\right\}\langle\hat{m}_1(t)\rangle. \tag{2.71}$$

Hence in view of this result, Eq. (2.34) takes the form

$$\frac{d}{dt}\langle\hat{m}_1(t)\rangle = -\nu\langle\hat{m}_1(t)\rangle, \tag{2.72}$$

where

$$\nu = \Big(\gamma_c + \gamma\Big) + \frac{1}{2}\left\{\frac{\Omega^2}{(\gamma_c + \gamma)}\right\}. \tag{2.73}$$

The solution of Eq. (2.72) is given by

$$\langle\hat{m}_1(t)\rangle = \langle\hat{m}_1(0)\rangle e^{-\nu t}. \tag{2.74}$$

Similarly, applying the large-time approximation scheme to Eq. (2.34), we find

$$\langle\hat{m}_1(t)\rangle = \frac{1}{2}\left\{\frac{\Omega}{\gamma_c + \gamma}\right\}\langle\hat{m}_2^\dagger(t)\rangle, \tag{2.75}$$

and the complex conjugate of this result yields

$$\langle\hat{m}_1^\dagger(t)\rangle = \frac{1}{2}\left\{\frac{\Omega}{\gamma_c + \gamma}\right\}\langle\hat{m}_2(t)\rangle. \tag{2.76}$$

Now on substituting Eq. (2.76) into Eq. (2.35), we readily get

$$\frac{d}{dt}\langle\hat{m}_2(t)\rangle = -\frac{1}{2}\nu\langle\hat{m}_2(t)\rangle, \tag{2.77}$$

in which ν is given by Eq. (2.73). Upon adding Eqs. (2.72) and (2.77), we arrive at

$$\frac{d}{dt}\langle\hat{m}_a(t)\rangle = -\frac{1}{2}\nu\langle\hat{m}_a(t)\rangle - \frac{1}{2}\nu\langle\hat{m}_1(t)\rangle. \tag{2.78}$$

We note that the solution of Eq. (2.78) is expressible as

$$\langle\hat{m}_a(t)\rangle = \langle\hat{m}_a(0)\rangle e^{-\nu t/2} + e^{-\nu t/2}\int_0^t dt' e^{\nu t'/2}\left\{-\frac{1}{2}\nu\langle\hat{m}_1(t')\rangle\right\}. \tag{2.79}$$

Thus upon substituting Eq. (2.74) into Eq. (2.79) and carrying out the integration, we arrive at

$$\langle\hat{m}_a(t)\rangle = \langle\hat{m}_a(0)\rangle e^{-\nu t/2} + \langle\hat{m}_1(0)\rangle\Big\{e^{-\nu t} - e^{-\nu t/2}\Big\}. \tag{2.80}$$

On account of Eq. (2.55), Eq. (2.80) takes the form

$$\langle \hat{m}_a(t)\rangle = \langle \hat{m}_1(0)\rangle e^{-\nu t} + \langle \hat{m}_2(0)\rangle\, e^{-\nu t/2}, \tag{2.81}$$

so that on introducing this result into Eq. (2.69), there emerges

$$\begin{aligned}\langle \hat{a}(t)\rangle = \langle \hat{a}(0)\rangle e^{-kt/2} + \frac{g}{\sqrt{N}} e^{-kt/2}\bigg\{\langle \hat{m}_1(0)\rangle \int_0^t e^{(k-2\nu)t'/2} dt' \\ + \langle \hat{m}_2(0)\rangle \int_0^t e^{(k-\nu)t'/2} dt'\bigg\},\end{aligned} \tag{2.82}$$

and on carrying out the integration over t', we readily find

$$\begin{aligned}\langle \hat{a}(t)\rangle = \langle \hat{a}(0)\rangle e^{-kt/2} + \frac{2g}{\sqrt{N}(k-2\nu)}\langle \hat{m}_1(0)\rangle \Big\{e^{-\nu t} - e^{-kt/2}\Big\} \\ + \frac{2g}{\sqrt{N}(k-\nu)}\langle \hat{m}_2(0)\rangle \Big\{e^{-\nu t/2} - e^{-kt/2}\Big\}.\end{aligned} \tag{2.83}$$

Assuming that cavity mode a is initially in a vacuum state and with the three-level atoms considered to be initially at the bottom level, we have

$$\langle \hat{a}(t)\rangle = 0. \tag{2.84}$$

We observe from Eqs. (2.68) and (2.84) that the operator $\hat{a}(t)$ is a Gaussian variable with zero mean.

We finally wish to determine the expectation values of the atomic operators at steady state. We note that the steady-state solution of Eqs. (2.34) and (2.35) are given by

$$\langle \hat{m}_1\rangle = \frac{1}{2}\left\{\frac{\Omega}{\gamma_c + \gamma}\right\}\langle \hat{m}_2^\dagger\rangle \tag{2.85}$$

and

$$\langle \hat{m}_2\rangle = -\left\{\frac{\Omega}{\gamma_c + \gamma}\right\}\langle \hat{m}_1^\dagger\rangle. \tag{2.86}$$

The adjoint of these equations are

$$\langle \hat{m}_1^\dagger\rangle = \frac{1}{2}\left\{\frac{\Omega}{\gamma_c + \gamma}\right\}\langle \hat{m}_2\rangle \tag{2.87}$$

and

$$\langle \hat{m}_2^\dagger \rangle = -\left\{ \frac{\Omega}{\gamma_c + \gamma} \right\} \langle \hat{m}_1 \rangle. \tag{2.88}$$

Upon substituting Eq. (2.88) into (2.85), we get

$$\langle \hat{m}_1 \rangle = 0. \tag{2.89}$$

Similarly, on account of Eq. (2.87) along with (2.86), one easily finds

$$\langle \hat{m}_2 \rangle = 0. \tag{2.90}$$

Moreover, the steady-state solution of Eqs. (2.36), (2.37), and (2.38) are found to be

$$\langle \hat{m}_3 \rangle = \left\{ \frac{\Omega}{\gamma_c + \gamma} \right\} \left\{ \langle \hat{N}_c \rangle - \langle \hat{N}_a \rangle \right\}, \tag{2.91}$$

$$\langle \hat{N}_a \rangle = \frac{1}{2} \left\{ \frac{\Omega}{\gamma_c + \gamma} \right\} \left\{ \langle \hat{m}_3^\dagger \rangle + \langle \hat{m}_3 \rangle \right\}, \tag{2.92}$$

$$\langle \hat{N}_b \rangle = \langle \hat{N}_a \rangle. \tag{2.93}$$

Furthermore, with the aid of Eq. (2.47), we obtain

$$\langle \hat{N}_c \rangle = N - \langle \hat{N}_a \rangle - \langle \hat{N}_b \rangle. \tag{2.94}$$

Now employing the relation described by Eq. (2.93), one can put (2.94) in the form

$$\langle \hat{N}_c \rangle = N - 2\langle \hat{N}_a \rangle. \tag{2.95}$$

On account of Eq. (2.95) , Eq. (2.91) can be expressed as

$$\langle \hat{m}_3 \rangle = \left\{ \frac{\Omega}{\gamma_c + \gamma} \right\} \left\{ N - 3\langle \hat{N}_a \rangle \right\}, \tag{2.96}$$

Then in view of Eq. (2.96), we see that

$$\langle \hat{m}_3^\dagger \rangle = \langle \hat{m}_3 \rangle. \tag{2.97}$$

On account of this result, one can put Eq. (2.92) in the form

$$\langle \hat{N}_a \rangle = \left\{ \frac{\Omega}{\gamma_c + \gamma} \right\} \langle \hat{m}_3 \rangle. \tag{2.98}$$

Hence, on substituting Eq. (2.96) into (2.98), we get

$$\langle \hat{N}_a \rangle = \left\{ \frac{\Omega^2}{(\gamma_c + \gamma)^2 + 3\Omega^2} \right\} N. \tag{2.99}$$

In view of this result, one can write (2.93) in the form

$$\langle \hat{N}_b \rangle = \left\{ \frac{\Omega^2}{(\gamma_c + \gamma)^2 + 3\Omega^2} \right\} N. \tag{2.100}$$

Upon substituting (2.99) into Eq. (2.95), we get

$$\langle \hat{N}_c \rangle = \left\{ \frac{(\gamma_c + \gamma)^2 + \Omega^2}{(\gamma_c + \gamma)^2 + 3\Omega^2} \right\} N. \tag{2.101}$$

Finally, on account of (2.99), Eq. (2.96) takes the form

$$\langle \hat{m}_3 \rangle = \left\{ \frac{\Omega(\gamma_c + \gamma)}{(\gamma_c + \gamma)^2 + 3\Omega^2} \right\} N. \tag{2.102}$$

Upon setting $\gamma = 0$ (i.e., in the absence of spontaneous emission) into Eqs. (2.99)-(2.102), we have

$$\langle \hat{N}_a \rangle = \left\{ \frac{\Omega^2}{\gamma_c^2 + 3\Omega^2} \right\} N, \tag{2.103}$$

$$\langle \hat{N}_b \rangle = \left\{ \frac{\Omega^2}{\gamma_c^2 + 3\Omega^2} \right\} N, \tag{2.104}$$

$$\langle \hat{N}_c \rangle = \left\{ \frac{\gamma_c^2 + \Omega^2}{\gamma_c^2 + 3\Omega^2} \right\} N, \tag{2.105}$$

$$\langle \hat{m}_3 \rangle = \left\{ \frac{\Omega\gamma_c}{\gamma_c^2 + 3\Omega^2} \right\} N. \tag{2.106}$$

These results are exactly the same as the one which has been obtained in [1]. Moreover, for $\Omega = 0$, we obtain

$$\langle \hat{N}_a \rangle = 0, \tag{2.107}$$

$$\langle \hat{N}_b \rangle = 0, \tag{2.108}$$

$$\langle \hat{N}_c \rangle = N, \tag{2.109}$$

$$\langle \hat{m}_3 \rangle = 0. \tag{2.110}$$

On the basis of this, all atoms are found initially in the bottom level. Furthermore, for $\gamma_c = \gamma = 0$, we can put Eqs. (2.99)-(2.102) in the form

$$\langle \hat{N}_a \rangle = \frac{1}{3} N, \tag{2.111}$$

$$\langle \hat{N}_a \rangle = \frac{1}{3} N, \tag{2.112}$$

$$\langle \hat{N}_a \rangle = \frac{1}{3} N \tag{2.113}$$

and

$$\langle \hat{m}_3 \rangle = 0. \tag{2.114}$$

In line of this, we observe that equal number of atoms are available in the top, middle, and bottom levels, respectively.

3

Two-Mode Laser Light Beam

In this chapter we seek to study the statistical properties of the light produced by a coherently pumped degenerate three-level laser. Employing the solutions of the quantum Langevin equations for the cavity modes and the pertinent equations of evolution, we calculate the global mean photon number and the global variance of the photon number for the two-mode cavity (output) light. We also determine the local mean photon number and the local variance of the photon number for this light mode.

3.1 Photon statistics

In this section we seek to obtain the statistical properties of the two-mode cavity (output) light produced by the degenerate three-level laser pumped by coherent light and coupled to a two-mode vacuum reservoir.

3.1.1 The global mean photon number

Here we wish to calculate the mean photon number of the two-mode cavity light in the entire frequency interval. The mean photon number of the two-mode cavity light is defined by

$$\bar{n} = \langle \hat{a}^{\dagger}\hat{a} \rangle. \tag{3.1}$$

We note that the steady-state solution of Eq. (2.68) is

$$\hat{a} = \frac{2g}{k\sqrt{N}}\hat{m}_a. \tag{3.2}$$

Then with the help of this result and Eq. (2.56), we get

$$\bar{n} = \frac{\gamma_c}{k}\left\{\langle\hat{N}_a\rangle + \langle\hat{N}_b\rangle\right\}. \tag{3.3}$$

We see that the mean photon number of the two-mode cavity light is the sum of the mean photon numbers of the separate single-mode cavity light beams. With the aid of (2.93), one can put Eq. (3.3) in the form

$$\bar{n} = 2\frac{\gamma_c}{k}\langle\hat{N}_a\rangle. \tag{3.4}$$

Thus on account of (2.99), the mean photon number of the two-mode cavity light turns out to be

$$\bar{n} = \frac{\gamma_c}{k}N\left\{\frac{2\Omega^2}{(\gamma_c+\gamma)^2+3\Omega^2}\right\}. \tag{3.5}$$

Finally, we note that the global mean photon number of the two-mode cavity light beam takes for $\Omega \gg \gamma_c$ and $\Omega \gg \gamma$ the form

$$\bar{n} = \frac{2}{3}\frac{\gamma_c}{k}N. \tag{3.6}$$

We immediately observe from this result that the global mean photon number of the two-mode cavity light beam is independent of the spontaneous and stimulated emissions.

In addition, for $\gamma = 0$, Eq. (3.5) reduces to

$$\bar{n} = \frac{\gamma_c}{k}N\left\{\frac{2\Omega^2}{\gamma_c^2+3\Omega^2}\right\}. \tag{3.7}$$

We see from this result that the global mean photon number of the two-mode cavity light beam is just twice of the mean photon number of one of the single-mode cavity light beams. We observe from the plots in Fig. 3.1 that the presence of spontaneous emission leads to a decrease in the global mean photon number of the two-mode cavity light beam. We also note that the global mean photon number increases with

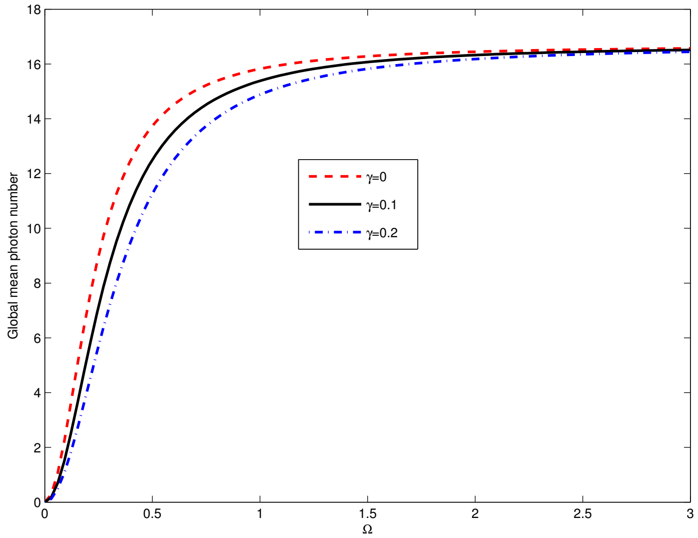

Figure 3.1: Plots of the global mean photon number [Eq. (3.5)] versus Ω at steady state for $\gamma_c = 0.4$, $k = 0.8$, $N = 50$, and different values of γ.

increasing Ω.

On the other hand, the laser cavity is coupled to a two-mode vacuum reservoir and we carry out our analysis by putting the input mode operators in normal order. Therefore, one can write

$$\hat{a}_{out}(t) = \sqrt{k}\hat{a}(t). \tag{3.8}$$

Then substituting Eq. (3.8) into (3.1), the mean photon number for the two-mode output light is expressible as

$$\langle \hat{a}^{\dagger}_{out}\hat{a}_{out}\rangle = k\langle \hat{a}^{\dagger}\hat{a}\rangle, \tag{3.9}$$

which indicates that the global mean photon number of the two-mode output light is just k times that of the two-mode cavity light.

3.1.2 Local mean photon number

We seek to obtain the mean photon number in a given frequency interval, employing the power spectrum for the two-mode cavity light. The power spectrum of a two-mode cavity light with central common frequency ω_0 is defined as

$$P(\omega) = \frac{1}{\pi} Re \int_0^\infty d\tau e^{i(\omega-\omega_0)\tau} \langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss}. \tag{3.10}$$

We now proceed to determine the two-time correlation function that appear in Eq. (3.10) for the two-mode cavity light. To this end, we realize that the solution of Eq. (2.68) can be written as

$$\hat{a}(t+\tau) = \hat{a}(t)e^{-k\tau/2} + \frac{g}{\sqrt{N}} e^{-k\tau/2} \int_0^\tau e^{k\tau'/2} \hat{m}_a(t+\tau')d\tau'. \tag{3.11}$$

On the other hand, one can put Eq. (2.78) in the form

$$\frac{d}{dt}\hat{m}_a(t) = -\frac{1}{2}\nu\hat{m}_a(t) - \frac{1}{2}\nu\hat{m}_1(t) + \hat{F}_a(t), \tag{3.12}$$

in which $\hat{F}_a(t)$ is a noise operator with vanishing mean. The solution of Eq. (3.12) can be put in the form

$$\hat{m}_a(t+\tau) = \hat{m}_a(t)e^{-\nu\tau/2} + e^{-\nu\tau/2} \int_0^\tau e^{\nu\tau'/2} \left\{ -\frac{1}{2}\nu\hat{m}_1(t+\tau') + \hat{F}_a(t+\tau') \right\} d\tau'. \tag{3.13}$$

In addition, one can write Eq. (2.72) in the form

$$\frac{d}{dt}\hat{m}_1(t) = -\nu\hat{m}_1(t) + \hat{F}_1(t), \tag{3.14}$$

in which $\hat{F}_1(t)$ is a noise operator with zero mean. Now applying the large-time approximation scheme to Eq. (3.14), we get

$$\hat{m}_1(t+\tau) = \frac{1}{\nu}\hat{F}_1(t+\tau), \tag{3.15}$$

so that on introducing this result into Eq. (3.13), there follows

$$\hat{m}_a(t+\tau) = \hat{m}_a(t)e^{-\nu\tau/2} + e^{-\nu\tau/2} \int_0^\tau e^{\nu\tau'/2} \left\{ -\frac{1}{2}\hat{F}_1(t+\tau') + \hat{F}_a(t+\tau') \right\} d\tau'. \tag{3.16}$$

Now combination of Eqs. (3.16) and (3.11) yields

$$\hat{a}(t+\tau) = \hat{a}(t)e^{-k\tau/2} + \frac{g}{\sqrt{N}} e^{-k\tau/2} \Bigg\{ \hat{m}_a(t) \int_0^\tau d\tau' e^{(k-\nu)\tau'/2}$$

$$+\int_0^{\tau} d\tau' e^{(k-\nu)\tau'/2}\int_0^{\tau'} d\tau'' e^{\nu\tau''/2}\Big[-\frac{1}{2}\hat{F}_1(t+\tau'')+\hat{F}_a(t+\tau'')\Big]\Big\}. \tag{3.17}$$

On multiplying both sides on the left by $\hat{a}^\dagger(t)$ and taking the expectation value of the resulting equation, we get

$$\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle e^{-k\tau/2} + \frac{g}{\sqrt{N}}e^{-k\tau/2}\Big\{\langle\hat{a}^\dagger(t)\hat{m}_a(t)\rangle\int_0^{\tau} d\tau' e^{(k-\nu)\tau'/2}$$
$$+\int_0^{\tau} d\tau' e^{(k-\nu)\tau'/2}\int_0^{\tau'} d\tau'' e^{\nu\tau''/2}\Big[-\frac{1}{2}\langle\hat{a}^\dagger(t)\hat{F}_1(t+\tau'')\rangle$$
$$+\langle\hat{a}^\dagger(t)\hat{F}_a(t+\tau'')\rangle\Big]\Big\}. \tag{3.18}$$

Moreover, applying the large-time approximation scheme to Eq. (2.68), we find

$$\hat{m}_a(t) = \frac{k\sqrt{N}}{2g}\hat{a}(t). \tag{3.19}$$

With this substituted into Eq. (3.18), there emerges

$$\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle e^{-k\tau/2} + \frac{g}{\sqrt{N}}e^{-k\tau/2}\Big\{\frac{k\sqrt{N}}{2g}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle\int_0^{\tau} d\tau' e^{(k-\nu)\tau'/2}$$
$$+\int_0^{\tau} d\tau' e^{(k-\nu)\tau'/2}\int_0^{\tau'} d\tau'' e^{\nu\tau''/2}\Big[-\frac{1}{2}\langle\hat{a}^\dagger(t)\hat{F}_1(t+\tau'')\rangle$$
$$+\langle\hat{a}^\dagger(t)\hat{F}_a(t+\tau'')\rangle\Big]\Big\}. \tag{3.20}$$

Since the cavity mode operator and the noise operator of the atomic are not correlated, we see that

$$\langle\hat{a}^\dagger(t)\hat{F}_1(t+\tau'')\rangle = \langle\hat{a}^\dagger(t)\rangle\langle\hat{F}_1(t+\tau'')\rangle = 0, \tag{3.21}$$

$$\langle\hat{a}^\dagger(t)\hat{F}_a(t+\tau'')\rangle = \langle\hat{a}^\dagger(t)\rangle\langle\hat{F}_a(t+\tau'')\rangle = 0, \tag{3.22}$$

we have

$$\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle e^{-k\tau/2} + \frac{k}{2}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle e^{-k\tau/2}\int_0^{\tau} d\tau' e^{(k-\nu)\tau'/2}. \tag{3.23}$$

On carrying out the integration over τ', we readily get

$$\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle\left\{\frac{k}{k-\nu}e^{-\nu\tau/2} - \frac{\nu}{k-\nu}e^{-k\tau/2}\right\}. \tag{3.24}$$

Finally, on substituting (3.24) into Eq. (3.10), we have

$$P(\omega) = \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle\frac{1}{\pi}Re\bigg\{\left[\frac{k}{k-\nu}\right]\int_0^\infty d\tau e^{-\left(\nu/2-i[\omega-\omega_0]\right)\tau}$$
$$-\left[\frac{\nu}{k-\nu}\right]\int_0^\infty d\tau e^{-\left(k/2-i[\omega-\omega_0]\right)\tau}\bigg\} \tag{3.25}$$

and then carrying out the integration over τ, the power spectrum of the two-mode cavity light is found to be

$$P(\omega) = \bar{n}\bigg[\left\{\frac{k}{k-\nu}\right\}\left\{\frac{\nu/2\pi}{(\omega-\omega_0)^2+(\nu/2)^2}\right\} - \left\{\frac{\nu}{k-\nu}\right\}$$
$$\times\left\{\frac{k/2\pi}{(\omega-\omega_0)^2+(k/2)^2}\right\}\bigg]. \tag{3.26}$$

We next wish to calculate the mean photon number in a given frequency interval. Upon integrating both sides of (3.26) over ω, we readily get

$$\int_{-\infty}^{+\infty} P(\omega)d\omega = \bar{n}, \tag{3.27}$$

in which $\bar{n}$ is the steady-state mean photon number. On the basis of Eq. (3.27), we observe that $P(\omega)d\omega$ represents the steady-state mean photon number in the interval between ω and $\omega + d\omega$ [1]. We thus realize that the mean photon number in the interval between $\omega' = -\lambda$ and $\omega' = +\lambda$ is expressible as [1]

$$\bar{n}_{\pm\lambda} = \int_{-\lambda}^{+\lambda} P(\omega')d\omega', \tag{3.28}$$

in which $\omega' = \omega - \omega_0$. Thus on introducing (3.26) into Eq. (3.28), we find

$$\bar{n}_{\pm\lambda} = \left[\frac{k\bar{n}}{k-\nu}\right]\int_{-\lambda}^{+\lambda}\left[\frac{\nu/2\pi}{\omega'^2+(\nu/2)^2}\right]d\omega'$$
$$-\left[\frac{\nu\bar{n}}{k-\nu}\right]\int_{-\lambda}^{+\lambda}\left[\frac{k/2\pi}{\omega'^2+(k/2)^2}\right]d\omega', \tag{3.29}$$

and on carrying out the integration over ω', applying the relation

$$\int_{-\lambda}^{+\lambda}\frac{dx}{x^2+a^2} = \frac{2}{a}\tan^{-1}\left(\frac{\lambda}{a}\right), \tag{3.30}$$

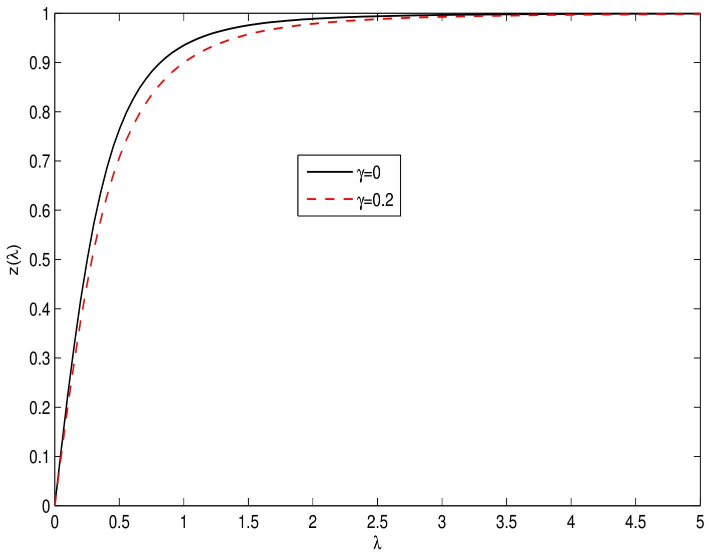

Figure 3.2: Plots of Eq. (3.32) versus λ for $\gamma_c = 0.4$, $k = 0.8$, $\Omega = 3$, and different values of γ.

we arrive at

$$\bar{n}_{\pm\lambda} = \bar{n}z(\lambda), \tag{3.31}$$

where

$$z(\lambda) = \left[\frac{2k/\pi}{k-\nu}\right]\tan^{-1}\left(\frac{2\lambda}{\nu}\right) - \left[\frac{2\nu/\pi}{k-\nu}\right]\tan^{-1}\left(\frac{2\lambda}{k}\right). \tag{3.32}$$

One can readily get from Fig. 3.2 that $z(0.5) = 0.7671$, $z(1) = 0.9362$, $z(1.5) = 0.9761$, and $z(2) = 0.9889$ for $\gamma = 0$. And for $\gamma = 0.2$, we find $z(0.5) = 0.7102$, $z(1) = 0.9015$, $z(1.5) = 0.958$, and $z(2) = 0.979$. Then combination of these results with Eq. (3.31) yields $\bar{n}_{\pm 0.5} = 0.7671\bar{n}$, $\bar{n}_{\pm 1} = 0.9362\bar{n}$, $\bar{n}_{\pm 1.5} = 0.9761\bar{n}$, and $\bar{n}_{\pm 2} = 0.9889\bar{n}$ for $\gamma = 0$. And for $\gamma = 0.2$, we have $\bar{n}_{\pm 0.5} = 0.7102\bar{n}$, $\bar{n}_{\pm 1} = 0.9015\bar{n}$, $\bar{n}_{\pm 1.5} = 0.958\bar{n}$, and $\bar{n}_{\pm 2} = 0.979\bar{n}$. We therefore observe that a large part of the total mean photon number is confined in a relatively small frequency interval. Moreover, we have

$$\bar{n}^{out}_{\pm\lambda} = k\bar{n}_{\pm\lambda}. \tag{3.33}$$

Now on account of Eqs. (3.9) and (3.31), we easily see that

$$k\bar{n}_{\pm\lambda} = \bar{n}_{out}z(\lambda). \tag{3.34}$$

Finally, in view of (3.33) and (3.34), the local mean photon number of the two-mode output laser light turns out to be

$$\bar{n}^{out}_{\pm\lambda} = \bar{n}_{out} z(\lambda). \tag{3.35}$$

We see that the local mean photon number of the two-mode output laser light is $z(\lambda)$ times the global mean photon number of the two-mode output light.

