Topological and dynamical properties of Volterra-type integral and differential operators on Fock-type spaces

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Declaration

I, Mafuz Humer Worku, with student ID number GSR/9650/09, hereby declare that this thesis is my own work and that it has not been previously submitted for assessment or completion of any post graduate qualification to another university or for another qualification.

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This is to certify that the thesis prepared by Mafuz Humer Worku, entitled: **Topological and Dynamical Properties of Volterra-type Integral and Differential Operators on Fock-type Spaces** and submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy(in Mathematics) compiles with the regulations of the University and meets the accepted standards with respect to originality and quality.

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Abstract

Many real world problems can be modelled using different forms of differential and integral operators. Due to this, studying various properties of these class of operators have been an important area of research in mathematics. The study has attracted lots of interest especially on function related operator theory in the past two decades. In this thesis, we consider these operators and study several topological and dynamical properties on Fock-type spaces.

The thesis is organized into four chapters. In the first chapter we discuss some preliminary results on the Volterra-type integral operators. In particular, properties like boundedness and compactness of the operators on some spaces are reviewed. In the second chapter, we study various properties of the generalized Volterra-type integral and the generalized composition operators acting on the classical Fock spaces. We describe some properties in terms of growth and integrability conditions which are simpler to apply than those already known Berezin-type integral transform characterizations.

In the third chapter we apply those simple conditions obtained in Chapter 2 and study some topological structures of the space of the operators under the operator norm. In particular, we prove that the difference of two non-trivial Volterra-type integral operators is compact if and only if both are compact. We also consider the structures on other Fock-type spaces with

weight function growing faster than the classical Gaussian function. It is proved that while the space contains each noncompact operator as its isolated points, it fails to admit essentially isolated points. We have also shown that the space of all bounded Volterra companion and multiplication operators are path connected.

In the last chapter, various dynamical properties of the Volterra-type integral, differentiation and Hardy operators on Fock-type spaces are studied. More specifically, properties like cyclicity, supercyclicity, hypercyclicity, power boundedness and uniformly mean ergodicity are characterized in terms of workable conditions. As an application, we prove that the differentiation operator satisfies the Ritt's resolvent condition if and only if it is power bounded and uniformly mean ergodic, whereas the Hardy operator always satisfies such a condition.

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Chapter 1

The Volterra-type integral operators

Integral operators arise in many branches of mathematics, physics, engineering, biology and economics [18, 22, 26, 57, 33], as often a means in modeling real-world situations. A typical examples of these operators include the integral operators of Fredholm, Volterra, Hammerstein and Urysohn. Ever since introduced by C. Pommerenke [61] in 1977, the Volterra-type integral operator is one of these class of operators which has been studied a lot by several authors. In this chapter, we will discuss some preliminary results about this operators on Fock spaces.

1.1 Introduction

Given a space $\mathcal{H}(U)$ of holomorphic functions on U, the Volterra-type integral operator on $\mathcal{H}(U)$ induced by a holomorphic symbol g is defined by

$$V_g f(z) = \int_0^z f(w)g'(w)dw.$$

The operator is known by several names which includes Volterra operator, generalized Cesáro operator, Riemann-Stieltjes operator, and integration operator. The various names arises since the operator is a more general form that include other operators as special cases. For example, when $g(z) = z$, the operator reduces to the integration operator,

$$Jf(z) = \int_0^z f(w)dw$$

and for the case $g(z) = \log(\frac{1}{1-z})$, we obtain the Cesáro operator,

$$C(f)(z) = \int_0^z \frac{f(w)}{1-w} dw.$$

The Volterra-type integral operator V_g for holomorphic function on the unit disc was first introduced by C. Pommerenke [61] to study exponentials of BMOA functions. He proved that V_g is bounded on the Hardy space H^2 if and only if g is a BMOA function. Later in 1995, A. Aleman and A. Siskakis [3] extended this result to other Hardy spaces H^p for $1 \leq p < \infty$. Moreover, they have proved that V_g is compact on H^p if and only if g is in the VMOA class. In [4] they also proved an analogous results on weighted Bergman spaces. These important results of A. Aleman and A. Siskakis has motivated a number of other researchers to study various properties of the operator V_g including boundedness, compactness, essential norm, Schatten class membership and spectral on different spaces of holomorpic functions. For instance, on Bergman spaces [4, 58, 59, 60], Dirichlet spaces [29, 30], Model spaces [50], Bloch spaces [72], Fock-Sobolev spaces [44, 48], Fock spaces [19, 20, 21, 42, 46, 49, 51]. One may also consult the surveys in [2, 64] and the references therein. In most perspectives, the study of the operator aims at describing various of its operator theoretic properties in terms of the functions theoretic properties of the inducing map g.

1.2 Volterra-type integral operators on Fock spaces

Most of the earlier works on the Volterra-type integral operator V_g had been on spaces of holomprphic functions defined on the unit disc; see for example [61, 3, 4, 58, 59, 60]. In 2012, O. Constantin [19] studied the operator on spaces of entire functions defined on the whole complex plane, namely on the Fock spaces $\mathcal{F}_p$. We recall that the classical Fock spaces $\mathcal{F}_p$, $0 < p < \infty$ consist of entire functions for which

$$\|f\|_p = \left(\frac{p}{2\pi}\int_{\mathbb{C}} |f(z)|^p d\mu_p(z)\right)^{\frac{1}{p}} < \infty$$

where $d\mu_p(z) = e^{-\frac{p}{2}|z|^2} dA(z)$ is called the Gaussian measure on $\mathbb{C}$ and dA is the usual Lebesgue area measure.

Then the study was continued by T. Mengestie [49] on the growth type Fock space $\mathcal{F}_\infty$ which consists of entire functions for which

$$\|f\|_\infty = \sup_{z\in\mathbb{C}} |f(z)|e^{-\frac{1}{2}|z|^2} < \infty.$$

We note that the space $\mathcal{F}_p$ is a Banach space with norm $\|f\|_p$ for all $1 \leq p \leq \infty$ and it is a complete metric space for $0 < p < 1$ with distance function $d(f, g) = \|f - g\|_p^p$.

For $0 < p \leq \infty$ and $f \in \mathcal{F}_p$, the Vukotic's inequality

$$|f(z)| \leq e^{\frac{1}{2}|z|^2}\|f\|_p,$$

asserts that point evaluation functionals are bounded on classical Fock spaces. Thus, $\mathcal{F}_2$ is a reproducing kernel Hilbert space with kernel and normalized reproducing kernel functions given by the explicit formulas

$$K_w(z) = e^{\overline{w}z} \quad \text{and} \quad k_w(z) = e^{\overline{w}z - \frac{|w|^2}{2}}.$$

The Fock kernel $K_w(z)$ is neither bounded above nor bounded below, even when one of the two variables is fixed. On the Hardy and Bergman spaces, it is known that the kernel function is both bounded above and bounded below when one of the two variables is fixed. This makes many estimates in the Fock space setting much more difficult. On the other hand, the exponential decay of $e^{-|z|^2}$ makes it easier to prove the convergence of certain integrals and infinite series on the Fock space setting than their Hardy and Bergman space counterparts.

The kernel function K_w belongs to all the Fock space $\mathcal{F}_p$ with norms

$$\|K_w\|_p = e^{\frac{1}{2}|w|^2} \tag{1.2.1}$$

for all $w \in \mathbb{C}$ and $0 < p \leq \infty$. This follows from a simple computation

$$\begin{aligned} \|K_w\|_p^p = \frac{p}{2\pi}\int_{\mathbb{C}} e^{p\Re(\overline{w}z) - \frac{p}{2}|z|^2} dA(z) &= e^{\frac{p}{2}|w|^2} \frac{p}{2\pi}\int_{\mathbb{C}} e^{\frac{p}{2}\left(2\Re(\overline{w}z) - |w| - |z|^2\right)} dA(z) \\ &= e^{\frac{p}{2}|w|^2}\left(\frac{p}{2\pi}\int_{\mathbb{C}} e^{-\frac{p}{2}|w-z|^2} dA(z)\right) = e^{\frac{p}{2}|w|^2} \end{aligned}$$

for $0 < p < \infty$. The result holds for $p = \infty$ just by definition.

It follows that k_w constitutes a unit norm sequence of functions $\mathcal{F}_p$ and converges to zero in compact subset of $\mathbb{C}$. We refer to the book of K. Zhu [70] for more details on these class of spaces.

The results of O. Constantin [19] and T. Mengestie [49] gives an interesting characterization of the bounded and compact Volterra-type integral operator V_g in terms of the function theoretic properties of the inducing symbol g. Combining their results we record the following two important theorems in our further considerations.

Theorem 1.2.1. *Let $0 < p \leq q \leq \infty$. Then the Volterra-type integral operator $V_g : \mathcal{F}_p \to \mathcal{F}_q$ is*

a) bounded if and only if $g(z) = az^2 + bz + c$ for $a, b, c \in \mathbb{C}$.

b) compact if and only if $g(z) = az + b$ for $a, b \in \mathbb{C}$.

It should be noted that the result in the above theorem is independent of the exponent p in the space except that $p \leq q$. This is manifested due to the fact that the classical Fock spaces are nested in the sense that $\mathcal{F}_p \subseteq \mathcal{F}_q$ whenever $p \leq q$.
For the case when $0 < q < p \leq \infty$, that is when the operators map the larger space into smaller, we have a stronger condition in which boundedness and compactness are equivalent.

Theorem 1.2.2. *Let $0 < q < p \leq \infty$. Then the following are equivalent.*

a) $V_g : \mathcal{F}_p \to \mathcal{F}_q$ is bounded,

b) $V_g : \mathcal{F}_p \to \mathcal{F}_q$ is compact,

c) $q > \begin{cases} \frac{2p}{p+2}, p < \infty \\ 2, p = \infty \end{cases}$ *and $g(z) = az + b$ for some $a, b \in \mathbb{C}$.*

To prove the results above, the authors first obtained Littlewood–Paley type estimates on the respective spaces. The estimates play an important role in the study of the operator since it helps eliminate the integral that emanates from the definition of the operator. We state the estimates as follows.

Lemma 1.2.3. *Let f be an entire function on the complex plane and $0 < p \leq \infty$. Then*

$$\|f\|_p \simeq \begin{cases} \left(|f(0)|^P + \int_{\mathbb{C}} \frac{|f'(z)|^p}{\left(1+|z|\right)^p} e^{-\frac{p}{2}|z|^2}\, dA(z)\right)^{\frac{1}{p}}, & 0 < p < \infty \\ |f(0)| + \sup_{z\in\mathbb{C}} \frac{|f'(z)|}{1+|z|} e^{-\frac{1}{2}|z|^2}, & p = \infty. \end{cases}$$

Here and in the rest of the thesis, the notation $q(z) \lesssim r(z)$ (or equivalently $r(z) \gtrsim q(z)$) means there exists a positive constant k independent of the argument such that $q(z) \leq kr(z)$ and $q(z) \simeq r(z)$ if both $q(z) \lesssim r(z)$ and $q(z) \gtrsim r(z)$.

Chapter 2

The generalized Volterra-type integral operators on Fock spaces

In this chapter we study various properties of the generalized Volterra-type integral operator and generalized composition operators on the classical Fock spaces. We in particular describe the operators bounded, compact, essential norm and Schatten class memebership properties. Most of the results in this chapter are from our published paper in [53].

2.1 Introduction

The generalized Volterra-type integral operator $V_{(g,\psi)}$ and generalized composition operator $J_{(g,\psi)}$ on $\mathcal{H}(U)$ induced by a pair of holomorphic symbols (g, ψ) on U is defined by

$$V_{(g,\psi)}f(z) = \int_0^z f(\psi(w))g'(w)dw, \text{ and } J_{(g,\psi)}f(z) = \int_0^z f'(\psi(w))g(w)dw.$$

In particular, if we set $\psi(z) = z$, then the operators $V_{(g,\psi)}$ and $J_{(g,\psi)}$ become respectively the operators V_g and the companion operators,

$$I_g f(z) = \int_0^z f'(w)g(w)dw.$$

On the other hand, the choice $g = \psi'$ reduces the operator $J_{(g,\psi)}$ to the composition operator, $C_\psi f(z) = f(\psi(z))$ taking f to $f(\psi)$ up to a constant.

It was S. Li and S. Stević [37, 38] who introduced the generalized operators $V_{(g,\psi)}$ and $J_{(g,\psi)}$ in 2008 and studied some of their operator-theoretic properties in terms of properties of the inducing pairs (g, ψ) on spaces defined over the unit disk. Such operators have found applications for example in the study of linear isometries of spaces of analytic functions. If S^p denotes the space of all analytic functions f in the unit disc for which its derivative f' belongs to the Hardy space H^p, then it has been shown that for $p \neq 2$, any surjective isometry T of S^p under the norm $\|f\|_{S^p} = |f(0)| + \|f'\|_{H^p}$ is of the form

$$Tf = \lambda f(0) + \lambda J_{(g,\psi)} f$$

for some unimodular λ in $\mathbb{C}$, a nonconstant inner function ψ and a function g belonging to the space H^p [28].

In 2012, S. Ueki [68] studied the operators when they act from weighted Bergman spaces into α-Zygmund spaces. Later in [43, 46] T. Mengestie took the study of the operators on the Fock spaces $\mathcal{F}_p$ and characterized several properties in terms of Berezin-type integral transforms

$$B_{(|g|^p,\psi)}(w) = \int_{\mathbb{C}} \big|k_w(\psi(z))\big|^p \frac{|g'(z)|^p e^{-\frac{p}{2}|z|^2}}{(1+|z|)^p} dA(z) \quad \text{and}$$
$$\widetilde{B}_{(|g|^p,\psi)}(w) = \int_{\mathbb{C}} \big|k_w(\psi(z))\big|^p \frac{(|w|+1)^p |g(z)|^p e^{-\frac{p}{2}|z|^2}}{(1+|z|)^p} dA(z).$$

The study here has required to estimate the $L^q, 0 < q \leq \infty$ norms of the integral transforms $B_{(|g|^p,\psi)}$ and $\widetilde{B}_{(|g|^p,\psi)}$. Such types of characterizations have been also referred as reproducing kernel thesis properties for the operators. One of the main purposes of this chapter is to substantially improve these conditions and provide characterizations which are simpler to apply. Our new results will be expressed in terms of the functions

$$M_{(g,\psi)}(z) = \frac{|g'(z)|}{1+|z|} e^{\frac{1}{2}\left(|\psi(z)|^2 - |z|^2\right)} \text{ and } \widetilde{M}_{(g,\psi)}(z) = \frac{|g(z)|\big(1+|\psi(z)|\big)}{1+|z|} e^{\frac{1}{2}\left(|\psi(z)|^2 - |z|^2\right)},$$

which are easier to handle than those class of Berezin-type integral transforms.

2.2 Bounded and Compact $V_{(g,\psi)}$ and $J_{(g,\psi)}$

In this section we characterize the bounded and compact properties of the operators $V_{(g,\psi)}$ and $J_{(g,\psi)}$ in terms of the functions $M_{(g,\psi)}$ and $\widetilde{M}_{(g,\psi)}$. We may first give a key proposition that will be used repeatedly in our subsequent considerations. The proposition is interesting by its own right, and states that the linear form $\psi(z) = az + b$ for some $|a| \leq 1$ is a necessary condition for boundedness of $V_{(g,\psi)}$ and $J_{(g,\psi)}$ while $\psi(z) = az + b$ for some $|a| < 1$ is necessary for compactness of the operators.

Proposition 2.2.1. *Let (g, ψ) be a pair of nonconstant entire functions on $\mathbb{C}$ and $0 < p < \infty$. Then if any of* *(i)* $M_{(g,\psi)}$, $\widetilde{M}_{(g,\psi)}, B_{(|g|^p,\psi)}$ *or* $\widetilde{B}_{(|g|^p,\psi)}$ *belongs to* $L^\infty(\mathbb{C}, dA)$, *then* $\psi(z) = az + b$ *for some* $|a| \leq 1$.

(ii) $M_{(g,\psi)}(z)$, $\widetilde{M}_{(g,\psi)}(z), B_{(|g|^p,\psi)}(z)$, *or* $\widetilde{B}_{(|g|^p,\psi)}(z)$ *tends to zero as* $|z| \to \infty$, *then* $\psi(z) = az + b$ *with* $|a| < 1$.

Proof. The proof of the proposition for the parts $M_{(g,\psi)}$ and $\widetilde{M}_{(g,\psi)}$ follows from a variant of the proof of Proposition 2.1 in [36] or Lemma 2.3 of [47]. But, we will prove it here for the sake of completeness. First we consider the case when

$$\sup_{z\in\mathbb{C}} M_{(g,\psi)} < \infty. \tag{2.2.1}$$

If g' is constant, then the assertion is clear. Thus, we suppose that g' is nonconstant. From (2.2.1), we have

$$M_\infty(g', |z|) \lesssim \frac{1 + |z|}{e^{\frac{1}{2}\left(|\psi(z)|^2 - |z|^2\right)}}$$

where $M_\infty(g', |z|)$ is the integral mean (maximum modulus) of the function g'. Since $M_\infty(g\psi, |z|)$ is a nondecreasing function of $|z|$, we have

$$\lim_{|z|\to\infty} \sup(|\psi(z)| - |z|) \leq 0 \tag{2.2.2}$$

Otherwise there would be a sequence $\{z_j\}$ such that $|z_j| \to \infty$ as $j \to \infty$ and $\lim_{j\to\infty} \sup(|\psi(z_j)| - |z_j|) > 0$. This along with the fact that ψ is an entire function implies that

$$M_\infty(g', |z_j|) \lesssim \frac{1 + |z_j|}{e^{\frac{\alpha}{2}\left(|\psi(z_j)|^2 - |z_j|^2\right)}}$$

is bounded which gives a contradiction. Thus, from equation (2.2.2) we deduce $\psi(z) = az + b$ for some a and b in $\mathbb{C}$ and $|a| \leq 1$ and clearly if $M_{(g,\psi)}(z)$ tends to zero as $|z| \to \infty$, then $\psi(z) = az + b$ with $|a| < 1$. The case for $\widetilde{M}_{(g,\psi)}$ follows in a similar manner.

On the other hand, for the integral transform $B_{(|g|^p,\psi)}$, we have

$$B_{(|g|^p,\psi)}(w) = \int_{\mathbb{C}} \frac{|k_w(\psi(z))|^p |g'(z)|^p}{(1+|z|)^p e^{\frac{p}{2}|z|^2}} dA(z) \geq \int_{D(\zeta,1)} \frac{|k_w(\psi(z))|^p |g'(z)|^p}{(1+|z|)^p e^{\frac{p}{2}|z|^2}} dA(z)$$

for all $w, \zeta \in \mathbb{C}$, where $D(\zeta, 1)$ is a disc of radius 1 and center ζ.

Since $1 + z \simeq 1 + \zeta$ for all $z \in D(\zeta, 1)$, we estimate the integral above as

$$\begin{aligned}\int_{D(\zeta,1)} \frac{|k_w(\psi(z))|^p |g'(z)|^p}{(1+|z|)^p e^{\frac{p}{2}|z|^2}} dA(z) &\simeq \frac{1}{(1+|\zeta|)^p} \int_{D(\zeta,1)} \frac{|k_w(\psi(z))|^p |g'(z)|^p}{e^{\frac{p}{2}|z|^2}} dA(z) \\ &\gtrsim \frac{1}{(1+|\zeta|)^p} |k_w(\psi(\zeta)) g'(\zeta)|^p e^{-\frac{p}{2}|\zeta|^2}\end{aligned}$$

for all $\zeta \in \mathbb{C}$. Setting $w = \psi(\zeta)$ in particular and applying (1.2.1) gives

$$\begin{aligned}B_{(|g|^p,\psi)}(\psi(\zeta)) &\gtrsim \frac{1}{(1+|\zeta|)^p} |k_{\psi(\zeta)}(\psi(\zeta))|^p |g'(\zeta)|^p e^{-\frac{p}{2}|\zeta|^2} \\ &= \frac{|g'(\zeta)|^p}{(1+|\zeta|)^p} e^{\frac{p}{2}|\psi(\zeta)|^2 - \frac{p}{2}|\zeta|^2} = M^p_{(g,\psi)}(\zeta),\end{aligned} \tag{2.2.3}$$

and the assertion follows from the boundedness condition on $M_{(g,\psi)}$.

