SUPER LINEAR PROBLEM WITH P – LAPLACIAN.

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The undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled **Super linear problem with p-laplacian** by Nigussie Dedefi in partial fulfillment of the requirements for the degree of master of Science.

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abstract

This project is concerned with elliptic eigenvalue problem(Super linear) on a bounded domain $\Omega \subset \mathbb{R}^n$ $\big(n \geq 2\big)$ with smooth boundary $\partial\Omega$. Existence result for non-trivial solution of the problem is established.

Notations

a.e	Almost everywhere.
Ω	An open subset of $\mathbb{R}^N$.
$\overline{\Omega}$	The closure of Ω in $\mathbb{R}^N$.
$\partial\Omega$	The boundary of Ω i.e $\partial\Omega = \overline{\Omega} \backslash \Omega$.
$\Omega' \Subset \Omega$	If $\overline{\Omega'} \subset \Omega$ and $\overline{\Omega'}$ is compact.
$\partial_i u$	$u_{x_i} = \dfrac{\partial u}{\partial x_i}$.
$\lvert\alpha\rvert$	$\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n$.
D^α	$\dfrac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \dfrac{\partial^{\alpha_n}}{\partial x_1^{\alpha_n}}$ for $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) \in \mathrm{N}_0^n$ is multi-index.
D_w^α	The weak derivatives.
$\lvert \cdot \rvert$	A seminorm on a Banach space, or an Euclidean norm in $\mathbb{R}^n$.
∇u	$(\dfrac{\partial^2 u}{\partial x_1^2}, \cdots, \dfrac{\partial^2 u}{\partial x_n^2})$.
Δ	The laplace operator: $\Delta u = div(\nabla u) = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}$.
div	The divergence of vector field i.e $div(\nabla u) = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}$.
Δ_p	The p-laplace operator: $\Delta_p u = div(\lvert\nabla u\rvert^{p-2} \nabla u)\ with\ p > 1$.
$\Delta_{p(x)}$	The p(x)-laplace operator: $\Delta_{p(x)} u = div(\lvert\nabla u\rvert^{p(x)-2} \nabla u)$.
X^*	The topological dual of the space X.
p^+	$ess \sup_{x\in\Omega} p(x)$
p^-	$ess \inf_{x\in\Omega} p(x)$
p'	The conjugate of p given by $\frac{1}{p} + \frac{1}{p'} = 1$.
$L^1_{loc}(\Omega)$	Locally integrable function over Ω.
$supp f$	The support of a function $f : \Omega \to \mathbb{R}$ is defined by $\overline{\{x \in \Omega : f(x) \neq 0\}}$.
$C^\infty(\Omega)$	The space of continuous infinitely differentiable functions on Ω.
$C_0^\infty(\Omega)$	The space of continuous infinitely often differentiable functions with compact support on Ω.
$L^p(\Omega)$	$\{f : \Omega \to \mathbb{R} \big\vert \int_\Omega \lvert f(x)\rvert^p dx < \infty\}$
$D(\Omega)$	The set of all test functions.
$W^{k,p}(\Omega)$	The Sobolev space of functions whose distributional derivatives up to k^{th} order belongs to $L^p(\Omega)$.
$W^{1,p}(\Omega)$	First order Sobolev space on Ω.
$W_0^{k,p}(\Omega)$	The closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$.
$H_0^k(\Omega)$	$W_0^{k,2}(\Omega)$.
$H^{-1}(\Omega)$	The dual space of $H_0^1(\Omega)$.

Introduction

The theory of partial differential equations oftentimes involves function spaces that are defined in terms of properties of pertinent functions and their derivatives. In this regard, Sobolev space turn out to be one with suitable settings as compered to the classical Banach space of smooth functions $C^n(\Omega)$.

In this paper we study elliptic eigenvalue problem of the form

$$\Delta_p u + \lambda |\nabla u|^{p-2} u = 0$$

where the scalar λ is an eigenvalue of the p-Laplace equation for $1 \leq p < \infty$. A non-zero C^2- function is an eigenfunction corresponding to λ.

The p-Laplace equation which is the Euler-Lagrange equation of the Dirichlet integral

$$J(u) = \int_\Omega |\nabla u|^p dx$$

reduces to the $2-$Laplace equation for $p = 2$. For $p \neq 2$, the p-Laplace equation is non-linear and for critical deformation ($\nabla u = 0$) it is singular in the event that $1 \leq p < 2$. Solutions of the p-Laplace equation are named p-harmonic functions. The study of p-Laplace equation and hence of p-harmonic functions attracted significant attention in recent years. The $2-$Laplace equation or simply Laplace's equation is the prototype of linear PDE of second order.

In order to determine unique solution of a PDE we need to prescribe additional conditions which can either be initial, boundary or initial-boundary condition. For instance, the following poisson's equation with homogeneous condition on the boundary of a domain $\Omega \subset \mathbb{R}^n$

$$\begin{cases} -\Delta u = f \ \ in \ \Omega \\ u = 0 \ \ on \ \partial\Omega \end{cases} \tag{1}$$

is the Dirichlet problem. The solution of (1) is established by using divergence theorem for smooth functions u and f.

In the preparation of this seminar paper the following sequence of narratives are adopted. In chapter 1, we will see description of partial differential operator using multi-index notation followed by the notion of test functions and weak derivatives of the functions. In chapter 2, we drive Euler-Lagrange equations for variational (Dirichlet) integral and present formulation of boundary value problem of Dirichlet type. In chapter 3, we take a closer look at eigenvalue problem with p-Laplace operator. In particular, for linear problems we show that the eigenvalue λ is real and non-negative. Finally in chapter 4, we present weak formulation of the Dirichlet problem on bounded domain and establish existence of non-trivial weak solution.

Chapter 1

Overview of Function Spaces

1.1 Lebesgue Spaces

If $\Omega \subset \mathbb{R}^n$ is a domain and $1 \leq p < \infty$, then

$$L^p(\Omega) = \big\{ f : \Omega \to \mathbb{R} \big| \int_\Omega |f(x)|^p dx < \infty \big\}$$

is space of p-integrable functions on Ω, while for $p = \infty$

$$L^\infty(\Omega) = \big\{ f : \Omega \longrightarrow \mathbb{R} \big| |f(x)| < M \; a.e \; on \; \Omega, \; M > 0 \big\}$$

is the space of essentially bounded functions on Ω. We recall, for $f \in L^p(\Omega), 1 \leq p \leq \infty$

$$||f||_{L^p(\Omega)} = \begin{cases} \big(\int_\Omega |f(x)|^p dx \big)^{\frac{1}{p}}, \; 1 \leq p < \infty \\ ess \sup_\Omega |f(x)|, \; p = \infty \end{cases}$$

is a norm[1] and $\big(L^p(\Omega), || \cdot ||_{L^p(\Omega)}\big)$ is a normed space which is complete i.e a Banach Space.

Definition 1.1.1. *(Locally integrable functions) For $1 \leq p < \infty$, the space of locally integrable functions on Ω,*

$$L^p_{loc}(\Omega) = \big\{ f : \Omega \to \mathbb{R} \big| \int_K |f(x)|^p dx < \infty, K \subset\subset \Omega \big\}^2$$

is the set of functions on Ω which are p-integrable on every $K \subset \Omega$ compact.

Proposition 1.1.1. *For $1 \leq p < \infty, \; L^p(\Omega) \subset L^1_{loc}(\Omega)$*

Proof. Let $f \in L^p(\Omega)$.

$$If \;\; \mathrm{II}_k = \begin{cases} 1, & x \in K \\ 0, & x \notin K \end{cases}$$

[1]A norm $\|.\|$ on a linear (vector) space V is a functional on V with the following properties:
i) $\|u\| \geq 0, \|u\| = 0 \Leftrightarrow u = 0, \; \forall u \in V$
ii) $\|c.u\| = |c|\|u\|, \forall c \in \mathbb{R}, \; \forall u \in V$
iii) $\|u + w\| \leq \|u\| + \|w\|, \; \forall u, w \in V$

[2]$\mathrm{K} \subset\subset \Omega \overset{def}{=} \; \overline{k} \subset \Omega \;\; and \;\; \overline{k} \neq \Omega$

is an indicator function of $K \subset\subset \Omega$, then for $1 \leq q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned} \int_\Omega |f(x)|^p dx &= \int_\Omega \amalg_k |f(x)|^p dx \\ &\leq \; || \amalg_k ||_{L^q} ||f||_{L^p}, \quad (H\ddot{o}lder's\ inequality) \\ &< \; \infty \end{aligned}$$

Therefore $f \in L^1_{loc}(\Omega)$ □

Definition 1.1.2. *The support of the function f on Ω is*

$$supp\ f = \overline{\{x \in \Omega : f(x) \neq 0\}}$$

Evidently, the space $C_0(\Omega)$ of functions with compact support on Ω is a linear space, that is

i. $f \in C_0(\Omega) \Longrightarrow \alpha f \in C_0(\Omega), \;\; \alpha \in \mathbb{R}$

ii. $f, g \in C_0(\Omega) \Longrightarrow f + g \in C_0(\Omega)$

1.2 Multi-index and Derivatives

The use of multi-index notation for partial differential operators renders an efficient and ultra convenient settings for description of partial differential equations.

1.2.1 Multi-index

An array $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}_0^n$ of non-negative integers is what we call n-dimensional multi-index. With multi-index α we associate the following scalars,

i. $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$

ii. $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!$

The operation of addition (subtraction) on the product set $\mathbb{N}_0^n$ is introduced component wise, that is

$$\alpha \pm \beta = (\alpha_1 \pm \beta_1, \cdots, \alpha_n \pm \beta_n)$$

Proposition 1.2.1. *The relation $\sim$ defined on $\mathbb{N}_0^n$ by*

$$\alpha \sim \beta \;\; iff \;\; \alpha_i \leq \beta_i \; ; \;\; i = 1, 2, \cdots, n$$

is partial ordering.