3.1.3 The global variance of the photon number

We now seek to calculate the variance of the photon number for the two-mode cavity light in the entire frequency interval produced by the system under consideration. The variance of the photon number for the two-mode cavity light is defined as

$$(\Delta n)^2 = \langle \left(\hat{a}^\dagger \hat{a}\right)^2 \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2 \tag{3.36}$$

and using the fact that $\hat{a}$ is a Gaussian variable with zero mean, we arrive at

$$(\Delta n)^2 = \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a} \hat{a}^\dagger \rangle + \langle \hat{a}^{\dagger 2} \rangle \langle \hat{a}^2 \rangle. \tag{3.37}$$

Furthermore, with the aid of (3.2) along with (2.57), we obtain

$$\langle \hat{a} \hat{a}^\dagger \rangle = \frac{\gamma_c}{k} \left\{ \langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle \right\}. \tag{3.38}$$

On account of the identity given by 2.47, Eq. (3.38) can be put in the form

$$\langle \hat{a} \hat{a}^\dagger \rangle = \frac{\gamma_c}{k} \left\{ N - \langle \hat{N}_a \rangle \right\}. \tag{3.39}$$

With the aid of (3.4), one can rewrite Eq. (3.39) as

$$\langle \hat{a} \hat{a}^\dagger \rangle = \frac{\gamma_c}{k} N - \frac{1}{2} \bar{n}. \tag{3.40}$$

On the other hand, using (3.2) along with (2.58), we easily get

$$\langle \hat{a}^2 \rangle = \frac{\gamma_c}{k} \langle \hat{m}_3 \rangle, \tag{3.41}$$

so that in view of (2.102) and (3.5), there follows

$$\langle \hat{a}^2 \rangle = \frac{1}{2}\left\{\frac{(\gamma_c + \gamma)}{\Omega}\right\}\bar{n}. \tag{3.42}$$

Now on account of (3.1), (3.40), and (3.42), we readily find Eq. (3.37) to be

$$(\Delta n)^2 = \bar{n}\left\{\frac{\gamma_c}{k}N - \frac{1}{2}\bar{n}\right\} + \frac{1}{4}\left\{\frac{(\gamma_c + \gamma)^2}{\Omega^2}\right\}\bar{n}^2. \tag{3.43}$$

This can be rewritten as

$$(\Delta n)^2 = \left\{\frac{\gamma_c}{k}N - \frac{1}{2}\left[\frac{2\Omega^2 - (\gamma_c + \gamma)^2}{2\Omega^2}\right]\bar{n}\right\}\bar{n}. \tag{3.44}$$

On account of (3.5), we arrive at

$$(\Delta n)^2 = \left(\frac{\gamma_c}{k}N\right)^2\left\{\frac{3\Omega^2(\gamma_c + \gamma)^2}{[(\gamma_c + \gamma)^2 + 3\Omega^2]^2} + \frac{4\Omega^4}{[(\gamma_c + \gamma)^2 + 3\Omega^2]^2}\right\}. \tag{3.45}$$

Finally, we note that the variance of the photon number takes for $\Omega \gg \gamma_c$ and $\Omega \gg \gamma$ the form

$$\begin{aligned}(\Delta n)^2 &= \left(\frac{2}{3}\frac{\gamma_c}{k}N\right)^2 \\ &= \bar{n}^2.\end{aligned} \tag{3.46}$$

This represents the normally-ordered variance of the photon number for a chaotic light. Moreover, we consider the case in which the spontaneous emission is absent (i.e., $\gamma = 0$). We then see that Eq. (3.45) reduces to

$$(\Delta n)^2 = \left(\frac{\gamma_c}{k}N\left[\frac{2\Omega^2}{\gamma_c^2 + 3\Omega^2}\right]\right)^2 + \left(\frac{\gamma_c}{k}N\right)^2\left[\frac{3\Omega^2\gamma_c^2}{(\gamma_c^2 + 3\Omega^2)^2}\right]. \tag{3.47}$$

We clearly observe from the plots in Fig. 3.3 that the presence of spontaneous emission leads to a decrease in the global variance of the photon number for the two-mode cavity light. We also note that the global variance of the photon number for the two-mode cavity light increases with increasing Ω.

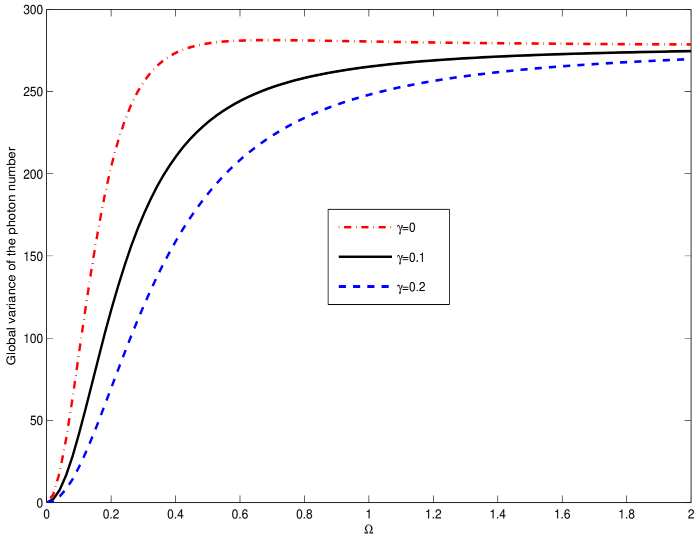

Figure 3.3: Plots of the global variance of the photon number [Eq. (3.45)] versus Ω at steady-state for $\gamma_c = 0.4$, $k = 0.8$, $N = 50$, and different values of γ.

On the other hand, the global variance of the photon number for the two-mode output laser light can be defined as

$$(\Delta n)^2_{out} = \left\langle \left(\hat{a}^{\dagger}_{out}(t)\hat{a}_{out}(t)\right)^2 \right\rangle - \left\langle \hat{a}^{\dagger}_{out}(t)\hat{a}_{out}(t) \right\rangle^2 . \quad (3.48)$$

Employing Eq. (3.8) along with (3.48), the global variance of the photon number for the two-mode output laser light can be written as

$$(\Delta n)^2_{out} = k^2 \left(\Delta n\right)^2 . \quad (3.49)$$

We clearly see that the global variance of the photon number for the two-mode output laser light is k^2 times that of the two-mode cavity light.

3.1.4 Local variance of the photon number

Here we wish to obtain the variance of the photon number in a given frequency interval, employing the spectrum of the photon number fluctuations for the superposition of light modes a_1 and a_2. We denote the central common frequency of these modes by ω_0. The spectrum of the photon number fluctuations for the superposed light modes can be expressed as

$$R(\omega) = \frac{1}{\pi} Re \int_0^{+\infty} d\tau e^{i(\omega-\omega_0)\tau} \langle \hat{n}(t), \hat{n}(t+\tau) \rangle_{ss}, \tag{3.50}$$

where

$$\hat{n}(t) = \hat{a}^\dagger(t)\hat{a}(t) \tag{3.51}$$

and

$$\hat{n}(t+\tau) = \hat{a}^\dagger(t+\tau)\hat{a}(t+\tau), \tag{3.52}$$

in which $\hat{a}(t)$ is given by Eq. (2.60) and we have used the notation $\langle A, B\rangle = \langle AB\rangle - \langle A\rangle\langle B\rangle$. We next proceed to calculate $\langle \hat{n}(t), \hat{n}(t+\tau)\rangle$ that appears in Eq. (3.50). To this end, on account of Eqs. (3.51) and (3.52), we readily obtain

$$\begin{aligned}\langle \hat{n}(t), \hat{n}(t+\tau)\rangle &= \langle \hat{a}^\dagger(t)\hat{a}(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\rangle \\ &\quad -\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle\langle \hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\rangle. \end{aligned} \tag{3.53}$$

In view of the fact that $\hat{a}(t)$ is a Gaussian variable with zero mean, we obtain

$$\begin{aligned}\langle \hat{a}^\dagger(t)\hat{a}(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\rangle &= \langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle\langle \hat{a}(t)\hat{a}^\dagger(t+\tau)\rangle \\ &\quad +\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\rangle\langle \hat{a}(t)\hat{a}(t+\tau)\rangle \\ &\quad +\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle\langle \hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\rangle. \end{aligned} \tag{3.54}$$

With the aid of this result, we can put (3.53) in the form

$$\begin{aligned}\langle \hat{n}(t), \hat{n}(t+\tau)\rangle &= \langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle\langle \hat{a}(t)\hat{a}^\dagger(t+\tau)\rangle \\ &\quad +\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\rangle\langle \hat{a}(t)\hat{a}(t+\tau)\rangle. \end{aligned} \tag{3.55}$$

We now proceed to determine the two-time correlation functions that appear in Eq. (3.55) for the two-mode cavity light. To this end, we observe that the adjoint of Eq. (3.11) has the form

$$\hat{a}^\dagger(t+\tau) = \hat{a}^\dagger(t)e^{-k\tau/2} + \frac{g}{\sqrt{N}} e^{-k\tau/2} \int_0^\tau e^{k\tau'/2} \hat{m}_a^\dagger(t+\tau')d\tau'. \tag{3.56}$$

Using the adjoint of (3.16), one can put Eq. (3.56) in the form

$$\hat{a}^\dagger(t+\tau) = \hat{a}^\dagger(t)e^{-k\tau/2} + \frac{g}{\sqrt{N}}e^{-k\tau/2}\Big\{\hat{m}_a^\dagger(t)\int_0^\tau d\tau' e^{(k-\nu)\tau'/2}$$
$$+ \int_0^\tau d\tau' e^{(k-\nu)\tau'/2}\int_0^{\tau'} d\tau'' e^{\nu\tau''/2}\Big[-\frac{1}{2}\hat{F}_1^\dagger(t+\tau'') + \hat{F}_a^\dagger(t+\tau'')\Big]\Big\}. \qquad (3.57)$$

On multiplying both sides on the left by $\hat{a}(t)$ and taking the expectation value of the resulting equation, we get

$$\langle\hat{a}(t)\hat{a}^\dagger(t+\tau)\rangle = \langle\hat{a}(t)\hat{a}^\dagger(t)\rangle e^{-k\tau/2} + \frac{g}{\sqrt{N}}e^{-k\tau/2}\Big\{\langle\hat{a}(t)\hat{m}_a^\dagger(t)\rangle\int_0^\tau d\tau' e^{(k-\nu)\tau'/2}$$
$$+ \int_0^\tau d\tau' e^{(k-\nu)\tau'/2}\int_0^{\tau'} d\tau'' e^{\nu\tau''/2}\Big[-\frac{1}{2}\langle\hat{a}(t)\hat{F}_1^\dagger(t+\tau'')\rangle + \langle\hat{a}(t)\hat{F}_a^\dagger(t+\tau'')\rangle\Big]\Big\}. \qquad (3.58)$$

Moreover, one can put the adjoint of Eq. (3.19) as

$$\hat{m}_a^\dagger(t) = \frac{k\sqrt{N}}{2g}\hat{a}^\dagger(t). \qquad (3.59)$$

With this substituted into Eq. (3.58), there follows

$$\langle\hat{a}(t)\hat{a}^\dagger(t+\tau)\rangle = \langle\hat{a}(t)\hat{a}^\dagger(t)\rangle e^{-k\tau/2} + \frac{g}{\sqrt{N}}e^{-k\tau/2}\Big\{\frac{k\sqrt{N}}{2g}\langle\hat{a}(t)\hat{a}^\dagger(t)\rangle\int_0^\tau d\tau' e^{(k-\nu)\tau'/2}$$
$$+ \int_0^\tau d\tau' e^{(k-\nu)\tau'/2}\int_0^{\tau'} d\tau'' e^{\nu\tau''/2}\Big[-\frac{1}{2}\langle\hat{a}(t)\hat{F}_1^\dagger(t+\tau'')\rangle$$
$$+ \langle\hat{a}(t)\hat{F}_a^\dagger(t+\tau'')\rangle\Big]\Big\}, \qquad (3.60)$$

Since the cavity mode operator and the noise operator of the atomic are not correlated, we see that

$$\langle\hat{a}(t)\hat{F}_1^\dagger(t+\tau'')\rangle = \langle\hat{a}(t)\rangle\langle\hat{F}_1^\dagger(t+\tau'')\rangle = 0, \qquad (3.61)$$

$$\langle\hat{a}(t)\hat{F}_a^\dagger(t+\tau'')\rangle = \langle\hat{a}(t)\rangle\langle\hat{F}_a^\dagger(t+\tau'')\rangle = 0, \qquad (3.62)$$

so that Eq. (3.60) become

$$\langle\hat{a}(t)\hat{a}^\dagger(t+\tau)\rangle = \langle\hat{a}(t)\hat{a}^\dagger(t)\rangle e^{-k\tau/2} + \frac{k}{2}\langle\hat{a}(t)\hat{a}^\dagger(t)\rangle e^{-k\tau/2}\int_0^\tau d\tau' e^{(k-\nu)\tau'/2}. \qquad (3.63)$$

Then on carrying out the integration over τ', we readily find

$$\langle \hat{a}(t)\hat{a}^{\dagger}(t+\tau)\rangle = \langle \hat{a}(t)\hat{a}^{\dagger}(t)\rangle \left\{ \frac{k}{k-\nu}e^{-\nu\tau/2} - \frac{\nu}{k-\nu}e^{-k\tau/2} \right\}. \tag{3.64}$$

In can be established in a similar manner that

$$\langle \hat{a}(t)\hat{a}(t+\tau)\rangle = \langle \hat{a}^{2}(t)\rangle \left\{ \frac{k}{k-\nu}e^{-\nu\tau/2} - \frac{\nu}{k-\nu}e^{-k\tau/2} \right\} \tag{3.65}$$

and

$$\langle \hat{a}^{\dagger}(t)\hat{a}^{\dagger}(t+\tau)\rangle = \langle \hat{a}^{\dagger 2}(t)\rangle \left\{ \frac{k}{k-\nu}e^{-\nu\tau/2} - \frac{\nu}{k-\nu}e^{-k\tau/2} \right\}. \tag{3.66}$$

On account of Eqs. (3.24), and (3.64)-(3.66), one can write (3.55) as

$$\begin{aligned}\langle \hat{n}(t), \hat{n}(t+\tau)\rangle &= \Big[\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle\langle \hat{a}(t)\hat{a}^{\dagger}(t)\rangle + \langle \hat{a}^{\dagger 2}(t)\rangle\langle \hat{a}^{2}(t)\rangle\Big] \\ &\times \left(\frac{k}{k-\nu}e^{-\nu\tau/2} - \frac{\nu}{k-\nu}e^{-k\tau/2} \right)^{2} \\ &= (\Delta n)^{2} \left\{ \frac{k^{2}}{(k-\nu)^{2}}e^{-\nu\tau} + \frac{\nu^{2}}{(k-\nu)^{2}}e^{-k\tau} - \frac{2k\nu}{(k-\nu)^{2}}e^{-(k+\nu)\tau/2} \right\}.\end{aligned} \tag{3.67}$$

Thus on substituting this result into (3.50), we have

$$\begin{aligned} R(\omega) = (\Delta n)^{2}\,\frac{1}{\pi} Re\Bigg\{ &\left[\frac{k^{2}}{(k-\nu)^{2}}\right] \int_{0}^{+\infty} d\tau e^{-\Big(\nu - i[\omega-\omega_{0}]\Big)\tau} \\ &+ \left[\frac{\nu^{2}}{(k-\nu)^{2}}\right] \int_{0}^{+\infty} d\tau e^{-\Big(k - i[\omega-\omega_{0}]\Big)\tau} \\ &- \left[\frac{2k\nu}{(k-\nu)^{2}}\right] \int_{0}^{+\infty} d\tau e^{-\Big([k+\nu]/2 - i[\omega-\omega_{0}]\Big)\tau} \Bigg\} \end{aligned} \tag{3.68}$$

and on carrying out the integration over τ, the spectrum of the photon number fluctuations for the two-mode cavity light is found to be

$$\begin{aligned} R(\omega) = (\Delta n)^{2} \Bigg\{ &\left[\frac{k^{2}}{(k-\nu)^{2}}\right]\left[\frac{\nu/\pi}{(\omega-\omega_{0})^{2}+\nu^{2}}\right] \\ &+ \left[\frac{\nu^{2}}{(k-\nu)^{2}}\right]\left[\frac{k/\pi}{(\omega-\omega_{0})^{2}+k^{2}}\right] \\ &- \left[\frac{2k\nu}{(k-\nu)^{2}}\right]\left[\frac{(k+\nu)/2\pi}{(\omega-\omega_{0})^{2}+(k+\nu)^{2}/4}\right] \Bigg\}. \end{aligned} \tag{3.69}$$

We next seek to calculate the variance of the photon number in a given frequency interval. Upon integrating both sides of (3.69) over ω, one easily finds

$$\int_{-\infty}^{+\infty} R(\omega)d\omega = (\Delta n)^2, \tag{3.70}$$

in which $(\Delta n)^2$ is the steady-state global variance of the photon number for the two-mode cavity light. On the basis of (3.70), we observe that $R(\omega)d\omega$ represents the steady-state variance of the photon number for the two-mode cavity light in the interval between ω and $\omega + d\omega$ [1]. We thus realize that the photon-number variance in the interval between $\omega' = -\lambda$ and $\omega' = +\lambda$ can be written as [1]

$$(\Delta n)^2_{\pm\lambda} = \int_{-\lambda}^{+\lambda} R(\omega')d\omega', \tag{3.71}$$

in which $\omega' = \omega - \omega_0$. Thus upon substituting (3.69) into Eq. (3.71), we have

$$\begin{aligned}(\Delta n)^2_{\pm\lambda} = (\Delta n)^2_{ss} \Bigg\{ &\left[\frac{k^2}{(k-\nu)^2}\right] \int_{-\lambda}^{+\lambda} \left[\frac{\nu/\pi}{\omega'^2 + \nu^2}\right] d\omega' \\ &+ \left[\frac{\nu^2}{(k-\nu)^2}\right] \int_{-\lambda}^{+\lambda} \left[\frac{k/\pi}{\omega'^2 + k^2}\right] d\omega' \\ &- \left[\frac{2k\nu}{(k-\nu)^2}\right] \int_{-\lambda}^{+\lambda} \left[\frac{(k+\nu)/2\pi}{\omega'^2 + (k+\nu)^2/4}\right] d\omega' \Bigg\},\end{aligned} \tag{3.72}$$

on carrying out the integration over ω', applying the relation described by Eq. (3.30), we readily get

$$(\Delta n)^2_{\pm\lambda} = (\Delta n)^2 z'(\lambda), \tag{3.73}$$

where $z'(\lambda)$ is given by

$$\begin{aligned}z'(\lambda) = &\left[\frac{2k^2/\pi}{(k-\nu)^2}\right] \tan^{-1}\left(\frac{\lambda}{\nu}\right) + \left[\frac{2\nu^2/\pi}{(k-\nu)^2}\right] \tan^{-1}\left(\frac{\lambda}{k}\right) \\ &- \left[\frac{4k\nu/\pi}{(k-\nu)^2}\right] \tan^{-1}\left(\frac{2\lambda}{k+\nu}\right).\end{aligned} \tag{3.74}$$

From the plots in Fig. 3.4, $z'(0.5) = 0.7625$, $z'(1) = 0.9483$, $z'(1.5) = 0.984$, and $z'(2) = 0.9934$ for $\gamma = 0$. And for $\gamma = 0.2$, we find $z'(0.5) = 0.5204$, $z'(1) = 0.7893$, $z'(1.5) =$

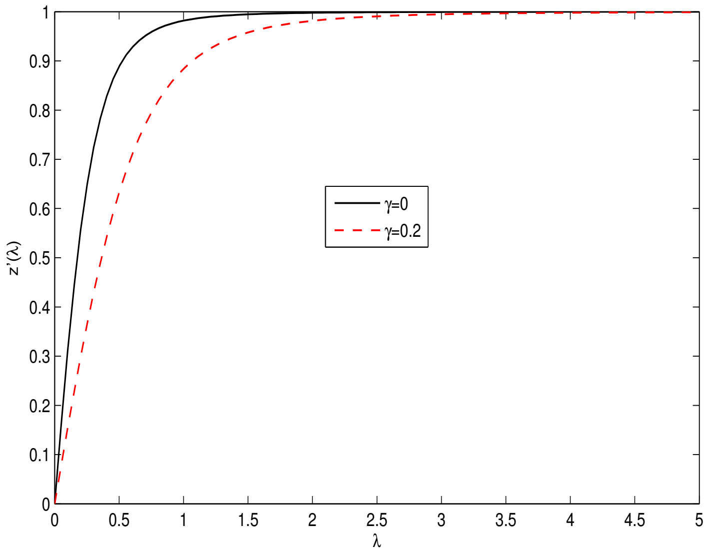

Figure 3.4: Plots of Eq. (3.74) versus λ for $\gamma_c = 0.4$, $k = 0.8$, $\Omega = 2$, and different values of γ.

0.9015, and $z'(2) = 0.9496$. Then combination of these results with Eq. (3.73) yields $(\Delta n)^2_{\pm 0.5} = 0.7625(\Delta n)^2$, $(\Delta n)^2_{\pm 1} = 0.9483(\Delta n)^2$, $(\Delta n)^2_{\pm 1.5} = 0.984(\Delta n)^2$, and $(\Delta n)^2_{\pm 2} = 0.9934(\Delta n)^2$ for $\gamma = 0$. And for $\gamma = 0.2$, we have $(\Delta n)^2_{\pm 0.5} = 0.5204(\Delta n)^2$, $(\Delta n)^2_{\pm 1} = 0.7893(\Delta n)^2$, $(\Delta n)^2_{\pm 1.5} = 0.9015(\Delta n)^2$, and $(\Delta n)^2_{\pm 2} = 0.9496(\Delta n)^2$. In light of this, we immediately observe that a large part of the total variance of the photon number is confined in a relatively small frequency interval.

On the other hand, the local variance of the photon number for the two-mode output laser light is given by

$$(\Delta n_{\pm\lambda})^2_{out} = k^2(\Delta n)^2_{\pm\lambda}. \tag{3.75}$$

On account of Eqs. (3.49) and (3.73), one easily finds

$$k^2(\Delta n)^2_{\pm\lambda} = (\Delta n)^2_{out} z'(\lambda), \tag{3.76}$$

in which $z'(\lambda)$ is given by Eq. (3.74). Now in view of (3.75) and (3.76), the local variance of the photon number for the two-mode output light turns out to be

$$(\Delta n_{\pm\lambda})^2_{out} = (\Delta n)^2_{out} z'(\lambda). \tag{3.77}$$

3.2 Quadrature Squeezing

In this chapter we wish to analyze the squeezing properties of the two-mode cavity light produced by the degenerate three-level laser pumped by coherent light and coupled to a vacuum reservoir.

