DYNAMICS IN A MAXIMALLY SYMMETRIC UNIVERSE

By

Asnakew Bewketu

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ADDIS ABABA UNIVERSITY
DEPARTMENT OF
PHYSICS

Supervisor: ______________________________
Dr. Legesse Wotro

Examiners: ______________________________
Prof. P. Singh

____
Dr. S. Bhatnagar

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Author: **Asnakew Bewketu**

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Abstract

Our present understanding of the evolution of the universe relies upon the Friedmann-Robertson- Walker cosmological models. This model is so successful that it is now being considered as the Standard Model of Cosmology. So in this work we derive the Friedmann equations using the Friedmann-Robertson-Walker metric together with Einstein field equation and then we give a simple method to reduce Friedmann equations to a second order linear differential equation when it is supplemented with a time dependent equation of state. Furthermore, as illustrative examples, we solve this equation for some specific time dependent equation of states. And also by using the Friedmann equations with some time dependent equation of state we try to determine the cosmic scale factor(the rate at which the universe expands) and age of the Friedmann universe, for the matter dominated era, radiation dominated era and for both matter and radiation dominated era by considering different cases. We have finally discussed the observable quantities that can be evidences for the accelerated expansion of the Friedmann universe.

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To: My family and Ethiopian burnt face woman(Shewaye) by Gaddafi's son wife.

Introduction

The universe is a huge wide-open space that holds everything from the smallest particle to the biggest galaxy, or it refers to all the matter and energy that exist. As the Big Bang, the universe is originated from an infinitely dense point and evolved in to our present universe[4]. On the very largest scale, the universe consists of clusters of galaxies. About 10^{11} galaxies can be seen in their largest optical telescopes. The universe also contains isotropic background radiation, that is, radiation which is not directional[9,11].

Our present understanding of the evolution of the universe relies upon the Friedmann-Robertson- Walker (FRW) cosmological models[4,15]. This model is so successful that it is now being considered as the Standard Model of Cosmology. The standard model of cosmology has successfully predicted the nucleosynthesis of the light elements, the existance of the cosmic background radiation, and the dynamics of an expanding universe, i.e. the Hubble expansion[16,19]. When modern cosmologies were first formulated, they required to obey the "cosmological principle", that the universe should have a uniform matter distribution on the largest scales("homogeneity"), and look essentially the same for all observers viewing in all direction ("isotropy")[9,11,18]. With this background Edwin Hubble found that the light from galaxies appeared redshift; and that the fainter(and therefore farther away, on average) a galaxy was, the more its light was redshifted[30]. The discovery of the expansion of the universe by Edwin Hubble in 1929 heralded the dawn of observational cosmology. If we mentally rewind the expansion, we find that the universe was hotter and denser in its past[30]. At very early times the temperature was high enough to ionize the material that filled the universe. The universe therefore consisted of a plasma of nuclei, electrons and photons, and the number density of free electrons was so high that the mean free path for the Thomson scattering of photons was extremely short[15]. As the universe expanded, it cooled, and the mean photon energy diminished. The universe transitioned from being dominated by radiation to being matter-dominated. Eventually, at a temperature of about 3000 K, the photon energies became too low to keep the universe ionized. At this time, known as recombination, the primordial plasma coalesced into neutral atoms, and the mean free path of the photons increased to roughly the size of the observable universe[4,30]. Initial inhomogeneities present in the primordial plasma grew under the action of gravitational instability during the matter-dominated era into all the bound structures we observe in the universe today. Now, 13.7 billion years later, it appears that the universe has entered an epoch of accelerated expansion, with its energy density dominated by the mysterious “dark energy”[17,19].

Currently, most scientists believe that the universe is approximately 13.4 billion years old. There are multiple approaches measuring the age of the universe[18,27,28]. One approach is the observation of cosmic microwave background or CMB. By measuring the CMB scientists have studied the radiation of the universe as far back in time as the Big Bang. The universe is so large that when scientists observe far away materials and radiation, they are in fact looking back in time. This is due to the time taken for light to travel from these places to Earth. Therefore, by using the amount of radiation of the material to find the distance to these sources, scientists can estimate the age of the universe.

Another method used to find the age of the universe is the rate of expansion of the elements in it. The discovery that distant galaxies are all receding from us with velocities that are proportional to their distances shows that all the galaxies came from one single point and expanded at the same time. By measuring the rate of expansion, assuming that it is constant, and how far apart the objects are today, scientists can estimate the amount of time needed for these astronomical bodies to become as far apart as they are currently. Using this method the universe is roughly 14 billion years old.

The third method is finding specific microwaves from the beginning of the universe and using them to calculate the universes age. The detection of the 3 K microwave background by Penzias and Wilson, which represents the red shifted radiation of the primeval fireball provides evidence of a single starting point and also evidence for the Big Bang, which many scientists believe is how the universe started[16,18,30]. Despite how the universe started, most agree that it has a definite beginning.

There is also other evidence that makes scientists think that there is a single beginning to the universe. The universe could also be open, in which there would be no beginning and it would be indefinitely old[27]. However, there is not as much evidence supporting this argument.

Though these methods give an approximate age of the universe, the most exact method is solving Friedmanns equation. Aleksandr Friedmann was a Russian physicist and mathematician who, after studying Einsteins Friedmann derived his equation using the Einstein field equations, assuming "a homogeneous and isotropic universe". Friedmann equations lead to a relation between the energy density of the universe with its size when an equation of state is supplemented. We review this behavior for a general equation of state of the form $P = w\rho$. Here, ρ and P are energy density and pressure of the universe respectively. This equation of state includes the cases of radiation, matter and vacuum dominated era. In general, in this thesis, the fundamentals of modern cosmology for an isotropic and homogeneous space-time, which is naturally motivated by observation, will be reviewed. The Friedmann equations are derived and the consequences for the dynamics of the universe are discussed.

Chapter 1

Space-Time Geometry and Gravitation

1.1 Introduction to Space-Time Geometry

The Special theory of relativity abandons the Newtonian separation of Space and time and introduces the concept of Space-time[2,3]. Space-time is a 4-dimensional manifold, with points referred to as events. Space-time also is provided with a notion of distance, or length, between pairs of events, often referred to as the interval. It is usual to denote this quantity by $\Delta\tau^2$, when the two events are well-separated; in the infinitesimal case, we will refer to it as $d\tau^2$. When we study the geometry of special relativity and then of space-times with gravity, we will of course have to use coordinates (such as t, x, y, and z) to describe events in the space-time. The coordinate system (t, x, y, z) provided by an inertial frame is sometimes called an inertial coordinate system, and sometimes a Minkowski coordinate system, and sometimes a Lorentz coordinate system [because it was Lorentz (1904) who first studied the relationship of one such coordinate system to another, the Lorentz trans- formation][2,3]. The geometry of space-time is determined by the space-time-intervals between events. An event is a precise location in space at a precise moment of time; i.e., a precise location (or "point") in 4-dimensional space-time. Events in Space-time are specified by four coordinates x^α with $\alpha = 0, 1, 2, 3$. Where the zero component of α corresponds to the time coordinate and the rest of three components are the spacial coordinates x, y and z respectively. The Special theory of relativity postulates the existence of Minkowski coordinates x'^{α}[23] in which the proper time τ can be expressed in a simple form as

$$\tau^2 = c^2(x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2. \tag{1.1.1}$$

Similarly it is possible to write this as

$$d\tau^2 = c^2(dx'^0)^2 - (dx'^1)^2 - (dx'^2)^2 - (dx'^3)^2. \tag{1.1.2}$$

Or in a compact form with ($c = 1$), the above equation can be rewritten as

$$d\tau^2 = dt^2 - d\vec{x}^2. \tag{1.1.3}$$

Where τ is the proper time between two neighboring events along the world line of the clock considered, with coordinates x'^{α} and $x'^{\alpha}+dx'^{\alpha}$ or it is the time read by a standard clock traveling with the body (for example, the ticks might be successive crests of light waves emitted during a certain transition between energy levels of the atoms of the body), t is the coordinate of the zero component, $\vec{x}$ corresponds to the three spacial coordinates(x^1, x^2, x^3) and c in equation(1.1.2) is a fundamental velocity which experiment identifies with the speed of light in vacuum. Perhaps the most central of special relativistic laws is the one stating that the speed of light c in vacuum is frame-independent, i.e., is a constant, independent of the inertial reference frame in which it is measured. Considering the equation that governs the rate at which clocks tick in a gravitational field, special relativity tells that if a clock ticks once in every time interval $d\tau$ when at rest in the absence of a gravitational field, then the separation dx^{α} between the space-time locations of successive ticks when the clock is moving in the absence of a gravitational field is governed by[2,8] the equation

$$\eta_{\alpha\beta}dx^{\alpha}dx^{\beta} = -d\tau^2. \tag{1.1.4}$$

This is from the fact that in flat space time the Minkowski metric tensor $\eta_{\alpha\beta}$ is given by[2,4]

$$\eta_{\alpha\beta} = \begin{cases} 1, & \alpha, \beta = 1, 2, 3, \\ -1, & \alpha, \beta = 0, \\ 0, & \alpha \neq \beta. \end{cases} \tag{1.1.5}$$

The Minkowski metric is the diagonal metric with $\eta_{11}=\eta_{22}=\eta_{33}=1$ and $\eta_{00}=-1$. All the rest components are zero. Therefore, in flat space-time coordinate system, i.e. in the absence of a gravitational field the space-time element(line element) is given by equation (1.1.4). The Cartesian space coordinates have units of length in which the speed of light c is unity.

1.2 Tensor in General Relativity

1.2.1 The Metric Tensor in a Gravitational Field

To describe the world line of a body in Newtonian mechanics the sensible procedure is to use absolute time coordinate as a parameter, and express the spatial position **r** as a function of t; thus we would write as **r**(t)[2]. In relativity, however, this procedure is unsymmetrical, because t is no longer an absolute parameter , but just another coordinate, namely x^0[3]. We would like an absolute parameter τ which increase smoothly along the particle's world line from its past to its future; then a 'covariant' specification of the history of the particle would be the four functions $x^\alpha(\tau)$[23,25]. To clarify this use of the parameter τ to specify world lines in space- time, it is better to examine the familiar simpler case of specifying curves in the ordinary Eculidean three dimensional space of positions. There a point is described by three coordinates x^μ where $\mu = 1, 2, 3$. Curves are parametrized by using the arc length τ, that is, they are specified by the three functions $x^\mu(\tau)$[2,14]. On any curve, the arc length ($\Delta\ s$) between the neighboring points x^μ and

x^μ $+\Delta x^\mu$ is given by Pythagoras' theorem[2]. In Cartesian coordinates x, y, z, this gives

$$\Delta\ s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2. \tag{1.2.1}$$

It is now important to find general expressions for the distance Δs in position space, and the separation $\Delta\tau$ in space-time. Suppose we locate a point by general coordinates corresponding to an arbitrary coordinates in this space. Then if the original Cartesian x, y and z are arbitrary functions of the new coordinates x^1, x^2 and x^3, we can write this as

$$x = x(x^1, x^2, x^3)\ ;\quad y = y(x^1, x^2, x^3)\ ;\quad z = z(x^1, x^2, x^3).$$

The coordinate differences appearing in (1.2.1) are therefore given by

$$\Delta x = \frac{\partial x}{\partial x^1}\Delta x^1 + \frac{\partial x}{\partial x^2}\Delta x^2 + \frac{\partial x}{\partial x^3}\Delta x^3,$$
$$\Delta y = \frac{\partial y}{\partial x^1}\Delta x^1 + \frac{\partial y}{\partial x^2}\Delta x^2 + \frac{\partial y}{\partial x^3}\Delta x^3,$$
$$\Delta z = \frac{\partial z}{\partial x^1}\Delta x^1 + \frac{\partial z}{\partial x^2}\Delta x^2 + \frac{\partial z}{\partial x^3}\Delta x^3.$$

Substituting these in to equation (1.2.1), to get

$$\Delta s^2 = \left[\left(\frac{\partial x}{\partial x^1}\right)^2 + \left(\frac{\partial y}{\partial x^1}\right)^2 + \left(\frac{\partial z}{\partial x^1}\right)^2\right](\Delta x^1)^2 + 2\left[\frac{\partial x}{\partial x^1}\frac{\partial x}{\partial x^2} + \frac{\partial y}{\partial x^1}\frac{\partial y}{\partial x^2} + \frac{\partial z}{\partial x^1}\frac{\partial z}{\partial x^2}\right]\Delta x^1\Delta x^2 + ...$$
$$= \sum_{\mu=1}^{3}\ \sum_{v=1}^{3}\ g_{\mu v}(x^1, x^2, x^3)\Delta x^\mu \Delta x^v \tag{1.2.2}$$

Where,

$$g_{\mu v} = \left(\frac{\partial x}{\partial x^\mu}\frac{\partial x}{\partial x^v} + \frac{\partial y}{\partial x^\mu}\frac{\partial y}{\partial x^v} + \frac{\partial z}{\partial x^\mu}\frac{\partial z}{\partial x^v}\right).$$

Finally, using Einstein's summation convention[3,8]: two identical indices in any expression are to be summed over. Both μ and v occur twice in the formula for Δs^2, so we can write it as

$$\Delta s^2 = g_{\mu v}(\mathbf{r})\Delta x^\mu \Delta x^v. \tag{1.2.3}$$

This expression is very important: it tell us how to obtain a physically significant quantity- the distance between two points from a knowledge of the coordinate differences between the points. The above argument may be applied in a precisely analogous manner to space-time: in arbitrary, possibly accelerating and rotating reference frames the separation $\Delta\tau$ between two events is given in terms of their coordinate differences by

$$\Delta\tau^2 = -g_{ij}\Delta x^i \Delta x^j = \quad -\sum_{i=0}^{3}\sum_{j=0}^{3} \; g_{ij}\Delta x^i \Delta x^j. \tag{1.2.4}$$

Where,

$$g_{ij} = \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^i}\frac{\partial\xi^\beta}{\partial x^j}.$$

The tensor (g_{ij}) is the metric tensor in a gravitational field[2,8]. It can be defined as the proper time interval between two events with a given infinitesimal coordinate separation. The indices i and j runs 0 up to 3. In general, however, the components of g_{ij} are functions of the event coordinates x^i. There are two important differences, apart from the number of dimensions, between Euclidean three-dimensional space and four-dimensional space-time[23,26]. The first difference is that in space-time it is possible to find non-coincident events with zero separation, so the distinction can be made between space-like and time-like separations. This distinction arises from the occurrence of negative signs in g_{ij}. The second difference is that while in Euclidean three- dimensional space $g_{\mu v}$ can always be reduced to $g^0_{\mu v}$ by an appropriate coordinate transformation, it is generally possible in space-time to reduce $g_{\mu v}$ to $g^0_{\mu v}$ only locally, according to the principle of equivalence.

