Two-Mode Coherent and Squeezed Light

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Abstract

Employing c-number Langevin equations and taking into considerations the correlation properties of the noise forces, we obtained the antinormally-ordered characteristic function, and with the aid of this result, we have determined the Q-function for a two-mode cavity light. Upon the pertinent Q-function, we have calculated the mean and the normally-ordered variance of the photon numbers sum and difference, the quadrature variance, the photon number correlation and the quadrature squeezing.

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Chapter 1

Introduction

Of all states of the radiation field, the coherent states play a fundamental role in quantum optics and has a practical significance both in scientific and nonscientific realms. Coherent light is produced by a two-level laser operating well above threshold. Perfectly coherent light has Poissonian photon statistics, with random time interval between the photons. A coherent state is a minimum-uncertainty state with equal noise in both quadratures [1-8].

A light mode is said to be in a squeezed state if the noise in one quadrature is below the coherent level such that the uncertainty principle is not violated.
Squeezed light has potential applications in the detection of weak signals and in low-noise communications [1, 2,3, 7, 8].

In this study, we seek to deal the statistical properties of a cavity-mode driven by a two-mode coherent light and coupled to a two-mode squeezed vacuum reservoir via a single port-mirror. To this end, with the aid of the pretinent master equation, we first obtain c-number Langevin equations. Then employing the solutions of the resulting c-number Langevin equations along with the correlation properties of the noise forces, we obtain the antinormally-ordered characterstic function and with the aid of which we obtain the Q-function. Applying the resulting Q-function, we calculate the mean of the photon number sum and difference, the normally-ordered variance of the photon number sum and difference, the quadrature variances. Finally, using the mean of the separate modes along with the expectation value of $\hat{n}_a\hat{n}_b$, we calculate the photon number correlation.

Chapter 2

Two-Mode Cavity Light

2.1 The Q-function

In this chapter we focuse on developing the c-number Langevin equations and the Q-function corrosponding to the antinormally-ordered characteristic function for a two-mode cavity light driven by a two-mode coherent light and coupled to a two-mode squeezed vacuum reservoir via a single-port mirror.

2.1.1 c-number Langevin equations

We consider a cavity mode driven by a two-mode coherent light and coupled to a two-mode squeezed vacuum reservoir. The interaction between the two-mode cavity light

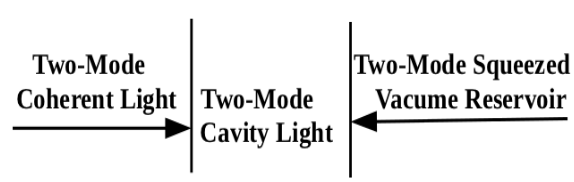

Figure 2.1: A cavity mode driven by coherent light and coupled to a squeezed vacuum reservoir

and the two-mode driving coherent light, can be described by the Hamiltonian

$$\hat{H} = i\varepsilon(\hat{a}^{\dagger} - \hat{a} + \hat{b}^{\dagger} - \hat{b}), \tag{2.1.1}$$

where $\hat{a}$ ($\hat{b}$) is the photon annihilation operator for the two-mode cavity (coherent) light and ε, considered to be real and constant, is proportional to the amplitude of the driving coherent light.

Using the above Hamiltonian, the equation of evolution of the density operator for the two-mode cavity light can be expressed as

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= -i[\hat{H},\hat{\rho}] + \frac{\kappa}{2}(N+1)(2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{a} + 2\hat{b}\hat{\rho}\hat{b}^{\dagger} - \hat{b}^{\dagger}\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^{\dagger}\hat{b})\\ &+\frac{\kappa}{2}N(2\hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{a}\hat{a}^{\dagger}\hat{\rho} - \rho\hat{a}\hat{a}^{\dagger} + 2\hat{b}^{\dagger}\hat{\rho}\hat{b} - \hat{b}\hat{b}^{\dagger}\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^{\dagger})\\ &+\kappa M(\hat{a}^{\dagger}\hat{\rho}\hat{b}^{\dagger} + \hat{b}^{\dagger}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{b}^{\dagger} + \hat{b}\hat{\rho}\hat{a} + \hat{a}\hat{\rho}\hat{b} - \hat{b}\hat{a}\hat{\rho} - \hat{\rho}\hat{b}\hat{a}).\end{aligned} \tag{2.1.2}$$

With the aid of the commutator relation, the master equation for the cavity mode, can be put in the form

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= -\varepsilon(\hat{a}\hat{\rho} - \hat{a}^{\dagger}\hat{\rho} - \hat{\rho}\hat{a} + \hat{\rho}\hat{a}^{\dagger} + \hat{b}\hat{\rho} - \hat{b}^{\dagger}\hat{\rho} - \hat{\rho}\hat{b} + \hat{\rho}\hat{b}^{\dagger})\\ &+\frac{\kappa}{2}(N+1)(2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{a} + 2\hat{b}\hat{\rho}\hat{b}^{\dagger} - \hat{b}^{\dagger}\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^{\dagger}\hat{b})\\ &+\frac{\kappa}{2}N(2\hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{a}\hat{a}^{\dagger}\hat{\rho} - \rho\hat{a}\hat{a}^{\dagger} + 2\hat{b}^{\dagger}\hat{\rho}\hat{b} - \hat{b}\hat{b}^{\dagger}\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^{\dagger})\\ &+\kappa M(\hat{a}^{\dagger}\hat{\rho}\hat{b}^{\dagger} + \hat{b}^{\dagger}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{b}^{\dagger} + \hat{b}\hat{\rho}\hat{a} + \hat{a}\hat{\rho}\hat{b} - \hat{b}\hat{a}\hat{\rho} - \hat{\rho}\hat{b}\hat{a}),\end{aligned} \tag{2.1.3}$$

where the parameters N and M are defined as

$$N = sinh^2 r = \left[\frac{e^r - e^{-r}}{2}\right]^2 = \frac{1}{4}[e^{2r} + e^{-2r} - 2], \tag{2.1.4}$$

,

$$M = coshrsinhr = [\frac{e^r + e^{-r}}{2}][\frac{e^r - e^{-r}}{2}] = \frac{1}{4}[e^{2r} - e^{-2r}], \tag{2.1.5}$$

and κ is called the cavity damping constant and the squeeze parameter r is taken for convenience to be real and positive [1].

We now seek to obtain the quantum Langevin equations applying the master equation. The time evolution of the expectation value of an operator $\hat{A}$ in Heisenberg picture can be expressed as [1]

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{A}). \tag{2.1.6}$$

Employing this relation along with Eq. (2.1.3), one can see that the annihilation operator for the cavity mode evolves in time according to the quantum Langevin equation

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\rangle \quad &= -\varepsilon(\hat{a}\hat{\rho}\hat{a} - \hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^2 + \hat{\rho}\hat{a}^\dagger\hat{a} + \hat{b}\hat{\rho}\hat{a} - \hat{b}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{b}\hat{a} + \hat{\rho}\hat{b}^\dagger\hat{a})\\ &+\frac{\kappa}{2}(N+1)(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2 + 2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a})\\ &+\frac{\kappa}{2}N(2\hat{a}^\dagger\hat{\rho}\hat{a}^2 - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a} - \rho\hat{a}\hat{a}^\dagger\hat{a} + 2\hat{b}^\dagger\hat{\rho}\hat{b}\hat{a} - \hat{b}\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{b}\hat{b}^\dagger\hat{a})\\ &+\kappa M(\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger\hat{a} + \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{+}\hat{b}\hat{\rho}\hat{a}^2\\ &+\hat{a}\hat{\rho}\hat{b}\hat{a} - \hat{b}\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}\hat{a}^2).\end{aligned} \tag{2.1.7}$$

Applying the cyclic property of the trace operation together with the commutation relations

$$[\hat{a},\hat{a}^\dagger] = [\hat{b},\hat{b}^\dagger] = 1, \tag{2.1.8}$$

$$[\hat{a},\hat{b}] = [\hat{a}^\dagger,\hat{b}^\dagger] = [\hat{a}^\dagger,\hat{b}] = [\hat{a},\hat{b}^\dagger] = 0, \tag{2.1.9}$$

and with the aid of the identity

$$[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}, \tag{2.1.10}$$

$$[\hat{A},\hat{B}\hat{C}] = [\hat{A},\hat{B}]\hat{C} + \hat{B}[\hat{A},\hat{C}], \tag{2.1.11}$$

we obtain

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -\frac{1}{2}\kappa\langle\hat{a}(t)\rangle + \varepsilon. \tag{2.1.12}$$

In similar procedure, one can show that

$$\frac{d}{dt}\langle\hat{b}(t)\rangle = -\frac{1}{2}\kappa\langle\hat{b}(t)\rangle + \varepsilon, \tag{2.1.13}$$

$$\frac{d}{dt}\langle \hat{a}^2(t)\rangle = -\kappa\langle \hat{a}^2(t)\rangle + 2\varepsilon\langle \hat{a}(t)\rangle, \tag{2.1.14}$$

$$\frac{d}{dt}\langle \hat{b}^2(t)\rangle = -\kappa\langle \hat{b}^2(t)\rangle + 2\varepsilon\langle \hat{b}(t)\rangle, \tag{2.1.15}$$

$$\frac{d}{dt}\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle = -\kappa\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle + \varepsilon\langle \hat{a}^\dagger(t)\rangle + \varepsilon\langle \hat{a}(t)\rangle + \kappa N, \tag{2.1.16}$$

$$\frac{d}{dt}\langle \hat{b}^\dagger(t)\hat{b}(t)\rangle = -\kappa\langle \hat{b}^\dagger(t)\hat{b}(t)\rangle + \varepsilon\langle \hat{b}^\dagger(t)\rangle + \varepsilon\langle \hat{b}(t)\rangle + \kappa N, \tag{2.1.17}$$

$$\frac{d}{dt}\langle \hat{a}(t)\hat{b}(t)\rangle = -\kappa\langle \hat{a}(t)\hat{b}(t)\rangle + \varepsilon\langle \hat{a}(t)\rangle + \varepsilon\langle \hat{b}(t)\rangle - \kappa M, \tag{2.1.18}$$

$$\frac{d}{dt}\langle \hat{a}^\dagger(t)\hat{b}(t)\rangle = -\kappa\langle \hat{a}^\dagger(t)\hat{b}(t)\rangle + \varepsilon\langle \hat{a}^\dagger(t)\rangle + \varepsilon\langle \hat{b}(t)\rangle. \tag{2.1.19}$$