3.2.1 The global quadrature squeezing

We now seek to calculate the global quadrature squeezing of the two-mode cavity (output) light at steady state. The squeezing properties of the two-mode cavity light are described by two quadrature operators defined as

$$\hat{a}_+(t) = \hat{a}^\dagger(t) + \hat{a}(t) \tag{3.78}$$

and

$$\hat{a}_-(t) = i(\hat{a}^\dagger(t) - \hat{a}(t)), \tag{3.79}$$

where $\hat{a}_+$ and $\hat{a}_-$ are Hermitian operators representing physical quantities called the plus and minus quadratures, respectively, while $\hat{a}^\dagger(\hat{a})$ is the creation (annihilation) operator for the superposition of light modes a_1 and a_2. It can be readily established that

$$[\hat{a}_-, \hat{a}_+] = 2i\frac{\gamma_c}{k}\left(\hat{N}_a - \hat{N}_c\right). \tag{3.80}$$

It then follows that

$$\Delta a_+ \Delta a_- \geqslant \frac{\gamma_c}{k}\left|\langle\hat{N}_a\rangle - \langle\hat{N}_c\rangle\right|. \tag{3.81}$$

With the aid of (2.99) and (2.101), Eq. (3.81) can be put in the form

$$\Delta a_+ \Delta a_- \geqslant \frac{\gamma_c}{k}\left|\frac{(\gamma_c+\gamma)^2 N}{(\gamma_c+\gamma)^2 + 3\Omega^2}\right|. \tag{3.82}$$

Now upon replacing the atomic operators that appear in Eq. (2.64) by their expectation values, the commutation relation for the two-mode cavity light produced by the three-level laser can be written as

$$[\hat{a}, \hat{a}^{\dagger}] = \lambda, \tag{3.83}$$

in which

$$\lambda = \frac{\gamma_c}{k}\Big(\langle \hat{N}_c\rangle - \langle \hat{N}_a\rangle\Big). \tag{3.84}$$

We define the quadrature variance of the two-mode cavity light by

$$\left(\Delta a_{\pm}\right)^2 = \langle \hat{a}_{\pm}(t), \hat{a}_{\pm}(t)\rangle \,. \tag{3.85}$$

On account of (3.83), we rewrite Eq. (3.85) as

$$\left(\Delta a_{\pm}\right)^2 = \lambda + \langle : \hat{a}_{\pm}(t), \hat{a}_{\pm}(t) :\rangle \,, \tag{3.86}$$

where :: stands for normal ordering. Then on account of (3.78) and (3.79), Eq. (3.86) takes the form

$$\left(\Delta a_{\pm}\right)^2 = \lambda + 2\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle \pm \langle \hat{a}^{\dagger 2}(t)\rangle \pm \langle \hat{a}^{2}(t)\rangle \mp \langle \hat{a}^{\dagger}(t)\rangle^2 \mp \langle \hat{a}(t)\rangle^2 - 2\langle \hat{a}^{\dagger}(t)\rangle\langle \hat{a}(t)\rangle. \tag{3.87}$$

In view of (2.84), Eq. (3.87) reduces to

$$\left(\Delta a_{\pm}\right)^2 = \lambda + 2\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle \pm \langle \hat{a}^{\dagger 2}(t)\rangle \pm \langle \hat{a}^{2}(t)\rangle. \tag{3.88}$$

Thus on substituting (3.3) and (3.39) into Eq. (3.88), we get

$$\left(\Delta a_{\pm}\right)^2 = \lambda + \frac{\gamma_c}{k}\left\{2\langle \hat{N}_b\rangle + 2\langle \hat{N}_a\rangle \pm \langle \hat{m}_3^{\dagger}\rangle \pm \langle \hat{m}_3\rangle\right\}. \tag{3.89}$$

On account of (3.84), one can write Eq. (3.89) as

$$\left(\Delta a_{\pm}\right)^2 = \frac{\gamma_c}{k}\left\{2\langle \hat{N}_b\rangle + \langle \hat{N}_a\rangle + \langle \hat{N}_c\rangle \pm \langle \hat{m}_3^{\dagger}\rangle \pm \langle \hat{m}_3\rangle\right\}, \tag{3.90}$$

so that in view of the identity given by (2.47), Eq. (3.90) can be put in the form

$$\left(\Delta a_{\pm}\right)^2 = \frac{\gamma_c}{k}\left\{N + \langle \hat{N}_b\rangle \pm \langle \hat{m}_3^{\dagger}\rangle \pm \langle \hat{m}_3\rangle\right\}. \tag{3.91}$$

Finally, on account of (2.100) and (2.102), the global quadrature variance of the two-mode cavity light turns out at steady state to be

$$(\Delta a_+)^2 = \frac{\gamma_c}{k}\left\{1 + \frac{\Omega^2 + 2\Omega(\gamma_c + \gamma)}{(\gamma_c + \gamma)^2 + 3\Omega^2}\right\} N, \tag{3.92}$$

$$(\Delta a_-)^2 = \frac{\gamma_c}{k}\left\{1 + \frac{\Omega^2 - 2\Omega(\gamma_c + \gamma)}{(\gamma_c + \gamma)^2 + 3\Omega^2}\right\} N. \tag{3.93}$$

We observe that the two-mode cavity light is in a squeezed state and the squeezing occurs in the minus quadrature. In addition, we note that the global quadrature variance of the two-mode cavity light takes for $\Omega \gg \gamma_c$ and $\Omega \gg \gamma$ the form

$$(\Delta a_+)^2 = 2\bar{n} \tag{3.94}$$

and

$$(\Delta a_-)^2 = 2\bar{n}, \tag{3.95}$$

where $\bar{n}$ is given by Eq. (3.6). These results represent the normally-ordered quadrature variance for a chaotic light. On the basis of these results, we assert that the light produced by the coherently pumped degenerate three-level laser operating under the conditions $\Omega \gg \gamma_c$ and $\Omega \gg \gamma$ is in a chaotic state. In addition, for $\Omega = 0$, Eqs. (3.92) and (3.93) reduce to

$$(\Delta a_+)^2_v = (\Delta a_-)^2_v = \frac{\gamma_c}{k} N. \tag{3.96}$$

This represents the quadrature variance of a two-mode vacuum state.

We next calculate the quadrature squeezing of the two-mode cavity light relative to the quadrature variance of the two-mode vacuum state. We thus define the quadrature squeezing of the two-mode cavity light by [1]

$$S = \frac{(\Delta a_-)^2_v - (\Delta a_-)^2}{(\Delta a_-)^2_v}. \tag{3.97}$$

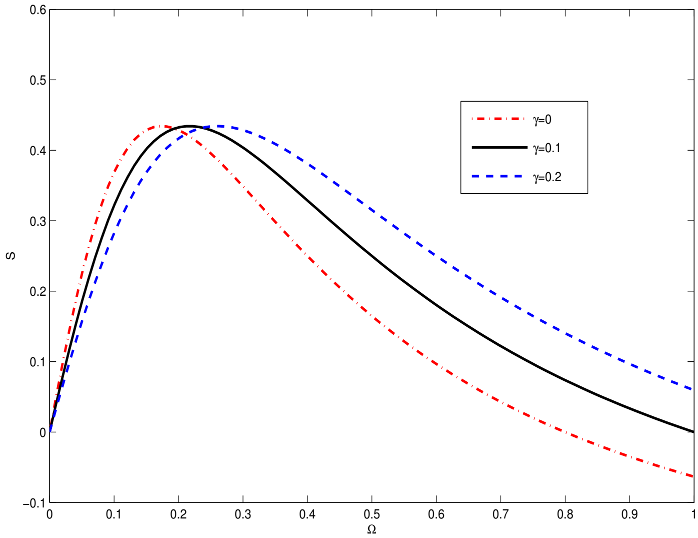

Figure 3.5: Plots of the global quadrature squeezing [Eq. (3.98)] versus Ω at steady state for $\gamma_c = 0.4$ and different value of γ.

Hence on account of (3.93) and (3.96), we arrive at

$$S = \left\{ \frac{2\Omega(\gamma_c + \gamma)}{(\gamma_c + \gamma)^2 + 3\Omega^2} - \frac{\Omega^2}{(\gamma_c + \gamma)^2 + 3\Omega^2} \right\}. \tag{3.98}$$

We note that, unlike the mean photon number and the variance of the photon number, the quadrature squeezing does not depend on the number of atoms. This implies that the quadrature squeezing of the two-mode cavity light is independent of the number of photons [1].

We clearly see from the plots in Fig. 3.5 that the presence of the spontaneous emission process leads to a decrease in the maximum global quadrature squeezing of the two-mode cavity light beam. In addition, we note that the maximum global quadrature squeezing for $\gamma = 0$ is 43.42% (and occurs at $\Omega = 0.1717$) and for $\gamma = 0.1$ is 43.41% (and occurs at $\Omega = 0.2222$). And for $\gamma = 0.2$, the maximum global quadrature squeezing is observed to be 43.4% (and occurs at $\Omega = 0.2525$).

On the other hand, the quadrature variance of the two-mode output laser light is defined by

$$\left(\Delta a_{\pm}^{out}\right)^2 = k\lambda + \left\langle : \hat{a}_{\pm}^{out}, \hat{a}_{\pm}^{out} : \right\rangle, \tag{3.99}$$

where

$$\hat{a}_{+}^{out} = \hat{a}_{out}^{\dagger} + \hat{a}_{out} \tag{3.100}$$

and

$$\hat{a}_{-}^{out} = i(\hat{a}_{out}^{\dagger} - \hat{a}_{out}) \tag{3.101}$$

are the plus and minus quadrature operators for the two-mode output light. Now we define the quadrature squeezing of the two-mode output laser light by [1]

$$S^{out} = \frac{\left(\Delta a_{-}^{out}\right)_v^2 - \left(\Delta a_{-}^{out}\right)^2}{\left(\Delta a_{-}^{out}\right)_v^2}, \tag{3.102}$$

where $\left(\Delta a_{-}^{out}\right)_v^2$ is the quadrature variance of the two-mode output vacuum state. On account of (3.8), we easily see that

$$\left(\Delta a_{-}^{out}\right)_v^2 = k\left(\Delta a_{-}\right)_v^2, \tag{3.103}$$

$$\left(\Delta a_{-}^{out}\right)^2 = k\left(\Delta a_{-}\right)^2. \tag{3.104}$$

Thus in view of (3.103) and (3.104) along with (3.97), we arrive at

$$S^{out} = S, \tag{3.105}$$

which indicates that the global quadrature squeezing of the two-mode output laser light is exactly the same as that of the two-mode cavity light.

3.2.2 Local quadrature squeezing

We next seek to determine the quadrature squeezing of the two-mode cavity (output) light in a given frequency interval. To this end, we first obtain the spectrum of quadrature fluctuations for the two-mode cavity light. This spectrum for the two-mode cavity light is defined by

$$S_{\pm}(\omega) = \frac{1}{\pi} Re \int_0^{\infty} d\tau e^{i(\omega-\omega_0)\tau} \left\langle \hat{a}_{\pm}(t), \hat{a}_{\pm}(t+\tau)\right\rangle_{ss}, \tag{3.106}$$

in which

$$\hat{a}_{+}(t+\tau) = \hat{a}^{\dagger}(t+\tau) + \hat{a}(t+\tau), \tag{3.107}$$

$$\hat{a}_{-}(t+\tau) = i\Big(\hat{a}^{\dagger}(t+\tau) - \hat{a}(t+\tau)\Big) \tag{3.108}$$

and ω_0 is the central common frequency of light mode a_1 and light mode a_2. On account of Eq. (2.84), we see that

$$\langle \hat{a}_{\pm}(t), \hat{a}_{\pm}(t+\tau)\rangle = \langle \hat{a}_{\pm}(t)\hat{a}_{\pm}(t+\tau)\rangle\,. \tag{3.109}$$

Then with the aid of (3.78), (3.79), (3.107), and (3.108), one can rewrite Eq. (3.109) as

$$\begin{aligned}\langle \hat{a}_{\pm}(t), \hat{a}_{\pm}(t+\tau)\rangle &= \langle \hat{a}^{\dagger}(t)\hat{a}(t+\tau)\rangle + \langle \hat{a}(t)\hat{a}^{\dagger}(t+\tau)\rangle \\ &\quad \pm\langle \hat{a}^{\dagger}(t)\hat{a}^{\dagger}(t+\tau)\rangle \pm \langle \hat{a}(t)\hat{a}(t+\tau)\rangle.\end{aligned} \tag{3.110}$$

Upon substituting (3.24) and (3.64)-(3.66) into Eq. (3.110), we arrive at

$$\begin{aligned}\langle \hat{a}_{\pm}(t), \hat{a}_{\pm}(t+\tau)\rangle &= \Big\{\langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle + \langle \hat{a}(t)\hat{a}^{\dagger}(t)\rangle \pm \langle \hat{a}^{\dagger 2}(t)\rangle \pm \langle \hat{a}^{2}(t)\rangle\Big\} \\ &\quad \times\left\{\frac{k}{k-\nu}e^{-\nu\tau/2} - \frac{\nu}{k-\nu}e^{-k\tau/2}\right\}.\end{aligned} \tag{3.111}$$

This can be rewritten as

$$\langle \hat{a}_{+}(t), \hat{a}_{+}(t+\tau)\rangle = (\Delta a_{+})^{2}\left\{\frac{k}{k-\nu}e^{-\nu\tau/2} - \frac{\nu}{k-\nu}e^{-k\tau/2}\right\} \tag{3.112}$$

and

$$\langle \hat{a}_{-}(t), \hat{a}_{-}(t+\tau)\rangle = (\Delta a_{-})^{2}\left\{\frac{k}{k-\nu}e^{-\nu\tau/2} - \frac{\nu}{k-\nu}e^{-k\tau/2}\right\}. \tag{3.113}$$

Now on introducing (3.113) into Eq. (3.106), we find

$$\begin{aligned}S_{-}(\omega) = (\Delta a_{-})^{2}_{ss}\,\frac{1}{\pi}Re\bigg\{&\left[\frac{k}{k-\nu}\right]\int_{0}^{+\infty} d\tau e^{-\Big(\nu/2 - i[\omega-\omega_0]\Big)\tau} \\ &-\left[\frac{\nu}{k-\nu}\right]\int_{0}^{+\infty} d\tau e^{-\Big(k/2 - i[\omega-\omega_0]\Big)\tau}\bigg\}\end{aligned} \tag{3.114}$$

and on carrying out the integration over τ, the spectrum of the minus quadrature fluctuations for the two-mode cavity light is found to be

$$S_{-}(\omega) = (\Delta a_{-})^2_{ss}\left\{\left[\frac{k}{k-\nu}\right]\left[\frac{\nu/2\pi}{(\omega-\omega_0)^2+(\nu/2)^2}\right] - \left[\frac{\nu}{k-\nu}\right]\left[\frac{k/2\pi}{(\omega-\omega_0)^2+(k/2)^2}\right]\right\}, \tag{3.115}$$

where $(\Delta a_{-})^2$ is given by Eq. (3.93).

Upon integrating both sides of Eq. (3.115) over ω, we get

$$\int_{-\infty}^{+\infty} S_{-}(\omega)d\omega = (\Delta a_{-})^2, \tag{3.116}$$

in which

$$(\Delta a_{-})^2 = \langle \hat{a}_{-}(t), \hat{a}_{-}(t)\rangle_{ss} \tag{3.117}$$

is the global variance of the minus quadrature for the two-mode cavity light at steady state. On the basis of Eq. (3.116), we observe that $S_{-}(\omega)d\omega$ is the steady-state variance of the minus quadrature in the interval between ω and $\omega + d\omega$ [1]. We thus realize that the variance of the minus quadrature in the interval between $\omega' = -\lambda$ and $\omega' = +\lambda$ is expressible as [1]

$$(\Delta a_{-})^2_{\pm\lambda} = \int_{-\lambda}^{+\lambda} S_{-}(\omega')d\omega', \tag{3.118}$$

in which $\omega' = \omega - \omega_0$. Now on introducing (3.115) into Eq. (3.118), we find

$$(\Delta a_{-})^2_{\pm\lambda} = (\Delta a_{-})^2\left\{\left[\frac{k}{k-\nu}\right]\int_{-\lambda}^{+\lambda}\left[\frac{\nu/2\pi}{\omega'^2+(\nu/2)^2}\right]d\omega' - \left[\frac{\nu}{k-\nu}\right]\int_{-\lambda}^{+\lambda}\left[\frac{k/2\pi}{\omega'^2+(k/2)^2}\right]d\omega'\right\} \tag{3.119}$$

and on carrying out the integration over ω', employing (3.30), we readily obtain

$$(\Delta a_{-})^2_{\pm\lambda} = (\Delta a_{-})^2 z(\lambda), \tag{3.120}$$

where

$$z(\lambda) = \left[\frac{2k/\pi}{k-\nu}\right]\tan^{-1}\left(\frac{2\lambda}{\nu}\right) - \left[\frac{2\nu/\pi}{k-\nu}\right]\tan^{-1}\left(\frac{2\lambda}{k}\right). \tag{3.121}$$

Upon setting $\Omega = 0$ in Eq. (3.120), we easily get

$$(\Delta a_-)^2_{v\pm\lambda} = (\Delta a_-)^2_v \, z_v(\lambda), \tag{3.122}$$

in which

$$(\Delta a_-)^2_v = \frac{\gamma_c}{k} N \tag{3.123}$$

and

$$z_v(\lambda) = \left[\frac{2k/\pi}{k - (\gamma_c + \gamma)}\right] \tan^{-1}\left(\frac{2\lambda}{\gamma_c + \gamma}\right) - \left[\frac{2(\gamma_c + \gamma)/\pi}{k - (\gamma_c + \gamma)}\right] \tan^{-1}\left(\frac{2\lambda}{k}\right). \tag{3.124}$$

We see that Eq. (3.122) is the local quadrature variance of the two-mode vacuum state.

We next proceed to calculate the local quadrature squeezing of the two-mode cavity light relative to the local quadrature variance of the two-mode vacuum state. We define the quadrature squeezing of the two-mode cavity light in the $\lambda_\pm$ frequency interval by [1]

$$S_{\pm\lambda} = \frac{(\Delta a_-)^2_{v\pm\lambda} - (\Delta a_-)^2_{\pm\lambda}}{(\Delta a_-)^2_{v\pm\lambda}}. \tag{3.125}$$

Then on account of Eqs. (3.93), (3.120), and (3.122)-(3.124), we readily get

$$S_{\pm\lambda} = \frac{1}{z_v(\lambda)}\left\{z_v(\lambda) - z(\lambda)\left[1 + \frac{\Omega^2 - 2\Omega(\gamma_c + \gamma)}{(\gamma_c + \gamma)^2 + 3\Omega^2}\right]\right\}. \tag{3.126}$$

We readily see from the plots in Fig. 3.6 that the maximum local quadrature squeezing for $\gamma = 0$ is 78.48% (and occurs at $\lambda = 0.0606$) and for $\gamma = 0.1$ is 77.42% (and occurs at $\lambda = 0.0606$). And for $\gamma = 0.2$, the maximum local quadrature squeezing is observed to be 71.02% (and occurs at $\lambda = 0.0606$). In line of this, the presence of the spontaneous emission process leads to a decrease in the local quadrature squeezing of the two-mode cavity light beam with increasing λ.

Finally, we define the local quadrature squeezing of the two-mode output laser

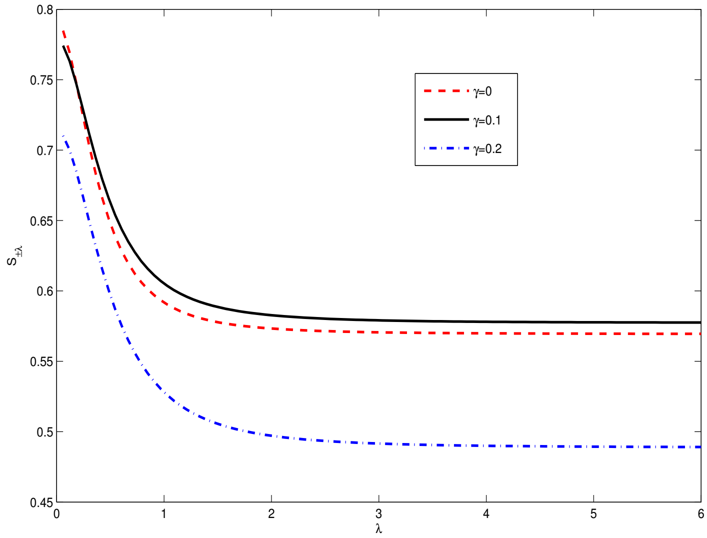

Figure 3.6: Plots of the local quadrature squeezing [Eq. (3.126)] versus λ at steady state for $\gamma = 0$, $\gamma_c = 0.4$, $k = 0.8$, and $\Omega = 0.1717$ [red, dashed line], for $\gamma = 0.1$, $\gamma_c = 0.4$, $k = 0.8$, and $\Omega = 0.2222$ [black, solid line], and for $\gamma = 0.2$, $\gamma_c = 0.4$, $k = 0.8$, and $\Omega = 0.2525$ [blue, dashed dot line].

light in the aforementioned frequency interval [1]

$$S^{out}_{\pm\lambda} = \frac{\left(\Delta a^{out}_{-}\right)^2_{v\pm\lambda} - \left(\Delta a^{out}_{-}\right)^2_{\pm\lambda}}{\left(\Delta a^{out}_{-}\right)^2_{v\pm\lambda}}, \tag{3.127}$$

and taking into account the fact that

$$\left(\Delta a^{out}_{-}\right)^2_{v\pm\lambda} = z_v(\lambda)\left(\Delta a^{out}_{-}\right)^2_{v} \tag{3.128}$$

and

$$\left(\Delta a^{out}_{-}\right)^2_{\pm\lambda} = z(\lambda)\left(\Delta a^{out}_{-}\right)^2, \tag{3.129}$$

along with (3.97), we arrive at

$$S^{out}_{\pm\lambda} = S_{\pm\lambda}. \tag{3.130}$$

We note that the quadrature squeezing of the two-mode output laser light in a given frequency interval is equal to the local quadrature squeezing of the two-mode cavity light.

4

Superposed Two-Mode Laser Light Beams

In this chapter we seek to investigate the squeezing and statistical properties of a pair of superposed two-mode laser light beams produced by coherently pumped degenerate three-level lasers with open cavities coupled to two-mode vacuum reservoirs via single-port mirrors. We arrange our system in such a way that the light beam (LB1) from one of the laser is incident on a side of perfectly transmitting mirror, while the light beam (LB2) from the other laser is incident on a side of perfectly reflecting mirror as depicted in the Fig. 4.1.

We wish to calculate, applying the density operator, the global (local) mean photon number, the global (local) variance of the photon number, and the global (local) quadrature squeezing of a pair of superposed two-mode cavity light beams. We then obtain the corresponding output photon statistics and quadrature squeezing.

4.1 The Q function

Here we seek to determine the Q function for the light generated by a coherently pumped degenerate three-level laser with an open cavity coupled to a two-mode vacuum reservoir. The Q function can be expressed as

$$Q(\alpha^*, \alpha, t) = \frac{\lambda}{\pi^2} \int d^2z \phi_a(z,t) e^{z^*\alpha - z\alpha^*}, \tag{4.1}$$

where $\phi_a(z,t)$ is the antinormally ordered characteristic function defined by

$$\phi_a(z,t) = Tr\Big(\hat{\rho} e^{-z^*\hat{a}(t)} e^{z\hat{a}^\dagger(t)}\Big). \tag{4.2}$$

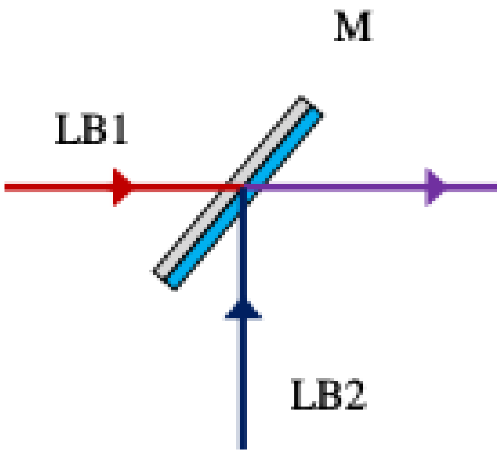

Figure 4.1: Schematic representation of a pair of superposed two-mode laser light beams, with $\kappa = 1$ and $\kappa = 0$ for the upper (lower) surface of the mirror, respectively.