Proof. i) Let $\alpha \in \mathbb{N}_0^n$ be arbitrary.
Since, $\alpha_i \leq \alpha_i \;\; for \;\; i = 1, 2, \cdots, n$
we have $\alpha \sim \alpha$
$\Rightarrow$ " $\sim$ " is reflexive.

ii) Let $\alpha, \beta \in \mathbb{N}_0^n$ such that

$$\alpha \sim \beta \quad \wedge \quad \beta \sim \alpha$$
$$\Rightarrow \alpha_i \leq \beta_i \quad \wedge \quad \beta_i \leq \alpha_i \;\; i = 1, 2, \cdots, n$$
$$\Rightarrow \alpha_i = \beta_i \;; \quad i = 1, 2, \cdots, n$$
$$\Rightarrow \alpha = \beta$$
$$\Rightarrow \quad " \sim " \text{ is antisymmetric.}$$

iii) Let $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ such that

$$\alpha \sim \beta \quad \wedge \quad \beta \sim \gamma$$
$$\Rightarrow \alpha_i \leq \beta_i \quad \wedge \quad \beta_i \leq \gamma_i \;\; i = 1, 2, \cdots, n$$
$$\Rightarrow \alpha_i \leq \gamma_i \;; \quad i = 1, 2, \cdots, n$$
$$\Rightarrow \alpha \sim \gamma$$
$$\Rightarrow \quad " \sim " \text{ is transitive.}$$

Consequently, $" \sim "$ is partial ordering, and $(\mathbb{N}_0^n, \sim)$ is partial ordered set. □

One can employ multi-indexes to describe a monomial in $\mathbb{R}^n$ and a polynomial of degree k with n-independent variables, that is if $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$ then a

i) Monomial

$$X^\alpha = \prod_{i=n}^{n} X_i^{\alpha_i}$$

ii) Polynomial of degree k

$$P(x) = \sum_{|\alpha| \leq k} C_\alpha X^\alpha$$

1.2.2 Derivatives and differential operators

We write D_k for the usual partial derivative with respect to the k^{th}-independent variable. As a result, the expression

$$D = (D_1, \cdots, D_n)$$

represents gradient.
If $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is n-dimensional multi-index with length $|\alpha| = k$ then the expression

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

stands for a partial differential operator(PDO), of order k.

obviously, $|\alpha| = 0 \quad \Rightarrow \alpha = (0, 0, \cdots, 0)$

$$\Rightarrow D^\alpha = D^0 = \text{identity operator}$$

$$|\alpha| = 1 \quad \Rightarrow \alpha \in \big\{(1, 0, \cdots, 0), (0, 1, \cdots, 0), \cdots, (0, \cdots, 1)\big\}$$

$$\Rightarrow D^\alpha \in \big\{D_1, D_2, \cdots, D_n\big\} \text{ that is one of the } n \text{ first partials.}$$

For $|\alpha| = 2$, D^α is one of the $\frac{n(n+1)}{2}$ second order partials.

A linear partial differential operator, L of order k can now be expressed as

$$L := \sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha, \quad x \in \mathbb{R}^n$$

consequently, a linear PDE of order k in n-independent variables is given by

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u = b(x)$$

and the corresponding quasi-linear PDE of order k is

$$\sum_{|\alpha| \leq k} a_\alpha\big(x, (D^\beta u)_{|\beta| \leq k-1}\big) D^\alpha u = b\big(x, (D^\beta u)_{|\beta| \leq k-1}\big)$$

1.3 Test Functions

Given a non-negative integer k, we say that a function

$$f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$$

is k-times continuously differentiable if

i) $D^\alpha f$ exists on Ω

ii) $D^\alpha f$ is continuous

for all multi-indexes $\alpha \in \mathbb{N}_0^n \; with \; |\alpha| \leq k$

Natation: $C^k(\Omega) = \big\{ f : \Omega \to \mathbb{R} : D^\alpha f \in C(\Omega), \;\; |\alpha| \leq k \big\}$
For $\Omega \subseteq \mathbb{R}^n$ a domain, the space of infinitely often continuously differentiable functions on Ω is given by

$$C^\infty(\Omega) = \big\{ f : \Omega \to \mathbb{R} : D^\alpha f \in C(\Omega), \;\; \forall \alpha \in \mathbb{N}_0^n \big\}$$

and the space of infinitely often continuously differentiable functions with compact support on Ω is given by

$$C_0^\infty(\Omega) = \big\{ f : \Omega \to \mathbb{R} : f \in C_0(\Omega) \cap C^\infty(\Omega) \big\}$$

Natation: $D(\Omega) := C_0^\infty(\Omega)$
The set $D(\Omega)$ as above is the set of test functions on Ω
For demonstration, consider the function

$$g(t) = \begin{cases} e^{\frac{-1}{t}}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

The fact that the exponential function $f_1(t) = e^{\frac{-1}{t}} \quad (t > 0)$ and constant function $f_2(t) = 0 \quad (t < 0)$ are infinitely often differentiable with continuous derivatives is plain. The derivative $g^{[n]}(t)$ of any order at t=0 can also be shown to be "0", that is

$g^{[n]}(t)\Big|_{t=0} = 0, \quad \forall n \in \mathbb{N}.$
Here, one only needs to change variable ($\frac{1}{t} \to h$) and then employ L'Hopitals rule.
If we set $f(x) = g(a^2 - ||x||^2)$, for any $a > 0$ then $f \in C^\infty(\mathbb{R}^n)$ with

$$supp\ f = \begin{cases} [-a, a], & n = 1 \\ \\ \overline{B_a}(0), & n > 1 \end{cases}$$

That is f is test function on $\mathbb{R}^n$.

1.4 Weak derivatives

In this part, we introduce the notion of weak derivatives, a situation whereby significant weakening of the classical notion of derivatives takes place. The definition of weak derivatives rests on the known concept of integration by parts,

$$\int_\Omega u_{x_i}\phi dx = -\int_\Omega u\phi_{x_i} dx, \quad \phi \in D(\Omega)$$

Definition 1.4.1. *(Weak derivatives) Let $f \in L^1_{loc}(\Omega)$ and $\alpha \in \mathbb{N}_0^n$. If there is $g \in L^1_{loc}(\Omega)$ such that*

$$\int_\Omega f D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega g\phi dx, \quad \phi \in D(\Omega)$$

then we say that g is the α^{th}-weak derivative of f on Ω.

Natation: $g = D^\alpha_w f$
Some of the typical properties of weak derivatives are stated in the lemma that follow.

Lemma 1.4.1. *(Uniqueness)*
Let $f \in L^1_{loc}(\Omega)$ such that $D^\alpha_w f$ exists for $\alpha \in \mathbb{N}_0^n$.
If g and h are α^{th}-weak derivative of f on Ω, then $g = h$ a.e on Ω.

Proof. By definition of weak derivatives; we have

$$g = D^\alpha_w f \ \Rightarrow \int_\Omega f D^\alpha \phi dx = (-)^{|\alpha|} \int_\Omega g\phi dx, \ \forall \phi \in D(\Omega) \tag{1.1}$$

$$h = D^\alpha_w f \ \Rightarrow \int_\Omega f D^\alpha \phi dx = (-)^{|\alpha|} \int_\Omega h\phi dx, \ \forall \phi \in D(\Omega) \tag{1.2}$$

From (1.1) and (1.2) we have

$$\textstyle\int_\Omega g(x)\phi(x) - \int_\Omega h(x)\phi(x) dx = 0, \ \forall \phi \in D(\Omega)$$

$$\Rightarrow \ \textstyle\int_\Omega (g(x) - h(x))\phi(x) dx = 0, \ \forall \phi \in D(\Omega)$$

$$\Rightarrow \ (g(x) - h(x))\phi = 0 \ a.e \ on \ \Omega, \ \forall \phi \in D(\Omega)$$

$$\Rightarrow \ g(x) - h(x) = 0 \ a.e \ on \ \Omega$$

$$g \Rightarrow \ (x) = h(x) \ a.e \ on \ \Omega$$

□

Lemma 1.4.2. *(Linearity)*
Let $f, g \in L^1_{loc}(\Omega)$ and $\alpha \in \mathbb{N}^n_0$. If f and g have α^{th}-weak derivative on Ω, then so does $f+g$, further more,

$$D^\alpha_w(c_1 f + c_2 g) = c_1 D^\alpha_w f + c_2 D^\alpha_w g$$

Proof. Suppose $h_1 = D^\alpha_w f,\ \ h_2 = D^\alpha_w g$, then

$$\begin{aligned}
\int_\Omega (c_1 f + c_2 g) D^\alpha_w \phi dx &= \int_\Omega c_1 f D^\alpha_w \phi dx + \int_\Omega c_2 g D^\alpha_w \phi dx \ \ \forall \phi \in D(\Omega) \\
&= c_1 \int_\Omega f D^\alpha_w \phi dx + c_2 \int_\Omega g D^\alpha_w \phi dx \\
&= c_1 (-1)^{|\alpha|} \int_\Omega h_1 \phi dx + c_2 (-1)^{|\alpha|} \int_\Omega h_2 \phi dx \\
&= (-1)^{|\alpha|} \int_\Omega (c_1 h_1 + c_2 h_2) \phi dx, \ \ \forall \phi \in D(\Omega) \\
&= c_1 D^\alpha_w f + c_2 D^\alpha_w g
\end{aligned}$$

Lemma 1.4.3. *(Commutativity)*
Let $f \in L^1_{loc}(\Omega)$ such that $D^\alpha_w f$ exists for $|\alpha| = k$.
If $\alpha, \beta \in \mathbb{N}^n_0$ with $|\alpha + \beta| = |\alpha| + |\beta| = k$ then

$$D^\alpha(D^\beta f) = D^\beta(D^\alpha f)$$

Proof.

$$\begin{aligned}
\int_\Omega D^\alpha(D^\beta f)\phi dx &= (-1)^{|\alpha|} \int_\Omega (D^\beta f)(D^\alpha \phi) dx \\
&= (-1)^k \int_\Omega f D^\beta (D^\alpha \phi) dx \\
&= (-1)^k \int_\Omega f D^{\beta+\alpha} \phi dx \\
&= (-1)^k \int_\Omega f D^\alpha (D^\beta \phi) dx \\
&= (-1)^{k+|\alpha|} \int_\Omega D^\alpha f D^\beta \phi dx \\
&= (-1)^{2k} \int_\Omega D^\beta (D^\alpha f) \phi dx \\
&= \int_\Omega D^\beta (D^\alpha f) \phi dx
\end{aligned}$$

□

Function spaces visited thus far are not adequate for indepth treatment of partial differential equations. Treatment of the theory of partial differential equations, especially, approximation (interior) of solutions by smooth functions requires a space with suitable settings. Hence, the need for Sobolev Space.

1.5 Sobolev Spaces

If $\Omega \subset \mathbb{R}^n$ is a domain and k is non-negative integer, the space of functions on Ω whose α^{th}-weak derivative exists for various orders (up to k) is

$$W^k(\Omega) = \left\{ f : \Omega \to \mathbb{R} \middle| D_w^\alpha f \ exists \ \forall \alpha, \ |\alpha| \leq k \right\}^3$$

The space $W^k(\Omega)$ has the following properties,
i) $W^k(\Omega)$ is a linear space.
ii) $C^k(\Omega) \subset W^k(\Omega)$
Property "i)" is an immediate consequence of the linearity property of weak derivatives while "ii)" follows from the fact that classical differentiability implies weak differentiability.