We note that the normally-ordered c-number equations corrosponding to Eqs. (2.1.12 - 2.1.19) are

$$\frac{d}{dt}\langle \alpha(t)\rangle = -\frac{1}{2}\kappa\langle \alpha(t)\rangle + \varepsilon, \tag{2.1.20}$$

$$\frac{d}{dt}\langle \beta(t)\rangle = -\frac{1}{2}\kappa\langle \beta(t)\rangle + \varepsilon, \tag{2.1.21}$$

$$\frac{d}{dt}\langle \alpha^2(t)\rangle = -\kappa\langle \alpha^2(t)\rangle + 2\varepsilon\langle \alpha(t)\rangle, \tag{2.1.22}$$

$$\frac{d}{dt}\langle \beta^2(t)\rangle = -\kappa\langle \beta^2(t)\rangle + 2\varepsilon\langle \beta(t)\rangle, \tag{2.1.23}$$

$$\frac{d}{dt}\langle \alpha^*(t)\alpha(t)\rangle = -\kappa\langle \alpha^*(t)\alpha(t)\rangle + \varepsilon\langle \alpha^*(t)\rangle + \varepsilon\langle \alpha(t)\rangle + \kappa N, \tag{2.1.24}$$

$$\frac{d}{dt}\langle \beta^*(t)\beta(t)\rangle = -\kappa\langle \beta^*(t)\beta(t)\rangle + \varepsilon\langle \beta^*(t)\rangle + \varepsilon\langle \beta(t)\rangle + \kappa N, \tag{2.1.25}$$

$$\frac{d}{dt}\langle \alpha(t)\beta(t)\rangle = -\kappa\langle \alpha(t)\beta(t)\rangle + \varepsilon\langle \alpha(t)\rangle + \varepsilon\langle \beta(t)\rangle - \kappa M, \tag{2.1.26}$$

$$\frac{d}{dt}\langle \alpha^*(t)\beta(t)\rangle = -\kappa\langle \alpha^*(t)\beta(t)\rangle + \varepsilon\langle \alpha^*(t)\rangle + \varepsilon\langle \beta(t)\rangle. \tag{2.1.27}$$

On the basis of Eq. (2.1.20) and Eq. (2.1.21), one can write

$$\frac{d}{dt}\alpha(t) = -\frac{\kappa}{2}\alpha(t) + \varepsilon + f_\alpha(t), \tag{2.1.28}$$

$$\frac{d}{dt}\beta(t) = -\frac{\kappa}{2}\beta(t) + \varepsilon + f_\beta(t), \tag{2.1.29}$$

where $f_\alpha(t)$ and $f_\beta(t)$ are noise forces associated with the normal-ordering and whose correlation properties remain to be dertmineed.

We note that Eqs. (2.1.12) and the expectation value of (2.1.28) as well as (2.1.21) and (2.1.29) will have the same form if

$$\langle f_\alpha(t)\rangle = \langle f_\beta(t)\rangle = 0. \tag{2.1.30}$$

Furthermore, using Eqs. (2.1.28) and (2.1.29) together with the relation

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = \left\langle \frac{d\alpha(t)}{dt}\beta(t)\right\rangle + \left\langle \alpha(t)\frac{d\beta(t)}{dt}\right\rangle, \tag{2.1.31}$$

it can be verified that

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = -\kappa\langle\alpha(t)\beta(t)\rangle + \varepsilon\langle\alpha(t)\rangle + \varepsilon\langle\beta(t)\rangle + \langle\alpha(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha(t)\rangle. \tag{2.1.32}$$

So that by comparing Eqs. (2.1.26) and (2.1.32), one can directly show that

$$\langle\alpha(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha(t)\rangle = -\kappa M. \tag{2.1.33}$$

Here we seek to obtain an explicit expressions for $\alpha(t)$ and $\beta(t)$. The solution of the equation having the form [1]

$$\frac{d}{dt}\alpha(t) = -\lambda\alpha(t) + F(t), \tag{2.1.34}$$

is expressible as

$$\alpha(t) = \alpha(t_0)e^{-\lambda(t-t^{'})} + \int_{t_0}^{t} e^{-\lambda(t-t^{'})}F(t^{'})dt^{'}. \tag{2.1.35}$$

Hence by employing this relation, the formal solutions of Eqs. (2.1.28) and (2.1.29) respectively can be written in the form

$$\alpha(t) = \alpha(0)e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2}[\varepsilon + f_\alpha(t^{'})]dt^{'}, \tag{2.1.36}$$

and

$$\beta(t) = \beta(0)e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2}[\varepsilon + f_\beta(t^{'})]dt^{'}. \tag{2.1.37}$$

Then in view of Eqs. (2.1.36) and (2.1.37), one obtains

$$\langle\alpha(t)f_\beta(t)\rangle = \langle\alpha(0)f_\beta(t)\rangle e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2}[\varepsilon\langle f_\beta(t)\rangle + \langle f_\beta(t)f_\alpha(t^{'})\rangle]dt^{'}, \tag{2.1.38}$$

and

$$\langle\beta(t)f_\alpha(t)\rangle = \langle\beta(0)f_\alpha(t)\rangle e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2}[\varepsilon\langle f_\alpha(t)\rangle + \langle f_\alpha(t)f_\beta(t^{'})\rangle]dt^{'}. \quad (2.1.39)$$

On account of the assertion that a noise force at a certain time t should not affect the system variables at earlier times, we note

$$\langle\alpha(0)f_\beta(t)\rangle = \langle\alpha(0)\rangle\langle f_\beta(t)\rangle = 0, \quad (2.1.40)$$

and

$$\langle\beta(0)f_\alpha(t)\rangle = \langle\beta(0)\rangle\langle f_\alpha(t)\rangle = 0. \quad (2.1.41)$$

Therefore, on substituting Eqs. (2.1.38) and (2.1.39) together with (2.1.30), (2.1.40) and (2.1.41) into (2.1.33), we obtain

$$\int_0^t e^{-\kappa(t-t')/2}[\langle f_\alpha(t)f_\beta(t')\rangle + \langle f_\beta(t)f_\alpha(t')\rangle]dt' = -\kappa M. \quad (2.1.42)$$

Thus in view of Eq. (2.1.42) together with the assumption

$$\langle f_\alpha(t)f_\beta(t')\rangle = \langle f_\beta(t)f_\alpha(t')\rangle, \quad (2.1.43)$$

we obtain

$$\int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t)f_\beta(t')\rangle dt' = -\frac{1}{2}\kappa M. \quad (2.1.44)$$

Now on the basis of the relation [1]

$$\int_0^t e^{-a(t-t')}\langle f(t)g(t')\rangle dt' = D, \quad (2.1.45)$$

we assert that

$$\langle f(t)g(t')\rangle = 2D\delta(t-t'), \quad (2.1.46)$$

where a and D are constants, or D is some function of the time t.

So that on account of this relation, expression (2.1.44) can be written as

$$\langle f_\alpha(t)f_\beta(t')\rangle = \langle f_\beta(t)f_\alpha(t')\rangle = -\kappa M\delta(t-t^{'}). \quad (2.1.47)$$

Furthermore, using Eq. (2.1.28) along with the relation

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = \frac{d}{dt}\langle\alpha(t)\alpha(t)\rangle = \left\langle\frac{d\alpha(t)}{dt}\alpha(t)\right\rangle + \left\langle\alpha(t)\frac{d\alpha(t)}{dt}\right\rangle, \tag{2.1.48}$$

we get

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -\kappa\langle\alpha^2(t)\rangle + 2\varepsilon\langle\alpha(t)\rangle + 2\langle\alpha(t)f_\alpha(t)\rangle. \tag{2.1.49}$$

Now comparision of Eqs. (2.1.22) and (2.1.49) leads to

$$\langle\alpha(t)f_\alpha(t)\rangle = 0. \tag{2.1.50}$$

On account of Eq. (2.1.36) along with (2.1.50), we can see that

$$\langle\alpha(t)f_\alpha(t)\rangle = \langle\alpha(0)f_\alpha(t)\rangle e^{-\kappa t/2} + \int_0^t e^{-\kappa(t-t')/2}[\varepsilon\langle f_\alpha(t)\rangle + \langle f_\alpha(t)f_\alpha(t')\rangle]dt'. \tag{2.1.51}$$

Taking into account Eq. (2.1.30) along with (2.1.50) and on the fact that a noise force at a certain time does not affect the system operator at an earlier time, we have

$$\int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t)f_\alpha(t')\rangle dt' = 0, \tag{2.1.52}$$

and considering the application of Eqs. (2.1.45) and (2.1.46), Eq. (2.1.52) leads to

$$\langle f_\alpha(t)f_\alpha(t')\rangle = 0. \tag{2.1.53}$$

Using a similar procedure, we can easily estabilish that

$$\langle f_\beta(t)f_\beta(t')\rangle = 0. \tag{2.1.54}$$

Moreover, on the basis of Eq. (2.1.28) and its complex conjugate along with the relation

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = \left\langle\frac{d\alpha^*(t)}{dt}\alpha(t)\right\rangle + \left\langle\alpha^*(t)\frac{d\alpha(t)}{dt}\right\rangle, \tag{2.1.55}$$

we readily obtain

$$\begin{aligned}\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle &= -\kappa\langle\alpha^*(t)\alpha(t)\rangle + \varepsilon\langle\alpha(t)\rangle \\ &+\varepsilon\langle\alpha^*(t)\rangle + \langle\alpha(t)f^*_\alpha(t)\rangle + \langle\alpha^*(t)f_\alpha(t)\rangle.\end{aligned} \tag{2.1.56}$$

Comparision of Eqs. (2.1.24) and (2.1.56) indicates that

$$\langle\alpha(t)f^*_\alpha(t)\rangle + \langle\alpha^*(t)f_\alpha(t)\rangle = \kappa N. \tag{2.1.57}$$

In addition, employing Eq. (2.1.36) and its complex conjugate along with (2.1.30), it can easily verified respectively that

$$\langle\alpha(t)f^*_\alpha(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t')f^*_\alpha(t)\rangle dt^{'}, \tag{2.1.58}$$

and

$$\langle\alpha^*(t)f_\alpha(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f^*_\alpha(t')f_\alpha(t)\rangle dt^{'}. \tag{2.1.59}$$

Hence, substituting Eqs. (2.1.58) and (2.1.59) into (2.1.57) along with the assumption

$$\langle f_\alpha(t^{'})f^*_\alpha(t)\rangle = \langle f^*_\alpha(t')f_\alpha(t)\rangle, \tag{2.1.60}$$

and with the aid of Eqs. (2.1.45) and (2.1.46), one readily obtains

$$\int_0^t e^{-\kappa(t-t')/2}\langle f_\alpha(t')f^*_\alpha(t)\rangle dt^{'} = \frac{1}{2}\kappa M, \tag{2.1.61}$$

which implies

$$\langle f_\alpha(t')f^*_\alpha(t)\rangle = \langle f^*_\alpha(t')f_\alpha(t)\rangle = \kappa N\delta(t-t'). \tag{2.1.62}$$

Using a similar fasion, it can also be estabilished that

$$\langle f_\beta(t')f^*_\beta(t)\rangle = \langle f^*_\beta(t')f_\beta(t)\rangle = \kappa N\delta(t-t'). \tag{2.1.63}$$