Similarly, if $\widetilde{B}_{(|g|^p,\psi)}$ is bounded, then

$$\begin{aligned}\widetilde{B}_{(|g|^p,\psi)}(w) &= \int_{\mathbb{C}} \left|k_w(\psi(z))\right|^p \frac{(|w|+1)^p |g(z)|^p e^{-\frac{p}{2}|z|^2}}{(1+|z|)^p} dA(z) \\ &\geq \int_{D(\zeta,1)} |k_w(\psi(z))|^p \frac{(1+|w|)^p |g(z)|^p}{(1+|z|)^p} e^{-\frac{p}{2}|z|^2} dA(z) \\ &\gtrsim \frac{(1+|w|)^p}{(1+|\zeta|)^p} |k_w(\psi(\zeta)) g(\zeta)|^p e^{-\frac{p}{2}|\zeta|^2},\end{aligned}$$

and setting $w = \psi(\zeta)$ and applying (1.2.1) again results

$$\begin{aligned}\widetilde{B}_{(|g|^p,\psi)}(\psi(\zeta)) &\gtrsim \frac{(1+|\psi(\zeta)|)^p}{(1+|\zeta|)^p}|k_{\psi(\zeta)}(\psi(\zeta))|^p|g(\zeta)|^p e^{-\frac{p}{2}|\zeta|^2} \\ &= \frac{(1+|\psi(\zeta)|)^p|g(\zeta)|^p}{(1+|\zeta|)^p}e^{\frac{p}{2}|\psi(\zeta)|^2-\frac{p}{2}|\zeta|^2} = \widetilde{M}^p_{(g,\psi)}(\zeta), \qquad (2.2.4)\end{aligned}$$

and the assertion follows from the boundedness condition on $\widetilde{M}_{(g,\psi)}$ and completes the proof. □

Next, we state the corresponding results in terms of the Berezin-type integral transforms proved in [43, 46].

Theorem 2.2.2. *Let $0 < p \leq q < \infty$ and (g, ψ) be pair of entire functions. Then $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ (Respectively, $J_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$) is*

i) bounded if and only if $B_{(|g|^p,\psi)} \in L^\infty(\mathbb{C}, dA)$ (Respectively, $\widetilde{B}_{(|g|^p,\psi)} \in L^\infty(\mathbb{C}, dA)$).

ii) compact if and only if $\lim_{|z|\to\infty} B_{(|g|^p,\psi)}(z) = 0$ (Respectively, $\lim_{|z|\to\infty} \widetilde{B}_{(|g|^p,\psi)}(z) = 0$).

The result is different for the case when $0 < q < p < \infty$ and gives a stronger condition under which boundedness and compactness are equivalent.

Theorem 2.2.3. *Let $0 < q < p < \infty$ and (g, ψ) be pair of entire functions on $\mathbb{C}$. Then*

i) $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded(compact) is equivalent to the condition $B_{(|g|^p,\psi)} \in L^{\frac{p}{p-q}}(\mathbb{C}, dA)$.

ii) $J_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded(compact) is equivalent to the condition $\widetilde{B}_{(|g|^p,\psi)} \in L^{\frac{p}{p-q}}(\mathbb{C}, dA)$.

Now, we state our first theorem on boundedness and compactness of $V_{(g,\psi)}$ and $J_{(g,\psi)}$ in terms of the functions $M_{(g,\psi)}$ and $\widetilde{M}_{(g,\psi)}$. The case when the exponents are infinite, the corresponding results have been proved by T. Mengestie in [43, 49], where partly the idea to simplify those conditions now emanates from.

Theorem 2.2.4. *Let $0 < p \leq q < \infty$ and (g, ψ) be pairs of nonconstant entire functions. Then*

(i) $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ *is bounded if and only if* $M_{(g,\psi)} \in L^{\infty}(\mathbb{C}, dA)$.

(ii) $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ *is compact if and only if*

$$\lim_{|z|\to\infty} M_{(g,\psi)}(z) = 0.$$

(iii) $J_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ *is bounded if and only if* $\widetilde{M}_{(g,\psi)} \in L^{\infty}(\mathbb{C}, dA)$.

(iv) $J_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ *is compact if and only if*

$$\lim_{|z|\to\infty} \widetilde{M}_{(g,\psi)}(z) = 0.$$

Proof. By Theorem 2.2.2, the operator $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded if and only if the Berezin-type integral transforms $B_{(|g|^q,\psi)}$ is bounded, and compact if and only if $B_{(|g|^q,\psi)}(z) \to 0$ as $|z| \to \infty$. Seemingly, by Theorem 2.2.2 we have also, $J_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded if and only if $\widetilde{B}_{(|g|^q,\psi)}$ is bounded and compact if and only if $\widetilde{B}_{(|g|^q,\psi)} \to 0$ as $|z| \to \infty$. Due to these, the proof of our theorem will be complete if we prove the following two results which we formulate them as lemmas as they are interesting by their own.

Lemma 2.2.5. *Let (g, ψ) be a pair of nonconstant entire functions on $\mathbb{C}$. Then* *(i)* $M_{(g,\psi)} \in L^\infty(\mathbb{C}, dA)$ *if and only if* $B_{(|g|^p,\psi)} \in L^\infty(\mathbb{C}, dA)$ *for some* $0 < p < \infty$.

(ii) $\widetilde{M}_{(g,\psi)} \in L^\infty(\mathbb{C}, dA)$ *if and only if* $\widetilde{B}_{(|g|^p,\psi)} \in L^\infty(\mathbb{C}, dA)$ *for some* $0 < p < \infty$.

Proof. (i). If the Berezin-type integral transform $B_{(|g|^p,\psi)}$ is bounded for some $0 < p < \infty$, then the series of estimates in (2.2.3) implies that $M_{(g,\psi)}$ is bounded. On the other hand, if $M_{(g,\psi)}$ is bounded, then by Proposition 2.2.1, we may set $\psi(z) = az + b$ and argue

$$\begin{aligned}
B_{(|g|^p,\psi)}(w) &= \int_{\mathbb{C}} |k_w(\psi(z))|^p \frac{|g'(z)|^p}{(1+|z|)^p} e^{-\frac{p}{2}|z|^2} dA(z) \\
&= \int_{\mathbb{C}} |k_w(\psi(z))|^p e^{-\frac{p}{2}|\psi(z)|^2} \left(\frac{|g'(z)|^p}{(1+|z|)^p} e^{\frac{p}{2}(|\psi(z)|^2-|z|^2)} \right) dA(z) \\
&\le \left(\sup_{z\in\mathbb{C}} \frac{|g'(z)|^p}{(1+|z|)^p} e^{\frac{p}{2}(|\psi(z)|^2-|z|^2)} \right) \int_{\mathbb{C}} |k_w(\psi(z))|^p e^{-\frac{p}{2}|\psi(z)|^2} dA(z) \\
&= \left(\sup_{z\in\mathbb{C}} M^p_{(g,\psi)}(z) \right) \frac{1}{|a|^2} \int_{\mathbb{C}} |k_w(z)|^p e^{-\frac{p}{2}|z|^2} dA(z) = \frac{1}{|a|^2} \sup_{z\in\mathbb{C}} M^p_{(g,\psi)}(z),
\end{aligned}$$

where the last equality follows by (1.2.1) and hence the claim.

Similarly, one side of the statement in part (ii) follows from the series of the estimations made leading to (2.2.4). On the other hand, applying Proposition 2.2.1 and (1.2.1) again,

$$\begin{aligned}
\widetilde{B}_{(|g|^p,\psi)}(w) &= \int_{\mathbb{C}} \frac{|k_w(\psi(z))|^p}{e^{\frac{p}{2}|\psi(z)|^2}} \left(\frac{(1+|w|)^p(1+|\psi(z)|)^p |g(z)|^p}{(1+|\psi(z)|)^p(1+|z|)^p} e^{\frac{p}{2}(|\psi(z)|^2-|z|^2)} \right) dA(z) \\
&\lesssim \left(\sup_{z\in\mathbb{C}} \frac{(1+|\psi(z)|)^p |g(z)|^p}{(1+|z|)^p} e^{\frac{p}{2}(|\psi(z)|^2-|z|^2)} \right) \int_{\mathbb{C}} |k'_w(\psi(z))|^p e^{-\frac{p}{2}|\psi(z)|^2} dA(z) \\
&= \left(\sup_{z\in\mathbb{C}} \widetilde{M}^p_{(g,\psi)}(z) \right) \frac{1}{|a|^2} \int_{\mathbb{C}} \frac{|k'_w(z)|^p}{(1+|\psi(z)|)^p} e^{-\frac{p}{2}|z|^2} dA(z).
\end{aligned}$$

Using Lemma 1.2.3, (1.2.1) and Proposition 2.2.1 we estimate the above last integral by

$$\int_{\mathbb{C}} \frac{|k_w'(\psi(z))|^p}{(1+|\psi(z)|)^p} e^{-\frac{p}{2}|\psi(z)|^2} dA(z) \simeq \|k_w\|_p^p = 1,$$

from which we have that

$$\widetilde{B}_{(|g|^p,\psi)}(w) \lesssim \sup_{z\in\mathbb{C}} \widetilde{M}^p_{(g,\psi)}(z)\|k_w\|_p^p = \sup_{z\in\mathbb{C}} \widetilde{M}^p_{(g,\psi)}(z),$$

and completes the proof.

Lemma 2.2.6. *Let (g,ψ) be a pair of nonconstant entire functions on $\mathbb{C}$. Then*

(i) $M_{(g,\psi)}(w) \to 0$ *as* $|w| \to \infty$ *if and only if* $B_{(|g|^p,\psi)}(w) \to 0$ *as* $|w| \to \infty$ *for some* $0 < p < \infty$.

(ii) $\widetilde{M}_{(g,\psi)}(w) \to 0$ *as* $|w| \to \infty$ *if and only if* $\widetilde{B}_{(|g|^p,\psi)}(w) \to 0$ *as* $|w| \to \infty$ *for some* $0 < p < \infty$.

Proof. (i) One side of the statement follows easily from the estimates in (2.2.3). We shall proceed to show the other side, and assume that $M_{(g,\psi)}(w) \to 0$ as $|w| \to \infty$. Then we estimate

$$B_{(|g|^p,\psi)}(w) = \int_{\mathbb{C}} \frac{|k_w(\psi(z))|^p |g'(z)|^p}{(1+|z|)^p e^{\frac{p}{2}|z|^2}} dA(z) = \int_{|z|\leq|w|} \frac{|k_w(\psi(z))|^p}{e^{\frac{p}{2}|\psi(z)|^2}} M^p_{(g,\psi)}(z) dA(z)$$
$$+ \int_{|z|>|w|} \frac{|k_w(\psi(z))|^p M^p_{(g,\psi)}(z)}{e^{\frac{p}{2}|\psi(z)|^2}} dA(z) \lesssim \left(\sup_{z:|z|\leq|w|} |k_w(\psi(z))|^p \right) \int_{|z|\leq|w|} \frac{M^p_{(g,\psi)}(z)}{e^{\frac{p}{2}|\psi(z)|^2}} dA(z)$$
$$+ \left(\sup_{z:|z|>|w|} M^p_{(g,\psi)}(z) \right) \int_{|z|>|w|} |k_w(\psi(z))|^p e^{-\frac{p}{2}|\psi(z)|^2} dA(z)$$

By Proposition 2.2.1 and the assumption that ψ is nonconstant, it follows

$$\int_{|z|\le|w|} M^p_{(g,\psi)}(z)e^{-\frac{p}{2}|\psi(z)|^2}dA(z) \le \Big(\sup_{z\in\mathbb{C}} M^p_{(g,\psi)}(z)\Big)\int_{|z|\le|w|} e^{-\frac{p}{2}|\psi(z)|^2}dA(z) < \infty$$

from which we deduce

$$B_{(|g|^p,\psi)}(w) \lesssim \sup_{z:|z|\le|w|} |k_w(\psi(z))|^p + \sup_{z:|z|>|w|} M^p_{(g,\psi)}(z).$$

The first term in the last sup converges to zero as $|w|\to\infty$ since the normalized reproducing kernel converges to zero on compact subsets. On the other hand, by hypothesis, we have

$$\sup_{z:|z|>|w|} M_{(g,\psi)}(z) \to 0 \;\text{ as } |w|\to\infty.$$

and hence the assertion.
(ii) From the inequalities in (2.2.4), we easily deduce one of the implications for this part as well. On the other hand, if $\widetilde{M}_{(g,\psi)}(w)\to 0$ as $|w|\to\infty$, then arguing as above and eventually applying Lemma 1.2.3 and (1.2.1)

$$\begin{aligned}
\widetilde{B}_{(|g|^p,\psi)}(w) &= \int_{\mathbb{C}} |k_w(\psi(z))|^p \frac{(1+|w|)^p|g(z)|^p}{(1+|z|)^p} e^{-\frac{p}{2}|z|^2}dA(z)\\
&\simeq \int_{|z|\le|w|} \frac{|k'_w(\psi(z))|^p}{(1+|\psi(z)|)^p} e^{-\frac{p}{2}|\psi(z)|^2}\widetilde{M}^p_{(g,\psi)}(z)dA(z)\\
&\quad + \int_{|z|>|w|} \frac{|k'_w(\psi(z))|^p}{(1+|\psi(z)|)^p} e^{-\frac{p}{2}|\psi(z)|^2}\widetilde{M}^p_{(g,\psi)}(z)dA(z)\\
&\lesssim \Big(\sup_{z:|z|\le|w|} |k'_w(\psi(z))|^p\Big)\int_{|z|\le|w|} \widetilde{M}^p_{(g,\psi)}(z)e^{-\frac{p}{2}|\psi(z)|^2}dA(z)\\
&\quad + \Big(\sup_{z:|z|>|w|} \widetilde{M}^p_{(g,\psi)}(z)\Big)\int_{|z|>|w|} \frac{|k'_w(\psi(z))|^p}{(1+|\psi(z)|)^p} e^{-\frac{p}{2}|\psi(z)|^2}dA(z)\\
&\lesssim \sup_{z:|z|\le|w|} |k'_w(\psi(z))|^p + \Big(\sup_{z:|z|>|w|} \widetilde{M}^p_{(g,\psi)}(z)\Big)\|k_w\|_p^p
\end{aligned}$$

from which the assertion follows as before since k'_w converges uniformly on compact subsets and $\sup_{z:|z|>|w|} \widetilde{M}_{(g,\psi)}(z) \to 0$ when $|w| \to \infty$.

As one would expect, our results are different for the cases $p \leq q$ and $q < p$. For the latter case, we have rather a stronger condition under which the boundedness implies compactness as formulated in our next main result.

Theorem 2.2.7. *Let $0 < q < p < \infty$ and (g, ψ) be pairs of nonconstant entire functions on $\mathbb{C}$. Then*

(i) the following statements are equivalent.

(a) $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded;

(b) $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is compact;

(c) $M_{(g,\psi)} \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$.

(ii) the following statements are also equivalent.

(a) $J_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded;

(b) $J_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is compact;

(c) $\widetilde{M}_{(g,\psi)} \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$.

Proof. By Theorem 2.2.3, $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded (compact) if and only if the integral transform $B_{(|g|^q,\psi)}$ belongs to $L^{\frac{p}{p-q}}(\mathbb{C}, dA)$ and the operator $J_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is also bounded (compact) if and only if $\widetilde{B}_{(|g|^q,\psi)}$ belongs to $L^{\frac{p}{p-q}}(\mathbb{C}, dA)$. Thus, our theorem will be proved once we prove the following key lemma which is again interesting by its own right.

Lemma 2.2.8. *Let $0 < q < p < \infty$ and (g, ψ) be pairs of nonconstant entire functions on $\mathbb{C}$. Then* *(i)* $M_{(g,\psi)} \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$ *if and only if* $B_{(|g|^q,\psi)} \in L^{\frac{p}{p-q}}(\mathbb{C}, dA)$.

(ii) $\widetilde{M}_{(g,\psi)} \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$ *if and only if* $\widetilde{B}_{(|g|^q,\psi)} \in L^{\frac{p}{p-q}}(\mathbb{C}, dA)$.

Proof. (i). Making use of the estimate in (2.2.3) and Proposition 2.2.1 namely that $\psi(z) = az + b$, we estimate

$$\int_{\mathbb{C}} M_{(g,\psi)}^{\frac{pq}{p-q}}(z)dA(z) \lesssim \int_{\mathbb{C}} B_{(|g|^q,\psi)}^{\frac{p}{p-q}}(\psi(z))dA(z) = \frac{1}{|a|^2}\int_{\mathbb{C}} B_{(|g|^q,\psi)}^{\frac{p}{p-q}}(z)dA(z),$$

and the necessity of the conditions follows.

To prove the sufficiency, we may consider

$$\int_{\mathbb{C}} B_{(|g|^q,\psi)}^{\frac{p}{p-q}}(w)dA(w) = \int_{\mathbb{C}}\left(\int_{\mathbb{C}} |k_w(\psi(z))|^q \frac{|g'(z)|^q}{(1+|z|)^q} e^{-\frac{q}{2}|z|^2} dA(z)\right)^{\frac{p}{p-q}} dA(w).$$

Applying Hölder's inequality to the inner integral above gives

$$\begin{aligned} H(w) &:= \left(\int_{\mathbb{C}} |k_w(\psi(z))|^q \frac{|g'(z)|^q}{(1+|z|)^q} e^{-\frac{q}{2}|z|^2} dA(z)\right)^{\frac{p}{p-q}} \\ &\le \int_{\mathbb{C}} |k_w(\psi(z))|^q \frac{|g'(z)|^{\frac{qp}{p-q}}}{(1+|z|)^{\frac{qp}{p-q}}} e^{-\frac{qp}{2(p-q)}|z|^2} e^{\frac{q}{2}\left(\frac{p}{p-q}-1\right)|\psi(z)|^2} dA(z) \\ &\qquad \times \left(\int_{\mathbb{C}} |k_w(\psi(z))|^q e^{-\frac{q}{2}|\psi(z)|^2} dA(z)\right)^{\frac{q}{p-q}} \\ &\lesssim \int_{\mathbb{C}} |k_w(\psi(z))|^q \left|\frac{g'(z)}{1+|z|}\right|^{\frac{qp}{p-q}} e^{-\frac{qp}{2(p-q)}|z|^2} e^{\frac{q}{2}\left(\frac{p}{p-q}-1\right)|\psi(z)|^2} dA(z). \end{aligned}$$

Making use of Fubini's Theorem and (1.2.1), we further estimate

$$\int_{\mathbb{C}} B^{\frac{p}{p-q}}_{(|g|^q,\psi)}(w)dA(w) = \int_{\mathbb{C}} H(w)dA(w)$$
$$= \int_{\mathbb{C}} \left|\frac{g'(z)}{1+|z|}\right|^{\frac{qp}{p-q}} e^{-\frac{qp}{2(p-q)}|z|^2} e^{\frac{q}{2}\left(\frac{p}{p-q}-1\right)|\psi(z)|^2} \int_{\mathbb{C}} |K_{\psi(z)}(w)|^q e^{-\frac{q}{2}|w|^2} dA(w)dA(z)$$
$$\simeq \int_{\mathbb{C}} \left|\frac{g'(z)}{(1+|z|)}\right|^{\frac{qp}{p-q}} e^{\frac{qp}{2(p-q)}\left(|\psi(z)|^2-|z|^2\right)} dA(z) = \int_{\mathbb{C}} M^{\frac{pq}{p-q}}_{(g,\psi)}(z)dA(z) < \infty.$$

(ii). The necessity of the condition follows from the inequalities in (2.2.4) and the fact that $\psi(z) = az + b$ as ensured by Proposition 2.2.1.

To prove the sufficiency, we may set

$$T(w) := \left(\int_{\mathbb{C}} |k_w(\psi(z))|^q \frac{(1+|w|)^q |g(z)|^q}{(1+|z|)^q} e^{-\frac{q}{2}|z|^2} dA(z)\right)^{\frac{p}{p-q}},$$

and write

$$\int_{\mathbb{C}} \widetilde{B}^{\frac{p}{p-q}}_{(|g|^q,\psi)}(w)dA(w) = \int_{\mathbb{C}} T(w)dA(w).$$

Apply Hölder's inequality and subsequently Fubini's Theorem, Lemma 1.2.3, and (1.2.1) we observe that the above integral is bounded by

$$\int_{\mathbb{C}} \left|\frac{(1+|\psi(z)|)g(z)}{1+|z|}\right|^{\frac{qp}{p-q}} e^{-\frac{qp}{2(p-q)}|z|^2} e^{\frac{q}{2}\left(\frac{p}{p-q}-1\right)|\psi(z)|^2}$$
$$\times \int_{\mathbb{C}} \frac{|k'_{\psi(z)}(w)|^q}{(1+|\psi(w)|)^q} e^{-\frac{q}{2}|w|^2} dA(w)dA(z)$$
$$\simeq \int_{\mathbb{C}} \left|\frac{(1+|\psi(z)|)g(z)}{1+|z|}\right|^{\frac{qp}{p-q}} e^{\frac{qp}{2(p-q)}\left(|\psi(z)|^2-|z|^2\right)} dA(z) = \int_{\mathbb{C}} \widetilde{M}^{\frac{pq}{p-q}}_{(g,\psi)}(z)dA(z),$$

and completes the proof.