1.2.2 The Metric Tensor and Affine Connection

From the Metric tensor that we defined above, the infinitesimal line element in a gravitational field[3,8] can be written as

$$d\tau^2 = -g_{\mu v}dx^\mu dx^v. \tag{1.2.5}$$

Again, from this, as we try to defined above, $g_{\mu v}$ can be written as

$$g_{\mu v} = \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^v}.$$

Which is the metric tensor in a gravitational field. ξ^α and ξ^β are local inertial coordinates. Now differentiating the metric tensor in a gravitational field with respect to the general coordinate system x^λ we have,

$$\frac{\partial g_{\mu v}}{\partial x^\lambda} = \frac{\partial}{\partial x^\lambda}\left(\eta_{\alpha\beta}\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^v}\right) = \eta_{\alpha\beta}\left(\frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu}\frac{\partial \xi^\beta}{\partial x^v} + \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial^2 \xi^\beta}{\partial x^\lambda \partial x^v}\right).$$

This can be written as

$$\frac{\partial g_{\mu v}}{\partial x^\lambda} = \Gamma^p_{\lambda\mu}\frac{\partial \xi^\alpha}{\partial x^p}\frac{\partial \xi^\beta}{\partial x^v}\eta_{\alpha\beta} + \Gamma^p_{\lambda v}\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^p}\eta_{\alpha\beta}. \tag{1.2.6}$$

Where,

$$\Gamma^p_{\lambda\mu} = \frac{\partial x^p}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu},$$
$$\Gamma^p_{\lambda v} = \frac{\partial x^p}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^v}.$$

$\Gamma^p_{\lambda\mu}$ and $\Gamma^p_{\lambda v}$ are called affine connections. Considering freely falling particles affine connection is field that determines the gravitational force. They are symmetric with the exchange of lower indices, i.e. $\Gamma^p_{\lambda\mu}= \Gamma^p_{\mu\lambda}$. Now using this definition of affine connection and the relation given by

$$g_{pv} = \frac{\partial \xi^\alpha}{\partial x^p}\frac{\partial \xi^\beta}{\partial x^v}\eta_{\alpha\beta},$$
$$g_{\mu p} = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^p}\eta_{\alpha\beta}.$$

Equation (1.2.6) can be rewritten as

$$\frac{\partial g_{\mu v}}{\partial x^\lambda} = \Gamma^p_{\lambda\mu}g_{pv} + \Gamma^p_{\lambda v}g_{p\mu}. \tag{1.2.7}$$

Which is for the general case. Here the task is to solve for the affine connection. To solve, it is a matter of adding to equation (1.2.7) the same equation with μ and λ interchanged and subtract the same equation with v and λ interchanged. It is shown as

$$\frac{\partial g_{\mu v}}{\partial x^\lambda}+\frac{\partial g_{\lambda v}}{\partial x^\mu}-\frac{\partial g_{\mu\lambda}}{\partial x^v} = \Gamma^k_{\lambda\mu}g_{kv}+\Gamma^k_{\lambda v}g_{k\mu}+\Gamma^k_{\mu\lambda}g_{kv}+\Gamma^k_{\mu v}g_{k\lambda}-\Gamma^k_{v\mu}g_{k\lambda}-\Gamma^k_{v\lambda}g_{k\mu} = 2\Gamma^k_{\lambda\mu}g_{kv} = 2\Gamma^k_{\mu\lambda}g_{kv}.$$

This results from the symmetric property of affine connections.

From the fact expressed in equation (1.2.7) it is possible to write the following from the above expression as

$$\frac{\partial g_{\mu v}}{\partial x^{\lambda}} = \Gamma^{k}_{\lambda\mu} g_{kv} + \Gamma^{k}_{\lambda v} g_{k\mu},$$
$$\frac{\partial g_{\lambda v}}{\partial x^{\mu}} = \Gamma^{k}_{\mu\lambda} g_{kv} + \Gamma^{k}_{\mu v} g_{k\lambda},$$
$$\frac{\partial g_{\mu\lambda}}{\partial x^{v}} = \Gamma^{k}_{v\mu} g_{k\lambda} + \Gamma^{k}_{v\lambda} g_{k\mu}.$$

Now adding the first two terms and subtracting the third term yields

$$\frac{\partial g_{\mu v}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda v}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{v}} = 2\Gamma^{k}_{\mu\lambda} g_{kv}. \tag{1.2.8}$$

Multiplying both sides of equation (1.2.8) by $\frac{1}{2}g^{v\sigma}$ gives

$$\frac{1}{2} g^{v\sigma}\left(\frac{\partial g_{\mu v}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda v}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{v}}\right) = \frac{1}{2} g^{v\sigma}(2 g_{kv}\Gamma^{k}_{\lambda\mu}),$$
$$\frac{1}{2} g^{v\sigma}\left(\frac{\partial g_{\mu v}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda v}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{v}}\right) = g^{v\sigma} g_{kv}\Gamma^{k}_{\lambda\mu} = \delta^{\sigma}_{k}\Gamma^{k}_{\lambda\mu}.$$

Where δ^{k}_{σ} is the Kronecker delta with the following properties.

$$g^{v\sigma} g_{kv} = \delta^{\sigma}_{k} = \begin{cases} 1, & \sigma = k, \\ 0, & \sigma \neq k. \end{cases}$$

Applying this property $(\sigma = k)$ of Kronecker delta to the above equation one can write it as

$$\frac{1}{2} g^{v\sigma}\left(\frac{\partial g_{\mu v}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda v}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{v}}\right) = \Gamma^{\sigma}_{\lambda\mu}. \tag{1.2.9}$$

Which is the relation developed between the metric tensor in a gravitational field $(g_{\alpha\beta})$ and the affine connection $(\Gamma^{\sigma}_{\lambda\mu})$[14,26]. And from their definitions, both of them represents the presence of gravitational effect.

1.2.3 Curvature Tensor

Using the definition of affine connection it is possible to write the transformation rule as

$$\Gamma'^{\lambda}_{\mu v} = \frac{\partial x'^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x'^{\mu} \partial x'^{v}},$$

or it can be written as

$$\Gamma'^{\lambda}_{\mu v} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \frac{\partial}{\partial x'^{\mu}} \left(\frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial x'^{v}} \right). \tag{1.2.10}$$

But,

$$\frac{\partial}{\partial x'^{\mu}} \left(\frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial x'^{v}} \right) = \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \left(\frac{\partial^2 x^{\sigma}}{\partial x'^{\mu} \partial x'^{v}} \right) + \frac{\partial x^{\sigma}}{\partial x'^{v}} \left(\frac{\partial^2 \xi^{\alpha}}{\partial x'^{\mu} \partial x^{\sigma}} \right).$$

Now replacing,

$$\frac{\partial^2 \xi^{\alpha}}{\partial x'^{\mu} \partial x^{\sigma}} \quad by \quad \frac{\partial^2 \xi^{\alpha}}{\partial x^{T} \partial x^{\sigma}} \frac{\partial x^{T}}{\partial x'^{\mu}}$$

and substituting in to equation (1.2.10) to get

$$\Gamma'^{\lambda}_{\mu v} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{T}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial \xi^{\alpha}} \left(\frac{\partial x^{\rho}}{\partial x'^{v}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{T} \partial x^{\sigma}} \right) + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \left(\frac{\partial^2 x^{\sigma}}{\partial x'^{\mu} \partial x'^{v}} \right). \tag{1.2.11}$$

Making use of the relation given by affine connection and Kronecker delta in to equation(1.2.11) which are;

$$\Gamma^{\rho}_{T\sigma} = \frac{\partial x'^{\rho}}{\partial \xi^{\alpha}} \left(\frac{\partial^2 \xi^{\alpha}}{\partial x^{T} \partial x^{\sigma}} \right).$$

and

$$\frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} = \delta^{\rho}_{\sigma} = \begin{cases} 1, & \rho = \sigma, \\ 0, & \rho \neq \sigma. \end{cases}$$

We can write equation(1.2.11) as

$$\Gamma'^{\lambda}_{\mu v} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{T}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{v}} \Gamma^{\rho}_{T\sigma} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \left(\frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{v}} \right). \tag{1.2.12}$$

This implies that $\Gamma^{\lambda}_{\mu v}$ is not a tensor according to the statement given by general coordinate transformation. If $\Gamma^{\lambda}_{\mu v}$ is a tensor the expected term will be

$$\frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{T}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{v}} \Gamma^{\rho}_{T\sigma}.$$

Equation (1.2.12) may be inverted to the following form

$$\Gamma^{\lambda}_{\mu v} = \frac{\partial x^{\lambda}}{\partial x'^{T}} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{v}} \Gamma'^{T}_{\rho\sigma} + \frac{\partial x^{\lambda}}{\partial x'^{T}} \left(\frac{\partial^2 x'^{T}}{\partial x^{\mu} \partial x^{v}} \right).$$

Thus,

$$\left(\frac{\partial^2 x'^{T}}{\partial x^{\mu} \partial x^{v}} \right) = \frac{\partial x'^{T}}{\partial x^{\lambda}} \Gamma^{\lambda}_{\mu v} - \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{v}} \Gamma'^{T}_{\rho\sigma}. \tag{1.2.13}$$

Differentiating this with respect to x^k to get

$$\frac{\partial}{\partial x^k} \left(\frac{\partial^2 x'^{T}}{\partial x^{\mu} \partial x^{v}} \right) = \frac{\partial}{\partial x^k} \left(\frac{\partial x'^{T}}{\partial x^{\lambda}} \Gamma^{\lambda}_{\mu v} - \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{v}} \Gamma'^{T}_{\rho\sigma} \right).$$

$$\frac{\partial^3 x'^{T}}{\partial x^k \partial x^{\mu} \partial x^{v}} = \frac{\partial^2 x'^{T}}{\partial x^k \partial x^{\lambda}} \Gamma^{\lambda}_{\mu v} + \frac{\partial x'^{T}}{\partial x^{\lambda}} \frac{\partial}{\partial x^k} (\Gamma^{\lambda}_{\mu v}) - \frac{\partial^2 x'^{\rho}}{\partial x^k \partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{v}} \Gamma'^{T}_{\rho\sigma} - \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial^2 x'^{\sigma}}{\partial x^{v} \partial x^k} \Gamma'^{T}_{\rho\sigma} - \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{v}} \frac{\partial}{\partial x^k} (\Gamma'^{T}_{\rho\sigma}).$$

From the relation given by equation(1.2.13) it is possible to write the following

$$\frac{\partial^2 x'^{T}}{\partial x^k \partial x^{\lambda}} = \frac{\partial x'^{T}}{\partial x^{\eta}} \Gamma^{\eta}_{k\lambda} - \frac{\partial x'^{\rho}}{\partial x^k} \frac{\partial x'^{\sigma}}{\partial x^{\lambda}} \Gamma'^{T}_{\rho\sigma},$$

$$\frac{\partial^2 x'^{\rho}}{\partial x^k \partial x^{\mu}} = \frac{\partial x'^{\rho}}{\partial x^{\eta}} \Gamma^{\eta}_{k\mu} - \frac{\partial x'^{\eta}}{\partial x^k} \frac{\partial x'^{\varepsilon}}{\partial x^{\mu}} \Gamma'^{\rho}_{\eta\varepsilon},$$

$$\frac{\partial^2 x'^{\sigma}}{\partial x^k \partial x^{v}} = \frac{\partial x'^{\sigma}}{\partial x^{\eta}} \Gamma^{\eta}_{kv} - \frac{\partial x'^{\eta}}{\partial x^k} \frac{\partial x'^{\varepsilon}}{\partial x^{v}} \Gamma'^{\sigma}_{\eta\varepsilon}.$$