For $1 \leq p \leq \infty$ and "k" a non-negative integer, the space of k-times weakly differentiable functions with p-integrable weak derivatives on Ω,

$$W^{k,p}(\Omega) = \left\{ f \in W^k(\Omega) : D_w^\alpha f \in L^p(\Omega), \ \ |\alpha| \leq k \right\}$$

is a linear space.
We define a norm on the space $W^{k,p}(\Omega)$ by

$$||f||_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha| \leq k} \int_\Omega |D_w^\alpha f|^p dx \right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \sum_{|\alpha| \leq k} ess \sup_\Omega |D_w^\alpha f|, & p = \infty \end{cases}$$

We make the following observation,

i) Since $|D_w^\alpha f| \geq 0, \quad \forall \alpha \in \mathbb{N}_0^n$
we have $||f||_{W^{k,p}(\Omega)} \geq 0$ and $||f||_{W^{k,p}(\Omega)} = 0$ if $f = 0$ a.e on Ω.

ii) For any scalar, $D_w^\alpha(\lambda f) = \lambda D_w^\alpha f$
$\Rightarrow |D_w^\alpha(\lambda f)| = |\lambda||D_w^\alpha f|$
$\Rightarrow ||\lambda f||_{W^{k,p}(\Omega)} = |\lambda|||f||_{W^{k,p}(\Omega)}$

iii) For any $f, g \in W^{k,p}(\Omega)$ and $1 \leq p < \infty$,

$$\begin{aligned} ||f+g||_{W^{k,p}(\Omega)} &= \left(\sum_{|\alpha| \leq k} \int_\Omega |D_w^\alpha (f+g)|^p dx \right)^{\frac{1}{p}} \\ &= \left(\sum_{|\alpha| \leq k} \int_\Omega |D_w^\alpha f + D_w^\alpha g|^p dx \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{|\alpha| \leq k} \left(||D_w^\alpha f||_{L^p(\Omega)} + ||D_w^\alpha g||_{L^p(\Omega)} \right)^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{|\alpha| \leq k} ||D_w^\alpha f||_{L^p(\Omega)}^p \right)^{\frac{1}{p}} + \left(\sum_{|\alpha| \leq k} ||D_w^\alpha g||_{L^p(\Omega)}^p \right)^{\frac{1}{p}} \\ &= ||f||_{W^{k,p}(\Omega)} + ||g||_{W^{k,p}(\Omega)} \end{aligned}$$

[3] f is k-times weakly differentiable if $D_w^\alpha f \ exists, \ |\alpha| \leq k$

and convinced that $||\cdot||_{W^{k,p}(\Omega)}$ is indeed a norm
As a result, the space

$$(W^{k,p}(\Omega), ||\cdot||_{W^{k,p}(\Omega)})$$

is a normed space. We proceed to show that this space is complete. To this end, let $\{f_n\}_{n=1}^{\infty} \subseteq W^{k,p}(\Omega)$ be a Cauchy sequence[4]

$\Rightarrow \{D_w^{\alpha}(f_n)\}_{n=1}^{\infty} \subseteq L^p(\Omega)$ is a Cauchy sequence for any α with $|\alpha| \leq k$.

$\Rightarrow \lim_{n\to\infty} D_w^{\alpha}(f_n) = f_\alpha \in L^p(\Omega)$ (Since $L^p(\Omega)$ is complete.)

$\Rightarrow \lim_{n\to\infty} f_n = f \in L^p$; $(\alpha = (0,0,\cdots,0) = 0 \;\; D^\alpha = identity)$

It remains to show $f \in W^{k,p}(\Omega)$
Now, for $\phi \in D(\Omega)$ fixed,

$$\begin{aligned}
\int_\Omega f D^\alpha \phi dx &= \int_\Omega \lim_{n\to\infty} f_n D^\alpha \phi dx \\
&= \lim_{n\to\infty} \int_\Omega f_n D^\alpha \phi dx \\
&= \lim_{n\to\infty} (-1)^{|\alpha|} \int_\Omega D^\alpha f_n \phi dx \\
&= (-1)^{|\alpha|} \int_\Omega \lim_{n\to\infty} D^\alpha f_n \phi dx \\
&= (-1)^{|\alpha|} \int_\Omega f_\alpha \phi dx
\end{aligned}$$

$\Rightarrow D_w^\alpha f = f_\alpha \in L^p(\Omega)$

$\Rightarrow f \in W^{k,p}(\Omega)$

Definition 1.5.1. *(Sobolev space) For $1 \leq p \leq \infty$ and a non-negative integer k, the Banach space of functions on Ω,*

$$(W^{k,p}(\Omega), ||\cdot||_{W^{k,p}(\Omega)})$$

is called a Sobolev space.

Sobolev space with $k = 1$ i.e $W^{1,p}(\Omega)$ are often called first order spaces.
For $1 \leq p < n$, the number $p^* = \frac{np}{n-p}$ is the Sobolev conjugate of p.
Since $\frac{1}{p^*} = \frac{n-p}{np} = \frac{1}{p} - \frac{1}{n} < \frac{1}{p}$ we have $1 \leq p < p^*$

[4]Sequence $\{f_n\}_{n=1}^{\infty} \in V$ *is called Cauchy sequence if and only if* $\forall \epsilon > 0$, $\exists N$ such that $\|f_n - f_m\| < \epsilon, \;\; \forall n, m \geq N$

Chapter 2

Harmonic and p-harmonic functions

2.1 The Dirichlet integral

Our aim in this section is to establish Euler-Lagrange equations for the Dirichlet integral, i.e

$$J(u) = \int_\Omega |\nabla u|^p dx$$

where $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ is a domain.

2.2 Euler-Lagrange equations

The above Dirichlet integral is also referred to as the (potential) energy functional of a hyperelastic or Cauchy-elastic material. With in the confines of the principles of minimum potential energy we proceed to find stationary states (minimizers) of the potential among admissible functions.

$$u : \Omega \to \mathbb{R}$$

with $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that u satisfies prescribed condition on the boundary. In view of this, if u is a minimizer then a oneparameter family of functions

$$v(x) = u(x) + \alpha\phi(x), \quad \phi \in D(\Omega)$$

is admissible (α is a parameter). Thus, as based on variational principles, the stationary integral as a function of the parameter α is expressed as

$$J(\alpha) = \int_\Omega |\nabla(u + \alpha\phi)|^p dx$$

and minimality condition is given by $\left.\frac{dJ(\alpha)}{d\alpha}\right|_{\alpha=0} = 0$

2.3 Harmonic functions

Consider the variational integral for $p = 2$, i.e

$$J(\alpha) = \int_\Omega |\nabla(u + \alpha\phi)|^2 dx$$

the stationary value (minima) of J can determined from $\left.\dfrac{dJ(\alpha)}{d\alpha}\right|_{\alpha=0} = 0$

But

$$\begin{aligned}
\frac{dJ(\alpha)}{d\alpha} &= \frac{d}{d\alpha}\int_\Omega |\nabla(u+\alpha\phi)|^2 dx \\
&= \int_\Omega \frac{d}{d\alpha}|\nabla u + \alpha\nabla\phi|^2 dx \\
&= \int_\Omega \frac{d}{d\alpha}\Big(\sum_{i=1}^{n}(u_{x_i}+\alpha\phi_{x_i})^2\Big)dx \\
&= \int_\Omega \Big(\sum_{i=1}^{n} 2(u_{x_i}+\alpha\phi_{x_i})\phi_{x_i}\Big)dx \\
&= 2\int_\Omega (|\nabla u + \alpha\nabla\phi|)\cdot\nabla\phi dx \\
\Rightarrow \left.\frac{dJ(\alpha)}{d\alpha}\right|_{\alpha=0} &= 2\int_\Omega \nabla u\cdot\nabla\phi dx
\end{aligned}$$

Thus, $\left.\dfrac{dJ(\alpha)}{d\alpha}\right|_{\alpha=0} = 0 \Rightarrow \int_\Omega \nabla u\cdot\nabla\phi dx = 0$

Recall, Gauss theorem[1] and Green's identity[2], respectively

$$u, v \in C^2(\Omega)\cap C(\overline{\Omega}) \Rightarrow v\Delta u = div(v\nabla u) - \nabla u\cdot\nabla v$$

and

$$\int_\Omega div(u)dx = \int_{\partial\Omega} u(y)\cdot n(y)ds(y),$$

$n(y)$ outward normal.

Let $\phi \in D(\Omega) \Rightarrow \phi \in C^2(\Omega)\cap C(\overline{\Omega})$ we have $\phi\Delta u = div(\phi\nabla u) - \nabla u\cdot\nabla\phi, \quad \forall\phi\in D(\Omega)$

$$\begin{aligned}
\int_\Omega \phi\Delta u dx &= \int_\Omega div(\phi\nabla u)dx - \int_\Omega \nabla u\cdot\nabla\phi dx \\
&= \int_\Omega div(\phi\nabla u)dx \\
&= \int_{\partial\Omega} \phi\nabla u(y)\cdot n(y)ds(y), \quad \forall\phi\in D(\Omega) \\
&= 0
\end{aligned}$$

$\Rightarrow \Delta u = 0 \quad (\Delta u = div(\nabla u))$ i.e

$$\sum_{i=1}^{n} u_{x_i}^2 = 0$$

[1] Let $u : \overline{\Omega} \to \mathbb{R}^n$ be a C^1-boundary $\partial\Omega$. Then $\int_\Omega div(u)dx = \int_{\partial\Omega} u(y)\cdot n(y)ds(y)$

[2] If Ω is bounded C^1 open set in $\mathbb{R}^n$ and $u, v \in C^2(\overline{\Omega})$, then

i. $\int_\Omega u\Delta v dx = \int_{\partial\Omega} u\frac{\partial v}{\partial n}ds - \int_\Omega \nabla u\cdot\nabla v dx$

ii. $\int_\Omega u\Delta v dx = \int_\Omega v\Delta u dx + \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds$

Therefore, harmonic(Laplace's) equation is the Euler-Lagrange equation for the Dirichlet integral with $p = 2$.

In the following subsection we will establish Euler-Lagrange equation for the Dirichlet integral with $p \neq 2$.