2.3 Essential norm of $V_{(g,\psi)}$

First we recall the definition of essential norm of an operator. Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces and T is a bounded linear operator from $\mathcal{X}$ to $\mathcal{Y}$. Then the essential norm $\|T\|_e$ of T is defined as the distance from T to the compact operators. That is

$$\|T\|_e = \inf\{\|T-K\| : K \text{ is compact operator from } \mathcal{X} \text{ to } \mathcal{Y}\}$$

From this and since the set of all compact operators is a closed subset of the set of bounded operators, it follows that an operator from $\mathcal{X}$ to $\mathcal{Y}$ is compact if and only if its essential norm is zero.

Computing essential norms of operators has been an interesting research problem in operator related function theory, and different estimates have been made for instance for the composition operator and Volterra-type integral operator on various functional spaces: See for example composition operator on Hardy space [63], weighted composition operator on the Hardy and Bergman space [23, 65], Volterra-type integral operator on Fock-type spaces [41, 42, 46] and generalized Volterra-type integral operators on Fock spaces [46].

In [46], T. Mengestie gave the following characterization of the essential norm of $V_{(g,\psi)}$ in terms of the Berezin-type integral transform.

Theorem 2.3.1. *Let $1 < p \leq q < \infty$ and (ψ, g) be a pair of entire functions on $\mathbb{C}$. If $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded, then*

$$\|V_{(g,\psi)}\|_e \simeq \left(\limsup_{|w|\to\infty} B_{(|g|^q,\psi)}(w)\right)^{\frac{1}{q}}$$

Our aim in this section is to improve the characterization given above by the following.

Theorem 2.3.2. *Let $1 < p \leq q < \infty$ and (g, ψ) be a pair of nonconstant entire functions on $\mathbb{C}$. If $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded, then*

$$\|V_{(g,\psi)}\|_e \simeq \limsup_{|z|\to\infty} M_{(g,\psi)}(z).$$

Proof. From Theorem 2.3.1, we have

$$\|V_{(g,\psi)}\|_e \lesssim \left(\limsup_{|w|\to\infty} B_{(|g|^q,\psi)}(w) \right)^{\frac{1}{q}}.$$

On the other hand,

$$\begin{aligned} B_{(|g|^q,\psi)}(w) &= \int_{\mathbb{C}} \left|k_w(\psi(z))\right|^q \frac{|g'(z)|^q e^{-\frac{q}{2}|z|^2}}{(1+|z|)^q} dA(z) \\ &\leq \sup_{z\in\mathbb{C}} \left(\frac{|g'(z)|^q e^{q\frac{|\psi(z)|^2}{2} - \frac{q}{2}|z|^2}}{(1+|z|)^q} \right) \int_{\mathbb{C}} \left|k_w(\psi(z))\right|^q e^{-\frac{q}{2}|\psi(z)|^2} dA(z) \end{aligned}$$

Since $V_{(g,\psi)}$ is bounded by Theorem 2.2.4 and Proposition 2.2.1, we have that $\psi(z) = az + b$ with $|a| \leq 1$. Furthermore, a can not be zero. Thus, we deduce the estimate

$$\int_{\mathbb{C}} \left|k_w(\psi(z))\right|^q e^{-\frac{q}{2}|\psi(z)|^2} dA(z) \simeq 1$$

and it follows from this that

$$B_{(|g|^q,\psi)}(z) \lesssim \sup_{z\in\mathbb{C}} \frac{|g'(z)|^q e^{\left(q\frac{|\psi(z)|^2}{2} - \frac{q}{2}|z|^2\right)}}{(1+|z|)^q} = \sup_{z\in\mathbb{C}} M^q_{(g,\psi)}(z).$$

Thus,

$$\|V_{(g,\psi)}\|_e \lesssim \limsup_{|z|\to\infty} M_{(g,\psi)}(z).$$

On the other hand, assume that $Q : \mathcal{F}_p \to \mathcal{F}_q$ to be a compact operator. Then since k_w converges weakly on $\mathcal{F}_p$ as $|w| \to \infty$, we have

$$\begin{aligned}\|V_{(g,\psi)} - Q\| \geq \limsup_{|w|\to\infty} \|V_{(g,\psi)}k_w - Qk_w\|_q &\geq \limsup_{|w|\to\infty} \|V_{(g,\psi)}k_w\|_q - \|Qk_w\|_q \\ &= \limsup_{|w|\to\infty} \|V_{(g,\psi)}k_w\|_q.\end{aligned}$$

Then applying monotonocity of the classical Fock spaces and the estimate in Lemma 1.2.3

$$\begin{aligned}\limsup_{|w|\to\infty} \|V_{(g,\psi)}k_w\|_q &\geq \limsup_{|w|\to\infty} \|V_{(g,\psi)}k_w\|_\infty \\ &\geq \limsup_{|w|\to\infty} \frac{\left|k_w(\psi(z))g'(z)\right|}{1+|z|} e^{-\frac{|z|^2}{2}}.\end{aligned}$$

Setting, in particular $w = \psi(z)$ in the above estimates, we have

$$\|V_{(g,\psi)} - Q\| \gtrsim \limsup_{|z|\to\infty} \frac{\left|k_{\psi(z)}(\psi(z))g'(z)\right|}{1+|z|} e^{-\frac{|z|^2}{2}} = \limsup_{|z|\to\infty} M_{(g,\psi)}(z).$$

2.4 Schatten class membership of $V_{(g,\psi)}$

The singular values of a compact operator T on a Hilbert space $\mathcal{H}$ are the square roots of the positive eigen values of the operator T^*T where T^* denotes the adjoint of T (i.e sequence of positive eigen values of $|T| = (T^*T)^{\frac{1}{2}}$). Given $0 < p < \infty$ and a Hilbert space $\mathcal{H}$, we denoted by $\mathcal{S}_p(\mathcal{H})$ the space of all compact operators T on $\mathcal{H}$ with its singular value sequence (λ_n) belonging to ℓ^p. We also recall that $\mathcal{S}_p$ is a Banach space for $1 \leq p < \infty$ with the norm

$$\|T\|_{\mathcal{S}_p} = \left(\sum_n |\lambda_n|^p\right)^{\frac{1}{p}}$$

For $p = 1$, the set $\mathcal{S}_1$ is called the trace class and for $p = 2$, the set $\mathcal{S}_2$ is called the Hilbert-Schmidt class. We refer the readers to the book of K. Zhu [71] for more details.

In [46], T. Mengestie studied the Schatten class membership of $V_{(g,\psi)}$ on the Fock spaces, and expressed the condition in terms of the Berezin-type integral transform. We state the result as follows.

Theorem 2.4.1. *Let $0 < p < \infty$ and (g, ψ) be pair of entire functions on $\mathbb{C}$. Then a bounded map $V_{(g,\psi)} : \mathcal{F}_2 \to \mathcal{F}_2$ belongs to $\mathcal{S}_p$ if and only if $B_{(|g|^2,\psi)} \in L^{\frac{p}{2}}(\mathbb{C}, dA)$.*

We now plan to simplify and replace the integral transform above by the function $M_{(g,\psi)}$. The analogous result for the operator $J_{(g,\psi)}$ is obtained in [47], where partly a source of inspiration for this simplified result also stems from.

Theorem 2.4.2. *Let $0 < p < \infty$ and (g, ψ) be pair of entire functions on $\mathbb{C}$ and the map $V_{(g,\psi)} : \mathcal{F}_2 \to \mathcal{F}_2$ be bounded. Then $V_{(g,\psi)}$ belongs to the algebra $\mathcal{S}_p(\mathcal{F}_2)$ if and only if $M_{(g,\psi)} \in L^p(\mathbb{C}, dA)$.*

Proof. By Theorem 2.4.1, the operator $V_{(g,\psi)} : \mathcal{F}_2 \to \mathcal{F}_2$ belongs to the Schatten $S_p(\mathcal{F}_2)$ class if and only if the Berezin-type integral transform $B_{(|g|^2,\psi)}$ belongs to $L^{\frac{p}{2}}(\mathbb{C}, dA)$. Thus, we only need to establish the following lemma.

Lemma 2.4.3. *Let $0 < p < \infty$ and (g, ψ) be a pair of nonconstant entire functions on $\mathbb{C}$. Then $M_{(g,\psi)} \in L^p(\mathbb{C}, dA)$ if and only if $B_{(|g|^2,\psi)} \in L^{\frac{p}{2}}(\mathbb{C}, dA)$.*

We remark in passing that Lemma 2.4.3 does not follow from Lemma 2.2.8.

Proof. Applying the inequalities in (2.2.3) and Proposition 2.2.1 we have

$$\int_{\mathbb{C}} M^p_{(g,\psi)}(z)dA(z) \leq \int_{\mathbb{C}} B^{\frac{p}{2}}_{(|g|^2,\psi)}(\psi(z))dA(z) = \frac{1}{|a|^2}\int_{\mathbb{C}} B^{\frac{p}{2}}_{(|g|^2,\psi)}(z)dA(z),$$

from which one side of the assertion in the lemma follows.

For the remaining part, we consider two different cases.

Case 1. If $0 < p < 2$, then using Lemma 1.2.3 and the fact that $\mathcal{F}_p \subset \mathcal{F}_2$ for $0 < p \leq 2$,

$$\begin{aligned}
&\int_{\mathbb{C}} |k_w(\psi(\zeta))|^2 \frac{|g'(\zeta)|^2 e^{-|\zeta|^2}}{(1+|\zeta|)^2} dA(\zeta) \\
&\qquad \simeq \int_{\mathbb{C}} \left| \int_0^z k_w(\psi(\zeta))g'(\zeta)dA(\zeta)\right|^2 e^{-|z|^2} dA(z) \\
&\qquad \lesssim \left(\int_{\mathbb{C}} \left| \int_0^z k_w(\psi(\zeta))g'(\zeta)dA(\zeta)\right|^p e^{-\frac{p}{2}|z|^2} dA(z)\right)^{\frac{2}{p}} \\
&\qquad \simeq \left(\int_{\mathbb{C}} |k_w(\psi(\zeta))|^p \frac{|g'(\zeta)|^p e^{-\frac{p}{2}|\zeta|^2}}{(1+|\zeta|)^p} dA(\zeta)\right)^{\frac{2}{p}}.
\end{aligned}$$

Using this, Fubini's Theorem and (1.2.1), we further estimate

$$\begin{aligned}
\int_{\mathbb{C}} B^{\frac{p}{2}}_{(|g|^2,\psi)}(w)dA(w) &\lesssim \int_{\mathbb{C}}\int_{\mathbb{C}} |k_w(\psi(\zeta))|^p \frac{|g(\zeta)|^p e^{-\frac{p}{2}|\zeta|^2}}{(1+|\zeta|)^p} dA(\zeta)dA(w) \\
&\int_{\mathbb{C}} \frac{|g'(\zeta)|^p e^{-\frac{p}{2}|\zeta|^2}}{(1+|\zeta|)^p} \int_{\mathbb{C}} |K_{\psi(\zeta)}(w)|^p e^{-\frac{p}{2}|w|^2} dA(w)dA(\zeta) \\
&\simeq \int_{\mathbb{C}} \frac{|g'(\zeta)|^p e^{\frac{p}{2}(\psi(\zeta)-|\zeta|^2}}{(1+|\zeta|)^p} dA(\zeta) = \int_{\mathbb{C}} M^p_{(g,\psi)}(\zeta)dA(\zeta) < \infty.
\end{aligned}$$

Case 2. If $p > 2$, then applying Hölder's inequality we get

$$S(w) := \left(\int_{\mathbb{C}} |k_w(\psi(z))|^2 \frac{|g'(z)|^2 e^{-|z|^2}}{(1+|z|)^2} dA(z) \right)^{\frac{p}{2}}$$

$$\leq \left(\int_{\mathbb{C}} |k_w(\psi(z))|^2 \frac{|g'(z)|^p e^{-\frac{p}{2}|z|^2}}{(1+|z|)^p} e^{(\frac{p}{2}-1)|\psi(z)|^2} dA(z) \right)$$

$$\times \left(\int_{\mathbb{C}} |k_w(\psi(z))|^2 e^{-|\psi(z)|^2} dA(z) \right)^{\frac{p-2}{2}}.$$

Making a change of variables and applying (1.2.1) again yields

$$\int_{\mathbb{C}} |k_w(\psi(z))|^2 e^{-|\psi(z)|^2} dA(z) \simeq 1,$$

which implies that

$$S(w) \lesssim \int_{\mathbb{C}} |k_w(\psi(z))|^2 \frac{|g'(z)|^p e^{-\frac{p}{2}|z|^2}}{(1+|z|)^p} e^{(\frac{p}{2}-1)|\psi(z)|^2} dA(z).$$

From this, Fubini's Theorem and (1.2.1) we have the estimate

$$\int_{\mathbb{C}} B^{\frac{p}{2}}_{(|g|^2,\psi)}(w) dA(w) = \int_{\mathbb{C}} S(w) dA(w)$$

$$\lesssim \int_{\mathbb{C}} \int_{\mathbb{C}} |k_w(\psi(z))|^2 \frac{|g'(z)|^p e^{-\frac{p}{2}|z|^2}}{(1+|z|)^p} e^{(\frac{p}{2}-1)|\psi(z)|^2} dA(z) dA(w).$$

$$= \int_{\mathbb{C}} \frac{|g'(z)|^p e^{-\frac{p}{2}|z|^2}}{(1+|z|)^p} e^{(\frac{p}{2}-1)|\psi(z)|^2} \int_{\mathbb{C}} |K_{\psi(z)}(w)|^2 e^{-|w|^2} dA(w) dA(z)$$

$$\simeq \int_{\mathbb{C}} \frac{|g'(z)|^p}{(1+|z|)^p} e^{\frac{p}{2}(|\psi(z)|^2 - |z|^2)} dA(z) = \int_{\mathbb{C}} M^p_{(g,\psi)}(z) dA(z) < \infty.$$

Chapter 3

Topological properties of the generalized Volterra-type integral operators on Fock spaces

In this chapter, using our results from the proceeding chapter we study some topological structures of the space of bounded $V_{(g,\psi)}$ and $J_{(g,\psi)}$ equipped with the operator norm topology. We note that over the past few decades much effort has been paid to study various operator-theoretic properties of these operators in terms of the function-theoretic properties of the inducing pairs (g, ψ). On the contrary, there has been no effort devoted to understand the topological structures of the space of bounded operators $V_{(g,\psi)}$ and $J_{(g,\psi)}$ equipped with the operator norm topology. It is worth noting that such structures have been studied for many other operators including compact differences of composition operators. We refer to [54] for references and historical remarks in this topic.

By considering a class of generalized Fock spaces where the inducing weight function grows faster than the classical Gaussian weight function $\frac{|z|^2}{2}$ we plan to study some topological structures like connected and path connected components, isolated points and essentially isolated points for the spaces of the operator. Most of the results in this chapter have been published by T. Mengestie and the author in [53, 52].

3.1 Topological Structures of $V_{(g,\psi)}$ and $J_{(g,\psi)}$

We first prove the following compact difference property.

Theorem 3.1.1. *Let $0 < p \leq q < \infty$, (g_1, ψ_1) and (g_2, ψ_2) be pairs of nonconstant entire functions on $\mathbb{C}$, and $V_{(g_1,\psi_1)}$, $V_{(g_2,\psi_2)}$, $J_{(g_1,\psi_1)}$ and $J_{(g_2,\psi_2)}$ be bounded operators from $\mathcal{F}_p$ into $\mathcal{F}_q$. Then the operator*

(i) $V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)}$ is compact if and only if either both $V_{(g_1,\psi_1)}$ and $V_{(g_2,\psi_2)}$ are compact or $\psi_1 = \psi_2$ and

$$\lim_{|z|\to\infty} M_{(g_1-g_2,\psi_1)}(z) = 0. \tag{3.1.1}$$

(ii) $J_{(g_1,\psi_1)} - J_{(g_2,\psi_2)}$ is compact if and only if either both $J_{(g_1,\psi_1)}$ and $J_{(g_2,\psi_2)}$ are compact or $\psi_1 = \psi_2$ and

$$\lim_{|z|\to\infty} \widetilde{M}_{(g_1-g_2,\psi_1)}(z) = 0. \tag{3.1.2}$$

Proof. The sufficiency of the condition in the theorem follows from Theorem 2.2.4. Thus, we may assume that the difference $V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)}$ is compact and proceed to show the necessity. If one of either $V_{(g_1,\psi_1)}$ or $V_{(g_2,\psi_2)}$ is compact, so is the other one which follows from the algebra of compact operators. It follows that either both operators are compact or both are non-compact. Thus, we may assume the later. To this end, by Theorem 2.2.4, there exist two positive numbers α_1 and α_2 such that

$$\alpha_1 = \limsup_{|z|\to\infty} M_{(g_1,\psi_1)}(z) \quad \text{and}$$

$$\alpha_2 = \limsup_{|z|\to\infty} M_{(g_2,\psi_2)}(z).$$

Since k_w is weakly convergent, compactness of $V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)}$ implies that

$$\|\big(V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)}\big)k_w\|_q \to 0 \quad \text{as} \quad |w| \to \infty. \tag{3.1.3}$$

On the other hand, using the Lemma 1.2.3 and the techniques leading to (2.2.3), we have

$$\begin{aligned}\|\big(V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)}\big)k_w\|_q &\simeq \bigg(\int_{\mathbb{C}} \frac{|g_1'(z)k_w(\psi_1(z)) - g_2'(z)k_w(\psi_2(z))|^q}{(1+|z|)^q e^{\frac{q}{2}|z|^2}} dA(z)\bigg)^{\frac{1}{q}}\\ &\gtrsim \frac{1}{1+|z|}\big|g_1'(z)k_{\psi_1(z)}(\psi_1(z)) - g_2'(z)k_{\psi_1(z)}(\psi_2(\psi_1))\big|e^{-\frac{1}{2}|z|^2}\end{aligned} \tag{3.1.4}$$

after setting $w = \psi_1(z)$.The right-hand side above can be estimated further

$$\begin{aligned}&\frac{1}{1+|z|}\big|g_1'(z)k_{\psi_1(z)}(\psi_1(z)) - g_2'(z)k_{\psi_1(z)}(\psi_2(\psi_1))\big|e^{-\frac{1}{2}|z|^2}\\ &\qquad\qquad \geq \big|M_{(g_1,\psi_1)}(z) - M_{(g_2,\psi_2)}(z)e^{-\frac{1}{2}|\psi_1(z)-\psi_2(z)|^2}\big|.\end{aligned} \tag{3.1.5}$$

Since $V_{(g_1,\psi_1)}$ and $V_{(g_2,\psi_2)}$ are bounded operators, by Theorem 2.2.4 and Proposition 2.2.1, ψ_1 and ψ_2 have the linear forms $\psi_1(z) = az + b$ and

$\psi_2(z) = cz + d$ where $|a| \leq 1$ and $|c| \leq 1$. It follows from this that if $a \neq c$, then

$$\lim_{|z|\to\infty} e^{-\frac{1}{2}|\psi_1(z)-\psi_2(z)|^2} = 0.$$

Combining this with (3.1.3), (3.1.4), (3.1.5) and the triangle inequality, we deduce

$$M_{(g_1,\psi_1)}(z) \leq \frac{M_{(g_2,\psi_2)}(z)}{e^{\frac{1}{2}|\psi_1(z)+\psi_2(z)|^2}} + \left| M_{(g_1,\psi_1)}(z) - \frac{M_{(g_2,\psi_2)}(z)}{e^{\frac{1}{2}|\psi_1(z)+\psi_2(z)|^2}} \right|$$

which implies

$$\lim_{|z|\to\infty} M_{(g_1,\psi_1)}(z) = 0.$$

It follows from this and Theorem 2.2.4 that $V_{(g_1,\psi_1)}$ is compact which contradicts our assumption. Thus, we must have $a = c$. Taking this into account,

$$\begin{aligned} \alpha_1 - \alpha_2 e^{-\frac{1}{2}|b-d|^2} &\leq \limsup_{|z|\to\infty} \left| M_{(g_1,\psi_1)}(z) - M_{(g_2,\psi_2)}(z) e^{-\frac{1}{2}|b-d|^2} \right| \\ &\lesssim \limsup_{|z|\to\infty} \|\big(V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)}\big) k_{\psi_1(z)}\|_q = 0 \end{aligned}$$

from which we get

$$\alpha_1 \leq \alpha_2 e^{-\frac{1}{2}|b-d|^2}. \tag{3.1.6}$$

On the other hand, if we repeat the above process by setting $w = \psi_2(z)$, we get

$$\alpha_2 \leq \alpha_1 e^{-\frac{1}{2}|b-d|^2}.$$

From this and (3.1.6), we find

$$\alpha_1 \leq \alpha_2 e^{-\frac{1}{2}|b-d|^2} \leq \alpha_1 e^{-|b-d|^2} \leq \alpha_1,$$

which holds only if $b = d$. This shows that $\psi_1 = \psi_2$ and hence the necessity of the condition follows from Theorem 2.2.4.

The proof of part (ii) follows in a similar fashion.