Substitution of these in to the above equation gives

$$\frac{\partial^3 x'^{T}}{\partial x^k \partial x^{\mu} \partial x^{v}} = \left(\frac{\partial x'^{T}}{\partial x^{\eta}} \Gamma^{\eta}_{k\lambda} - \frac{\partial x'^{\rho}}{\partial x^k} \frac{\partial x'^{\sigma}}{\partial x^{\lambda}} \Gamma'^{T}_{\rho\sigma} \right) \Gamma^{\lambda}_{\mu v} + \frac{\partial}{\partial x^k} \Gamma^{\lambda}_{\mu v} \frac{\partial x'^{T}}{\partial x^{\lambda}} - \Gamma'^{T}_{\rho\sigma} \frac{\partial x'^{\sigma}}{\partial x^{v}} \left(\frac{\partial x'^{\rho}}{\partial x^{\eta}} \Gamma^{\eta}_{k\mu} - \frac{\partial x'^{\eta}}{\partial x^k} \frac{\partial x'^{\varepsilon}}{\partial x^{\mu}} \Gamma'^{\rho}_{\eta\varepsilon} \right) -$$

$$\Gamma'^{T}_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \left(\frac{\partial x'^{\sigma}}{\partial x^{\eta}} \Gamma^{\eta}_{kv} - \frac{\partial x'^{\eta}}{\partial x^k} \frac{\partial x'^{\varepsilon}}{\partial x^{v}} \Gamma'^{\sigma}_{\eta\varepsilon} \right) - \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{v}} \frac{\partial}{\partial x^k} \Gamma'^{T}_{\rho\sigma}.$$

$$\frac{\partial^3 x'^{T}}{\partial x^k \partial x^{\mu} \partial x^{v}} = \Gamma^{\lambda}_{\mu v} \left(\frac{\partial x'^{T}}{\partial x^{\eta}} \Gamma^{\eta}_{k\lambda} - \frac{\partial x'^{\rho}}{\partial x^k} \frac{\partial x'^{\sigma}}{\partial x^{\lambda}} \Gamma'^{T}_{\rho\sigma} \right) - \Gamma'^{T}_{\rho\sigma} \frac{\partial x'^{\sigma}}{\partial x^{v}} \left(\frac{\partial x'^{\rho}}{\partial x^{\eta}} \Gamma^{\eta}_{k\mu} - \frac{\partial x'^{\eta}}{\partial x^k} \frac{\partial x'^{\varepsilon}}{\partial x^{\mu}} \Gamma'^{\rho}_{\eta\varepsilon} \right) + \frac{\partial x'^{T}}{\partial x^{\lambda}} \frac{\partial}{\partial x^k} (\Gamma^{\lambda}_{\mu v}) -$$

$$\Gamma'^{T}_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \left(\frac{\partial x'^{\sigma}}{\partial x^{\eta}} \Gamma^{\eta}_{kv} - \frac{\partial x'^{\eta}}{\partial x^k} \frac{\partial x'^{\varepsilon}}{\partial x^{v}} \Gamma'^{\sigma}_{\eta\varepsilon} \right) - \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{v}} \frac{\partial x'^{\eta}}{\partial x^k} \frac{\partial}{\partial x'^{\eta}} (\Gamma'^{T}_{\rho\sigma}).$$

Now collect similar terms and juggle indices a bit gives

$$\frac{\partial^3 x'^{T}}{\partial x^k \partial x^{\mu} \partial x^{v}} = \frac{\partial x'^{T}}{\partial x^{\lambda}} \left(\frac{\partial}{\partial x^k} \Gamma^{\lambda}_{\mu v} + \Gamma^{\eta}_{\mu v} \Gamma^{\lambda}_{k\mu} \right) - \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{v}} \frac{\partial x'^{\eta}}{\partial x^k} \left(\frac{\partial}{\partial x'^{\eta}} \Gamma'^{T}_{\rho\sigma} - \Gamma'^{T}_{\rho\lambda} \Gamma'^{\lambda}_{\eta\sigma} - \Gamma'^{T}_{\lambda\sigma} \Gamma'^{\lambda}_{\eta\rho} \right) -$$

$$\Gamma'^{T}_{\rho\sigma} \frac{\partial x'^{\sigma}}{\partial x^{\lambda}} \left(\Gamma^{\lambda}_{\mu v} \frac{\partial x'^{\rho}}{\partial x^k} + \Gamma^{\lambda}_{kv} \frac{\partial x'^{\rho}}{\partial x^{\mu}} + \Gamma^{\lambda}_{k\mu} \frac{\partial x'^{\rho}}{\partial x^{v}} \right).$$

By subtracting the same equation with $v \leftrightarrow k$ to find the product of Γ and Γ' drop out and so that

$$0 = \frac{\partial x'^T}{\partial x^\lambda}\left(\frac{\partial}{\partial x^k}(\Gamma^\lambda_{\mu v}) - \frac{\partial}{\partial x^v}(\Gamma^\lambda_{\mu k}) + \Gamma^\eta_{\mu v}\Gamma^\lambda_{k\eta} - \Gamma^\eta_{\mu k}\Gamma^\lambda_{v\eta}\right) -$$
$$\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^v}\frac{\partial x'^\eta}{\partial x^k}\left(\frac{\partial}{\partial x'^\eta}\Gamma'^T_{\rho\sigma} - \frac{\partial}{\partial x'^\sigma}\Gamma'^T_{\rho\eta} - \Gamma'^T_{\lambda\sigma}\Gamma'^\lambda_{\eta\rho} + \Gamma'^T_{\lambda\eta}\Gamma'^\lambda_{\sigma\rho}\right).$$

Now defining the term in the first bracket using the curvature tensor notation as,

$$R^\lambda_{\mu vk} = \frac{\partial}{\partial x^k}(\Gamma^\lambda_{\mu v}) - \frac{\partial}{\partial x^v}(\Gamma^\lambda_{\mu k}) + \Gamma^\eta_{\mu v}\Gamma^\lambda_{k\eta} - \Gamma^\eta_{\mu k}\Gamma^\lambda_{v\eta}. \tag{1.2.14}$$

This is named as **Riemann-Christoffel** curvature tensor[3,23,26]. This tensor plays an important role in specifying the geometrical properties of space-time. The space-time is considered flat if the Riemann tensor vanishes every where.

Considering the covariant form it is possible to write the Riemann curvature tensor in its fully covariant form as

$$R_{\lambda\mu vk} = g_{\lambda\sigma}g^\sigma_{\mu vk}.$$

Recall the definition given by affine connection, that was written as

$$\Gamma^\sigma_{\mu v} = \frac{1}{2}g^{\sigma\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^v} + \frac{\partial g_{\rho v}}{\partial x^\mu} - \frac{\partial g_{\mu v}}{\partial x^\rho}\right).$$

Using this definition of affine connection, Riemann curvature tensor can also be written directly in terms of the space-time metric. This shows as the following ,

$$R_{\lambda\mu vk} = \frac{1}{2}g_{\lambda\sigma}\frac{\partial}{\partial x^k}g^{\sigma\rho}\left[\frac{\partial g_{\rho\mu}}{\partial x^v} + \frac{\partial g_{\rho v}}{\partial x^\mu} - \frac{\partial g_{\mu v}}{\partial x^\rho}\right] - \frac{1}{2}g_{\lambda\sigma}\frac{\partial}{\partial x^v}g^{\sigma\rho}\left[\frac{\partial g_{\rho\mu}}{\partial x^k} + \frac{\partial g_{\rho k}}{\partial x^\mu} - \frac{\partial g_{\mu k}}{\partial x^\rho}\right] +$$
$$g_{\lambda\sigma}\left[\Gamma^\eta_{\mu v}\Gamma^\sigma_{k\eta} - \Gamma^\eta_{\mu k}\Gamma^\sigma_{v\eta}\right].$$

Applying the properties of Kronecker delta ($\delta^\rho_\lambda = 1$ for $\rho = \lambda$) and the relation given by

$$g_{\lambda\sigma}\frac{\partial}{\partial x^k}g^{\sigma\rho} = -g^{\sigma\rho}\frac{\partial}{\partial x^k}g_{\lambda\sigma} = -g^{\sigma\rho}(\Gamma^\eta_{k\lambda}g_{\eta\sigma} + \Gamma^\eta_{k\sigma}g_{\eta\lambda}).$$

In to the above equation it can be rewritten as

$$R_{\lambda\mu vk} = \frac{1}{2}\left[\frac{\partial^2 g_{\lambda v}}{\partial x^k\partial x^\mu} - \frac{\partial^2 g_{\mu v}}{\partial x^k\partial x^\lambda} - \frac{\partial^2 g_{\lambda k}}{\partial x^v\partial x^\mu} + \frac{\partial^2 g_{\mu k}}{\partial x^v\partial x^\lambda}\right] - [\Gamma^\eta_{k\lambda}g_{\eta\sigma} + \Gamma^\eta_{k\sigma}g_{\eta\lambda}]\Gamma^\sigma_{\mu v} +$$

$$[\Gamma^{\eta}_{v\lambda} g_{\eta\sigma} + \Gamma^{\eta}_{v\sigma} g_{\eta\lambda}]\Gamma^{\sigma}_{\mu k} + g_{\lambda\sigma}[\Gamma^{\eta}_{\mu v}\Gamma^{\sigma}_{k\eta} - \Gamma^{\eta}_{\mu k}\Gamma^{\sigma}_{v\eta}].$$

From this it is possible to see that most of $\Gamma\Gamma$ terms cancel, leaving only

$$R_{\lambda\mu vk} = \frac{1}{2}\left[\frac{\partial^2 g_{\lambda v}}{\partial x^k \partial x^\mu} - \frac{\partial^2 g_{\mu v}}{\partial x^k \partial x^\lambda} - \frac{\partial^2 g_{\lambda k}}{\partial x^v \partial x^\mu} + \frac{\partial^2 g_{\mu k}}{\partial x^v \partial x^\lambda}\right] + g_{\eta\sigma}[\Gamma^{\eta}_{v\lambda}\Gamma^{\sigma}_{\mu k} - \Gamma^{\eta}_{k\lambda}\Gamma^{\sigma}_{\mu v}]. \quad (1.2.15)$$

This is the covariant form of Riemann-Christoffel Curvature tensor and it has the following symmetries.

$$R_{\lambda\mu vk} = -R_{\mu\lambda vk} = -R_{\lambda\mu kv} = R_{vk\lambda\mu}.$$

Because of these symmetries, the Riemann tensor in 4-dimensional space-time has only 20 independent components. This is true from the general rule for computing the number of independent components in an N-dimensional space-time is $\frac{N^2(N^2-1)}{12}$[24].

1.2.4 Ricci Tensor, Ricci Scalar and Einstein Field Tensor

Ricci Tensor is obtained from the Riemann curvature tensor[3,8] simply by contracting over two of the indices:

$$R_{\mu k} = g^{\lambda v} R_{\lambda\mu vk}.$$

This implies;

$$R_{\mu k} = g^{\lambda v} R_{\lambda\mu vk} = \frac{1}{2} g^{\lambda v}\left[\frac{\partial^2 g_{\lambda v}}{\partial x^k \partial x^\mu} - \frac{\partial^2 g_{\mu v}}{\partial x^k \partial x^\lambda} - \frac{\partial^2 g_{\lambda k}}{\partial x^v \partial x^\mu} + \frac{\partial^2 g_{\mu k}}{\partial x^v \partial x^\lambda}\right] + g^{\lambda v} g_{\eta\sigma}[\Gamma^{\eta}_{v\lambda}\Gamma^{\sigma}_{\mu k} - \Gamma^{\eta}_{k\lambda}\Gamma^{\sigma}_{\mu v}].$$

Now referring back the definition given by the affine connection and making use of it in the above expression, one can write the Ricci tensor as

$$R_{\mu k} = \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^k} - \frac{\partial \Gamma^{\lambda}_{\mu k}}{\partial x^\lambda} + \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{k\eta} - \Gamma^{\eta}_{\mu k}\Gamma^{\lambda}_{\lambda\eta}. \quad (1.2.16)$$

It is symmetric ($R_{\mu k} = R_{k\mu}$), which means that it has at most 10 independent quantities.
Ricci Scalar is obtained by contracting the Ricci tensor[2,3] over the remaining two indices:

$$R \equiv g^{\mu k} R_{\mu k} = R^{\mu}_{\mu}.$$

Einstein Field Tensor is defined interms of the Ricci tensor and Ricci scalar[23,25] as

$$G_{\mu k} \equiv R_{\mu k} - \frac{1}{2} g_{\mu k} R. \tag{1.2.17}$$

Where G is a linear combination of the metric tensor and its first and second derivatives. The derivation of equation(1.2.18) will follow soon when we will discuse Einstein field equation in this chapter.

