STUDIES ON PION COMPTON SCATTERING AND PION ELECTROPRODUCTION IN THE FRAMEWORK OF SCALAR ELECTRODYNAMICS

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ABSTRACT

In this thesis we first discuss the S-matrix expansion in the interaction representation. We employ this to illustrate the calculational techniques in the study of one of the QED processes-i.e., electron Compton scattering. We then formulate Scalar Electrodynamics which is the relativistic theory for the interaction of scalar particles (such as $\pi^0, \pi^{\pm}$) with the electromagnetic fields A_μ. In this process we work out the mathematical forms of the pion propagator. We then use Scalar Electrodynamics to study the processes such as pion Compton scattering and pion Electroproduction. We give explicit evaluation of the transition amplitudes and the scattering cross sections for both these processes. We then compare the pion Compton scattering results with the corresponding results for the electron Compton scattering .

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Contents

CHAPTER 1

Introduction

During the past thirty years many ingenious and powerful accelerators have been developed to probe the physics deep into the core of the nucleus of the atom. The phenomenon at the most fundamental level can be understood if we understand the physics governing the interactions of elementary particles and the fields. Without this it is impossible to appreciate and understand nuclear and particle physics.

In this thesis we will first develop the framework of covariant perturbation theory and obtain the matrix elements of the S-matrix in the interaction representation. We then see how the same matrix elements can be obtained much more elegantly from the electron propagator in the language of Feynman diagrams. We will employ these Feynman diagrams as a convenient pictorial device that enables us to keep track of the various terms in the matrix elements of the S-matrix between specified initial and final states of some QED processes which we will consider initially. This theory will then be generalized to study the relativistic interactions of scalar particles (such as pions etc.) with electromagnetic fields–the Scalar Electrodynamics. Using this theory of Scalar Electrodynamics we will then calculate the transition amplitudes and cross sections of some processes such as pion Compton scattering and Pion Electroproduction. Throughout this thesis we will employ the natural system of units (i.e., $\hbar = c = 1$).

To familiarize with the theoretical apparatus, chapter 2 deals with the scattering matrix ($S-$ matrix) expansion in the interaction picture in which one can determine the overall change of the state of the system due to interaction. We then develop the mathematical forms of the electron propagator which enable us to write down the S- matrix element and hence the transition amplitude corresponding to any given physical process represented by Feynman diagrams. Finally, we will apply the above rules to one of the QED processes i.e., the electron Compton scattering just as an illustration of the techniques developed.

In chapter 3, we will be dealing with scalar particles (such as pions). We develop the relativistic framework for the study of interactions of scalar (spin 0) particles with an electromagnetic field. In this process we drive the pion propagator. This will enable us to calculate the S-matrix elements to arbitrary orders of interaction strength. Next we will apply the above formulation of Scalar Electrodynamics to the process of pion Compton scattering and calculate the transition amplitude and the scattering cross section for this process. We further compare the pion Compton scattering results with that of the electron Compton scattering results we already know of.

In chapter 4 we first develop the mathematical form of the photon propagator and next we calculate the transition amplitude and the differential cross section for the Pion Electroproduction. In this chapter we will study the process $e^+e^- \rightarrow \pi^+\pi^-$ which proceeds through the production of a virtual photon. This process involves the production of hadrons (π^+, π^-) due to interactions of two leptons. We calculate the transition amplitude and the scattering cross section for this process in the framework of Scalar Electrodynamics. The angular distribution of the cross section is then studied.

Chapter 5 is devoted to discussions.

CHAPTER 2

Electron Compton Scattering

Compton scattering is one of the most fundamental processes describing the scattering of electromagnetic radiation with matter. In this chapter we will consider first the field theoretic treatment of electron Compton scattering just as an illustration of the techniques employed in the solution of this problem. These techniques will then be generalized to the process of pion Compton scattering in chapter 3. This chapter is organized as follows. In section 2.1 we will consider S-matrix expansion in the interaction representation. 2.2 The electron propagator. Finally in section 2.3 we will consider electron Compton scattering cross section.

2.1 S-matrix expansion in the interaction representation

Let Φ denotes the wave function that describes the state of a system. The Hamiltonian of the system is $H=H_o+H_I$, where H_o is the unperturbed Hamiltonian and H_I is the interaction Hamiltonian.

Let Φ_n be the eigenfunctions of the unperturbed Hamiltonian, each corresponding to certain definite values of all the occupation numbers.Then any function Φ can be expanded as $\Phi = \sum C_n \Phi_n$. Then the exact wave equation

$$i\frac{\partial}{\partial t}\Phi = (H_o + H_I)\Phi, \tag{2.1}$$

becomes a set of equations for the coefficient C_n [1]:

$$i\frac{\partial}{\partial t}C_n = \sum_m (H_I)_{nm} e^{i(E_n-E_m)t} C_m, \tag{2.2}$$

where $(H_I)_{nm}$ are the time independent matrix elements of the operator H_I, and E_n the energy levels of the unperturbed system.

By definition, the operator H_I does not depend explicitly on the time. The quantities

$$(H_I)_{nm}(t) = (H_I)_{nm} e^{i(E_n-E_m)t}, \tag{2.3}$$

on the other hand, may be regarded as matrix elements of the time dependent operator

$$H_I(t) = e^{iH_o t} H_I e^{-iH_o t}. \tag{2.4}$$

This is said to be an operator in the interaction representation, as opposed to the original time independent Schrodinger operator H_I. Now denoting the wave function in this new representation by the same letter Φ,we can write equation (2.2) symbolically as

$$i\frac{\partial}{\partial t}\Phi = H_I(t)\Phi. \tag{2.5}$$

The change in the wave function in this representation is due entirely to the action of the perturbation, i.e, it corresponds to processes which results from the interaction of the particles. A formal solution of Eq.(2.5) can be written as

$$\Phi(t) = U_I(t, t_o)\Phi(t_o) \ , \tag{2.6}$$

where $\Phi(t_o)$ is the state vector that characterizes the system at some fixed time t_0 and $U_I\ (t,t_o)$ is the time evolution operator which satisfies

$$U_I(t_o,t_o) = 1. \tag{2.7}$$

It then follows that

$$i\frac{\partial}{\partial t}U_I(t,t_o) = H_I(t)U_I(t,t_o). \tag{2.8}$$

But the differential equation(2.8) and the initial condition (2.7) can be combined to give a single integral equation [2]

$$U_I(t,t_o) = 1 - i\int_{t_o}^{t} dtH_I(t)U_I(t,t_o). \tag{2.9}$$

Let us solve this by the iteration procedure. We have

$$\begin{aligned} U_I(t,t_o) &= 1 - i\int_{t_o}^{t} dt_1 H_I(t_1)[1 - i\int_{t_o}^{t_1} dt_2 H_I(t_2)U_I(t_2,t_o)] \\ &= 1 - i\int_{t_o}^{t} dt_1 H_I(t_1) + (-i)^2\int_{t_o}^{t} dt_1\int_{t_o}^{t_1} dt_2 H_I(t_1)H_I(t_2) + ... \\ &\quad +(-i)^n\int_{t_o}^{t} dt_1\int_{t_o}^{t_1} dt_2 ... \int_{t_o}^{t_{n-1}} dt_n H_I(t_1)H_I(t_2)...H_I(t_n) \\ &\quad +... \end{aligned} \tag{2.10}$$

The physical meaning of $U_I(t,t_o)$ is as follows. Suppose the system is known to be in the state $\mid i\rangle$ at time t_o. The probability of finding the system in the state $\mid f\rangle$ at some later time t is given by

$$P_{fi} = \mid \langle f \mid U_I(t,t_o) \mid i\rangle \mid^2 = \mid (U_I)_{fi}(t,t_o)|^2. \tag{2.11}$$

The transition probability per unit time from $| i\rangle \rightarrow | f\rangle$ is

$$w = \frac{1}{t-t_o} \mid (U_I)_{fi}(t,t_o) \mid^2, \tag{2.12}$$

Although this ratio has a rather complicated time dependence when the time interval t-t$_o$ is small, it approaches a definite limit as $t_0 \rightarrow -\infty$ and $t \rightarrow +\infty$, as we shall see explicitly in working out actual physical problems in the next section. This motivates us to define what is known as the S-matrix by

$$S = U_I(+\infty, -\infty) \ . \tag{2.13}$$

From (2.6) we see that the S-matrix connects $\Phi(+\infty)$ and $\Phi(-\infty)$ according to

$$\Phi(\infty) = S\Phi(-\infty). \tag{2.14}$$

On account of (2.10), one can express the S-matrix as

$$\begin{aligned} S &= S^{(o)} + S^{(1)} + S^{(2)} + ... \\ &= 1 - i\int_{-\infty}^{\infty} dt_1 H_I(t_1) + (-i)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 H_I(t_1) H_I(t_2) + ... \end{aligned}$$

$$+(-i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 ... \int_{-\infty}^{t_{n-1}} dt_n H_I(t_1) H_I(t_2) ... H_I(t_n) + ... \tag{2.15}$$

Whether or not the expansion (2.15) is useful depends on the strength of the interaction. In quantum electrodynamics the S-matrix expansion converges rapidly so that we obtain results that agree extremely well with observation just by considering the first few terms. This is because the dimensionless coupling constant

$\alpha = \frac{1}{137}$ is small compared to unity. The S-matrix is unitary

$$SS^\dagger = S^\dagger S = 1. \tag{2.16}$$

This guarantees that the sum of probabilities of finding the system in various states $|f\rangle$ at t=∞ if it was initially in a given state $|i\rangle$ at t=-∞ is 1. i.e.,

$$\sum_f |S_{fi}|^2 = 1. \tag{2.17}$$

2.2 The electron propagator

The relativistic equation of motion of a free Dirac particle is given by

$$(\gamma_\mu \partial_\mu + m)\Psi = 0 \;, \tag{2.18}$$

where Ψ is a 4 component wave function of the particle,

γ_μ are Dirac matrices,

$\partial_\mu = (\partial_i, \partial_4)$, i=1,2,3.

$\gamma_\mu = (\gamma_i, \gamma_4)$,

where γ_i = $-i\beta\alpha_i$ and $\gamma_4 = \beta$, and in matrix form, β and α_i are:

$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$ where 1 is 2×2 unit matrix and σ_i are the pauli matrices. i.e.,

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

When a relativistic particle of charge e and mass m is placed in an electromagnetic field $A_\mu = (A_i, i\Phi)$, the Hamiltonian is obtained by replacing:

$P_i \to P_i - eA_i$ or $\partial_i \to \partial_i - ieA_i$ and $E \to E - e\Phi$ or $\partial_t \to \partial_t + ie\Phi$

i.e,

$$p_\mu \to p_\mu - eA_\mu, \tag{2.19a}$$

or

$$\partial_\mu \to \partial_\mu - ieA_\mu \;, \tag{2.19b}$$

where $\partial_\mu = (\partial_i, -i\partial_t)$ and $p_\mu = (p_i, iE)$.

Substituting (2.19b) into (2.18) yields

$$[\gamma_\mu(\partial_\mu + ieA_\mu) - m]\Psi = 0, \tag{2.20a}$$

where $\mu = 1, 2, 3, 4$.

$$[\gamma_\mu \partial_\mu + m]\Psi = ie\gamma_\mu A_\mu \Psi, \tag{2.20b}$$

which is the Dirac equation in the presence of a 4 vector potential, A_μ.

Since Feynman's approach is based directly on the wave equation we first review the way of solving an inhomogeneous differential equation in mathematical physics. Consider the Poisson's equation

$$\nabla^2 \Phi(x) = -\rho(x) \; . \tag{2.21}$$

The solution to this equation with the boundary condition $\Phi(x) \to 0$ as $| x | \to \infty$ is given by

$$\Phi(x) = \int G(x, x')\rho(x')d^3x' \; , \tag{2.22}$$

where

$$G(x, x') = \frac{1}{4\pi \mid x - x' \mid}. \tag{2.23}$$

The function $G(x, x')$ is called the Green's function for Poisson's equation and satisfies

$$\nabla^2 G(x,x') = -\delta^3(x-x'). \tag{2.24}$$

In other words, instead of solving (2.21) directly, we first solve the unit source problem (2.24). Once we obtain the solution to this simpler problem, the solution to the more general problem (2.21) can be written immediately.

In relativistic electron theory we are interested in solving the Dirac equation, for an electron moving in an electromagnetic field [3]

$$(\gamma_\mu \frac{\partial}{\partial x_\mu} + m)\Psi = ie\gamma_\mu A_\mu \Psi, \tag{2.25}$$

where Ψ is the electron field. The right hand side of (2.25) is very much like the source $\rho(x)$ in (2.21). Let's consider a unit source problem in analogy to electrostatics:

$$(\gamma_\mu \frac{\partial}{\partial x_\mu} + m)K(x,x') = -i\delta^4(x-x'). \tag{2.26}$$

Writing the Dirac indices explicitly, we have

$$[(\gamma_\mu)_{\alpha\beta} \frac{\partial}{\partial x_\mu} + m\delta_{\alpha\beta}]K_{\beta\gamma}(x,x') = -i\delta^4(x-x')_{\alpha\gamma}. \tag{2.27}$$

The function $K(x,x')$ is the Green's function for the free particle Dirac equation just as G(x,x') is the Green's function for Poisson's equation. In complete analogy with (2.22) we expect that a solution Ψ to the more complex differential equation (2.25) satisfies

$$\Psi(x) = -\int K(x,x')e\gamma_\mu A_\mu(x')\Psi(x')d^4x'. \tag{2.28}$$

That this is indeed the case can be proved by direct substitution:

$$\begin{aligned}
&(\gamma_\mu \tfrac{\partial}{\partial x_\mu} + m)[-\textstyle\int K(x,x')e\gamma_\mu A(x')\Psi(x')d^4x'] \\
&\quad = \int i\delta^4(x-x')e\gamma_\mu A_\mu(x')\Psi(x')d^4x' \ , \\
&\quad = ie\gamma_\mu A_\mu \Psi.
\end{aligned} \tag{2.29}$$

Note, however, that even if we add to (2.28) any solution to the free wave equation (e=0), the differential equation (2.25) will still be satisfied. Thus, we can write

$$\Psi(x) = \Psi_0(x) - \int K(x,x')e\gamma_\mu A_\mu(x')\Psi(x')d^4x', \tag{2.30}$$

where $\Psi_0(x)$ is a free particle solution to the Dirac equation . We can easily verify that (2.30) satisfies (2.25) just as well. In a scattering problem, for instance, $\Psi_0(x)$ may represent an incident plane wave which would be present even if there were no interaction.