2.4 P-Harmonic functions

Once again consider the stationary integral

$$J(\alpha) = \int_\Omega |\nabla(u + \alpha\phi)|^p dx, \quad p \neq 2$$

Since,

$$\begin{aligned}
\frac{dJ(\alpha)}{d\alpha} &= \frac{d}{d\alpha}\int_\Omega |\nabla(u + \alpha\phi)|^p dx \\
&= \int_\Omega \frac{d}{d\alpha}|\nabla u + \alpha\nabla\phi|^p dx \\
&= \int_\Omega \frac{d}{d\alpha}\left[\sqrt{\sum_{i=1}^n (u_{x_i} + \alpha\phi_{x_i})^2}\right]^p dx \\
&= \int_\Omega \frac{d}{d\alpha}\left((\sum_{i=1}^n \left(u_{x_i} + \alpha\phi_{x_i}\right)^2)^{\frac{1}{2}}\right)^p dx \\
&= \int_\Omega \frac{d}{d\alpha}\left(\sum_{i=1}^n \left(u_{x_i} + \alpha\phi_{x_i}\right)^2\right)^{\frac{p}{2}} dx \\
&= \int_\Omega \frac{p}{2}\left(\sum_{i=1}^n \left(u_{x_i} + \alpha\phi_{x_i}\right)^2\right)^{\frac{p}{2}-1} \left(\sum_{i=1}^n 2\big(u_{x_i} + \alpha\phi_{x_i}\big)\phi_{x_i}\right) dx \\
&= p\int_\Omega \left(\sum_{i=1}^n \left(u_{x_i} + \alpha\phi_{x_i}\right)^2\right)^{\frac{p-2}{2}} \left(\sum_{i=1}^n \big(u_{x_i} + \alpha\phi_{x_i}\big)\phi_{x_i}\right) dx \\
&= p\int_\Omega \left(\sqrt{\sum_{i=1}^n \left(u_{x_i} + \alpha\phi_{x_i}\right)^2}\right)^{p-2} \Big(\big(\nabla u + \alpha\nabla\phi\big)\Big)\cdot\nabla\phi dx \\
&= p\int_\Omega |\nabla u + \alpha\nabla\phi|^{p-2}\big[\nabla u + \alpha\nabla\phi\big]\cdot\nabla\phi dx \\
\left.\frac{dJ(\alpha)}{d\alpha}\right|_{\alpha=0} &= p\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\phi dx
\end{aligned}$$

$\therefore \quad \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi dx = 0$

From Green's identity we get

$$\phi \; div(|\nabla u|^{p-2}\nabla u) = div(\phi \; |\nabla u|^{p-2}\nabla u) - |\nabla u|^{p-2}\nabla u \cdot \nabla\phi$$

$$\begin{aligned}\int_\Omega \phi \ div(|\nabla u|^{p-2}\nabla u)dx &= \int_\Omega div(\phi \ |\nabla u|^{p-2}\nabla u)dx - \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi dx \\ &= \int_\Omega div(\phi \ |\nabla u|^{p-2}\nabla u)dx \\ &= \int_{\partial\Omega} \phi(y) \ |\nabla u(y)|^{p-2}\nabla u(y) \cdot n(y)d\sigma(y) \\ &= 0 \\ \therefore \ div(|\nabla u|^{p-2}\nabla u) &= 0\end{aligned}$$

Hence we have established the fact that the p-laplace equation, i.e

$$\Delta_p u = div(|\nabla u|^{p-2}\nabla u) = 0$$

is the Euler-Lagrange equation for the Dirichlet integral with $p \neq 2$.

To come up with the unique solution of the p-laplace equation on a bounded domain $\Omega \subset \mathbb{R}^n$, we need to prescribe values of u on the boundary, $\partial\Omega$.
In this regard, we mention two type of boundary conditions and hence two types of boundary value problems,

$$\begin{cases}\Delta_p u = 0 \ \ in \ \ \Omega \\ u = g \ \ on \ \ \partial\Omega\end{cases}$$

which is the Dirichlet problem and

$$\begin{cases}\Delta_p u = 0 \ \ in \ \ \Omega \\ \frac{\partial u}{\partial n} = g \ \ on \ \ \partial\Omega\end{cases}$$

the Neuman problem ($\frac{\partial u}{\partial n}$ = normal derivative). As we shall show for the Neuman problem the boundary function g can not be arbitrary. We shall prove this after the following auxiliary fact, Green's second identity.
If $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$, then

$$v \ div(\nabla u) = div(v\nabla u) - \nabla u \cdot \nabla v$$

$$v \ div(\nabla u) - u \ div(\nabla v) = div(v\nabla u) - div(u\nabla v)$$

$$v\Delta u - u\Delta v = div(v\nabla u - u\nabla v)$$

Lemma 2.4.1. *If $\Delta u = 0$ in Ω, then $\int_{\partial\Omega} \frac{\partial u}{\partial n} d\sigma(y) = 0$*

Proof. Take $v = 1$ the Green's second identity becomes

$$div(\nabla u) = \Delta u = 0$$

$$\Rightarrow \ \int_{\partial\Omega} \frac{\partial u}{\partial n} d\sigma(y) = \int_{\partial\Omega} \nabla u \cdot n(y) ds(y) = \int_\Omega div(\nabla u)dx = 0$$

Now, $\frac{\partial u}{\partial n} = g$

$$\Rightarrow \ \int_{\partial\Omega} g(y)ds(y) = \int_{\partial\Omega} \frac{\partial u}{\partial n} ds(y) = 0$$

Hence, the boundary function should satisfy this constraint. □

Chapter 3

Dirichlet Eigenvalue problem

3.1 The p-harmonic operator

The p-harmonic operator, with the corresponding p-harmonic equation is

$$\Delta_p u = div(|\nabla u|^{p-2}\nabla u) = 0$$

which was shown in the earlier section to the Euler-Lagrange equation of the Dirchlet integral operator appears in variations contexts in theory of partial differential equations. We now take a close look at one such context, the notion of eigenvalue problem. If $\Omega \subset \mathbb{R}^n$ is abounded domain with smooth boundary and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ then,

$$\begin{aligned}
\Delta_p u &= div(|\nabla u|^{p-2}\nabla u) \\
&= \sum_{j=1}^{n} \partial_{x_j}\big(|\nabla u|^{p-2}u_{x_j}\big) \\
&= \sum_{j=1}^{n} \Big[\big(\partial_{x_j}\big(|\nabla u|^{p-2}\big)\big)u_{x_j} + |\nabla u|^{p-2}u_{x_j x_j}\Big] \\
&= \sum_{j=1}^{n} \Bigg[\Bigg(\partial_{x_j}\Bigg(\sqrt{\sum_{i=1}^{n} u_{x_i}^2}\Bigg)^{p-2}\Bigg)\Bigg)u_{x_j} + |\nabla u|^{p-2}u_{x_j x_j}\Bigg] \\
&= \sum_{j=1}^{n} \Big[\big(\partial_{x_j}\big(\sum_{i=1}^{n} u_{x_i}^2\big)^{\frac{p-2}{2}}\big)\big)u_{x_j} + |\nabla u|^{p-2}u_{x_j x_j}\Big] \\
&= \sum_{j=1}^{n} \Big[\big(\frac{p-2}{2}\sum_{i=1}^{n} u_{x_i}^2\big)^{\frac{p-2}{2}-1}\big)\big(\sum_{i=1}^{n} 2u_{x_i}u_{x_j}u_{x_i x_j}\big) + |\nabla u|^{p-2}u_{x_j x_j}\Big] \\
&= \sum_{j=1}^{n} \Big[(p-2)\big(\sum_{i=1}^{n} u_{x_i}^2\big)^{\frac{p-4}{2}}\big)\big(\sum_{i=1}^{n} u_{x_i}u_{x_j}u_{x_i x_j}\big) + |\nabla u|^{p-2}u_{x_j x_j}\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n}\left[(p-2)|\nabla u|^{p-4}\sum_{i=1}^{n} u_{x_i}u_{x_j}u_{x_ix_j} + |\nabla u|^{p-2}u_{x_jx_j}\right] \\
&= (p-2)|\nabla u|^{p-4}\sum_{j=1}^{n}\sum_{i=1}^{n} u_{x_i}u_{x_j}u_{x_ix_j} + |\nabla u|^{p-2}\sum_{j=1}^{n} u_{x_jx_j} \\
&= (p-2)|\nabla u|^{p-4}\sum_{j=1}^{n}\sum_{i=1}^{n} u_{x_i}u_{x_j}u_{x_ix_j} + |\nabla u|^{p-2}\Delta u \\
&= |\nabla u|^{p-4}\left[(p-2)\sum_{j,i=1}^{n} u_{x_i}u_{x_j}u_{x_ix_j} + |\nabla u|^{2}\Delta u\right]
\end{aligned}
$$

3.2 Eigenvalue problem

For scalar λ and $1 \leq p < \infty$, the homogeneous equation,

$$\Delta_p u + \lambda|\nabla u|^{p-2}u = 0$$

is called an eigenvalue problem. A non-trivial C^2- function u is an eigenfunction corresponding to λ, if the pair $\{\lambda, u\}$ satisfies the above partial differential equation. One usually calls

$$\{\lambda, u\}$$

an eigenpair corresponding to an eigenvalue problem.
For $p \neq 2$, the PDE

$$-\Delta_p u = \lambda|\nabla u|^{p-2}u$$

is a non-linear eigenvalue problem while for $p = 2$,

$$-\Delta_2 u = \lambda u$$

is a linear eigenvalue problem.

3.2.1 Linear Eigenvalue Problem

Given the linear eigenvalue problem as above, if one prescribes the unknown function u on the boundary of Ω, $\partial\Omega$ then the resulting PDE,

$$\begin{cases} -\Delta_2 u = \lambda u, & in \ \ \Omega \\ u = 0, & on \ \ \partial\Omega \end{cases}$$

is called Dirichlet eigenvalue problem (DEVP).

Lemma 3.2.1. *Let $\{\lambda, u\}$ and $\{\mu, v\}$ be eigenpairs of the DEVP.*
If $\lambda \neq \mu$ then u is orthogonal to v.