Finally, on account of Eq. (2.1.29) and the complex conjugate of Eq. (2.1.28) along with the relation

$$\frac{d}{dt}\langle\alpha^*(t)\beta(t)\rangle = \left\langle\frac{d\alpha^*(t)}{dt}\beta(t)\right\rangle + \left\langle\alpha^*(t)\frac{d\beta(t)}{dt}\right\rangle, \tag{2.1.64}$$

we can find

$$\begin{aligned}\frac{d}{dt}\langle\alpha^*(t)\beta(t)\rangle \quad &= -\kappa\langle\alpha^*(t)\beta(t)\rangle + \varepsilon\langle\alpha^*(t)\rangle \\ &+\varepsilon\langle\beta(t)\rangle + \langle\beta(t)f^*_\alpha(t)\rangle + \langle\alpha^*(t)f_\beta(t)\rangle,\end{aligned} \tag{2.1.65}$$

and comparision of this expression with Eq. (2.1.27) provides that

$$\langle \alpha^*(t) f_\beta(t)\rangle + \langle \beta(t) f^*_\alpha(t)\rangle = 0. \tag{2.1.66}$$

In view of (2.1.37) and the complex conjugate of (2.1.36), one obtains

$$\langle \alpha^*(t) f_\beta(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f^*_\alpha(t') f_\beta(t)\rangle dt^{'}, \tag{2.1.67}$$

and

$$\langle \beta(t) f^*_\alpha(t)\rangle = \int_0^t e^{-\kappa(t-t')/2}\langle f_\beta(t') f^*_\alpha(t)\rangle dt^{'}. \tag{2.1.68}$$

Upon substituting Eqs. (2.1.67) and (2.1.68) into (2.1.66) with the assumption

$$\langle f_\beta(t') f^*_\alpha(t)\rangle = \langle f^*_\alpha(t') f_\beta(t)\rangle, \tag{2.1.69}$$

and using the application of Eqs. (2.1.45) and (2.1.46), expression (2.1.66) becomes

$$\langle f_\beta(t') f^*_\alpha(t)\rangle = \langle f^*_\alpha(t') f_\beta(t)\rangle = 0. \tag{2.1.70}$$

We can also estabilish in a similar manner that

$$\langle f_\alpha(t') f^*_\beta(t)\rangle = \langle f^*_\beta(t') f_\alpha(t)\rangle = 0. \tag{2.1.71}$$

It is worth mentioning that expressions (2.1.30), (2.1.47), (2.1.53), (2.1.54), (2,1.62), (2.1.63), (2.1.70) and (2.1.71) describe the correlation properties of the noise forces $f_\alpha(t)$ and $f_\beta(t)$ associated with the normal-ordering.

In order to determine a coupled differential equation and its corrosponding solution on the basis of Eqs. (2.1.28) and (2.1.29), we now introduce a new variable defined by

$$\gamma_\pm(t) = \alpha(t) \pm \beta^*(t). \tag{2.1.72}$$

So that upon carrying out the differenciation with respect to time, one can get

$$\frac{d}{dt}\gamma_\pm(t) = \frac{d}{dt}\alpha(t) \pm \frac{d}{dt}\beta^*(t), \tag{2.1.73}$$

and taking into account (2.1.28) together with the complex conjugate of (2.1.29) this expression yields

$$\frac{d}{dt}\gamma_{\pm}(t) = -\frac{\kappa}{2}\gamma_{\pm}(t) + [\varepsilon \pm \varepsilon] + [f_{\alpha}(t) \pm f^{*}_{\beta}(t^{'})]. \tag{2.1.74}$$

Now here we seek to obtain the solution of the time evolution of this coupled differntial equation. Hence, using the application of Eqs. (2.1.34) and (2.1.35), the solution of this expression can be put in the form

$$\gamma_{\pm}(t) = \gamma_{\pm}(0)e^{-\kappa t/2} + \int_{0}^{t} e^{-\kappa(t-t')/2}[\varepsilon \pm \varepsilon + f_{\alpha}(t^{'}) \pm f^{*}_{\beta}(t^{'})]dt^{'}. \tag{2.1.75}$$

Using the definition (2.1.72), it can be rewritten as

$$\begin{aligned} \alpha(t) \pm \beta^{*}(t) \quad &= [\alpha(0) \pm \beta(0)]e^{-\kappa t/2} \\ &+ \int_{0}^{t} e^{-\kappa(t-t')/2}[\varepsilon \pm \varepsilon + f_{\alpha}(t^{'}) \pm f^{*}_{\beta}(t^{'})]dt^{'}. \end{aligned} \tag{2.1.76}$$

Inspection of Eq. (2.1.76) reveals that

$$\alpha(t) + \beta^{*}(t) = [\alpha(0) + \beta^{*}(0)]e^{-\kappa t/2} + \int_{0}^{t} e^{-\kappa(t-t')/2}[2\varepsilon + f_{\alpha}(t') + f^{*}_{\beta}(t^{'})]dt^{'}, \tag{2.1.77}$$

$$\alpha(t) - \beta^{*}(t) = [\alpha(0) - \beta^{*}(0)]e^{-\kappa t/2} + \int_{0}^{t} e^{-\kappa(t-t')/2}[f_{\alpha}(t') - f^{*}_{\beta}(t^{'})]dt^{'}. \tag{2.1.78}$$

Now by solving Eqs. (2.1.77) and (2.1.78) simultaneously, we find

$$\begin{aligned} \alpha(t) \quad &= \alpha(0)e^{-\kappa t/2} + \varepsilon\int_{0}^{t} e^{-\kappa(t-t')/2}dt^{'} + \frac{1}{2}\int_{0}^{t} e^{-\kappa(t-t')/2}[f_{\alpha}(t^{'}) + f^{*}_{\beta}(t^{'})]dt^{'} \\ &+\frac{1}{2}\int_{0}^{t} e^{-\kappa(t-t')/2}[f_{\alpha}(t^{'}) - f^{*}_{\beta}(t^{'})]dt^{'}, \end{aligned} \tag{2.1.79}$$

$$\begin{aligned} \beta(t) \quad &= \beta(0)e^{-\kappa t/2} + \varepsilon\int_{0}^{t} e^{-\kappa(t-t')/2}dt^{'} + \frac{1}{2}\int_{0}^{t} e^{-\kappa(t-t')/2}[f^{*}_{\alpha}(t^{'}) + f_{\beta}(t^{'})]dt^{'} \\ &-\frac{1}{2}\int_{0}^{t} e^{-\kappa(t-t')/2}[f^{*}_{\alpha}(t^{'}) - f_{\beta}(t^{'})]dt^{'}. \end{aligned} \tag{2.1.80}$$

We notice that

$$\varepsilon\int_{0}^{t} e^{-\kappa(t-t')/2}dt^{'} = \frac{2\varepsilon}{\kappa}[1 - e^{-\kappa t/2}]. \tag{2.1.81}$$

In view of this result, Eqs. (2.1.79) and (2.1.80) can be rewritten as

$$\alpha(t) = A(t)\alpha(0) + E(t) + F_{+}(t) + F_{-}(t), \tag{2.1.82}$$

$$\beta(t) = A(t)\beta(0) + E(t) + F_{+}^{*}(t) - F_{-}^{*}(t). \tag{2.1.83}$$

Where

$$A(t) = e^{-\kappa t/2}, \tag{2.1.84}$$

$$E(t) = \frac{2\varepsilon}{\kappa}[1 - e^{-\kappa t/2}], \tag{2.1.85}$$

$$F_{\pm}(t) = \frac{1}{2}\int_{0}^{t} e^{-\kappa(t-t')/2}[f_{\alpha}(t^{'}) \pm f_{\beta}^{*}(t^{'})]dt^{'}. \tag{2.1.86}$$

Furthermore, Eqs. (2.1.82) and (2.1.83) can be rewritten in the form

$$\alpha(t) = \alpha^{'}(t) + E(t), \tag{2.1.87}$$

$$\beta(t) = \beta^{'}(t) + E(t), \tag{2.1.88}$$

where

$$\alpha^{'}(t) = A(t)\alpha(0) + F_{+}(t) + F_{-}(t), \tag{2.1.89}$$

$$\beta^{'}(t) = A(t)\beta(0) + F_{+}^{*}(t) + F_{-}^{*}(t). \tag{2.1.90}$$

2.1.2 The Q-function of the cavity light

We next seek to obtain the Q-function for a coherently driven two-mode cavity light. The Q-function for a two-mode coherent light is expressible as

$$Q(\alpha,\beta,t) = \frac{1}{\pi^{4}}\int d^{2}zd^{2}\eta\phi_{a}(z,\eta,t)exp(z^{*}\alpha - z\alpha^{*} + \eta^{*}\beta - \eta\beta^{*}), \tag{2.1.91}$$

where the antinormally-ordered characteristic function $\phi_(z,\eta,t)$ is defined in the Heisenberg picture by

$$\phi_{a}(z,\eta,t) = Tr[\hat{\rho}(0)e^{-z^{*}\hat{a}(t)}e^{z\hat{a}^{\dagger}(t)}e^{-\eta^{*}(t)\hat{b}(t)}e^{\eta\hat{b}^{\dagger}(t)}]. \tag{2.1.92}$$

With the aid of the Baker-Hausdorff identity [1]

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{B}}e^{\hat{A}}e^{[\hat{A},\hat{B}]}, \tag{2.1.93}$$

one can check that

$$e^{-z^*\hat{a}(t)}e^{z\hat{a}^\dagger(t)} = e^{-zz^*}e^{z\hat{a}^\dagger(t)}e^{-z^*\hat{a}(t)}, \tag{2.1.94}$$

$$e^{-\eta^*\hat{b}(t)}e^{\eta\hat{b}^\dagger(t)} = e^{-\eta\eta^*}e^{\eta\hat{b}^\dagger(t)}e^{-\eta^*\hat{b}(t)}. \tag{2.1.95}$$

In view of these results, expression (2.1.92) can be put in the form

$$\phi_a(z,\eta,t) = e^{-zz^*-\eta\eta^*}Tr[\hat{\rho}(0)e^{z\hat{a}^\dagger(t)}e^{-z^*\hat{a}(t)}e^{\eta\hat{b}^\dagger(t)}e^{-\eta^*(t)\hat{b}(t)}]. \tag{2.1.96}$$

The expectation value of a given operator function $A(\hat{a}^\dagger, \hat{a})$ is expressible in terms of the density operator as [1]

$$\langle\hat{A}\rangle = Tr(\hat{\rho}\hat{A}). \tag{2.1.97}$$

In view of this relation, expression (2.1.96) can be put in such a way that

$$\phi_a(z,\eta,t) = e^{-zz^*-\eta\eta^*}\langle e^{z\hat{a}^\dagger(t)}e^{-z^*\hat{a}(t)}e^{\eta\hat{b}^\dagger(t)}e^{-\eta^*(t)\hat{b}(t)}\rangle. \tag{2.1.98}$$

This function also can be expressed in terms of the c-number variables associated with the normal-order as

$$\phi_a(z,\eta,t) = e^{-zz^*-\eta\eta^*}\langle exp[z\alpha^*(t) - z^*\alpha(t) + \eta\beta^*(t) - \eta^*\beta(t)]\rangle. \tag{2.1.99}$$