Theorem 3.1.2. *Let $0 < p < \infty$, (g_1, ψ_1) and (g_2, ψ_2) be pairs of nonconstant entire functions on $\mathbb{C}$, and $V_{(g_1,\psi_1)}$, $V_{(g_2,\psi_2)}$, $J_{(g_1,\psi_1)}$ and $J_{(g_2,\psi_2)}$ be bounded operators on $\mathcal{F}_2$. Then the operator*

(i) $V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)}$ belongs to the Schatten $\mathcal{S}_p(\mathcal{F}_2)$ class if and only if either both $V_{(g_1,\psi_1)}$ and $V_{(g_2,\psi_2)}$ belong to the $\mathcal{S}_p(\mathcal{F}_2)$ class or $\psi_1 = \psi_2 = \psi$ and

$$\int_{\mathbb{C}} \frac{|g_1'(z) - g_2'(z)|^p}{(1 + |z|)^p} e^{\frac{p}{2}(|\psi(z)|^2 - |z|^2)} dA(z) < \infty. \tag{3.1.7}$$

(ii) $J_{(g_1,\psi_1)} - J_{(g_2,\psi_2)}$ belongs to the Schatten $\mathcal{S}_p(\mathcal{F}_2)$ class if and only if either both $J_{(g_1,\psi_1)}$ and $J_{(g_2,\psi_2)}$ belong to the $\mathcal{S}_p(\mathcal{F}_2)$ class or $\psi_1 = \psi_2$ and

$$\int_{\mathbb{C}} \frac{|g_1(z) - g_2(z)|^p}{(1 + |z|)^p} e^{\frac{p}{2}(|\psi(z)|^2 - |z|^2)} dA(z) < \infty. \tag{3.1.8}$$

Proof. Since all Schatten $S_p(\mathcal{F}_2)$ class operators are compact, we can assume that $V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)}$ is compact. Then by Theorem 3.1.1, the difference is compact if and only if either both $V_{(g_1,\psi_1)}$ and $V_{(g_2,\psi_2)}$ are compact or $\psi_1 = \psi_2 = \psi$ and condition (3.1.1) holds. If both are compact and the difference is in the $\mathcal{S}_p$ class, then either both are in the $S_p(\mathcal{F}_2)$ class or both are not. In the latter case, following the same argument as in the proof of Theorem 3.1.1, the assumption that $V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)}$ belongs to $\mathcal{S}_p$ implies

$\psi_1 = \psi_2 = \psi$. On the other hand, since $V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)} = V_{(g_1-g_2,\psi)}$ is itself a generalized Volterra-type integral operator induced by the pair of symbols $(g_1 - g_2, \psi)$, by Theorem 2.4.2, it belongs to the $\mathcal{S}_p$ class if and only $M_{(g_1-g_2,\psi)} \in L^p(\mathbb{C}, dA)$. That is

$$\int_{\mathbb{C}} \frac{|g'(z)|^p}{(1+|z|)^p} e^{\frac{p}{2}(|\psi(z)|^2-|z|^2)} dA(z) < \infty$$

which completes the proof of part (i) in the theorem.

The proof of part (ii) follows in a similar manner.

As noted earlier, if we set $\psi(z) = z$, then the operators $V_{(g,\psi)}$ becomes the operators V_g. By Theorem 3.1.1, it turns out that a Volterra-type integral difference is compact if and only if both operators are compact. We record this as part of the following corollary since it is interests of its own.

Corollary 3.1.3. *(i) Let $0 < p \leq q < \infty$, $g_1 \neq g_2$, and V_{g_1} and V_{g_2} be bounded operators from $\mathcal{F}_p$ into $\mathcal{F}_q$. Then the difference operator $V_{g_1} - V_{g_2}$ is compact if and only if both V_{g_1} and V_{g_2} are compact.*

(ii) Let $g_1 \neq g_2$, $0 < p < \infty$, and $V_{g_1} - V_{g_2}$ be compact operators on $\mathcal{F}_2$. Then the operator $V_{g_1} - V_{g_2}$ belongs to the Schatten $S_p(\mathcal{F}_2)$ class if and only if $p > 2$.

(iii) Let $1 \leq p < \infty$, and V_{g_1} and V_{g_2} be bounded operators on $\mathcal{F}_p$. That is $g_1(z) = a_1z^2 + b_1z + c_1$ and $g_2(z) = a_2z^2 + b_2z + c_2$. Then the spectrum of $V_{g_1} - V_{g_2}$ on $\mathcal{F}_p$ is given by

$$\sigma\big(V_{g_1} - V_{g_2}\big) = \Big\{\lambda \in \mathbb{C} : |\lambda| \leq 2|a_1 - a_2|\Big\}.$$

Proof. Part (i) of the corollary follows once by setting $\psi_1(z) = \psi_2(z) = z$ in

Theorem 3.1.1. Thus, we shall verify part (ii). By part (i), if the difference $V_{g_1} - V_{g_2}$ is compact, then both V_{g_1} and V_{g_2} are compact and hence this gives the representation $g_1(z) = az + b$ and $g_2(z) = cz + d$ (see Corollary 2 of [46]). Since the integral is linear, we also have the relation

$$V_{g_1} - V_{g_2} = V_{g_1 - g_2}.$$

On the other hand, $V_{g_1-g_2} = V_{(g_1-g_2,z)}$. Then by Theorem 2.4.2, the difference operator $V_{g_1-g_2}$ belongs to the Schatten S_p class if and only if

$$\int_{\mathbb{C}} \frac{|(g_1' - g_2')(z)|^p}{(1+|z|)^p} dA(z) = \int_{\mathbb{C}} \frac{|a+b-c-d|^p}{(1+|z|)^p} dA(z) < \infty. \tag{3.1.9}$$

Since $g_1 \neq g_2$, we easily see that (3.1.9) holds only if $p > 2$.

Part (iii). By linearity of the integral, $V_{g_1} - V_{g_2} = V_{g_1-g_2} = V_{g_3}$ where

$$g_3(z) = (a_1 - a_2)z^2 + (b_1 - b_2)z + c_1 + c_2.$$

Then by Theorem 1 of [21],

$$\sigma(V_{g_3}) = \{0\} \cup \overline{\left\{\lambda \in \mathbb{C} \setminus \{0\} : e^{\frac{g_3}{\lambda}} \in \mathcal{F}_p\right\}} \\ = \left\{\lambda \in \mathbb{C} : |\lambda| \leq 2|a_1 - a_2|\right\} = \sigma\big(V_{g_1} - V_{g_2}\big).$$

This shows that the difference of two Volterra-type integral operators can not be nontrivially compact. If $\psi_1 = \psi_2 = \psi$, then because of the additive of the integral we observe that the difference operator $V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)}$ reduces to the integration operator $V_{(g_1-g_2,\psi)}$ for which several of its properties have been established above. Thus, in the rest of this section, we will assume that $\psi_1 \neq \psi_2$.

Theorem 3.1.4. *Let $0 < p \leq q \leq \infty$, and (g_1, ψ_1) and (g_2, ψ_2) be pairs of nonconstant entire functions on $\mathbb{C}$. If $\psi_1 \neq \psi_2$, then*

(i) $V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded if and only if both $V_{(g_1,\psi_1)}$ and $V_{(g_2,\psi_2)}$ are bounded.

(ii) $V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)} : \mathcal{F}_p \to \mathcal{F}_q$ is compact if and only if both $V_{(g_1,\psi_1)}$ and $V_{(g_2,\psi_2)}$ are compact.

The result shows that cancellation property plays no roll in the boundedness and compactness properties of the difference of generalized Volterra-type integral operators on the classical Fock spaces. On the other hand, part (ii) of the result has been already proved in Theorem 3.1.1 for finite exponent cases, and the same arguments works for infinite case. We will provide a new and simpler proof using the notion of essential norm in right after the proof of Theorem 3.1.5.

Proof. i) The boundedness of $V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)} : \mathcal{F}_p \to \mathcal{F}_q$ follows easily whenever both $V_{(g_1,\psi_1)}$ and $V_{(g_2,\psi_2)}$ are bounded. Thus we shall give a proof for the other side implication and assume that $V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded in the scale of $0 < p \leq q \leq \infty$. Then applying monotonocity of the classical Fock spaces and the estimate in Lemma 1.2.3

$$\begin{aligned}
&\|V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)}\| \geq \|V_{(g_1,\psi_1)}k_w - V_{(g_2,\psi_2)}k_w\|_q \\
&\geq \|V_{(g_1,\psi_1)}k_w - V_{(g_2,\psi_2)}k_w\|_\infty \simeq \sup_{z\in\mathbb{C}} \frac{\left|V'_{(g_1,\psi_1)}k_w(z) - V'_{(g_2,\psi_2)}k_w(z)\right|}{1+|z|} e^{-\frac{|z|^2}{2}} \\
&\qquad\qquad \geq \frac{\left|k_w(\psi_1(z))g_1'(z) - k_w(\psi_2(z))g_2'(z)\right|}{1+|z|} e^{-\frac{|z|^2}{2}}.
\end{aligned} \tag{3.1.10}$$

Here afterwards, we will simply modify the arguments used in [67] to our settings. Thus, taking in particular $w = \psi_1(z)$ in the above series of estimates, we have

$$\begin{aligned}\|V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)}\| &\gtrsim \frac{\left|k_{\psi_1(z)}(\psi_1(z))g_1'(z) - k_{\psi_1(z)}(\psi_2(z))g_2'(z)\right|}{1+|z|}e^{-\frac{|z|^2}{2}} \\ &\geq \frac{\left|k_{\psi_1(z)}(\psi_1(z))\right||g_1'(z)|e^{-\frac{|z|^2}{2}} - \left|k_{\psi_1(z)}(\psi_2(z))\right||g_2'(z)|e^{-\frac{|z|^2}{2}}}{1+|z|} \\ &= \frac{|g_1'(z)|}{1+|z|}e^{\left(\frac{|\psi_1(z)|^2}{2}-\frac{|z|^2}{2}\right)} - \frac{|g_2'(z)|\left|e^{\psi_2(z)\overline{\psi_1(z)}}\right|}{1+|z|}e^{\left(-\frac{|\psi_1(z)|^2}{2}-\frac{|z|^2}{2}\right)}.\end{aligned} \tag{3.1.11}$$

Similarly, setting $w = \psi_2(z)$ in (3.1.10), we also have

$$\begin{aligned}\|V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)}\| &\gtrsim \frac{\left|k_{\psi_2(z)}(\psi_2(z))g_2'(z) - k_{\psi_2(z)}(\psi_1(z))g_1'(z)\right|}{1+|z|}e^{-\frac{|z|^2}{2}} \\ &\geq \frac{|g_2'(z)|}{1+|z|}e^{\left(\frac{|\psi_2(z)|^2}{2}-\frac{|z|^2}{2}\right)} - \frac{|g_1'(z)|\left|e^{\psi_1(z)\overline{\psi_2(z)}}\right|}{1+|z|}e^{\left(-\frac{|\psi_2(z)|^2}{2}-\frac{|z|^2}{2}\right)}.\end{aligned} \tag{3.1.12}$$

Adding the two estimates in (3.1.11) and (3.1.12) we further have

$$\begin{aligned}&\|V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)}\| \\ &\gtrsim \left(\frac{|g_1'(z)|}{1+|z|}e^{\left(\frac{|\psi_1(z)|^2}{2}-\frac{|z|^2}{2}\right)} + \frac{|g_2'(z)|}{1+|z|}e^{\left(\frac{|\psi_2(z)|^2}{2}-\frac{|z|^2}{2}\right)}\right)\left(1 - e^{-\frac{|\psi_1(z)-\psi_2(z)|^2}{2}}\right) \\ &\geq \left(\frac{|g_1'(z)|}{1+|z|}e^{\left(\frac{|\psi_1(z)|^2}{2}-\frac{|z|^2}{2}\right)} + \frac{|g_2'(z)|}{1+|z|}e^{\left(\frac{|\psi_2(z)|^2}{2}-\frac{|z|^2}{2}\right)}\right)\left(\frac{|\psi_1(z)-\psi_2(z)|^2}{2(2+|\psi_1(z)-\psi_2(z)|^2)}\right),\end{aligned}$$

where for the last inequality we used the fact that $1 - e^{-x} \geq \frac{x}{1+x}, x \geq 0$. It follows that

$$\|V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)}\| \geq C\frac{|\psi_1(z) - \psi_2(z)|^2}{2+|\psi_1(z)-\psi_2(z)|^2}M_{(g_i,\psi_i)}(z), \tag{3.1.13}$$

with $i = 1, 2$ and for some positive constant C. To arrive at our claim, it is enough to show that

$$\lim_{|z|\to\infty} \frac{|\psi_1(z)-\psi_2(z)|^2}{2+|\psi_1(z)-\psi_2(z)|^2} \geq c > 0 \tag{3.1.14}$$

for some positive constant c. To this end, denoting $V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)}$ by T and taking logarithm on both sides of the inequality in (3.1.13), we deduce

$$\log C + \log\left(\frac{|g_1'(z)|}{1+|z|}\right) + 2\log|\psi_1(z)-\psi_2(z)| + \frac{|\psi_1(z)|^2-|z|^2}{2} \\ \leq \log(2+|\psi_1(z)-\psi_2(z)|^2) + \log\|T\|$$

and

$$\log C + \log\left(\frac{|g_2'(z)|}{1+|z|}\right) + 2\log|\psi_1(z)-\psi_2(z)| + \frac{|\psi_2(z)|^2-|z|^2}{2} \\ \leq \log(2+|\psi_1(z)-\psi_2(z)|^2) + \log\|T\|$$

Adding again the above two inequalities, we further obtain

$$\log\left(\frac{|g_1'(z)g_2'(z)|}{(1+|z|)^2}\right) + 4\log|\psi_1(z)-\psi_2(z)| + \frac{|\psi_1(z)|^2+|\psi_2(z)|^2}{2} - |z|^2 \\ \leq 2\log(2+|\psi_1(z)-\psi_2(z)|^2) + 2\log\|T\| - 2\log C. \tag{3.1.15}$$

On the other hand, $\log(2+x) \leq \frac{x}{16}+2$ for all $x \geq 0$. Applying this, for all $z \in \mathbb{C}$

$$\log\left(\frac{|g_1'(z)g_2'(z)(\psi_1(z)-\psi_2(z))^4|}{(1+|z|)^2}\right) + \frac{|\psi_1(z)-\psi_2(z)|^2}{8} - |z|^2 \\ \leq 4 - 2\log C + 2\log\|T\|.$$

Multiplying by 2 on both sides gives

$$\log\left(\frac{|g_1'(z)g_2'(z)(\psi_1(z)-\psi_2(z))^4|^2}{(1+|z|)^4}\right)+\frac{|\psi_1(z)-\psi_2(z)|^2}{4}-|\sqrt{2}z|^2$$
$$\leq 8-4\log C+4\log\|T\|.$$

Using this and proposition 2.1 of [36] after modifying the functions there to

$$f(\eta):=\frac{g_1'\big(\frac{\eta}{\sqrt{2}}\big)g_2'\big(\frac{\eta}{\sqrt{2}}\big)\big(\psi_1(\frac{\eta}{\sqrt{2}})-\psi_2(\frac{\eta}{\sqrt{2}})\big)^4}{(1+|\frac{\eta}{\sqrt{2}}|)^2}$$

and $\psi(\eta):=\frac{\psi_1(\frac{\eta}{\sqrt{2}})-\psi_2(\frac{\eta}{\sqrt{2}})}{2}$, we have that $\psi_1(z)-\psi_2(z)=az+b$ for some $a,b\in\mathbb{C}$. Consequently

$$\lim_{|z|\to\infty}\frac{|\psi_1(z)-\psi_2(z)|^2}{2+|\psi_1(z)-\psi_2(z)|^)}=\begin{cases}1 & a\neq 0\\ \frac{|b|^2}{2+|b|^2} & a=0\end{cases}$$

Observe that since $\psi_1\neq\psi_2$, both a and b can not be zero at once and hence the limit above is positive as asserted in (3.1.13). Hence $M_{(g_1,\psi_1)}$ and $M_{(g_2,\psi_2)}$ are uniformly bounded from which and Theorem 2.2.4 finite p and q, and Theorem 1 of [49] for $q=\infty$, the operators $V_{(g_1,\psi_1)}$ and $V_{(g_2,\psi_2)}$ are bounded.

Theorem 3.1.5. *Let $1<p\leq q\leq\infty$ and (g_1,ψ_1) and (g_2,ψ_2) be pairs of nonconstant entire functions on $\mathbb{C}$, and $\psi_1\neq\psi_2$. If $V_{(g_1,\psi_1)}-V_{(g_2,\psi_2)}:\mathcal{F}_p\to\mathcal{F}_q$ is bounded, then*

$$\limsup_{|z|\to\infty}\big(M_{(g_1,\psi_1)}(z)+M_{(g_2,\psi_2)}(z)\big)\lesssim\|V_{(g_1,\psi_1)}-V_{(g_2,\psi_2)}\|_e$$
$$\lesssim\limsup_{|z|\to\infty}M_{(g_1,\psi_1)}(z)+\limsup_{|z|\to\infty}M_{(g_2,\psi_2)}(z).$$

Proof. Since $\|V_{(g_1,\psi_1)}-V_{(g_2,\psi_2)}\|_e \leq \|V_{(g_1,\psi_1)}\|_e+\|V_{(g_2,\psi_2)}\|_e$ the upper estimate follows from Theorem (2.3.2).

Let $Q : \mathcal{F}_p \to \mathcal{F}_q$ be a compact operator. Since k_w converges weakly on $\mathcal{F}_p$ as $|w| \to \infty$, then

$$\begin{aligned}\|(V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)}) - Q\| &\geq \limsup_{|w|\to\infty} \|(V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)})k_w - Qk_w\|_q \\ &\geq \limsup_{|w|\to\infty} \|V_{(g_1,\psi_1)}k_w - V_{(g_2,\psi_2)}k_w\|_q - \|Qk_w\|_q \\ &= \limsup_{|w|\to\infty} \|V_{(g_1,\psi_1)}k_w - V_{(g_2,\psi_2)}k_w\|_q.\end{aligned}$$

Following the same arguments as those leading to (3.1.15) we have

$$\begin{aligned}&\limsup_{|w|\to\infty} \|V_{(g_1,\psi_1)}k_w - V_{(g_2,\psi_2)}k_w\|_q \\ &\gtrsim \limsup_{|z|\to\infty} \left(\frac{|g_1'(z)|}{1+|z|}e^{\frac{|\psi_1(z)|^2}{2}-\frac{|z|^2}{2}} + \frac{|g_2'(z)|}{1+|z|}e^{\frac{|\psi_2(z)|^2}{2}-\frac{|z|^2}{2}}\right)\left(\frac{|\psi_1(z)-\psi_2(z)|^2}{2+|\psi_1(z)-\psi_2(z)|^2}\right) \\ &\gtrsim \limsup_{|z|\to\infty}(M_{(g_1,\psi_1)}(z) + M_{(g_2,\psi_2)}(z))\end{aligned}$$

and completes the proof.

As noticed before, part (ii) of Theorem 2.3.2 follows Theorem 3.1.5. In deed, if $V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)}$ is compact, then $\|V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)}\|_e = 0$ which implies

$$\limsup_{|z|\to\infty}(M_{(g_1,\psi_1)}(z) + M_{(g_2,\psi_2)}(z)) = 0.$$

From this and Theorem 2.2.4, the assertion follows.

3.2 Topological properties of V_g on generalized Fock spaces

As pointed earlier the boundedness, compactness, Schatten-class membership and spectra structures of the Volterra-type integral operators V_g have been studied by O. Constantin [19] and T. Mengestie [49] on the classical Fock spaces. Recently, these properties were further studied in [20, 21, 51] on generalized Fock spaces where the weight functions generating the spaces grow faster than the classical Gaussian weight function $\frac{|z|^2}{2}$. Their main results showed that the operators experience richer structure on such spaces than on the classical Fock spaces.

The main goal of this section is to investigate whether the faster growth (with unbounded Laplacian) of the weight function has an impact on the topological structures like it did for the boundedness and other operator-theoretic properties. To this end we will first precise our working spaces $\mathcal{F}^p_{(\alpha,m)}$ as follows.