1.2.5 Energy-Momentum Tensor

Energy-Momentum(Stress-energy) tensor $T^{\alpha\beta}$ describes the density and flows of the 4 momentum $(E, -P_1, -P_2, -P_3)$[4]. The component $T^{\alpha\beta}$ is the flux or flow of the α component of the 4 momentum crossing the surface of constant x^{β}.The energy-momentum tensor is symmetric($T^{\alpha\beta} = T^{\beta\alpha}$).
T^{00} represents energy density,
T^{0i} represents the flow(flux) of energy in the x^i direction(energy flux),
T^{i0} represents the density of the i-component of momentum,
T^{ij} represents the flow of the i-component of momentum in the j-direction(momentum flux).

$$\begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix}$$

There are two types of energy-momentum tensor[2,3] frequently used in general relativity: these are dust and perfect fluid that we have to discussed in chapter two.

1.3 Geodesic Equation

A particle's equation of motion is such that the path followed by the particle under the influence of gravity is minimum[8]. Consider the curve of shortest distance between two points. Let the curve, parametrized by τ, connect two points P_1 and P_2 of space-time with parameters τ_1 and τ_2 respectively.

Then the "distance" of P_2 from P_1 is defined as

$$S(p_2, p_1) = \int_{p_1}^{p_2} \delta d\tau. \tag{1.3.1}$$

Where $d\tau$ is the space-time line element given by equation(1.2.5). Using the definition of $d\tau$ in to equation(1.3.1) this can be rewritten as

$$S(p_2, p_1) = \int_{\tau_1}^{\tau_2} \delta \left(g_{ik} \frac{dx^i}{d\tau} \frac{dx^k}{d\tau} \right)^{1/2} d\tau.$$

Defining the term in the bracket as

$$L = \left(g_{ik} \frac{dx^i}{d\tau} \frac{dx^k}{d\tau} \right)^{1/2}.$$

Where L is the Lagrange. With this definition equation(1.3.1) again written as

$$S(p_2, p_1) = \int_{\tau_1}^{\tau_2} L d\tau. \tag{1.3.2}$$

Now $S(p_2, p_1)$ demand to be stationary for small displacements of the curve connecting P_1 and P_2, these displacements vanishing at P_1 and P_2. So that

$$\delta \int_{p_1}^{p_2} d\tau = \delta \int_{\tau_1}^{\tau_2} \left(g_{ik} \frac{dx^i}{d\tau} \frac{dx^k}{d\tau} \right)^{1/2} d\tau = 0.$$

This implies that

$$\delta \int_{\tau_1}^{\tau_2} L d\tau = 0.$$

The solution is given by the Euler-Lagrange equation

$$\frac{d}{d\tau} \left(\frac{\partial L}{\partial (dx^i/d\tau)} \right) - \frac{\partial L}{\partial x^i} = 0.$$

L is a function of x^i and $dx^i/d\tau$. Substituting the value given for L in to the Euler-Lagrange equation the equation get the form

$$\frac{d}{d\tau} \left(\frac{1}{L} g_{ik} \frac{dx^k}{d\tau} \right) - \frac{1}{2} g_{mn;i} \frac{1}{L} \frac{dx^m}{d\tau} \frac{dx^n}{d\tau} = 0. \tag{1.3.3}$$

From the equation(1.3.2) we can write as

$$dS = Ld\tau \Rightarrow L = \frac{dS}{d\tau}.$$

Substituting this in to equation(1.3.3) gives

$$\frac{d}{d\tau}\left(g_{ik}\frac{dx^k}{dS}\right) - \frac{1}{2}g_{mn;i}\frac{dx^m}{dS}\frac{dx^n}{d\tau} = 0.$$

For L=1, dS=$d\tau$ this implies S=τ. With this relation the above equation rewritten as

$$\frac{d}{d\tau}\left(g_{ik}\frac{dx^k}{d\tau}\right) - \frac{1}{2}g_{mn;i}\frac{dx^m}{d\tau}\frac{dx^n}{d\tau} = 0.$$

Now applying the property $g_{mn;i} = -g_{mi;n}$ and multiplying the above equation both side by g^{li} to get

$$g^l_k\frac{d^2x^k}{d\tau^2} + \frac{1}{2}g^{li}g_{mi;n}\frac{dx^m}{d\tau}\frac{dx^n}{d\tau} = 0.$$

Recall the relation given by $\Gamma^\beta_{\sigma\mu} = \frac{1}{2}g^{\alpha\beta}g_{\sigma\alpha;\mu}$. Making use of this relation and puting $l = k$ in the above equation, it can be written as

$$\frac{d^2x^l}{d\tau^2} + \Gamma^l_{mn}\frac{dx^m}{d\tau}\frac{dx^n}{d\tau} = 0. \tag{1.3.4}$$

Equation(1.3.4) is the geodesic equation or the equation of motion of a particle's in the gravitational field[20,23]. From this we can conclude that the equation of motion of the particle in a gravitational field is determined by the Christoffel symbol Γ^l_{mn}. The derivative $d^2x^l/d\tau^2$ is the four-acceleration of the particle therefore, it is possible to call the quantity $-m\Gamma^l_{mn}(dx^m/d\tau)(dx^n/d\tau)$ is the 4-force acting on the particle.

1.4 Einstein's Field Equation

The stage is now set for deriving and understanding Einstein's field equations[3,8]. General relativity must present appropriate analogues of the two parts of the dynamical picture: 1) how particles move in response to gravity ; and 2) how particles generate gravitational effects. The first part was answered when the geodesic equation derived. The second part requires finding the analogue of the Poisson equation[24,25]

$$\nabla^2\Phi(x) = 4\pi G\rho(\vec{x}). \tag{1.4.1}$$

Which specifies how matter curves space-time. Before the derivation of the field equation, it is better to consider the case where a particle is moving slowly in a weak stationary gravitational field. For sufficiently slow motion of a particle, the equation of motion of a particle can be written as

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0. \tag{1.4.2}$$

This is from the equation of motion of the particle in a gravitational field given by the equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\lambda v}\frac{dx^\lambda}{d\tau}\frac{dx^v}{d\tau} = 0.$$

And for $\lambda = v = 0$ and $dx^0 = dt$,

Recall the relation given by

$$\Gamma^\sigma_{\lambda\mu} = \frac{1}{2}g^{v\sigma}\left(\frac{\partial g_{\mu v}}{\partial x^\lambda} + \frac{\partial g_{\lambda v}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^v}\right).$$

But since the field is stationary, all time derivatives of $g_{\mu v}$ vanish; so that

$$\Gamma^\lambda_{00} = -\frac{1}{2}g^{\lambda v}\frac{\partial g_{00}}{\partial x^v}. \tag{1.4.3}$$

For a weak static field produced by non-relativistic mass density ρ,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}.$$

Where $|h_{\alpha\beta}| << 1$ and $\eta_{\alpha\beta}$ is Minkowski metric tensor.

For $\alpha = \beta = 0$ and applying the relation $\eta_{00} = -1$ gives

$$g_{00} = -1 + h_{00}. \tag{1.4.4}$$

Substitution of equation(1.4.4) in to equation(1.4.3) for first order in $h_{\alpha\beta}$ yields

$$\Gamma^{\alpha}_{00} = -\frac{1}{2}\eta^{\alpha\beta}\frac{\partial h_{00}}{\partial x^{\beta}}. \tag{1.4.5}$$

Making use of equation(1.4.5) in to equation(1.4.2) with $\mu = \alpha$,

$$\frac{\partial^2 x^{\alpha}}{\partial \tau^2} = -\Gamma^{\alpha}_{00}\left(\frac{\partial t}{\partial \tau}\right)^2 = -\left(-\frac{1}{2}\eta^{\alpha\beta}\frac{\partial h_{00}}{\partial x^{\beta}}\right).$$

For $\alpha = \beta = 1, 2, 3$ the Minkowski metric tensor, $\eta_{\alpha\beta} = \eta^{\alpha\beta} = 1$ and the above equation take the form

$$\frac{\partial^2 x}{\partial \tau^2} = \frac{1}{2}\left(\frac{\partial t}{\partial \tau}\right)^2 \frac{\partial h_{00}}{\partial x}.$$

It is possible to write as

$$\frac{\partial^2 x}{\partial \tau^2} = \frac{1}{2}\left(\frac{\partial t}{\partial \tau}\right)^2 \nabla h_{00}. \tag{1.4.6}$$

Now dividing equation(1.4.6) by

$$\left(\frac{\partial t}{\partial \tau}\right)^2$$

to get

$$\frac{\partial^2 x}{\partial t^2} = \frac{1}{2}\nabla h_{00}. \tag{1.4.7}$$

The corresponding Newtonian result is

$$\frac{\partial^2 x}{\partial t^2} = -\nabla\Phi. \tag{1.4.8}$$

Where Φ is the Newtonian potential determined from the equation given by (1.4.1).

With the comparison of equation(1.4.7) and equation(1.4.8) one can write the following result

$$h_{00} = -2\Phi + constant.$$

Further more, the coordinates system must become Minkowskian at great distance so h_{00} vanish at infinity, and if Φ defined to vanish at infinity(where $\Phi = \frac{-GM}{r}$), r is the distance from the center of a spherical body of mass M.

Recall the relation for a weak static field given by

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}.$$

This implies

$$g_{00} = -1 + h_{00}.$$

Using the value of h_{00} for zero constant in to the above equation

$$g_{00} = -(1 + 2\Phi). \tag{1.4.9}$$

Now we start to derive Einstein field equation under the approximation of a weak static field produced by a non-relativistic mass density ρ.

Further more the energy density for non-relativistic matter is

$$T_{00} = \rho = T^0_0.$$

With this result equation(1.4.1) can be written as

$$\nabla^2 \Phi = 4\pi G T_{00}.$$

Or from equation(1.4.9) it is also possible to write as

$$\nabla^2 g_{00} = -8\pi G T_{00}.$$

From this fact the weak field equation for a general distribution of energy and momentum $T_{\alpha\beta}$ will take the form

$$G_{\alpha\beta} = -8\pi G T_{\alpha\beta}. \tag{1.4.10}$$

Where $G_{\alpha\beta}$ is a linear combination of the metric tensor and its first and second derivatives. It then follows from the principle of equivalence that the equations which govern gravitational fields of arbitrary strength must take the form

$$G_{\mu v} = -8\pi G T_{\mu v}. \tag{1.4.11}$$

i.e. equation(1.4.10) is the approximated form of equation(1.4.11) in a weak static gravitational field as equivalence principle states. Here $G_{\mu v}$ is a tensor which reduce to $G_{\alpha\beta}$ for a weak fields. And since $T_{\mu v}$ is symmetric $G_{\mu v}$ also.

To go further consider the nature of $G_{\mu v}$;

(a) By definition $G_{\mu v}$ is a tensor.

(b) By assumption $G_{\mu v}$ contain terms that are linear in the second derivative of the metric tensor or quadratic in the first derivative of the metric.

(c) Since $T_{\mu v}$ is symmetric so does $G_{\mu v}$.

(d) Since $T_{\mu v}$ is conserved in the absence of external forces, so does $G_{\mu v}$.