And upon substituting Eqs. (2.1.87) and (2.1.88) along with their complex conjugate into (2.1.99), one can get

$$\begin{aligned}\phi_a(z,\eta,t) \quad &= exp[-zz^* - \eta\eta^* + (z - z^* + \eta - \eta^*)E(t)] \\ &\times\langle exp[z\alpha^{'*}(t) - z^*\alpha^{'}(t) + \eta\beta^{'*}(t) - \eta^*\beta^{'}(t)]\rangle.\end{aligned} \tag{2.1.100}$$

Now we seek to show $\alpha^{'}(t)$ and $\beta^{'}(t)$ are Gaussian variables. With the aid of Eqs. (2.1.89) and (2.1.90) along with Eqs. (2.1.84) and (2.1.86), the equation of evolution of the expectation values of $\alpha^{'}(t)$ and $\beta^{'}(t)$ can be written as

$$\langle\alpha^{'}(t)\rangle = \langle\alpha(0)\rangle e^{-\kappa t/2}, \tag{2.1.101}$$

$$\langle\beta^{'}(t)\rangle = \langle\beta(0)\rangle e^{-\kappa t/2}. \tag{2.1.102}$$

By carrying out the differentiation on both expressions with respect to time, it can be shown that

$$\frac{d}{dt}\langle\alpha^{'}(t)\rangle = -\frac{\kappa}{2}\langle\alpha^{'}(t)\rangle, \tag{2.1.103}$$

$$\frac{d}{dt}\langle\beta^{'}(t)\rangle = -\frac{\kappa}{2}\langle\beta^{'}(t)\rangle, \tag{2.1.104}$$

which are linear equations. Which in turn shows that $\alpha^{'}(t)$ and $\beta^{'}(t)$ are Gaussian variables. In adition, using (2.1.89) and (2.1.90) along with the assumption that the cavity light is initially in a two-mode vacuum state, we get

$$\langle\alpha^{'}(t)\rangle = \langle\beta^{'}(t)\rangle = 0, \tag{2.1.105}$$

which also shows that $\alpha^{'}(t)$ and $\beta^{'}(t)$ are Gaussian variables with vanishing means. Furthermore, consider the relation [1]

$$\langle exp(\alpha+\beta)\rangle = exp[\frac{1}{2}\langle(\alpha+\beta)^2\rangle], \tag{2.1.106}$$

where $\alpha^{'}(t)$ and $\beta^{'}(t)$ are Gaussian variables with vanishing means.

Applying this relation to the last term, at the right side of expression (2.1.100) yields

$$\begin{aligned}\langle exp[z\alpha^{'*} - z^*\alpha^{'} + \eta\beta^{'*} - \eta^*\beta^{'}]\rangle \quad &= exp[\frac{1}{2}\langle(z\alpha^{'*} - z^*\alpha^{'} + \eta\beta^{'*} - \eta^*\beta^{'})^2\rangle] \\ &= exp[-zz^*\langle\alpha^{'}\alpha^{'*}\rangle - z^*\eta\langle\alpha^{'}\beta^{'*}\rangle + z\eta\langle\alpha^{'*}\beta^{'}\rangle \\ &-z\eta^*\langle\alpha^{'*}\beta^{'}\rangle + z^*\eta^*\langle\alpha^{'}\beta^{'}\rangle - \eta\eta^*\langle\beta^{'}\beta^{'*}\rangle \\ &+\frac{z^2}{2}\langle\alpha^{'*2}\rangle + \frac{z^{*2}}{2}\langle\alpha^{'2}\rangle + \frac{\eta^2}{2}\langle\beta^{'*2}\rangle + \frac{\eta^{*2}}{2}\langle\beta^{'2}\rangle].\end{aligned} \tag{2.1.107}$$

Now on account of Eqs. (2.1.89) and (2.1.90) along with their complex conjugate, and assuming the cavity mode is initially in a two-mode vacuum state, one readily obtains

$$\langle\alpha^{'}\alpha^{'*}\rangle = \langle\beta^{'}\beta^{'*}\rangle = N[1-e^{-\kappa t}], \tag{2.1.108}$$

$$\langle\alpha^{'}\beta^{'*}\rangle = \langle\alpha^{'*}\beta^{'}\rangle = 0, \tag{2.1.109}$$

$$\langle\alpha^{'}\beta^{'}\rangle = \langle\alpha^{'*}\beta^{'*}\rangle = -M[1-e^{-\kappa t}], \tag{2.1.110}$$

$$\langle\alpha^{'2}\rangle = \langle\alpha^{'*2}\rangle = 0, \tag{2.1.111}$$

$$\langle\beta^{'2}\rangle = \langle\beta^{'*2}\rangle = 0. \tag{2.1.112}$$

In view of these results,expression (2.1.107) becomes

$$\langle exp[z\alpha^{'*} - z^*\alpha^{'} + \eta\beta^{'*} - \eta^*\beta^{'}]\rangle = exp[(-Nzz^* - MZ\eta - Mz^*\eta^* - N\eta\eta^*)(1-e^{-\kappa t})]. \tag{2.1.113}$$

Now with the aid of this result, the characteristic function given by Eq. (2.1.100) can thus become

$$\begin{aligned}\phi_a(z,\eta,t) \quad &= exp[-(1+N-Ne^{-\kappa t})zz^* + Ez - Ez^* + E\eta - E\eta^* \\ &-M(1-e^{-\kappa t})z\eta - M(1-e^{-\kappa t})z^*\eta^* - (1+N-Ne^{-\kappa t})\eta\eta^*].\end{aligned} \tag{2.1.114}$$

Finally, according to this expression, the Q-function for the two-mode light, given by Eq. (2.1.91) takes the form

$$\begin{aligned}Q(\alpha,\beta,t) \quad &= \frac{1}{\pi^4}\int d^2z exp[-lzz^* + (E-\alpha^*)z + (\alpha - E)z^*] \\ &\times \int d^2\eta exp[-l\eta\eta^* + (E-\beta^*)\eta + (\beta - E)\eta^* + mz\eta + mz^*\eta^*].\end{aligned} \tag{2.1.115}$$

or

$$\begin{aligned}Q(\alpha,\beta,t) \quad &= \frac{1}{\pi^4}\int d^2z exp[-lzz^* + (E-\alpha^*)z + (\alpha - E)z^*] \\ &\times \int d^2\eta exp[-l\eta\eta^* + (E-\beta^* + mz)\eta + (\beta - E + mz^*)\eta^*],\end{aligned} \tag{2.1.116}$$

where

$$l = 1 + N[1-e^{-\kappa t}], \tag{2.1.117}$$

$$m = -M[1 - e^{-\kappa t}]. \tag{2.1.118}$$

Furthermore, using the relation [1]

$$\int \frac{d^2 z}{\pi} exp[-az^* z + bz + cz^* + Az^2 + Bz^{*2}]$$
$$= \left[\frac{1}{a^2 - 4AB}\right]^{1/2} exp\left(\frac{abc + Ac^2 + Bb^2}{a^2 - 4AB}\right), a > 0, \tag{2.1.119}$$

and performing the integration, there follows

$$Q(\alpha, \beta, t) \quad = \frac{R}{\pi^2} exp[-a\alpha\alpha^* + u\alpha + u\alpha^* - a\beta\beta^* + u\beta + u\beta^* + v\alpha\beta + v\alpha^*\beta^* + C], \tag{2.1.120}$$

where

$$R = \frac{1}{l^2 - m^2}, \tag{2.1.121}$$

$$a = \frac{l}{l^2 - m^2}, \tag{2.1.122}$$

$$u = \frac{E}{l + m}, \tag{2.1.123}$$

$$v = \frac{m}{l^2 - m^2}, \tag{2.1.124}$$

$$C = -\frac{2E^2}{l + m}. \tag{2.1.125}$$

Now by employing (2.1.119) and taking into account (2.1.120), one arrives at

$$\int d^2\alpha d^2\beta Q(\alpha, \beta, t) = 1, \tag{2.1.126}$$

which indicates that the Q-function is normalized.

Chapter 3

photon statistics

The statistical properties of a light beam is described in terms of the mean and variance of the photon number. In this section, applying the Q-function, we seek to determine the mean, and the variance of the photon numbers sum and difference for modes a and b as well as the photon number correlation.

3.1 The mean of the photon number sum and difference

Here, employing the Q-function, we proceed to study the quantum and statistical properties of the two-mode cavity light coupled to a two-mode squeezed vacuum reservoir via a single port-mirror.

The photon annihilation and creation operators obey the commutation rule

$$[\hat{a}, \hat{a}^{\dagger}] = 1, \tag{3.1.1}$$

and their normally-ordered product

$$\hat{N} = \hat{a}^{\dagger}\hat{a}, \tag{3.1.2}$$

has the meaning of the photon number operator [4].

On the basis of this definition, the photon number operators for mode a and mode b are defined by

$$\hat{n}_a = \hat{a}^\dagger \hat{a}, \tag{3.1.3}$$

$$\hat{n}_b = \hat{b}^\dagger \hat{b}. \tag{3.1.4}$$

On the other hand, the photon number sum and difference is defined by

$$\hat{n}_\pm = \hat{n}_a \pm \hat{n}_b. \tag{3.1.5}$$

The mean of the photon number sum and difference can be expressed in terms of the Q-function as

$$\bar{n}_\pm = \bar{n}_a \pm \bar{n}_b = \int d^2\alpha Q(\alpha^*, \alpha, t) n_a(\alpha) \pm \int d^2\beta Q(\beta^*, \beta, t) n_b(\beta), \tag{3.1.6}$$

where $n_a(\alpha) = \alpha\alpha^* - 1$ and $n_b(\beta) = \beta\beta^* - 1$ are the c-number variables corrosponding to the operators $\hat{n}_a$ and $\hat{n}_b$ in the antinormal order.

With the aid of (2.1.119) and integrating (2.1.120) over β, one obtains

$$Q(\alpha^*, \alpha, t) = \int d^2\beta Q(\alpha, \beta, t) = \frac{1}{l\pi} exp[-\frac{1}{l}\alpha\alpha^* + \frac{E}{l}\alpha + \frac{E}{l}\alpha^* + \frac{u^2}{a} + C]. \tag{3.1.7}$$

We immediately notice that, the mean photon number for mode a is given by

$$\bar{n}_a = \int d^2\alpha Q(\alpha^*, \alpha) n_a(\alpha) = \int d^2\alpha Q(\alpha^*, \alpha, t)[\alpha\alpha^* - 1]. \tag{3.1.8}$$

It then follows

$$\bar{n}_a = l + E^2 - 1. \tag{3.1.9}$$

Substituting Eqs. (2.1.85) and (2.1.117) into (3.1.9) directly shows that

$$\bar{n}_a = N - Ne^{-\kappa t} + \frac{4\varepsilon^2}{\kappa^2}[1 - e^{-\kappa t}]^2. \tag{3.1.10}$$

Therefore, the steady-state mean photon number for mode a turns out to be

$$\bar{n}_a = N + \frac{4\varepsilon^2}{\kappa^2}. \tag{3.1.11}$$

Similarly, upon integrating Eq. (2.1.120) over α and taking into account (2.1.119), we readily obtain

$$Q(\beta^*,\beta,t)=\int d^2\alpha Q(\alpha,\beta,t)=\frac{1}{l\pi}exp[-\frac{1}{l}\beta\beta^*+\frac{E}{l}\beta+\frac{E}{l}\beta^*+\frac{u^2}{a}+C], \quad (3.1.12)$$

and on the basis of this result, the mean photon number for mode b can be expressed as

$$\bar{n}_b=\int d^2\beta Q(\beta^*,\beta,t)n_b(\beta)=\int d^2\beta Q(\beta^*,\beta,t)[\beta\beta^*-1]. \quad (3.1.13)$$

As before carying out the necessary integration yeilds

$$\bar{n}_b=l+E^2-1, \quad (3.1.14)$$

and using Eq. (2.1.85) along with (2.1.117), it can be verified that

$$\bar{n}_b=N[1-e^{-\kappa t}]+\frac{4\varepsilon^2}{\kappa^2}[1-e^{-\kappa t}]^2. \quad (3.1.15)$$

Moreover, using the steady-state condition, there follows

$$\bar{n}_b=N+\frac{4\varepsilon^2}{\kappa^2}, \quad (3.1.16)$$

We observe that mode a and mode b have the same mean photon numbers.