Proof. $\{\lambda, u\}$ and $\{\mu, v\}$ are eigenpairs of the DEVP.

$$\Rightarrow \begin{cases} -\Delta_2 u = \lambda u \\ -\Delta_2 v = \mu v \end{cases}$$

$$\Rightarrow \begin{cases} -v\Delta_2 u = \lambda uv \\ -u\Delta_2 v = \mu uv \end{cases}$$

$\Rightarrow \big(\lambda - \mu\big)uv = v\Delta u - u\Delta v$
$\Rightarrow \big(\lambda - \mu\big)uv = div\big(v\Delta u - u\Delta v\big)$ (Green's Identity)

$$\begin{aligned} \big(\lambda - \mu\big)\int_\Omega uv\ dx &= \int_\Omega div\big(v\nabla u - u\nabla v\big)\ dx \\ &= \int_{\partial\Omega} \big(v\nabla u - u\nabla v\big)\cdot n(y)\ ds(y)\ \ (\textit{by divergence theorem}) \\ &= \int_{\partial\Omega} \big(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\big)\ d\sigma(y) \\ &= 0 \end{aligned}$$

$\Rightarrow \big(\lambda - \mu\big)\int_\Omega uv\ dx = 0$

$\overset{\lambda\neq\mu}{\Rightarrow}\quad \int_\Omega uv\ dx = 0$

$\therefore\ \ u$ is orthogonal to v. □

Theorem 3.2.1. *If $\{\lambda, u\}$ is an eigenpair of the linear Dirichlet problem then $\lambda \in \mathbb{R}$ and u is a real valued function.*

Proof. -$\Delta_2 u = \lambda u \Rightarrow -\Delta_2\overline{u} = \overline{\lambda u} = \overline{\lambda}\overline{u}$

$$\begin{cases} -\overline{u}\Delta_2 u = \overline{\lambda}\overline{u}u \\ -u\Delta_2\overline{u} = \lambda u\overline{u} \end{cases}$$

$\big(\lambda - \overline{\lambda}\big)\overline{u}u = -\overline{u}\Delta_2 u - u\Delta_2\overline{u} = div\big(\overline{u}\nabla u - u\nabla\overline{u}\big)$

$$\begin{aligned} \big(\lambda - \overline{\lambda}\big)\int_\Omega \overline{u}u dx &= \int_\Omega div\big(\overline{u}\nabla u - u\nabla\overline{u}\big)dx \\ &= \int_\Omega \big(\overline{u}\frac{\partial u}{\partial n} - u\frac{\partial\overline{u}}{\partial n}\big)dx \\ &= 0 \\ \big(\lambda - \overline{\lambda}\big)||u||^2 &= 0 \\ \lambda - \overline{\lambda} &= 0 \\ \lambda = \overline{\lambda} & \\ \therefore\ \ \lambda \in \mathbb{R} & \end{aligned}$$

□

Corollary 3.2.1. *If $\{\lambda, u\}$ is eigenpair of the linear Dirichlet problem then $\lambda > 0$.*

Proof. Let $\{\lambda, u\}$ is eigenpair of the DEVP.

$$\begin{aligned} -\Delta_2 u &= \lambda u\ \ in\ \ \Omega \\ \lambda uv &= -v\Delta_2 u \end{aligned}$$

$$
\begin{aligned}
\lambda \int_\Omega uv\ dx &= -\int_\Omega v\Delta_2 u\ dx \\
&= \int_\Omega \big(\nabla u\cdot\nabla v - div(v\nabla u)\big)\ dx \\
&= \int_\Omega \nabla u\cdot\nabla v\ dx - \int_\Omega div(v\nabla u)\ dx \\
&= \int_\Omega \nabla u\cdot\nabla v\ dx - \int_{\partial\Omega} v\frac{\partial u}{\partial n} d\sigma(y) \\
&= \int_\Omega \nabla u\cdot\nabla v\ dx \\
\overset{u=v}{\Rightarrow}\quad \lambda\int_\Omega u^2 dx &= \int_\Omega \big(\nabla u\big)^2 dx \\
\Rightarrow \lambda &= \frac{\int_\Omega \big(\nabla u\big)^2 dx}{\int_\Omega u^2 dx} \cdots\cdots\cdots (*)
\end{aligned}
$$

For if $u = constant$, then $\nabla u = 0$ and hence $\Delta_2 u = 0$

$\Rightarrow\ 0 = -\Delta_2 u = \lambda u \neq 0$. Which is a contradiction

Hence $u \neq constant \Rightarrow\quad \nabla u \neq 0$

$\Rightarrow\quad \int_\Omega u^2 dx$ and $\int_\Omega \big(\nabla u\big)^2 dx$ are all positive

$\Rightarrow\quad \lambda > 0$ □

Definition 3.2.1. *(Rayleigh Quotient) The expression in (*) is called the Rayleigh quotient.*

Definition 3.2.2. *(Principal Eigenvalue) The smallest eigenvalue of the Dirichlet eigenvalue problem is called the principal(or smallest) eigenvalue and is denoted* λ_1.

Lemma 3.2.2. *(Rayleigh Ritz) The principal eigenvalue of the Dirichlet eigenvalue problem*

$$
\begin{cases} -\Delta_2 u = \lambda u & in\ \Omega \\ u = 0 & on\ \partial\Omega \end{cases}
$$

is given by

$$
\lambda_1 = \inf \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}
$$

Proof. proceeding as in the above Corollary, we have

$$
\lambda = \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}
$$

$$
\begin{aligned}
\inf \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx} &\leq \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}, \quad \forall u \\
\lambda_1 &= \inf \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}
\end{aligned}
$$

$\Rightarrow\ \lambda_1$ is the smallest. □

Chapter 4

Weak solution of Dirichlet problem

4.1 Critical points,Deformations

Hereafter H denotes a real Hilbert space, with norm $||\cdot||$ and inner product $(,)$. Let $I : H \to \mathbb{R}$ be non-linear functional on H.

Definition 4.1.1. *We say I is differentiable at $u \in H$ if there exists $v \in H$ such that*

$$I(w) = I(u) + (v, w - u) + o(||w - u||), \quad w \in H$$

the element v, if it exists, is unique. We then write

$$I'(u) = v$$

Definition 4.1.2. *Let $U \subset \mathbb{R}^n$ be open. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be Lipschits continuous on U if and only if there exists $L \in \mathbb{R}$ such that*

$$||f(x) - f(y)|| \leq L||x - y||, \quad \forall x, y \in U$$

Definition 4.1.3. *We say I belongs to $C^1(H, \mathbb{R})$ if $I'(u)$ exists for each $u \in H$, and the mapping $I' : H \to H$ is continuous.*

Notation: i. We denotes by $\mathcal{C}$ the collection of functions $I \in C^1(H, \mathbb{R})$ and satisfy $I' : H \to H$ is Lipschits continuous on bounded sub sets of H.
ii. If $c \in \mathbb{R}$, we write

$$\begin{aligned} k_c &= \{u \in H : I(u) = c, I'(u) = 0\} \\ A_c &= \{u \in H : I(u) \leq c\} \end{aligned}$$

Definition 4.1.4. *i. We say $u \in H$ is critical point if $I'(u) = 0$*
ii. The real number c is critical value if $k_c \neq \emptyset$

Definition 4.1.5. *A functional $I \in C^1(H, \mathbb{R})$ satisfies the palais-smale compactness condition if each sequence $\{u_k\}_{k=1}^{\infty} \subset H$ such that*

i. $\{I(u_k)\}_{k=1}^{\infty}$ is bounded

ii.$I'(u_k) \to 0$ in H

Lemma 4.1.1. *(Deformation). Let $I : H \longrightarrow \mathbb{R}$ be a C^1 functional satisfying palais-smale(PS) conditions. Suppose also $k_c = \emptyset$.*
Then there exist an $\epsilon > 0$, there exists a constant $0 < \delta < \epsilon$ and a continuous function $\eta : [0,1] \times H \longrightarrow H$ such that the mapping

$$\eta_t(u) = \eta(t,u), \quad \forall t \in [0,1], \quad u \in H$$

satisfying the following conditions

i. $\eta_0(u) = u, \quad \forall\, u \in H$

ii. $\eta_t(u) = u, \quad \forall\, t \in [0,1] \;, u \notin I^{-1}\big([c-\epsilon, c+\epsilon]\big)$

iii. $I(\eta_t(u)) \le I(u), \quad \forall\, t \in [0,1], u \in H$

iv. $\eta_1(A_{c+\epsilon}) \subset A_{c-\epsilon}$

See the proof of this lemma on [1].

Remark: *The above lemma shows that if c is not critical value, then the set $A_{c+\epsilon}$ is deformed in to $A_{c-\epsilon}$ for some $\epsilon > 0$.*

Definition 4.1.6. *Let X be a normed space. A sequence $x_n \in X$ is said to be weakly converges to $x \in X$, written $x_n \rightharpoonup x$, if $f(x_n) \to f(x), \;\; \forall f \in X^*$*

Definition 4.1.7. *Let $X = W^{1,p}(\Omega, \mathbb{R}^n)$. A functional $I : X \to \mathbb{R}$ is weakly lower semi-continuous on X if for every $u \in X$ and every sequence $\{u_n\}$ is weakly convergent to u in X such that*

$$I(u) \le \lim_{n\to\infty} \inf I(u_n)$$

Theorem 4.1.1. *(The Mountain pass theorem) Assume $I \in \mathcal{C}$ satisfies Palais-Smale condition. Suppose that*

i.$I(0) = 0$

ii.There exist $r, \rho > 0$ such that $I(u) \ge \rho$ for all $\|u\| = r$

iii.There exist $v \in H$ such that $\|v\| > r$

Set $K = \{p : [0,1] \longrightarrow H : \; p(0) = 0, p(1) = v\}$ and let

$$c = \inf_{p \in k} \sup_{t\in[0,1]} I(p(t)),$$

then c is the critical value of I.

Proof. If, by contradiction, there is no critical point at level c, then $k_c = \emptyset$ if we choose ϵ small enough so that $0 < \delta < \epsilon$. Let $p \in k$ and define the path
$\beta : [0,1] \to H$ by $\beta(t) = \eta_1(p(t))$.
Since $p(0) = 0$ *and* $p(1) = u$, it follows by choice of ϵ, that $\beta(0) = \eta_1(0) = 0,\; \beta(1) = \eta_1(u) = u$ using condition (ii) of lemma.

Now we can choose $p \in k$ such that

$$\max_{t\in[0,1]} I(p(t)) < c + \epsilon, \; p(t) \in A_{c+\epsilon}$$

by condition (iv) of lemma, $p(t) \in A_{c-\epsilon}$.
Thus

$$\max_{t\in[0,1]} I(p(t)) \leq c - \epsilon.$$

Which is contradiction to the definition of c , Hence $k_c \neq \emptyset$. Therefore c is the critical value of I. □

Definition 4.1.8. *Let H be real Hilbert space.*
i. $B : H \times H \to \mathbb{R}$ is called bilinear form if

$$a. \; B(\alpha u + \beta v, w) = \alpha B(u, w) + \beta B(v, w)$$

$$b. \; B(w, \alpha u + \beta v) = \alpha B(w, u) + \beta B(w, v)$$

for all $u, v, w \in H$ and $\alpha, \beta \in \mathbb{R}$

Theorem 4.1.2. *(Lax-Milgram theorem) Let $B : H \times H \to \mathbb{R}$ be bilinear form. Suppose there exist positive constants α and β such that*

i. $|B(u, v)| \leq \alpha||u||||v||$, and

ii. $B(u, u) \geq \beta||u||^2$

Then for every $f \in H^*$ there exists a unique $u \in H$ such that

$$B(u, v) = (f, v), \;\; \forall v \in H$$

See the proof of this theorem on [1, 2].