(e) For a weak stationary field produced by non-relativistic matter, the 00 component must satisfy

$$G_{00} \simeq \nabla^2 g_{00}.$$

Hence (a) and (b) require $G_{\mu v}$ to take the form

$$G_{\mu v} = C_1 R_{\mu v} + C_2 g_{\mu v} R.$$

Where C_1 and C_2 are constants. Since this is symmetric condition (c) is authomatically satisfied. It follows from the above relation that

$$g^{\sigma\mu} G_{\mu v} = C_1 g^{\sigma\mu} R_{\mu v} + C_2 g^{\sigma\mu} g_{\mu v} R.$$

This is equivalent to

$$G^{\sigma}_{v} = C_1 R^{\sigma}_{v} + C_2 \delta^{\sigma}_{v} R.$$

This follows as

$$G^{\sigma}_{v;\sigma} = C_1 R^{\sigma}_{v;\sigma} + C_2 \delta^{\sigma}_{v} R_{;\sigma}.$$

Using the result

$$R^{\sigma}_{v;\sigma} = \frac{1}{2} \delta^{\sigma}_{v} R_{;\sigma}.$$

that is obtained from the contraction of Biachi identity in to the preeciding equation

$$G^{\sigma}_{\mu;\sigma} = \left(\frac{C_1}{2} + C_2 \right) R_{;\sigma}.$$

But by the conservation of $G_{\mu v}$ we have $G^{\sigma}_{\mu;\sigma} = 0$, this yields the relation

$$\left(\frac{C_1}{2} + C_2 \right) R_{;\sigma} = 0.$$

This implies

$$\frac{C_1}{2} = -C_2.$$

Then with this result we can rewrite $G_{\mu v}$ as

$$G_{\mu v} = C_1 \left(R_{\mu v} - \frac{1}{2} g_{\mu v} R \right).$$

Note that from

$$G_{\mu v} = C_1 R_{\mu v} + C_2 g_{\mu v} R = -8\pi G T_{\mu v}.$$

Follows

$$G^{\mu}_{\mu} = (C_1 + 4C_2) R = -8\pi G T^{\mu}_{\mu}.$$

Thus if $R_{;v} \equiv \frac{\partial R}{\partial x^v}$ vanishes, then so must $\frac{\partial T^{\mu}_{\mu}}{\partial x^v}$ and this is not the case in the presence of inhomogeneous non-relativistic matter. From this it is possible to conclude that $C_2 = \frac{-C_1}{2}$ then the above relation becomes

$$G_{\mu v} = C_1 \left(R_{\mu v} - \frac{1}{2} g_{\mu v} R \right) = -8\pi G T_{\mu v}. \tag{1.4.12}$$

Finally, I use the property (e) to fix the constant C_1. A non-relativistic system always has $|T_{ij}| << |T_{00}|$ here I look the case where $|G_{ij}| << |G_{00}|$.

$$|G_{ij}| << |G_{00}|.$$

This implies that

$$G_{ij} \approx 0.$$

Putting this in to equation(1.4.12) to get

$$R_{ij} - \frac{1}{2} g_{ij} R = 0.$$

$$R_{ij} \cong \frac{1}{2} g_{ij} R.$$

Since we deal here with a weak field approximation (i.e. $g_{\alpha\beta} \simeq \eta_{\alpha\beta}$),

$$\eta_{ij} \cong g_{ij}.$$

This lead to write as

$$R_{ij} \cong \frac{1}{2}\eta_{ij}R.$$

Applying the property of metric tensor($\eta_{ij} = 1$ for i=j=1,2,3) and taking the sum over each indices in the above relation gives

$$R_{ij} \cong \sum_{i,j=1}^{3} \frac{1}{2}\eta_{ij}R \cong \frac{3}{2}R.$$

$$R_{kk} \cong \frac{3}{2}R.$$

The curvature scalar is therefore given by

$$R \simeq R_{kk} - R_{00} \simeq \frac{3}{2}R - R_{00}.$$

This implies that

$$R \cong 2R_{00}.$$

So in the weak field approximation we have the following information

$$\begin{cases} R \cong 2R_{00}, \\ g_{\alpha\beta} \simeq \eta_{\alpha\beta}, \\ G_{\mu v} = C_1(R_{\mu v} - \frac{1}{2}g_{\mu v}R). \end{cases} \tag{1.4.13}$$

For the 00 component $G_{\mu v}$ equals to

$$G_{00} = C_1\left(R_{00} - \frac{1}{2}\eta_{00}R\right) = C_1\left(\frac{R}{2} - \frac{1}{2}(-1)R\right).$$

$$G_{00} = 2C_1R_{00} \cong C_1R. \tag{1.4.14}$$

Now the task is to calculate R_{00}. Recall the expression given by the Riemann curvature tensor $R_{\lambda\mu vk}$ that is

$$R_{\lambda\mu vk} = \frac{1}{2}\left[\frac{\partial^2 g_{\lambda v}}{\partial x^k \partial x^\mu} - \frac{\partial^2 g_{\mu v}}{\partial x^k \partial x^\lambda} - \frac{\partial^2 g_{\lambda k}}{\partial x^v \partial x^\mu} + \frac{\partial^2 g_{\mu k}}{\partial x^v \partial x^\lambda}\right] + g_{\eta\sigma}[\Gamma^{\eta}_{v\lambda}\Gamma^{\sigma}_{\mu k} - \Gamma^{\eta}_{k\lambda}\Gamma^{\sigma}_{\mu v}].$$

But since we are looking for a weak field approximation it is better to use the linear part of $R_{\lambda\mu vk}$, given by

$$R_{\lambda\mu vk} = \frac{1}{2}\left[\frac{\partial^2 g_{\lambda v}}{\partial x^k \partial x^\mu} - \frac{\partial^2 g_{\mu v}}{\partial x^k \partial x^\lambda} - \frac{\partial^2 g_{\lambda k}}{\partial x^v \partial x^\mu} + \frac{\partial^2 g_{\mu k}}{\partial x^v \partial x^\lambda}\right].$$

When the field is static all time derivatives vanish, and the components that we need are

$$R_{0000} \simeq 0 \quad , \qquad R_{i0j0} \simeq \frac{1}{2}\frac{\partial^2 g_{00}}{\partial x^i \partial x^j} \simeq \frac{1}{2}\nabla^2 g_{00}.$$

And also from the contraction of curvature tensor over the two indices

$$R_{00} = g^{\lambda v} R_{\lambda 0 V 0} = \eta^{\lambda v} R_{\lambda 0 v 0} = R_{i0j0} - R_{0000}.$$

Making use of this relation in to equation(1.4.14) for $G_{\mu v}$,

$$G_{00} = 2C_1(R_{i0j0} - R_{0000}).$$

$$G_{00} = 2C_1\left(\frac{1}{2}\nabla^2 g_{00} - 0\right) = C_1\nabla^2 g_{00}.$$

Comparing this with

$$G_{00} = \nabla^2 g_{00}.$$

$$G_{00} = C_1\nabla^2 g_{00} = \nabla^2 g_{00}.$$

This gives the value of $C_1 = 1$. With this value equation(1.4.12) can be rewritten as

$$G_{\mu v} = \left(R_{\mu v} - \frac{1}{2}g_{\mu v}R\right) = -8\pi G T_{\mu v}.$$

$$R_{\mu v} - \frac{1}{2}g_{\mu v}R = -8\pi G T_{\mu v}. \qquad (1.4.15)$$

Equation(1.4.15) is Einstein's field equation. This equation shows that the metric of space-time is dependent upon the matter present in that space-time.

1.5 Homogeneous and Isotropic Spaces

From the cosmological principle, that says we are not located at any special location in the Universe[4]. The other way to put it is that the universe is homogeneous and isotropic. We will now make the assumption that space is homogeneous and isotropic. This is a reasonable assumption at the largest scale of the whole universe, where galaxies are viewed as particles. Homogeneous means that at any given instant of time each point of space should look like any other point and isotropic means that at each time it looks the same in every space direction at each point[4,15]. For a foliation of space-time into hype-surfaces Σ_t such that t is the proper time along the flow of the normal vectors to Σ_t and hence the space-time metric can be written as:

$$g_{ab} = -dt^2 + h_{ab}(t,x)dx^a dx^b.$$

Where $h_{ab}(t)$ is the restriction of the metric to g_{ab} to Σ_t. Homogeneous then means that for each t and any points p, q$\in \Sigma_t$ there is an isometry of the space-time that also is an isometry of Σ_t that takes p to q. Isotropy means that for any given point p $\in \Sigma_t$ and two tangent vectors s_1 and s_2 at p tangential to Σ_t there is an isometry of space-time that also is an isometry of Σ_t that leaves p and the normal at p fixed and takes s_1 to s_2.
For any point in our space-time we can always find, using Riemann normal coordinates, an open neighborhood of this point and some foliation of this neighborhood by spatial slices.

1.6 Maximally Symmetric Spaces

Maximally symmetric spaces are spaces with constant curvature[12]. A given space is said to be maximally symmetric if there exist $\frac{1}{2}m(m+1)$ independent killing vector fields on M. Where M is Pseudo-Riemannian manifold and m is the dimension of M.

1.6.1 The Metric of a Maximally Symmetric Space III

For our work, let us assume that maximally symmetric space III is both homogeneous and isotropic. Where the definitions for homogeneous and isotropic spaces were given in section 1.5. Consider a flat $(N+1)$-dimensional space with metric given by

$$-d\tau^2 = g_{ab}dx^a dx^b.$$

$$-d\tau^2 = C_{\mu v}dx^\mu dx^v + k^{-1}dz^2.$$

Where $C_{\mu v}$ is a constant $N * N$ matrix and k is some constant.

Restricting the variable x^μ and z to the surface of a pseudo-sphere, that means

$$kC_{\mu v}x^\mu x^v + z^2 = 1. \tag{1.6.1}$$

On this surface then

$$2zdz + 2kC_{\mu v}x^\mu dx^v = 0.$$

$$zdz = -kC_{\mu v}x^\mu dx^v.$$

If one can square both side and divid by z^2 to get

$$dz^2 = k^2 \frac{(C_{\mu v}x^\mu dx^v)^2}{z^2}.$$

Substituting the value of z^2 from equation(1.6.1) in to the above equation yields

$$dz^2 = k^2 \frac{(C_{\mu v}x^\mu dx^v)^2}{(1 - kC_{\mu v}x^\mu x^v)}.$$

Substituting this in to $d\tau^2$ to find

$$-d\tau^2 = C_{\mu v}dx^\mu dx^v + k\frac{(C_{\mu v}x^\mu dx^v)^2}{(1 - kC_{\rho\sigma}x^\rho x^\sigma)},$$

$$-d\tau^2 = C_{\mu v}dx^\mu dx^v + \frac{k}{(1 - kC_{\rho\sigma}x^\rho x^\sigma)}C_{\mu\lambda}x^\lambda C_{vk}x^k dx^\mu dx^v,$$

$$-d\tau^2 = \left[C_{\mu v} + \frac{k}{(1 - kC_{\rho\sigma}x^\rho x^\sigma)}C_{\mu\lambda}x^\lambda C_{vk}x^k\right] dx^\mu dx^v.$$

Comparing this to

$$d\tau^2 = -g_{\mu v}dx^\mu dx^v.$$

It is possible to conclude that

$$g_{\mu v} = C_{\mu v} + \frac{k}{(1 - kC_{\rho\sigma}x^\rho x^\sigma)}C_{\mu\lambda}x^\lambda C_{vk}x^k. \tag{1.6.2}$$

Hence k is the same as the curvature constant and is an invariant parameter, it is impossible to convert equation(1.6.2) by coordinate transformation in to a similar metric with a different k. Recall the relation given by

$$-d\tau^2 = C_{\mu v}dx^\mu dx^v + k\frac{(C_{\mu v}x^\mu dx^v)^2}{(1 - kC_{\rho\sigma}x^\rho x^\sigma)}.$$

Now make a linear transformation of the type

$$x^{\mu} = A^{\mu}_{v} x'^{v}.$$

for the first part of the RHS of $-d\tau^2$ that are

$$x^{\mu} = A^{\mu}_{\rho} x'^{\rho}, \qquad x^{v} = A^{v}_{\sigma} x'^{\sigma}.$$

This implies, $dx^{\mu} dx^{v}$ can be written as

$$dx^{\mu} dx^{v} = A^{\mu}_{\rho} A^{v}_{\sigma} dx'^{\rho} dx'^{\sigma}.$$

Hence

$$-d\tau^2 = C_{\mu v} A^{\mu}_{\rho} A^{v}_{\sigma} dx'^{\rho} dx'^{\sigma}.$$

Let $C_{\rho\sigma} = C^{'}_{\mu v} A^{\mu}_{\rho} A^{v}_{\sigma}$. From this we can write the following

$$C_{\rho\sigma} = C^{'}_{\mu v} A^{\mu}_{\rho} A^{v}_{\sigma} \Longrightarrow C^{'}_{\mu v} = C_{\rho\sigma} A^{\rho}_{\mu} A^{\sigma}_{v}.$$

The same applies to the second term on the RHS of $-d\tau^2$. Such a linear transformation converts the metric in to a similar metric with the same constant k. The number of eigenvalues of each of the matrix $C_{\mu v}$ are the same as for the metric $g_{\mu v}$ at the point $x = 0$ and hence the same every where since all points are equivalent. An N-dimensional matrix allows the introduction of locally Euclidean coordinate systems will have all its eigenvalue values positive. So for $k \neq 0$ we can set

$$C_{\mu v} = |k^{-1}| \times I.$$

Where I is the unit matrix. So in this case

$$-d\tau^2 = C_{\mu v} dx^{\mu} dx^{v} + k \frac{(C_{\mu v} x^{\mu} dx^{v})^2}{(1 - k C_{\rho\sigma} x^{\rho} x^{\sigma})}.$$

For the Euclidean space take the form

$$ds^2 = |k^{-1}| dx^2 + \frac{k|k|^{-2}(x \cdot dx)^2}{(1 - k|k|^{-1} x^2)}. \tag{1.6.3}$$

If $k > 0$ then

$$ds^2 = \frac{1}{k} d\vec{x}^2 + \frac{1}{k} \frac{(\vec{x} \cdot d\vec{x})^2}{(1 - \vec{x}^2)} = k^{-1} \left[d\vec{x}^2 + \frac{(\vec{x} \cdot d\vec{x})^2}{(1 - \vec{x}^2)} \right].$$

For $k < 0$ then

$$ds^2 = \frac{1}{k} dx^2 - \frac{1}{k} \frac{(x \cdot dx)^2}{(1 + x^2)} = |k|^{-1} \left[dx^2 - \frac{(x \cdot dx)^2}{(1 + x^2)} \right].$$