There is an important difference between the differential equation in electrostatics(2.21) and the Dirac equation (2.25). The right hand side of (2.21) does not contain Φ. So we could write the solution (2.22) immediately in a closed form. In contrast, in the case of (2.25) the function Ψ itself appears on the right hand side. As a result (2.30) is an integral equation.

So, even if we obtain an explicit form of $K(x, x')$ as we shall in a moment, we cannot write the exact solution to (2.25) in a closed form. However, if a perturbation expansion in powers of e can be justified, we can obtain an approximate solution to (2.25) accurate to any desired power of e by the iteration method

$$\begin{aligned}\Psi(x) &= \Psi_0(x) + \int d^4x' K(x,x')[-e\gamma_\mu A_\mu(x')]\Psi_0(x') \\ &+ \int d^4x' \int d^4x'' K(x,x')[-e\gamma_\mu A_\mu(x')]K(x',x'')[-e\gamma_\nu A_\nu(x'')\Psi_0(x'')] \\ &+ \int d^4x' \int d^4x'' \int d^4x''' K(x,x')[-e\gamma_\mu A_\mu(x')]K(x',x'') \\ &\times [-e\gamma_\nu A_\nu(x'')]K(x'',x''')[-e\gamma_\lambda A_\lambda(x''')]\Psi_0(x''') + ... \qquad (2.31)\end{aligned}$$

Our next task is to obtain the explicit form of $K(x, x')$. First, because of the translation... both space and time) it is clear that $K(x, x')$ is a ...ut loss of generality we can set $x' = 0$. We solve ...ourier transform method. Let us define $\widetilde{K}(p)$

$$\ldots \int \widetilde{K}(p)\, e^{ip.x} d^4p. \qquad (2.32)$$

...) in (2.26) with $x' = 0$, we get

$$\ldots = -i\delta^4(x),$$

$$\ldots = -\frac{i}{(2\pi)^4} \int e^{ip.x} d^4p. \qquad (2.33)$$

So the ... to a simple algebraic equation

in mome...

$$(i\gamma.p + m)\widetilde{K}(p) = -i, \tag{2.34a}$$

$$\widetilde{K}(p) = \frac{-i}{i\gamma.p + m}, \tag{2.34b}$$

where $\frac{1}{i\gamma.p+m}$ is understood to be a 4x4 matrix such that if we multiply with $i\gamma.p+m$ from the right or from the left, we obtain the 4x4 identity matrix.

Clearly we can write (2.34b) as

$$\widetilde{K}(p) = -i\frac{(-i\gamma.p + m)}{(i\gamma.p + m)(-i\gamma.p + m)} = \frac{-i\gamma.p + m}{i(p^2 + m^2)}, \tag{2.35}$$

since $(i\gamma.p)^2 = -p^2$. Going back to coordinate (x) space, we get using Eq. (2.35)

$$K(x, 0) = \frac{-i}{(2\pi)^4}\int d^4p\frac{(-i\gamma.p + m)}{p^2 + m^2}e^{ip.x}. \tag{2.36}$$

Or , more generally

$$K(x, x') = \frac{-i}{(2\pi)^4}\int d^4p\frac{(-i\gamma.p + m)}{p^2 + m^2}e^{ip.(x-x')}. \tag{2.37}$$

When we integrate the integrand of (2.37) along the real p_0-axis, there are poles at $p_0 = \pm E$ where $E = (|\vec{p}|^2 + m^2)^{\frac{1}{2}}$

$$K(x, x') = \frac{-i}{(2\pi)^4}\int d^3\vec{p}\, e^{i\vec{p}.(\vec{x}-\vec{x}')}\int_{-\infty}^{\infty} dp_0\frac{(-i\gamma.p + m)}{(-p_0 + E)(p_0 + E)}e^{-ip_0(x_0 - x_0')}. \tag{2.38}$$

The particular form of $K(x, x')$ depends on the particular manner in which we go around the poles in the complex p_0−plane. That this kind of ambiguity exists is to be expected on physical grounds. Since we have not yet specified the boundary conditions to be used in connection with the differential equation (2.38).

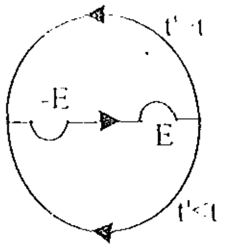

Figure 2.1: Prescription for the integration (2.38) in the complex p_0-plane.

We shall now consider the electron propagator $K_F(x, x')$ corresponding to different conditions.

In the complex p_0−plane, $K_F(x, x')$ corresponds to the choice of contour indicated in fig. (2.1) when $t > t'$ and $t < t'$. The Green's function obtained by the following prescription shown in figure (2.1) is denoted by $K_F(x, x')$. In other words

$$K_F(x, x') = \frac{-i}{(2\pi)^4} \int d^4p \frac{(-i\gamma.p + m)}{p^2 + m^2 - i\epsilon} e^{ip.(x-x')}, \qquad (2.39)$$

since

$$p^2 + m^2 = \mid \overrightarrow{p} \mid^2 + p_4^2 + m = \mid \overrightarrow{p} \mid^2 + m^2 - p_0^2 = E^2 - p_0^2 = (E - p_0)(E + p_0),$$

the integrand of Eq.(2.39) has two poles at $P_0 = \pm E$

To integrate the time component of (2.39), the contour of integration is as shown in figure (2.1) for $t > t'$ and $t < t'$.

For $t > t'$:

$$\oint dp_0 \frac{(-i\gamma.p+m)}{(E-P_0)(E+P_0)-i\epsilon} e^{-iP_0(t-t')}$$

$$= \lim_{R\to\infty\epsilon\to 0} \int_{-R}^{R} dp_0 \frac{(-i\gamma.p+m)}{(E-p_0)(E+p_0)-i\epsilon} e^{-ip_0(t-t')}$$

$$+ \lim_{R\to\pm\infty\epsilon\to 0} \int_C dp_0 \frac{(-i\gamma.p+m)}{(E-p_0)(E+p_0)-i\epsilon} e^{-ip_0(t-t')}. \quad (2.40)$$

But the second integral vanishes as $R \to \pm\infty$,and since the contour of integration enclosed in the negative sense:

$$\lim_{R\to\infty,\epsilon\to 0} \int_{-R}^{R} dp_0 \frac{(-i\gamma.p+m)}{(E-p_0)(E+p_0)-i\epsilon} e^{-ip_0(t-t')} = -2\pi i a_{-1}, \quad (2.41)$$

$$\text{where } a_{-1} = \lim_{p_0\to E\epsilon\to 0} \frac{(-i\gamma.p+m)(p_0-E)}{(E-p_0)(E+p_0)-i\epsilon} e^{-ip_0(t-t')}, \quad (2.42a)$$

$$= -\frac{(-i\gamma.\overrightarrow{p}+\gamma_4 E+m)}{2E} e^{-iE(t-t')}. \quad (2.42b)$$

Therefore, substituting $(2.42b)$ into (2.41), Eq.(2.39) equals

$$K_F(x,x') = \frac{-i(2\pi i)}{(2\pi)^4} \int d^3 \overrightarrow{p} \; \frac{(-i\gamma.\overrightarrow{p}+\gamma_4 E+m)}{2E} \times e^{ip.(x-x')}, \quad (2.43a)$$

$$K_F(x,x') = \frac{1}{(2\pi)^3} \int d^3 \overrightarrow{p} \frac{(-i\gamma.\overrightarrow{p}+\gamma_4 E+m)}{2E} e^{ip.(x-x')}, t > t'. \quad (2.43b)$$

According to hole theory, Eq.$(2.43b)$ defined as the propagator for an electron with positive energy state to propagate forward in time.

Similarly for $t < t'$,the contour of integration is the upper half circle as shown in fig.(2.1). Therefore, the residue, a_{-1}

$$a_{-1} = \lim_{p_0\to -E,\epsilon\to 0} \frac{(-i\gamma.p+\gamma_4 p_0+m)(E+p_0)}{(E-p_0)(E+p_0)-i\epsilon} e^{-ip_0(t-t')}, \quad (2.44a)$$

$$= -\frac{(i\gamma.p+\gamma_4 E-m)}{2E} e^{iE(t-t')}. \quad (2.44b)$$

Eq. (2.41) is equal to $2\pi i a_{-1}$. Thus

$$K_F(x,x') = \frac{i(2\pi i)}{(2\pi)^4} \int d^3\overrightarrow{p} \frac{(i\gamma.p + \gamma_4 E - m)}{2E} e^{ip.(x-x')+iE(t-t')}, t < t' \quad (2.45a)$$

$$= \frac{-1}{(2\pi)^3} \int d^3\overrightarrow{p} \frac{(i\gamma.p + \gamma_4 E - m)}{2E} e^{i[p.(x-x')+E(t-t')]}, \quad (2.45b)$$

or

$$K_F(x,x') = \frac{-1}{(2\pi)^3} \int d^3\overrightarrow{p} \frac{(i\gamma.p + \gamma_4 E - m)}{2E} e^{i[p.(x-x')-E_{neg}(t-t')]}. \quad (2.45c)$$

But Eq.(2.45c) can be defined as the propagator over all the negative energy states. But a particle of mass m charge,-e and energy,-E propagating backward in time with momentum,p is the same as that of the antiparticle with mass m charge,+e and energy,+E propagating forward in time with momentum,-p. Therefore Eq.(2.45c) can be rewritten as

$$K_F(x,x') = \frac{-1}{(2\pi)^3} \int d^3p \frac{(-i\gamma.p + \gamma_4 E - m)}{2E} e^{i[-p(x-x')+E(t-t')]} \quad (2.45d)$$

But using the projection operators

$$\sum_s u^{(s)}(\overrightarrow{p})\overline{u}^{(s)}(\overrightarrow{p}) = \frac{-i\gamma.p + m}{2m}, \quad (2.46a)$$

and

$$\sum_s v^{(s)}(\overrightarrow{p})\overline{v}^{(s)}(\overrightarrow{p}) = -\frac{(i\gamma.p + m)}{2m} \quad (2.46b)$$

where $u^{(s)}(\overrightarrow{p})$ and $v^{(s)}(\overrightarrow{p})$ are the bispinor amplitudes for a free Dirac particle in the positive energy and in the negative energy states respectively and s lables the spin states.

And using the formula

$$\frac{1}{(2\pi)^3}\int d^3\overrightarrow{p} = \lim_{V\to\infty}\frac{1}{V}\sum_{\overrightarrow{p}}. \tag{2.47}$$

$Eq.$ (2.39) can be expressed as

$$\begin{aligned} K_F(x,x') &= \lim_{V\to\infty}\sum_{\overrightarrow{p},s}(\frac{m}{EV})u^{(s)}(\overrightarrow{p})\overline{u}^{(s)}(\overrightarrow{p})e^{ip.(x-x')}, t > t' & \text{(2.48a)} \\ &= -\lim_{V\to\infty}\sum_{\overrightarrow{p},s}(\frac{m}{EV})v^{(s)}(\overrightarrow{p})\overline{v}^{(S)}(\overrightarrow{p})e^{-ip.(x-x')}, t < t' & \text{(2.48b)} \end{aligned}$$

From equation (2.31) in lowest order in e, we have

$$\Psi(x) = \Psi_0(x) + \int d^4x' K(x,x')\left[-e\gamma_\mu A_\mu(x')\right]\Psi_0(x'), \tag{2.49}$$

$$\Psi(x) = \Psi_0(x) + \int d^4x' K_F(x,x')\left[-e\gamma_\mu A_\mu(x')\right]\Psi_0(x'). \tag{2.50}$$

To go from (2.49) to (2.50), we use $K_F(x,x')$.