Employing the Baker-Hausdorff identity

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}, \tag{4.3}$$

one can be put Eq. (4.2) in the form

$$\phi_a(z,t) = e^{-\frac{1}{2}\lambda z^* z} Tr\Big(\hat{\rho} e^{z\hat{a}^\dagger(t) - z^*\hat{a}(t)}\Big). \tag{4.4}$$

This can be rewritten as

$$\phi_a(z,t) = e^{-\frac{1}{2}\lambda z^* z}\left\langle \exp\Big(z\hat{a}^\dagger(t) - z^*\hat{a}(t)\Big)\right\rangle. \tag{4.5}$$

Using the fact that the operator $\hat{a}(t)$ is a Gaussian variable with zero mean, one can show that [1]

$$\phi_a(z,t) = e^{-\frac{1}{2}\lambda z^* z}\exp\left[\left\langle \frac{1}{2}\Big(z\hat{a}^\dagger(t) - z^*\hat{a}(t)\Big)^2\right\rangle\right]. \tag{4.6}$$

It then follows that

$$\phi_a(z,t) = \exp\left[\frac{1}{2}z^2\langle\hat{a}^{\dagger 2}(t)\rangle + \frac{1}{2}z^{*2}\langle\hat{a}^2(t)\rangle - \frac{1}{2}z^* z\Big(\lambda + \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle + \langle\hat{a}(t)\hat{a}^\dagger(t)\rangle\Big)\right]. \tag{4.7}$$

On account of (3.3), (3.38), and (3.41), the antinormally ordered characteristic function can put in the form

$$\phi_a(z,t) = \exp\left[\frac{1}{2}z^2\left\{\frac{\gamma_c}{k}\langle\hat{m}_3^\dagger\rangle\right\} + \frac{1}{2}z^{*2}\left\{\frac{\gamma_c}{k}\langle\hat{m}_3\rangle\right\}\right.$$

$$-\frac{z^*z}{2}\left\{\lambda+\frac{\gamma_c}{k}\Big(\langle\hat{N}_a\rangle+\langle\hat{N}_c\rangle+2\langle\hat{N}_b\rangle\Big)\right\}\Bigg], \tag{4.8}$$

so that with aid of Eqs. (2.97) and (3.84), we readily find

$$\phi_a(z,t)=\exp\Bigg[-pz^*z+\frac{q}{2}(z^2+z^{*2})\Bigg], \tag{4.9}$$

in which

$$p=\frac{\gamma_c}{k}\Bigg\{\langle\hat{N}_c\rangle+\langle\hat{N}_b\rangle\Bigg\} \tag{4.10}$$

and

$$q=\frac{\gamma_c}{k}\langle\hat{m}_3\rangle. \tag{4.11}$$

Finally, upon introducing (4.9) into Eq. (4.1), we get

$$Q(\alpha^*,\alpha)=\frac{\lambda}{\pi^2}\int d^2z\exp\Big[-pz^*z+z^*\alpha-z\alpha^*+\frac{q}{2}(z^2+z^{*2})\Big], \tag{4.12}$$

so that on carrying out the integration over z, employing the relation [1]

$$\int\frac{d^2z}{\pi}\exp\Big[-az^*z+bz+cz^*+A'z^2+B'z^{*2}\Big]=\Big[\frac{1}{a^2-4A'B'}\Big]^{1/2}$$

$$\times\exp\Big[\frac{abc+A'c^2+B'b^2}{a^2-4A'B'}\Big], a>0 \tag{4.13}$$

we readily find

$$Q(\alpha^*,\alpha)=\frac{\lambda}{\pi}\left[u^2-v^2\right]^{1/2}\exp\left[-u\alpha^*\alpha+\frac{v}{2}(\alpha^2+\alpha^{*2})\right], \tag{4.14}$$

in which

$$u=\frac{p}{p^2-q^2}, \tag{4.15}$$

$$v=\frac{q}{p^2-q^2}. \tag{4.16}$$

Furthermore, employing Eq. (4.14), we have

$$\int d^2\alpha Q(\alpha^*,\alpha)=\lambda\left[u^2-v^2\right]^{1/2}\int\frac{d^2\alpha}{\pi}\exp\left[-u\alpha^*\alpha+\frac{v}{2}(\alpha^2+\alpha^{*2})\right] \tag{4.17}$$

and on carrying out the integration using Eq. (4.13), we arrive at

$$\int d^2\alpha Q(\alpha^*, \alpha) = \lambda. \tag{4.18}$$

This shows that the Q function is normalized to λ.

Now we use convenient notations for the Q functions of the constituent two-mode light beams. Thus with the aid of Eq. (4.14), the Q function of the first two-mode light beam can be written as

$$Q(\gamma_0^*, \gamma_0) = \frac{\lambda_a}{\pi}\left[u_a^2 - v_a^2\right]^{1/2} \exp\left[- u_a\gamma_0^*\gamma_0 + \frac{v_a}{2}\left(\gamma_0^2 + \gamma_0^{*2}\right)\right], \tag{4.19}$$

where

$$\lambda_a = \frac{\gamma_c}{k}\bigg\{\langle\hat{N}_c\rangle_a - \langle\hat{N}_a\rangle_a\bigg\}, \tag{4.20}$$

$$u_a = \frac{p_a}{p_a^2 - q_a^2}, \tag{4.21}$$

$$v_a = \frac{q_a}{p_a^2 - q_a^2}, \tag{4.22}$$

with

$$p_a = \frac{\gamma_c}{k}\bigg\{\langle\hat{N}_c\rangle_a + \langle\hat{N}_b\rangle_a\bigg\} \tag{4.23}$$

and

$$q_a = \frac{\gamma_c}{k}\langle\hat{m}_3\rangle_a. \tag{4.24}$$

Now on the basis of Eqs. (2.99)-(2.102), we have

$$\langle\hat{N}_a\rangle_a = \left\{\frac{\Omega^2}{(\gamma_c + \gamma)^2 + 3\Omega^2}\right\} N_1, \tag{4.25}$$

$$\langle\hat{N}_b\rangle_a = \left\{\frac{\Omega^2}{(\gamma_c + \gamma)^2 + 3\Omega^2}\right\} N_1, \tag{4.26}$$

$$\langle\hat{N}_c\rangle_a = \left\{\frac{(\gamma_c + \gamma)^2 + \Omega^2}{(\gamma_c + \gamma)^2 + 3\Omega^2}\right\} N_1, \tag{4.27}$$

$$\langle\hat{m}_3\rangle_a = \left\{\frac{\Omega(\gamma_c + \gamma)}{(\gamma_c + \gamma)^2 + 3\Omega^2}\right\} N_1. \tag{4.28}$$

And the Q function for the second two-mode light beam can be expressed as

$$Q(\eta^*,\eta) = \frac{\lambda_b}{\pi}\left[u_b^2 - v_b^2\right]^{1/2} \exp\Big[-u_b\eta^*\eta + \frac{v_b}{2}\left(\eta^2 + \eta^{*2}\right)\Big], \tag{4.29}$$

in which

$$\lambda_b = \frac{\gamma_c}{k}\Big\{\langle \hat{N}_c\rangle_b - \langle \hat{N}_a\rangle_b\Big\}, \tag{4.30}$$

$$u_b = \frac{p_b}{p_b^2 - q_b^2}, \tag{4.31}$$

$$v_b = \frac{q_b}{p_b^2 - q_b^2}, \tag{4.32}$$

with

$$p_b = \frac{\gamma_c}{k}\Big\{\langle \hat{N}_c\rangle_b + \langle \hat{N}_b\rangle_b\Big\}, \tag{4.33}$$

$$q_b = \frac{\gamma_c}{k}\langle \hat{m}_3\rangle_b \tag{4.34}$$

and

$$\langle \hat{N}_a\rangle_b = \left\{\frac{\Omega^2}{(\gamma_c+\gamma)^2 + 3\Omega^2}\right\} N_2, \tag{4.35}$$

$$\langle \hat{N}_b\rangle_b = \left\{\frac{\Omega^2}{(\gamma_c+\gamma)^2 + 3\Omega^2}\right\} N_2, \tag{4.36}$$

$$\langle \hat{N}_c\rangle_b = \left\{\frac{(\gamma_c+\gamma)^2 + \Omega^2}{(\gamma_c+\gamma)^2 + 3\Omega^2}\right\} N_2, \tag{4.37}$$

$$\langle \hat{m}_3\rangle_b = \left\{\frac{\Omega(\gamma_c+\gamma)}{(\gamma_c+\gamma)^2 + 3\Omega^2}\right\} N_2. \tag{4.38}$$

4.2 The density operator

Here we want to express the expectation value of a given operator $\hat{A}(\hat{a}^\dagger,\hat{a})$ in terms of the Q-function and the c-number function corresponding to $\hat{A}(\hat{a}^\dagger,\hat{a})$ in the normal order. Suppose $\hat{\rho}(\hat{a}^\dagger,\hat{a},t)$ is the density operator for a two-mode laser light beam. Then upon expanding this density operator in the normal order

$$\hat{\rho}(t) = \sum_{lm} C_{lm}(t)\hat{a}^{\dagger l}\hat{a}^m \tag{4.39}$$

and employing the completeness relation for coherent states [1]

$$\frac{\lambda}{\pi}\int d^2\gamma_0 \mid \gamma_0\rangle\langle\gamma_0 \mid = \hat{I}, \tag{4.40}$$

one easily finds

$$\hat{\rho}(t) = \frac{\lambda}{\pi}\int d^2\gamma_0 \sum_{lm} C_{lm}(t)\big(\lambda\gamma_0^*\big)^l \mid \gamma_0\rangle\langle\gamma_0 \mid \hat{a}^m. \tag{4.41}$$

Then on account of the relation

$$\mid \gamma_0\rangle\langle\gamma_0 \mid \hat{a}^m = \Big(\lambda\gamma_0 + \frac{\partial}{\partial\gamma_0^*}\Big)^m \mid \gamma_0\rangle\langle\gamma_0 \mid, \tag{4.42}$$

we have

$$\hat{\rho}(t) = \lambda\int d^2\gamma_0 Q\Big(\lambda\gamma_0^*, \lambda\gamma_0 + \frac{\partial}{\partial\gamma_0^*}, t\Big) \mid \gamma_0\rangle\langle\gamma_0 \mid, \tag{4.43}$$

where

$$Q\Big(\lambda\gamma_0^*, \lambda\gamma_0 + \frac{\partial}{\partial\gamma_0^*}, t\Big) = \frac{1}{\pi}\sum_{lm} C_{lm}\big(\lambda\gamma_0^*\big)^l\Big(\lambda\gamma_0 + \frac{\partial}{\partial\gamma_0^*}\Big)^m. \tag{4.44}$$

We recall that the expectation value of a given operator $\hat{A}\big(\hat{a}^\dagger, \hat{a}, t\big)$ can be expressed as

$$\langle\hat{A}\big(\hat{a}^\dagger, \hat{a}, t\big)\rangle = Tr\left[\hat{\rho}(t)\hat{A}\big(\hat{a}^\dagger, \hat{a}\big)\right]. \tag{4.45}$$

Upon substituting (4.43) into Eq. (4.45), we readily find

$$\langle\hat{A}\big(\hat{a}^\dagger, \hat{a}, t\big)\rangle = \lambda\int d^2\gamma_0 Q\Big(\lambda\gamma_0^*, \lambda\gamma_0 + \frac{\partial}{\partial\gamma_0^*}, t\Big)A_n\big(\lambda\gamma_0^*, \lambda\gamma_0\big), \tag{4.46}$$

where $A_n\big(\lambda\gamma_0^*, \lambda\gamma_0\big)$ is the c-number function corresponding to $\hat{A}\big(\hat{a}^\dagger, \hat{a}\big)$ in the normal order. Now on introducing new variables $\alpha = \lambda\gamma_0$, we find

$$\langle\hat{A}\big(\hat{a}^\dagger, \hat{a}, t\big)\rangle = \int \frac{d^2\alpha}{\lambda} Q\Big(\alpha^*, \alpha + \lambda\frac{\partial}{\partial\alpha^*}, t\Big)A_n\big(\alpha^*, \alpha\big). \tag{4.47}$$

This represents the expectation value of an operator representing a two-mode light beam.

Furthermore, we seek to derive an alternative expression for the expectation value of a given operator for a one-mode as well as a two-mode light beam. To this end, applying the completeness relation given by Eq. (4.40) twice, we obtain

$$\hat{\rho}\big(\hat{a}^\dagger, \hat{a}, t\big) = \lambda^2\int \frac{d^2\eta}{\pi}\frac{d^2\gamma_0}{\pi} \mid \eta\rangle\langle\eta \mid \hat{\rho}\big(\hat{a}^\dagger, \hat{a}, t\big) \mid \gamma_0\rangle\langle\gamma_0 \mid. \tag{4.48}$$

This can be rewritten as

$$\hat{\rho}\big(\hat{a}^\dagger, \hat{a}, t\big) = \frac{\lambda^2}{\pi}\int d^2\eta d^2\gamma_0 Q\big(\lambda\eta^*, \lambda\gamma_0, t\big) \mid \eta\rangle\langle\gamma_0 \mid, \tag{4.49}$$

in which

$$Q\big(\lambda\eta^*, \lambda\gamma_0, t\big) = \frac{1}{\pi}\langle\eta \mid \hat{\rho}(\hat{a}^\dagger, \hat{a}, t) \mid \gamma_0\rangle. \tag{4.50}$$

With the aid of (4.49), one can write Eq. (4.45) as

$$\begin{aligned}\langle\hat{A}\big(\hat{a}^\dagger, \hat{a}, t\big)\rangle &= \frac{\lambda^2}{\pi}\int d^2\eta d^2\gamma_0 Q\big(\lambda\eta^*, \lambda\gamma_0, t\big)\langle\eta \mid \gamma_0\rangle\langle\gamma_0 \mid \hat{A} \mid \eta\rangle \\ &= \frac{\lambda^2}{\pi}\int d^2\eta d^2\gamma_0 Q\big(\lambda\eta^*, \lambda\gamma_0, t\big) \mid \langle\eta \mid \gamma_0\rangle \mid^2 A_n\big(\lambda\gamma_0^*, \lambda\eta\big).\end{aligned} \tag{4.51}$$

The expansion of the coherent states $\mid \eta\rangle$ and $\mid \gamma_0\rangle$ in terms of the number state is given by [1]

$$\mid \eta\rangle = e^{-\frac{\lambda}{2}|\eta|^2}\sum_{n=0}^{\infty}\frac{\eta^n\sqrt{\lambda^n}}{\sqrt{n!}} \mid \lambda n\rangle, \tag{4.52}$$

$$\mid \gamma_0\rangle = e^{-\frac{\lambda}{2}|\gamma_0|^2}\sum_{m=0}^{\infty}\frac{\gamma_0^m\sqrt{\lambda^m}}{\sqrt{m!}} \mid \lambda m\rangle, \tag{4.53}$$

so that with the aid of (4.52) and (4.53), we easily get

$$\langle\eta \mid \gamma_0\rangle = \exp\left[-\frac{\lambda}{2} \mid \eta \mid^2 - \frac{\lambda}{2} \mid \gamma_0 \mid^2 + \lambda\eta^*\gamma_0\right]. \tag{4.54}$$

On account of this result, one can be put Eq. (4.51) in the form

$$\begin{aligned}\langle\hat{A}\big(\hat{a}^\dagger, \hat{a}, t\big)\rangle &= \frac{\lambda^2}{\pi}\int d^2\eta d^2\gamma_0 Q\big(\lambda\eta^*, \lambda\gamma_0, t\big) \\ &\times \exp\Big[-\lambda\eta^*\eta - \lambda\gamma_0^*\gamma_0 + \lambda\eta^*\gamma_0 + \lambda\eta\gamma_0^*\Big]A_n\left(\lambda\gamma_0^*, \lambda\eta\right),\end{aligned} \tag{4.55}$$

where $A_n\big(\lambda\gamma_0^*, \lambda\eta\big)$ is the c-number function corresponding to the operator $\hat{A}\big(\hat{a}^\dagger, \hat{a}\big)$ in the normal order. Now on introducing new variables $\xi_1 = \lambda\gamma_0$ and $\xi_2 = \lambda\eta$, we have

$$\begin{aligned}\langle\hat{A}\big(\hat{a}^\dagger, \hat{a}, t\big)\rangle &= \frac{1}{\pi}\int\frac{d^2\xi_1 d^2\xi_2}{\lambda^2}Q\left(\xi_2^*, \xi_1, t\right) \\ &\times \exp\left[-\frac{1}{\lambda}\xi_1^*\xi_1 - \frac{1}{\lambda}\xi_2^*\xi_2 + \frac{1}{\lambda}\xi_1^*\xi_2 + \frac{1}{\lambda}\xi_1\xi_2^*\right]A_n\big(\xi_1^*, \xi_2\big).\end{aligned} \tag{4.56}$$

We now wish to derive the density operator for a pair of superposed two-mode laser light beams. In view of (4.43), the density operator for the first two-mode laser light beam can be written as

$$\hat{\rho}'(t) = \lambda_a \int d^2\gamma_0 Q\Big(\lambda_a \gamma_0^*, \lambda_a \gamma_0 + \frac{\partial}{\partial \gamma_0^*}\Big) \mid \gamma_0\rangle\langle\gamma_0 \mid . \tag{4.57}$$

This expression for the density operator can be put in the form

$$\hat{\rho}'(t) = \lambda_a \int d^2\gamma_0 Q\Big(\lambda_a \gamma_0^*, \lambda_a \gamma_0 + \frac{\partial}{\partial \gamma_0^*}\Big) \hat{D}(\gamma_0)\hat{\rho}_0 \hat{D}(-\gamma_0), \tag{4.58}$$

where $\hat{D}(\gamma_0)$ is the displacement operator and

$$\hat{\rho}_0 = \mid 0\rangle\langle 0 \mid \tag{4.59}$$

represents the density operator for the cavity light at the initial time. We now realize that the density operator for the superposition of the first two-mode light beam and the second one can be expressed as

$$\hat{\rho}(t) = \lambda_b \int d^2\eta Q\Big(\lambda_b \eta^*, \lambda_b \eta + \frac{\partial}{\partial \eta^*}\Big) \hat{D}(\eta)\hat{\rho}'(t) \hat{D}(-\eta), \tag{4.60}$$

so that on account of (4.58), we readily get

$$\hat{\rho}(t) = \lambda_a\lambda_b \int d^2\eta d^2\gamma_0 Q\Big(\lambda_a \gamma_0^*, \lambda_a \gamma_0 + \frac{\partial}{\partial \gamma_0^*}\Big) Q\Big(\lambda_b \eta^*, \lambda_b \eta + \frac{\partial}{\partial \eta^*}\Big)$$
$$\times\, \hat{D}(\eta) \mid \gamma_0\rangle\langle\gamma_0 \mid \hat{D}(-\eta), \tag{4.61}$$

where

$$Q\Big(\lambda_a \gamma_0^*, \lambda_a \gamma_0 + \frac{\partial}{\partial \gamma_0^*}\Big) = \frac{1}{\pi}\sum_{kl} C_{kl}\Big(\lambda_a {\gamma_0}^*\Big)^k \Big(\lambda_a \gamma_0 + \frac{\partial}{\partial \gamma_0^*}\Big)^l, \tag{4.62}$$

$$Q\Big(\lambda_b \eta^*, \lambda_b \eta + \frac{\partial}{\partial \eta^*}\Big) = \frac{1}{\pi}\sum_{mn} C_{mn}\Big(\lambda_b \eta^*\Big)^m \Big(\lambda_b \eta + \frac{\partial}{\partial \eta^*}\Big)^n \tag{4.63}$$

are the Q functions associated with the constituent two-mode light beams. On account of the fact that

$$\hat{D}(\eta) \mid \gamma_0\rangle = \exp\Big[\frac{(\lambda_a + \lambda_b)}{2}\big(\eta\gamma_0^* - \eta^*\gamma_0\big)\Big] \mid \gamma_0 + \eta\rangle, \tag{4.64}$$

the density operator for the superposed two-mode light beams has the form

$$\hat{\rho}(t) = \lambda_a\lambda_b \int d^2\eta d^2\gamma_0 Q\Big(\lambda_a\gamma_0^*, \lambda_a\gamma_0 + \frac{\partial}{\partial\gamma_0^*}\Big) Q\Big(\lambda_b\eta^*, \lambda_b\eta + \frac{\partial}{\partial\eta^*}\Big)$$

$$\times \mid \gamma_0 + \eta\rangle\langle\eta + \gamma_0 \mid . \tag{4.65}$$

4.3 Photon statistics

In this section we wish to analyze the statistical properties of a pair of superposed two-mode laser light beams produced by the system under consideration.

4.3.1 The global mean photon number

We now seek to obtain the mean photon number for the superposed two-mode cavity (output) light beams in the entire frequency interval, using the density operator. The mean photon number can be expressed in terms of the density operator as

$$\bar{n}_s = Tr\left[\hat{\rho}(t)\hat{c}^\dagger\hat{c}\right], \tag{4.66}$$

where $\hat{c}$ represents the annihilation operator for the superposed two-mode laser light beams. Now we want to express in terms of $\hat{a}(t)$ and $\hat{b}(t)$, the annihilation operator $\hat{c}(t)$ for the superposed two-mode laser light beams using the quantum Langevin equation. On account of Eq. (2.68), the equation of evolution of the second two-mode light beam can be written as

$$\frac{d}{dt}\hat{b}(t) = -\frac{k}{2}\hat{b}(t) + \frac{g}{\sqrt{N}}\hat{m}_b, \tag{4.67}$$

where

$$\hat{m}_b = \hat{m}_1' + \hat{m}_2'. \tag{4.68}$$

Upon adding Eqs. (2.68) and (4.67), the equation of evolution for the superposed two-mode cavity light beams has the form

$$\frac{d}{dt}\hat{c}(t) = -\frac{k}{2}\hat{c}(t) + \frac{g}{\sqrt{N}}\hat{m}, \tag{4.69}$$

where

$$\hat{m} = \hat{m}_a + \hat{m}_b \tag{4.70}$$

and

$$\hat{c}(t) = \hat{a}(t) + \hat{b}(t). \tag{4.71}$$

We next wish to determine the global mean photon number for the superposed two-mode laser light beams. Applying the density operator given by (4.65) in (4.66) with the value of (4.71), the global mean photon number for the superposed two-mode laser light beams can be put in the form

$$\bar{n}_s = \lambda_a\lambda_b \int d^2\eta d^2\gamma_0 Q\Big(\lambda_a\gamma_0^*, \lambda_a\gamma_0 + \frac{\partial}{\partial\gamma_0^*}\Big) Q\Big(\lambda_b\eta^*, \lambda_b\eta + \frac{\partial}{\partial\eta^*}\Big)$$
$$\times \Big[\lambda_a^2\gamma_0^*\gamma_0 + \lambda_b^2\eta^*\eta + \lambda_a\lambda_b\gamma_0^*\eta + \lambda_a\lambda_b\gamma_0\eta^*\Big]. \tag{4.72}$$

Now on introducing the variables $\alpha = \lambda_a\gamma_0$ and $\beta = \lambda_b\eta$, Eq. (4.72) can be rewritten as

$$\bar{n}_s = \int \frac{d^2\alpha}{\lambda_a}\frac{d^2\beta}{\lambda_b} Q\Big(\alpha^*, \alpha + \lambda_a\frac{\partial}{\partial\alpha^*}\Big) Q\Big(\beta^*, \beta + \lambda_b\frac{\partial}{\partial\beta^*}\Big)$$
$$\times \Big[\alpha^*\alpha + \beta^*\beta + \alpha^*\beta + \alpha\beta^*\Big], \tag{4.73}$$

from which follows

$$\begin{aligned}
\bar{n}_s = &\int \frac{d^2\alpha}{\lambda_a} Q\Big(\alpha^*, \alpha + \lambda_a\frac{\partial}{\partial\alpha^*}\Big)\alpha^*\alpha \\
&+ \int \frac{d^2\beta}{\lambda_b} Q\Big(\beta^*, \beta + \lambda_b\frac{\partial}{\partial\beta^*}\Big)\beta^*\beta \\
&+ \int \frac{d^2\alpha}{\lambda_a} Q\Big(\alpha^*, \alpha + \lambda_a\frac{\partial}{\partial\alpha^*}\Big)\alpha^* \\
&\times \int \frac{d^2\beta}{\lambda_b} Q\Big(\beta^*, \beta + \lambda_b\frac{\partial}{\partial\beta^*}\Big)\beta \\
&+ \int \frac{d^2\alpha}{\lambda_a} Q\Big(\alpha^*, \alpha + \lambda_a\frac{\partial}{\partial\alpha^*}\Big)\alpha \\
&\times \int \frac{d^2\beta}{\lambda_b} Q\Big(\beta^*, \beta + \lambda_b\frac{\partial}{\partial\beta^*}\Big)\beta^*.
\end{aligned} \tag{4.74}$$

On account of Eq. (4.47), we see that

$$\bar{n} = \langle \hat{a}^{\dagger}\hat{a}\rangle + \langle \hat{b}^{\dagger}\hat{b}\rangle + \langle \hat{a}^{\dagger}\rangle\langle \hat{b}\rangle + \langle \hat{a}\rangle\langle \hat{b}^{\dagger}\rangle, \tag{4.75}$$

in which

$$[\hat{a}, \hat{a}^{\dagger}] = \lambda_a, \tag{4.76}$$

$$[\hat{b}, \hat{b}^{\dagger}] = \lambda_b, \tag{4.77}$$

where λ_a and λ_b are given by Eqs. (4.20) and (4.30), respectively.