Let $0 < p \leq \infty$. Then for $m, \alpha > 0$, a class of generalized Fock space $\mathcal{F}^p_{(\alpha,m)}$ consist of all entire functions f for which

$$\|f\|_{(p,\alpha,m)} = \begin{cases} \left(\int_{\mathbb{C}} |f(z)|^p e^{-p\alpha|z|^m} dA(z) \right)^{\frac{1}{p}} < \infty, & p < \infty \\ \sup_{z\in\mathbb{C}} |f(z)| e^{-\alpha|z|^m} < \infty, & p = \infty \end{cases}$$

where dA again denotes the usual Lebesgue area measure on $\mathbb{C}$. Let $\mathcal{L}^p_{(\alpha,m)}$ be the space of Lebesgue measurable functions f in $\mathbb{C}$ such that the function $f(z)e^{-\alpha|z|^m}$ is in $L^p(\mathbb{C}, dA)$. Then, $\mathcal{F}^p_{(\alpha,m)}$ is a closed subspace of $\mathcal{L}^p_{(\alpha,m)}$. Therefore, $\mathcal{F}^p_{(\alpha,m)}$ is a Banach space when $1 \leq p \leq \infty$, and it is a complete metric space when $0 < p < 1$. We note that, the norm in $\mathcal{F}^p_{(\alpha,m)}$ can be rewritten in terms of the integral mean

$$M_p^p(f,r) = \int_0^{2\pi} |f(re^{it})|^p \frac{dt}{2\pi} \quad \text{and} \quad M_\infty(f,r) = \sup_{|z|=r} |f(z)|.$$

That is

$$\|f\|_{(p,\alpha,m)} = \begin{cases} \left(2\pi \int_0^\infty M_p^p(f,r) r e^{-p\alpha r^m} dr\right)^{\frac{1}{p}}, & p < \infty \\ \sup_{r>0} e^{-\alpha r^m} M_\infty(f,r), & p = \infty. \end{cases}$$

Next, we set

$$\tau_m(z) = \begin{cases} 1, & 0 \leq |\alpha(m^2-m)z| < 1 \\ \frac{|z|^{\frac{2-m}{2}}}{|\alpha(m^2-m)|^{\frac{1}{2}}}, & |\alpha(m^2-m)z| \geq 1. \end{cases} \tag{3.2.1}$$

Then, the pointwise estimate in Lemma 2.2 of [58], for subharmonic $|f|^p$

$$|f(z)|^p e^{-p\alpha|z|^m} \leq \frac{M}{\sigma^2 \tau_m^2(z)} \int_{D(z,\sigma\tau_m(z))} |f(w)|^p e^{-p\alpha|w|^m} dA(w) \tag{3.2.2}$$

holds for all finite exponent p, a small positive number σ and some constant M. This asserts that the point evaluation functionals are bounded and hence $\mathcal{F}^2_{(\alpha,m)}$ is a reproducing kernel Hilbert space. But, an explicit expression for the kernel function is still an open problem except for $m = 2$ and $\alpha = \frac{1}{2}$, which is a classical Fock space. Thus, a sequence of test functions have been constructed to play the role of kernel function. Its construction goes back to [17] and is used further by several authors for instance in [20, 51]. Following Proposition A of [20], for a sufficiently large positive number R, there exists a number $\eta(R)$ such that for any $w \in \mathbb{C}$ with $|w| > \eta(R)$, there exists an entire function $f_{(w,R)}$ such that

$$|f_{(w,R)}(z)|e^{-\alpha|z|^m} \leq C \min\left\{1, \left(\frac{\min\{\tau_m(w), \tau_m(z)\}}{|z-w|}\right)^{\frac{R^2}{2}}\right\} \tag{3.2.3}$$

for all $z \in \mathbb{C}$ and for some constant C that depends on $\alpha|z|^m$ and R. In particular when $z \in D(w, R\tau_m(w))$, the estimate becomes

$$|f_{(w,R)}(z)|e^{-\alpha|z|^m} \simeq 1. \tag{3.2.4}$$

The sequence $f_{(w,R)}$ belong to $\mathcal{F}^p_{(\alpha,m)}$ with norm estimate

$$\|f_{(w,R)}\|_{(p,\alpha,m)} \simeq \begin{cases} \tau_m^{2/p}(w), & \text{with } \eta(R) \leq |w|, \quad 0 < p < \infty \\ 1, & p = \infty. \end{cases} \tag{3.2.5}$$

In [20, 51], the spaces have been characterized in terms of the following Littlewood–Paley type derivative formula.

$$\|f\|^p_{(p,\alpha,m)} \simeq |f(0)|^p + \int_{\mathbb{C}} \frac{|f'(z)|^p e^{-p\alpha|z|^m}}{(1+|z|)^{p(m-1)}} dA(z), \quad 0 < p < \infty$$

$$\|f\|_{(\infty,\alpha,m)} \simeq |f(0)| + \sup_{z\in\mathbb{C}} \frac{|f'(z)|e^{-\alpha|z|^m}}{(1+|z|)^{m-1}}, \qquad p = \infty.$$

Such descriptions play key roles in the study of integration operators especially on spaces where the norm is defined in terms of integrals. Following Corollary 2 of [20], the family of generalized Fock spaces $\{\mathcal{F}^p_{(\alpha,m)}, m > 2\}_p$ is not nested. In fact, $\mathcal{F}^p_{(\alpha,m)} \backslash \mathcal{F}^q_{(\alpha,m)} = \emptyset$ and $\mathcal{F}^q_{(\alpha,m)} \backslash \mathcal{F}^p_{(\alpha,m)} = \emptyset$ for all $p, q > 0$ with $p \neq q$. This, makes difficult to study operators on the space and it is a major difference between the classical Fock space and the space. We refer the reader to [8, 17, 20, 21, 40, 51, 52] and related references therein for more details on this type of spaces.

In the remaining part of this chapter, we in particular consider $\alpha = 1$ and our weight function $|z|^m$ to be $m > 2$. We will denote the space $\mathcal{F}^p_{(1,m)}, m > 2$ shortly by $\mathcal{F}^p_m$ and investigate whether the faster growth of the weight function has an impact on the topological structures of space of Volterra-type integral operators compared to the results on the classical Fock spaces, which is studied by T. Mengestie [45]. We begin by considering the compact and Schatten-class differences of two Volterra-type integral operators.

3.2.1 Difference of Volterra-type integral operators

By the additivity of the integral, observe that the difference of two Volterra-type integral operators, $(V_{g_1} - V_{g_2})f = V_{g_1 - g_2}f$, is itself a Volterra-type integral operator with symbol $g_1 - g_2$. An immediate consequence of this, Theorem 3 of [20], and Theorem 1.1 of [51] gives the following useful lemma in our further consideration.

Lemma 3.2.1. *Let g_1 and g_2 be in $\mathcal{H}(\mathbb{C})$ and $0 < p, q \leq \infty$. Then if*

(i) $p \leq q$, then $V_{g_1} - V_{g_2} : \mathcal{F}^p_m \to \mathcal{F}^q_m$ is

(a) bounded if and only if $g_1 - g_2$ is a complex polynomial of degree,

$$deg(g_1 - g_2) \leq \begin{cases} m - (m-2)\left(\frac{q-p}{pq}\right), & q < \infty \\ m, & p = q = \infty \\ \frac{m(p-1)+2}{p}, & p < q = \infty. \end{cases} \tag{3.2.6}$$

(b) compact if and only if $g_1 - g_2$ is a complex polynomial of degree,

$$deg(g_1 - g_2) < \begin{cases} m - (m-2)\left(\frac{q-p}{pq}\right), & q < \infty \\ m, & p = q = \infty \\ \frac{m(p-1)+2}{p}, & p < q = \infty. \end{cases} \tag{3.2.7}$$

(ii) $q < p \leq \infty$ then $V_{g_1 - g_2} : \mathcal{F}_m^p \to \mathcal{F}_m^q$ is bounded (compact) if and only if $g_1 - g_2$ a complex polynomial of degree

$$deg(g_1 - g_2) < \begin{cases} m - 2\left(\frac{p-q}{pq}\right), & p < \infty \\ m - \frac{2}{q}, & p = \infty. \end{cases} \tag{3.2.8}$$

It follows that the difference of two unbounded Volterra-type integral operators can be bounded or compact. A suitable cancellation of each term with degree bigger than those described respectively in (3.2.6) and (3.2.7) of each of the operators in the difference makes it possible to obtain the structures even if the operators in the difference fail to possess them. Said differently, the cancellation property plays a key role in the difference. Furthermore, observe that while part (i) of the lemma gives a condition under which two Volterra-type integral operators become equal under modulo the class of bounded Volterra-type integral operators, part (ii) provides the corresponding equivalency condition under the compact class.

Before proceeding further, let us remark that this lemma will be referred several times in the rest of this chapter even in situation where we are dealing with a single Volterra-type integral operator. In this case, it shall be assumed that the second operator in the difference is zero.

A similar related question is when the difference of two Volterra-type integral operators belongs to the Schatten $S_p(\mathcal{F}_m^2)$ class. Because of the additivity of the integral again, simplifying Theorem 4 of [20] gives the following stronger topological property.

Lemma 3.2.2. *Let g_1 and g_2 be entire functions on $\mathbb{C}$ such that $g_1 \neq g_2$. Then if*

(i) $1 < p < \infty$, then $V_{g_1} - V_{g_2} \in \mathcal{S}_p(\mathcal{F}_m^2)$ if and only if $g_1 - g_2$ is a complex polynomial of

$$deg(g_1 - g_2) < m - \frac{m}{p}. \tag{3.2.9}$$

(ii) $0 < p \leq 1$, then $V_{g_1} - V_{g_2} \in \mathcal{S}_p(\mathcal{F}_m^2)$ if and only if $g_1 - g_2$ is a constant.

In contrast to the classical Fock spaces setting where it was known that all compact operators V_g belong to the Schatten S_p classes for all $p > 2$ [19, 49], from (3.2.8) and (3.2.9), this fails to hold on generalized Fock spaces $\mathcal{F}_m^2$. On the other hand, the structure gets richer with fast growing weights than on the classical setting.

3.2.2 Path connected components and isolated points in $\mathbf{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$

Many properties like boundedness and compactness of Volterra-type integral operators V_g have been studied on different spaces including Fock spaces. But much progress is not made to understand some topological structures of the space of the operators, except the works of T. Mengestie [45] and T. Mengestie with the author [53] on the classical Fock spaces. In this chapter we study these properties further on generalized Fock spaces. We denote by

$\mathbf{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$ the space of all bounded Volterra-type integral operators V_g : $\mathcal{F}_m^p \to \mathcal{F}_m^q$. Then our result reads as follows.

Theorem 3.2.3. *Let $0 < p, q \leq \infty$. Then the set of all compact operators $V_g : \mathcal{F}_m^p \to \mathcal{F}_m^q$ forms a path connected component in the space $\boldsymbol{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$.*

Proof. It suffices to show that if $V_g : \mathcal{F}_m^p \to \mathcal{F}_m^q$ a compact operator, then V_g and $V_{g(0)}$ belong to the same path connected components of the space $\mathbf{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$. The sufficiency of this can be justified as follows. If V_{g_1} and V_{g_2} are compact, then by our claim it follows that V_{g_1} and $V_{g_1(0)}$ belong to the same path connected component. The same holds true for V_{g_2} and $V_{g_2(0)}$. But $V_{g_1(0)} = V_{g_2(0)}$ is the zero operator and hence V_{g_1} and V_{g_2} are path connected. Hence, we shall proceed to show the claim.

If g is constant, then $V_g = V_{g(0)} = 0$ and the assertion of the result holds trivially. Thus, assume g is non constant, and consider scaling functions, $g_t : [0, 1] \to \mathbb{C}$, $g_t(z) = g(tz)$. Then $V_{g_t} : \mathcal{F}_m^p \to \mathcal{F}_m^q$ constitutes a sequence of compact operators for all t and satisfies

$$V_g = V_{g_1} \qquad \text{and} \qquad V_{g(0)} = V_{g_0}.$$

We need to show that the map $t \mapsto V_{g_t}$ is continuous. For this, it suffices again to show that for every $s \in [0, 1]$,

$$\lim_{t \to s} \|V_{g_t} - V_{g_s}\| = 0. \tag{3.2.10}$$

Let $g(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0$ where k is the degree of g satisfying condition on compactness cases of Lemma 3.2.1 depending on p and q. Then, for $0 < p \leq q < \infty$ and each $f \in \mathcal{F}_m^p$, with $\|f\|_{(m,p)} \leq 1$, an application of the Littlewood–Paley type estimate gives the estimate

$$\|V_{g_t}f - V_{g_s}f\|_{(m,q)}^q \simeq \int_{\mathbb{C}} \frac{|f(z)|^q |g_t'(z) - g_s'(z)|^q e^{-q|z|^m}}{(1+|z|)^{q(m-1)}} dA(z)$$
$$= |t-s|^q \int_{\mathbb{C}} \frac{|f(z)|^q |g_{(t,s)}'(z)|^q e^{-q|z|^m}}{(1+|z|)^{q(m-1)}} dA(z)$$

where

$$g_{(t,s)}'(z) = \sum_{l=1}^{k} a_l l z^{l-1} \sum_{i=0}^{l-1} s^i t^{l-1-i}.$$

It follows from Littlewood–Paley type estimate again

$$\|V_{g_t}f - V_{g_s}f\|_{(m,q)}^q \simeq |t-s|^q \int_{\mathbb{C}} \frac{|f(z)|^q |g_{(t,s)}'(z)|^q e^{-q|z|^m}}{(1+|z|)^{q(m-1)}} dA(z)$$
$$\simeq |t-s|^q \|V_{g_{(t,s)}}f\|_{(m,q)}^q \leq |t-s|^q \|V_{g_{(t,s)}}\|^q \|f\|_{(m,p)}^q \qquad (3.2.11)$$

where in the last inequality we used the fact that $V_{g_{(t,s)}}$ is bounded for every possible pair (t,s) of numbers in the interval $[0,1]$. Next, we need to show that $\|V_{g_{(t,s)}}\|$ is indeed uniformly bounded independent of the pairs (s,t). This follows easily as Lemma 3.2.1 implies

$$\|V_{g_{(t,s)}}\| \lesssim \sup_{z\in\mathbb{C}} \frac{|g_{(t,s)}'(z)||z|^{(m-2)\frac{q-p}{pq}}}{1+|z|^{m-1}} \lesssim |a_k|k^2 2^{k-2} \sup_{z\in\mathbb{C}} \frac{|z|^{k-1+(m-2)\frac{q-p}{pq}}}{1+|z|^{m-1}} \lesssim |a_k|k^2 2^{k-2}$$

Similarly, if $p < q = \infty$, then

$$\|V_{g_t}f - V_{g_s}f\|_{(m,\infty)} \simeq \sup_{z\in\mathbb{C}} \frac{|f(z)||g_t'(z) - g_s'(z)|}{(1+|z|)^{m-1}} e^{-|z|^m}$$
$$\leq |t-s| \sup_{z\in\mathbb{C}} \frac{|f(z)||g_{(t,s)}'(z)|}{(1+|z|)^{m-1}} e^{-|z|^m} \simeq |t-s| \|V_{g_{(t,s)}}f\|_{(m,\infty)}$$
$$\leq |t-s| \|V_{g_{(t,s)}}\| \|f\|_{(m,p)} \qquad (3.2.12)$$

Here again, the sequence of the norms $\|V_{g_{(t,s)}}\|$ are uniformly bounded over the pairs (s,t) since the estimate in Lemma 3.2.1 implies

$$\|V_{g_{(t,s)}}\| \lesssim \sup_{z\in\mathbb{C}} \frac{|g'_{(t,s)}(z)||z|^{\frac{m-2}{p}}}{1+|z|^{m-1}} \lesssim |a_k k^2| 2^{k-2} \sup_{z\in\mathbb{C}} \frac{|z|^{k-1+\frac{(m-2)}{p}}}{1+|z|^{m-1}} \lesssim |a_k| k^2 2^{k-2}.$$

When $0<q<p<\infty$, applying Hölder's inequality

$$\begin{aligned} \|V_{g_t} f - V_{g_s} f\|_{(m,q)}^q &\simeq \int_{\mathbb{C}} \frac{|f(z)|^q |g'_t(z) - g'_s(z)|^q e^{-q|z|^m}}{(1+|z|)^{q(m-1)}} dA(z) \\ &= |t-s|^q \int_{\mathbb{C}} \frac{|f(z)|^q |g'_{(t,s)}(z)|^q e^{-q|z|^m}}{(1+|z|)^{q(m-1)}} dA(z) \\ &\le |t-s|^q \Big(\int_{\mathbb{C}} |f(z)|^p e^{-p|z|^m} dA(z)\Big)^{\frac{q}{p}} \Big(\int_{\mathbb{C}} \Big(\frac{|g'_{(t,s)}(z)|}{(1+|z|)^{m-1}}\Big)^{\frac{pq}{p-q}} dA(z)\Big)^{\frac{p-q}{p}} \\ &= |t-s|^q \|f\|_{(m,p)}^q \Big(\int_{\mathbb{C}} \Big(\frac{|g'_{(t,s)}(z)|}{(1+|z|)^{m-1}}\Big)^{\frac{pq}{p-q}} dA(z)\Big)^{\frac{p-q}{p}} \\ &\lesssim |t-s|^q \|f\|_{(m,p)}^q \end{aligned} \tag{3.2.13}$$

where the last estimate follows from Lemma 3.2.1 since

$$\begin{aligned} \big(\deg(g'_{(t,s)})\big)^{\frac{pq}{p-q}} = (k-1)\Big(\frac{pq}{p-q}\Big) &< \Big((m-1) - 2\Big(\frac{p-q}{pq}\Big)\Big)\Big(\frac{pq}{p-q}\Big) \\ &= (m-1)\Big(\frac{pq}{p-q}\Big) - 2 \end{aligned}$$

which ensures that

$$\int_{\mathbb{C}} \Big(\frac{|g'_{(t,s)}(z)|}{(1+|z|)^{m-1}}\Big)^{\frac{pq}{p-q}} dA(z) < \infty.$$

It remains to verify the cases when $p=q=\infty$ and $0<q<p=\infty$. For the first, applying Littlewood–Paley type estimate for the infinite exponent

case,

$$\|V_{g_t}f - V_{g_s}f\|_{(m,\infty)} \simeq \sup_{z\in\mathbb{C}} \frac{|f(z)||g_t'(z)-g_s'(z)|e^{-|z|^m}}{(1+|z|)^{m-1}}$$
$$\leq \sup_{z\in\mathbb{C}} |f(z)|e^{-|z|^m} \sup_{z\in\mathbb{C}} \frac{|g_t'(z)-g_s'(z)|}{(1+|z|)^{m-1}} = \|f\|_{(m,\infty)} \sup_{z\in\mathbb{C}} \frac{|g_t'(z)-g_s'(z)|}{(1+|z|)^{m-1}}$$
$$\leq \|f\|_{(m,\infty)}|t-s| \sup_{z\in\mathbb{C}} \frac{|g'_{(t,s)}(z)|}{(1+|z|)^{m-1}} \lesssim \|f\|_{(m,p)}|t-s| \tag{3.2.14}$$

where the last inequality follows as $\deg(g'_{(t,s)}) = k-1 < m-1$.

For $0 < q < p = \infty$, by the same argument as above, we have

$$\|V_{g_t}f - V_{g_s}f\|^q_{(m,q)} \lesssim |t-s|^q \|f\|^q_{(m,\infty)} \int_{\mathbb{C}} \frac{|g'_{(t,s)}(z)|^q}{(1+|z|)^{q(m-1)}} dA(z)$$
$$\lesssim |t-s|^q \|f\|^q_{(m,\infty)}, \tag{3.2.15}$$

here the integral convergence since

$$\big(\deg(g'_{(t,s)})\big)^q = (k-1)q < \big((m-1) - \frac{2}{q}\big)q = q(m-1)-2.$$

Now, considering the series of estimates made in (3.2.11)-(3.2.15), the claim in (3.2.10) achieved and completes the proof.

An immediate consequence of this result and part (ii) of Lemma 3.2.1 is that the whole space $\mathbf{V}(\mathcal{F}^p_m, \mathcal{F}^q_m)$ is path connected whenever $0 < q < p \leq \infty$. Having grouped all the compact ones in one component, the next natural question to ask is whether there exists another component consisting of some noncompact operators in $\mathbf{V}(\mathcal{F}^p_m, \mathcal{F}^q_m)$. Our next main result claims that each of the noncompact operators in $\mathbf{V}(\mathcal{F}^p_m, \mathcal{F}^q_m)$ is in fact isolated.

Theorem 3.2.4. *Let $0 < p \leq q \leq \infty$ and $V_g : \mathcal{F}^p_m \to \mathcal{F}^q_m$ where $m > 2$ be a* *bounded operator. Then*

(i) if $q = \infty$, then the following statements are equivalent:

(a) V_g is isolated in $\boldsymbol{V}(\mathcal{F}_m^p, \mathcal{F}_m^\infty)$;

(b) g is a complex polynomial of degree

$$deg(g) = \begin{cases} m, & p = \infty \\ \frac{m(p-1)+2}{p}, & p < \infty. \end{cases} \tag{3.2.16}$$

(ii) if $q < \infty$, then the following statements are equivalent:

(a) V_g is isolated in $\boldsymbol{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$;

(b) g is a complex polynomial of degree

$$deg(g) = m - (m-2)\Big(\frac{q-p}{pq}\Big).$$

(iii) the space $\boldsymbol{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$ has the same connected and path connected components which is only the set of all compact operators in $\boldsymbol{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$.