Definition 4.1.9. *We denote by $H^{-1}(\Omega)$ the dual space to $H_0^1(\Omega)$. In other words f belongs to $H^{-1}(\Omega)$ provided f is bounded linear functional on $H_0^1(\Omega)$.*
Notation: *We will write $\langle , \rangle$ to denote the pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.*

Definition 4.1.10. *If $f \in H^{-1}(\Omega)$, we define the norm*

$$||f||_{H^{-1}(\Omega)} = \sup \left\{ \langle f, u \rangle : u \in H_0^1(\Omega), ||u||_{H_0^1(\Omega)} \leq 1 \right\}$$

4.2 Weak formulation of Dirichlet problem

Let us consider the Dirichlet problem for the laplacian with homogeneous boundary condition on bounded domain Ω in $\mathbb{R}$,

$$-\Delta u = f \;\; in \; \Omega \tag{4.1}$$

$$u = 0 \;\; on \; \partial\Omega \tag{4.2}$$

First, suppose that the boundary of Ω is smooth and $u, f : \overline{\Omega} \to \mathbb{R}$ are smooth functions. Multiplying (4.1) by $\phi \in C_0^\infty(\Omega)$, integrating the result over Ω, and using the divergence theorem, we get

$$\int_\Omega Du \cdot D\phi dx \quad = \quad \int_\Omega f\phi dx, \quad \forall \phi \in D(\Omega) \tag{4.3}$$

The boundary term vanish because $\phi = 0$ on the boundary. Conversely, if f and Ω are smooth, then any smooth function u satisfying (4.3) is a solution of (4.1).

Definition 4.2.1. *Let Ω be an open set in $\mathbb{R}^n$ and $f \in H^{-1}(\Omega)$. A function $u : \Omega \to \mathbb{R}$ is weak solution of* (4.1) – (4.2) *if*

i. $u \in H_0^1(\Omega)$

ii. $\int_\Omega Du \cdot D\phi dx = \int_\Omega f\phi dx, \ \ \forall \phi \in H_0^1(\Omega)$

4.3 Application to quasi-linear Elliptic PDE

To illustrate the utility of mountain pass theorem, let us investigate now the semi linear boundary value problem

$$\begin{cases} -\Delta u = f(u) \ \ in \ \Omega \\ u = 0 \ \ on \ \ \partial\Omega \end{cases} \tag{4.4}$$

we assume f is smooth, and for some $1 < p < \dfrac{n+p}{n-p}$ we have

$$|f(z)| \leq c(1 + |z|^p), \quad |f'(z)| \leq c(1 + |z|^{p-1}), \quad z \in \mathbb{R}, \tag{4.5}$$

where c is constant. We will suppose also

$$0 \leq F(z) \leq \gamma f(z) z \tag{4.6}$$

for some constant $\gamma < \frac{1}{2}$, where $F(z) := \int_0^z f s ds, \quad z \in \mathbb{R}$. We hypothesize finally for constants $0 < a \leq A$ that

$$a|z|^{p+1} \leq |F(z)| \leq A|z|^{p+1}, \quad z \in \mathbb{R} \tag{4.7}$$

Now (4.7) implies $f(0) = 0$, and so that $u \equiv 0$ is a trivial solution of (4.4). We want to find another.

Theorem 4.3.1. *(Existence) The boundary value problem* (4.4) *has at least one weak solution $u \not\equiv 0$.*

Proof. Define

$$I(u) := \int_\Omega \frac{1}{2}|Du|^2 - F(u) dx$$

for $u \in H_0^1(\Omega)$.

We set $H = H_0^1(\Omega)$, with the norm $||u|| = \left(\int_\Omega |Du|^2\right)^{\frac{1}{2}}$ and inner product $(u, v) = \int_\Omega Du \cdot Dv dx$. Then

$$I(u) = \frac{1}{2}||u||^2 - \int_\Omega F(u) dx =: I_1(u) - I_2(u)$$

We first claim

$$I \textit{ belongs to } \mathcal{C} \tag{4.8}$$

To see this, note first that for each $u, v \in H$

$$I_1(w) = \frac{1}{2}||w||^2 = \frac{1}{2}||u + w - u||^2 = \frac{1}{2}||u||^2 + (u, w - u) + \frac{1}{2}||w - u||^2$$

Hence I_1 is differentiable at u, with $I_1'(u) = u$. Thus, $I_1 \in \mathcal{C}$
We must next examine the term I_2. Recall from Lax-Milgram theorem that for each $v^* \in H^{-1}(\Omega)$, the problem

$$\begin{cases} -\Delta u = v^* \;\; in\ \Omega \\ v = 0 \;\; on\ \ \partial\Omega \end{cases} \tag{4.9}$$

has unique solution $v \in H_0^1(\Omega)$. We write $v = kv^*$; so that

$$K : H^{-1}(\Omega) \to H_0^1(\Omega) \textit{ is an isometry} \tag{4.10}$$

Note in particular that if $w \in L^{\frac{n+p}{n-p}}(\Omega)$, then the linear functional w^* defined by

$$(w^*, u) := \int_\Omega wudx, \;\; u \in H_0^1(\Omega) \textit{ belongs to } H^{-1}(\Omega)$$

Observe next that $p(\frac{np}{n+p}) < \frac{n+p}{n-p} \cdot \frac{np}{n+p} = p^*$, and so

$$f(u) \in L^{\frac{n+p}{n-p}}(\Omega) \subset H^{-1}(\Omega) \; if \; u \in H_0^1(\Omega)$$

We know demonstrate that if $u \in H_0^1(\Omega)$, then

$$I_2'(u) = k[f(u)] \tag{4.11}$$

to see this note, first that

$$F(a + b) = F(a) + f(a)b + \int_0^1 (1 - s)f'(a + bs)dsb^2$$

Thus for each $w \in H_0^1(\Omega)$

$$\begin{aligned} I_2(w) &= \int_\Omega F(w)dx = \int_\Omega F(u + w - u)dx \\ &= \int_\Omega F(u)dx + f(u)(w - u)dx + R \\ &= I_2(u) + \int_\Omega Dk[f(u)] \cdot D(w - u) + R, \end{aligned}$$

Where the remainder term R satisfies, according to (4.5),

$$\begin{aligned} |R| &\le c\int_\Omega (1 + |u|^{p-1} + |w - u|^{p-1})|w - u|^2 dx \\ &\le c\Big(\int_\Omega |w - u|^2 + |w - u|^{p+1})dx\Big) + c\Big(\int_\Omega (|u|^{p+1}dx\Big)^{\frac{p-1}{p+1}} c\Big(\int_\Omega (|w - u|^{p+1}dx\Big)^{\frac{2}{p+1}} \end{aligned}$$

Since $p+1 < p^*$, Sobolev inequalities show $R = o(||w-u||)$. Thus we see from (4.10) that

$$I_2(w) = I_2(u) + (k[f(u)], w) + o(||w-u||)$$

as required.
Finally, we note that if $u, \overline{u} \in H_0^1(\Omega)$ with norm $||u||, ||\overline{u}|| \leq L$, then

$$\begin{aligned} ||I_2'(u) - I_2'(\overline{u})|| &= ||[f(u)] - k[f(\overline{u})]||_{H_0^1(\Omega)} \\ &= ||f(u) - f(\overline{u})||_{H^{-1}(\Omega)} \\ &\leq ||f(u) - f(\overline{u})||_{L^{\frac{np}{n+p}}(\Omega)} \end{aligned}$$

But

$$\begin{aligned} ||f(u) - f(\overline{u}||_{L^{\frac{np}{n+p}}(\Omega)} &\leq c\Big(\int_\Omega \big((1+|u|^{p-1}+|\overline{u}|^{p-1})|u-\overline{u}|^2\big)^{\frac{np}{n+p}}dx\Big)^{\frac{n+p}{np}} \\ &\leq c\Big(\int_\Omega \big((1+|u|^{p-1}+|\overline{u}|^{p-1})|u-\overline{u}|^2\big)^{\frac{np}{n+p}\cdot\frac{n+p}{p}}\Big)^{\frac{1}{n}}||u-\overline{u}||_{L^{p^*}(\Omega)} \\ &\leq c(L)||u-\overline{u}||_{L^{p^*}(\Omega)} \\ &\leq c(L)||u-\overline{u}|| \ \ \textit{where we used } (4.5) \end{aligned}$$

Thus $I_2' : H_0^1(\Omega) \to H_0^1(\Omega)$ is Lipschitz continuous on bounded sets. Hence $I_2 \in \mathcal{C}$.
Now we verify the Palis-Smale condition. For this purpose $\{u_k\}_{k=1}^\infty \subset H_0^1(\Omega)$, with

$$\{I(u_k)\}_{k=1}^\infty \ \textit{bounded} \tag{4.12}$$

and

$$I'(u_k) \to 0 \ \ \textit{in } H_0^1(\Omega) \tag{4.13}$$

According to the foregoing

$$u_k - k(f(u_k)) \to 0, \ \ \textit{in } H_0^1(\Omega) \tag{4.14}$$

Thus for each $\epsilon > 0$ we have

$$|(I'(u_k), v)| = \Big|\int_\Omega Du_k \cdot Dv - f(u_k)v dx\Big| \leq \epsilon||v||, \ \ v \in H_0^1(\Omega)$$

for k large enough. Let $v = u_k$ above to find

$$\left|\int_\Omega \big|Du_k\big|^2 - f(u_k)u_k dx\right| \leq \epsilon||u_k||$$

for each $\epsilon > 0$ and for all k sufficiently large. For $\epsilon = 1$ in particular, we see that

$$\int_\Omega f(u_k)u_k dx \leq ||u_k||^2 + ||u_k|| \tag{4.15}$$

for all k sufficiently large. But since (4.12) says

$$\Big(\frac{1}{2}||u_k||^2 - \Big|\int_\Omega f(u_k)dx\Big) \leq c < \infty$$

for all k and some constant c, we deduce

$$\begin{aligned} ||u_k||^2 &\leq c + 2\int_\Omega f(u_k)dx \\ &\leq c + 2\gamma\big(||u_k||^2 + ||u_k||\big) \ \textit{by (4.15) and (4.6)} \end{aligned}$$