Using Eqs. (3.1.10), (3.1.15), and taking the expectation value of Eq. (3.1.5), the mean of the photon number sum and difference can be expressed in the form

$$\bar{n}_\pm=[N(1-e^{-\kappa t})+\frac{4\varepsilon^2}{\kappa^2}(1-e^{-\kappa t})^2]\pm[N(1-e^{-\kappa t})+\frac{4\varepsilon^2}{\kappa^2}(1-e^{-\kappa t})^2]. \quad (3.1.17)$$

Hence, the sum of the mean photon number is found to be

$$\bar{n}_+=2[N(1-e^{-\kappa t})+\frac{4\varepsilon^2}{\kappa^2}(1-e^{-\kappa t})^2], \quad (3.1.18)$$

and at the steady-state, the sum of the mean photon number reduces to

$$\bar{n}_+=2[N+\frac{4\varepsilon^2}{\kappa^2}]. \quad (3.1.19)$$

Similarly, the mean of the photon number difference turns out to be

$$\bar{n}_-=0. \quad (3.1.20)$$

3.2 The variance of the photon number sum and difference

We now proceed to determine the variance of the photon number sum and difference. The variance of the photon number sum and difference is expressible in the form

$$(\Delta n_{\pm})^2 = \langle \hat{n}_{\pm}^2 \rangle - \langle \hat{n}_{\pm} \rangle^2. \tag{3.2.1}$$

Upon taking the expectation value for expression (3.1.5), we obtain

$$\langle \hat{n}_{\pm}^2 \rangle = \langle (\hat{n}_a \pm \hat{n}_b)^2 \rangle = \langle \hat{n}_a^2 \rangle + \langle \hat{n}_b^2 \rangle \pm 2\langle \hat{n}_a \hat{n}_b \rangle, \tag{3.2.2}$$

and

$$\langle \hat{n}_{\pm} \rangle^2 = \langle \hat{n}_a \pm \hat{n}_b \rangle^2 = \bar{n}_{\pm}^2 = \bar{n}_a^2 + \bar{n}_b^2 \pm 2\bar{n}_a \bar{n}_b. \tag{3.2.3}$$

Substituting of Eqs. (3.2.2) and (3.2.3) into (3.2.1), one gets

$$(\Delta n_{\pm})^2 = \langle \hat{n}_a^2 \rangle - \langle \hat{n}_a \rangle^2 + \langle \hat{n}_b^2 \rangle - \langle \hat{n}_b \rangle^2 \pm 2[\langle \hat{n}_a \hat{n}_b \rangle - \langle \hat{n}_a \rangle \langle \hat{n}_b \rangle]. \tag{3.2.4}$$

This can be also rewritten in the form

$$(\Delta n_{\pm})^2 = (\Delta n_a)^2 + (\Delta n_b)^2 \pm 2[\langle \hat{n}_a \hat{n}_b \rangle - \bar{n}_a \bar{n}_b], \tag{3.2.5}$$

where

$$(\Delta n_a)^2 = \langle \hat{n}_a^2 \rangle - \langle \hat{n}_a \rangle^2, \tag{3.2.6}$$

$$(\Delta n_b)^2 = \langle \hat{n}_b^2 \rangle - \langle \hat{n}_b \rangle^2, \tag{3.2.7}$$

are the photon number variance for the separate modes. On the basis of Eq. (3.1.1), the photon number operator for mode a in the antinormal-order is obtained as

$$\hat{n}_a = \hat{a}^\dagger \hat{a} = \hat{a}\hat{a}^\dagger - 1, \tag{3.2.8}$$

which implies

$$\langle \hat{n}_a^2 \rangle = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - 3\langle \hat{a}\hat{a}^\dagger \rangle + 1. \tag{3.2.9}$$

On account of (3.1.1), the above expression can be rewritten as

$$\langle \hat{n}_a^2 \rangle = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - 3\langle \hat{a}^\dagger \hat{a} \rangle - 2, \tag{3.2.10}$$

or

$$\langle \hat{n}_a^2 \rangle = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - 3\bar{n}_a - 2. \tag{3.2.11}$$

Substituting of Eqs. (3.2.6) and (3.2.11) implies

$$(\Delta n_a)^2 = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - \bar{n}_a^2 - 3\bar{n}_a - 2. \tag{3.2.12}$$

The first term on the right side of Eq. (3.2.12) is expressible in terms of the Q-function for the two-mode coherent state as

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \int d^2\alpha d^2\beta Q(\alpha, \beta, t)\alpha^2 \alpha^{*2}, \tag{3.2.13}$$

where α^2 and α^{*^2} are c-number variables corrosponding to the operators $\hat{a}^2$ and $\hat{a}^{\dagger 2}$. On the basis of (2.1.120), we have

$$\begin{aligned} \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle \quad &= \frac{R}{\pi^2} e^C \int d^2\alpha \Big[exp(-a\alpha\alpha^* + u\alpha + u\alpha^*)\alpha^2\alpha^{*2} \\ &\times \int d^2\beta exp[-a\beta\beta^* + (u + v\alpha)\beta + (u + v\alpha^*)\beta^*] \Big]. \end{aligned} \tag{3.2.14}$$

Now with the aid of (2.1.119), first carying out the integration over β implies

$$\begin{aligned} \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle \quad &= \frac{R}{\pi} e^{C + \frac{u^2}{a}} \int d^2\alpha exp\Big[- [a - \frac{v^2}{a}]\alpha\alpha^* + [u + \frac{uv}{a}]\alpha \\ &+ [u + \frac{uv}{a}]\alpha^* \Big] \alpha^2\alpha^{*2}. \end{aligned} \tag{3.2.15}$$

Upon substituting Eqs. (2.1.121), (2.1.122), (2.1.123), (2.1.124), (2.1.125) into (3.2.15), we can rewrite the above expression as

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{1}{l\pi} e^{-E^2/l} \int d^2\alpha exp[-\frac{1}{l}\alpha\alpha^* + \frac{E}{l}\alpha + \frac{E}{l}\alpha^*]\alpha^2\alpha^{*2}, \tag{3.2.16}$$

or, in other way it can also be written as

$$\langle \hat{a}^2\hat{a}^{\dagger 2}\rangle = \frac{l}{\pi}e^{-E^2/l}\frac{d^2}{d\omega^2}\int d^2\alpha exp[-\frac{1}{l}\omega\alpha\alpha^* + \frac{E}{l}\alpha + \frac{E}{l}\alpha^*]\Big|_{\omega=1}. \tag{3.2.17}$$

Considering the relation (2.1.119) and performing the integration over α shows that

$$\langle \hat{a}^2\hat{a}^{\dagger 2}\rangle = l^2\frac{d^2}{d\omega^2}\left[\frac{1}{\omega}exp[-\frac{E^2}{l} + \frac{E^2}{l\omega}]\right]\Big|_{\omega=1}. \tag{3.2.18}$$

Carying out the differentiation and applying the condition ω=1 yeilds

$$\langle \hat{a}^2\hat{a}^{\dagger 2}\rangle = 2l^2 + 4lE^2 + E^4. \tag{3.2.19}$$

Similarly

$$\langle \hat{b}^2\hat{b}^{\dagger 2}\rangle = 2l^2 + 4lE^2 + E^4. \tag{3.2.20}$$

If we again substitute Eqs. (2.1.85) and (2.1.117), this expression takes the form

$$\begin{aligned}\langle \hat{a}^2\hat{a}^{\dagger 2}\rangle \quad &= 2 + 4N[1 - e^{-\kappa t}] + 2N^2[1 - e^{-\kappa t}] + \frac{16\varepsilon^2}{\kappa^2}[1 - e^{-\kappa t}]^2 \\ &+\frac{16\varepsilon^2}{\kappa^2}N[1 - e^{-\kappa t}]^3 + \frac{16\varepsilon^4}{\kappa^4}[1 - e^{-\kappa t}]^4.\end{aligned} \tag{3.2.21}$$

At the steady-state condition, this result reduces to

$$\langle \hat{a}^2\hat{a}^{\dagger 2}\rangle \quad = 2 + 4N + 2N^2 + \frac{16\varepsilon^2}{\kappa^2} + \frac{16\varepsilon^2}{\kappa^2}N + \frac{16\varepsilon^4}{\kappa^4}. \tag{3.2.22}$$

Therefore, upon substituting Eqs. (3.1.9)and (3.2.19) into (3.2.12), the variance of the photon number for mode a takes the form

$$(\Delta n_a)^2 = l^2 + 2lE^2 - l - E^2. \tag{3.2.23}$$

In view of Eqs. (2.1.85) and (2.1.117), the above expression becomes

$$(\Delta n_a)^2 = N[1 - e^{-\kappa t}] + N^2[1 - e^{-\kappa t}]^2 + \frac{8\varepsilon^2}{\kappa^2}N[1 - e^{-\kappa t}]^3 + \frac{4\varepsilon^2}{\kappa^2}[1 - e^{-\kappa t}]^2, \tag{3.2.24}$$

and at the steady-state condition, the above equation reduces to

$$(\Delta n_a)^2 = N + N^2 + \frac{8\varepsilon^2}{\kappa^2}N + \frac{4\varepsilon^2}{\kappa^2}. \tag{3.2.25}$$

In similar way, following the same procedure and condition, one can obtain the photon number variance for mode b as

$$(\Delta n_b)^2 = N + N^2 + \frac{8\varepsilon^2}{\kappa^2}N + \frac{4\varepsilon^2}{\kappa^2}, \tag{3.2.26}$$

which also shows that the separate modes have the same photon number variance. Furthermore, the expectation value of $\hat{n}_a\hat{n}_b$ in terms of the $Q(\alpha,\beta,t)$ function is expressed as

$$\langle\hat{n}_a\hat{n}_b\rangle = \int d^2\alpha d^2\beta Q(\alpha,\beta,t)n_a(\alpha)n_b(\beta), \tag{3.2.27}$$

where

$$n_a(\alpha) = \alpha\alpha^* - 1, \tag{3.2.28}$$

$$n_b(\beta) = \beta\beta^* - 1, \tag{3.2.29}$$

are the c-number functions corrosponding to the operators $\hat{n}_a$ and $\hat{n}_b$ in the antinormal-order. It then follows

$$\begin{aligned}\langle\hat{n}_a\hat{n}_b\rangle \quad &= \int d^2\alpha d^2\beta Q(\alpha,\beta,t)\alpha\alpha^*\beta\beta^* - \int d^2\alpha d^2\beta Q(\alpha,\beta,t)\alpha\alpha^* \\ &- \int d^2\alpha d^2\beta Q(\alpha,\beta,t)\beta\beta^* - \int d^2\alpha d^2\beta Q(\alpha,\beta,t).\end{aligned} \tag{3.2.30}$$