Proof. Part (i). The assertion (a) implies (b) follows easily by an application of Theorem 3.2.3. On the other hand, if (b) holds, by Lemma 3.2.1, g has the form

$$g(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0, \quad \text{where} \quad a_k \neq 0 \quad \text{and}$$

k is a positive integer given by (3.2.16). Then we plan to show that there exists a positive number α such that

$$\|V_g - V_{g_1}\| \gtrsim \alpha$$

for all V_{g_1} in $\mathbf{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$ such that $g \neq g_1$. Note that V_{g_1} can be a compact operator here. Without loss of generality, we may set $g_1(z) = b_k z^k + b_{k-1}z^{k-1} + \cdots + b_1 z + b_0$. We shall first consider when $p = \infty$. Then k= m and using (3.2.5), (3.2.4) and the Littlewood–Paley type formula, we have

$$\begin{aligned}
\|V_g - V_{g_1}\| \geq \|V_g f_{(w,R)} - V_{g_1} f_{(w,R)}\|_{(m,\infty)} &\simeq \sup_{z\in\mathbb{C}} \frac{|f_{(w,R)}(z)||g'(z) - g_1'(z)|}{(1+|z|)^{m-1}} e^{-|z|^m} \\
&= \sup_{z\in\mathbb{C}} \frac{|f_{(w,R)}(z)||k(a_k - b_k)z^{m-1} + \cdots + (a_1 - b_1)|}{(1+|z|)^{m-1}} e^{-|z|^m} \\
&\geq \frac{|f_{(w,R)}(z)||k(a_k - b_k)z^{m-1} + \cdots + (a_1 - b_1)|}{(1+|z|)^{m-1}} e^{-|z|^m} \\
&\simeq \frac{|k(a_k - b_k)z^{m-1} + \cdots + (a_1 - b_1)|}{(1+|z|)^{m-1}}
\end{aligned}$$

where the last estimate follows after in particular setting $z = w$. Now, if $a_1 \neq b_1$, then we set $z = 0$ in in the above estimate to deduce

$$\|V_g - V_{g_1}\| \gtrsim |a_1 - b_1|.$$

On the other hand, if $a_1 = b_1$, since $g_1 \neq g$, there exists some t such that $a_t \neq b_t$, and let j be the smallest of such t. We may then rewrite

$$\begin{aligned}
\frac{|k(a_k - b_k)z^{m-1} + \cdots + (a_1 - b_1)|}{(1+|z|)^{m-1}} &= |z|^j \frac{|k(a_k - b_k)z^{m-1-j} + \cdots + (a_j - b_j)|}{(1+|z|)^{m-1}} \\
&\simeq \frac{|k(a_k - b_k)z^{m-1-j} + \cdots + (a_j - b_j)|}{(1+|z|)^{m-1-j}}
\end{aligned}$$

and setting $z = 0$ in this again leads to

$$\|V_g - V_{g_1}\| \gtrsim |a_j - b_j|.$$

Now set $\alpha = |a_j - b_j|$ or $\alpha = |a_1 - b_1|$ depending on whether $a_1 - b_1 = 0$ or not.

For $1 \leq p < \infty$, arguing similarly as in the preceding, and using (3.2.5) in addition

$$\|V_g - V_{g_1}\| \geq \frac{\|V_g f_{(w,R)} - V_{g_1} f_{(w,R)}\|_{(m,\infty)}}{\|f_{(w,R)}\|_{(m,p)}} \simeq \tau_m^{-\frac{2}{p}}(w)\|V_g f_{(w,R)} - V_{g_1} f_{(w,R)}\|_{(m,\infty)}$$
$$\simeq \tau_m^{-\frac{2}{p}}(w) \sup_{z\in\mathbb{C}} \frac{|f_{(w,R)}(z)||g'(z) - g_1'(z)|}{(1+|z|)^{m-1}} e^{-|z|^m}$$

where we estimate $\tau_m(z) \simeq |z|^{\frac{2-m}{2}}, \quad |z| \geq 1$ and $\tau_m(z) \simeq 1$ else.

Putting $z = w$ and using (3.2.4), the above quantity is estimated further

$$\gtrsim |z|^{\frac{m-2}{p}} \frac{|g'(z) - g_1'(z)|}{(1+|z|)^{m-1}} \simeq \frac{|k(a_k - b_k)z^{m-1} + \cdots + (a_1 - b_1)|}{(1+|z|)^{m-1-\frac{m-2}{p}}}.$$

Then we repeat the arguments leading to choose $z = 0$ and arrive at $\alpha = |a_j - b_j|$ or $\alpha = |a_1 - b_1|$.

Part (ii). The statement (a) implies (b) follows by an application of Theorem 3.2.3 again. On the other hand, if (b) holds, by Lemma 3.2.1, g has the form

$$g(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_0, \quad \text{where} \quad a_k \neq 0 \quad \text{and}$$

k is a positive integer given by

$$k = m - (m-2)\Big(\frac{q-p}{pq}\Big)$$

Then as before we estimate a lower bound for $\|V_g - V_{g_1}\|$ for each V_{g_1} in $\mathbf{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$ and $g \neq g_1$. We aim to show that there exists a positive number

α such that

$$\|V_g - V_{g_1}\| \gtrsim \alpha$$

for all complex polynomials of the form $g_1(z) = b_k z^k + b_{k-1}z^{k-1} + \cdots + b_0$ again. Then using Littlewood–Paley type estimate, (3.2.4) and (3.2.5), for some small positive number σ

$$\begin{aligned}
\|V_g - V_{g_1}\|^q &\geq \frac{\|V_{g-g_1} f_{(w,R)}\|^q_{(m,q)}}{\|f_{(w,R)}\|^q_{(m,p)}} \\
&\gtrsim \tau_m^{-\frac{2q}{p}}(w) \int_{D(w,\sigma\tau_m(w))} \frac{|f_{(w,R)}(z)|^q |g'(z) - g_1'(z)|^q}{(1+|z|)^{q(m-1)}} e^{-q|z|^m} dA(z) \\
&\simeq \tau_m^{-\frac{2q}{p}}(w) \int_{D(w,\sigma\tau_m(w))} \frac{|g'(z) - g_1'(z)|^q}{(1+|z|)^{q(m-1)}} dA(z).
\end{aligned}$$

Applying the pointwise estimate in (3.2.2),

$$\begin{aligned}
\tau_m^{-\frac{2q}{p}}(w) \int_{D(w,\sigma\tau_m(w))} \frac{|g'(z) - g_1'(z)|^q}{(1+|z|)^{q(m-1)}} dA(z) &\gtrsim \frac{|w|^{\frac{q(m-2)}{p}}}{(1+|w|)^{q(m-1)}} |g'(w) - g_1'(w)|^q \\
\simeq \frac{|g'(w) - g_1'(w)|^q}{(1+|w|)^{q(m-1)-\frac{q(m-2)}{p}}} &= \frac{|k(a_k - b_k)w^{m-1} + \cdots + (a_1 - b_1)|^q}{(1+|w|)^{q(m-1)-\frac{q(m-2)}{p}}}.
\end{aligned}$$

Where we estimate $\tau_m(z) \simeq |z|^{\frac{2-m}{2}}$, $|z| \geq 1$ and $\tau_m(z) \simeq 1$ else.
Then we repeat the arguments in part(i) to arrive at

$$\|V_g - V_{g_1}\| \gtrsim \alpha$$

where $\alpha = |a_j - b_j|$ or $\alpha = |a_1 - b_1|$ as before. *Part (iii).* Since path connectedness implies connectedness, by Theorem 3.2.3, the set of compact operators is a connected component in $\mathbf{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$. On the other hand, by part (i) and (ii) of Theorem 3.2.4, all non-compact operators in $\mathbf{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$

are isolated. Hence, the assertion and complete the proof.

3.2.3 Essentially isolated points in $\mathbf{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$

Having completely identified that only the set of compact operators in $\mathbf{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$ constitutes a connected component, the next natural question is whether weakening the topology in $\mathbf{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$ leads to a wider class of such components. The main result of this section confirms this affirmatively.

We in particular study the essentially isolated Volterra-type integral operators in $\mathbf{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$. We shall first estimate such norms for V_g. An important observation worthwhile making is that since g is an entire function from Lemma 3.2.1, if

$$k = \begin{cases} m - (m-2)\left(\frac{q-p}{pq}\right), & 0 < p \leq q < \infty \\ m, & p = q = \infty \\ \frac{m(p-1)+2}{p}, & 0 < p < \infty, \quad q = \infty. \end{cases} \tag{3.2.17}$$

is not a positive integer, then boundedness and compactness are equivalent, and $\|V_g\|_e = 0$ in this case. Thus, the only interesting case is when k is a positive integer.

Proposition 3.2.5. *Let $1 \leq p \leq q \leq \infty$, $V_g \in \mathbf{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$ where $k = deg(g)$ as in (3.2.17) is an integer, then $\|V_g\|_e \simeq |a_k|$ where $g(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0$.*

Proof. Simplifying Theorem 1.1 of [42] for the case $\psi(z) = |z|^m$, $m > 2$, we have

$$\|V_g\|_e \simeq \begin{cases} \limsup_{|z|\to\infty} \frac{|g'(z)|}{1+m|z|^{m-1}}, & p=q=\infty \\ \limsup_{|z|\to\infty} \frac{|g'(z)||z|^{(m-2)\frac{1}{p}}}{1+m|z|^{m-1}}, & p<q=\infty \\ \limsup_{|z|\to\infty} \frac{|g'(z)||z|^{(m-2)\frac{q-p}{pq}}}{1+m|z|^{m-1}}, & p\le q<\infty. \end{cases}$$

Substituting $g'(z) = a_k k z^{k-1} + a_{k-1}(k-1)z^{k-2} + \cdots + 2a_2 z + a_1$, we get

$$\|V_g\|_e \simeq \begin{cases} \limsup_{|z|\to\infty} \frac{|a_k k z^{k-1} + a_{k-1}(k-1)z^{k-2}+\cdots+2a_2z+a_1|}{1+m|z|^{m-1}}, & p=q=\infty \\ \limsup_{|z|\to\infty} \frac{|a_k k z^{k-1} + a_{k-1}(k-1)z^{k-2}+\cdots+2a_2z+a_1||z|^{(m-2)\frac{1}{p}}}{1+m|z|^{m-1}}, & p<q=\infty \\ \limsup_{|z|\to\infty} \frac{|a_k k z^{k-1} + a_{k-1}(k-1)z^{k-2}+\cdots+2a_2z+a_1||z|^{(m-2)\frac{q-p}{pq}}}{1+m|z|^{m-1}}, & p\le q<\infty, \end{cases}$$

from which and (3.2.17), the assertion follows.

Since the essential norm topology is weaker than the operator norm topology, Theorem 3.2.3 assured that no compact operator is essentially isolated. The next result shows further that there exists no essentially isolated points in the space $\mathbf{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$ when equipped with the essential norm topology.

Theorem 3.2.6. *Let $1 \le p \le q \le \infty$. Then there exists no essentially isolated Volterra-type integral operator in the space $\mathbf{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$.*

Proof. Consider an isolated operator $V_g \in \mathbf{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$ in the operator norm topology. By Theorem 3.2.3 and Theorem 3.2.4, V_g is not compact and hence $g(z) = a_k z^k + a_{k-1}z^{k-1} + \cdots + a_0$ where $a_k \ne 0$ and k is a positive integer given by

$$k = \begin{cases} m, & ,p = q = \infty \\ \frac{m(p-1)+2}{p}, & p < \infty, q = \infty \\ m - (m-2)(\frac{q-p}{pq}), & q < \infty. \end{cases}$$

We need to show V_g is not essentially isolated. That is

$$B(V_g, r) \nsubseteq \{V_g\} \qquad \forall r > 0$$

where $B(V_g, r) = \{V_{g_1} \in \mathbf{V}(\mathcal{F}_m^p, \mathcal{F}_m^q) : \|V_g - V_{g_1}\|_e < r\}$. To this end, consider $V_h \in \mathbf{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$ with an inducing symbol $h(z) = b_k z^k + b_{k-1} z^{k-1} + \cdots + b_0$ such that $a_k = b_k$ and $V_g \neq V_h$. Then, by Proposition 3.2.5, we have

$$\|V_g - V_h\|_e = \|V_{g-h}\|_e \simeq |a_k - b_k| = 0$$

which implies $V_h \in B(V_g, r)$ and confirms the assertion.

Following the absence of essentially isolated Volterra-type integral operators, given a noncompact $V_g \in \mathbf{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$ we further ask in which essentially connected components it belongs. Aiming this, let $P(V)$ denotes the set of complex polynomials g such that $V_g \in \mathbf{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$. Then for each such g we set

$$V(g) = \Big\{V_{g_1} \in \mathbf{V}(\mathcal{F}_m^p, \mathcal{F}_m^q) : V_g - V_{g_1} \text{is compact}\Big\}.$$

Notice here that if V_g itself is compact, since

$$|V_{g_1} f| \leq |V_{g_1} f - V_g f| + |V_g f|$$

for any V_{g_1} in $V(g)$, it follows $V(g)$ represents the set of all compact operators in $\mathbf{V}(\mathcal{F}_m^p, \mathcal{F}_m^q)$ which is known to be a connected component. Now we may decompose the whole space into the following essentially connected components:

$$\mathbf{V}(\mathcal{F}_m^p, \mathcal{F}_m^q) = \bigcup_{g \in P(V)} V(g).$$

3.2.4 The multiplication and the Volterra companion integral operators

Two operators closely related to the operator V_g are its companion operator I_g and the multiplication operator M_g where

$$I_g f(z) = \int_0^z f'(w)g(w)dw \quad \text{and} \quad M_g(f) = gf.$$

Applying integration by part in the above integral relates the three operators

$$M_g(f) = f(0)g(0) + V_g(f) + I_g(f).$$

These operators have been studied by T. Mengestie [49] on the classical Fock spaces, where boundedness of $M_g : \mathcal{F}_p \to \mathcal{F}_q$ ($I_g : \mathcal{F}_p \to \mathcal{F}_q$) for the case $0 < p \leq q \leq \infty$, is expressed in terms of g is a polynomial of degree atmost linear, while compactness is expressed in terms of g being a complex constant. For the case $0 < q < p \leq \infty$, boundedness (compactness) of $M_g : \mathcal{F}_p \to \mathcal{F}_q$ or $I_g : \mathcal{F}_p \to \mathcal{F}_q$ is equivalent to g is constant. Recently, T. Mengestie and S. Ueki [51] for the finite exponent and J. Bonet and J. Taskinen [16] for the infinite exponent studied boundedness and compactness of the operators on the generalized Fock spaces $\mathcal{F}_m^p$, which we will state it as follows.

Theorem 3.2.7. *Let $0 < p, q \leq \infty$, $m > 2$ and g be an entire function on $\mathbb{C}$. Then*

(i) if $p \neq q$, then the following are equivalent.

a) $I_g : \mathcal{F}_m^p \to \mathcal{F}_m^q$ is bounded; (or $M_g : \mathcal{F}_m^p \to \mathcal{F}_m^q$ is bounded;)

b) g is a zero function.

(ii) if $0 < p \leq \infty$, then the following are equivalent.

a) $I_g : \mathcal{F}_m^p \to \mathcal{F}_m^p$ is bounded; (or $M_g : \mathcal{F}_m^p \to \mathcal{F}_m^p$ is bounded;)

b) g is a constant function.

(iii) if $0 < p \leq \infty$, then the following are equivalent.

a) $I_g : \mathcal{F}_m^p \to \mathcal{F}_m^p$ is compact; (or $M_g : \mathcal{F}_m^p \to \mathcal{F}_m^p$ is compact;)

b) g is a zero function.

These results differ significantly between the cases when $p = q$ and $p \neq q$. It has been known that this difference does not exist in the corresponding classical setting. On the other hand, the appearance of such a difference has not been totally unexpected since in the classical Fock spaces, the monotonicity property in the sense of inclusion $\mathcal{F}_p \subset \mathcal{F}_q$ whenever $0 < p \leq q \leq \infty$, this property fails to hold for the family of generalized Fock spaces $\mathcal{F}_m^p$.
In contrast to the structure of V_g, a result in [51, Theorem 1.2], which is stated above, revealed that the structures of I_g and M_g get poorer when they act between two different generalized Fock spaces with fast growing generating weight functions than on the classical setting. Consequently, the

spaces of the operators under the operator norm are path connected which we formulate it as follows.

Theorem 3.2.8. *Let* $0 < p, q \leq \infty$*. Then both* $\boldsymbol{I}(\mathcal{F}_m^p, \mathcal{F}_m^q)$ *and* $\boldsymbol{M}(\mathcal{F}_m^p, \mathcal{F}_m^q)$ *are path connected spaces where* $\boldsymbol{I}(\mathcal{F}_m^p, \mathcal{F}_m^q)$ *and* $\boldsymbol{M}(\mathcal{F}_m^p, \mathcal{F}_m^q)$ *are respectively the spaces of all bounded* J_g *and* M_g *from* $\mathcal{F}_m^p$ *to* $\mathcal{F}_m^q$ *.*

Proof. Since the proofs on the two spaces $\mathbf{I}(\mathcal{F}_m^p, \mathcal{F}_m^q)$ and $\mathbf{M}(\mathcal{F}_m^p, \mathcal{F}_m^q)$ are similar, we shall only provide a simple proof for the space $\mathbf{I}(\mathcal{F}_m^p, \mathcal{F}_m^q)$. If $p \neq q$, then from Theorem 3.2.7 boundedness of J_g implies g is zero function, I_g becomes the zero operator and the result holds trivially. Thus, suppose $p = q$, and let $g_1 \neq g_2$ such that $I_{g_1}, I_{g_2} \in \mathbf{I}(\mathcal{F}_m^p, \mathcal{F}_m^q)$. Then, by Theorem 3.2.7 we have g_1 and g_2 are constants. If one of them is zero, then the result holds trivially again, and we shall assume that both g_1 and g_2 are nonzero. Consider a map $\gamma(t) = tg_2 + (1-t)g_1$, $t \in [0,1]$, then

$$J_{\gamma(0)} = J_{g_1} \qquad \text{and} \qquad J_{\gamma(1)} = J_{g_2}.$$

We need to show that the map $t \mapsto J_{\gamma(t)}$ is continuous. That is

$$\lim_{t \to s} \|J_{\gamma(t)} - J_{\gamma(s)}\| = 0 \qquad \forall s \in [0,1]. \tag{3.2.18}$$

For each $f \in \mathcal{F}_m^p$ with $\|f\|_{(m,p)} \leq 1$ applying the Littlewood-Paley type formula, for $0 < p < \infty$, we estimate

$$\|J_{\gamma(t)} f - J_{\gamma(s)} f\|_{(m,p)}^p \simeq |t-s|^p \int_{\mathbb{C}} \frac{|f'(z)|^p e^{-p|z|^m}}{(1+|z|)^{p(m-1)}} dA(z)$$

$$\simeq |t-s|^p \|f\|_{(m,p)}^p \leq |t-s|^p \tag{3.2.19}$$

and similarly applying Littlewood–Paley type formula for the case infinite case, we have

$$\|J_{\gamma(t)}f - J_{\gamma(s)}f\|_{(m,\infty)} \lesssim |t-s|$$

from which and (3.2.19), we conclude (3.2.18).

Chapter 4

Dynamics of the Volterra-type integral and differentiation operators on generalized Fock spaces

This chapter is devoted to the study of dynamical properties of Volterra-type integral operator, Hardy operator and differentiation operator acting on a class of generalized Fock spaces $\mathcal{F}^p_{(\alpha,m)}$. Many of their basic properties including boundedness, compactness and spectra have been extensively studied when acting on several function spaces over various domains; see for example [1, 7, 40, 44, 48, 51] and the references therein.

Understanding the dynamical structures of these operators is another important and basic problem in operator related function theory. The main purpose of this chapter is to study such structures on generalized Fock spaces $\mathcal{F}^p_{(\alpha,m)}$. We are especially interested in identifying their various forms of cyclicity, power boundedness, and uniform mean ergodic properties. Most of the results in this chapter have been published by J. Bonet, T. Mengestie and the author in [15].

4.1 Introduction

We recall some definitions related to iterates of an operator. Given a Banach space X, we denote by $\mathcal{L}(X)$ the space of continuous linear operators T on X. An operator $T \in \mathcal{L}(X)$ is said to be hypercyclic if there exists a vector x in X such that its orbit, $\{T^n x; n \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}\}$, is dense in X. The operator is called cyclic if the linear span of an orbit is dense in X, and supercyclic whenever the projective orbit, $\{\lambda T^n x; n \in \mathbb{N}_0, \lambda \in \mathbb{C}\}$, is dense in X. Obviously, hypercyclicity is a stronger property than supercyclicity which in turn is stronger than cyclicity. Good references on this subject are the books in [7, 31].

An operator $T \in \mathcal{L}(X)$ is said to be power bounded if there exists a positive number M such that $\|T^n\| \leq M$ for all $n \in \mathbb{N}_0$. It is said to be mean ergodic if there exists an operator $P \in \mathcal{L}(X)$ such that

$$Px := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} T^k x, \;\; x \in X$$

exists in X. If the convergence is in the operator norm, then T is called uniformly mean ergodic. The standard references about studies on mean ergodic operators are the books of U. Krengel [35] and K. Yosida [69].