The general result that governs the structure of the space with maximally symmetric(spherically symmetric) subspaces[9,18] is contained in the following theory. It is always possible to choose the u-coordinates so that the metric of the whole space is given by

$$-d\tau^2 = g_{\mu v} dx^\mu dx^v.$$

$$-d\tau^2 = g_{ab}(v) dv^a dv^b + f(v) g_{ij}(u) du^i du^j.$$

Where $g_{\alpha\beta}(v)$ and $f(v)$ are functions of the v-coordinates and $g_{ij}(u)$ is a function of the u-coordinate alone (which is the metric of a subspace in a maximally symmetric space). In all cases of interest, the maximally symmetric subspaces are spaces, as opposed to space-time, so all eigenvalues of the sub matrix g_{ij} are positive. In this case we can use the case where $k > 0$,

$$ds^2 = k^{-1} \left[dx^2 + \frac{(x \cdot dx)^2}{(1 - x^2)} \right].$$

This is the metric of the curved space embedded by $kC_{\mu v} x^\mu x^v + z^2 = 1$. To evaluate $g_{ij} du^i du^j$, notice that

$$-d\tau^2 = g_{ab}(v) dv^a dv^b + f(v) g_{ij}(u) du^i du^j.$$

If v are the temporal components and u are the spatial components, then this reduce to a single statement, since we are interested in the curved space the line element take the form

$$-d\tau^2 = g_{ab}(v) dv^a dv^b + f(v) \left[d\vec{u}^2 + \frac{k(\vec{u} \cdot d\vec{u})^2}{(1 - k\vec{u}^2)} \right]. \quad (1.6.4)$$

Where $f(v)$ is a positive function and $g(v)$ is a negative function of v. k have the following properties

$$k = \begin{cases} +1, & if \;\; MSS \;\; has \;\; k > 0, \\ -1, & if \;\; MSS \;\; has \;\; k < 0, \\ 0, & if \;\; MSS \;\; has \;\; k = 0. \end{cases}$$

Since we are interested in a Maximally symmetric space III, we consider a space with N=4 case. Suppose three of the eigenvalues of its metric are positive and one of the eigenvalues is negative. Again suppose that it has maximally symmetric 3-D subspaces whose metric has positive eigenvalues and arbitrary curvature. Out of the four coordinates 1 v-coordinate and 3 u-coordinates. Making use of the relation

$$\int -[g(v)]^2 dv = t \Longrightarrow -dt^2 = g(v)dv^2.$$

In to equation(1.6.4) it can be rewritten as

$$-d\tau^2 = -dt^2 + f(v)\left[d\vec{u}^2 + \frac{k(\vec{u}\cdot d\vec{u})^2}{(1-k\vec{u}^2)}\right]. \tag{1.6.5}$$

Now the task is to evaluate the term in the bracket, that is

$$\left[d\vec{u}^2 + \frac{k(\vec{u}\cdot d\vec{u})^2}{(1-k\vec{u}^2)}\right].$$

The three components of a Maximally symmetric(spherically symmetric) space that are represented by u_1, u_2 and u_3 as usuall defined by

$$u_1 = rsin\theta cos\phi, \quad u_2 = rsin\theta sin\phi, \quad u_3 = rcos\theta.$$

From the definition given by dot product

$$\vec{u}^2 = u_1^2 + u_2^2 + u_3^2, \quad d\vec{u}^2 = du_1^2 + du_2^2 + du_3^2, \quad (\vec{u}\cdot d\vec{u})^2 = (u_1du_1 + u_2du_2 + u_3du_3)^2.$$

Substitution of the values u_1, u_2 and u_3 in to the above relation yields the following values

$$\vec{u}^2 = r^2, \quad d\vec{u}^2 = dr^2 + r^2d\theta^2 + r^2sin^2\theta d\phi^2, \quad \vec{u}\cdot d\vec{u} = rdr.$$

Making use of all these in to equation(1.6.5) the line element become

$$d\tau^2 = dt^2 - f(v)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2\right].$$

If we use $a^2(t)$ instead of $f(v)$ the line element rewritten as

$$d\tau^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2\right]. \tag{1.6.6}$$

Which is the metric in a maximally symmetric space III or it is called the **Friedmann-Robertson-Walker metric**[4,10,12]. This metric can describe an expanding, spatially homogeneous and isotropic universe in accord with the cosmological principle.

Note that, in this equation, $d\tau$ is the differential of the invariant distance, a is the dimensionless cosmic scale factor at which the Friedmann universe expands, r, θ, ϕ are co-moving spherical coordinates with an arbitrarily chosen origin, t is the clock time of an observer that is co- moving with the cosmic expansion and k is the curvature takes on the value 0 if space is Euclidean (flat), $k = +1$ if the space is positively curved, and $k = -1$ for negative curvature. To preserve dimensionality with our conventions, it must be that $k\ \alpha\ (length)^{-2}$. However, if the scale factor is taken to have dimensions of length, then k can be dimensionless. The quantity in brackets represents the spatial component of the metric. The radial and angular components of the metric are separated, and the angular component is familiar from spherical polar coordinates.

Chapter 2

The Friedmann Universe

2.1 Structure and Formation of the Universe

The standard theory of cosmic structure formation posits that the present day rich structure of the universe developed through gravitational amplification of tiny matter density fluctuations generated in its very early history[4]. One of the most accepted model that explain the origin of the universe is the Big-bang cosmology. The Big-bang model proposed that the universe originated from an infinitely dense point and evolved into our present universe[18].

During the early part of its existence, after one times ten to the minus 12th(1×10^{-12}) of a second, our universe was so small and dense that light and matter intertwined; space was hot, dark, and ionized-filled with a plasma of charged particles[15,16]. By the time the universe was one second old, the temperatures and densities had dropped enough for protons and neutrons to form from quarks. Within the next few minutes, the nuclei of the light elements, hydrogen, helium, and lithium, were created in a process called primal or Big Bang nucleosynthesis[12]. The universe at this point was cooling rapidly enough to shut off the process of nucleosynthesis before elements heavier than boron could form. About four hundred thousand years after the Big Bang the cosmos had grown large enough for matter and energy to move through space without immediately colliding-ending the plasma state of the early universe[1]. The universe had cooled to about 3,000 degrees

Celsius (5,400 degrees Fahrenheit) allowing electrons, protons, and neutrons to come together to form neutral atoms-the basic building blocks of all visible matter in the universe. This marked the "Decoupling" of matter and energy that we detect today as the cosmic microwave background radiation[7,16]. This radiation has been stretched and cooled by the expansion of the universe from three thousand degrees to minus 270.42 degrees Celsius, or just three degrees above absolute zero.

At this point the universe was made up mostly of clouds of hydrogen and helium atoms. As the universe expanded and cooled, some regions of space amassed slightly higher densities of hydrogen. As millions of years passed, the slight differences grew large, as dense areas drew in material because they had more gravity.

2.2 Constituting Matters of the Universe

Here we need to know how much of the constitutes the universe contains to day. Those amounts are generally expressed as a fractions of the critical density(critical energy density). The geometry and evolution of the universe are determined by the fractional contribution of various types of matters[4,9]. Since both energy density and pressure contribute to the strength of gravity in general relativity, cosmologists classify types of matter by its "equation of state" the relation ship between its pressure and energy density[12,15]. The basic classifications are:

Radiation: Composed of massless or nearly massless particles that move at the speed of light. Example; photons (light) and neutrinos. This form of matter is characterized by having a large positive pressure. At early epoch when the temperature is $K_BT \geq 3eV$ and the total energy density is made up of more radiations and relativistic particles, our universe was radiation-dominated[4].

Baryonic Matter: This is "ordinary matter" composed primarily of protons, neutrons and electrons. This form of matter has essentially no pressure of cosmological importance.

The baryonic matter is made of atoms and it contributes about 4 percent to the matter content of the universe[15].

Dark Matter: This generally refer to "exotic" non-baryonic matter that interacts only weakly with ordinary matter and very important in the formation of structures in the universe since it makes up for most of the matter content in the universe. While no such matter has ever been directly observed in the laboratory, its existence has long been suspected for reasons that by making accurate measurement of the cosmic microwave background fluctuations using an instrument Wilkinson Microwave Anisotropy Probe which is able to measure the interactions of the non-baryonic matter with ordinary matter all affect the details of the cosmic microwave background fluctuation spectrum. This form of matter also has no cosmologically significant pressure. Dark matter contributes about 25 percent of the mass-energy content of our universe[15,9].

Dark Energy: This is a truly bizarre form of matter, or perhaps a property of the vacuum itself, that is characterized by a large negative pressure. This is the only form of matter that can cause the expansion of the universe to accelerate or speed up. The contribution of dark energy is about 70 percent of the mass-energy content of the universe. When the universe is composed mainly of vacuum energy and its temperature is less than $K_B T < 3eV$, it is a dark energy dominated universe[7,12,16].

2.3 Derivation of the Friedmann Equations

To derive the Friedmann equations, we need to use the assumptions that the Friedmann universe is homogeneous and isotropic in its largest scale. Using the metric(Friedmann-Robertson-Walker metric) given by equation(1.6.6) we need to calculate the energy tensor T^i_k to describe the contents of the universe and then we compute the Einstein tensor and there by complete the Einstein field equation. To do these first let us evaluate the non vanishing components of the metric tensor(g_{ik}), affine connection(Γ^λ_{ik}) and the Ricci tensor(R_{ik}). Now we start with the metric tensor(g_{ik}). Referring back equation(1.6.6) that is

$$d\tau^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2sin^2\theta d\phi^2\right].$$

The metric tensor (g_{ik}), holds the coefficients to the coordinates in the line element:

$$d\tau^2 = g_{ik}dx^idx^k.$$

The Robertson-Walker line element is diagonal[15] i.e. $g_{ik} = 0$ if $i \neq k$ where the diagonal elements are functions of some of the 4-coordinates $x^i = x^0, x^1, x^2, x^3$. So the non vanishing components of g_{ik} are:

$$g_{00} = 1, \quad g_{11} = -\frac{a^2}{(1-kr^2)}, \quad g_{22} = -a^2r^2, \quad g_{33} = -a^2r^2sin^2\theta.$$

Since $g^{ik} = g_{ik}^{-1}$, the non vanishing components of g^{ik} are the inverse of g_{ik} these are:

$$g^{00} = 1, \quad g^{11} = -\frac{(1-kr^2)}{a^2}, \quad g^{22} = -\frac{1}{a^2r^2}, \quad g^{33} = -\frac{1}{a^2r^2sin^2\theta}.$$

Recall the definition given by the Christoffel symbol(Γ^λ_{ik}) that is

$$\Gamma^\lambda_{ik} = \frac{1}{2}g^{\sigma\lambda}\left(\frac{\partial g_{k\sigma}}{\partial x^i} + \frac{\partial g_{i\sigma}}{\partial x^k} - \frac{\partial g_{ki}}{\partial x^\sigma}\right).$$

Consider the very useful relations: $g^{ik}g_{ik} = 1$ and $\Gamma^\lambda_{ik} = \Gamma^\lambda_{ki}$, where the second relation implies that the Christoffel symbols are thus symmetric with respect to the exchange of indices $i \leftrightarrow k$. For convenience it is better to calculate for each $\lambda = 0, 1, 2$ and 3. Note for $\lambda = 0$ one can see that $\sigma = 0$, other wise $g^{\sigma\lambda} = 0(g_{ik} = 0$ for $i \neq k)$ and there by $\Gamma^0_{ik} = 0$. Because of the diagonality of the R-W metric this holds for all four cases: $\sigma = \lambda = 0, 1, 2, 3$. The non vanishing components of the Christoffel symbol(Γ^λ_{ik}) are:

The case $\lambda = 0$:

$$\Gamma^0_{ik} = \frac{1}{2}g^{00}\left(\frac{\partial g_{k0}}{\partial x^i} + \frac{\partial g_{i0}}{\partial x^k} - \frac{\partial g_{ki}}{\partial x^0}\right).$$

$$\Gamma^0_{11} = \frac{a\dot{a}}{(1-kr^2)}, \quad \Gamma^0_{22} = a\dot{a}r^2, \quad \Gamma^0_{33} = a\dot{a}r^2sin^2\theta.$$

The case $\lambda = 1$:

$$\Gamma^1_{ik} = \frac{1}{2}g^{11}\left(\frac{\partial g_{k1}}{\partial x^i} + \frac{\partial g_{i1}}{\partial x^k} - \frac{\partial g_{ki}}{\partial x^1}\right).$$

$$\Gamma^1_{01} = \Gamma^1_{10} = \frac{\dot{a}}{a}\ ,\quad \Gamma^1_{11} = \frac{kr}{(1-kr^2)}\ ,\quad \Gamma^1_{22} = -r(1-kr^2)\ ,\quad \Gamma^1_{33} = -r(1-kr^2)sin^2\theta.$$

The case $\lambda = 2$:

$$\Gamma^2_{ik} = \frac{1}{2}g^{22}\left(\frac{\partial g_{k2}}{\partial x^i} + \frac{\partial g_{i2}}{\partial x^k} - \frac{\partial g_{ki}}{\partial x^2}\right).$$

$$\Gamma^2_{02} = \Gamma^2_{20} = \frac{\dot{a}}{a}\ ,\quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}\ ,\quad \Gamma^2_{33} = -sin\theta cos\theta.$$