Now using Eq. (2.1.120), solving this expression separately

$$\begin{aligned}\int d^2\alpha d^2\beta Q(\alpha,\beta,t)\alpha\alpha^*\beta\beta^* \quad &= \frac{R}{\pi^2}\int d^2\alpha d^2\beta exp[-a\alpha\alpha^* + u\alpha + u\alpha^* - a\beta\beta^* \\ &+u\beta + u\beta^* + v\alpha\beta + v\alpha\beta^* + C]\alpha\alpha^*\beta\beta^*.\end{aligned} \tag{3.2.31}$$

For simplicity, first solve for β and β^* related expressions

$$\int d^2\alpha d^2\beta Q(\alpha,\beta,t)\alpha\alpha^*\beta\beta^* = -\frac{R}{a\pi^2}\int d^2\alpha\Big[exp[-a\alpha\alpha^* + u\alpha + u\alpha^* + C] \times\frac{d}{d\omega}\int d^2\beta exp[-\omega a\beta\beta^* + (u+v\alpha)\beta + (u+v\alpha^*)\beta^*]\alpha\alpha^*\Big]_{\omega=1}. \quad (3.2.32)$$

On the basis of (2.1.119), this expression becomes

$$\int d^2\alpha d^2\beta Q(\alpha,\beta,t)\alpha\alpha^*\beta\beta^* = -\frac{R}{a^2\pi}\int d^2\alpha\Big[exp[-a\alpha\alpha^* + u\alpha + u\alpha^* + C]\alpha\alpha \times\frac{d}{d\omega}[\frac{1}{\omega}exp(\frac{u^2 + uv\alpha + uv\alpha^* + v^2\alpha\alpha^*}{\omega a})]\Big]_{\omega=1}. \quad (3.2.33)$$

Peforming the differentition and apply the condition $\omega = 1$ provides

$$\int d^2\alpha d^2\beta Q(\alpha,\beta,t)\alpha\alpha^*\beta\beta^* = \frac{R}{a^3\pi}\int d^2\alpha\Big[exp[-a\alpha\alpha^* + u\alpha + u\alpha^* + C]\alpha\alpha^* \times[exp(\frac{u^2 + uv\alpha + uv\alpha^* + v^2\alpha\alpha^*}{a})] \times[a + u^2 + uv\alpha + uv\alpha^* + v^2\alpha\alpha^*]\Big]. \quad (3.2.34)$$

By collecting related terms, one obtains

$$\int d^2\alpha d^2\beta Q(\alpha,\beta,t)\alpha\alpha^*\beta\beta^* = \frac{R}{a^3\pi}\int d^2\alpha\Big[exp[-(a - \frac{v^2}{a})\alpha\alpha^* + (u + \frac{uv}{a})\alpha + (u + \frac{uv}{a})\alpha^* + C]\alpha\alpha^* \times[a + u^2 + uv\alpha + uv\alpha^* + v^2\alpha\alpha^*]\Big]. \quad (3.2.35)$$

We can also rewrite this as

$$\int d^2\alpha d^2\beta Q(\alpha,\beta,t)\alpha\alpha^*\beta\beta^* = \frac{R}{a^3\pi}\int d^2\alpha exp[-x\alpha\alpha^* + b\alpha + b\alpha^* + C]\alpha\alpha^* \times[a + u^2 + uv\alpha + uv\alpha^* + v^2\alpha\alpha^*], \quad (3.2.36)$$

which also can be rewritten as

$$\int d^2\alpha d^2\beta Q(\alpha,\beta,t)\alpha\alpha^*\beta\beta^* = -\frac{R}{xa^3\pi}e^{C+\frac{u^2}{a}}\frac{d}{d\omega}\Bigg[(a+u^2)\int d^2\alpha exp[-\omega x\alpha\alpha^*+b\alpha+b\alpha^*] +uv\int d^2\alpha exp[-\omega x\alpha\alpha^*+b\alpha+b\alpha^*]\alpha+uv\int d^2\alpha exp[-\omega x\alpha\alpha^* +b\alpha+b\alpha^*]\alpha^*+v^2\int d^2\alpha exp[-\omega x\alpha\alpha^*+b\alpha+b\alpha^*]\alpha\alpha^* \times[a+u^2+uv\alpha+uv\alpha^*+v^2\alpha\alpha^*]\Bigg]_{\omega=1}, \tag{3.2.37}$$

where

$$x = a - \frac{v^2}{a}, \tag{3.2.38}$$

$$b = u + \frac{uv}{a}. \tag{3.2.39}$$

Or

$$\int d^2\alpha d^2\beta Q(\alpha,\beta,t)\alpha\alpha^*\beta\beta^* = -\frac{R}{xa^3\pi}e^{C+\frac{u^2}{a}}\frac{d}{d\omega}\Bigg[(a+u^2)\int d^2\alpha exp[-\omega x\alpha\alpha^*+b\alpha+b\alpha^*]+ \frac{uv}{b}\frac{d}{d\lambda}\int d^2\alpha exp[-\omega x\alpha\alpha^*+\lambda b\alpha+b\alpha^*]\alpha +\frac{uv}{b}\frac{d}{d\lambda}\int d^2\alpha exp[-\omega x\alpha\alpha^*+b\alpha+\lambda b\alpha^*] -\frac{v^2}{x\omega}\frac{d}{d\lambda}\int d^2\alpha exp[-\lambda\omega x\alpha\alpha^*+b\alpha+b\alpha^*] \times[a+u^2+uv\alpha+uv\alpha^*+v^2\alpha\alpha^*]\Bigg]_{\omega=\lambda=1}. \tag{3.2.40}$$

On the basis of Eq. (2.1.119) carying out the necessary integration, differenciatin respectively and apply the condition $\omega = \lambda = 1$ and finally equate the values of a, b, C, x and R gives

$$\int d^2\alpha d^2\beta Q(\alpha,\beta,t)\alpha\alpha^*\beta\beta^* = l^2+2lE^2+m^2+2mE^2+E^4. \tag{3.2.41}$$

We can also see from the second term of Eq. (3.2.30) that

$$\int d^2\alpha d^2\beta Q(\alpha,\beta,t)\alpha\alpha^* = \frac{R}{\pi^2}\int d^2\alpha d^2\beta exp[-a\alpha\alpha^* + u\alpha + u\alpha^* - a\beta\beta^* + (u+v\alpha)\beta + (u+v\alpha^*)\beta^* + C]\alpha\alpha^*, \tag{3.2.42}$$

or

$$\int d^2\alpha d^2\beta Q(\alpha,\beta,t)\alpha\alpha^* = -\frac{R}{a\pi^2}\int d^2\beta\Big[exp(-a\beta\beta^* + u\beta + u\beta^* + C) \frac{d}{d\omega}\int d^2\alpha exp[-\omega a\alpha\alpha^* + (u+v\beta)\alpha + (u+v\beta^*)\alpha^*\Big]_{\omega=1},$$

$$= -\frac{R}{a^2\pi}\int d^2\beta\Big[exp[-a\beta\beta^* + u\beta + u\beta^* + C] \times\frac{d}{d\omega}[\frac{1}{\omega}exp(\frac{u^2+uv\alpha+uv\alpha^*+v^2\alpha\alpha}{a\omega})]\Big]_{\omega=1}, \tag{3.2.43}$$

$$\int d^2\alpha d^2\beta Q(\alpha,\beta,t)\alpha\alpha^* = \frac{R}{a^3\pi}e^{C+\frac{u^2}{a}}\int d^2\beta\Big[exp[-x\beta\beta^* + b\beta + b\beta^*] \times[a+u^2+uv\beta+uv\beta^*+v^2\beta\beta^*]\Big]. \tag{3.2.44}$$

Employing the differentiation with respect to ω where necessary and collecting similar terms respectively yeilds

$$\int d^2\alpha d^2\beta Q(\alpha,\beta,t)\alpha\alpha^* = \frac{R}{a^3\pi}e^{C+\frac{u^2}{a}}\Big[(a+u^2)\int d^2\beta exp(-x\beta\beta^* + b\beta + b\beta^*) + \frac{uv}{b}\frac{d}{d\omega}\int d^2\beta exp(-x\beta\beta^* + \omega b\beta + b\beta^*) + \frac{uv}{b}\frac{d}{d\omega}\int d^2\beta exp(-x\beta\beta^* + b\beta + \omega b\beta^*) + \frac{d}{d\omega}\int d^2\alpha exp[-\omega a\alpha\alpha^* + (u+v\beta) - \frac{v^2}{x}\frac{d}{d\omega}\int d^2\beta exp(-\omega x\beta\beta^* + b\beta + b\beta^*)\Big]_{\omega=1}. \tag{3.2.45}$$

Using the relation (2.1.119), carying out the integration, differentiation and applying the condition $\omega = 1$ simplifies the above expression to

$$\int d^2\alpha d^2\beta Q(\alpha, \beta, t)\alpha\alpha^* = l + E^2. \tag{3.2.46}$$

Similarly

$$\int d^2\alpha d^2\beta Q(\alpha, \beta, t)\beta\beta^* = l + E^2. \tag{3.2.47}$$

Now inserting Eqs. (3.2.41), (3.2.46), (3.2.47) and (2.1.126) into (3.2.30) leads to

$$\langle \hat{n}_a \hat{n}_b \rangle = l^2 + 2lE^2 + 2mE^2 + m^2 + E^4 - 2(l + E^2) + 1. \tag{3.2.48}$$

By substituting Eqs. (2.1.85), (2.1.117), and (2.1.118), this result could also be rewritten as

$$\langle \hat{n}_a \hat{n}_b \rangle = (N^2 + M^2)[1 - e^{-\kappa t}]^2 + \frac{8\varepsilon^2}{\kappa^2}(N - M)[1 - e^{-\kappa t}]^3 + \frac{16\varepsilon^4}{\kappa^4}[1 - e^{-\kappa t}]^4. \tag{3.2.49}$$

And on the basis of the steady-state condition, one can directly show

$$\langle \hat{n}_a \hat{n}_b \rangle = N^2 + M^2 + \frac{8\varepsilon^2}{\kappa^2}(N - M) + \frac{16\varepsilon^4}{\kappa^4}, \tag{3.2.50}$$

and using Eqs. (3.1.11) and (3.1.16), we have

$$\bar{n}_a \bar{n}_b = N^2 + \frac{8\varepsilon^2}{\kappa^2}N + \frac{16\varepsilon^4}{\kappa^4}. \tag{3.2.51}$$