Since $2\gamma < 1$, we discover that $\{u_k\}_{k=1}^\infty$ is bounded in $H_0^1(\Omega)$. Hence there exists a subsequence $\{u_{k_i}\}_{k=1}^\infty$ and $u \in H_0^1(\Omega)$, with $u_{k_i} \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $u_{k_i}u$ in $L^{p+1}(\Omega)$, the latter assertion holds since $p+1 < p^*$.
But then $f(u_k) \to f(u)$ in $H^{-1}(\Omega)$. Hence $k[f(u_k)] \to k[f(u)]$ in $H_0^1(\Omega)$. Consequently (4.14) implies

$$u_{k_i}u$$

in $H_0^1(\Omega)$We finally verify the remaining hypotheses of mountain pass theorem. Suppose now that $u \in H_0^1(\Omega)$, with $||u|| = r$, for $r > 0$. Then

$$I(u) = I_1(u) - I_2(u) = \frac{r^2}{2} - I_2(u) \tag{4.16}$$

Now hypotheses (4.7) implies, since $p+1 < p^*$, that

$$\begin{aligned} \big|I_2(u)\big| &\leq c\int_\Omega |u|^{p+1}dx \leq c\Big(\int_\Omega |u|^{p^*}dx\Big)^{\frac{p+1}{p^*}} \\ &\leq c||u||^{p+1} \leq cr^{p+1} \end{aligned}$$

In view of (4.16), then

$$I(u) \geq \frac{r^2}{2} - cr^{p+1} \geq \frac{r^2}{4} = a > 0$$

provided $r > 0$ is small enough, since $p+1 > p$. Now fix some element $u \in H$, u not identically 0. Write $v = tu$ for $t > 0$. Then

$$\begin{aligned} I(v) &= I_1(tu) - I_2(tu) \\ &= t^2 I_1(u) - \int_\Omega F(tu)dx \\ &\leq t^2 I_1(u) - at^{p+1}\int_\Omega |u|^{p+1}dx \ \textit{by (4.7)} \\ &< 0 \end{aligned}$$

for $t > 0$ large enough.
We have at last checked all hypotheses of mountain pass theorem. There must consequently exist a function $u \in H_0^1(\Omega)$, u is not identically 0 with

$$I'(u) = u - k[f(u)] = 0$$

in particular for each $v \in H_0^1(\Omega)$ we have

$$\int_\Omega Du \cdot Dv dx = \int_\Omega f(u)v dx$$

Hence u is weak solution of (4.4) □

4.4 Existence of solution via mountain pass theorem

The Ordinary eigenvalue problem for $p(x)$-laplacian is defined as

$$\begin{cases} -\Delta_{p(x)}u - \lambda|u|^{p(x)-2}u = 0, \quad in\ \Omega \\ u = 0\ on\ \ \partial\Omega \end{cases} \tag{4.17}$$

The value of parameter $\lambda \in \mathbb{R}$ for which (4.17) has non zero solution ($u \neq 0$) is called eigenvalue of (4.17) and corresponding u is called eigenfunction associated with λ.
If we consider non-linear eigenvalue problem involving $p(x)$-laplacian:

$$\begin{cases} -div\big(|\nabla u|^{p(x)-2}\nabla u\big) = \lambda f(x,u), \quad in\ \Omega \\ u = 0\ \ on\ \partial\Omega, \end{cases} \tag{4.18}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is bounded domain with smooth boundary, $\partial\Omega$, $1 < p(x) \in C(\overline{\Omega})$, $f \in C(\overline{\Omega} \times \mathbb{R})$ is super linear.
The nontrivial solution for problem (4.18) is recognized by assuming the following conditions:
$(\mathbf{f_0})$ $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies Caratheodory condition and

$$|f(x,t)| \leq C_1 + C_2|t|^{\alpha(x)-1}, \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $\alpha(x) \in C_+(\overline{\Omega}) = \{h \mid h \in C(\overline{\Omega}), h(x) > 1\}$ for any $x \in \overline{\Omega}$ and $\alpha(x) < p^*(x)$.

$(\mathbf{f_1})$ There is $a \in [0,\infty)$ such that $f(x,s)s^{-1} \longrightarrow a$ if $s \longrightarrow \infty$, uniformly in $x \in \mathbb{R}^{\mathrm{N}}$.
Let $G : \mathbb{R}^{\mathrm{N}} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ be defined by

$$G(x,s) = \frac{1}{2}f(x,s)s - F(x,s)$$

where $F(x,s) = \int_0^s f(x,t)dt$.We shall also use

$$A = G(x,s) \geq 0, \quad \forall s \geq 0, \quad a.e\ x \in \mathbb{R}^{\mathrm{N}}\ and\ \exists\delta > 0\ such\ that\ G(x,s) \geq \delta$$

$(\mathbf{f_2})$ $f(x,t) = o\big(|t|^{p^+-1}\big)^1$, t$\rightarrow 0$ for $x \in \Omega$ uniformly and

$$\alpha^- := \min_{\overline{\Omega}} \alpha(x) > p^+.$$

$(\mathbf{f_3})$ $F(x,t) \leq c_1|t|^\theta - c_2$, $x \in \Omega$, $t \in \mathbb{R}$ and $\theta > p^+$

$(\mathbf{f_4})$ $\lim_{|t|\to\infty} \dfrac{F(x,t)}{|t|^{p^+}} = +\infty$,uniformly a.e $x \in \Omega$

$(\mathbf{f_4'})$ There is $C_* > 0$ such that

$$tf(x,t) - p^-F(x,t) \leq sf(x,s) - p^+F(x,s) + C_*$$

for all $\quad 0 < t < s \quad$ or $\quad s < t < 0.$

[1]We write f =o(g) as x$\rightarrow x_0$, *provided* $\lim_{x\to x_0} \dfrac{|f(x)|}{|g(x)|} = 0$

Definition 4.4.1. *Let f be a function on an open subset U of Banach space X in to the Banach space Y. We say f is Gataeux differentiable at $x \in U$ if there is bounded linear operator $T : X \longrightarrow Y$ such that*

$$\lim_{t \to 0} \frac{f(x+th) - f(x)}{t} = T_x(h)$$

for every $h \in X$.The operator T is called the Gataeux derivative of f at x.

Now we introduce the energy functional $I_\lambda : W_0^{1,p(x)}(\Omega) \longrightarrow \mathbb{R}$ associated with problem (4.18), defined by

$$I_\lambda(u) = \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx - \lambda \int_\Omega F(x,u) dx$$

and Gateaux derivative of I_λ is

$$I'_\lambda(u) \cdot v = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v dx - \lambda \int_\Omega f(x,u) v dx, \quad u, v \in W_0^{1,p(x)}(\Omega)$$

Thus the critical points of I_λ are the weak solutions of problem (4.18)

Theorem 4.4.1. *Under hypothesis $(f_0), (f_2), (f_3)$ and (f_4) there is $t_0 > 0$ such that*

$$\frac{f(x,t)}{t^{p^+ - 1}} \text{ is increasing in } t \geq t_0 \text{ and decreasing in } t \leq -t_0, \ \forall x \in \Omega.$$

Moreover, $f \in C(\overline{\Omega} \times \mathbb{R})$,then problem (4.18) has a non-trivial weak solution, for all $\lambda > 0$.

Lemma 4.4.1. (1) *Under the condition (f_3) the functional I_λ is unbounded from below.*
(2) *Under the condition (f_0) and $(f_2), u = 0$ is strictly local minimum for the functional I_λ.*

***Proof*. (1)**. From (f_3) we have that, for all $M > 0$ there exists $C_M > 0$, such that

$$F(x,t) \geq M|t|^{p^+} - C_M, \ \ \forall x \in \Omega, \ \forall t > 0. \tag{4.19}$$

Take $\phi \in W_0^{1,p(x)}(\Omega)$ with $\phi > 0$, from (4.19) we obtain

$$I_\lambda(t\phi) \leq t^{p^+} \left(\int_\Omega \frac{|\nabla \phi|^{p(x)}}{p(x)} - \lambda M \int_\Omega |\phi|^{p^+} \right) + \int_\Omega C_M$$

$$I_\lambda(t\phi) \leq t^{p^+} \left(\int_\Omega \frac{|\nabla \phi|^{p(x)}}{p(x)} - \lambda M \int_\Omega |\phi|^{p^+} \right) + C_M |\Omega|$$

Where $t \geq 1$ and $|\Omega|$ denotes the lebesgue measure of Ω. If M is large, then

$$\lim_{t \to \infty} I_\lambda(t\phi) = -\infty$$

This proves (1)
proof of (2). From f_0 and f_2, we have

$$F(x,t) \leq \epsilon |t|^{p^+} + C(\epsilon)|t|^{\alpha(x)}, \ \ \forall (x,t) \in \Omega \times \mathbb{R}$$

Then

$$\begin{aligned} I_\lambda(u) &\geq \int_\Omega \frac{1}{p^+}|\nabla u|^{p^+}dx - \epsilon\lambda\int_\Omega |u|^{p^+}dx - C(\epsilon)\lambda\int_\Omega |u|^{\alpha(x)}dx] \\ &\geq \frac{1}{p^+}||u||^{p^+} - \epsilon\lambda C_0^{p^+}||u||^{p^+} - C(\epsilon)\lambda||u||^{\alpha^-} \\ &\geq \frac{1}{2p^+}||u||^{p^+} - \lambda C(\epsilon)||u||^{\alpha^-}, when\ ||u|| \leq 1, \end{aligned}$$

There exist $r > 0$ and by ($\mathbf{f_1}$) there exist $\delta > 0$ such that $I_\lambda(u) \geq \delta > 0$ for every $u \in W_0^{1,p(x)}(\Omega)$ and $||u|| = r$. The proof is complete.

Fix $0 < \lambda_0 < \mu_0$. Now we can see that the geometry on I_λ works uniformly on $[\lambda_0, \mu_0]$. From the proof of lemma 4.4.1, we obtain

$$I_\lambda(u) \geq \frac{1}{2p^+}||u||^{p^+} - \lambda C(\epsilon)||u||^{\alpha^-}, when\ ||u|| \leq 1, 0 < \lambda \leq \lambda_0$$

That is , there exist $r > 0$ and $\delta > 0$ such that $I_\lambda(u) \geq \delta > 0$ for every $u \in W_0^{1,p(x)}(\Omega)$, and $||u|| = r$ and $\forall \lambda \leq \mu_0$.
By choosing $e \in W_0^{1,p(x)}(\Omega)$ such that $I_{\lambda_0}(e) < 0$, we infer that

$$\frac{I_\lambda(e)}{\lambda} \leq \frac{I_{\lambda_0}(e)}{\lambda_0} < 0, \quad \lambda_0 \leq \lambda \leq \mu_0$$

We also have

$$\frac{I_\lambda(u)}{\lambda} \leq \frac{I_\mu(u)}{\mu}, \quad \forall u \in W_0^{1,p(x)}(\Omega),\ \mu < \lambda \tag{4.20}$$

Define

$$P = \{\gamma : [0,1] \longrightarrow W_0^{1,p}(\Omega) : \gamma\ is\ continuous\ and\ \gamma(0) = 0\ \ and\ \ \gamma(1) = e\},\ and\ for\ \lambda_0 \leq \lambda \leq \mu_0,$$

Let

$$c_\lambda = \inf_{\gamma \in P} \max_{t \in [0,1]} I_\lambda(\gamma(t)).$$

We recall that the map $c : [\lambda_0, \mu_0] \longrightarrow R_+$, given by $c(\lambda) = c_\lambda$ is such that $\frac{c_\lambda}{\lambda}$ is decreasing, left semi-continuous and bounded from below by $c_{\mu_0} > 0$.