The case $\lambda = 3$:

$$\Gamma^3_{ik} = \frac{1}{2}g^{33}\left(\frac{\partial g_{k3}}{\partial x^i} + \frac{\partial g_{i3}}{\partial x^k} - \frac{\partial g_{ki}}{\partial x^3}\right).$$

$$\Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{a}}{a}\ ,\quad \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}\ ,\quad \Gamma^3_{23} = \Gamma^3_{32} = cot\theta.$$

Recall the expression given by the Ricci tensor:

$$R_{ik} = \frac{\partial \Gamma^\lambda_{i\lambda}}{\partial x^k} - \frac{\partial \Gamma^\lambda_{ik}}{\partial x^\lambda} + \Gamma^\eta_{i\lambda}\Gamma^\lambda_{k\eta} - \Gamma^\eta_{ik}\Gamma^\lambda_{\lambda\eta}.$$

Here also all non diagonal components vanish ($R_{ik} = 0$) for $i \neq k$. The non vanshing components of the Ricci tensor are:

The time-time component(i.e. $ik = 00$):

$$R_{00} = \frac{\partial \Gamma^\lambda_{0\lambda}}{\partial x^0} - \frac{\partial \Gamma^\lambda_{00}}{\partial x^\lambda} + \Gamma^\eta_{0\lambda}\Gamma^\lambda_{0\eta} - \Gamma^\eta_{00}\Gamma^\lambda_{\lambda\eta} = \frac{\partial \Gamma^\lambda_{0\lambda}}{\partial x^0} + \Gamma^\eta_{0\lambda}\Gamma^\lambda_{0\eta}.$$

Writing the components explicitly for($\eta = \lambda = 0, 1, 2, 3$) and substituting the corresponding components of affine connection gives

$$R_{00} = 3\frac{\ddot{a}}{a}.$$

The space-space components(i.e. $ik = 11, 22, 33$):

The $ik = 11$ component:

$$R_{11} = \frac{\partial \Gamma^\lambda_{1\lambda}}{\partial x^1} - \frac{\partial \Gamma^\lambda_{11}}{\partial x^\lambda} + \Gamma^\eta_{1\lambda}\Gamma^\lambda_{1\eta} - \Gamma^\eta_{11}\Gamma^\lambda_{\lambda\eta}.$$

Writing the components explicitly for$(\eta = \lambda = 0, 1, 2, 3)$ and substituting the corresponding components of the Christoffel symbol gives

$$R_{11} = \frac{1}{(1 - kr^2)}(-a\ddot{a} - 2\dot{a}^2 - 2k).$$

The $ik = 22$ component:

$$R_{22} = \frac{\partial \Gamma^{\lambda}_{2\lambda}}{\partial x^2} - \frac{\partial \Gamma^{\lambda}_{22}}{\partial x^{\lambda}} + \Gamma^{\eta}_{2\lambda}\Gamma^{\lambda}_{2\eta} - \Gamma^{\eta}_{22}\Gamma^{\lambda}_{\lambda\eta}.$$

Here also writing the components explicitly for$(\eta = \lambda = 0, 1, 2, 3)$ and substituting the corresponding components of the Christoffel symbol gives

$$R_{22} = -(2\dot{a}^2 + a\ddot{a} + 2k)r^2.$$

The $ik = 33$ component:

$$R_{33} = \frac{\partial \Gamma^{\lambda}_{3\lambda}}{\partial x^3} - \frac{\partial \Gamma^{\lambda}_{33}}{\partial x^{\lambda}} + \Gamma^{\eta}_{3\lambda}\Gamma^{\lambda}_{3\eta} - \Gamma^{\eta}_{33}\Gamma^{\lambda}_{\lambda\eta}.$$

Similarly by writing the components explicitly for$(\eta = \lambda = 0, 1, 2, 3)$ and substituting the corresponding components of the Christoffel symbol gives

$$R_{33} = -(2\dot{a}^2 + a\ddot{a} + 2k)r^2 sin^2\theta.$$

The space-space components of the Ricci tensor(R_{ik}) can be expressed by the space-space components of the metric tensor g_{ik} as

$$R_{11} = \left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2 + 2k}{a^2}\right) g_{11} \ , \quad R_{22} = \left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2 + 2k}{a^2}\right) g_{22} \ , \quad R_{33} = \left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2 + 2k}{a^2}\right) g_{33}.$$

This implies that

$$R_{ik} = \left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2 + 2k}{a^2}\right) g_{ik}.$$

This symmetry reflects the isotropy of space. Recall the Ricci scaler obtained by contracting the Ricci tensor and note that $g^{ik}g_{ik} = 1$

$$R = g^{ik}R_{ik} \Longrightarrow R = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33}.$$

Substituting the values of $R_{00}, R_{11}, R_{22}, R_{33}$ in to the above relation to obtain

$$R = 3\frac{\ddot{a}}{a} + 3\left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2 + 2k}{a^2}\right) = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right). \tag{2.3.1}$$

Referring back the Einstein tensor that was written as

$$G_{ik} = R_{ik} - \frac{1}{2}g_{ik}R.$$

The time-time component is:

$$G_{00} = R_{00} - \frac{1}{2}g_{00}R = -3\left(\frac{\dot{a}^2 + k}{a^2}\right). \tag{2.3.2}$$

The space-space components are:

$$G_{ik} = R_{ik} - \frac{1}{2}g_{ik}R = \left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2 + 2k}{a^2}\right)g_{ik} - 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right)g_{ik},$$

$$G_{ik} = -\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right)g_{ik}. \tag{2.3.3}$$

Where $ik = 11, 22, 33$. Using Einstein's field equation(1.4.15) that is

$$G_{ik} = R_{ik} - \frac{1}{2}g_{ik}R = -8\pi G T_{ik}.$$

This gives that

$$R_{00} - \frac{1}{2}g_{00}R = -3\left(\frac{\dot{a}^2 + k}{a^2}\right) = -8\pi G T_{00}.$$

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8}{3}\pi G T_{00}. \tag{2.3.4}$$

Similarly the space components part of Einstein's field equation take the form

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} = 8\pi G T_{11} = 8\pi G T_{22} = 8\pi G T_{33}. \tag{2.3.5}$$

Then equation(2.3.4) and equation(2.3.5) are called **Friedmann's equations**.

Where T_{00} and $T_{11} = T_{22} = T_{33}$ are energy density and pressure of the Friedmann universe respectively, which are determine in the next section.

2.4 Energy-Momentum Tensor of the Source

The energy-momentum tensor is needed to describe the mass and energy contents of the universe[15]. Isotropy requires the mean value of any three-tensor t^{ij} at $x = 0$ to be proportional to δ_{ij} and hence to g^{ij} which equals $a^{-2}\delta_{ij}$ at $x = 0$. Homogeneity requires the proportionality coefficient to be some function only of time. Since this is a proportionality between two three tensors t^{ij} and g^{ij}, it must remain unaffected by an arbitrary transformation of space coordinates including those transformations that preserve the form of g^{ij} while taking the origin in to any other point. Now let us consider the two types of momentum-energy tensor frequently used in GR: dust and perfect fluid. **Dust** is the simplest possible energy-momentum tensor. It is given by

$$T^{\alpha\beta} = \rho u^{\alpha} u^{\beta}.$$

Dust is an approximation of the Universe at later times, when radiation is negligible[4]. **Perfect fluid** is a fluid that has no heat conduction or viscosity. It is fully parametrized by its mass density ρ and the pressure P . It is given by

$$T^{\alpha\beta} = (\rho + P)u^{\alpha} u^{\beta} + P g^{\alpha\beta}.$$

In the limit of $P \longrightarrow 0$, the perfect fluid approximation reduces to that of dust. Perfect fluid is an approximation of the universe at earlier times, when radiation dominates[9]. Conservation equations for the energy-momentum tensor $T^{\alpha\beta}$ are simply given by[4]

$$T^{\alpha\beta}_{;\beta} = 0. \tag{2.4.1}$$

This expression incorporates both energy and momentum conservations in a general metric. Conservation of energy given in equation(2.4.1) can be used to determine how components of the energy-momentum tensor evolve with time. Hence, isotropy and homogeneity require the components of the energy-momentum tensor every where to take the form

$$T^{i}_{0} = 0 \ , \quad T^{00} = \rho(t) \ , \quad T^{ik} = T_{ik} = g^{ik} a^{-2}(t) P(t). \tag{2.4.2}$$

The metric tensor g_{ik} is scaled by $a^2(t)$ from η_{ik}. As shown by the matrix representation

$$g^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(t) & 0 & 0 \\ 0 & 0 & -a^2(t) & 0 \\ 0 & 0 & 0 & -a^2(t) \end{pmatrix}$$

Applying the diagonal elements of g^{ik} in to equation(2.4.2) to get

$$T_{00} = \rho(t)\ , \quad T_{11} = T_{22} = T_{33} = -P(t).$$

This implies that

$$T^i_k = diag(\rho(t), -P(t), -P(t), -P(t)).$$

Therefore, with these results the Friedmann equations, equation(2.3.4) and equation(2.3.5) can be rewritten as

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8}{3}\pi G\rho(t) = \frac{8}{3}\pi G\rho, \tag{2.4.3}$$

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} = -8\pi GP(t) = -8\pi GP. \tag{2.4.4}$$

The first equation states that the energy of expansion is proportional to the sum of matter energy, cosmological(vaccum) energy, radiation energy and curvature energy.

From equation(2.4.3) we have

$$\frac{\dot{a}^2}{a^2} = \frac{8}{3}\pi G\rho - \frac{k}{a^2}.$$

This equation describes how the expansion velocity($\dot{a}$) depends on energy density and curvature. Again from the above equation for $k = 0$ we can see that

$$\rho_c = \frac{3}{8\pi G}\left(\frac{\dot{a}}{a_o}\right)^2 \equiv \frac{3H_o^2}{8\pi G} \cong 1.88 \times 10^{-29} h^2 gcm^{-3}.$$

Where the critical density(ρ_c) is defined to be the density required for the universe to be flat, $k = 0$ and $H_o = (\dot{a}/a)_o$ is the Hubble definition from his observation[4,18], which is observed to have the value $100hkms^{-1}Mpc^{-1}$ with $0.5 < h < 1$. The parameter h quantifies the uncertainty in the measurements of the Hubble constant. The ratio of the actual density(ρ) of the universe to the critical density(ρ_c) is given by omega(Ω) i.e. $\Omega = \rho/\rho_c$. Using the definition of Hubble in to equation(2.4.3) it is possible to write as

$$a^2 = -\frac{3k}{3H^2 - 8\pi G\rho} = -\frac{3k}{3\left(H^2 - \frac{8\pi G\rho}{3}\right)}.$$

We can rewrite this as

$$a^2 = -\frac{k}{\left(H^2 - \frac{8\pi G\rho H^2}{3H^2}\right)}.$$

Now making use of the relation given by ρ_c in to the above equation yields

$$a^2 = -\frac{k}{H^2\left(1 - \frac{\rho}{\rho_c}\right)} = -\frac{k}{H^2(1-\Omega)} = \frac{k}{H^2(\Omega - 1)}.$$

From this we have

$$a = \frac{1}{H}\left[\frac{(\Omega - 1)}{k}\right]^{-1/2} \Longrightarrow \dot{a}(t_0) = \left[\frac{(\Omega - 1)}{k}\right]^{-1/2}. \quad (2.4.5)$$

Depending on equation(2.4.5) for different values of k we have the following information.

For $k = -1$, $\Omega < 1$ and a is imaginary. The universe expands forever, there not being sufficient density for gravitational attraction to stop the expansion, this implies open universe with hyperbolic geometry.

For $k = 0$, $\Omega = 1$ and $a \longrightarrow \infty$. In this case the density is equal to a critical value at which the universe will expand forever at a decreasing rate, this implies flat universe with euclidean geometry.

For $k = 1$, $\Omega > 1$ and a is real. Here the density is high enough that the gravitational attraction will eventually stop the expansion and it will collapse backward to a "big crunch", this implies closed universe with spherical geometry .

Referring back equation(2.4.4) it is possible to write as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P). \quad (2.4.6)$$

Equation(2.4.6) describes the acceleration($\ddot{a}$) of the scale factor depending on pressure and density. In this equation if the term in the bracket is negative, then the RHS of equation(2.4.6) is positive, and this entails that d^2a/dt^2 is positive, i.e., the rate of expansion is increasing. If the term in the bracket is positive, d^2a/dt^2 must be negative, which means that the rate of expansion is decreasing.