The norms of the monomials will play an important role in studying the dynamical properties of both differentiation and integration operators on the spaces $\mathcal{F}^p_{(\alpha,m)}$. Thus, we estimate them using Striling's formulas,

$$n! \simeq \sqrt{n}n^n e^{-n} \quad \text{and} \quad \Gamma(x+1) \simeq \sqrt{x}x^x e^{-x}, \ x > 0, \tag{4.1.1}$$

where Γ denotes the Gamma function, as

$$\begin{aligned}\|z^n\|_{(p,\alpha,m)} &= \left(2\pi \int_0^\infty r^{pn} e^{-p\alpha r^m} r dr\right)^{1/p} = \frac{\left(2\pi\Gamma\left(\frac{pn}{m} + \frac{2}{m}\right)\right)^{1/p}}{m(p\alpha)^{\frac{n}{m}+\frac{2}{mp}}} \\ &\simeq \frac{(pn+2-m)^{\frac{n}{m}+\frac{2}{mp}-\frac{1}{2p}}}{(p\alpha)^{\frac{n}{m}+\frac{2}{mp}} m^{\frac{n}{m}+\frac{2}{mp}+\frac{1}{2p}} e^{\frac{n}{m}+\frac{2}{mp}-\frac{1}{p}}} \\ &\simeq \left(\frac{n}{me\alpha}\right)^{\frac{n}{m}+\frac{2}{mp}-\frac{1}{2p}}.\end{aligned} \tag{4.1.2}$$

From the preceding estimate we have in particular, for all $n \in \mathbb{N}$,

$$\|z^n\|_{(p,\alpha,1)} \simeq n!\alpha^{-n} n^{\frac{3-p}{2p}}. \tag{4.1.3}$$

4.2 Dynamical properties of the Volterra-type integral operators

As discussed in the previous chapters, various aspects of the Volterra- type integral operator V_g which includes boundedness, compactness, and spectra have been well studied in large class of function spaces. But, much less is known about its dynamical and mean ergodic properties, except in the special case when the symbol g is the identity map. The fact that the iterates of the operator involve multiple integrals makes it difficult to get best possible estimates of the norms.

In this section, we begin the study of the dynamical properties of V_g on the spaces $\mathcal{F}^p_{(\alpha,m)}$. It was shown in [20] (see also [21]), and in [16] for $p=\infty$, that V_g is bounded on $\mathcal{F}^p_{(\alpha,m)}$ if and only if g is a complex polynomial of degree l not bigger than m ($l \leq m$), and V_g is compact in this space if and only if the degree l of g is strictly smaller than m or m is not a positive integer.

Proposition 4.2.1. *Let $1 \leq p < \infty$. Let V_g be bounded on $\mathcal{F}^p_{(\alpha,m)}$ and hence $g(z) = a_l z^l + a_{l-1}z^{l-1} + ... + a_1 z + a_0,\ l \leq m$. Then*

(i) V_g is not supercyclic on $\mathcal{F}^p_{(\alpha,m)}$ and hence not hypercyclic.

(ii) If $g(z) = az^l + b, a \neq 0$, then V_g is cyclic if and only if $l = 1$.

Proof. (i) Since $V_g f(0) = 0$ for every f in $\mathcal{F}^p_{(\alpha,m)}$, the projective orbit of f under V_g contains only functions that vanish at zero. Thus, it cannot be dense in $\mathcal{F}^p_{(\alpha,m)}$, which implies that V_g is not supercyclic on $\mathcal{F}^p_{(\alpha,m)}$ and hence not hypercyclic either.

(ii) If $l = 1$, then $(V_g)^n(\mathbf{1})(z) = \frac{a^n z^n}{n!}$. Hence

$$\{(V_g)^n(\mathbf{1}), n \geq 0\} = \{1, az, \frac{a^2z^2}{2}, \cdots, \frac{a^n z^n}{n!}, \cdots\}.$$

The linear span of the latter is known to be dense in $\mathcal{F}^p_{(\alpha,m)}$.

If $l > 1$ and f belongs to $\mathcal{F}^p_{(\alpha,m)}$, using a functional that annihilates the $(l-1)$-th Taylor polynomial of f, together with the monomials $z^k, k \geq l$, we conclude that f cannot be a cyclic vector for V_g. □

The class of Volterra-type integral operators include the classical integration operator $Jf(z) = \int_0^z f(w)dw$ in particular when $g(z) = z$. By [20, 51, 1] or [52, Lemma 1.1], it follows that J is bounded on $\mathcal{F}^p_{(\alpha,m)}$ if and only if $m \geq 1$

and compact if and only if $m > 1$. Thus, from the above proposition, the space $\mathcal{F}^p_{(\alpha,m)}$ supports no supercyclic and hypercyclic integration operator J, while J is cyclic on $\mathcal{F}^p_{(\alpha,m)}$ with cyclic vector 1.

To state our next main result, we first recall some definitions. Denote by $\mathcal{H}(\mathbb{C})$ the set of entire functions on $\mathbb{C}$. For $r \geq 0$ and each $f \in \mathcal{H}(\mathbb{C})$, set

$$M_\infty(f, r) = \sup_{|z|=r} |f(z)|,$$

and define the growth type space $\mathcal{F}^\infty_{(\alpha,m)}$ as the space of functions $f \in \mathcal{H}(\mathbb{C})$ such that

$$\|f\|_{(\infty,\alpha,m)} = \sup_{r>0} e^{-\alpha r^m} M_\infty(f, r) < \infty.$$

The estimate corresponding to (4.1.2) becomes

$$\|z^n\|_{(\infty,\alpha,m)} \simeq \left(\frac{n}{me\alpha}\right)^{\frac{n}{m}}. \tag{4.2.1}$$

For $m = 1$, the space $\mathcal{F}^p_{(\alpha,1)}$ is denoted by $B_{p,p}(\alpha)$ in [8]. To simplify the notation below, we write the symbol $g(z) = a_l z^l + a_{l-1} z^{l-1} + \ldots + a_1 z + a_0$, with $l \leq m$, as $g = g_l + g_{l-1}$ where $g_l(z) = a_l z^l$ and $g_{l-1}(z) = a_{l-1} z^{l-1} + \ldots + a_1 z + a_0$. Then the operators on $\mathcal{F}^\infty_{(\alpha,m)}$ satisfy

$$V_g = V_{g_l} + V_{g_{l-1}},$$

of which $V_{g_{l-1}}$ is always compact and quasi-nilpotent.

Following [6], for each $\lambda \in \mathbb{C}$, $a \in \mathbb{C}$ and $m \in \mathbb{N}$, the operator K_λ is defined on $\mathcal{H}(\mathbb{C})$ by

$$K_\lambda f(z) = ame^{\lambda z^m} \int_0^z e^{-\lambda w^m} w^{m-1} f(w) dw = amz^m \int_0^1 e^{\lambda z^m(1-t^m)} t^{m-1} f(tz) dt$$

Thus, when $\lambda = 0$ $m = l \in \mathbb{N}$ and $a = a_l$, then K_λ is just the Volterra-type integral operator V_{g_l}.

Lemma 4.2.2. *Let $m \geq 1, m \in \mathbb{N}$.*

(i) If $|\lambda| < \alpha$, then the operator K_λ is continuous on $\mathcal{F}^\infty_{(\alpha,m)}$ with operator norm $\|K_\lambda\| \leq \frac{|a|}{\alpha - |\lambda|}$.

(ii) If $l = m \in \mathbb{N}$, then V_{g_l} is continuous on $\mathcal{F}^\infty_{(\alpha,m)}$ and its operator norm satisfies $\|V_{g_l}\| \leq \frac{|a_l|}{\alpha}$.

Proof. (i) For $r > 0$, we have

$$\begin{aligned} e^{-\alpha r^m} M_\infty(K_\lambda f, r) &= e^{-\alpha r^m} |a| m r^m \int_0^1 e^{|\lambda| r^m(1-t^m)} t^{m-1} M_\infty(f, rt) dt \\ &= |a| m r^m \int_0^1 t^{m-1} e^{-\alpha(tr)^m} e^{(\alpha - |\lambda|) r^m (t^m - 1)} M_\infty(f, rt) dt \end{aligned}$$

$$\leq |a| m r^m \|f\|_{(\infty,\alpha,m)} \int_0^1 t^{m-1} e^{(\alpha-|\lambda|) r^m (t^m-1)} dt \leq \frac{|a|}{\alpha - |\lambda|} \|f\|_{(\infty,\alpha,m)}.$$

Therefore

$$\|K_\lambda f\|_{(\infty,\alpha,m)} = \sup_{r>0} e^{-\alpha r^m} M_\infty(K_\lambda f, r) \leq \frac{|a|}{\alpha - |\lambda|} \|f\|_{(\infty,\alpha,m)}.$$

(ii) This is a direct consequence of part (i) for $l = m$, $a = a_l$ and $\lambda = 0$.

Theorem 4.2.3. *Let $1 \leq p \leq \infty$, $m \geq 1$ and $l \in \mathbb{N}$. Assume that the operator V_g is bounded on $\mathcal{F}^p_{(\alpha,m)}$, with $g(z) = g_l(z) + g_{l-1}(z)$, $g_l(z) = a_l z^l$, $l \leq m$. Then* *(i) If $m > 1$ and $l < m$, then V_g is compact, quasi-nilpotent, hence power bounded, and uniformly mean ergodic on $\mathcal{F}^p_{(\alpha,m)}$.*

(ii) If $l = m \in \mathbb{N}$ and $p = \infty$, then V_g is power bounded if and only if $|a_l| \le \alpha$.

(iii) If $l = m \in \mathbb{N}$, $p = \infty$ and $|a_l| \le \alpha$, then V_{g_l} is uniformly mean ergodic if and only if $|a_l| < \alpha$.

(iv) If $l = m \in \mathbb{N}$ and $1 \le p < \infty$ and V_g is power bounded, then $|a_l| \le \alpha$.

(v) If $l = m \in \mathbb{N}$ and $1 \le p < \infty$ and V_{g_l} is power bounded and uniformly mean ergodic, then $|a_l| < \alpha$.

Proof. (i) From results in [16, 51] for $p = \infty$ and in [20] for $1 \le p < \infty$, the operator V_g is compact on $\mathcal{F}^p_{(\alpha,m)}$ for all $1 \le p \le \infty$, since $l < m$. Moreover, by [13, Theorem] and [21, Theorem 1], we have $\sigma(V_g) = \{0\}$, hence V_g is quasi-nilpotent. By the spectral radius formula, there exist $\beta < 1$ and $N \in \mathbb{N}$ such that

$$\|V_g^n\| \le \beta^n \quad \text{for all } n \ge N. \tag{4.2.2}$$

This shows that V_g is power bounded.

The operator V_g is also uniformly mean ergodic in this case. Indeed, an application of (4.2.2) yields, for some $C > 0$ depending on N,

$$\frac{1}{n}\left\|\sum_{k=1}^{n} V_g^k\right\| \le \frac{1}{n}\sum_{k=1}^{n} \|V_g^k\| \le \frac{C}{n} + \frac{\beta}{n(1-\beta)} \to 0 \text{ as } n \to \infty.$$

(ii) Now $p = \infty$ and $l = m \in \mathbb{N}$. Assume that $|a_l| \le \alpha$. By Lemma 4.2.2 (ii) we have, for each $n \in \mathbb{N}$,

$$\|V_{g_l}^n\| \leq |a_l|^n/\alpha^n \leq 1.$$

This implies that V_{g_l} is power bounded. Since the sum of a power bounded operator and a quasi-nilpotent operator is power bounded, we conclude that V_g is power bounded.

Conversely, suppose that V_g is power bounded. Since $V_{g_{l-1}}$ is quasi-nilpotent by part (i), we get that V_{g_l} is power bounded. However, for each positive integer n, a straightforward integral computation gives

$$V_{g_l}^n(\mathbf{1})(z) = \frac{l^n(a_l)^n z^{ln}}{\prod_{j=1}^n (jl)} = \frac{(a_l)^n z^{ln}}{n!}$$

from which and (4.2.1) we have

$$\|V_{g_l}^n\| \geq \frac{\|V_{g_l}^n \mathbf{1}\|_{(\infty,\alpha,l)}}{\|\mathbf{1}\|_{(\infty,\alpha,l)}} \simeq \frac{|a_l|^n}{n!(\alpha e l)^n}(nl)^n \simeq \frac{|a_l|^n}{\alpha^n}\frac{1}{\sqrt{n}} \to \infty$$

as $n \to \infty$ whenever $|a_l| > \alpha$, a contradiction. Thus $|a_l| \leq \alpha$.

(iii) In this part we assume $p = \infty$, $l = m \in \mathbb{N}$ and $|a_l| \leq \alpha$. We first suppose that $|a_l| < \alpha$. By Lemma 4.2.2 we have, for each $n \in \mathbb{N}$,

$$\|V_{g_l}^n\| \leq |a_l|^n/\alpha^n.$$

Hence

$$\frac{1}{n}\left\|\sum_{k=1}^n V_{g_l}^k\right\| \leq \frac{1}{n}\sum_{k=1}^n \|V_{g_l}^k\| \leq \frac{|a_l|}{n(\alpha - |a_l|)} \to 0 \text{ as } \ n \to \infty,$$

and V_{g_l} is uniformly mean ergodic.

Conversely, suppose that V_{g_l} is uniformly mean ergodic and that $|a_l| \leq \alpha$.

Part (ii) implies that V_{g_l} is power bounded. From [13] it follows that the spectrum $\sigma(V_{g_l}) = \{\lambda \in \mathbb{C} : |\lambda| \leq \frac{|a_l|}{\alpha}\}$. Thus, if $|a_l| = \alpha$, then 1 is an accumulation point of $\sigma(V_{g_l})$. By [27, Theorem 3.16] (see also [35, Theorem 2.7]), the operator V_{g_l} is not uniformly mean ergodic. Therefore $|a_l| < \alpha$.

(iv) If $l = m \in \mathbb{N}$ and $1 \leq p < \infty$ and V_g is power bounded, then V_{g_l} is power bounded since $V_{g_{l-1}}$ is quasi-nilpotent by part (i). Now, an integral computation again gives, for each positive integers k and n,

$$V_{g_l}^n(z^k) = \frac{(a_l)^n l^n z^{ln+k}}{\prod_{j=1}^n (jl+k)}$$

and hence

$$\begin{aligned} \|V_{g_l}^n\| \gtrsim \limsup_{k\to\infty} \frac{\|V_{g_l}^n(z^k)\|_{(p,\alpha,l)}}{\|z^k\|_{(p,\alpha,l)}} &= \limsup_{k\to\infty} \frac{|a_l|^n l^n \|z^{nl+k}\|_{(p,\alpha,l)}}{\prod_{j=1}^n (jl+k)\|z^k\|_{(p,\alpha,l)}} \\ &\geq \limsup_{k\to\infty} \frac{|a_l|^n l^n \|z^{nl+k}\|_{(p,\alpha,l)}}{(nl+k)^n \|z^k\|_{(p,\alpha,l)}}, \end{aligned} \tag{4.2.3}$$

where the last inequality follows since

$$\prod_{j=1}^n (jl+k) = e^{\sum_{j=1}^n \log(jl+k)} \leq e^{n\log(nl+k)} = (nl+k)^n.$$

Applying the norm estimate in (4.1.2),

$$\frac{\|z^{nl+k}\|_{(p,\alpha,l)}}{\|z^k\|_{(p,\alpha,l)}} \simeq \frac{(nl+k)^n}{(e\alpha)^n}\Big(1+\frac{nl}{k}\Big)^{\frac{k}{l}+\frac{2}{pl}-\frac{1}{2p}}$$

and plugging this in (4.2.3) and making further simplifications

$$\|V_{g_l}^n\| \geq \frac{|a_l|^n}{\alpha^n} \limsup_{k\to\infty} \frac{\left(1+\frac{nl}{k}\right)^{\frac{k}{l}}}{e^n} \left(1+\frac{nl}{k}\right)^{\frac{2}{pl}-\frac{1}{2p}}$$
$$\geq \frac{|a_l|^n}{\alpha^n} \limsup_{k\to\infty} \frac{\left(1+\frac{nl}{k}\right)^{\frac{k}{l}}}{e^n} = \frac{|a_l|^n}{\alpha^n},$$

which yields $\alpha \geq |a_l|$.

(v) Suppose that V_{g_l} is power bounded and uniformly mean ergodic. It follows from [21] that the spectrum $\sigma(V_{g_l}) = \{\lambda \in \mathbb{C} : |\lambda| \leq \frac{|a_l|}{\alpha}\}$. Hence, if $|a_l| = \alpha$, then 1 is an accumulation point of $\sigma(V_{g_l})$. By [27, Theorem 3.16] (see also [35, Theorem 2.7]), the operator V_{g_l} is not uniformly mean ergodic. This implies $|a_l| < \alpha$.

Corollary 4.2.4. *Let $1 \leq p \leq \infty$ and $m \geq 1$. Then the integration operator J on $\mathcal{F}^p_{(\alpha,m)}$ satisfies*

(i) If $m > 1$ and $1 \leq p \leq \infty$, then J is compact, quasi-nilpotent, hence power bounded, and uniformly mean ergodic on $\mathcal{F}^p_{(\alpha,m)}$.

(ii) If $m = 1$ and $p = \infty$, then J is power bounded if and only if $\alpha \geq 1$.

(iii) If $m = 1$ and $p = \infty$, then J is uniformly mean ergodic if and only if $\alpha > 1$.

(iv) If $m = 1$ and $1 \leq p < \infty$ and J is power bounded, then $\alpha \geq 1$.

(v) If $m = 1$ and $1 \leq p < \infty$ and J is power bounded and uniformly mean ergodic, then $\alpha > 1$.

The proof of the corollary follows immediately from Theorem 4.2.3.

4.3 Dynamical properties of the Hardy operator

The classical Hardy operator defined by,

$$Hf(z) = \frac{1}{z}\int_0^z f(w)dw = \frac{1}{z}Jf(z),$$

has been studied on various spaces by several authors. A. Arvantidis and A. Siskakis [5] studied on the Hardy spaces of the upper half-plane, M. Beltran, J. Bonet and C. Fernandez [9] studied boundedness, spectrum, power boundedness and uniform mean ergodicity of the operator acting on weighted Banach spaces of entire functions. The study was continued by M. Beltran in [8] and she characterized different properties of the operator including dynamical properties on weighted Bergman spaces. In this section, we study the dynamics of the operator on the generalized Fock spaces $\mathcal{F}^p_{(\alpha,m)}$.

Theorem 4.3.1. *Let $1 \leq p < \infty$. Then the Hardy operator H is both power bounded and uniformly mean ergodic on $\mathcal{F}^p_{(\alpha,m)}$. Furthermore, $\|H\| = 1$.*

Proof. Let us first show that H is bounded and $\|H\| = 1$. Proceeding as in [8], for $r > 0$, we get

$$\begin{aligned} M_p^p(Hf,r) &= \frac{1}{2\pi}\int_0^{2\pi} \Big|\frac{1}{re^{i\theta}}\int_0^{re^{i\theta}} f(w)dw\Big|^p d\theta = \frac{1}{2\pi}\int_0^{2\pi} \Big|\int_0^1 f(tre^{i\theta})dt\Big|^p d\theta \\ &\leq \int_0^1 \frac{1}{2\pi}\int_0^{2\pi} |f(tre^{i\theta})|^p d\theta dt = M_p^p(f,tr) \leq M_p^p(f,r). \end{aligned}$$

Thus, we have

$$M_p^p(Hf,r) \leq M_p^p(f,r).$$

Multiplying both sides by $2\pi r e^{-p\alpha r^m}$ and integrating over r yields

$$\|Hf\|^p_{(p,\alpha,m)} \le \|f\|^p_{(p,\alpha,m)},$$

which implies that H is bounded and $\|H\| \le 1$.
On the other hand, if H is bounded, then

$$\|H\mathbf{1}\|^p_{(p,\alpha,m)} = \int_{\mathbb{C}} |H\mathbf{1}(z)|^p e^{-p\alpha|z|^m} dA(z) = \|\mathbf{1}\|^p_{(p,\alpha,m)}$$

and hence $\|H\| = 1$.
Next fix $n > 1$. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be the Taylor series expansion of $f \in \mathcal{F}^p_{(\alpha,m)}$. A simple integral computation shows that

$$H^n f(z) = \sum_{k=0}^{\infty} \frac{a_k z^k}{(k+1)^n}. \tag{4.3.1}$$

On the other hand, for each $r > 0$, applying Cauchy inequalities

$$|a_k| r^k = \frac{r^k}{2\pi}\left| \int_{|\zeta|=r} \frac{|f(\zeta)|}{\zeta^{k+1}} d\zeta \right| \le \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})| d\theta \le M_p(f,r),$$

from which we get $|a_k|\|z^k\|_{(p,\alpha,m)} \le \|f\|_{(p,\alpha,m)}$ holds for $k \ge 0$. This and (4.3.1) imply

$$\|H^n f\|_{(p,\alpha,m)} \le \sum_{k=0}^{\infty} \frac{|a_k|\|z^k\|_{(p,\alpha,m)}}{(k+1)^n} \le \|f\|_{(p,\alpha,m)} \sum_{k=0}^{\infty} \frac{1}{(k+1)^n} \simeq \frac{\|f\|_{(p,\alpha,m)}}{n-1}$$

for all $n > 1$. Therefore, H is power bounded. Observe also that,

$$\Big\|\frac{1}{n}\sum_{j=1}^{n} H^j\Big\| \leq \frac{1}{n}\sum_{j=1}^{n}\|H^j\| = \frac{1}{n}\Big(\|H\| + \sum_{j=2}^{n}\|H^j\|\Big) = \frac{1}{n} + \frac{1}{n}\sum_{j=2}^{n}\|H^j\|$$
$$\lesssim \frac{1}{n} + \frac{1}{n}\sum_{j=2}^{n}\frac{1}{j-1} \simeq \frac{1}{n} + \frac{\log|n-1|}{n} \to 0$$

as $n \to \infty$. Therefore, H is also uniformly mean ergodic, which completes the proof.