Substituting (2.48a) and (2.48b) into ((2.50) yields

$$\begin{aligned} \Psi(x) &= \Psi_0(x) + [\lim_{V\to\infty}\int d^4x' \sum_{\overrightarrow{p}}\sum_{s=1}^{2}\sqrt{\frac{m}{E'V}}\overline{u}^{(s')}(\overrightarrow{p}') \\ &\times \quad \left[-e\gamma_\mu A_\mu(x')\right]\Psi_0(x')e^{-ip'.x'}]\sqrt{\frac{m}{E'V}}u^{(s')}(\overrightarrow{p}')e^{ip.x} \\ &+(-)[\lim_{V\to\infty}\int d^4x' \sum_{\overrightarrow{p}}\sum_{s=1}^{2}\sqrt{\frac{m}{E'V}}\overline{v}^{(s')}(\overrightarrow{p}')\left[-e\gamma_\mu A_\mu(x')\right] \end{aligned}$$

$$\times\Psi_0(x')e^{ip'.x'}]\sqrt{\frac{m}{E'V}}v^{(s')}\left(\overrightarrow{p}'\right)e^{-ip'.x}. \tag{2.51}$$

$$\begin{aligned}\Psi\left(x\right) &= \Psi_0\left(x\right)+\sum_{\overrightarrow{p}'}\sum_{s'=1}^{2}C^{(+)}_{p',s'}\left(t\right)\sqrt{\frac{m}{EV}}u^{(s')}\left(\overrightarrow{p}'\right)e^{ip'.x}\\ &\quad+\sum_{\overrightarrow{p}'}\sum_{s'=1}^{2}C^{(-)}_{p',s'}\left(t\right)\sqrt{\frac{m}{EV}}v^{(s)}\left(\overrightarrow{p}\right)e^{-ip.x}.\end{aligned} \tag{2.52}$$

where

$$C^{(+)}_{p',s'}\left(t\right)=\int d^3x'\int_{-\infty}^{t}dt'\sqrt{\frac{m}{E'V}}\overline{u}^{(s')}\left(\overrightarrow{p}'\right)e^{-ip'.x'}\left[-e\gamma_\mu A_\mu\left(x'\right)\right]\Psi_0\left(x'\right), \tag{2.53a}$$

and

$$C^{(-)}_{p',s'}\left(t\right)=-\int d^3x'\int_{t}^{\infty}dt'\sqrt{\frac{m}{E'V}}\overline{v}^{(s')}\left(\overrightarrow{p}'\right)e^{ip'.x'}\left[-e\gamma_\mu A_\mu(x')\right]\Psi_0\left(x'\right). \tag{2.53b}$$

Let us assume that $\Psi_0\left(x\right)$ is the normalized wave function for a positive energy plane wave, characterized by p and s. According to the usual interpretation of wave mechanics, $\left|C^{(+)}_{p',s'}\left(t\right)\right|^2$ gives the probability for finding the electron at t in a positive energy state characterized by p', s' (assumed to be different from p,s) when the electron is known with certainty to be in state p,s in the remote past. In fact, from section 2.1, we see that $C^{(+)}_{p',s'}\left(\infty\right)$ is precisely the first order correction to the S-matrix for the transition of an electron characterized by (p, s) into another state characterized by (p', s'). This is reasonable because a positive energy electron state is known to be able to make a transition into some other positive energy state in the presence of an external potential. However $C^{(-)}_{p',s'}\left(t\right)$ does go to zero as $t\rightarrow\infty$. Physically this means that the positive energy electron cannot make an energy conserving transition into a negative energy state even in the presence of a time dependent external potential. On the other hand $C^{(-)}_{p',s'}(-\infty)$ does not in general vanish whereas $C^{(+)}_{p',s'}(-\infty)$ goes to zero. As a result $\Psi(t=-\infty)$ is not equal to Ψ_0.

In the S-matrix expansion $S^{(1)}_{fi} = -i\int_{-\infty}^{\infty} dt_1 \langle i|H_I(t_1)|f\rangle$. But for field theoretic application $\mathcal{H}_I = -ie\overline{\Psi}\gamma_\mu\Psi A_\mu$ and $H_I(t_1) = -ie\int d^3x_1\overline{\Psi}\gamma_\mu\Psi A_\mu$. Therefore

$$S^{(1)}_{fi} = (-i)^2\int d^4x e\overline{\Psi}_f\gamma_\mu\Psi_i A_\mu$$

$$= \int d^4x\sqrt{\tfrac{m}{EV}}\overline{u}(\overrightarrow{p_f})\, e^{-ip_f.x}\left[-e\gamma_\mu A_\mu\right]\Psi_i,$$

where $\overline{\Psi}_f(x) = \sqrt{\frac{m}{EV}}\overline{u}(\overrightarrow{p}_f)\, e^{-ip_f.x}$.

Comparing this expression with Eq.(2.53a), we see that $S^{(1)}_{fi}$ is exactly identical to $C^{(+)}(\infty)$. In analogous manner, one can obtain the matrix element of the S-matrix for the higher order corrections using the iteration procedure.

2.3 The electron Compton scattering cross section

The scattering of a photon by a free electron is known as Compton scattering. This calculation of the transition amplitude for Compton scattering is being given merely as an illustration of the calculational techniques employed. These techniques will later be generalized to the study of one of the Scalar Electrodynamics processes-the pion Compton scattering in chapter 3.

There are two Feynman diagrams for this process in lowest order as shown in fig.2.3. In fig.2.3(a), the electron is initially in state which is characterized with momentum $p_1$and spin s_1 absorbs a quantum of photon of momentum $k_1$and

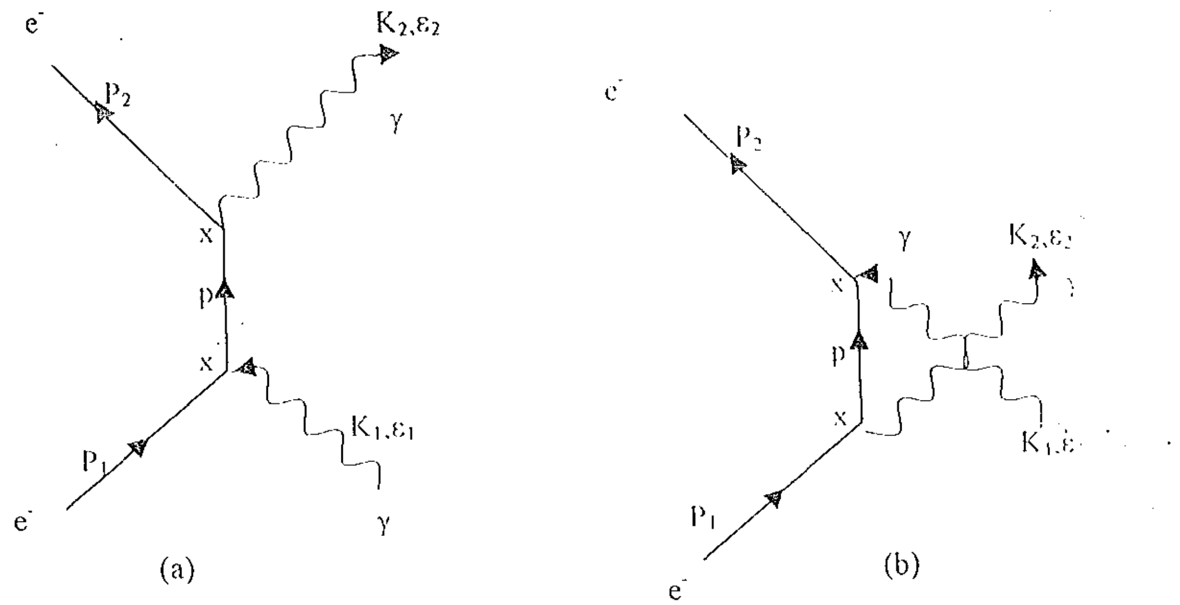

Figure 2.2: Feynman diagrams (a) for direct process (b) for exchange process in lowest order

polarization vector ϵ_1 and emits a quantum of photon of momentum k_2 and polariztion vector ϵ_2 to arrive at a final state of momentum p_2 and spin s_2. Similarly, in fig. 2.3(b) the act of emission precedes that of absorption. The 4-momentum conservation holds at each of the vertices in fig. 2.3(a) and 2.3(b) .

The invariant transition amplitude $S_{fi}^{(2)}$ is

$$S_{fi}^{(2)} = S_{fi}^{(2)} \mid_d + S_{fi}^{(2)} \mid_e, \tag{2.54}$$

where $S_{fi}^{(2)} \mid_d$ is the transition amplitude from state p_1, s_1 to p_2, s_2 for the direct process, and $S_{fi}^{(2)} \mid_e$ is the transition amplitude from state p_1, s_1 to p_2, s_2 for the exchange process. And

$$S_{fi}^{(2)}\mid_d = C_{p,s}^{(+)(2)}(\infty), \tag{2.55a}$$

$$\begin{aligned} S_{fi}^{(2)}|_d &= \int\int d^4x d^4x' \sqrt{\frac{m}{E_2 V}} \overline{u}^{(s_2)}\left(\overrightarrow{p}_2\right) e^{-ip_2.x}\left[-e\gamma_\mu A_\mu(x)\right] K(x,x') \\ &\times \left[-e\gamma_\mu A_\mu(x') \sqrt{\frac{m}{E_1 V}} u^{(s_1)}\left(\overrightarrow{p}_1\right) e^{ip_1.x'}\right], \end{aligned} \tag{2.55b}$$

where

$$A_\mu(x) = \frac{1}{\sqrt{2\omega_2 V}} \epsilon_{\mu 2}(\alpha) e^{-ik_2.x} \text{ -- wave function of the outgoing photon,} \tag{2.56a}$$

and

$$A_\mu(x') = \frac{1}{\sqrt{2\omega_1 V}} \epsilon_{\mu 1}(\alpha) e^{ik_1.x'} \text{ -- wave function of the incoming photon.} \tag{2.56b}$$

Substituting $(2.56a)$ and $(2.56b)$ into $(2.55b)$ yields

$$\begin{aligned} S_{fi}^{(2)}|_d &= -ie^2 \frac{m}{(2\pi)^4 \sqrt{E_1 E_2} V} \int\int d^4x d^4x' \overline{u}_2 \gamma.\epsilon_2 \left[\frac{-i\gamma.p + m}{p^2 + m^2}\right] \gamma.\epsilon_1 \\ &\times u_1 \frac{1}{2\sqrt{\omega_1\omega_2} V} \quad e^{i(p_1+k_1).x'} e^{-i(p_2+k_2).x} e^{ip.(x-x')}, \end{aligned} \tag{2.57a}$$

where for the sake of simplification we put $u^{(s_1)}\left(\overrightarrow{p}_1\right)$ by u_1, $\overline{u}^{(s_2)}\left(\overrightarrow{p}_2\right)$ by $\overline{u}_2$ and $\epsilon_{\mu 1}(\alpha)$ by ϵ_1, $\epsilon_{\mu 2}(\alpha)$ by ϵ_2.

After carrying out the integrations in Eq.$(2.57a)$ one obtains

$$\begin{aligned} S_{fi}^{(2)}|_d &= \frac{-ie^2 m}{2\sqrt{E_1 E_2 \omega_1 \omega_2} V^2} \overline{u}_2 \gamma.\epsilon_2 \left[\frac{-i\gamma.(p_1+k_1) + m}{(p_1+k_1)^2 + m^2}\right] \gamma.\epsilon_1 u_1 \\ &\times (2\pi)^4 \delta^4(p_1 + k_1 - p_2 - k_2). \end{aligned} \tag{2.57b}$$

Similarly $S_{fi}^{(2)}|_e$ is equivalent to $S_{fi}^{(2)}|_d$ by replacing $k_1 \rightleftarrows -k_2$, $\epsilon_1 \rightleftarrows \epsilon_2$. Thus

$$S_{fi}^{(2)}|_e = \frac{-ie^2 m}{2\sqrt{E_1 E_2 \omega_1 \omega_2 V^2}} \overline{u}_2 \gamma.\epsilon_1 \left[\frac{-i\gamma.(p_1 - k_2) + m}{(p_1 - k_2)^2 + m^2} \right] \gamma.\epsilon_2 u_1 \times (2\pi)^4 \delta^4 (p_1 + k_1 - p_2 - k_2). \quad (2.58)$$

The total transition amplitude is therefore

$$S_{fi}^{(2)} = \frac{-ie^2 m}{2\sqrt{E_1 E_2 \omega_1 \omega_2 V^2}} \overline{u}_2 \{\gamma.\epsilon_2 \left[\frac{-i\gamma.(p_1 + k_1) + m}{(p_1 + k_1)^2 + m^2} \right] \gamma.\epsilon_1 + \gamma.\epsilon_1 \left[\frac{-i\gamma.(p_1 - k_2) + m}{(p_1 - k_2)^2 + m^2} \right] \gamma.\epsilon_2 \} u_1 (2\pi)^4 \delta^4 (p_1 + k_1 - p_2 - k_2) \quad (2.59)$$

Since the electron and photon are free

$$p_1^2 + m^2 = p_2^2 + m^2 = 0 \text{ and } k_1^2 = k_2^2 = 0. \quad (2.60)$$

If we choose the rest frame of the initial electron and also choose the polarization vectors to have only space component then by a suitable gauge transformation we have

$$p_1.\epsilon_1 = p_1.\epsilon_2 = 0, \quad (2.61)$$

where

$$p_1 = (m, 0),\ \epsilon = (0, \overrightarrow{\epsilon}), \quad (2.62a)$$

and

$$k_1.\epsilon_1 = k_2.\epsilon_2 = 0, \text{ transversality condition.} \quad (2.62b)$$