Lastly, in view of Eqs. (3.2.25), (3.2.26), (3.2.50), (3.2.51) and (3.2.5), the variance of the photon number sum and difference becomes

$$(\Delta n_{\pm})^2 = 2\left[N + N^2 + \frac{4\varepsilon^2}{\kappa^2}N \pm [M^2 - \frac{8\varepsilon^2}{\kappa^2}M]\right]. \tag{3.2.52}$$

3.3 The photon number correlation

The photon number correlation is defined by

$$g^2_{ab}(0) = \frac{\langle \hat{n}_a \hat{n}_b \rangle}{\bar{n}_a \bar{n}_b}. \tag{3.3.1}$$

Thus, on account of (3.2.50) and (3.2.51), we obtain

$$g_{ab}^{2}(0) = \frac{N^2 + M^2 + \frac{8\varepsilon^2}{\kappa^2}(N - M) + \frac{16\varepsilon^4}{\kappa^4}}{N^2 + \frac{8\varepsilon^2}{\kappa^2}N + \frac{16\varepsilon^4}{\kappa^4}}. \tag{3.3.2}$$

In general, the photon numbers $\hat{n}_a$ and $\hat{n}_b$ are not correlated if

$$g_{ab}^{2}(0) = 1, \tag{3.3.3}$$

and they are correlated if

$$g_{ab}^{2}(0) \neq 1. \tag{3.3.4}$$

Chapter 4

The quadrature variance

4.1 The plus and minus quadratures

The squeezing properties of a two-mode light are described by two quadrature operators defined by

$$\hat{C}_+ = \hat{C} + \hat{C}^\dagger, \tag{4.1.1}$$

$$\hat{C}_- = i(\hat{C}^\dagger - \hat{C}). \tag{4.1.2}$$

Where

$$\hat{C} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{b}), \tag{4.1.3}$$

$$\hat{C}^\dagger = \frac{1}{\sqrt{2}}(\hat{a}^\dagger + \hat{b}^\dagger). \tag{4.1.4}$$

In view of (4.1.3) and (4.1.4), one can rewrite (4.1.1) and (4.2.2) as

$$\hat{C}_+ = \frac{1}{\sqrt{2}}(\hat{a}_+ + \hat{b}_+), \tag{4.1.5}$$

$$\hat{C}_- = \frac{1}{\sqrt{2}}(\hat{a}_- + \hat{b}_-), \tag{4.1.6}$$

in which $\hat{a}_+$,$\hat{a}_-$,$\hat{b}_+$ and $\hat{b}_-$ are defined by

$$\hat{a}_+ = \hat{a} + \hat{a}^\dagger, \tag{4.1.7}$$

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}), \tag{4.1.8}$$

and

$$\hat{b}_+ = \hat{b} + \hat{b}^\dagger, \tag{4.1.9}$$

$$\hat{b}_{-} = i(\hat{b}^{\dagger} - \hat{b}). \tag{4.1.10}$$

On the basis of (2.1.3), we can show that

$$[\hat{C}, \hat{C}^{\dagger}] = 1, \tag{4.1.11}$$

and the operators $\hat{C}_{+}$ and $\hat{C}_{-}$ are Hermitian and satisfy the commutation relation

$$[\hat{C}_{+}, \hat{C}_{-}] = 2i. \tag{4.1.12}$$

A two-mode light is said to be in a squeezed state if $\Delta C_{+} < 1$ or $\Delta C_{-} < 1$ such that $\Delta C_{+} \Delta C_{-} \geq 1$.

We now prceed to determine the quadrature variance. The quadrature variance for the plus and minus quadrature is defined as

$$(\Delta C_{\pm})^2 = \langle C_{\pm}^2 \rangle - \langle C_{\pm} \rangle^2. \tag{4.1.13}$$

Hence in view of Eqs. (4.1.5) and (4.1.6), we see that

$$\begin{aligned} \langle \hat{C}_{\pm}^2 \rangle \quad &= \pm\frac{1}{2}[\langle \hat{a}^2 \rangle + \langle \hat{a}^{\dagger 2} \rangle + \langle \hat{b}^2 \rangle + \langle \hat{b}^{\dagger 2} \rangle] + \langle \hat{a}^{\dagger} \hat{a} \rangle + \langle \hat{b}^{\dagger} \hat{b} \rangle + \langle \hat{a}^{\dagger} \hat{b} \rangle \\ &+ \langle \hat{a} \hat{b}^{\dagger} \rangle \pm \langle \hat{a}^{\dagger} \hat{b}^{\dagger} \rangle \pm \langle \hat{a} \hat{b} \rangle + 1, \end{aligned} \tag{4.1.14}$$

and

$$\begin{aligned} \langle \hat{C}_{\pm} \rangle^2 \quad &= \pm\frac{1}{2}[\langle \hat{a}^2 \rangle + \langle \hat{a}^{\dagger 2} \rangle + \langle \hat{b}^2 \rangle + \langle \hat{b}^{\dagger 2} \rangle] + \langle \hat{a} \rangle \langle \hat{a}^{\dagger} \rangle + \langle \hat{b} \rangle \langle \hat{b}^{\dagger} \rangle + \langle \hat{a}^{\dagger} \rangle \langle \hat{b} \rangle \\ &+ \langle \hat{a} \rangle \langle \hat{b}^{\dagger} \rangle + \pm \langle \hat{a}^{\dagger} \rangle \langle \hat{b}^{\dagger} \rangle \pm \langle \hat{a} \rangle \langle \hat{b} \rangle. \end{aligned} \tag{4.1.15}$$

So that combination of (4.1.13), (4.1.14) and (4.1.15) leads to

$$\begin{aligned} (\Delta C_{\pm})^2 \quad &= \frac{1}{2}\left[(\Delta a_{\pm})^2 + (\Delta b_{\pm})^2\right] + \langle \hat{a} \hat{b}^{\dagger} \rangle - \langle \hat{a} \rangle \langle \hat{b}^{\dagger} \rangle + \langle \hat{a}^{\dagger} \hat{b} \rangle - \langle \hat{a}^{\dagger} \rangle \langle \hat{b} \rangle \\ &\pm \langle \hat{a} \hat{b} \rangle \mp \langle \hat{a} \rangle \langle \hat{b} \rangle \pm \langle \hat{a}^{\dagger} \hat{b}^{\dagger} \rangle \mp \langle \hat{a}^{\dagger} \rangle \langle \hat{b}^{\dagger} \rangle. \end{aligned} \tag{4.1.16}$$

Where

$$(\Delta a_+)^2 = \langle \hat{a}^2 \rangle - \langle \hat{a} \rangle^2 + \langle \hat{a}^{\dagger 2} \rangle - \langle a^\dagger \rangle^2 + 2\langle \hat{a}^\dagger \hat{a} \rangle - 2\langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle + 1, \tag{4.1.17}$$

$$(\Delta a_-)^2 = \langle \hat{a} \rangle^2 - \langle \hat{a}^2 \rangle + \langle \hat{a}^\dagger \rangle^2 - \langle a^{\dagger 2} \rangle + 2\langle \hat{a}^\dagger \hat{a} \rangle - 2\langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle + 1, \tag{4.1.18}$$

with similar definitions for $(\Delta b_+)^2$ and $(\Delta b_-)^2$.

We next seek to calculate, with the aid of the Q-function, the expectation value of $\hat{a}$ for the two-mode coherent state. It is expressible as

$$\langle \hat{a} \rangle = \int d^2\alpha d^2\beta Q(\alpha, \beta, t)\alpha. \tag{4.1.19}$$

This can be rewriten on account of Eq. (2.1.120) as

$$\begin{aligned} \langle \hat{a} \rangle \quad &= \frac{R}{\pi^2} \int d^2\alpha \Big[exp(-a\alpha\alpha^* + u\alpha + u\alpha + C)\alpha \\ &\times \int d^2\beta exp[-a\beta\beta^* + (u + v\alpha)\beta + (u + v\alpha^*)\beta^*] \Big]. \end{aligned} \tag{4.1.20}$$

And with the aid of (2.1.119), by integrating this expression first over β and collecting similar terms, we can find

$$\langle \hat{a} \rangle = \frac{R}{a\pi^2} e^{C + \frac{u^2}{a}} \int d^2\alpha exp[-a\alpha\alpha^* + b\alpha + b\alpha^*]\alpha. \tag{4.1.21}$$

This result can also be rewriten as

$$\langle \hat{a} \rangle = \frac{R}{ab\pi} e^{C + \frac{u^2}{a}} \frac{d}{d\omega} \int d^2\alpha exp[-\omega a\alpha\alpha^* + b\alpha + b\alpha^*] \Big|_{\omega=1}. \tag{4.1.22}$$

Furthermore, integrating this expression over α and carrying out the differentiation with respect to ω and finally applying the condition $\omega = 1$ indicates that

$$\langle \hat{a} \rangle = \frac{Rb}{ax^2} exp[C + \frac{u^2}{a} + \frac{b^2}{x}]. \tag{4.1.23}$$

Finally, substituting Eqs. (2.1.121), (2.1.122), (2.1.123), (2.1.125), (3.2.39) into this result and making use of (2.1.85), there follows

$$\langle \hat{a} \rangle = E = \frac{2\varepsilon}{\kappa}[1 - e^{-\kappa t}]. \tag{4.1.24}$$

Similarly

$$\langle\hat{b}\rangle = E = \frac{2\varepsilon}{\kappa}[1 - e^{-\kappa t}], \tag{4.1.25}$$

$$\langle\hat{a}^{\dagger}\rangle = E = \frac{2\varepsilon}{\kappa}[1 - e^{-\kappa t}], \tag{4.1.26}$$

$$\langle\hat{b}^{\dagger}\rangle = E = \frac{2\varepsilon}{\kappa}[1 - e^{-\kappa t}]. \tag{4.1.27}$$

In adition, with the aid of the Q-function, we can also write

$$\langle\hat{a}^2\rangle = \int d^2\alpha d^2\beta Q(\alpha,\beta,t)\alpha^2. \tag{4.1.28}$$

Where in view of (2.1.120), this result can be written in the form

$$\begin{aligned}\langle\hat{a}^2\rangle \qquad &= \frac{R}{\pi^2}\int d^2\alpha\Big[exp(-a\alpha\alpha^* + u\alpha + u\alpha + C)\alpha^2 \\ &\times \int d^2\beta exp[-a\beta\beta^* + (u+v\alpha)\beta + (u+v\alpha^*)\beta^*]\Big].\end{aligned} \tag{4.1.29}$$

As before, carrying out the integration over β and collecting similar terms yields

$$\langle\hat{a}^2\rangle = \frac{R}{a\pi}e^{C+\frac{u^2}{a}}\int d^2\alpha exp[-a\alpha\alpha^* + b\alpha + b\alpha^*]\alpha^2. \tag{4.1.30}$$