We now mention consequences of Theorem 4.3.1. Since all orbits $\{T^n f; n = 0, 1, 2, ...\}$ of any power bounded operator T are bounded, it cannot be hypercyclic. This conclusion fails to hold for the supercyclicity property in general. But if the operator satisfies in addition for example $Tf(\zeta) = f(\zeta)$ for all f in the space and at least one point $\zeta \in \mathbb{C}$, then T is not supercyclic either. We will prove this for the operator H, and the same proof works in general for any other power bounded operator T.

Corollary 4.3.2. *Let $1 \leq p < \infty$. Then the Hardy operator H is not supercyclic on $\mathcal{F}^p_{(\alpha,m)}$.*

Proof. For each f and each n we have

$$Hf(0) = f(0) = H^n f(0). \tag{4.3.2}$$

Proceeding by contradiction, suppose that f is a supercyclic vector for H. Then for the constant function $\mathbf{1}$, there exits a sequence $(\lambda_k H^{n_k} f)$ in the projective orbit such that $\lambda_k H^{n_k} f \to \mathbf{1}$ as $k \to \infty$. Consequently, by (4.3.2),

$$\lim_{k\to\infty} \lambda_k H^{n_k} f(0) = \lim_{k\to\infty} \lambda_k f(0) = 1,$$

hence $f(0) \neq 0$ and (λ_k) is not a null sequence either.

Similarly, for the function $h(z) = z$, there exists again a sequence $(\theta_j H^{n_j} f)$ in the projective orbit such that $\theta_j H^{n_j} f \to h$ as $j \to \infty$, and

$$\lim_{j\to\infty} \theta_j H^{n_j} f(0) = \lim_{j\to\infty} \theta_j f(0) = h(0) = 0$$
$$\lim_{j\to\infty} \theta_j H^{n_j} f(1) = \lim_{j\to\infty} \theta_j f(1) = h(1) = 1, \tag{4.3.3}$$

from which, since $f(0) \neq 0$, we deduce $\theta_j \to 0$ as $j \to \infty$. This and power boundedness in Theorem 4.3.1 yield

$$\lim_{j\to\infty} \|\theta_j H^{n_j} f\|_{(p,\alpha,m)} \lesssim \|f\|_{(p,\alpha,m)} \lim_{j\to\infty} |\theta_j| = 0.$$

Therefore, $\theta_j H^{n_j} f \to \mathbf{0}$ which implies $\theta_j H^{n_j} f(1) \to 0$ as $j \to \infty$. This contradicts (4.3.3).

4.4 Dynamics properties of the differential operator

Various order differential operators play fundamental roll in many part of mathematics including in the study of differential equations. The differentiation operator $Df = f'$ has been studied on Banach spaces of analytic functions by several authors.

A. Harutyunyan and W. Lusky [32] identified conditions under which the operator becomes bounded when acting between weighted spaces of holomorphic functions endowed with the supremum norm; see also [1]. J. Bonet [12] studied various dynamical properties of the operator on these weighted spaces, and the study was continued jointly with M. Beltrán, A. Bonilla and C. Fernández in [9, 14]. Later in 2014, M. Beltrán [8] studied the dynamics of the operator on a wider class of generalized weighted Bergman spaces.

On the other hand, the operator appears as a canonical example of unbounded operators in many Banach spaces including the very classical Hilbert space $L^2(\mathbb{R})$ and the space of continuous functions $C([a,b])$ with the supremum norm. In [68] S. Ueki showed its unboundedness on the classical growth type Fock spaces. T. Mengestie and S. Ueki [51] verified its unboundedness on all classical Fock spaces and generalized Fock spaces where the weight function grows faster than the Gaussian weight function $|z|^2/2$. The same conclusion was later drawn in [48] on the Fock–Sobolev spaces which are typical examples of generalized Fock spaces with weight function growing slower than the Gaussian function.

Inspired by all these, T. Mengestie [40] asked the question of how fast should the weight function need to grow in order that the corresponding generalized Fock spaces support a continuous differentiation operator. He further considered the spaces $\mathcal{F}^p_{(1,m)}$ and showed that the weight function should actually grow much slower than the classical Gaussian weight function. More specifically, it was proved that the operator D is bounded on $\mathcal{F}^p_{(1,m)}$, $0 < p < \infty$, if and only if $m \leq 1$, and compact if and only if $m < 1$. See also [8, Section 5]. These conditions are opposite to the corresponding conditions for the integration operator J except when $m = 1$, in which case both J and D are bounded. Clearly $DJf = f$ and $JDf(z) = f(z) - f(0)$ for all $z \in \mathbb{C}$ and f in $\mathcal{F}^p_{(\alpha,m)}$. Note that, there exists no vector whose projective orbit under J is dense in $\mathcal{F}^p_{(\alpha,m)}$, while our next result shows that D is supercyclic.

Proposition 4.4.1. *Let $1 \leq p < \infty$ and let the differentiation operator D be bounded on $\mathcal{F}^p_{(\alpha,m)}$. Then*

(i) D is hypercyclic on $\mathcal{F}^p_{(\alpha,m)}$ if and only if either $m = 1$ and $\alpha > 1$ or

$m = 1$, $\alpha = 1$ and $p > 3$.

(ii) D is supercyclic and hence cyclic on $\mathcal{F}^p_{(\alpha,m)}$.

The result reflects an interesting interplay between the power m of the weight function and the weight multiplier factor α, and the exponent p.

Proof. (i) First note that since D is bounded, $m \leq 1$. In addition, since no compact operator is hypercyclic on a non zero complex Banach space [7, Corollary 1.22], it follows that D is not hypercyclic on $\mathcal{F}^p_{(\alpha,m)}$ whenever $m < 1$. On the other hand, for $m = 1$, using the relation in (4.1.3), we have

$$\liminf_{n\to\infty} \frac{\|z^n\|_{(p,\alpha,1)}}{n!} \simeq \liminf_{n\to\infty} \frac{n^{\frac{3}{2p}}}{\alpha^n\sqrt{n}} = \begin{cases} 0, & \text{for } \alpha = 1 \text{ and } p > 3 \text{ or } \alpha > 1, \\ 1, & \text{for } \alpha = 1 \text{ and } p = 3, \\ \infty, & \text{for } \alpha < 1 \text{ or } \alpha = 1 \text{ and } p < 3. \end{cases}$$

Then, by Theorem 5.2 of [8], D is hypercyclic if and only if either $\alpha = 1$ and $p > 3$ or $\alpha > 1$.

(ii) For this part, we follow the arguments used in the proof of [12, Proposition 2.7]. Since for each $n \in \mathbb{N}$ the monomial z^n belongs to the kernel $\mathrm{Ker}D^{n+1}$ of D^{n+1}, the generalized kernel set

$$G_{\mathrm{Ker}} := \bigcup_{n=0}^{\infty} \mathrm{Ker}D^n$$

contains all the polynomials. Since the polynomials are dense in $\mathcal{F}^p_{(\alpha,m)}$, it follows that G_{Ker} is dense in $\mathcal{F}^p_{(\alpha,m)}$. Moreover, the range of the operator D contains polynomials and therefore it is dense in $\mathcal{F}^p_{(\alpha,m)}$. Then our conclusion follows after an application of [10, Corollary 3.3]. □

We note that the proof of part (i) depends on the hypercyclicity criterion due to J. Bés and A. Peris [11], where the original idea goes back to the work of Kitai in her Ph.D. thesis [31, Theorem 3.4]. The aforementioned Theorem 5.2 of [8] ensures that D is hypercyclic on generalized Bergman spaces if and only if it satisfies the hypercyclicity criterion. This was further shown to be equivalent to a condition like

$$\liminf_{n\to\infty}(n!)^{-1}\|z^n\|_{(p,\alpha,1)} = 0$$

which remains valid in our setting. Similarly, the proof of part (ii) was based on a density condition in [10]. This condition is equivalent to the known supercyclicity criterion; see [10, Lemma 3.1]. Therefore, D satisfies the supercyclicity criterion if and only if it is supercyclic. We note that not all supercyclic operators satisfy this criterion; see [25] for an example.

Having completely identified conditions under which D is hypercyclic, we next consider the question of when D can be topologically mixing on $\mathcal{F}^p_{(\alpha,m)}$. Recall that an operator T on a Banach space X is topologically mixing if for every pair of non-empty open subsets U and V of X, there exists an $N \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq N$. Note that topologically mixing is a stronger operator theoretic condition than hypercyclicity in general. Following the discussions above and the arguments used back in the proof of Theorem 2.4 and Corollary 2.6 of [12], the differentiation operator D is topologically mixing on $\mathcal{F}^p_{(\alpha,m)}$ whenever it is hypercyclic. On the other hand, by Proposition 4.4.1, the operator is hypercyclic on $\mathcal{F}^p_{(\alpha,m)}$ for $m = 1$, $\alpha = 1$ and $p > 3$ or $m = 1$ and $\alpha > 1$. Thus, it cannot be power bounded for such possible values.

Now we investigate when the differentiation operator is power bounded and uniformly mean ergodic.

Theorem 4.4.2. *Let $1 \leq p < \infty$ and the differentiation operator D be bounded on $\mathcal{F}^p_{(\alpha,m)}$. Then the following statements are equivalent.*

(i) D is power bounded and uniformly mean ergodic on $\mathcal{F}^p_{(\alpha,m)}$.

(ii) Either $m < 1$ or $m = 1$ and $\alpha < 1$.

Proof. We first show that (ii) implies (i). Since D is bounded, we have $m \leq 1$. If $m < 1$, then the operator is compact and by [40, Theorem 1.2], its spectrum $\sigma(D)$ contains only the zero element. By the spectral formula, there exist $\delta < 1$ and $N \in \mathbb{N}$ such that

$$\|D^n\| \leq \delta^n \quad \text{for all } n \geq N, \tag{4.4.1}$$

and therefore, D is power bounded and uniformly mean ergodic in this case. For the case when $m = 1$ and $\alpha < 1$, arguing as in (5.3) in [8, Proposition 5.9], applying Jensen's inequality and Fubini's Theorem, for $R > r$ we get

$$\begin{aligned}
M_p(D^k f, r) &= \left(\frac{1}{2\pi}\int_0^{2\pi} |f^{(k)}(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}} \\
&= \left(\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{n!}{2\pi i}\int_0^{2\pi} \frac{f(Re^{i\beta})iRe^{i\beta}}{(Re^{i\beta}-re^{i\theta})^{k+1}} d\beta\right|^p d\theta\right)^{\frac{1}{p}} \\
&\leq n!\left(\frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{|f(Re^{i\beta})|R}{\left|Re^{i\beta}-re^{i\theta}\right|^{k+1}} d\beta\right)^p d\theta\right)^{\frac{1}{p}} \\
&= \frac{n!R}{R^2-r^2}\left(\frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{|f(Re^{i\beta})|}{\left|Re^{i\beta}-re^{i\theta}\right|^{k-1}} P_{\frac{r}{R}}(\theta-\beta) d\beta\right)^p d\theta\right)^{\frac{1}{p}} \\
&\leq \left(\frac{n!R}{(R^2-r^2)(R-r)^{k-1}}\right)\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2\pi}\int_0^{2\pi} |f(Re^{i\beta})|^p P_{\frac{r}{R}}(\theta-\beta) d\beta d\theta\right)^{\frac{1}{p}}
\end{aligned}$$

$$= \left(\frac{n!R}{(R^2-r^2)(R-r)^{k-1}}\right)\left(\frac{1}{2\pi}\int_0^{2\pi}|f(Re^{i\beta})|^p\frac{1}{2\pi}\int_0^{2\pi}P_{\frac{r}{R}}(\theta-\beta)d\theta d\beta\right)^{\frac{1}{p}}$$
$$= \left(\frac{n!R}{(R^2-r^2)(R-r)^{k-1}}\right)M_p(f,R)$$

where $P_s(t) = \frac{1-s^2}{1-2s\cos t+s^2}$, $0 \leq s < 1$, is the Poisson kernel for the unit disc. Thus, we have

$$M_p(D^kf,r) \leq \frac{k!R}{(R^2-r^2)(R-r)^{k-1}}M_p(f,R).$$

Letting $R = r+k$, $k \geq 1$, we further get

$$M_p(D^kf,r) \leq \frac{k!(r+k)}{k^{k-1}(k^2+2rk)}M_p(f,r+k) \leq \frac{k!}{k^k}M_p(f,r+k),$$

and thus, we get

$$\|D^kf\|^p_{(p,\alpha,1)} \leq \frac{(k!)^pe^{p\alpha k}}{k^{pk}}\|f\|^p_{(p,\alpha,1)}.$$

Using this along with Striling's formula (4.1.1), we deduce

$$\|D^k\| \lesssim \frac{k^{\frac{p}{2}}}{e^{(1-\alpha)k}} \to 0 \tag{4.4.2}$$

as $k \to \infty$ for $\alpha < 1$, and hence the operator is power bounded in this case as well. Now, applying the estimate in (4.4.2),

$$\left\|\frac{1}{n}\sum_{k=1}^n D^k\right\| \leq \frac{1}{n}\sum_{k=1}^n\|D^k\| \lesssim \frac{1}{n}\sum_{k=1}^n\frac{k^{\frac{p}{2}}}{e^{(1-\alpha)k}} \to 0$$

as $n \to \infty$, from which it follows that D is uniformly mean ergodic.

Now we show that (i) implies (ii). Assume that $m = 1$. Using exponential functions $e_\beta(z) = e^{\beta z}$, $|\beta| < \alpha$, we get $\overline{D(0,\alpha)} \subset \sigma(D)$, where $D(0,\alpha)$ is a disc with center 0 and radius α. Thus, 1 is an accumulation point of $\sigma(D)$ whenever $\alpha \geq 1$. Since D is assumed to be power bounded, we apply [27, Theorem 3.16] (see also [35, Theorem 2.7]), to get that the operator cannot be uniformly mean ergodic in this case. Therefore, we must have $\alpha < 1$.

4.5 The Ritt's resolvent growth condition

A classical operator theoretic problem, for any given bounded operator T on a complex Banach space X, is to identify the relation between the size of the resolvent $(T - \lambda I)^{-1}$ when λ is near to the spectrum of T and the asymptotic behaviour of the orbits $\{T^n x : x \in X\}$. In this perspective, recall that the Ritt's resolvent condition [62] for T states that there exists a positive constant C such that

$$\|(T - \lambda I)^{-1}\| \leq \frac{C}{|\lambda - 1|}$$

for each $\lambda \in \mathbb{C}$ and $|\lambda| > 1$. Also, the Kreiss resolvent condition for T states that there exists a positive constant C such that

$$\|(T - \lambda I)^{-1}\| \leq \frac{C}{|\lambda| - 1}$$

for each $\lambda \in \mathbb{C}$ and $|\lambda| > 1$. Since Kreiss resolvent condition is weaker than Ritt's resolvent condition, any operator which satisfies Ritt's resolvent condition satisfies Kreiss resolvent condition. Both Ritt and Kreiss resolvent conditions are related to the behavior of the powers and their various means. For instance, in [66, Theorem 6.1] Kreiss condition have been characterized interms of certain functions of the operator T. The characterization for the Ritt's condition is given by B. Nagy and J. Zemanek in [55]. They have proved that a bounded operator T on a complex Banach space satisfies the

Ritt's resolvent condition if and only if it is power bounded and

$$\sup_{n\geq 1} n\|T^{n+1}-T^n\|<\infty. \tag{4.5.1}$$

As an immediate consequence of Theorem 4.3.1, it turns out that the Hardy operator H on generalized Fock spaces belongs to the class of operators satisfying such condition.

Proposition 4.5.1. *Let $1\leq p<\infty$. Then*

(i) the Hardy operator H satisfies the Ritt's resolvent condition on $\mathcal{F}^p_{(\alpha,m)}$.

(ii) the differentiation operator D satisfies the Ritt's resolvent condition on $\mathcal{F}^p_{(\alpha,m)}$ if and only if it is power bounded and uniformly mean ergodic.

Proof. (i) By Theorem 4.3.1, it is enough to show that the operator H satisfies condition (4.5.1). To this goal, applying (4.3.1)

$$\begin{aligned}\|H^{n+1}f-H^nf\|_{(p,\alpha,m)} &\leq \sum_{k=0}^{\infty}\frac{|a_k|\|z^k\|_{(p,\alpha,m)}}{(k+1)^n}\left|\frac{1}{k+1}-1\right|\\ &\leq \|f\|_{(p,\alpha,m)}\sum_{k=0}^{\infty}\frac{k}{(k+1)^{n+1}}\simeq\frac{\|f\|_{(p,\alpha,m)}}{n},\end{aligned}$$

and the conclusion easily follows.

(ii) Assume first that D is power bounded and uniformly mean ergodic. Then by Theorem 4.4.2, either $m<1$ or $m=1$ and $\alpha<1$. For $m<1$, arguing as in (4.4.1) there exist $N\in\mathbb{N}$ and $0<\delta<1$ such that

$$n\|D^{n+1}-D^n\|\leq n\|D^{n+1}\|+n\|D^n\|\leq 2n\delta^n \text{ for all } n\geq N. \tag{4.5.2}$$

Similarly, if $m = 1$ and $\alpha < 1$, then (4.4.2) implies

$$n\|D^{n+1} - D^n\| \leq n\|D^{n+1}\| + n\|D^n\| \leq n\frac{(n+1)^{\frac{p}{2}}}{e^{(1-\alpha)(n+1)}} + n\frac{n^{\frac{p}{2}}}{e^{(1-\alpha)n}}. \qquad (4.5.3)$$

Now we take the supremum with respect to $n \in \mathbb{N}$ both in (4.5.2) and (4.5.3) to see that condition (4.5.1) is satisfied.

For the other implication, by a result of B. Nagy and J. Zemanek [55], it is enough to show that D is uniformly mean ergodic. We arrive at this conclusion if we show that the Ritt's resolvent condition fails when $m = 1$ and $\alpha \geq 1$. If $m = 1$, then using again the exponential functions, $e_\beta(z) = e^{\beta z}$, $|\beta| < \alpha$, we get $\overline{D(0, \alpha)} \subset \sigma(D)$. Thus, then the spectrum $\sigma(D)$ contains the unit circle $\mathbb{T}$ whenever if $\alpha \geq 1$. This is a contradiction since the spectrum of an operator which satisfies the Ritt's resolvent condition contains only 1 from the unit circle; see [39] and [56, Theorem 4.5.4] for more details. □

Conclusion and further research plans

The thesis includes a number of results which characterize various topological and dynamical structures of integral and differential operators on classical and generalized Fock spaces. Most of the characterizations are easy to apply as compared to previously existing results in the subject. Our working topic has been at the interface of various disciplines in mathematics including Complex analysis, Functional analysis, Operator theory and Harmonic Analysis. This certainly assures that our findings are set to contribute to all these disciplines in particular and mathematics in general.

Yet there are still a number of structures that require further invitations about this class of operators on various settings. The following are some of them

- Our results in chapter three deals with various topological properties of the generalized Volterra-type integral operators when they act between the Fock spaces F_p and F_q when $p \leq q$. The corresponding problems when the operator acts from a larger spaces to smaller spaces are still interesting open problems.

- The essential norms of the generalized Volterra type integral operators are described either in terms of the Berezin type integral transforms or as limits of the other simple functions considered in [53]. On the other hand, given the fact that the Fock spaces are described in terms of derivatives, the generalized Volterra type integral transforms behave like weighted composition operators in a quantitative way. Such behaviour makes it possible to pose an interesting research problem namely that: Is it possible to describe the essential norm of this class of operators in terms of counting functions? The researcher would like to have a hand on this post PhD work.

- Another problem related to the dynamics of an operator is its Resolvent growth condition. For the Volterra-type integral operator, such property on the Fock spaces has been already described in our work [15]. How the structure behaves for the generalized Volterra-type integral operator remains open.

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