We next proceed to calculate the expectation values that appear in Eq. (4.75). Thus employing Eq. (4.56), the global mean photon number of the first two-mode laser light beam can be expressed as

$$\langle \hat{a}^{\dagger}\hat{a}\rangle = \frac{1}{\pi}\int \frac{d^2\xi_1 d^2\xi_2}{\lambda_a^2} Q\left(\xi_2^*, \xi_1, t\right) \times \exp\left[-\frac{1}{\lambda_a}\xi_1^*\xi_1 - \frac{1}{\lambda_a}\xi_2^*\xi_2 + \frac{1}{\lambda_a}\xi_1^*\xi_2 + \frac{1}{\lambda_a}\xi_1\xi_2^*\right]\xi_1^*\xi_2. \tag{4.78}$$

With the aid of Eq. (4.19), we have

$$Q(\xi_2^*, \xi_1) = \frac{\lambda_a}{\pi}\Big[u_a^2 - v_a^2\Big]^{1/2} \exp\Big[-u_a\xi_2^*\xi_1 + \frac{v_a}{2}\left(\xi_1^2 + \xi_2^{*2}\right)\Big]. \tag{4.79}$$

Using this Q function, one can put Eq. (4.78) in the form

$$\langle \hat{a}^{\dagger}\hat{a}\rangle = \frac{1}{\lambda_a}\Big[u_a^2 - v_a^2\Big]^{1/2}\int \frac{d^2\xi_1}{\pi}\frac{d^2\xi_2}{\pi}\exp\Bigg[-\frac{1}{\lambda_a}\xi_1^*\xi_1 - \frac{1}{\lambda_a}\xi_2^*\xi_2 + \frac{1}{\lambda_a}\xi_1^*\xi_2 + \xi_1\left(\Big[\frac{1}{\lambda_a} - u_a\Big]\xi_2^*\right) + \frac{v_a}{2}\xi_1^2 + \frac{v_a}{2}\xi_2^{*2}\Bigg]\xi_1^*\xi_2. \tag{4.80}$$

This can be rewritten as

$$\langle \hat{a}^{\dagger}\hat{a}\rangle = \frac{1}{\lambda_a}\Big[u_a^2 - v_a^2\Big]^{1/2}\int \frac{d^2\xi_2}{\pi}\exp\left[-\frac{1}{\lambda_a}\xi_2^*\xi_2 + \frac{v_a}{2}\xi_2^{*2}\right] \times \frac{d}{da}\int \frac{d^2\xi_1}{\pi}\exp\Bigg[-\frac{1}{\lambda_a}\xi_1^*\xi_1 + \xi_1\left(\Big[\frac{1}{\lambda_a} - u_a\Big]\xi_2^*\right) + \xi_1^*\left(\Big[\frac{1}{\lambda_a} + a\Big]\xi_2\right) + \frac{v_a}{2}\xi_1^2\Bigg]_{|a=0}, \tag{4.81}$$

so that on carrying out the integration over ξ_1, employing the relation given by Eq. (4.13), we readily obtain

$$\langle \hat{a}^\dagger \hat{a}\rangle = \Big[u_a^2 - v_a^2\Big]^{1/2} \int \frac{d^2\xi_2}{\pi} \exp\left[-\frac{1}{\lambda_a}\xi_2^*\xi_2 + \frac{v_a}{2}\xi_2^{*2}\right] \times \frac{d}{da}\exp\Bigg[\xi_2^*\xi_2\Big(\frac{1}{\lambda_a} - a u_a\lambda_a + a - u_a\Big) + \frac{v_a}{2}\left(\xi_2^2 + \xi_2^2 a^2\lambda_a^2 + 2a\lambda_a\xi_2^2\right)\Bigg]_{|a=0}. \tag{4.82}$$

Upon performing the differentiation and applying the condition $a = 0$, we get

$$\langle \hat{a}^\dagger \hat{a}\rangle = \Big[u_a^2 - v_a^2\Big]^{1/2} \int \frac{d^2\xi_2}{\pi} \exp\left[-u_a\xi_2^*\xi_2 + \frac{v_a}{2}\left(\xi_2^{*2} + \xi_2^2\right)\right] \times \left[\Big(1 - u_a\lambda_a\Big)\xi_2^*\xi_2 + v_a\lambda_a\xi_2^2\right]. \tag{4.83}$$

This can be put in the form

$$\langle \hat{a}^\dagger \hat{a}\rangle = \Big[u_a^2 - v_a^2\Big]^{1/2} \Bigg\{\Big(1 - u_a\lambda_a\Big) \int \frac{d^2\beta_2}{\pi} \exp\left[-u_a\xi_2^*\xi_2 + \frac{v_a}{2}\left(\xi_2^{*2} + \xi_2^2\right)\right]\xi_2^*\xi_2 + v_a\lambda_a \int \frac{d^2\xi_2}{\pi} \exp\left[-u_a\xi_2^*\xi_2 + \frac{v_a}{2}\left(\xi_2^{*2} + \xi_2^2\right)\right]\xi_2^2\Bigg\}. \tag{4.84}$$

In order to carry out the integration, one can rewrite Eq. (4.84) as

$$\langle \hat{a}^\dagger \hat{a}\rangle = \big[u_a^2 - v_a^2\big]^{1/2} \Bigg\{\Big(1 - u_a\lambda_a\Big)\frac{d^2}{dadb} \int \frac{d^2\xi_2}{\pi} \exp\Bigg[- u_a\xi_2^*\xi_2 + a\xi_2 + b\xi_2^* + \frac{v_a}{2}\left(\xi_2^{*2} + \xi_2^2\right)\Bigg]_{|a=b=0} + v_a\lambda_a\frac{d^2}{dc^2} \int \frac{d^2\xi_2}{\pi} \exp\left[-u_a\xi_2^*\xi_2 + c\xi_2 + \frac{v_a}{2}\left(\xi_2^{*2} + \xi_2^2\right)\right]_{|c=0}\Bigg\}, \tag{4.85}$$

so on carrying out the integration over ξ_2, we find

$$\langle \hat{a}^\dagger \hat{a}\rangle = \Big[1 - u_a\lambda_a\Big]\frac{d^2}{dadb}\exp\left[\frac{u_a}{u_a^2 - v_a^2}ab + \frac{v_a}{2\left(u_a^2 - v_a^2\right)}\left(a^2 + b^2\right)\right]_{|a=b=0} + v_a\lambda_a\frac{d^2}{dc^2}\exp\left[\frac{v_a}{2\left(u_a^2 - v_a^2\right)}c^2\right]_{|c=0}. \tag{4.86}$$

Upon performing the differentiation and applying the condition $a = b = c = 0$, we have

$$\langle \hat{a}^\dagger \hat{a}\rangle = \Big\{1 - u_a\lambda_a\Big\}\Big\{\frac{u_a}{u_a^2 - v_a^2}\Big\} + v_a\lambda_a\Big\{\frac{v_a}{u_a^2 - v_a^2}\Big\}$$

$$= \frac{u_a}{u_a^2 - v_a^2} - \lambda_a. \tag{4.87}$$

On account of Eqs. (4.21) and (4.22), we obtain

$$\langle \hat{a}^\dagger \hat{a} \rangle = p_a - \lambda_a. \tag{4.88}$$

With the aid of Eqs. (4.20) and (4.23), one can easily find

$$\langle \hat{a}^\dagger \hat{a} \rangle = \frac{\gamma_c}{k} \left\{ \langle \hat{N}_a \rangle_a + \langle \hat{N}_b \rangle_a \right\}. \tag{4.89}$$

Following a similar procedure, it is not difficult to verify that

$$\langle \hat{b}^\dagger \hat{b} \rangle = \frac{u_b}{u_b^2 - v_b^2} - \lambda_b$$

$$= p_b - \lambda_b. \tag{4.90}$$

In view of Eqs. (4.30) and (4.33), one can write the global mean photon number of the other two-mode laser light beam as

$$\langle \hat{b}^\dagger \hat{b} \rangle = \frac{\gamma_c}{k} \left\{ \langle \hat{N}_a \rangle_b + \langle \hat{N}_b \rangle_b \right\}. \tag{4.91}$$

In addition, in view of Eq. (4.56) one can write

$$\langle \hat{a} \rangle = \frac{1}{\pi} \int \frac{d^2\xi_1 d^2\xi_2}{\lambda_a^2} Q\left(\xi_2^*, \xi_1, t\right)$$

$$\times \exp\left[-\frac{1}{\lambda_a}\xi_1^*\xi_1 - \frac{1}{\lambda_a}\xi_2^*\xi_2 + \frac{1}{\lambda_a}\xi_1^*\xi_2 + \frac{1}{\lambda_a}\xi_1\xi_2^* \right] \xi_2. \tag{4.92}$$

On account of Eq. (4.79), we have

$$\langle \hat{a} \rangle = \frac{1}{\lambda_a}\left[u_a^2 - v_a^2\right]^{1/2} \int \frac{d^2\xi_1}{\pi}\frac{d^2\xi_2}{\pi} \exp\left[-\frac{1}{\lambda_a}\xi_1^*\xi_1 - \frac{1}{\lambda_a}\xi_2^*\xi_2 + \frac{1}{\lambda_a}\xi_1^*\xi_2 \right.$$

$$\left. + \xi_1 \left(\left[\frac{1}{\lambda_a} - u_a\right]\xi_2^* \right) + \frac{v_a}{2}\xi_1^2 + \frac{v_a}{2}\xi_2^{*2} \right] \xi_2. \tag{4.93}$$

This can be rewritten as

$$\langle \hat{a} \rangle = \frac{1}{\lambda_a}\left[u_a^2 - v_a^2\right]^{1/2} \int \frac{d^2\xi_1}{\pi} \exp\left[-\frac{1}{\lambda_a}\xi_1^*\xi_1 + \frac{v_a}{2}\xi_1^2 \right]$$

$$\times \frac{d}{db} \int \frac{d^2\xi_2}{\pi} \exp\left[-\frac{1}{\lambda_a}\xi_2^*\xi_2 + \xi_2\left(\frac{\xi_1^*}{\lambda_a} + b\right) + \xi_2^*\left(\left[\frac{1}{\lambda_a} - u_a\right]\xi_1\right) + \frac{v_a}{2}\xi_2^{*2}\right]_{|b=0}, \tag{4.94}$$

so that on carrying out the integration over ξ_2, employing the relation given by Eq. (4.13), we readily get

$$\langle \hat{a} \rangle = \left[u_a^2 - v_a^2\right]^{1/2} \int \frac{d^2\xi_1}{\pi} \exp\left[-\frac{1}{\lambda_a}\xi_1^*\xi_1 + \frac{v_a}{2}\xi_1^2\right] \times \frac{d}{db} \exp\left[\xi_1^*\xi_1\Big(\frac{1}{\lambda_a} - u_a\Big) + b\xi_1\Big(1 - u_a\lambda_a\Big) + \frac{v_1}{2}\Big(\xi_1^{*2} + b^2\lambda_a^2 + 2b\lambda_a\xi_1^*\Big)\right]_{|b=0}. \tag{4.95}$$

Upon performing the differentiation and applying the condition $b = 0$, we obtain

$$\langle \hat{a} \rangle = \left[u_a^2 - v_a^2\right]^{1/2} \int \frac{d^2\xi_1}{\pi} \exp\left[-u_a\xi_1^*\xi_1 + \frac{v_a}{2}\left(\xi_1^{*2} + \xi_1^2\right)\right] \times \left[\Big(1 - u_a\lambda_a\Big)\xi_1 + v_1\lambda_a\xi_1^*\right]. \tag{4.96}$$

This can be put in the form

$$\langle \hat{a} \rangle = \left[u_a^2 - v_a^2\right]^{1/2} \Bigg\{ \Big(1 - u_a\lambda_a\Big)\frac{d}{dm} \int \frac{d^2\xi_1}{\pi} \exp\left[-u_a\xi_1^*\xi_1 + m\xi_1 + \frac{v_a}{2}\Big(\xi_1^{*2} + \xi_1^2\Big)\right]_{|m=0} + v_a\lambda_a \frac{d^2}{dc^2} \int \frac{d^2\xi_1}{\pi} \exp\left[-u_a\xi_1^*\xi_1 + c\xi_1 + \frac{v_a}{2}\Big(\xi_1^{*2} + \xi_1^2\Big)\right]_{|c=0} \Bigg\}, \tag{4.97}$$

so on carrying out the integration over ξ_1, we get

$$\langle \hat{a} \rangle = \Big[1 - u_a\lambda_a\Big] \frac{d}{dm} \exp\left[\frac{v_a}{2\,(u_a^2 - v_a^2)} m^2\right]_{|m=0} + v_a\lambda_a \frac{d}{dc} \exp\left[\frac{v_a}{2\,(u_a^2 - v_a^2)} c^2\right]_{|c=0}. \tag{4.98}$$

Hence performing the differentiation and applying the condition $m = c = 0$, we arrive at

$$\langle \hat{a} \rangle = 0. \tag{4.99}$$

We also find

$$\langle \hat{b} \rangle = 0. \tag{4.100}$$

Thus upon substituting Eqs. (4.89), (4.91), (4.99), and (4.100) into Eq. (4.75), the global mean photon number for the superposed two-mode light beams turns out to be

$$\bar{n}_s = \frac{\gamma_c}{k}\bigg\{\langle\hat{N}_a\rangle_a + \langle\hat{N}_b\rangle_a\bigg\} + \frac{\gamma_c}{k}\bigg\{\langle\hat{N}_a\rangle_b + \langle\hat{N}_b\rangle_b\bigg\}. \tag{4.101}$$

We see that the global mean photon number of the superposed two-mode laser light beams is the sum of the global mean photon numbers of the constituent two-mode laser light beams. With the aid of Eqs. (4.25),(4.26), (4.35), and (4.36), one can put (4.101) in the form

$$\bar{n}_s = \frac{\gamma_c}{k}\bigg\{N_1 + N_2\bigg\}\left\{\frac{2\Omega^2}{(\gamma_c+\gamma)^2 + 3\Omega^2}\right\}. \tag{4.102}$$

Finally, we note that the global mean photon number of the superposed two-mode laser light beams takes for $\Omega \gg \gamma_c$ and $\Omega \gg \gamma$ the form

$$\bar{n}_s = \frac{2}{3}\frac{\gamma_c}{k}\bigg\{N_1 + N_2\bigg\}. \tag{4.103}$$

We observe from the plots in Fig. 4.2 that the presence of the spontaneous emission process leads to a decrease in the global mean photon number of the superposed two-mode cavity light beams. Moreover, for $N_1 = N_2 = N$, we see that

$$\bar{n}_s = \frac{4}{3}\frac{\gamma_c}{k}N, \tag{4.104}$$

which indicates that the global mean photon number of the superposed two-mode laser light beams is twice that of one of the two-mode laser light beams described by Eq. (3.6). In addition, for $\gamma = 0$, Eq. (4.102) reduces to

$$\bar{n}_s = \frac{\gamma_c}{k}\bigg\{N_1 + N_2\bigg\}\left\{\frac{2\Omega^2}{\gamma_c^2 + 3\Omega^2}\right\}. \tag{4.105}$$

We next wish to determine the global mean photon number of the superposed two-mode output laser light beam. On account of Eq. (3.8), the output annihilation

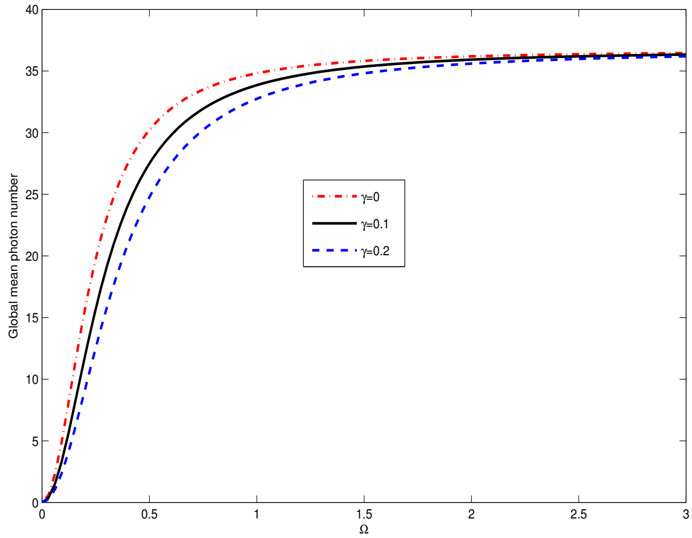

Figure 4.2: Plots of the global mean photon number [Eq. (4.102)] versus Ω at steady state for $\gamma_c = 0.4$, $k = 0.8$, $N_1 = 50$, $N_2 = 60$, and different values of γ.

operator $\hat{b}_{out}(t)$ for the second two-mode light can be written as

$$\hat{b}_{out}(t) = \sqrt{k}\hat{b}(t). \tag{4.106}$$

Now in view of Eqs. (3.8), (4.99), (4.100) and (4.106) along with (4.75), the global mean photon number of the superposed two-mode output laser light beams can be expressed at steady state as

$$\bar{n}_{out} = k\left\{\langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{b}^\dagger\hat{b}\rangle\right\},$$
$$= k\left\{\frac{\gamma_c}{k}\left[N_1 + N_2\right]\left[\frac{2\Omega^2}{(\gamma_c+\gamma)^2 + 3\Omega^2}\right]\right\}, \tag{4.107}$$

which indicates that the global mean photon number of the superposed two-mode output laser light beams is just k times that of the superposed two-mode cavity light beams.

4.3.2 Local mean photon number

Here we wish to calculate the mean photon number for the superposed two-mode laser light beams in a given frequency interval. To this end, we recall that the power spectrum for the superposed two-mode cavity light with central common frequency ω_0 is defined by

$$P_s(\omega) = \frac{1}{\pi} Re \int_0^{+\infty} d\tau e^{i(\omega-\omega_0)\tau} \langle \hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle_{ss}. \tag{4.108}$$

We now seek to obtain the two-time correlation function that appear in Eq. (4.108). The solution of Eq. (4.69) can be written as

$$\hat{c}(t+\tau) = \hat{c}(t)e^{-k\tau/2} + \frac{g}{\sqrt{N}} e^{-k\tau/2} \int_0^\tau d\tau' e^{k\tau'/2} \hat{m}(t+\tau'). \tag{4.109}$$

We next proceed to determine the explicit form of $\hat{m}(t+\tau')$ that appear in Eq. (4.109). In view of Eq. (3.12), one can write the equation of evolution of operator $\hat{m}_b(t)$ for the second two-mode laser light beam as

$$\frac{d}{dt}\hat{m}_b(t) = -\frac{1}{2}\nu\hat{m}_b(t) - \frac{1}{2}\nu\hat{m}'_1(t) + \hat{F}_b(t), \tag{4.110}$$

in which $\hat{F}_b(t)$ is a noise operator with zero mean. Adding Eqs. (3.12) and (4.110), we have

$$\frac{d}{dt}\hat{m}(t) = -\frac{1}{2}\nu\hat{m}(t) - \frac{1}{2}\nu\Big(\hat{m}_1(t) + \hat{m}'_1(t)\Big) + \Big(\hat{F}_a(t) + \hat{F}_b(t)\Big). \tag{4.111}$$

The solution of Eq. (4.111) can be expressed as

$$\hat{m}(t+\tau) = \hat{m}(t)e^{-\nu\tau/2} + e^{-\nu\tau/2} \int_0^\tau e^{\nu\tau'/2} \Big\{ -\frac{1}{2}\nu\Big(\hat{m}_1(t+\tau') + \hat{m}'_1(t+\tau')\Big) \\ \Big(\hat{F}_a(t+\tau') + \hat{F}_b(t+\tau')\Big)\Big\} d\tau'. \tag{4.112}$$

With the aid of Eq. (3.14), we find

$$\frac{d}{dt}\hat{m}'_1(t) = -\nu\hat{m}'_1(t) + \hat{F}'_1(t), \tag{4.113}$$

in which $\hat{F}'_1(t)$ is a noise operator with vanishing mean. Upon adding Eqs. (3.14) and (4.113), we get

$$\frac{d}{dt}\Big(\hat{m}_1(t) + \hat{m}'_1(t)\Big) = -\nu\Big(\hat{m}_1(t) + \hat{m}'_1(t)\Big) + \Big(\hat{F}_1(t) + \hat{F}'_1(t)\Big). \tag{4.114}$$

Now applying the large-time approximation scheme to Eq. (4.114), we easily obtain

$$\Big(\hat{m}_1(t) + \hat{m}'_1(t)\Big) = \frac{1}{\nu}\Big(\hat{F}_1(t) + \hat{F}'_1(t)\Big). \tag{4.115}$$

With the aid of this result, one can put Eq. (4.112) in the form

$$\hat{m}(t+\tau) = \hat{m}(t)e^{-\nu\tau/2} + e^{-\nu\tau/2}\int_0^\tau e^{\nu\tau'/2}\Big\{-\frac{1}{2}\Big(\hat{F}_1(t+\tau') + \hat{F}'_1(t+\tau')\Big) + \Big(\hat{F}_a(t+\tau') + \hat{F}_b(t+\tau')\Big)\Big\}d\tau'. \tag{4.116}$$

Now combination of Eqs. (4.109) and (4.116) yields

$$\hat{c}(t+\tau) = \hat{c}(t)e^{-k\tau/2} + \frac{g}{\sqrt{N}}e^{-k\tau/2}\Big\{\hat{m}(t)\int_0^\tau d\tau' e^{(k-\nu)\tau'/2} + \int_0^\tau d\tau' e^{(k-\nu)\tau'/2}\int_0^{\tau'} d\tau'' e^{\nu\tau''/2}\Big[-\frac{1}{2}\Big(\hat{F}_1(t+\tau'') + \hat{F}'_1(t+\tau'')\Big) + \Big(\hat{F}_a(t+\tau'') + \hat{F}_b(t+\tau'')\Big)\Big]\Big\}. \tag{4.117}$$

On multiplying both sides of Eq. (4.117) on the left by $\hat{c}^\dagger(t)$ and taking the expectation value of the resulting equation, one can readily get

$$\langle\hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle = \langle\hat{c}^\dagger(t)\hat{c}(t)\rangle e^{-k\tau/2} + \frac{g}{\sqrt{N}}e^{-k\tau/2}\Big\{\langle\hat{c}^\dagger(t)\hat{m}(t)\rangle\int_0^\tau d\tau' e^{(k-\nu)\tau'/2} + \int_0^\tau d\tau' e^{(k-\nu)\tau'/2}\int_0^{\tau'} d\tau'' e^{\nu\tau''/2}\Big[-\frac{1}{2}\Big(\langle\hat{c}^\dagger(t)\hat{F}_1(t+\tau'')\rangle + \langle\hat{c}^\dagger(t)\hat{F}'_1(t+\tau'')\rangle\Big) + \Big(\langle\hat{c}^\dagger(t)\hat{F}_a(t+\tau'')\rangle + \langle\hat{c}^\dagger(t)\hat{F}_b(t+\tau'')\rangle\Big)\Big]\Big\}. \tag{4.118}$$

Since the cavity mode operator and the noise operator of the atomic are not correlated, we see that

$$\langle\hat{c}^\dagger(t)\hat{F}_1(t+\tau'')\rangle = \langle\hat{c}^\dagger(t)\hat{F}'_1(t+\tau'')\rangle = 0, \tag{4.119}$$

$$\langle\hat{c}^\dagger(t)\hat{F}_a(t+\tau'')\rangle = \langle\hat{c}^\dagger(t)\hat{F}_b(t+\tau'')\rangle = 0. \tag{4.120}$$

It then follows that

$$\langle\hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle = \langle\hat{c}^\dagger(t)\hat{c}(t)\rangle e^{-k\tau/2} + \frac{g}{\sqrt{N}}e^{-k\tau/2}\Big\{\langle\hat{c}^\dagger(t)\hat{m}(t)\rangle\int_0^\tau d\tau' e^{(k-\nu)\tau'/2}\Big\}. \tag{4.121}$$

Moreover, applying the large-time approximation scheme to Eq. (4.69), we have

$$\hat{m}(t) = \frac{k\sqrt{N}}{2g}\hat{c}(t). \tag{4.122}$$

With the aid of this result, we can rewrite Eq. (4.121) in the form

$$\langle \hat{c}^{\dagger}(t)\hat{c}(t+\tau)\rangle = \langle \hat{c}^{\dagger}(t)\hat{c}(t)\rangle\Big\{e^{-k\tau/2} + \frac{1}{2}ke^{-k\tau/2}\int_0^{\tau} d\tau' e^{(k-\nu)\tau'/2}\Big\}, \tag{4.123}$$

so on carrying out the integration over τ', we easily obtain

$$\langle \hat{c}^{\dagger}(t)\hat{c}(t+\tau)\rangle = \langle \hat{c}^{\dagger}(t)\hat{c}(t)\rangle\left\{\frac{k}{k-\nu}e^{-\nu\tau/2} - \frac{\nu}{k-\nu}e^{-k\tau/2}\right\}. \tag{4.124}$$