And moreover from the property of γ-matrices we have

$$(\gamma.p)(\gamma.\epsilon) = -(\gamma.\epsilon)(\gamma.p) + 2p.\epsilon, \quad (2.63)$$

and a free Dirac spinor satisfies the equation

$$[-i\gamma.p + m]\, u\,(p) = 0. \tag{2.64}$$

Thus

$$\gamma.\epsilon_2\,[-i\gamma.\,(p_1+k_1)+m]\,\gamma.\epsilon_1 u(p_1) = -i\gamma.\epsilon_2\,(2p_1.\epsilon_1)\,u(p_1) + i\gamma.\epsilon_2\gamma.\epsilon_1\gamma.k_1 u\,(p_1)\,. \tag{2.65}$$

Using equations (2.60) , (2.61) ,and (2.65) , equation (2.59) reduces to

$$\begin{aligned} S_{fi}^{(2)} &= \frac{-ie^2 m}{2\sqrt{E_1E_2\omega_1\omega_2}V^2}\overline{u}_2\left(\frac{i\,(\gamma.\epsilon_2)\,(\gamma.\epsilon_1)\,(\gamma.k_1)}{2p_1.k_1}+\frac{i\,(\gamma.\epsilon_1)\,(\gamma.\epsilon_2)\,(\gamma.k_2)}{2p_1.k_2}\right)u_1 \\ &\quad \times (2\pi)^4\,\delta^4\,(p_1+k_1-p_2-k_2)\,. \end{aligned} \tag{2.66}$$

To obtain the transition probability, we shall have to square $S_{fi}^{(2)}$. If we are not interested in the spin of the final electron, and if we restrict ourselves to the case of unpolarized initial electron, then summing $|S_{fi}^{(2)}|^2$over s_2 and averaging over s_1 corresponds to carrying out a certain trace operation. This calculation then yields the result

$$\begin{aligned} \frac{1}{2}\sum_{s_1,s_2}|S_{fi}^{(2)}|^2 &= \frac{1}{2}\frac{e^4m^2}{4E_1E_2\omega_1\omega_2V^4}Tr\left[\left(\frac{-i\gamma.p_2+m}{2m}\right)\overline{O}\left(\frac{-i\gamma.p_1+m}{2m}\right)O\right] \\ &\quad \times[(2\pi)^4\,\delta^4\,(p_1+k_1-p_2-k_2)]^2, \end{aligned} \tag{2.67a}$$

where O is the matrix elements sandwiched between $\overline{u}_2$ and u_1 and $\overline{O} = \gamma_4 O^+\gamma_4$. Thus we can express $\frac{1}{2}\sum\limits_{s_1,s_2}|S_{fi}^{(2)}|^2$ as

$$\frac{1}{2}\sum_{s_1,s_2}|S_{fi}^{(2)}|^2 = \frac{e^4m^2}{16m^2E_1E_2\omega_1\omega_2V^4}\left\{\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}-2+4\,(\epsilon_1.\epsilon_2)^2\right\}$$

$$\times \left[(2\pi)^4 \delta^4 (p_1 + k_1 - p_2 - k_2)\right]^2, \tag{2.67b}$$

where we have used the fact that in the laboratory frame $p_1 = (m, 0, 0, 0)$.

The transition probability per unit time

$$\frac{1}{2}\frac{\sum_{s_1,s_2} |S_{fi}^{(2)}|^2}{T} = \frac{e^4 m^2}{E_1 E_2 \omega_1 \omega_2 V^4} \frac{V}{16m^2} \left[\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} - 2 + 4 (\epsilon_1.\epsilon_2)^2\right]$$
$$\times \quad (2\pi)^4 \delta^4 (p_1 + k_1 - p_2 - k_2). \tag{2.68}$$

The differential cross-section $d\sigma$ is thus

$$d\sigma = \frac{1}{2}\frac{\sum_{s_1,s_2} |S_{fi}^{(2)}|^2/T}{|F|} \frac{V}{(2\pi)^3} d^3\overrightarrow{p_2} \frac{V}{(2\pi)^3} d^3\overrightarrow{k_2}, \tag{2.69}$$

where F is the flux of the incident particles. Therefore

$$d\sigma = \frac{e^4 m^2}{E_1 E_2 \omega_1 \omega_2 V^3} \frac{1}{16m^2} \left[\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} - 2 + 4 (\epsilon_1.\epsilon_2)^2\right] / |j_e + j_\gamma|$$
$$\times (2\pi)^4 \delta^4 (p_1 + k_1 - p_2 - k_2) \frac{V^2}{(2\pi)^6} d^3\overrightarrow{p_2} d^3\overrightarrow{k_2}, \tag{2.70}$$

where F$= |j_e + j_\gamma|$ is the total incident flux i.e., the incident current density of the electron and photon. And

$$|j_e + j_\gamma| = \frac{1}{V} \left|\frac{\overrightarrow{p}_{1,i}}{E_1} - \frac{\overrightarrow{k_1}}{\omega_1}\right|. \tag{2.71a}$$

But in the lab. frame of the initial electron $\overrightarrow{p}_{1,i} = 0$, and since $|\overrightarrow{k_1}| = \omega_1$, then

$$|F| = \frac{1}{V}. \tag{2.71b}$$

Putting (2.71a) and (2.71b) in (2.70) and integrating over $\vec{p}_2$ yields

$$d\sigma = \frac{e^4}{16E_1E_2\omega_1\omega_2}\left[\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}-2+4\left(\epsilon_1.\epsilon_2\right)^2\right] \times\delta\left(E_1+\omega_1-E_2-\omega_2\right)d^3\vec{k_2}/\left(2\pi\right)^2. \quad (2.72)$$

We note that

$$d^3\vec{k_2} = |\vec{k_2}|^2 d|\vec{k_2}|d\Omega_2 = \omega_2^2 d\omega_2 d\Omega_2 = \omega_2^2\frac{d\omega_2}{dE_f}dE_f d\Omega_2, \quad (2.73)$$

where

$$E_f = \omega_2 + E_2 = \omega_2 + \sqrt{m^2+\left(\omega_1-\omega_2\right)^2} \quad (2.74)$$

$$E_f = \omega_2 + \sqrt{m^2+\omega_1^2+\omega_2^2-2\omega_1\omega_2\cos\theta}, \quad (2.75a)$$

with θ being the angle between the initial and final momenta of the photon. Therefore, using Eq. (2.75a) one can obtain an expression for $\frac{dE_f}{d\omega_2}$ as

$$\frac{dE_f}{d\omega_2} = 1+\frac{2\left(\omega_2-\omega_1\cos\theta\right)}{2E_2} = \frac{E_2+\omega_2-\omega_1\cos\theta}{E_2}. \quad (2.75b)$$

Substituting (2.73) into (2.72) and integrating it over E_f yields

$$d\sigma = \frac{e^4}{16E_1E_2\omega_1\omega_2}\left[\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}-2+4\left(\epsilon_1.\epsilon_2\right)^2\right]\omega_2^2/\left(2\pi\right)^2 d\omega_2/dE_f d\Omega_2. \quad (2.76)$$

Substituting (2.75b) into (2.76) leads to

$$d\sigma = \frac{e^4}{64\pi^2E_1E_2}\left(\frac{\omega_2}{\omega_1}\right)\left[\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}-2+4\left(\epsilon_1.\epsilon_2\right)^2\right]\frac{E_2}{E_2+\omega_2-\omega_1\cos\theta}d\Omega_2. \quad (2.77)$$

In the lab. frame of the initial electron, $\frac{E_2}{E_2+\omega_2-\omega_1\cos\theta} = \frac{\omega_2E_2}{m\omega_1}$. Therefore

$$d\sigma = \frac{e^4}{64\pi^2m^2}\left(\frac{\omega_2}{\omega_1}\right)^2\left[\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}-2+4\left(\epsilon_1.\epsilon_2\right)^2\right]d\Omega_2, \quad (2.78)$$

which is the Klein Nishina formula.

For non- relativistic limit, $\omega_1,\omega_2 \ll m, \Longrightarrow \omega_1 \simeq \omega_2$,

$$d\sigma = \frac{e^4}{64\pi^2 m^2} 4\left(\epsilon_1.\epsilon_2\right)^2 d\Omega_2, \quad (2.79a)$$

$$= \alpha^2/m^2 \left(\epsilon_1.\epsilon_2\right)^2 d\Omega_2, \quad (2.79b)$$

where $\alpha^2 = \frac{e^4}{16\pi^2}$ and since $\frac{\alpha^2}{m^2} = r_0^2$ [the classical electron radius].

$$d\sigma_{(NR)} = r_0^2 \left(\epsilon_1.\epsilon_2\right)^2 d\Omega_2, \quad (2.80)$$

which is the classical Thomson formula for the scattering of low energy radiation by an atomic electron.

From conservation of 4 momentum we have

$$p_2 = p_1 + k_1 - k_2. \quad (2.81a)$$

Squaring both sides of Eq.(2.81a) leads to

$$p_1 k_1 = p_1 k_2 + k_1 k_2, \quad (2.81b)$$

where we have used the fact that $p_1^2 = p_2^2 = -m^2$ and $k_1^2 = k_2^2 = 0$.

After some simplification (2.81b) reduces to

$$\frac{\omega_2}{\omega_1} = \frac{m}{m + \omega_1 (1 - \cos\theta)}, \quad (2.81c)$$

where θ is the angle between the momentum of the incident and the outgoing photons. When $\theta = 0, \frac{\omega_2}{\omega_1} = 1$, therefore Eq.(2.78) reduces to Thomson's formula whatever the energy of the incident photon may be.

We have found that the differential cross section for electron Compton scattering depends on the incident photon energy. There is also a shift in frequency of the photon when it scatters off a free electron. In the non-relativistic limits the electron Compton scattering cross section reduces to Thomson formula for the scattering of a low energy photon by an atomic electron. We see that near the forward direction, (i.e., when $\theta = 0$) the scattering cross sections of electron Compton scattering reduces to the non relativistic limit whatever the incident photon energy may be.

CHAPTER 3

Scalar Electrodynamics

Pions are unstable particles of mass around 270 times the electron mass. There are three different types of pions: π^+, π^-, and π^0. The charged pions ($\pi^\pm$) decay into a muon and a neutrino, while the neutral pion decays into two photons. The decay of the neutral pion into two photons prove that its spin $\overrightarrow{s} = 0$. Thus pions are scalar particles. This chapter is devoted to developing the mathematical techniques needed for the study of some of the fundamental processes in Scalar Electrodynamics which is the relativistic theory governing the electromagnetic interactions of scalar particles e.g π^0, $\pi^\pm$ [4].

Now for a relativistic equation for scalar particles, we must start with the correct relativistic energy-momentum relation. Energy and momentum appears as the 'time' and 'space' components of the momentum 4-vector.

$$p_\mu = (E, \overrightarrow{p}), \tag{3.1}$$

which satisfy the condition

$$p^2 = p_\mu^2 = E^2 - |\overrightarrow{p}|^2 = M^2, \tag{3.2}$$

where M is the rest mass of the particle and $\mu = 0, 1, 2, 3$.

Klein and Gordon attempted to build relativistic mechanics from the squared energy relation. Let Φ is the wave function of the klein Gordon particle which satisfies the mass shell condition

$$\left(E^2 - |\overrightarrow{p}|^2\right)\Phi = M^2\Phi. \tag{3.3}$$

Replacing $p \rightarrow -i\overrightarrow{\nabla}$ and $E \rightarrow i\frac{\partial}{\partial t}$ in (3.3), the free particle Klein-Gordon equation is :

$$\left[(\frac{i\partial}{\partial t})^2 - \left(-i\overrightarrow{\nabla}\right)^2\right]\Phi = M^2\Phi, \tag{3.4}$$

$$\left(\Box^2 - M^2\right)\Phi = 0, \tag{3.5}$$

where $\Box^2 = \left(\overrightarrow{\nabla}^2 - \frac{\partial^2}{\partial t^2}\right) = -\partial_\mu^2$ is the 4-dimensional Laplacian . We now proceed to obtain the equation of motion of a relativistic Klein-Gordon particle interacting with electromagnetic field $A_\mu = \left(\Phi, \overrightarrow{A}\right)$.

In the presence of an electromagnetic field, the equation of motion of the Klein-Gordon particle is modified by making the usual gauge invariant replacements. Therefore

$$\partial_\mu \rightarrow \partial_\mu + ieA_\mu, \tag{3.6}$$

where $\partial_\mu = (\partial_t, -\partial_i)$ and i=1,2,3.

Substituting (3.6) into (3.5) yields

$$\left(\Box^2 - M^2\right)\Phi = \left(ie\partial_\mu A_\mu + ieA_\mu\partial_\mu - e^2A_\mu^2\right)\Phi, \tag{3.7}$$

which is the KG equation in the presence of a potential $A_\mu = (\Phi, A)$.

Since the Klein Gordon equation is second order differential equation [5], equation (3.7) in an electromagnetic field contains an extra $e^2 A_\mu^2\ \Phi$ term. Thus if we study a process like pion Compton scattering, the scattering amplitude for this process will receive contribution from an additional diagram arising from this $e^2 A_\mu^2 \Phi$ term in Eq.(3.7). This chapter is organized as follows. In section 3.1 we will obtain the form of the pion propagator and in sec.3.2 we will consider pion Compton scattering cross section.

3.1 The Pion Propagator

Now let's denote by $D\left(x - x'\right)$ the pion propagator which satisfies(like spin 1/2 Dirac particles) the differential equation

$$\left[\Box^2 - M^2\right] D\left(x - x'\right) = -\delta^4\left(x - x'\right). \tag{3.8}$$

If we set x' =0, $Eq.$ (3.8) takes the form

$$\left(\Box^2 - M^2\right) D\left(x\right) = -\delta^4\left(x\right), \tag{3.9}$$

where $\Box^2 = -\partial_\mu^2$.