It then follows

$$\langle\hat{a}^2\rangle = \frac{R}{ab^2\pi}e^{C+\frac{u^2}{a}}\frac{d^2}{d\omega^2}\int d^2\alpha exp[-a\alpha\alpha^* + \omega b\alpha + b\alpha^*]\Big|_{\omega=1}. \tag{4.1.31}$$

Upon carrying out the integration, differentiation and applying $\omega = 1$ condition and finally inserting the values of the variables, we obtain

$$\langle\hat{a}^2\rangle = E^2 = \frac{4\varepsilon^2}{\kappa^2}[1 - e^{-\kappa t}]^2. \tag{4.1.32}$$

In the same fashion, using the same procedures, one reaily obtains

$$\langle\hat{b}^2\rangle = E^2 = \frac{4\varepsilon^2}{\kappa^2}[1 - e^{-\kappa t}]^2, \tag{4.1.33}$$

$$\langle\hat{a}^{\dagger 2}\rangle = E^2 = \frac{4\varepsilon^2}{\kappa^2}[1 - e^{-\kappa t}]^2, \tag{4.1.34}$$

$$\langle\hat{b}^{\dagger 2}\rangle = E^2 = \frac{4\varepsilon^2}{\kappa^2}[1-e^{-\kappa t}]^2. \tag{4.1.35}$$

Again, the expectation value of $\hat{a}^\dagger\hat{a}$ in terms of its corrosponding c-number variables in the antinormal-order and the Q-function is expressed as

$$\langle\hat{a}^\dagger\hat{a}\rangle = \int d^2\alpha d^2\beta Q(\alpha,\beta,t)(\alpha\alpha^* - 1). \tag{4.1.36}$$

Thus, in view of (2.1.120), which also implies

$$\begin{aligned}\langle\hat{a}^\dagger\hat{a}\rangle \quad &= \frac{R}{\pi^2}\int d^2\alpha\Bigg[exp(-a\alpha\alpha^* + u\alpha + u\alpha^* + C)\\ &\times\int d^2\beta exp[-a\beta\beta^* + (u+v\alpha)\beta + (u+v\alpha^*)\beta^*]\alpha^*\alpha\Bigg] - 1.\end{aligned} \tag{4.1.37}$$

Performing the integration over β and collecting similar terms gives

$$\langle\hat{a}^\dagger\hat{a}\rangle = \frac{R}{a\pi}e^{C+\frac{u^2}{a}}\int d^2\alpha exp[-a\alpha\alpha^* + b\alpha + b\alpha^*]\alpha\alpha^* - 1, \tag{4.1.38}$$

or

$$\langle\hat{a}^\dagger\hat{a}\rangle = -\frac{R}{ax\pi}e^{C+\frac{u^2}{a}}\frac{d}{d\omega}\int d^2\alpha exp[-\omega a\alpha\alpha^* + b\alpha + b\alpha^*]\Bigg|_{\omega=1} - 1. \tag{4.1.39}$$

After integrating, carrying out the differentiation and applying $\omega = 1$ condition and finally substituting the values of R, C, a, b, x, and u implies

$$\langle\hat{a}^\dagger\hat{a}\rangle = l + E^2 - 1 = N[1-e^{-\kappa t}] + \frac{4\varepsilon^2}{\kappa^2}[1-e^{-\kappa t}]^2, \tag{4.1.40}$$

$$\langle\hat{b}^\dagger\hat{b}\rangle = l + E^2 - 1 = N[1-e^{-\kappa t}] + \frac{4\varepsilon^2}{\kappa^2}[1-e^{-\kappa t}]^2. \tag{4.1.41}$$

Lastly, employing the Q-function, the expectation value of $\hat{a}\hat{b}$ along with their corrosponding c-number variables is expressed as

$$\langle\hat{a}\hat{b}\rangle = \int d^2\alpha d^2\beta Q(\alpha,\beta,t)\alpha\beta. \tag{4.1.42}$$

On the basis of (2.1.120), we get

$$\begin{aligned}\langle \hat{a}\hat{b}\rangle &= \frac{R}{\pi^2}\int d^2\alpha\Big[exp(-a\alpha\alpha^* + u\alpha + u\alpha + C)\\ &\times\int d^2\beta exp[-a\beta\beta^* + (u+v\alpha)\beta + (u+v\alpha^*)\beta^*]\alpha\beta\Big].\end{aligned} \tag{4.1.43}$$

This can be rewritten as

$$\begin{aligned}\langle \hat{a}\hat{b}\rangle &= \frac{R}{u\pi^2}\int d^2\alpha\Big[exp(-x\alpha\alpha^* + u\alpha + u\alpha + C)\\ &\times\frac{d}{d\omega}\int d^2\beta exp[-a\beta\beta^* + (\omega u+v\alpha)\beta + (u+v\alpha^*)\beta^*]\alpha\Big]\Big|_{\omega=1}.\end{aligned} \tag{4.1.44}$$

With the aid of Eq. (2.1.119), integrating the last expression on the right side of this result and differentiating over ω and then applying $\omega = 1$ condition, yields

$$\langle \hat{a}\hat{b}\rangle = \frac{R}{ua^2\pi}e^{C+\frac{u^2}{a}}\int d^2\alpha exp[-x\alpha\alpha^* + b\alpha + b\alpha^*][u^2\alpha + uv\alpha\alpha^*], \tag{4.1.45}$$

from which it follows

$$\begin{aligned}\langle \hat{a}\hat{b}\rangle &= \frac{R}{ua^2\pi}e^{C+\frac{u^2}{a}}\Big[u^2\int d^2\alpha exp[-x\alpha\alpha^* + b\alpha + b\alpha^*]\alpha\\ &+uv\int d^2\alpha exp[-x\alpha\alpha^* + b\alpha + b\alpha^*]\alpha\alpha^*\Big],\end{aligned}$$

$$\begin{aligned}\langle \hat{a}\hat{b}\rangle &= \frac{R}{ua^2\pi}e^{C+\frac{u^2}{a}}\Big[\frac{u^2}{b}\frac{d}{d\omega}\int d^2\alpha exp[-x\alpha\alpha^* + \omega b\alpha + b\alpha^*]\alpha\\ &-\frac{uv}{x}\frac{d}{d\eta}\int d^2\alpha exp[-\eta x\alpha\alpha^* + b\alpha + b\alpha^*]\Big]\Big|_{\omega=\eta=1}.\end{aligned} \tag{4.1.46}$$

Up on carrying out the integration, differentiation with respect to ω and η with the corrosponding integral results and then applying the condition $\omega = \eta = 1$ as well as inserting the values of the variables a, b, C, R, x, and u, one can directly obtain

$$\langle \hat{a}\hat{b}\rangle = m + E^2 = -M[1-e^{-\kappa t}] + \frac{4\varepsilon^2}{\kappa^2}[1-e^{-\kappa t}]^2. \tag{4.1.47}$$

In similar way

$$\langle \hat{a}^\dagger \hat{b}^\dagger \rangle = m + E^2 = -M[1 - e^{-\kappa t}] + \frac{4\varepsilon^2}{\kappa^2}[1 - e^{-\kappa t}]^2, \tag{4.1.48}$$

$$\langle \hat{a} \hat{b}^\dagger \rangle = m + E^2 = -M[1 - e^{-\kappa t}] + \frac{4\varepsilon^2}{\kappa^2}[1 - e^{-\kappa t}]^2, \tag{4.1.49}$$

$$\langle \hat{a}^\dagger \hat{b} \rangle = m + E^2 = -M[1 - e^{-\kappa t}] + \frac{4\varepsilon^2}{\kappa^2}[1 - e^{-\kappa t}]^2. \tag{4.1.50}$$

Now substitution of Eqs. (4.1.24), (4.1.26), (4.1.32), (4.1.34) and (4.1.40) into Eqs. (4.1.17) and (4.1.18) results in

$$(\Delta a_+)^2 = 1 + 2N[1 - e^{-\kappa t}], \tag{4.1.51}$$

$$(\Delta a_-)^2 = 1 + 2N[1 - e^{-\kappa t}]. \tag{4.1.52}$$

At the steady-state these follow

$$(\Delta a_+)^2 = 1 + 2N, \tag{4.1.53}$$

$$(\Delta a_-)^2 = 1 + 2N, \tag{4.1.54}$$

with similar definitions for mode b. Inspection of these results reveal that separately modes a and b are in chaotic states.

Therefore, up on substituting Eqs. (4.1.24 - 4.1.27), (4.1.32 - 4.1.35), (4.1.40), (4.1.41), (4.1.47 - 4.1.52) in to Eq. (4.1.16), the quadrature variance for the plus and minus quadrature take the form

$$(\Delta C_+)^2 = 1 + 2N[1 - e^{-\kappa t}] - 2M[1 - e^{-\kappa t}], \tag{4.1.55}$$

$$(\Delta C_-)^2 = 1 + 2N[1 - e^{-\kappa t}] + 2M[1 - e^{-\kappa t}]. \tag{4.1.56}$$

And at the steady-state, the plus and minus quadrature respecttvely become

$$(\Delta C_+)^2 = 1 + 2N - 2M, \tag{4.1.57}$$

$$(\Delta C_-)^2 = 1 + 2N + 2M. \tag{4.1.58}$$

4.2 Quadrature squeezing

A two-mode light is in a squeezed state if either $\Delta C_+ < 1$ or $\Delta C_- < 1$ such that $\Delta C_+ \Delta C_- \geq 1$. Hence by substituting Eqs. (2.1.4) and (2.1.5) into (4.1.57) and (4.1.58), we get

$$(\Delta C_+)^2 = e^{-2r}, \tag{4.2.1}$$

$$(\Delta C_-)^2 = e^{2r}. \tag{4.2.2}$$

From these results, we observe that $\Delta C_+ = e^{-r} < 1$ and $\Delta C_- = e^{r} > 1$.This shows that the two-mode light is in a squeezed state and the squeezing occurs in the plus quadrature. Thus, the quadrature squeezing of the two-mode light is given by

$$S = 1 - (\Delta C_+)^2 = 1 - e^{-2r}. \tag{4.2.3}$$

We note that the quadrature squeezing increases with increasing the squeeze parameter r, and we see that the coherent light has no contribution to the quadrature variance.

Chapter 5

Conclusion

Applying the pretinent master equation and following the procedure discussed in Ref. [1], we determined the c-number Langevin equations for the cavity-mode variables associated with the normal-ordering. Using the solutions of the resulting c-number Langevin equations and the correlation properties of the noise forces, we obtained the antinormally-ordered characterstic function. And with the aid of this characterstic function, we have determined the Q-function for the two-mode cavity light. The resulting Q-function of the two-mode cavity light is then used to calculate the mean and variance of the photon number sum and difference. Our result shows that the mean photon number and the variance of the photon number for mode a are the same as that of mode b. The mean of the photon number for the two-mode light is the sum of the individual mean photon numbers, and also we have seen that the mean of the photon number difference vanishes. Finally, we have calculated the quadrature squeezing and the squeezing occurs in the plus quadrature. The squeezing increases with increasing the squeeze parameter r.

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Declaration

This Project is my original work, has not been presented for a degree in any other University and that all the sources of material used for the project have been dully acknowledged.

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