Now upon substituting (4.124) into Eq. (4.108), the power spectrum of the cavity light mode takes the form

$$P_s(\omega) = \langle \hat{c}^{\dagger}(t)\hat{c}(t)\rangle\frac{1}{\pi}Re\Big\{\Big[\frac{k}{k-\nu}\int_0^{\infty} d\tau e^{-\Big(\nu/2-i(\omega-\omega_0)\Big)\tau}$$

$$- \frac{\nu}{k-\nu}\int_0^{\infty} d\tau e^{-\Big(k/2-i(\omega-\omega_0)\Big)\tau}\Big]\Big\}. \tag{4.125}$$

So on carrying out the integration over τ, the power spectrum of the superposed two-mode laser light beams turns out to be

$$P_s(\omega) = \bar{n}_s\Big\{\left[\frac{k}{k-\nu}\right]\left[\frac{\nu/2\pi}{(\omega-\omega_0)^2+(\nu/2)^2}\right]-\left[\frac{\nu}{k-\nu}\right]\left[\frac{k/2\pi}{(\omega-\omega_0)^2+(k/2)^2}\right]\Big\} \tag{4.126}$$

Furthermore, we seek to calculate the mean photon number in a given frequency interval. Upon integrating both sides of Eq. (4.126) over ω, we readily get

$$\int_{-\infty}^{+\infty} P_s(\omega)d\omega = \bar{n}_s, \tag{4.127}$$

in which $\bar{n}_s$ is the global mean photon number for the superposed two-mode laser light beams at steady state. On the basis of Eq. (4.127), we observe that $P_s(\omega)d\omega$ represents the steady-state mean photon number of the superposed two-mode laser light beams in the interval between ω and $\omega + d\omega$ [1]. We then realize that the mean

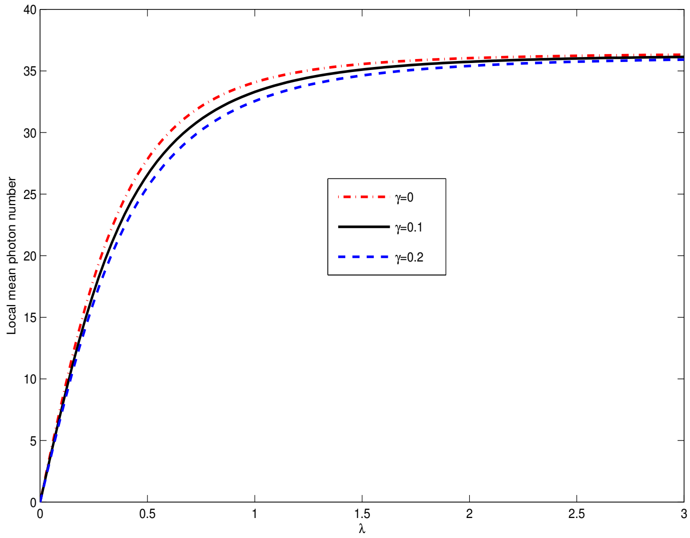

Figure 4.3: Plots of the local mean photon number [Eq. (4.131)] versus λ for $\gamma_c = 0.4$, $k = 0.8$, $\Omega = 3$, $N_1 = 50$, $N_2 = 60$, and different values of γ.

photon number in the interval between $\omega' = -\lambda$ and $\omega' = +\lambda$ can be expressed as [1]

$$\bar{n}_{s\pm\lambda} = \int_{-\lambda}^{+\lambda} P_s(\omega')d\omega', \tag{4.128}$$

in which $\omega' = \omega - \omega_0$. Thus on introducing (4.126) into Eq. (4.128), we find

$$\bar{n}_{s\pm\lambda} = \bar{n}_s \bigg\{ \left[\frac{k}{k-\nu}\right] \int_{-\lambda}^{+\lambda} \left[\frac{\nu/2\pi}{\omega'^2 + (\nu/2)^2}\right] d\omega' - \left[\frac{\nu}{k-\nu}\right] \int_{-\lambda}^{+\lambda} \left[\frac{k/2\pi}{\omega'^2 + (k/2)^2}\right] d\omega' \bigg\}, \tag{4.129}$$

so on carrying out the integration over ω', applying the relation described by Eq. (3.30), we arrive at

$$\bar{n}_{s\pm\lambda} = \bar{n}_s \bigg\{ \left[\frac{2k/\pi}{k-\nu}\right] \tan^{-1}\left(\frac{2\lambda}{\nu}\right) - \left[\frac{2\nu/\pi}{k-\nu}\right] \tan^{-1}\left(\frac{2\lambda}{k}\right) \bigg\}. \tag{4.130}$$

in which ν is given by (2.73). Moreover, with the aid of Eqs. (4.94), one can put Eq. (4.130) in the form

$$\bar{n}_{s\pm\lambda} = \frac{\gamma_c}{k} \bigg\{ N_1 + N_2 \bigg\} \bigg\{ \frac{2\Omega^2}{(\gamma_c+\gamma)^2 + 3\Omega^2} \bigg\} \bigg\{ \left[\frac{2k/\pi}{k-\nu}\right] \tan^{-1}\left(\frac{2\lambda}{\nu}\right)$$

$$-\left[\frac{2\nu/\pi}{k-\nu}\right]\tan^{-1}\left(\frac{2\lambda}{k}\right)\Bigg\}. \tag{4.131}$$

One can readily observe from the plots in Fig. 4.3 that the local mean photon number of the superposed two-mode laser light beams increases with increasing λ. We also see that the local mean photon number approaches the global mean photon number as λ increases.

4.3.3 The global variance of the photon number

We seek to calculate the global variance of the photon number for the superposed two-mode laser light beams. The variance of the photon number for the superposed two-mode laser light beams is defined by

$$(\Delta n)_s^2 = \langle\left(\hat{c}^\dagger\hat{c}\right)^2\rangle - \langle\hat{c}^\dagger\hat{c}\rangle^2. \tag{4.132}$$

We can put an arbitrary function of $\hat{c}$ and $\hat{c}^\dagger$ in the normal order by making use of the commutation relation [1]

$$\left[\hat{c},\hat{c}^\dagger\right] = \lambda_a + \lambda_b, \tag{4.133}$$

which holds for the superposed two-mode laser light beams. Applying (4.133), the global variance of the photon number can be expressed as

$$(\Delta n)_s^2 = \langle\hat{c}^{\dagger 2}\hat{c}^2\rangle + (\lambda_a+\lambda_b)\langle\hat{c}^\dagger\hat{c}\rangle - \langle\hat{c}^\dagger\hat{c}\rangle^2. \tag{4.134}$$

Now the expectation value of $\hat{c}^{\dagger 2}\hat{c}^2$ is expressible using the density operator as

$$\langle\hat{c}^{\dagger 2}\hat{c}^2\rangle = Tr\left[\hat{\rho}(t)\hat{c}^{\dagger 2}\hat{c}^2\right]. \tag{4.135}$$

Thus employing the density operator described by (4.65), we readily get

$$\langle\hat{c}^{\dagger 2}\hat{c}^2\rangle = \lambda_a\lambda_b\int d^2\gamma_0 d^2\eta Q\Big(\lambda_a\gamma_0^*,\lambda_a\gamma_0+\frac{\partial}{\partial\gamma_0^*}\Big)Q\Big(\lambda_b\eta^*,\lambda_b\eta+\frac{\partial}{\partial\eta^*}\Big)$$
$$\times\Big[\lambda_a^4\gamma_0^{*2}\gamma_0^2+\lambda_a^2\lambda_b^2\gamma_0^{*2}\eta^2+\lambda_a^2\lambda_b^2\gamma_0^2\eta^{*2}+2\lambda_a^3\lambda_b\gamma_0^{*2}\gamma_0\eta+2\lambda_a\lambda_2^3\gamma_0\eta^{*2}\eta$$

$$+ 2\lambda_a^3\lambda_b\gamma_0^*\gamma_0^2\eta^* + 2\lambda_a\lambda_b^3\gamma_0^*\eta^*\eta^2 + 4\lambda_a^2\lambda_b^2\gamma_0^*\gamma_0\eta^*\gamma_0 + \lambda_b^4\eta^{*2}\eta^2\Big]. \tag{4.136}$$

On introducing the variables $\alpha = \lambda_a\gamma_0$ and $\beta = \lambda_b\eta$, one can put Eq. (4.136) in the form

$$\langle\hat{c}^{\dagger 2}\hat{c}^2\rangle = \int \frac{d^2\alpha}{\lambda_a}\frac{d^2\beta}{\lambda_b} Q\Big(\alpha^*, \alpha + \lambda_a\frac{\partial}{\partial\alpha^*}\Big) Q\Big(\beta^*, \beta + \lambda_b\frac{\partial}{\partial\beta^*}\Big)$$
$$\times\Big[\alpha^{*2}\alpha^2 + \alpha^{*2}\beta^2 + \alpha^2\beta^{*2} + 2\alpha^{*2}\alpha\beta + 2\alpha\beta^{*2}\beta$$
$$+ 2\alpha^*\alpha^2\beta^* + 2\alpha^*\beta^*\beta^2 + 4\alpha^*\alpha\beta^*\beta + \beta^{*2}\beta^2\Big]. \tag{4.137}$$

This can be expressed as

$$\langle\hat{c}^{\dagger 2}\hat{c}^2\rangle = \int \frac{d^2\alpha}{\lambda_a} Q\Big(\alpha^*, \alpha + \lambda_a\frac{\partial}{\partial\alpha^*}\Big)\alpha^{*2}\alpha^2$$
$$+ \int \frac{d^2\beta}{\lambda_b} Q\Big(\beta^*, \beta + \lambda_b\frac{\partial}{\partial\beta^*}\Big)\beta^{*2}\beta^2$$
$$+\left\{\int \frac{d^2\alpha}{\lambda_a} Q\Big(\alpha^*, \alpha + \lambda_a\frac{\partial}{\partial\alpha^*}\Big)\alpha^{*2}\right\} \times \left\{\int \frac{d^2\beta}{\lambda_b} Q\Big(\beta^*, \beta + \lambda_b\frac{\partial}{\partial\beta^*}\Big)\beta^2\right\}$$
$$+\left\{\int \frac{d^2\alpha}{\lambda_a} Q\Big(\alpha^*, \alpha + \lambda_a\frac{\partial}{\partial\alpha^*}\Big)\alpha^2\right\} \times \left\{\int \frac{d^2\beta}{\lambda_b} Q\Big(\beta^*, \beta + \lambda_b\frac{\partial}{\partial\beta^*}\Big)\beta^{*2}\right\}$$
$$+2\left\{\int \frac{d^2\alpha}{\lambda_a} Q\Big(\alpha^*, \alpha + \lambda_a\frac{\partial}{\partial\alpha^*}\Big)\alpha^{*2}\alpha\right\} \times \left\{\int \frac{d^2\beta}{\lambda_b} Q\Big(\beta^*, \beta + \lambda_b\frac{\partial}{\partial\beta^*}\Big)\beta\right\}$$
$$+2\left\{\int \frac{d^2\alpha}{\lambda_a} Q\Big(\alpha^*, \alpha + \lambda_a\frac{\partial}{\partial\alpha^*}\Big)\alpha\right\} \times \left\{\int \frac{d^2\beta}{\lambda_b} Q\Big(\beta^*, \beta + \lambda_b\frac{\partial}{\partial\beta^*}\Big)\beta^{*2}\beta\right\}$$
$$+2\left\{\int \frac{d^2\alpha}{\lambda_a} Q\Big(\alpha^*, \alpha + \lambda_a\frac{\partial}{\partial\alpha^*}\Big)\alpha^*\alpha^2\right\} \times \left\{\int \frac{d^2\beta}{\lambda_b} Q\Big(\beta^*, \beta + \lambda_b\frac{\partial}{\partial\beta^*}\Big)\beta^*\right\}$$
$$+2\left\{\int \frac{d^2\alpha}{\lambda_a} Q\Big(\alpha^*, \alpha + \lambda_a\frac{\partial}{\partial\alpha^*}\Big)\alpha^*\right\} \times \left\{\int \frac{d^2\beta}{\lambda_b} Q\Big(\beta^*, \beta + \lambda_b\frac{\partial}{\partial\beta^*}\Big)\beta^*\beta^2\right\}$$
$$+ 4\left\{\int \frac{d^2\alpha}{\lambda_a} Q\Big(\alpha^*, \alpha + \lambda_a\frac{\partial}{\partial\alpha^*}\Big)\alpha^*\alpha\right\} \times \left\{\int \frac{d^2\beta}{\lambda_b} Q\Big(\beta^*, \beta + \lambda_b\frac{\partial}{\partial\beta^*}\Big)\beta^*\beta\right\}. \tag{4.138}$$

Now on the basis of (4.47), we can write Eq. (4.138) as

$$\langle\hat{c}^{\dagger 2}\hat{c}^2\rangle = \langle\hat{a}^{\dagger 2}\hat{a}^2\rangle + \langle\hat{b}^{\dagger 2}\hat{b}^2\rangle + \langle\hat{a}^{\dagger 2}\rangle\langle\hat{b}^2\rangle + \langle\hat{a}^2\rangle\langle\hat{b}^{\dagger 2}\rangle + 2\langle\hat{a}^{\dagger 2}\hat{a}\rangle\langle\hat{b}\rangle$$
$$+2\langle\hat{a}\rangle\langle\hat{b}^{\dagger 2}\hat{b}\rangle + 2\langle\hat{a}^\dagger\hat{a}^2\rangle\langle\hat{b}^\dagger\rangle + 2\langle\hat{a}^\dagger\rangle\langle\hat{b}^\dagger\hat{b}^2\rangle + 4\langle\hat{a}^\dagger\hat{a}\rangle\langle\hat{b}^\dagger\hat{b}\rangle. \tag{4.139}$$

On account of Eqs. (4.99) and (4.100), we can put Eq. (4.139) in the form

$$\langle \hat{c}^{\dagger 2}\hat{c}^2\rangle = \langle \hat{a}^{\dagger 2}\hat{a}^2\rangle + \langle \hat{a}^2\rangle\langle \hat{b}^{\dagger 2}\rangle + \langle \hat{a}^{\dagger 2}\rangle\langle \hat{b}^2\rangle + 4\langle \hat{a}^\dagger\hat{a}\rangle\langle \hat{b}^\dagger\hat{b}\rangle + \langle \hat{b}^{\dagger 2}\hat{b}^2\rangle. \tag{4.140}$$

In view of Eqs. (4.75) and (4.140) together with (4.134), we arrive at

$$(\Delta n)_s^2 = \langle \hat{a}^{\dagger 2}\hat{a}^2\rangle + \lambda_a\langle \hat{a}^\dagger\hat{a}\rangle - \langle \hat{a}^\dagger\hat{a}\rangle^2 + \langle \hat{b}^{\dagger 2}\hat{b}^2\rangle + \lambda_b\langle \hat{b}^\dagger\hat{b}\rangle - \langle \hat{b}^\dagger\hat{b}\rangle^2$$
$$+\langle \hat{a}^{\dagger 2}\rangle\langle \hat{b}^2\rangle + \langle \hat{a}^2\rangle\langle \hat{b}^{\dagger 2}\rangle + 2\langle \hat{a}^\dagger\hat{a}\rangle\langle \hat{b}^\dagger\hat{b}\rangle + \lambda_a\langle \hat{b}^\dagger\hat{b}\rangle + \lambda_b\langle \hat{a}^\dagger\hat{a}\rangle. \tag{4.141}$$

Furthermore, in view of the fact that $\hat{a}$ and $\hat{b}$ are Gaussian variables with zero mean, we see that

$$\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle = 2\langle \hat{a}^\dagger\hat{a}\rangle^2 + \langle \hat{a}^{\dagger 2}\rangle\langle \hat{a}^2\rangle \tag{4.142}$$

and

$$\langle \hat{b}^{\dagger 2}\hat{b}^2\rangle = 2\langle \hat{b}^\dagger\hat{b}\rangle^2 + \langle \hat{b}^{\dagger 2}\rangle\langle \hat{b}^2\rangle. \tag{4.143}$$

On account of Eqs. (4.142) and (4.143), one can write Eq. (4.141) as

$$(\Delta n)_s^2 = \langle \hat{a}^\dagger\hat{a}\rangle^2 + \lambda_a\langle \hat{a}^\dagger\hat{a}\rangle + \langle \hat{a}^{\dagger 2}\rangle\langle \hat{a}^2\rangle + \langle \hat{b}^\dagger\hat{b}\rangle^2 + \lambda_b\langle \hat{b}^\dagger\hat{b}\rangle + \langle \hat{b}^{\dagger 2}\rangle\langle \hat{b}^2\rangle$$
$$+\langle \hat{a}^2\rangle\langle \hat{b}^{\dagger 2}\rangle + \langle \hat{a}^{\dagger 2}\rangle\langle \hat{b}^2\rangle + 2\langle \hat{a}^\dagger\hat{a}\rangle\langle \hat{b}^\dagger\hat{b}\rangle + \lambda_a\langle \hat{b}^\dagger\hat{b}\rangle + \lambda_b\langle \hat{a}^\dagger\hat{a}\rangle. \tag{4.144}$$

With the aid of Eq. (4.56), the expectation value of $\hat{a}^2$ can express as

$$\langle \hat{a}^2\rangle = \frac{1}{\pi}\int \frac{d^2\xi_1 d^2\xi_2}{\lambda_a^2} Q(\xi_2^*, \xi_1)\exp\Big[-\frac{1}{\lambda_a}\xi_1^*\xi_1 - \frac{1}{\lambda_a}\xi_2^*\xi_2 + \frac{1}{\lambda_a}\xi_1^*\xi_2 + \frac{1}{\lambda_a}\xi_1\xi_2^*\Big]\xi_2^2. \tag{4.145}$$

In view of (4.79), one can be put Eq. (4.145) in the form

$$\langle \hat{a}^2\rangle = \frac{1}{\lambda_a}\Big[u_a^2 - v_a^2\Big]^{1/2}\int \frac{d^2\xi_1}{\pi}\frac{d^2\xi_2}{\pi}\exp\Big[-\frac{1}{\lambda_a}\xi_1^*\xi_1 - \frac{1}{\lambda_a}\xi_2^*\xi_2 + \frac{1}{\lambda_a}\xi_1^*\xi_2$$
$$+\frac{1}{\lambda_a}\xi_1\xi_2^* - u_a\xi_2^*\xi_1 + \frac{v_a}{2}\Big(\xi_1^2 + \xi_2^{*2}\Big)\Big]\xi_2^2$$
$$= \frac{1}{\lambda_a}\Big[u_a^2 - v_a^2\Big]^{1/2}\int \frac{d^2\xi_1}{\pi}\exp\Big[-\frac{1}{\lambda_a}\xi_1^*\xi_1 + \frac{v_a}{2}\xi_1^2\Big]$$
$$\times \int \frac{d^2\xi_2}{\pi}\exp\Big[-\frac{1}{\lambda_a}\xi_2^*\xi_2 + \frac{1}{\lambda_a}\xi_1^*\xi_2 + \frac{1}{\lambda_a}\xi_1\xi_2^*$$

$$- u_a\xi_2^*\xi_1 + \frac{v_a}{2}\xi_2^{*2} + b\xi_2\Big]_{|b=0}\xi_2^2. \tag{4.146}$$

This can be rewritten as

$$\langle \hat{a}^2 \rangle = \frac{1}{\lambda_a}\Big[u_a^2 - v_a^2\Big]^{1/2}\int \frac{d^2\xi_1}{\pi}\exp\Big[-\frac{1}{\lambda_a}\xi_1^*\xi_1 + \frac{v_a}{2}\xi_1^2\Big]$$
$$\times\frac{d^2}{db^2}\int\frac{d^2\xi_2}{\pi}\exp\Big[-\frac{1}{\lambda_a}\xi_2^*\xi_2 + \xi_2\Big(\frac{\xi_1^*}{\lambda_a} + b\Big)$$
$$+ \xi_2^*\Big(\xi_1\Big[\frac{1}{\lambda_a} - u_a\Big]\Big) + \frac{v_1}{2}\xi_2^{*2}\Big]_{|b=0}, \tag{4.147}$$

so that on carrying out the integration over ξ_2, using the relation given by Eq. (4.13), we get

$$\langle \hat{a}^2 \rangle = \Big[u_a^2 - v_a^2\Big]^{1/2}\int\frac{d^2\xi_1}{\pi}\exp\Big[-\frac{1}{\lambda_a}\xi_1^*\xi_1 + \frac{v_a}{2}\xi_1^2\Big]$$
$$\times\frac{d^2}{db^2}\exp\Big[\xi_1^*\xi_1\Big(\frac{1}{\lambda_a} - u_a\Big) + b\xi_1 - bu_a\lambda_a\xi_1$$
$$+ \frac{v_a}{2}\Big(\xi_1^{*2} + \lambda_a^2 b^2 + 2b\lambda_a\xi_1^*\Big)\Big]_{|b=0}. \tag{4.148}$$

Hence performing the differentiation and applying the condition $b = 0$, we find

$$\langle \hat{a}^2 \rangle = \Big[u_a^2 - v_a^2\Big]^{1/2}\int\frac{d^2\xi_1}{\pi}\exp\Big[-\frac{1}{\lambda_a}\xi_1^*\xi_1 + \frac{v_a}{2}\xi_1^2 + \frac{1}{\lambda_a}\xi_1^*\xi_1 - u_a\xi_1^*\xi_1 + \frac{v_a}{2}\xi_1^{*2}\Big]$$
$$\times\Big[v_a\lambda_a^2 + \Big(\xi_1\Big[1 - \lambda_a u_a\Big] + \lambda_a v_a\xi_1^*\Big)^2\Big]. \tag{4.149}$$

This can be rewritten as

$$\langle \hat{a}^2 \rangle = \Big[u_a^2 - v_a^2\Big]^{1/2}\int\frac{d^2\xi_1}{\pi}\exp\Big[-u_a\xi_1^*\xi_1 + \frac{v_a}{2}\Big(\xi_1^2 + \xi_1^{*2}\Big)\Big]$$
$$\times\Big[v_a\lambda_a^2 + \Big(1 - \lambda_a u_a\Big)^2\xi_1^2 + \lambda_a^2 v_a^2\xi_1^{*2} + 2\lambda_a v_a\Big(1 - \lambda_a u_a\Big)\xi_1^*\xi_1\Big]. \tag{4.150}$$

We can now put this equation in the form

$$\langle \hat{a}^2 \rangle = \Big[u_a^2 - v_a^2\Big]^{1/2}\Big\{v_a\lambda_a^2\int\frac{d^2\xi_1}{\pi}\exp\Big[-u_a\xi_1^*\xi_1 + \frac{v_a}{2}(\xi_1^2 + \xi_1^{*2})\Big]$$
$$+\Big(1 - \lambda_a u_a\Big)^2\int\frac{d^2\xi_1}{\pi}\exp\Big[-u_a\xi_1^*\xi_1 + \frac{v_a}{2}\Big(\xi_1^2 + \xi_1^{*2}\Big)\Big]\xi_1^2$$
$$+v_a^2\lambda_a^2\int\frac{d^2\xi_1}{\pi}\exp\Big[-u_a\xi_1^*\xi_1 + \frac{v_a}{2}\Big(\xi_1^2 + \xi_1^{*2}\Big)\Big]\xi_1^{*2}$$

$$+ 2\lambda_a v_a\Big(1-\lambda_a u_a\Big)\int \frac{d^2\xi_1}{\pi}\exp\Big[-u_a\xi_1^*\xi_1+\frac{v_a}{2}\Big(\xi_1^2+\xi_1^{*2}\Big)\Big]\,\xi_1^*\xi_1\Big\}. \tag{4.151}$$