Since the Fourier transform of $D\left(x\right)$ in 4 dimensional space is given by

$$D\left(x\right) = \frac{1}{\left(2\pi\right)^4}\int d^4p\widetilde{D}\left(p\right)e^{ip.x}, \text{ where } p.x = p_0x_0 - \overrightarrow{p}.\overrightarrow{x}. \tag{3.10}$$

Substituting (3.10) into (3.9) yields

$$\left(\Box^2 - M^2\right) \frac{1}{(2\pi)^4} \int d^4p \widetilde{D}(p) e^{ip.x} = -\delta^4(x), \tag{3.11a}$$

i.e.,

$$\frac{1}{(2\pi)^4} \int d^4p \left(p^2 - M^2\right) \widetilde{D}(p) e^{ip.x} = -\delta^4(x), \tag{3.11b}$$

since $\delta^4(x) = \frac{1}{(2\pi)^4} \int d^4p e^{ip.x}$ in 4 dimensional space. Eq. (3.11b) reduces to

$$\left(p^2 - M^2\right) \widetilde{D}(p) = -1, \tag{3.12a}$$

$$\widetilde{D}(p) = \frac{-1}{(p^2 - M^2)}, \tag{3.12b}$$

which is the pion propagator in momentum- energy space.

Going back to the space-time representation (3.10) becomes

$$D(x) = \frac{-1}{(2\pi)^4} \int d^4p \frac{1}{p^2 - M^2} e^{ip.x}. \tag{3.13a}$$

In general

$$D(x - x') = \frac{-1}{(2\pi)^4} \int d^4p \frac{1}{p^2 - M^2} e^{ip.(x-x')}. \tag{3.13b}$$

Like the Feynman approach to the electron propagator in section 2.3 one can define the wave function of KG particle using equation (3.7) and (3.8).

Let $\Phi(x')$ be the wave function of the KG particle at some time t'. The wave function of this particle $\Phi(x)$ at time t, $(t > t')$ is:

$$\begin{aligned}\Phi(x) &= \Phi_0(x) - \int d^4x' D(x-x') \\ &\times \left\{ ie\frac{\partial A_\mu(x')}{\partial x'_\mu} + ieA_\mu(x')\frac{\partial}{\partial x'_\mu} - e^2A_\mu^2(x') \right\} \Phi_0(x'), \qquad (3.14)\end{aligned}$$

where $D(x-x')$ is the Green's function of the KG equation and $\Phi_0(x)$ is the wave function in the absence of the perturbation.

We see in the above equation that it is not in a closed form like that of a spin $\frac{1}{2}$ particle in the previous section. Therefore, using iteration method, one can expand (3.14) as follows:

$$\begin{aligned}\Phi(x) &= \Phi_0(x) - \int d^4x' D(x-x') \left\{ ie\frac{\partial A_\mu(x')}{\partial x'_\mu} + ieA_\mu(x')\frac{\partial}{\partial x'_\mu} - e^2A_\mu^2(x') \right\} \\ &\times\Phi_0(x') + (-1)^2 \int\int d^4x'd^4x'' D(x-x') ie\frac{\partial A_\mu(x')}{\partial x'_\mu} + ieA_\mu(x')\frac{\partial}{\partial x'_\mu} \\ &-e^2A_\mu^2(x')\} D(x'-x'') \left\{ ie\frac{\partial A_\mu(x'')}{\partial_\mu x''} + ieA_\mu(x'')\frac{\partial}{\partial x''_\mu} - e^2A_\mu^2(x'') \right\} \\ &\times\Phi_0(x'') + ... \qquad (3.15)\end{aligned}$$

The physical meaning of this expansion is as follows. The first term on right hand side of Eq.(3.15) is unperturbed wave function , the second term is the first order correction to the wave function and the third term denotes the second order correction to the wave function , etc.

If we take the first order correction for $\Phi(x)$, we have

$$\Phi(x) = -\int d^4x' D(x-x') \left\{ ie\frac{\partial A_\mu(x')}{\partial x'_\mu} + ieA_\mu(x')\frac{\partial}{\partial x'_\mu} - e^2 A_\mu^2(x') \right\} \Phi_0(x'). \tag{3.16}$$

Separating space and time component of Eq. (3.13b) we obtain

$$D(x-x') = \frac{-1}{(2\pi)^4}\int\int d^3\overrightarrow{p}\, dp_0 \frac{1}{p^2-M^2} e^{i[E(t-t')-p.(x-x')]}. \tag{3.17}$$

And since

$$p^2 - M^2 = p_0^2 - |\overrightarrow{p}|^2 - M^2 = p_0^2 - E^2 = (p_0 - E)(p_0 + E),$$

$$\begin{aligned} D(x-x') &= \frac{-1}{(2\pi)^4}\int\int d^3\overrightarrow{p}\, dp_0 \frac{1}{(p_0-E)(p_0+E)} \\ &\quad \times e^{i[p_0.(t-t')-p.(x-x')]}. \end{aligned} \tag{3.18}$$

When we integrate the integrand of (3.18) along the real p_0-axis, there are two poles at $p_0 = \pm\sqrt{|\overrightarrow{p}|^2 + M^2}$. And if we give E has an infinitesimal negative imaginary part, the contour of integration in the complex p_0-plane for (3.18) appears as shown in fig. 3.1. For $t < t'$, the contour of integration is the upper semicircle as shown in fig. (3.1). Thus the integral over p_0 in (3.18) receives contribution only from the pole $p_0 = -E$. This integral in Eq.(3.18) for $t < t'$ is thus evaluated as

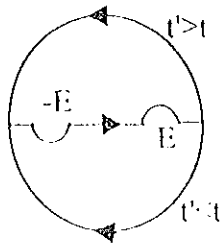

Figure 3.1: Prescription for the integration (3.18) in the complex p_0-plane.

$$\oint dp_0 \frac{1}{(p_0 - E)(p_0 + E) - i\epsilon} e^{ip_0(t-t')} = \lim_{R\to\infty,\epsilon\to 0} \int_{-R}^{R} dp_0 \frac{1}{(p_0 - E)(p_0 + E) - i\epsilon} \times e^{ip_0(t-t')} + \lim_{R\to\infty,\epsilon\to 0} \int_{c} dp_0 \frac{1}{(p_0 - E)(p_0 + E) - i\epsilon} \times e^{ip_0(t-t')}. \tag{3.19}$$

Integration over the curve vanishes as R→ ±∞,and Eq.(3.19) reduced to

$$\oint dp_0 \frac{1}{(p_0 - E)(p_0 + E) - i\epsilon} e^{ip_0(t-t')} = \lim_{R\to\infty,\epsilon\to 0} \int_{-R}^{R} dp_0 \frac{1}{(p_0 - E)(p_0 + E) - i\epsilon} \times e^{ip_0(t-t')}, \tag{3.20a}$$

since the contour enclosed in the positive sense

$$\lim_{R\to\infty,\epsilon\to 0} \int_{-R}^{R} dp_0 \frac{1}{(p_0 - E)(p_0 + E) - i\epsilon} e^{ip_0(t-t')} = 2\pi i a_{-1} \tag{3.20b}$$

where the residue, a_{-1} is:

$$a_{-1} = \lim_{p_0 \to -E, \epsilon \to 0} \frac{(p_0 + E)}{(p_0 - E)(p_0 + E) - i\epsilon} e^{ip_0 .(t-t')}, \tag{3.21a}$$

$$= -\frac{1}{2E} e^{-iE(t-t')}. \tag{3.21b}$$

After substituting (3.21*b*) into (3.20*b*), equation (3.18) becomes

$$D(x - x') = \frac{i}{(2\pi)^3} \int \frac{d^3 \overrightarrow{p}}{2E} e^{-ip.(x-x')}. \tag{3.22}$$

To arrive at Eq.(3.22) , we use the substitution p$\to -p$ for negative energy particle. Since according to the hole theory an antiparticle defined as a particle with negative energies propagating backwards in time. Similarly for $t > t'$, the contour of integration is the lower semicircle as shown in fig.(3.1) and the integral receives contribution only from the pole $p_0 = E$. The expression for $D(x - x')$ for $t > t'$ is similarly worked out as,

$$D(x - x') = \frac{i}{(2\pi)^3} \int \frac{d^3 \overrightarrow{p}}{2E} e^{ip.(x-x')}. \tag{3.23}$$

According to hole theory, Eq.(3.23) is the propagator for the particle with positive energies propagating forward in time and similarly (3.22) is the antiparticle propagator.

Using Eq. (2.47), Eq. (3.23) takes the form

$$D(x - x') = i \lim_{V \to \infty} \sum_{\overrightarrow{p}} \left[\frac{1}{\sqrt{2EV}} e^{ip.x} \right] \left[\frac{1}{\sqrt{2EV}} e^{-ip.x'} \right]. \tag{3.24}$$

In general

$$D\left(x-x'\right)=\left\{\begin{array}{c} i\lim\limits_{V\to\infty}\sum\limits_{\overrightarrow{p}}\frac{1}{\sqrt{2EV}}e^{ip.x}\frac{1}{\sqrt{2EV}}e^{-ip.x\prime},t>t' \\ i\lim\limits_{V\to\infty}\sum\limits_{\overrightarrow{p}}\frac{1}{\sqrt{2EV}}e^{-ip.x}\frac{1}{\sqrt{2EV}}e^{ip.x'},t<t' \end{array}\right\}. \tag{3.25}$$

where the first form is the propagator to all positive energy states of a particle and the second form is the propagator for the corresponding antiparticle.

Eq.(3.16) can be rewrite as

$$\begin{aligned}\Phi\left(x\right) &= -\int d^3x'\int_{-\infty}^{t}dt'D\left(x-x'\right) \\ &\times\left\{ie\frac{\partial A_\mu\left(x'\right)}{\partial x'_\mu}+ieA_\mu\left(x'\right)\frac{\partial}{\partial x'_\mu}-e^2A_\mu^2\left(x'\right)\right\}\Phi_0\left(x'\right) \\ &+\left(-1\right)\int d^3x'\int_{t}^{\infty}dt'D\left(x-x'\right) \\ &\times\left\{ie\frac{\partial A_\mu\left(x'\right)}{\partial x'_\mu}+ieA_\mu\left(x'\right)\frac{\partial}{\partial x'_\mu}-e^2A_\mu^2\left(x'\right)\right\}\Phi_0\left(x'\right),\end{aligned} \tag{3.26a}$$

If we substitute the first expression for $D\left(x-x'\right)$ in the first integral and the second expression for $D\left(x-x'\right)$ in the second integral from (3.25) in the above equation, one obtains

$$\begin{aligned}\Phi\left(x\right) &= -i\sum_{\overrightarrow{p}'}\int d^3x'\int_{-\infty}^{t}dt'\frac{1}{\sqrt{2E'V}}e^{-ip'\cdot x'}\left\{ie\frac{\partial A_\mu}{\partial x'_\mu}+ieA_\mu\left(x'\right)\frac{\partial}{\partial x'_\mu}-e^2A_\mu^2\left(x'\right)\right\} \\ &\times\Phi_0\left(x'\right)\left[\frac{1}{\sqrt{2EV}}e^{ip'\cdot x}\right]+\left(-i\right)\sum_{\overrightarrow{p}'}\int d^3x'\int_{t}^{\infty}dt'\frac{1}{\sqrt{2E'V}}e^{ip'.x'} \\ &\times\{ie\frac{\partial A_\mu}{\partial x'_\mu}+ieA_\mu\frac{\partial}{\partial x'_\mu}-e^2A_\mu^2\left(x'\right)\}\times\Phi_0\left(x'\right)\left[\frac{1}{\sqrt{2EV}}e^{-ip'.x}\right].\end{aligned} \tag{3.26b}$$

Or we can express $\Phi\left(x\right)$ as

$$\Phi\left(x\right)=C_{p'}^{(+)}\left(t\right)\frac{1}{\sqrt{2EV}}e^{ip'\cdot x}+C_{p'}^{(-)}\left(t\right)\frac{1}{\sqrt{2EV}}e^{-ip'.x}, \tag{3.26c}$$

where

$$C_{p'}^{(+)}(t) = (-i)\int d^3x' \int_{-\infty}^{t} dt' \frac{1}{\sqrt{2E'V}} e^{-ip'.x'}$$
$$\times \left\{ ie\frac{\partial A_\mu(x')}{\partial x'_\mu} + ieA_\mu(x')\frac{\partial}{\partial x'_\mu} - e^2 A_\mu^2(x') \right\} \times \Phi_0(x'), \quad (3.27a)$$

and

$$C_{p'}^{(-)}(t) = (-i)\int d^3x' \int_{t}^{\infty} dt' \frac{1}{\sqrt{2E'V}} e^{ip'.x'}$$
$$\times \{ ie\frac{\partial A_\mu(x')}{\partial x'_\mu} + ieA_\mu(x')\frac{\partial}{\partial x'_\mu} - e^2 A^2(x')\}\Phi_0(x'). \quad (3.27b)$$

But $C^{(+)}(\infty) = S_{fi}^{(1)}$ is the first order correction to the S-matrix expansion. We neglect the last term as compared with the first two terms. In analogous manner we can find the higher order corrections to the S-matrix expansion [6]. Now

$$S_{fi}^{(1)} = -i\int d^4x' \frac{1}{\sqrt{2E'V}} e^{-ip'.x'} \left\{ ie\frac{\partial A_\mu(x')}{\partial x'_\mu} + ieA_\mu(x')\frac{\partial}{\partial x'_\mu} \right\} \Phi_0(x'). \quad (3.28)$$