In fact,from (4.20) follows the monotonicity. While the estimate in lemma 4.4.1 (2) implies that $c_\lambda \geq \delta > 0$.

Now, we check the left semi-continuous of $\frac{c_\lambda}{\lambda}$. Fix $\mu \in [\lambda_0, \mu_0]$ and $\epsilon > 0$. Then fix $\gamma \in P$ such that

$$c_\mu \leq \max_{t \in [0,1]} I_\mu(\gamma(t)) \leq c(\mu) + \frac{\epsilon\mu}{4}$$

Let $R_0 = \max_{t\in[0,1]} \int_\Omega F(x, \gamma(t))dx$. Then , for $\lambda > \frac{\mu}{2}$ and such that $\frac{1}{\lambda} < \frac{1}{\mu} + \frac{\epsilon}{2\mu}$

$$\begin{aligned} I_\lambda(\gamma(t)) &= (I_\lambda(\gamma(t)) - I_\mu(\gamma(t))) + I_\mu(\gamma(t)) \\ &= I_\mu(\gamma(t)) + (\mu - \lambda)\int_\Omega F(x, \gamma(t))dx \\ &\leq R_0|\lambda - \mu| + c(\mu) + \frac{\epsilon\mu}{4} \quad \forall t \in [0,1], \end{aligned}$$

that is,

$$c_\lambda \le c(\mu) + \frac{\epsilon\mu}{2}, |\lambda - \mu| < \frac{\epsilon\mu}{4R_0}.$$

Hence,if $\mu > \lambda$, it follows that

$$\frac{c_\mu}{\mu} - \epsilon < \frac{c_\mu}{\mu} \le \frac{c_\lambda}{\lambda} \le \frac{c_\mu}{\mu} + \frac{2\epsilon}{3} \le \frac{c_\mu}{\mu} + \epsilon.$$

This proves the left semi-continuity of $\frac{c_\lambda}{\lambda}$ and c_λ. □

Lemma 4.4.2. *There exists $d > 0$, such that*

$$||I'_\mu(u) - I'_\lambda(u)||_* \le d\big(1 + ||u||^{\alpha^+ - 1}\big)|\mu - \lambda|, \ \forall \lambda, \mu > 0.$$

Proof. For $\alpha(x) \in C_+(\overline{\Omega})$, define $\alpha'(x)$ such that $\frac{1}{\alpha(x)} + \frac{1}{\alpha'(x)} = 1, \quad \forall x \in \overline{\Omega}$. From condition (f_0), we have

$$|f(x,t)|^{\alpha'(x)} = |f(x,t)|^{\frac{\alpha(x)}{\alpha(x)-1}} \le d_1 + d_2|t|^{\alpha(x)}, \ \ \forall x \in \Omega, \ \forall t \in R,$$

for some constants $d_1, d_2 > 0$ and then

$$\int_\Omega |f(x,u)|^{\alpha'(x)} \le d_1|\Omega| + d_2 \int_\Omega |u|^{\alpha(x)} dx.$$

Therefore, there exist positive constants d_3 and $d_4 > 0$, such that

$$\int_\Omega |f(x,u)|^{\alpha'(x)} \le d_3 + d_4||u||^{\alpha^+}, \ \ \forall u \in W_0^{1,p(x)}(\Omega).$$

Now, $\forall u \in W_0^{1,p(x)}(\Omega)$ with $||v|| \le 1$, we have

$$I'_\mu(u)v - I'_\lambda(u)v = (\lambda - \mu)\int_\Omega f(x,u)v dx$$

Moreover, we have

$$\begin{aligned} |I'_\mu(u)v - I'_\lambda(u)v| &\le |\lambda - \mu| \int_\Omega |f(x,u)v| dx \\ &\le 2|\lambda - \mu||f(x,u)|_{\alpha'(x)}|v|_{\alpha(x)} \\ &\le 2C_0|\lambda - \mu|\big(d_3 + d_4||u||^{\alpha^+}\big)^{\frac{\alpha^+ - 1}{\alpha^+}}||v||. \end{aligned}$$

So there exists constant $d > 0$ such that

$$||I'_\mu(u) - I'_\lambda(u)||_* \le d\big(1 + ||u||^{\alpha^+ - 1}\big)|\mu - \lambda|, \ \forall \lambda, \mu > 0.$$

□

Lemma 4.4.3. *Suppose the map $c : [\lambda_0, \mu_0] \longrightarrow R_+$, given by $c(\lambda) = c_\lambda$, is differentiable in μ, then there exist a sequence $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ such that*

$$I_\mu(u_n) \to c_\mu, \quad I'_\mu(u_n) \to 0, \ \textit{and} \ \ ||u_n||^{p^-} \le C'$$

as $n \to \infty$ and where $C' = p^+ c_\mu + p^+\mu(2 - c'(\mu)) + o_n(1)$.

You can see the proof of this lemma on [6]

Lemma 4.4.4. *For almost all $\lambda > 0$, c_λ is a critical value forI_λ.*

***Proof*.** By using above lemma's now we can proof **theorem** 4.4.1.
As c_λ is left semi-continuous , from lemma 4.4.4 , for each $\mu > 0$ we can fix sequence $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ and $\lambda_n \subset R$ such that $\lambda_n \to \mu, \quad c_{\lambda_n} \to c_\mu$ as $n \to \infty$,

$$I_{\lambda_n}(u_n) = c_{\lambda_n} \ \ and \ \ I'_{\lambda_n}(u_n) = 0.$$

For the proof of Theorem , it is enough to prove the sequence $\{u_n\}$ is bounded.For unboundedness we can define $\omega_n = \frac{u_n}{||u_n||}$. With out loss of generality , $\omega \in W_0^{1,p(x)}(\Omega)$ such that

$$\omega_n(x) \rightharpoonup \omega(x) \ \ in \ W_0^{1,p(x)}(\Omega), \quad n \to \infty,$$
$$\omega_n(x) \to \omega(x) \ \ in \ L^{\alpha(x)}(\Omega), \quad n \to \infty,$$
$$\omega_n(x) \to \omega(x) \ \ for \ a.e.x \in \Omega, \quad n \to \infty.$$

Let $\Omega_{\neq} = \{x \in \Omega : \omega(x) \neq 0\}$. If $x \in \Omega_{\neq}$, then

$$\lim_{n\to\infty} \frac{F(x, u_n(x))}{|u_n(x)|^{p^+}} |\omega_n(x)|^{p^+} = \infty.$$

By hypothesis $(\mathbf{f_1})$ we have

$$\lim_{s\to\infty} \frac{F(x,s)}{s} = a$$

and

$$\lim_{s\to\infty} \frac{F(x,s)}{s^2} = \frac{a}{2}$$

Now applying the Fatous lemma and the limit

$$\lim_{n\to\infty} \int_\Omega \frac{F(x, u_n(x))}{|u_n(x)|^{p^+}} |\omega_n(x)|^{p^+} \leq \frac{1}{\mu p^-}.$$

For $|\Omega_{\neq}| > 0$, that is $\omega = 0$ a.e in Ω.
Let $t_n \in [0, 1]$ such that

$$I_{\lambda_n}(t_n u_n) = \max_{t\in[0,1]} I_{\lambda_n}(tu_n).$$

If $t_n = 1$, $\ I_{\lambda_n}(tu_n)$ is bounded for all $t \in [0, 1]$. If $t_n < 1$, $\ I'_{\lambda_n}(t_n u_n)u_n = 0$. since $I'_{\lambda_n}(t_n u_n)(t_n u_n) = 0$, from (f'_4), we have

$$\begin{aligned}
I_{\lambda_n}(tu_n) &\leq I_{\lambda_n}(t_n u_n) - \frac{1}{p^+} I'_{\lambda_n}(t_n u_n)u_n \\
&= \int_\Omega (\frac{1}{p(x)} - \frac{1}{p^+}) |\nabla t_n u_n|^{p(x)} dx \\
&\quad + \lambda_n \int_\Omega \big(\frac{1}{p(x)} t_n u_n f(x, t_n u_n) - F(x, t_n u_n)\big) dx \\
&\leq \int_\Omega (\frac{1}{p(x)} - \frac{1}{p^+}) |\nabla u_n|^{p(x)} dx \\
&\quad + \lambda_n \int_\Omega (\frac{1}{p(x)} u_n f(x, u_n) - F(x, u_n) + \frac{C_*}{p^+}) dx
\end{aligned}$$

$$= \quad c_{\lambda_n} + \frac{C_* \lambda_n}{p^+}|\Omega|$$

For all $t \in [0,1]$.
Seeking for contradiction we assume that for all $R > 1$

$$\lim_{n\to\infty} \inf I_{\lambda_n}(t_n u_n) \leq R$$

Now, setting $R' = (2p^+R)^{\frac{1}{p^-}}$. Since $||\omega_n|| = 1$, we have

$$I_{\lambda_n}(R'\omega_n) = 2R - \lambda_n \int_\Omega F(x, R'\omega_n)dx \geq R.$$

which contradicts $I_{\lambda_n}(R'\omega_n) \leq c_{\lambda_n} + \frac{C_*\lambda_n}{p^+}|\Omega|$, for n large.

Now we have a bounded sequence $\{u_n\}$ such that

$$I_{\lambda_n}(u_n) = c_{\lambda_n} \;\; and \;\; I'_{\lambda_n}(u_n) = 0.$$

The proof is complete. □

Conclusion

The $p(x)$-laplacian processes more complicated non-linearity, for example, it is non-homogeneous, so in the discussions some special techniques will be needed.
We will use the notations such as p^- and p^+ where

$$p^- := ess \inf_{x\in\Omega} p(x) \leq p(x) \leq p^+ := ess \sup_{x\in\Omega} p(x) < \infty$$

and using variable exponent theory of Lebesgue and Sobolev spaces combined with variable method and mountain pass theorem, we show the existence of non trivial weak solution of problem (I_λ).

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