In order to carry out the integration, one can rewrite Eq. (4.151) as

$$\begin{aligned}\langle\hat{a}^2\rangle = \Big[u_a^2-v_a^2\Big]^{1/2}\Big\{v_a\lambda_a^2\int\frac{d^2\xi_1}{\pi}\exp\Big[-u_a\xi_1^*\xi_1+\frac{v_a}{2}\Big(\xi_1^2+\xi_1^{*2}\Big)\Big]\\ +\Big(1-\lambda_a u_a\Big)^2\frac{d^2}{da^2}\int\frac{d^2\xi_1}{\pi}\exp\Big[-u_a\xi_1^*\xi_1+a\xi_1+\frac{v_a}{2}\Big(\xi_1^2+\xi_1^{*2}\Big)\Big]_{|a=0}\\ +v_a^2\lambda_a^2\frac{d^2}{db^2}\int\frac{d^2\xi_1}{\pi}\exp\Big[-u_a\xi_1^*\xi_1+b\xi_1^*+\frac{v_a}{2}\Big(\xi_1^2+\xi_1^{*2}\Big)\Big]_{|b=0}\\ +2\lambda_a v_a\Big(1-\lambda_a u_a\Big)\frac{d^2}{dcdb}\int\frac{d^2\xi_1}{\pi}\exp\Big[-u_a\xi_1^*\xi_1+b\xi_1+c\xi_1^*+\frac{v_a}{2}\Big(\xi_1^2+\xi_1^{*2}\Big)\Big]_{|b=c=0}\Big\},\end{aligned} \tag{4.152}$$

so on carrying out the integration over ξ_1, one readily obtains

$$\begin{aligned}\langle\hat{a}^2\rangle = v_a\lambda_a^2+\Big(1-u_a\lambda_a\Big)^2\frac{d^2}{da^2}\exp\Big[\frac{a^2v_a}{2(u_a^2-v_a^2)}\Big]_{|a=0}\\ +v_a^2\lambda_a^2\frac{d^2}{db^2}\exp\Big[\frac{v_ab^2}{2(u_a^2-v_a^2)}\Big]_{|b=0}\\ +2v_a\lambda_a\Big(1-u_a\lambda_a\Big)\frac{d^2}{dcdb}\exp\Big[\frac{bcu_a+v_a(b^2+c)/2}{u_a^2-v_a^2}\Big]_{|b=c=0}.\end{aligned} \tag{4.153}$$

Hence performing the differentiation and applying the condition $a=b=c=0$, we easily get

$$\begin{aligned}\langle\hat{a}^2\rangle &= v_a\lambda_a^2+\frac{v_a}{u_a^2-v_a^2}\times\Big[1-u_a\lambda_a\Big]^2+\frac{v_a^3\lambda_a^2}{u_a^2-v_a^2}+\frac{2u_av_a\lambda_a}{u_a^2-v_a^2}\times\Big[1-u_a\lambda_a\Big]\\ &= v_a\lambda_a^2+\frac{v_a}{u_a^2-v_a^2}-\frac{u_a^2v_a\lambda_a^2}{u_a^2-v_a^2}+\frac{v_a^3\lambda_a^2}{u_a^2-v_a^2}\\ &= \frac{v_a}{u_a^2-v_a^2}.\end{aligned} \tag{4.154}$$

We also see that

$$\langle\hat{a}^{\dagger 2}\rangle = \frac{v_a}{u_a^2-v_a^2}. \tag{4.155}$$

And thus on account of Eqs. (4.21) and (4.22) along with (4.154) and (4.155), one readily finds

$$\langle\hat{a}^2\rangle = \langle\hat{a}^{\dagger 2}\rangle = q_a$$

$$= \frac{\gamma_c}{k}\langle \hat{m}_3\rangle_a. \tag{4.156}$$

Following a similar procedure, one can also verify that

$$\langle \hat{b}^2\rangle = \langle \hat{b}^{\dagger 2}\rangle = \frac{v_b}{u_b^2 - v_b^2}. \tag{4.157}$$

On account of (4.31) and (4.32) along with Eq. (4.157), we find

$$\langle \hat{b}^2\rangle = \langle \hat{b}^{\dagger 2}\rangle = q_b$$

$$= \frac{\gamma_c}{k}\langle \hat{m}_3\rangle_b. \tag{4.158}$$

Finally, with aid of Eqs. (4.89), (4.91), (4.156), (4.158) and Eq. (4.144), the global variance of the photon number for the superposed two-mode laser light beams can be expressed as

$$\begin{aligned}(\Delta n)_s^2 = {} & \left[\frac{\gamma_c}{k}\Big(\langle \hat{N}_a\rangle_a + \langle \hat{N}_b\rangle_a\Big)\right]^2 + \lambda_a\left[\frac{\gamma_c}{k}\Big(\langle \hat{N}_a\rangle_a + \langle \hat{N}_b\rangle_a\Big)\right] + \left[\frac{\gamma_c}{k}\langle \hat{m}_3\rangle_a\right]^2 \\ & + \left[\frac{\gamma_c}{k}\Big(\langle \hat{N}_a\rangle_b + \langle \hat{N}_b\rangle_b\Big)\right]^2 + \lambda_b\left[\frac{\gamma_c}{k}\Big(\langle \hat{N}_a\rangle_b + \langle \hat{N}_b\rangle_b\Big)\right] + \left[\frac{\gamma_c}{k}\langle \hat{m}_3\rangle_b\right]^2 \\ & + 2\left[\frac{\gamma_c}{k}\Big(\langle \hat{N}_a\rangle_a + \langle \hat{N}_b\rangle_a\Big)\right]\left[\frac{\gamma_c}{k}\Big(\langle \hat{N}_a\rangle_b + \langle \hat{N}_b\rangle_b\Big)\right] + 2\left[\frac{\gamma_c}{k}\langle \hat{m}_3\rangle_a\right]\left[\frac{\gamma_c}{k}\langle \hat{m}_3\rangle_b\right] \\ & + \lambda_a\left[\frac{\gamma_c}{k}\Big(\langle \hat{N}_a\rangle_b + \langle \hat{N}_b\rangle_b\Big)\right] + \lambda_b\left[\frac{\gamma_c}{k}\Big(\langle \hat{N}_a\rangle_a + \langle \hat{N}_b\rangle_a\Big)\right].\end{aligned} \tag{4.159}$$

With the help of Eqs. (4.89) and (4.91), we can write Eq. (4.159) in the form

$$\begin{aligned}(\Delta n)_s^2 = {} & \bar{n}_a^2 + \lambda_a\bar{n}_a + \left[\frac{\gamma_c}{k}\langle \hat{m}_3\rangle_a\right]^2 \\ & + \bar{n}_b^2 + \lambda_b\bar{n}_b + \left[\frac{\gamma_c}{k}\langle \hat{m}_3\rangle_b\right]^2 \\ & + 2\bar{n}_a\bar{n}_b + 2\left[\frac{\gamma_c}{k}\langle \hat{m}_3\rangle_a\right]\left[\frac{\gamma_c}{k}\langle \hat{m}_3\rangle_b\right] + \lambda_a\bar{n}_b + \lambda_b\bar{n}_a.\end{aligned} \tag{4.160}$$

We note that, unlike the mean photon number, the global variance of the photon number for the superposed two-mode laser light beams is not the sum of the photon number variances of the constituent two-mode laser light beams. Moreover, for the case in which the separate two-mode laser light beams are identical, one easily finds

$$(\Delta n)_s^2 = 4\left[\frac{\gamma_c}{k}\Big(\langle\hat{N}_a\rangle_a + \langle\hat{N}_b\rangle_a\Big)\right]^2 + 4\lambda_a\left[\frac{\gamma_c}{k}\Big(\langle\hat{N}_a\rangle_a + \langle\hat{N}_b\rangle_a\Big)\right] + 4\left[\frac{\gamma_c}{k}\langle\hat{m}_3\rangle_a\right]^2$$
$$= 4\bigg\{\bar{n}_a^2 + \lambda_a\bar{n}_a + \Big(\frac{\gamma_c}{k}\langle\hat{m}_3\rangle_a\Big)^2\bigg\}. \quad (4.161)$$

We immediately observe that the global variance of the photon number for the superposed identical two-mode laser light beams is four times that of one of a two-mode laser light beam. On the other hand, upon adding Eqs. (3.8) and (4.106) together with (4.71), the output annihilation operator $\hat{c}_{out}(t)$ for the superposed two-mode light beam can be expressed as

$$\hat{c}_{out}(t) = \sqrt{k}\hat{c}(t). \quad (4.162)$$

With the aid of this expression together with (4.132), the global variance of the photon number for the superposed two-mode output laser light beams have the form

$$(\Delta n)_{out}^2 = k^2\,(\Delta n)_s^2\,. \quad (4.163)$$

We see that the global variance of the photon number for the superposed two-mode output laser light beams is just k^2 times that of the superposed two-mode cavity light beams.

4.3.4 Local variance of the photon number

We wish to obtain the variance of the photon number for the superposed two-mode light beams in a given frequency interval, employing the spectrum of the photon number fluctuations for the superposed two-mode laser light beams. The spectrum of the photon number fluctuations for the superposed two-mode cavity light with central common frequency ω_0 is expressible as

$$R(\omega) = \frac{1}{\pi}Re\int_0^\infty d\tau e^{i(\omega-\omega_0)\tau}\langle\hat{n}(t),\hat{n}(t+\tau)\rangle_{ss}, \quad (4.164)$$

in which

$$\hat{n}(t) = \hat{c}^\dagger(t)\hat{c}(t), \tag{4.165}$$

$$\hat{n}(t+\tau) = \hat{c}^\dagger(t+\tau)\hat{c}(t+\tau) \tag{4.166}$$

and we have used the notation $\langle A, B\rangle = \langle AB\rangle - \langle A\rangle\langle B\rangle$. We now proceed to calculate $\langle \hat{n}(t), \hat{n}(t+\tau)\rangle$ that appears in Eq. (4.164). On account of Eqs. (4.165) and (4.166), we readily obtain

$$\langle \hat{n}(t), \hat{n}(t+\tau)\rangle = \langle \hat{c}^\dagger(t)\hat{c}(t)\hat{c}^\dagger(t+\tau)\hat{c}(t+\tau)\rangle$$
$$-\langle \hat{c}^\dagger(t)\hat{c}(t)\rangle\langle \hat{c}^\dagger(t+\tau)\hat{c}(t+\tau)\rangle. \tag{4.167}$$

On account of Eqs. (4.99) and (4.100), one can put the expectation value of Eq. (4.71) in the form

$$\langle \hat{c}(t)\rangle = 0. \tag{4.168}$$

In view of Eqs. (4.69) and (4.168) the annihilation operator $\hat{c}(t)$ is a Gaussian variable with zero mean. we find

$$\langle \hat{c}^\dagger(t)\hat{c}(t)\hat{c}^\dagger(t+\tau)\hat{c}(t+\tau)\rangle = \langle \hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle\langle \hat{c}(t)\hat{c}^\dagger(t+\tau)\rangle$$
$$+\langle \hat{c}^\dagger(t)\hat{c}^\dagger(t+\tau)\rangle\langle \hat{c}(t)\hat{c}(t+\tau)\rangle$$
$$+\langle \hat{c}^\dagger(t)\hat{c}(t)\rangle\langle \hat{c}^\dagger(t+\tau)\hat{c}(t+\tau)\rangle. \tag{4.169}$$

With the aid of this result, we can put (4.167) in the form

$$\langle \hat{n}(t), \hat{n}(t+\tau)\rangle = \langle \hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle\langle \hat{c}(t)\hat{c}^\dagger(t+\tau)\rangle$$
$$+\langle \hat{c}^\dagger(t)\hat{c}^\dagger(t+\tau)\rangle\langle \hat{c}(t)\hat{c}(t+\tau)\rangle. \tag{4.170}$$

We next seek to determine the two-time correlation functions that appear in Eq. (4.170) for the superposed two-mode cavity light beams. To this end, we observe that the adjoint of Eq. (4.117) has the form

$$\hat{c}^\dagger(t+\tau) = \hat{c}^\dagger(t)e^{-k\tau/2} + \frac{g}{\sqrt{N}}e^{-k\tau/2}\bigg\{\hat{m}^\dagger(t)\int_0^\tau d\tau' e^{(k-\nu)\tau'/2}$$

$$+\int_0^{\tau} d\tau' e^{(k-\nu)\tau'/2}\int_0^{\tau'} d\tau'' e^{\nu\tau''/2}\Big[-\frac{1}{2}\Big(\hat{F}_1^{\dagger}(t+\tau'')+\hat{F}_1'^{\dagger}(t+\tau'')\Big)$$
$$+\Big(\hat{F}_a^{\dagger}(t+\tau'')+\hat{F}_b^{\dagger}(t+\tau'')\Big)\Big]\Big\}. \quad (4.171)$$

On multiplying both sides of Eq. (4.171) on the left by $\hat{c}(t)$ and taking the expectation value of the resulting equation, one can readily get

$$\langle\hat{c}(t)\hat{c}^{\dagger}(t+\tau)\rangle = \langle\hat{c}(t)\hat{c}^{\dagger}(t)\rangle e^{-k\tau/2}+\frac{g}{\sqrt{N}}e^{-k\tau/2}\Big\{\langle\hat{c}(t)\hat{m}^{\dagger}(t)\rangle\int_0^{\tau} d\tau' e^{(k-\nu)\tau'/2}$$
$$+\int_0^{\tau} d\tau' e^{(k-\nu)\tau'/2}\int_0^{\tau'} d\tau'' e^{\nu\tau''/2}\Big[-\frac{1}{2}\Big(\langle\hat{c}(t)\hat{F}_1^{\dagger}(t+\tau'')\rangle+\langle\hat{c}(t)\hat{F}_1'^{\dagger}(t+\tau'')\rangle\Big)$$
$$+\Big(\langle\hat{c}(t)\hat{F}_a^{\dagger}(t+\tau'')\rangle+\langle\hat{c}(t)\hat{F}_b^{\dagger}(t+\tau'')\rangle\Big)\Big]\Big\}. \quad (4.172)$$

Since the cavity mode operator and the noise operator of the atomic are not correlated, we see that

$$\langle\hat{c}(t)\hat{F}_1^{\dagger}(t+\tau'')\rangle = \langle\hat{c}(t)\hat{F}_1'^{\dagger}(t+\tau'')\rangle = 0, \quad (4.173)$$

$$\langle\hat{c}(t)\hat{F}_a^{\dagger}(t+\tau'')\rangle = \langle\hat{c}(t)\hat{F}_b^{\dagger}(t+\tau'')\rangle = 0. \quad (4.174)$$

It then follows that

$$\langle\hat{c}(t)\hat{c}^{\dagger}(t+\tau)\rangle = \langle\hat{c}(t)\hat{c}^{\dagger}(t)\rangle e^{-k\tau/2}+\frac{g}{\sqrt{N}}e^{-k\tau/2}\Big\{\langle\hat{c}(t)\hat{m}^{\dagger}(t)\rangle\int_0^{\tau} d\tau' e^{(k-\nu)\tau'/2}\Big\}. \quad (4.175)$$

Introducing the adjoint of Eq. (4.122) into Eq. (4.175), we find

$$\langle\hat{c}(t)\hat{c}^{\dagger}(t+\tau)\rangle = \langle\hat{c}(t)\hat{c}^{\dagger}(t)\rangle\Big\{e^{-k\tau/2}+\frac{1}{2}ke^{-k\tau/2}\int_0^{\tau} d\tau' e^{(k-\nu)\tau'/2}\Big\}, \quad (4.176)$$

so that on carrying out the integration over τ', we readily get

$$\langle\hat{c}(t)\hat{c}^{\dagger}(t+\tau)\rangle = \langle\hat{c}(t)\hat{c}^{\dagger}(t)\rangle\left\{\frac{k}{k-\nu}e^{-\nu\tau/2}-\frac{\nu}{k-\nu}e^{-k\tau/2}\right\}. \quad (4.177)$$

It can be established in a similar manner that

$$\langle\hat{c}(t)\hat{c}(t+\tau)\rangle = \langle\hat{c}^2(t)\rangle\left\{\frac{k}{k-\nu}e^{-\nu\tau/2}-\frac{\nu}{k-\nu}e^{-k\tau/2}\right\}, \quad (4.178)$$

$$\langle\hat{c}^{\dagger}(t)\hat{c}^{\dagger}(t+\tau)\rangle = \langle\hat{c}^{\dagger 2}(t)\rangle\left\{\frac{k}{k-\nu}e^{-\nu\tau/2}-\frac{\nu}{k-\nu}e^{-k\tau/2}\right\}. \quad (4.179)$$

On account of Eq. (4.124) and Eqs. (4.177)-(4.179), one can write Eq. (4.170) as

$$\langle\hat{n}(t),\hat{n}(t+\tau)\rangle = \Big(\langle\hat{c}^\dagger(t)\hat{c}(t)\rangle\langle\hat{c}(t)\hat{c}^\dagger(t)\rangle + \langle\hat{c}^{\dagger 2}(t)\rangle\langle\hat{c}^2(t)\rangle\Big) \times\left(\frac{k}{k-\nu}e^{-\nu\tau/2} - \frac{\nu}{k-\nu}e^{-k\tau/2}\right)^2$$
$$= (\Delta n)_s^2\left\{\frac{k^2}{(k-\nu)^2}e^{-\nu\tau} + \frac{\nu^2}{(k-\nu)^2}e^{-k\tau} - \frac{2k\nu}{(k-\nu)^2}e^{-(k+\nu)\tau/2}\right\}, \tag{4.180}$$

Thus upon substituting this result into (4.164), we have

$$R(\omega) = (\Delta n)_s^2\frac{1}{\pi}Re\bigg\{\left[\frac{k^2}{(k-\nu)^2}\right]\int_0^{+\infty}d\tau e^{-\Big(\nu-i[\omega-\omega_0]\Big)\tau} + \left[\frac{\nu^2}{(k-\nu)^2}\right]\int_0^{+\infty}d\tau e^{-\Big(k-i[\omega-\omega_0]\Big)\tau} - \left[\frac{2k\nu}{(k-\nu)^2}\right]\int_0^{+\infty}d\tau e^{-\Big([k+\nu]/2-i[\omega-\omega_0]\Big)\tau}\bigg\}, \tag{4.181}$$

so on carrying out the integration over τ, we easily obtain

$$R(\omega) = (\Delta n)_s^2\bigg\{\left[\frac{k^2}{(k-\nu)^2}\right]\left[\frac{\nu/\pi}{(\omega-\omega_0)^2+\nu^2}\right] + \left[\frac{\nu^2}{(k-\nu)^2}\right]\left[\frac{k/\pi}{(\omega-\omega_0)^2+k^2}\right] - \left[\frac{2k\nu}{(k-\nu)^2}\right]\left[\frac{(k+\nu)/2\pi}{(\omega-\omega_0)^2+(k+\nu)^2/4}\right]\bigg\}. \tag{4.182}$$

Upon integrating both sides of (4.182) over ω, one find

$$\int_{-\infty}^{+\infty}R(\omega)d\omega = (\Delta n)_s^2, \tag{4.183}$$

in which $(\Delta n)_s^2$ is the steady-state global variance of the photon number for the superposed two-mode laser light beams given by Eq. (4.160). On the basis of (4.183), we observe that $R(\omega)d\omega$ represents the steady-state variance of the photon number for the superposed two-mode cavity light beams in the interval between ω and $\omega + d\omega$ [1]. We thus realize that the photon-number variance in the interval between $\omega' = -\lambda$ and $\omega' = +\lambda$ is expressible as [1]

$$(\Delta n)_{\pm\lambda}^2 = \int_{-\lambda}^{+\lambda}R(\omega')d\omega' \tag{4.184}$$

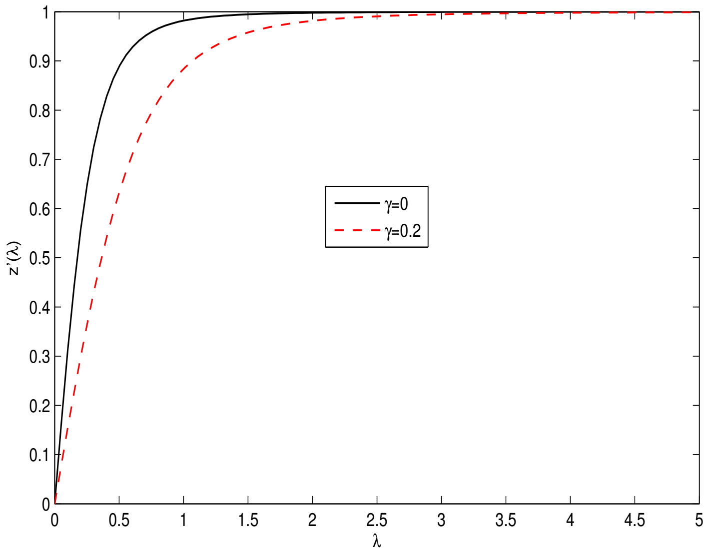

Figure 4.4: Plots of $z'(\lambda)$ [Eq. (4.187)] versus λ for $\gamma_c = 0.4$, $k = 0.8$, $\Omega = 2$, and different values of γ.

in which $\omega' = \omega - \omega_0$. Now on introducing (4.182) into Eq. (4.184), we have

$$(\Delta n)^2_{s\pm\lambda} = (\Delta n)^2_s \Bigg\{ \left[\frac{k^2}{(k-\nu)^2}\right] \int_{-\lambda}^{+\lambda} \left[\frac{\nu/\pi}{\omega'^2+\nu^2}\right] d\omega' + \left[\frac{\nu^2}{(k-\nu)^2}\right] \int_{-\lambda}^{+\lambda} \left[\frac{k/\pi}{\omega'^2+k^2}\right] d\omega' - \left[\frac{2k\nu}{(k-\nu)^2}\right] \int_{-\lambda}^{+\lambda} \left[\frac{(k+\nu)/2\pi}{\omega'^2+(k+\nu)^2/4}\right] d\omega' \Bigg\}, \tag{4.185}$$

so that on carrying out the integration over ω', applying the relation given by Eq. (3.30), we arrive at

$$(\Delta n)^2_{s\pm\lambda} = (\Delta n)^2_s\, z'(\lambda), \tag{4.186}$$

in which

$$z'(\lambda) = \left[\frac{2k^2/\pi}{(k-\nu)^2}\right] \tan^{-1}\left(\frac{\lambda}{\nu}\right) + \left[\frac{2\nu^2/\pi}{(k-\nu)^2}\right] \tan^{-1}\left(\frac{\lambda}{k}\right) - \left[\frac{4k\nu/\pi}{(k-\nu)^2}\right] \tan^{-1}\left(\frac{2\lambda}{k+\nu}\right). \tag{4.187}$$

We see from Eq. (4.186) along with the plots $z'(\lambda)$ that $(\Delta n)^2_{s\pm\lambda}$ increases with λ until it reaches the global variance of the photon number. In addition, we observe from the plots in Fig. 4.4 that $z'(\lambda)$ in the absence of spontaneous emission ($\gamma = 0$) is greater than in the presence of spontaneous emission ($\gamma = 2$). Moreover, we note that a large part of the total variance of the photon number for the superposed two-mode laser light beams is confined in a relatively small frequency interval.

4.4 Quadrature Squeezing

In this section we seek to investigate the squeezing properties of the superposed two-mode laser light beams.