But one can rewrite (3.28) as follows

$$S_{fi}^{(1)} = -i\int d^4x' \left\{ -\frac{ie}{\sqrt{2E'V}} A_\mu(x')\frac{\partial}{\partial x'_\mu}\left[e^{-ip'.x'}\right] + ieA_\mu(x')\frac{\partial}{\partial x'_\mu} \right\} \Phi_0(x'), \quad (3.29)$$

since $\overrightarrow{A} \to 0$ as $|x| \to \infty$ and $A_4 \to 0$ as $t \to \pm\infty$. Thus

$$\frac{1}{\sqrt{2E'V}} \int d^4x' e^{-ip'.x'} \frac{\partial A_\mu(x')}{\partial x'_\mu} = \frac{1}{\sqrt{2E'V}} e^{-ip'.x'} A_\mu(x') - \int d^4x' \frac{1}{\sqrt{2E'V}} A_\mu(x')$$
$$\times \frac{\partial}{\partial x'_\mu}\left[e^{-ip'.x'}\right]. \quad (3.30)$$

The right hand side of (3.30) is obtained by using integration by parts. But the first term on right hand side vanishes. Thus

$$\frac{1}{\sqrt{2E'V}}\int d^4x' e^{-ip'.x'}\frac{\partial A_\mu(x')}{\partial x'_\mu} = -\frac{1}{\sqrt{2E'V}}\int d^4x' A_\mu(x')\frac{\partial}{\partial x'_\mu}\left[e^{-ip'.x'}\right]. \quad (3.31)$$

As a result, Eq. (3.29) can be put in the following form

$$S^{(1)}_{fi} = -i\int d^4x'\left\{-ieA_\mu(x')\frac{\partial \Phi^*_f(x')}{\partial x'_\mu}\Phi_i(x') + ieA_\mu(x')\Phi^*_f(x')\frac{\partial \Phi_i(x')}{\partial x'_\mu}\right\}, \quad (3.32)$$

where $\Phi^*_f(x') = \frac{1}{\sqrt{2E'V}}e^{-ip'.x'}$,

which implies that

$$S^{(1)}_{fi} = -i\int d^4x' ie\left[-\frac{\partial \Phi^*_f(x')}{\partial x'_\mu}\Phi_i(x') + \Phi^*_f(x')\frac{\partial \Phi_i(x')}{\partial x'_\mu}\right]A_\mu(x'), \quad (3.33a)$$

$$= -i\int d^4x'\left\{[j_\mu(x')]_{fi}A_\mu(x')\right\}. \quad (3.33b)$$

Here $(j_\mu)_{fi} = ie\left[-\frac{\partial \Phi^*_f(x')}{\partial x'_\mu}\Phi_i(x') + \Phi^*_f(x')\frac{\partial \Phi_i(x')}{\partial x'_\mu}\right]$ is the 4 electromagnetic transition current for the KG particle.

3.2 The pion Compton scattering cross-section

There are three Feynman diagrams for $\gamma - \pi^-$ scattering in lowest order as shown in fig. 3.2.

Fig. 3.2 (a) and 3.2 (b) are interpreted like that of electron Compton scattering whereas fig. 3.2 (c) is due to the $e^2A^2_\mu$ term in the KG equation in a field and it

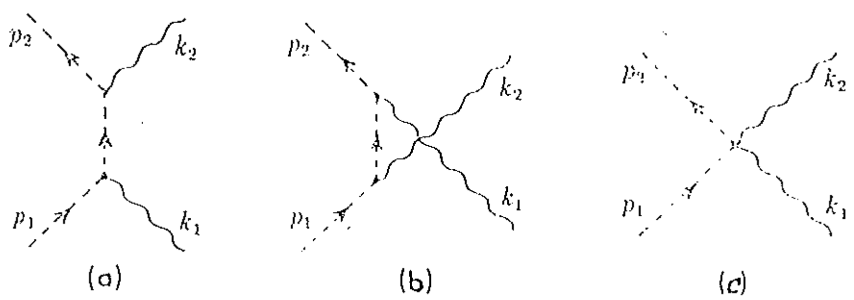

Figure 3.2: Feynman diagrams (a)direct process (b)exchange process (c) contact term in lowest order.

is interpreted as the simultaneous emission and absorption of photons without the intermediate state and it is called contact term. This diagram has no counterpart in the $e^- - \gamma$ scattering process (shown in fig. 2.1). Since the $\pi^- - \gamma$ scattering process is second order in e, the total transition amplitude is due to the sum of the contributions from the direct and exchange diagrams plus the contribution due to the occurrence of $e^2 A_\mu^2$ term in Eq.(3.27b) for $S_{fi}^{(1)}$. Thus

$$S_{fi}^{(2)} = S_{fi}^{(2)}|_d + S_{fi}^{(2)}|_e + S_{fi}^{(2)}|_c, \tag{3.34}$$

Since

$$\begin{aligned}\Phi^{(2)}(x) &= (-1)^2 \int\int d^4x' d^4x'' D(x-x') \\ &\times \left(ie\frac{\partial A_\mu(x')}{\partial x'_\mu} + ieA_\mu(x')\frac{\partial}{\partial x'_\mu} - e^2 A_\mu^2(x') \right) D(x'-x'')\end{aligned}$$

$$\times\left(ie\frac{\partial A_{\mu}\left(x''\right)}{\partial x''_{\mu}}+ieA_{\mu}\left(x''\right)\frac{\partial}{\partial x''_{\mu}}-e^{2}A_{\mu}^{2}\left(x''\right)\right)\Phi_{i}\left(x''\right), \quad (3.35a)$$

using Eq. (3.25), one can rewrite Eq. (3.35a) as

$$\Phi^{(2)}\left(x\right)=C^{(+)(2)}\left(t\right)\left[\frac{1}{\sqrt{2E'V}}e^{ip'.x}\right]+C^{(-)(2)}\left(t\right)\left[\frac{1}{\sqrt{2E'V}}e^{-ip'.x}\right], \quad (3.35b)$$

where

$$\begin{aligned}C^{(+)(2)}\left(t\right) &= -i\left(-1\right)^{2}\int\int d^{4}x'd^{4}x''\frac{1}{\sqrt{2E'V}}ie^{-ip'.x'}\\ &\times\left(ie\frac{\partial A_{\mu}\left(x'\right)}{\partial x'_{\mu}}+ieA_{\mu}\left(x'\right)\frac{\partial}{\partial x'_{\mu}}-e^{2}A_{\mu}^{2}\left(x'\right)\right)D\left(x'-x''\right)\\ &\times\left\{ie\frac{\partial A_{\mu}\left(x''\right)}{\partial x''_{\mu}}+ieA_{\mu}\left(x''\right)\frac{\partial}{\partial x''_{\mu}}-e^{2}A_{\mu}^{2}\left(x''\right)\right\}\Phi_{i}\left(x''\right), \quad (3.37a)\end{aligned}$$

and

$$\begin{aligned}C^{(-)(2)}\left(t\right) &= -i\left(-1\right)^{2}\int\int d^{4}x'd^{4}x''\frac{1}{\sqrt{2E'V}}ie^{ip'.x'}\\ &\times\left(ie\frac{\partial A_{\mu}\left(x'\right)}{\partial x'_{\mu}}+ieA_{\mu}\left(x'\right)\frac{\partial}{\partial x'_{\mu}}-e^{2}A_{\mu}^{2}\left(x'\right)\right)D\left(x'-x''\right)\\ &\times\left\{ie\frac{\partial A_{\mu}\left(x''\right)}{\partial x''_{\mu}}+ieA_{\mu}\left(x''\right)\frac{\partial}{\partial x''_{\mu}}-e^{2}A_{\mu}^{2}\left(x''\right)\right\}\Phi_{i}\left(x''\right), \quad (3.37b)\end{aligned}$$

Therefore, using the equality between $S_{fi}^{(2)}$ and $C^{(+)(2)}$ which we developed earlier, i.e.,$S_{fi}^{(2)}=C^{(+)(2)}\left(\infty\right)$, one can obtain an expression for $S_{fi}^{(2)}$ as

$$\begin{aligned}S_{fi}^{(2)} &= -i\left(-1\right)^{2}\int\int d^{4}x'd^{4}x''\frac{1}{\sqrt{2E'V}}A_{\mu}\left(x'\right)\left\{-ie\frac{\partial}{\partial x'_{\mu}}\left[e^{-ip'.x'}\right]+iee^{-ip'.x'}\frac{\partial}{\partial x'_{\mu}}\right\}\\ &\times A_{\mu}\left(x''\right)\left\{-ie\frac{\partial}{\partial x''_{\mu}}\left[D\left(x'-x''\right)\right]+ieD\left(x'-x''\right)\frac{\partial}{\partial x''_{\mu}}\right\}\Phi_{i}\left(x''\right), \quad (3.38)\end{aligned}$$

where we have neglected the contact term in lowest order. We know that the initial and final pion wave functions are

$$\Phi_i\left(x''\right)=\frac{1}{\sqrt{2E_1V}}e^{ip_1.x''},\text{ and }\Phi_f\left(x'\right)=\frac{1}{\sqrt{2E_2V}}e^{-ip_2.x'},\tag{3.39a}$$

and the emitted photon and the absorbed photon wave function are

$$A_\mu\left(x''\right)=\frac{\epsilon_{1\mu}}{\sqrt{2\omega_2V}}e^{-ik_2.x''},\text{ and }A_\mu\left(x'\right)=\frac{\epsilon_{2\mu}}{\sqrt{2\omega_1V}}e^{ik_1.x'}.\tag{3.39b}$$

Substituting $(3.23),(3.39a),(3.39b)$ into (3.38) yields

$$\begin{aligned}S_{fi}^{(2)}|_d &= -i\int\int d^4x'd^4x\frac{1}{\sqrt{2E_2V}}\left\{-ie(-ip_{2\mu})+ie\left(ip_\mu\right)\right\}\\ &\times\frac{\epsilon_{2\mu}}{\sqrt{2\omega_2V}}\int\frac{1}{(2\pi)^4}d^4p\frac{1}{p^2-M^2}e^{ip.(x'-x)}\\ &\times\frac{1}{2\sqrt{E_1\omega_1V}}\left\{-ie(-ip_\mu)+ie\left(ip_{1\mu}\right)\right\}\epsilon_{1\mu}\\ &\times e^{-ip_2.x'}e^{ip_1.x}e^{-ik_2.x'}e^{ik_1.x}.\end{aligned}\tag{3.40a}$$

Thus

$$\begin{aligned}S_{fi}^{(2)}|_d &= \frac{-i(i^2e)^2}{4\sqrt{E_1E_2\omega_1\omega_2V^2}}\int\int d^4xd^4x'\left[p_{2\mu}+p_\mu\right]\epsilon_{2\mu}\\ &\times\frac{1}{(2\pi)^4}\int d^4p\frac{1}{p^2-M^2}e^{ip.(x'-x)}\left[p_\mu+p_{1\mu}\right]\epsilon_{1\mu}\\ &\times e^{i(p-p_2-k_2).x'}e^{i(p_1+k_1-p).x},\end{aligned}\tag{3.40b}$$

Integration over x and x' leads to

$$\begin{aligned}S_{fi}^{(2)}|_d &= \frac{-i\left(i^2e\right)^2}{4\sqrt{E_1E_2\omega_1\omega_2V^2}}\left(p_2+p\right).\epsilon_2\frac{1}{(2\pi)^4}\int d^4p\frac{1}{p^2-M^2}\left(p+p_1\right).\epsilon_1\\ &\times(2\pi)^8\,\delta^4\left(p-p_2-k_2\right)\delta^4\left(p_1+k_1-p\right).\end{aligned}\tag{3.40c}$$

Integration the above equation over p leads to

$$S_{fi}^{(2)}|_d = \frac{-ie^2}{4\sqrt{E_1 E_2 \omega_1 \omega_2} V^2} (p_2 + p_1 + k_1).\epsilon_2 \frac{1}{(p_1+k_1)^2 - M^2} (2p_1 + k_1).\epsilon_1$$
$$\times (2\pi)^4 \delta^4 (p_1 + k_1 - p_2 - k_2). \qquad (3.40d)$$

Now $S_{fi}^{(2)}|_e$ is obtained from $S_{fi}^{(2)}|_d$ by replacing $k_1 \rightleftharpoons -k_2$ and $\epsilon_1 \rightleftharpoons \epsilon_2$. It is observed that if we take the contribution to $S_{fi}^{(2)}$ from direct and exchange terms alone (as in fig. 3.2 (a, b)), then $S_{fi}^{(2)}$ is not gauge invariant. However if one adds the contribution from the contact term to $S_{fi}^{(2)}$ then $S_{fi}^{(2)}$ is gauge invariant. Thus we take into account the effect of direct, exchange and contact terms in the construction of $S_{fi}^{(2)}$. The A.p term taken twice is of the same order as the A.A term so far as powers of e are concerned. Thus we see that $S_{fi}^{(2)}$ for the direct and exchange diagrams in fig. 3.2(a) and 3.2(b) is of the same power in e as $S_{fi}^{(1)}$ for the contact A.A term. Thus

$$S_{fi}^{(2)}|_c = \frac{2ie^2}{4\sqrt{E_1 E_2 \omega_1 \omega_2} V^2} \epsilon_1.\epsilon_2 (2\pi)^4 \delta^4 (p_1 + k_1 - p_2 - k_2). \qquad (3.40e)$$

The factor 2 in $S_{fi}|_c$ is due to the emission and absorption of either photons.