4.4.1 The global quadrature squeezing

Here we wish to calculate the quadrature squeezing of the superposed two-mode cavity (output) light beams in the entire frequency interval at steady state. The squeezing properties of the superposed two-mode laser light beams are described by two quadrature operators defined as

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c} \tag{4.188}$$

and

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}), \tag{4.189}$$

where $\hat{c}_+$ and $\hat{c}_-$ are Hermitian operators representing physical quantities called the plus and minus quadratures. Using Eqs. (4.188), (4.189), and (4.71) together with (2.56) and (2.57), one can readily verify that

$$[\hat{c}_-, \hat{c}_+] = 4i\frac{\gamma_c}{k}\left(\hat{N}_a - \hat{N}_c\right). \tag{4.190}$$

It then follows that

$$\Delta c_+ \Delta c_- \geqslant 2\frac{\gamma_c}{k}\left|\langle\hat{N}_a\rangle - \langle\hat{N}_c\rangle\right|. \tag{4.191}$$

Employing the commutation relation given by Eq. (4.133), the global quadrature variance of the superposed two-mode laser light beams can be expressed as

$$(\Delta c_\pm)^2 = \lambda_a + \lambda_b + \langle : \hat{c}_\pm(t), \hat{c}_\pm(t) : \rangle, \tag{4.192}$$

where :: stands for normal ordering. Then with the aid of (4.188) and (4.189), one can put Eq. (4.192) in the form

$$(\Delta c_{\pm})^2 = \lambda_a + \lambda_b + 2\langle\hat{c}^\dagger\hat{c}\rangle \pm \langle\hat{c}^{\dagger 2}\rangle \pm \langle\hat{c}^2\rangle \mp \langle\hat{c}^\dagger\rangle^2 \mp \langle\hat{c}\rangle^2 - 2\langle\hat{c}^\dagger\rangle\langle\hat{c}\rangle. \tag{4.193}$$

Moreover, with the aid of (4.45) and (4.65), the expectation value of $\hat{c}^2(t)$ can express as

$$\begin{aligned}\langle\hat{c}^2(t)\rangle = \lambda_a\lambda_b \int d^2\gamma_0 d^2\eta Q\Big(\lambda_a\gamma_0^*, \lambda_a\gamma_0 + \frac{\partial}{\partial\gamma_0^*}\Big)Q\Big(\lambda_b\eta^*, \lambda_b\eta + \frac{\partial}{\partial\eta^*}\Big)\\ \times\Big[\lambda_a^2\beta^2 + \lambda_b^2\gamma_0^2 + 2\lambda_a\lambda_b\beta\gamma_0\Big]\end{aligned}$$

$$\begin{aligned} &= \lambda_a^2 \int d^2\gamma_0 Q\Big(\lambda_a\gamma_0^*, \lambda_a\gamma_0 + \frac{\partial}{\partial\gamma_0^*}\Big)\gamma_0^2 \\ &+\lambda_b^2 \int d^2\eta Q\Big(\lambda_2\eta^*, \lambda_b\eta + \frac{\partial}{\partial\eta^*}\Big)\eta^2 \\ &+2\lambda_a^2 \int d^2\gamma_0 Q\Big(\lambda_a\gamma_0^*, \lambda_a\gamma_0 + \frac{\partial}{\partial\gamma_0^*}\Big)\gamma_0 \\ &\times\lambda_b^2 \int d^2\eta Q\Big(\lambda_b\eta^*, \lambda_b\eta + \frac{\partial}{\partial\eta^*}\Big)\eta. \end{aligned} \tag{4.194}$$

Now on introducing the variables $\alpha = \lambda_a\gamma_0$ and $\beta = \lambda_b\eta$, we arrive at

$$\begin{aligned}\langle\hat{c}^2(t)\rangle &= \int \frac{d^2\alpha}{\lambda_a} Q\Big(\alpha^*, \alpha + \lambda_a\frac{\partial}{\partial\alpha^*}\Big)\alpha^2 \\ &+ \int \frac{d^2\beta}{\lambda_b} Q\Big(\beta^*, \beta + \lambda_b\frac{\partial}{\partial\beta^*}\Big)\beta^2 \\ &+2 \int \frac{d^2\alpha}{\lambda_a} Q\Big(\alpha^*, \alpha + \lambda_a\frac{\partial}{\partial\alpha^*}\Big)\alpha \\ &\times \int \frac{d^2\beta}{\lambda_b} Q\Big(\beta^*, \beta + \lambda_b\frac{\partial}{\partial\beta^*}\Big)\beta.\end{aligned} \tag{4.195}$$

Then on account of (4.47), we can express Eq. (4.195) as

$$\langle\hat{c}^2(t)\rangle = \langle\hat{a}^2\rangle + \langle\hat{b}^2\rangle + 2\langle\hat{a}\rangle\langle\hat{b}\rangle. \tag{4.196}$$

On can also easily check that

$$\langle\hat{c}^{\dagger 2}(t)\rangle = \langle\hat{a}^{\dagger 2}\rangle + \langle\hat{b}^{\dagger 2}\rangle + 2\langle\hat{a}^\dagger\rangle\langle\hat{b}^\dagger\rangle. \tag{4.197}$$

Thus upon substituting Eqs. (4.168), (4.196), and (4.197) into Eq. (4.193), we readily get

$$(\Delta c_{\pm})^2 = \lambda_a + \lambda_b + 2\langle \hat{a}^\dagger \hat{a}\rangle + 2\langle \hat{b}^\dagger \hat{b}\rangle \pm \Big(\langle \hat{a}^{\dagger 2}\rangle + \langle \hat{a}^2\rangle + \langle \hat{b}^{\dagger 2}\rangle + \langle \hat{b}^2\rangle\Big). \tag{4.198}$$

This can be rewritten as

$$\begin{aligned}(\Delta c_{\pm})^2 &= \Big\{\lambda_a + 2\langle \hat{a}^\dagger \hat{a}\rangle \pm \langle \hat{a}^{\dagger 2}\rangle \pm \langle \hat{a}^2\rangle\Big\} + \Big\{\lambda_b + 2\langle \hat{b}^\dagger \hat{b}\rangle \pm \langle \hat{b}^{\dagger 2}\rangle \pm \langle \hat{b}^2\rangle\rangle\Big\} \\ &= (\Delta a_{\pm})^2 + (\Delta b_{\pm})^2 .\end{aligned} \tag{4.199}$$

In view of (3.91), the global quadrature variances of the constituent two-mode laser light beams are given by

$$(\Delta a_{\pm})^2 = \frac{\gamma_c}{k}\Big\{N_1 + \langle \hat{N}_b\rangle_a \pm 2\langle \hat{m}_3\rangle_a\Big\} \tag{4.200}$$

and

$$(\Delta b_{\pm})^2 = \frac{\gamma_c}{k}\Big\{N_2 + \langle \hat{N}_b\rangle_b \pm 2\langle \hat{m}_3\rangle_b\Big\}. \tag{4.201}$$

Thus on account of (4.200) and (4.201), one can put Eq. (4.199) in the form

$$(\Delta c_{\pm})^2 = \frac{\gamma_c}{k}\Big\{N_1 + N_2 + \langle \hat{N}_b\rangle_a + \langle \hat{N}_b\rangle_b \pm 2\Big(\langle \hat{m}_3\rangle_a + \langle \hat{m}_3\rangle_b\Big)\Big\}. \tag{4.202}$$

Finally, with the aid of Eqs. (4.26), (4.28), (4.36), and (4.38), we readily get

$$\begin{aligned}(\Delta c_{+})^2 &= \frac{\gamma_c}{k}\Big\{1 + \frac{\Omega^2 + 2\Omega(\gamma_c+\gamma)}{(\gamma_c+\gamma)^2 + 3\Omega^2}\Big\}N_1 \\ &+ \frac{\gamma_c}{k}\Big\{1 + \frac{\Omega^2 + 2\Omega(\gamma_c+\gamma)}{(\gamma_c+\gamma)^2 + 3\Omega^2}\Big\}N_2 \\ &= \frac{\gamma_c}{k}\Big\{N_1 + N_2\Big\}\Big\{1 + \frac{\Omega^2 + 2\Omega(\gamma_c+\gamma)}{(\gamma_c+\gamma)^2 + 3\Omega^2}\Big\}\end{aligned} \tag{4.203}$$

and

$$\begin{aligned}(\Delta c_{-})^2 &= \frac{\gamma_c}{k}\Big\{1 + \frac{\Omega^2 - 2\Omega(\gamma_c+\gamma)}{(\gamma_c+\gamma)^2 + 3\Omega^2}\Big\}N_1 \\ &+ \frac{\gamma_c}{k}\Big\{1 + \frac{\Omega^2 - 2\Omega(\gamma_c+\gamma)}{(\gamma_c+\gamma)^2 + 3\Omega^2}\Big\}N_2\end{aligned}$$

$$= \frac{\gamma_c}{k}\bigg\{N_1 + N_2\bigg\}\bigg\{1 + \frac{\Omega^2 - 2\Omega(\gamma_c+\gamma)}{(\gamma_c+\gamma)^2 + 3\Omega^2}\bigg\}. \tag{4.204}$$

We observe that the superposed two-mode laser light beams is in a squeezed state and the squeezing occurs in the minus quadrature. Moreover, we see that the global quadrature variance of the superposed two-mode laser light beams is the sum of the global quadrature variances of the constituent two-mode laser light beams.

Furthermore, for $N_1 = N_2 = N$, one can easily get

$$(\Delta c_+)^2 = 2\frac{\gamma_c}{k}N\bigg\{1 + \frac{\Omega^2 + 2\Omega(\gamma_c+\gamma)}{(\gamma_c+\gamma)^2 + 3\Omega^2}\bigg\} \tag{4.205}$$

and

$$(\Delta c_-)^2 = 2\frac{\gamma_c}{k}N\bigg\{1 + \frac{\Omega^2 - 2\Omega(\gamma_c+\gamma)}{(\gamma_c+\gamma)^2 + 3\Omega^2}\bigg\}. \tag{4.206}$$

Comparison of Eqs. (3.93) and (4.206) shows that the global quadrature variance of the superposed two-mode laser light beams is twice that of a single two-mode laser light beam. In addition, for $\Omega = 0$, we easily see from Eqs. (4.203) and (4.204) that the quadrature variance of the superposed two-mode vacuum states takes the form

$$(\Delta c_+)^2_v = (\Delta c_-)^2_v = \frac{\gamma_c}{k}\bigg\{N_1 + N_2\bigg\}. \tag{4.207}$$

Next we proceed to calculate the quadrature squeezing of the superposed two-mode laser light beams relative to the quadrature variance of the superposed two-mode vacuum states. Now we define the quadrature squeezing of the superposed two-mode cavity light beams by [1]

$$S_- = \frac{(\Delta c_-)^2_v - (\Delta c_-)^2}{(\Delta c_-)^2_v}. \tag{4.208}$$

Hence with the aid of Eqs. (4.204) and (4.207), we arrive at

$$S_- = \bigg\{\frac{2\Omega(\gamma_c+\gamma)}{(\gamma_c+\gamma)^2 + 3\Omega^2} - \frac{\Omega^2}{(\gamma_c+\gamma)^2 + 3\Omega^2}\bigg\}. \tag{4.209}$$

This shows that the global quadrature squeezing of the superposed two-mode laser light beams is equal to that of the single two-mode laser light beam given by Eq.

(3.98). In other words, we note that superposing the two-mode laser light beams does not affect the quadrature squeezing in the entire frequency interval. Moreover, upon setting $\gamma = 0$ in Eq. (4.209), we see that

$$S_{-} = \frac{2\chi - \chi^2}{1 + 3\chi^2}, \tag{4.210}$$

where $\chi = \Omega/\gamma_c$. Eq. (4.210) is the quadrature squeezing of the light produced by a three-level laser with the N three-level atoms available in a closed cavity and pumped by electron bombardment which has been reported by Fesseha [1].

On the other hand, the global quadrature variance of the superposed two-mode output laser light beams is given by

$$\left(\Delta c_{\pm}^{out}\right)^2 = k\left(\lambda_a + \lambda_b\right) + \left\langle : \hat{c}_{\pm}^{out}(t), \hat{c}_{\pm}^{out}(t) : \right\rangle, \tag{4.211}$$

where

$$\hat{c}_{+}^{out} = \hat{c}_{out}^{\dagger} + \hat{c}_{out} \tag{4.212}$$

and

$$\hat{c}_{-}^{out} = i\left(\hat{c}_{out}^{\dagger} - \hat{c}_{out}\right) \tag{4.213}$$

are the output plus and minus quadrature operators. With the aid of (4.212) and (4.213), one can put Eq. (4.211) in the form

$$\begin{aligned}\left(\Delta c_{\pm}^{out}\right)^2 &= k\left(\lambda_a + \lambda_b\right) + 2\langle \hat{c}_{out}^{\dagger}\hat{c}_{out}\rangle \pm \langle \hat{c}_{out}^{\dagger 2}\rangle \pm \langle \hat{c}_{out}^{2}\rangle \\ &\mp \langle \hat{c}_{out}^{\dagger}\rangle^2 \mp \langle \hat{c}_{out}\rangle^2 - 2\langle \hat{c}_{out}^{\dagger}\rangle\langle \hat{c}_{out}\rangle. \end{aligned} \tag{4.214}$$

On account of Eqs. (4.168), (4.196), and (4.197), we obtain

$$\langle \hat{c}_{out}\rangle = 0, \tag{4.215}$$

$$\langle \hat{c}_{out}^{2}\rangle = \left\{\langle \hat{a}_{out}^{2}\rangle + \langle \hat{b}_{out}^{2}\rangle\right\} \tag{4.216}$$

and

$$\langle\hat{c}^{\dagger 2}_{out}\rangle = \Big\{\langle\hat{a}^{\dagger 2}_{out}\rangle + \langle\hat{b}^{\dagger 2}_{out}\rangle\Big\}. \tag{4.217}$$

With the aid of Eqs. (3.8), (4.71),(4.106) and (4.215)-(4.217) together with (4.198), one can express Eq. (4.214) as

$$\left(\Delta c^{out}_{\pm}\right)^2 = k\left(\Delta c_{\pm}\right)^2. \tag{4.218}$$

Now in view of Eqs. (4.203) and (4.204), we see that

$$\left(\Delta c^{out}_{+}\right)^2 = k\left\{\frac{\gamma_c}{k}\left[N_1 + N_2\right]\left[1 + \frac{\Omega^2 + 2\Omega(\gamma_c+\gamma)}{(\gamma_c+\gamma)^2 + 3\Omega^2}\right]\right\}, \tag{4.219}$$

$$\left(\Delta c^{out}_{-}\right)^2 = k\left\{\frac{\gamma_c}{k}\left[N_1 + N_2\right]\left[1 + \frac{\Omega^2 - 2\Omega(\gamma_c+\gamma)}{(\gamma_c+\gamma)^2 + 3\Omega^2}\right]\right\}. \tag{4.220}$$

Upon setting $\Omega = 0$ in Eqs. (4.219) and (4.220), we easily obtain

$$\left(\Delta c^{out}_{+}\right)^2_v = \left(\Delta c^{out}_{-}\right)^2_v = k\left\{\frac{\gamma_c}{k}\left[N_1 + N_2\right]\right\}, \tag{4.221}$$

which represents the quadrature variance of the superposed two-mode output vacuum state. The global quadrature squeezing of the superposed two-mode output laser light beams is defined by [1]

$$S^{out}_{-} = \frac{\left(\Delta c^{out}_{-}\right)^2_v - \left(\Delta c^{out}_{-}\right)^2}{\left(\Delta c^{out}_{-}\right)^2_v}, \tag{4.222}$$

so that with the aid of Eqs. (4.220) and (4.221) along with (4.208), we readily find

$$S^{out}_{-} = S_{-}, \tag{4.223}$$

which shows that the quadrature squeezing of the superposed two-mode output laser light beams is exactly the same as that of the superposed two-mode cavity light beams.

4.4.2 Local quadrature squeezing

Here we seek to determine the quadrature squeezing for the superposed two-mode laser light beams in a given frequency interval. To this end, we first obtain the spectrum of the quadrature fluctuations for the superposed two-mode cavity light beams. We define the spectrum of quadrature fluctuations for the superposed two-mode laser light beams with central common frequency ω_0 by

$$S_{\pm}(\omega) = \frac{1}{\pi} Re \int_0^{\infty} d\tau e^{i(\omega-\omega_0)\tau} \langle \hat{c}_{\pm}(t), \hat{c}_{\pm}(t+\tau) \rangle_{ss}, \tag{4.224}$$

in which

$$\hat{c}_{+}(t+\tau) = \hat{c}^{\dagger}(t+\tau) + \hat{c}(t+\tau), \tag{4.225}$$

$$\hat{c}_{-}(t+\tau) = i\Big(\hat{c}^{\dagger}(t+\tau) - \hat{c}(t+\tau)\Big). \tag{4.226}$$

On account of Eqs. (4.168), (4.225) and (4.226), one can put Eq. (4.224) in the form

$$S_{\pm}(\omega) = \frac{1}{\pi} Re \int_0^{\infty} d\tau e^{i(\omega-\omega_0)\tau} \Big[\langle \hat{c}^{\dagger}(t)\hat{c}(t+\tau) \rangle \pm \langle \hat{c}^{\dagger}(t)\hat{c}^{\dagger}(t+\tau) \rangle + \langle \hat{c}(t)\hat{c}^{\dagger}(t+\tau) \rangle \pm \langle \hat{c}(t)\hat{c}(t+\tau) \rangle \Big]. \tag{4.227}$$

Thus upon substituting Eq. (4.124) and Eqs. (4.177)-(4.179) into Eq. (4.227), we have

$$S_{\pm}(\omega) = \frac{1}{\pi} Re \int_0^{\infty} d\tau e^{i(\omega-\omega_0)\tau} \Big\{ \Big[\langle \hat{c}^{\dagger}\hat{c} \rangle + \langle \hat{c}\hat{c}^{\dagger} \rangle \pm \langle \hat{c}^{\dagger 2} \rangle \pm \langle \hat{c}^{2} \rangle \Big] \times \Big[\frac{k}{k-\nu} e^{-\nu\tau/2} - \frac{\nu}{k-\nu} e^{-k\tau/2} \Big] \Big\}. \tag{4.228}$$

This can be put in the form

$$S_{\pm}(\omega) = (\Delta c_{\pm})^2 \frac{1}{\pi} Re \Big\{ \Big[\frac{k}{k-\nu} \Big] \int_0^{+\infty} d\tau e^{-\left(\nu/2 - i[\omega-\omega_0]\right)\tau} - \Big[\frac{\nu}{k-\nu} \Big] \int_0^{+\infty} d\tau e^{-\left(k/2 - i[\omega-\omega_0]\right)\tau} \Big\}, \tag{4.229}$$

so that on carrying out the integration over τ, the spectrum of the quadrature fluctuations for the superposed two-mode cavity light beams is found to be

$$S_{\pm}(\omega) = (\Delta c_{\pm})^2 \left\{ \left[\frac{k}{k-\nu}\right] \left[\frac{\nu/2\pi}{(\omega-\omega_0)^2+(\nu/2)^2}\right] - \left[\frac{\nu}{k-\nu}\right] \left[\frac{k/2\pi}{(\omega-\omega_0)^2+(k/2)^2}\right] \right\}. \tag{4.230}$$

Upon integrating (4.230) over ω, we find

$$\int_{-\infty}^{\infty} S_{\pm}(\omega) d\omega = (\Delta c_{\pm})^2 , \tag{4.231}$$

in which $(\Delta c_{\pm})^2$ is the steady-state global quadrature variance for the superposed two-mode laser light beams given by Eq. (4.198). On the basis of Eq. (4.231), we observe that $S_{\pm}(\omega)d\omega$ is the global quadrature variance of the superposed two-mode cavity light beams in the interval between ω and $\omega + d\omega$ [1]. We then realize that the quadrature variance of the superposed two-mode cavity light beams in the interval $\omega' = -\lambda$ and $\omega' = +\lambda$ can be written as [1]

$$(\Delta c_{\pm})^2_{\pm\lambda} = \int_{-\lambda}^{+\lambda} S_{\pm}(\omega') d\omega', \tag{4.232}$$

in which $\omega' = \omega - \omega_0$. Thus upon substituting (4.230) into Eq. (4.232), we have

$$(\Delta c_{\pm})^2_{\pm\lambda} = (\Delta c_{\pm})^2 \left\{ \left[\frac{k}{k-\nu}\right] \int_{-\lambda}^{+\lambda} \left[\frac{\nu/2\pi}{\omega'^2+(\nu/2)^2}\right] d\omega' - \left[\frac{\nu}{k-\nu}\right] \int_{-\lambda}^{+\lambda} \left[\frac{k/2\pi}{\omega'^2+(k/2)^2}\right] d\omega' \right\}, \tag{4.233}$$

so on carrying out the integration over ω', applying the relation described by Eq. (3.30), we readily get

$$(\Delta c_{\pm})^2_{\pm\lambda} = (\Delta c_{\pm})^2 \, z(\lambda), \tag{4.234}$$

in which

$$z(\lambda) = \left[\frac{2k/\pi}{k-\nu}\right] \tan^{-1}\left(\frac{2\lambda}{\nu}\right) - \left[\frac{2\nu/\pi}{k-\nu}\right] \tan^{-1}\left(\frac{2\lambda}{k}\right), \tag{4.235}$$

where ν is given by Eq. (2.73).

Moreover, upon setting $\Omega = 0$ in Eq. (4.234), we find

$$(\Delta c_{\pm})^2_{v\pm\lambda} = (\Delta c_{\pm})^2_v \, z_v(\lambda), \tag{4.236}$$

with

$$(\Delta c_{\pm})^2_v = \frac{\gamma_c}{k}\Big[N_1 + N_2\Big] \tag{4.237}$$

and

$$z_v(\lambda) = \left[\frac{2k/\pi}{k-(\gamma_c+\gamma)}\right]\tan^{-1}\left(\frac{2\lambda}{\gamma_c+\gamma}\right) - \left[\frac{2(\gamma_c+\gamma)/\pi}{k-(\gamma_c+\gamma)}\right]\tan^{-1}\left(\frac{2\lambda}{k}\right). \tag{4.238}$$

We immediately see that Eq. (4.236) is the local quadrature variance of the superposed two-mode vacuum state.

We next proceed to calculate the local quadrature squeezing of the superposed two-mode laser light beams relative to the local quadrature squeezing of the superposed two-mode vacuum state. According to Ref. [1], we define the local quadrature squeezing of the superposed two-mode cavity light beams in the $\lambda_{\pm}$ frequency interval by

$$S_{\pm\lambda} = \frac{(\Delta c_-)^2_{v\pm\lambda} - (\Delta c_-)^2_{\pm\lambda}}{(\Delta c_-)^2_{v\pm\lambda}}. \tag{4.239}$$

Then with the aid of Eqs. (4.204), (4.234), and (4.236)-(4.238), we easily get

$$S_{\pm\lambda} = \frac{1}{z_v(\lambda)}\left\{z_v(\lambda) - z(\lambda)\left[1 + \frac{\Omega^2 - 2\Omega(\gamma_c+\gamma)}{(\gamma_c+\gamma)^2 + 3\Omega^2}\right]\right\}. \tag{4.240}$$

We immediately observe that the local quadrature squeezing of the superposed two-mode laser light beams is exactly the same as that of the single two-mode laser light beams described by Eq. (3.126). In other words, we note that superposing the two-mode laser light beams does not affect the quadrature squeezing in a given frequency interval.

On the other hand, the quadrature squeezing of the superposed two-mode output laser light beam in the aforementioned frequency interval is defined by [1]

$$S_{\pm\lambda}^{out} = \frac{\left(\Delta c_{\pm}^{out}\right)_{v\pm\lambda}^{2} - \left(\Delta c_{\pm}^{out}\right)_{\pm\lambda}^{2}}{\left(\Delta c_{\pm}^{out}\right)_{v\pm\lambda}^{2}}, \tag{4.241}$$

and taking into account the fact that

$$\left(\Delta c_{\pm}^{out}\right)_{v\pm\lambda}^{2} = z_v(\lambda)\left(\Delta c_{\pm}^{out}\right)_{v}^{2} \tag{4.242}$$

and

$$\left(\Delta c_{\pm}^{out}\right)_{\pm\lambda}^{2} = z(\lambda)\left(\Delta c_{\pm}^{out}\right)^{2}, \tag{4.243}$$

we arrive at

$$\begin{aligned} S_{\pm\lambda}^{out} &= 1 - \frac{\left(\Delta c_{\pm}\right)_{\pm\lambda}^{2}}{\left(\Delta c_{\pm}\right)_{v\pm\lambda}^{2}} \\ &= S_{\pm\lambda}. \end{aligned} \tag{4.244}$$

We see that the quadrature squeezing of the superposed two-mode output laser light beams in a given frequency interval is equal to that of the superposed two-mode cavity light beams.

5

Conclusion

In this dissertation we have studied the quantum properties of the light produced by a degenerate three-level laser pumped by coherent light and in the presence of spontaneous emission. We have carried out our analysis by putting the noise operators associated with the vacuum reservoir in normal order and by taking into consideration the interaction of the three-level atoms with the vacuum reservoir outside the cavity. Employing the pertinent master equation for a coherently pumped degenerate three-level atom, we have obtained coupled equations of evolution for the expectation values of the atomic operators. Then applying the large-time approximation scheme to the quantum Langevin equations for the cavity mode operators, we have managed to decouple the equations of evolution for the expectation values of the atomic operators. Applying the solutions of the pertinent equations of evolution, we have calculated the local and global mean photon number, the local and global variance of the photon number, and the local and global quadrature squeezing of the two-mode cavity light.

We have found that a large part of the total mean photon number and the total variance of the photon number are confined in a relatively small frequency interval. Moreover, we have observed that the cavity light is in a squeezed state and the squeezing occurs in the minus qudrature. In addition, we have seen that the maximum global quadrature squeezing of the cavity light is 43.42% (and occurs at $\Omega = 0.1717$ for $\gamma = 0$). We have also observed that the cavity light produced by the laser operating under the conditions $\Omega \gg \gamma_c$ and $\Omega \gg \gamma$ is in a chaotic state. In addition, we have shown that the presence of the spontaneous emission process leads to a decrease in the mean photon number, the variance of the photon number, and the maximum quadrature squeezing. Furthermore, we would like to point out that, unlike the mean photon number, the variance of the photon number and the quadrature variance, the quadrature squeezing does not depend on the number of atoms. This implies that the quadrature squeezing of the two-mode cavity light is independent of the number of photons. On the other hand, we have found that the maximum local quadrature squeezing of the cavity light is 78.48% (and occurs at $\lambda = 0.0606$ for $\gamma = 0$).

We have also analyzed the squeezing and statistical properties of the superposed two-mode laser light beams. We have found that the global mean photon number of the superposed two-mode laser light beams is the sum of the global mean photon numbers of the constituent two-mode laser light beams. Furthermore, we have noted that unlike the mean photon number, the variance of the photon number for the superposed two-mode laser light beams is not the sum of the photon-number variances of the constituent two-mode laser light beams. However, it turns out to be four times that of a two-mode laser light beam, for the case in which the separate two-mode laser light beams are identical. Moreover, we have shown that the quadrature variance of the superposed two-mode cavity light beams is the sum of the quadrature variances of the constituent two-mode laser light beams. Finally, we have found that the superposition of the two-mode laser light beams does not affect the local and global quadrature squeezing, but it increases the global (local) mean photon number and the global (local) variance of the photon number. Thus, we note that the superposition of the two-mode laser light beams leads to a more bright squeezed light.

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DECLARATION

I here by declare that this PhD dissertation is my original work and has not been presented for a degree in any other universities, and that all sources of material used for the dissertation have been duly acknowledged.

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