Eq. (3.34) becomes

$$S_{fi}^{(2)} = \frac{-ie^2}{4\sqrt{E_1 E_2 \omega_1 \omega_2} V^2} \{(p_2 + p_1 + k_1).\epsilon_2 \frac{1}{(p_1+k_1)^2 - M^2} (2p_1 + k_1).\epsilon_1$$

$$+ (p_2 + p_1 - k_2) \epsilon_1 \frac{1}{(p_1 - k_2)^2 - M^2} (2p_1 - k_2).\epsilon_2 - 2\epsilon_1.\epsilon_2\} (2\pi)^4 \delta^4 (p_1 + k_1 - p_2 - k_2).$$
$$(3.41)$$

Integrating Eq. (3.46b) over $\overrightarrow{p}_2$ yields

$$d\sigma = \frac{e^4}{4E_1E_2\omega_1\omega_2}|\epsilon_1.\epsilon_2|^2\frac{1}{4\pi^2}\delta\left(E_1+\omega_1-E_2-\omega_2\right)d^3\overrightarrow{k_2}, \tag{3.47a}$$

and $d^3\overrightarrow{k}_2 = |k_{20}|^2 d|k_{20}|d\Omega_2 = \omega_2^2 d\omega_2 d\Omega_2$. Therefore

$$d\sigma = \frac{e^4}{16\pi^2E_1E_2\omega_1\omega_2}|\epsilon_1.\epsilon_2|^2\delta\left(E_1+\omega_1-E_1-\omega_2\right)k_{20}^2\frac{dk_{20}}{dE_f}dE_f d\Omega_2. \tag{3.47b}$$

But we know that

$$\frac{dE_f}{dk_{20}} = \frac{M\omega_1}{E_2\omega_2},$$

$$d\sigma = \frac{e^4}{16\pi^2E_1E_2\omega_1\omega_2}|\epsilon_1.\epsilon_2|^2\omega_2^2\frac{d\omega_2}{dE_f}d\Omega_2, \tag{3.47c}$$

where $E_i = E_1+\omega_1 = E_2+\omega_2 = E_f$. We have used integration over E_f to go from (3.47b) to (3.47c). Thus

$$d\sigma = \frac{e^4}{16\pi^2E_1E_2\omega_1}\frac{E_2\omega_2^2}{M\omega_1}|\epsilon_1.\epsilon_2|^2 d\Omega_2, \tag{3.48a}$$

$$= \frac{\alpha^2}{M^2}\left(\frac{\omega_2}{\omega_1}\right)^2|\epsilon_1.\epsilon_2|^2 d\Omega_2, \tag{3.48b}$$

which is the differential cross section in lab. frame.

From conservation of 4 momentum we have

$$p_2^2 = p_1^2+k_1^2+k_2^2+2p_1k_1-2p_1k_2-2k_1k_2, \tag{3.49a}$$

$$M^2 = M^2+2\left(M\omega_1-|p_1||k_1|\cos\theta\right)-2\left(M\omega_2-|p_1||k_2|\cos\theta\right)$$

$$-2\left(\omega_1\omega_2 - \omega_2\omega_2\cos\theta\right). \tag{3.49b}$$

After some simplification we get

$$\frac{\omega_1}{\omega_2} = 1 + \frac{\omega_1}{M}\left(1 - \cos\theta\right). \tag{3.49c}$$

Substituting $(3,49c)$ into $(3.48b)$ yields

$$d\sigma = \frac{\alpha^2}{M^2}\left[\frac{M^2}{\left[M + \omega_1\left(1 - \cos\theta\right)\right]^2}\right]|\epsilon_1.\epsilon_2|^2 d\Omega_2. \tag{3.50}$$

In non relativistic case, $\omega_1, \omega_2 \ll M \Rightarrow \omega_1 \approx \omega_2$,

$$d\sigma = \frac{\alpha^2}{M^2}|\epsilon_1.\epsilon_2|^2 d\Omega_2. \tag{3.51}$$

We have developed covariant formulation for relativistic scattering of "light" with "matter" in perturbative Scalar Electrodynamics in the language of Feynman diagrams. Like electron Compton scattering cross section, the pion Compton scattering cross section depends on the energy of the incident photon. Further there is also a shift in frequency of a photon when it scatters off a free pion and also near the forward direction (i.e., when $\theta = 0$), the pion Compton scattering cross section is reduced to the non relativistic result (3.51) whatever the incident energy may be. In non relativistic limit, the pion Compton scattering cross section is very similar to that of electron Compton scattering. In other words if we compare the differential cross-section in electron Compton scattering with that of pion Compton scattering we see that the scattering of a very low energy photon off a spin 0 particle is thus the same as off a spin $\frac{1}{2}$ particle. The explanation for this is that for very long wavelength the interaction takes place only through the total charge of the system from which the photon is scattered.

CHAPTER 4

Pion Electroproduction

In this chapter we study one more fundamental process in scalar electrodynamics. i.e., the pion Electroproduction [7].

Pions are hadrons and are heavy particles compared to electrons. However, when electron and positron with sufficient energy annihilate, they produce a high energy photon which in turn has finite probability for creating a $\pi^+\pi^-$ pair. Thus this process proceeds through the creation of a virtual photon. This process is extremely important since it involves the creation of hadrons due to the interaction of two leptons. This chapter is devoted to the study of this process, $e^+e^- \rightarrow \pi^+\pi^-$. We evaluate the transition amplitude and the cross section for this process [8]. The Feynman diagram for this process in lowest order is shown in figure 4.1. Let the electron and positron state be characterized by momenta and spin p_-, s_- and -p_+, s_+ respectively. And the π^-, π^+ pair by momenta p'_- and -p'_+ respectively. The $\pi^+$and π^- are spinless particles.

The transition amplitude for this process arises from the production of a virtual quanta. The transition amplitude for this process [8]

$$S_{fi}^{(1)} = -i \int j_\mu \left(\pi^\pm\right) A_\mu \left(x\right) d^4x, \tag{4.1}$$

where the electromagnetic potential

$$A_\mu \left(x\right) = \int j_\mu \left(e^\pm\right) D_F \left(x, x'\right) d^4x'. \tag{4.2a}$$

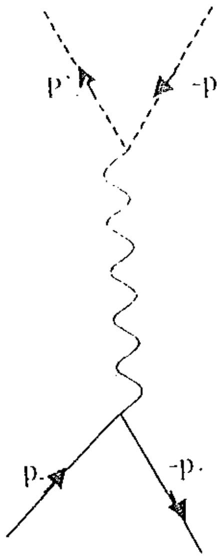

Figure 4.1: Feynman diagram for $e^{-}+e^{+} \rightarrow \pi^{-} + \pi^{+}$ scattering process in lowest order.

Now to obtain the photon propagator $D_F(x, x')$ in Eq.(4.2a), we start with Maxwell's equations

$$-\Box^2 A_\mu = j_\mu \tag{4.2b}$$

for electromagnetic current $j_\mu(e^{\pm}) = ie\overline{\Psi}_{e^+}\gamma_\mu\Psi_{e^-}$ produced due to the electron-positron pair annihilation at x'. Solving the unit source problem for (4.2b), we get

$$\Box_x^2 D_F(x, x') = -\delta^4(x, x') \tag{4.3}$$

In 4 dimensional space,if we set $x' = 0$ the photon propagator, $D_F(x)$ is given by

$$D_F(x) = \frac{1}{(2\pi)^4}\int d^4q\widetilde{D(q)}e^{iqx} \tag{4.4}$$

Using (4.3), one can obtain

$$D_F(x,x') = \frac{1}{(2\pi)^4}\int d^4q \frac{1}{q^2} e^{iq(x-x')} \tag{4.5}$$

If we consider the integrand of (4.5), the integration over p_0 is the same for $t > t'$ and $t < t'$. This result is because photon is its own antiparticle. Therefore the invariant transition amplitude for the process $e^+e^- \to \pi^+\pi^-$ is

$$\begin{aligned} S_{fi}^{(1)} &= -i\int d^4x\, j_\mu\left(\pi^\pm\right)\frac{1}{(2\pi)^4}\int d^4q \frac{1}{q^2+i\epsilon} e^{iq(x-x')} \\ &\times \int d^4x' j_\mu\left(e^\pm\right), \end{aligned} \tag{4.6}$$

where $j_\mu\left(\pi^\pm\right)$ is the transition electromagnetic current of the pion pair and $j_\mu\left(e^\pm\right)$ is the transition electromagnetic current of the electron-positron pair.

Since pions are Klein Gordon particles we have

$$j_\mu\left(\pi^\pm\right) = ie\left(-\frac{\partial\Psi_{\pi^-}^*}{\partial x_\mu}\Psi_{\pi^+} + \Psi_{\pi^-}^*\frac{\partial\Psi_{\pi^+}}{\partial x_\mu}\right), \tag{4.7a}$$

and

$$j_\mu\left(e^\pm\right) = ie\overline{\Psi}_{e^+}\gamma_\mu\Psi_{e^-}. \tag{4.7b}$$

Substituting $(4.7a)$ and $(4.7b)$ into (4.1) one obtains

$$S_{fi}^{(1)} = -i\left(ie\right)^2\int\int d^4x d^4x'\left\{-\frac{\partial\Psi_{\pi^-}^*}{\partial x_\mu}\Psi_{\pi^+} + \Psi_{\pi^-}^*\frac{\partial\Psi_{\pi^+}}{\partial x_\mu}\right\}$$

$$\times\frac{1}{(2\pi)^4}\int d^4q\frac{1}{q^2+i\epsilon}e^{iq.(x-x')}\left(\overline{\Psi}_{e^+}\gamma_\mu\Psi_{e^-}\right), \tag{4.8}$$

Since $\Psi_{\pi^+} = \frac{1}{\sqrt{2E'_+}} e^{-ip'_+.x}$, $\Psi^*_{\pi^-} = \frac{1}{\sqrt{2E'_-}} e^{-ip'_-.x}$, and

$\Psi_{e^-} = \sqrt{\frac{m}{E_- V}} u^{(s_-)} \left(\overrightarrow{p}_-\right) e^{ip_-.x'}$,

$\overline{\Psi}_{e^+} = \sqrt{\frac{m}{E_+ V}} \overline{v}^{(s_+)} \left(\overrightarrow{p}_+\right) e^{ip_+.x'}$.

Substituting these expressions into Eqs.(4.7a) ,and (4.7b) we obtain

$$j_\mu\left(\pi^{\pm}\right) = \frac{ie}{2\sqrt{E'_+ E'_-}} \left\{-\left(-ip'_{-\mu}\right) + \left(-ip'_{+\mu}\right)\right\} e^{-i\left(p'_- + p'_+\right).x}, \tag{4.9a}$$

and

$$j_\mu\left(e^{\pm}\right) = \frac{iem}{\sqrt{E_+ E_- V}} \left\{\overline{v}\left(\overrightarrow{p}_+\right) \gamma_\mu u\left(\overrightarrow{p}_-\right)\right\} e^{i(p_- + p_+).x'}, \tag{4.9b}$$

where in (4.9b) we replace $\overline{v}^{(s_+)}\left(\overrightarrow{p}_+\right)$ by $\overline{v}\left(\overrightarrow{p}_+\right)$ and $u^{(s_-)}\left(\overrightarrow{p}_-\right)$ by $u\left(\overrightarrow{p}_-\right)$.

Substituting (4.9a) and (4.9b) into (4.8) yields

$$\begin{aligned} S^{(1)}_{fi} &= \frac{(ie)^2 m}{2\sqrt{E_- E_+ E'_- E'_+} V} \int\int d^4x d^4x' \left\{\left(p'_- - p'_+\right)_\mu\right\} \frac{1}{(2\pi)^4} \\ &\times \int d^4q \frac{1}{q^2 + i\epsilon} \left[\overline{v}\left(\overrightarrow{p}_+\right) \gamma_\mu u\left(\overrightarrow{p}_-\right)\right] \\ &\times e^{iq(x-x')} e^{-i\left(p'_- + p'_+\right).x} \times e^{i(p_- + p_+).x'}. \end{aligned} \tag{4.10a}$$

After carrying out the integrations in the preceding equation and taking $|S^{(1)}_{fi}|^2$, the transition probability from state i to f is therefore

$$\begin{aligned} |S^{(1)}_{fi}|^2 &= \frac{e^4 m^2}{4E_+ E_- E'_+ E'_- V^2} \left[\left(p'_- - p'_+\right)_\mu \left(p'_- - p'_+\right)_\nu\right] \frac{1}{|p_- + p_+|^4} \\ &\times |\overline{v}\left(\overrightarrow{p}_+\right) \gamma_\mu u\left(\overrightarrow{p}_-\right)|^2 \\ &\times \left[(2\pi)^4 \delta^4\left(p_- + p_+ - p'_- - p'_+\right)\right]^2. \end{aligned} \tag{4.10b}$$

If the initial electron and positron are unpolarized, the total transition probability is the sum over all spin states of the positron and averaged over all spin states of the electron. Thus

$$\begin{aligned}\frac{1}{2}\sum_{s_-,s_+}|S_{fi}^{(1)}|^2 &= \frac{e^4m^2}{4E_+E_-E'_+E'_-V^2}\frac{1}{2}\sum_{s_+,s_-}\left[\left(p'_- - p'_+\right)_\mu\left(p'_- - p'_+\right)_\nu\right] \\ &\times\frac{1}{|p_- + p_+|^4}\left[\overline{u}\left(\overrightarrow{p}_-\right)\gamma_4\gamma_\mu^+\gamma_4 v\left(\overrightarrow{p}_+\right)\overline{v}\left(\overrightarrow{p}_+\right)\gamma_\nu u\left(\overrightarrow{p}_-\right)\right] \\ &\times\left[(2\pi)^4\delta^4\left(p_- + p_+ - p'_- - p'_+\right)\right]^2, \qquad (4.11a)\end{aligned}$$

Thus, we can express

$$\frac{1}{2}\sum_{s_-,s_+}|S_{fi}^{(1)}|^2 = \frac{e^4m^2}{4E_+E_-E'_+E'_-V^2}\frac{1}{2}\left[\left(p'_- - p'_+\right)_\mu\left(p'_- - p'_+\right)_\nu\right]\frac{1}{|p_- + p_+|^4}$$

$$Tr\left[\gamma_\mu\left\{\frac{i\gamma.p_+ + m}{2m}\right\}\gamma_\nu\left(\frac{-i\gamma.p_- + m}{2m}\right)\right]$$

$$\times\left[(2\pi)^4\delta^4\left(p_- + p_+ - p'_- - p'_+\right)\right]^2. \qquad (4.11b)$$

Or

$$\begin{aligned}\frac{1}{2}\sum_{s_-,s_+}|S_{fi}^{(1)}|^2 &= \frac{e^4m^2}{4E_+E_-E'_+E'_-V^2}\left\{p'_{-\mu}p'_{-\nu} - p'_{-\mu}p'_{+\nu} - p'_{+\mu}p'_{-\nu} + p'_{+\mu}p'_{+\nu}\right\} \\ &\times\frac{1}{|p_- + p_+|^4}\frac{1}{8m^2}Tr\left\{\gamma_\mu\left(\gamma.p_+\right)\left(\gamma.p_-\right)\gamma_\nu + m^2\gamma_\mu\gamma_\nu\right\} \\ &\times\left[(2\pi)^4\delta^4\left(p_- + p_+ - p'_- - p'_+\right)\right]^2. \qquad (4.11c)\end{aligned}$$

To go from (4.11*b*) to (4.11*c*), we have used the property of the γ matrices i.e.,

$\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\lambda...\gamma_n) = 0$,for n odd.

Therefore, the transition probability per unit time is

$$\frac{1}{2}\sum_{s_-,s_+}\frac{|S_{fi}^{(1)}|^2}{T} = \frac{1}{4m^2}\frac{e^4m^2}{4E_+E_-E'_+E'_-V^2}T_{\mu\nu}\frac{1}{|p_- + p_+|^4}L_{\mu\nu}V$$
$$\times(2\pi)^4\,\delta^4\left(p_- + p_+ - p'_- - p'_+\right) \tag{4.11d}$$

To go from (4.11*c*) to (4.11*d*), one of the δ–function has been eliminated by integrating at $p_- + p_+ = p'_- + p'_+$. $T_{\mu\nu}$ and $L_{\mu\nu}$ in equation (4.11*d*) are the pion and electron tensors which are given as

$$T_{\mu\nu} = \left\{p'_{-\mu}p'_{-\nu} - p'_{-\mu}p'_{+\nu} - p'_{+\mu}p'_{-\nu} + p'_{+\mu}p'_{+\nu}\right\}, \tag{4.12a}$$

and

$$L_{\mu\nu} == \frac{1}{2}Tr\{\gamma_\mu\left(\gamma.p_+\right)\left(\gamma.p_-\right)\gamma_\nu + m^2\gamma_\mu\gamma_\nu\}. \tag{4.12b}$$

But

$$L_{\mu\nu} = 2\left\{p_{+\mu}p_{-\nu} + p_{-\mu}p_{+\nu} - \delta_{\mu\nu}\left(p_+.p_-\right) + m^2\delta_{\mu\nu}\right\}. \tag{4.13a}$$

Or

$$L_{\mu\nu} = 2\left\{p_{+\mu}p_{-\nu} + p_{-\mu}p_{+\nu} - (p_+.p_- - m^2)\delta_{\mu\nu}\right\}. \tag{4.13b}$$

Therefore

$$T_{\mu\nu}L_{\mu\nu} = 2\{2\left(p_+.p'_-\right)\left(p_-.p'_-\right) - \left(p_+.p_- - m^2\right)\left(p'_-.p'_-\right)$$
$$-2\left(p_+.p'_-\right)\left(p_-.p'_+\right) - 2\left(p_-.p'_-\right)\left(p_+.p'_+\right) + 2\left(p_+.p_- - m^2\right)\left(p'_-.p'_+\right)$$

$$+2\left(p_+.p'_+\right)\left(p_-.p'_+\right)-(p_+.p_- - m^2)p_+^2\}. \tag{4.14}$$

In ultra relativistic case, E$\gg m, M \Rightarrow E \approx |\overrightarrow{p}|$

Thus

$$(p_- + p_+)^2 = p_-^2 + p_+^2 + 2p_-.p_+ = 2p_-.p_+ = \left(p'_- + p'_+\right)^2, \tag{4.15a}$$

$$\left(p_- - p'_-\right)^2 = \left(p'_+ - p_+\right)^2 = p_-^2 + p'^2_- - 2p_-.p'_- = -2p_-.p'_- = -2p'_+.p_+, \tag{4.15b}$$

and

$$\left(p_- - p'_+\right)^2 = \left(p'_- - p_+\right)^2 = p_-^2 + p'_+ - 2p_-.p'_+ = -2p_-.p'_+. \tag{4.15c}$$

Therefore (4.14) reduces to

$$\begin{aligned} T_{\mu\nu}L_{\mu\nu} &= 2\{2\left(p_+.p'_-\right)\left(p_-.p'_-\right) - 2\left(p_+.p'_-\right)\left(p_-.p'_+\right) - 2\left(p_-.p'_-\right)\left(p_+.p'_+\right) \\ &\quad +2\left(p_+.p_-\right)\left(p'_-.p'_+\right) + 2\left(p_+.p'_+\right)\left(p_-.p'_+\right)\} \end{aligned} \tag{4.16}$$

If we calculate in the c.m. frame of e^-e^+ where $p_- = -p_+ = p, E_- = E_+ = E, p'_- = -p'_+ = p', E'_- = E'_+ = E'$, then

$$2p_-.p_+ = 2p'_-.p'_+ = 2\left(E^2 + |\overrightarrow{p}|^2\right), \tag{4.17a}$$

$$-2p_-.p'_- = -2p_+.p'_+ = -2|\overrightarrow{p}||\overrightarrow{p}'|\left(1 - \cos\theta\right), \tag{4.17b}$$

$$-2p_{-}.p'_{+} = -2p_{+}.p'_{-} = -2|\overrightarrow{p}||\overrightarrow{p}'|(1+\cos\theta), \tag{4.17c}$$

and thus

$$T_{\mu\nu}L_{\mu\nu} = 8|\overrightarrow{p}|^2|\overrightarrow{p}'|^2\sin^2\theta. \tag{4.17d}$$

Substituting $(4.17d)$ into $(4.11d)$ yields

$$\begin{aligned}\frac{1}{2}\sum_{s_-,s_+}|S_{fi}^{(1)}|^2 &= \frac{e^4}{2E_+E_-E'_+E'_-V}\frac{|\overrightarrow{p}|^2|\overrightarrow{p}'|^2\sin^2\theta}{16|\overrightarrow{p}|^4} \\ &\times(2\pi)^4\,\delta^4\left(p_- + p_+ - p'_- - p'_+\right). \end{aligned} \tag{4.18}$$

The differential cross section can then be obtained as

$$d\sigma = \frac{1}{2}\sum_{s_-,s_+}\frac{|S_{fi}^{(1)}|^2}{flux}\times\frac{d^3\overrightarrow{p}'_-}{(2\pi)^3}\times\frac{d^3\overrightarrow{p}'_+}{(2\pi)^3}, \tag{4.19}$$

where flux=$|j_\mu(e^-) + j_\mu(e^+)|$

$=\frac{1}{V}|\frac{p_-}{E_-} - \frac{p_+}{E_+}| = \frac{2}{V}$ in the c.m frame.

Eq.(4.19) becomes

$$\begin{aligned}d\sigma &= \frac{e^4}{2E^2E'^2V}\frac{|\overrightarrow{p}|^2|\overrightarrow{p}'|^2\sin^2\theta}{16|\overrightarrow{p}|^4}\times V/2\times\frac{1}{(2\pi)^2} \\ &\times\delta^4\left(p_- + p_+ - p'_- - p'_+\right)d^3\overrightarrow{p}'_-d^3\overrightarrow{p}'_+. \end{aligned} \tag{4.20}$$

Integration over $\overrightarrow{p}'_+$ yields

$$d\sigma = \frac{e^4}{16\pi^2|p|^4}\frac{\sin^2\theta}{16}\delta\left(E_i - E_f\right)d^3\overrightarrow{p}'_-. \tag{4.21a}$$

Since $d^3\overrightarrow{p}'_- = \overrightarrow{p}'^2_- d\overrightarrow{p}'_- d\Omega'$ and EdE=pdp, $Eq.$ (4.21a) becomes

$$d\sigma = \frac{e^4}{16\pi^2|p|^2}\frac{\sin^2\theta}{16}\delta\left(E_i - E_f\right)dE'd\Omega, \tag{4.21b}$$

$$d\sigma = \frac{e^4}{16\pi^2|p|^2}\frac{\sin^2\theta}{16}\delta\left(E_i - E_f\right)\frac{dE'}{dE_f}dE_f d\Omega, \tag{4.21c}$$

$$d\sigma = \frac{e^4}{16\pi^2|p|^2}\frac{\sin^2\theta}{16}\frac{dE'}{dE_f}|_{E_f=E_i}d\Omega', \tag{4.21d}$$

$$d\sigma = \frac{e^4}{16\pi^2|p|^2}\frac{\sin^2\theta}{16}\frac{1}{2}d\Omega', \tag{4.21e}$$

$$d\sigma = \frac{\alpha^2}{32|p|^2}\sin^2\theta d\Omega', \tag{4.21f}$$

$$d\sigma = \frac{\alpha^2}{8q^2}\sin^2\theta d\Omega', \tag{4.21g}$$

where q=the c.m energy.

From Eq.(4.21g), we can see that $\frac{d\sigma}{d\Omega'}$ is maximum when $\theta = \frac{\pi}{2}$. This shows that the production of pion pairs is dominant when $q^2 = 4M^2$, where M is the rest energy of the pion. And the angular distribution of $\frac{d\sigma}{d\Omega'}$ decreases as $q^2 > 4M^2$. The explanation for this is that as q^2 increases the probability of other processes which involves the creation of extra particles becomes greater and the probability for the production of charged pion pairs is correspondingly reduced.

CHAPTER 5

Discussions

Chapter 1 deals with the introduction. In chapter 2 we have discussed the covariant perturbation theory which enables us to formulate the invariant S-matrix for the calculation of transition amplitudes for arbitrary orders of interaction strengths. We have also discussed the electron propagator which enables us to formulate the S-matrix using Feynman diagrams for arbitrary orders of interaction and we have performed derivation of the electron propagator. Then these rules have been generalized to study the interactions of scalar particles such as pions ($i.e., \pi^0, \pi^{\pm}$) with electromagnetic field in chapter 3. The underlying theory governing such processes is Scalar Electrodynamics.

Thus in chapter 3 we have calculated the pion propagator and described the basic techniques for the calculation of the Lorentz invariant transition amplitudes and differential cross sections of pion Compton scattering.

In chapter 4 we have discussed one more important process in Scalar Electrodynamics i.e., Pion Electroproduction ($e^+e^- \rightarrow \pi^+\pi^-$) which proceeds through the production of a virtual photon.

We have developed covariant formulation for relativistic scattering of "light" with "matter" and Pion Electroproduction in perturbative Scalar Electrodynamics in the language of Feynman diagrams. We have found that the differential cross section for both the electron Compton and pion Compton scatterings depend on the incident photon energy. There is also a shift in frequency of the photon when it scatters off a free electron and a free pion. In the non-relativistic limits the electron Compton and the pion Compton scatterings cross section reduce to Thomson formula for the scattering of a low energy photon by an atomic electron. It is seen that near the forward direction (i.e., when $\theta = 0$), the scattering cross sections of electron Compton and pion Compton scatterings reduce to their non relativistic limit for any range of values of the incident photon energy.

From the classical limit of electron Compton and pion Compton scatterings we have found that the scattering of a very low energy photon off a spin $\frac{1}{2}$ particle is thus the same as off a spin 0 particle. The explanation for this is that for very long wavelength the interaction takes place only through the total charge of the system from which the photon is scattered.

In the case of the Electroproduction process, the differential cross section for the production of charged pion pairs is dominant as the c.m. scattering angle of the system is equal to $\frac{\pi}{2}$. And, moreover, the angular distribution is large as $q^2 = 4M^2$, where M is the rest energy of the pion. Or in otherwords, the angular distribution decreases as $q^2 > 4M^2$. The explanation for this is that as q^2 increases the probability of other processes which involves the creation of extra particles becomes greater and the probability for the production of charged pion pairs is correspondingly reduced.

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