It is also possible to argue the following statement or predict a singular beginning of the universe. As we have seen above one amazing thing about the Friedmann universe is that all of them(provided that the matter content is reasonably physical) predict an initial singularity, commonly known as a **Big Bang**. Equation(2.4.5) shows that, as long as the RHS of the equation is positive, it is true that $q > 0$(positive decceleration parameter) i.e. $\ddot{a} < 0$ so that the universe is deccelerating due to gravitational attraction. Since $a > 0$ by definition, $\dot{a}(t_0) > 0$ because we observe a red-shift and $\ddot{a} < 0$ because $3P + \rho > 0$, it follows that there cannot have been a turning point in the past and $a(t)$ must be concave down wards. Therefore $a(t)$ must have reached $a = 0$ at some finite time in the past. The time is called as $t = 0 = t_{sing}$, $a_0 = 0$. As ρa^4 is constant for radiation that we have to discuss latter(an appropriate description of earlier periods of the universe), this shows that the energy density of radiation grows like $1/a^4$ as $a \longrightarrow 0$ so this leads to quite a singular situation. With the normalization $a_0 = 0$, it is fair to call t_0 as the age of the universe. If $\ddot{a}$ had been zero in the past for all $t \leq t_0$, then we would have $\ddot{a} = 0$ this implies that $a(t) = a_0 t/t_0$ and $\dot{a}(t) = a_0/t_0 = \dot{a}_0$, this would determine the age of the universe to be $t_0 = a_0/\dot{a}_0 = H_0^{-1}$ where H_0^{-1} is the Hubble time. In other word, since $t = t_{sing} = 0$, we can write as $t_0 - t_{sing} \equiv H_0^{-1}$. However, provided that $\ddot{a} < 0$ for $t < t_0$ the actual age of the universe must be smaller than this. So that with this idea we can write as for $\ddot{a} < 0$ it is true that $t_0 < H_0^{-1}$ this implies $t_0 - t_{sing} < t_{universe} \equiv H_0^{-1}$. Where $H_0^{-1} = 9.8 \times 10^9 h^{-1} yr$. Thus, the Hubble times sets an upper-bound on the age of the universe.

Chapter 3

Dynamics of The Friedmann Universe

3.1 General Equation of the Friedmann Universe

To study the Friedmann equations with a given equations of state, first let us write the equations in a compact form. Starting from the two Friedmann equations which are

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} = -8\pi G P,$$
$$\frac{\dot{a}^2 + k}{a^2} = \frac{8}{3}\pi G\rho.$$

Substitution of the RHS of the second equation in to the second term of the LHS of the first equation and after some mathematics we get the relation

$$\ddot{a} = -\frac{4}{3}\pi G(\rho + 3P)a. \tag{3.1.1}$$

And simply from the second Friedmann equation it is possible to write as

$$\frac{\dot{a}^2}{a^2} = \frac{8}{3}\pi G\rho - \frac{k}{a^2}. \tag{3.1.2}$$

Now differentiating equation(3.1.2) with respect to t as follows

$$\frac{d}{dt}\left(\frac{\dot{a}^2}{a^2}\right) = \frac{d}{dt}\left(\frac{8\pi G\rho}{3} - \frac{k}{a^2}\right),$$

$$\frac{2\dot{a}\ddot{a}a^2 - 2\dot{a}^2 a\dot{a}}{a^4} = \frac{8\pi G\dot{\rho}}{3} + \frac{2ka\dot{a}}{a^4},$$
$$\frac{2\dot{a}}{a^3}(\ddot{a}a - \dot{a}^2) = \frac{8\pi G\dot{\rho}}{3} + \frac{2k\dot{a}}{a^3}.$$

Making use of equation(3.1.1) in to the above equation so as to write as

$$\frac{2\dot{a}}{a^3}\left[-\frac{4}{3}\pi G(\rho + 3P)a^2 - \dot{a}^2\right] = \frac{8\pi G\dot{\rho}}{3} + \frac{2k\dot{a}}{a^3}.$$

This implies that

$$\frac{\dot{a}}{a}\left[-\frac{8}{3}\pi G(\rho + 3P) - \frac{2\dot{a}^2}{a^2}\right] = \frac{8\pi G\dot{\rho}}{3} + \frac{2k\dot{a}}{a^3}.$$

Again after substituting equation(3.1.2) in to the above equation and multiplying the whole expression by $a/\dot{a}$ we have the following result

$$\left[-\frac{8}{3}\pi G(\rho + 3P) - \frac{16\pi G\rho}{3} + \frac{2k}{a^2}\right] = \frac{8\pi G\dot{\rho}}{3}\left(\frac{a}{\dot{a}}\right) + \frac{2k}{a^2},$$
$$-\frac{8\pi G\rho}{3} - 8\pi GP - \frac{16\pi G\rho}{3} = \frac{8\pi G\dot{\rho}}{3}\left(\frac{a}{\dot{a}}\right),$$
$$-8\pi G\rho - 8\pi GP = \frac{8\pi G\dot{\rho}}{3}\left(\frac{a}{\dot{a}}\right),$$
$$-8\pi G(\rho + P) = 8\pi G\left(\frac{\dot{\rho}a}{3\dot{a}}\right).$$

After a little algebra, this gives that

$$-3\rho\dot{a} - 3P\dot{a} = \dot{\rho}a.$$

Now multiplying this both side by a^2 to write as

$$-3\rho\dot{a}a^2 - 3P\dot{a}a^2 = \dot{\rho}a^3,$$

$$\dot{\rho}a^3 + 3\rho\dot{a}a^2 = -3P\dot{a}a^2. \tag{3.1.3}$$

This can be rewritten as

$$\frac{d}{dt}(\rho a^3) = -P\frac{d}{dt}(a^3).$$

If the equation of state has the form $P = \omega\rho$, and for matter dominated universe (i.e. $\omega = 0 \Longrightarrow P = 0$) we can write the above equation as

$$\frac{d}{dt}(\rho_m a^3) = 0.$$

This implies that $\rho_m a^3 = \rho_{m_0} a_0^3$ =constant. Which is rewritten as

$$\rho_m = \left(\frac{a_0}{a}\right)^3 \rho_{m_0}. \tag{3.1.4}$$

In the same way for radiation dominated universe(i.e. $\omega = 1/3 \implies P = \rho/3$), from equation(3.1.3) we have

$$\dot{\rho}_r a^3 + 3\rho_r \dot{a} a^2 + \rho_r \dot{a} a^2 = 0,$$

$$\dot{\rho}_r + 3\left(\rho_r + \frac{\rho_r}{3}\right)\frac{\dot{a}}{a} = 0,$$

$$\dot{\rho}_r a + 4\rho_r \dot{a} = 0.$$

Now multiplying both side of the above equation by a^3 to get

$$\dot{\rho}_r a^4 + 4\rho_r \dot{a} a^3 = 0.$$

Which can be written as

$$\frac{d}{dt}(\rho_r a^4) = 0.$$

This implies that $\rho_r a^4 = \rho_{r_0} a_0^4$ =constant. And this leads us to write in the form

$$\rho_r = \left(\frac{a_0}{a}\right)^4 \rho_{r_0}. \tag{3.1.5}$$

And also for stiff matter (i.e. $\omega = 1 \implies P = \rho_{sm}$) equation(3.1.3) can be written as

$$\dot{\rho}_{sm} a^3 + 6\rho_{sm} \dot{a} a^2 = 0.$$

Now multiplying both side of the above equation by a^3 and rewrite as

$$\dot{\rho}_{sm} a^6 + 6\rho_{sm} \dot{a} a^5 = \frac{d}{dt}(\rho_{sm} a^6) = 0.$$

This implies that $\rho_{sm} a^6 = \rho_{sm_0} a_0^6$ =constant. Therefore, we have

$$\rho_{sm} = \left(\frac{a_0}{a}\right)^6 \rho_{sm_0}.$$

Referring back equation(3.1.3) to write the equation as

$$\frac{\dot{\rho}}{\rho} = -\frac{3\dot{a}}{a}\left(\frac{P}{\rho} + 1\right).$$

Making use of the equation of state ($P = \omega\rho$) in to the above relation yields

$$\frac{\dot{\rho}}{\rho} = -\frac{3\dot{a}}{a}(\omega + 1).$$

Integrating this with respect to t gives

$$\int \frac{\dot{\rho}}{\rho} dt = -3(\omega + 1) \int \frac{\dot{a}}{a} dt,$$

$$ln(\rho) = -3(\omega + 1) ln(a).$$

Taking exponential both side results

$$\rho \; \alpha \; a^{-3(\omega+1)}. \tag{3.1.6}$$

Where ω is constant in time. For different values of ω we have different energy density values.

For $\omega = 0$ [matter(baryonic + dark matter) dominated universe, i.e. $P = 0$],

$$\rho_m \; \alpha \; a^{-3}$$

With this relation, energy density of the matter can be calculated using $\Omega = \rho/\rho_c$, that means.

$$\Omega_m = \rho_m/\rho_c = \frac{8\pi G \rho_m}{3H^2} = \frac{8\pi G \rho_{m_0}}{3H^2} \left(\frac{a_0}{a}\right)^3.$$

For $\omega = 1/3$ (radiation dominated universe, i.e. $P = \rho/3$),

$$\rho_r \; \alpha \; a^{-4}$$

In the same way energy density of radiation is

$$\Omega_r = \rho_r/\rho_c = \frac{8\pi G \rho_r}{3H^2} = \frac{8\pi G \rho_{r_0}}{3H^2} \left(\frac{a_0}{a}\right)^4.$$

For $\omega = -1$ (vacuum dominated universe, i.e. $P = -\rho$),

$$\rho_v \; \alpha \; constant = \frac{\Lambda}{8\pi G}.$$

Similarly energy density of a vacuum is determined as

$$\Omega_v = \rho_v/\rho_c = \frac{8\pi G\rho_v}{3H^2} = \frac{8\pi G}{3H^2}\frac{\Lambda}{8\pi G} = \frac{\Lambda}{3H^2}.$$

For $\omega = 1$ (stiff matter dominated, i.e. $P = \rho$, known as 'stiff' equation of state),

$$\rho_{sm}\ \alpha\ a^{-6} = \left(\frac{a_0}{a}\right)^6 \rho_{sm_0}.$$

And also energy density due to stiff matter can be calculated as

$$\Omega_{sm} = \rho_{sm}/\rho_c = \frac{8\pi G\rho_{sm}}{3H^2} = \frac{8\pi G\rho_{sm_0}}{3H^2}\left(\frac{a_0}{a}\right)^6.$$

Here onwards we consider only the two kinds of matters(baryonic and non-baryonic). Recall the second Friedmann equation, that was given by

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8}{3}\pi G\rho.$$

Where $\rho = \rho_m(\rho_b + \rho_{dm}) + \rho_r + \rho_v$, which are the constituting densities of the universe that we consider here after. ρ_b and ρ_{dm} stands for baryonic matter and dark matter(non-baryonic) respectively. Then with this substitution the above Friedmann equation become

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8\pi G}{3}[\rho_r + \rho_m(\rho_b + \rho_{dm}) + \rho_v].$$

Now making use of the relation given by ρ_m, ρ_r and ρ_v in to the Friedmann equation given above yields

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8\pi G}{3}\left[\left(\frac{a_0}{a}\right)^4 \rho_{r_0} + \left(\frac{a_0}{a}\right)^3 \rho_{m_0} + \frac{\Lambda}{8\pi G}\right].$$

Multiplying the RHS of the above equation by ρ_{c_0}/ρ_{c_0} gives

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8\pi G\rho_{c_0}}{3}\left[\left(\frac{a_0}{a}\right)^4 \left(\frac{\rho_{r_0}}{\rho_{c_0}}\right) + \left(\frac{a_0}{a}\right)^3 \left(\frac{\rho_{m_0}}{\rho_{c_0}}\right) + \frac{\Lambda}{8\pi G\rho_{c_0}}\right].$$

But from $\rho_{c_0} = 3H_0^2/8\pi G$ we have $H_0^2 = 8\pi G\rho_{c_0}/3$ and we know that

$$\Omega_{r_0} = \rho_{r_0}/\rho_{c_0}\ ,\quad \Omega_{m_0} = \rho_{m_0}/\rho_{c_0}\ ,\quad \Omega_{v_0} = \rho_v/\rho_{c_0} = \frac{\Lambda}{8\pi G\rho_{c_0}} = \frac{\Lambda}{3H_0^2}.$$

Now substituting all these facts in to the above large equation to obtain

$$\frac{\dot{a}^2 + k}{a^2} = H_0^2 \left[\Omega_{r_0} \left(\frac{a_0}{a} \right)^4 + \Omega_{m_0} \left(\frac{a_0}{a} \right)^3 + \Omega_{v_0} \right]. \tag{3.1.7}$$

Where $\Omega_{m_0} = \Omega_{b_0} + \Omega_{dm_0}$ and $\Omega_{m_0} + \Omega_{r_0} + \Omega_{v_0} = 1$. Equation(3.1.7) is the Friedmann equation that describe the evolution of the universe that contains all the above constituting matters. As we can see from the equation the RHS shows that different kinds of matter will be influencing the dynamics of the universe at different epochs.

3.2 Time Dependant Solutions of the Friedmann Equations

The universe today is comprised of a mixture of matter(with zero pressure) and radiation[15]. Galaxies appear to be electrically neutral and so it is reasonable to assume that matter and radiation are effectively uncoupled, in the sense that they do not interact through electromagnetism[4]. Thus, both types of matter satisfy the conservation equation (with appropriate equation of state) and the total density is given by $\rho_{tot} = \rho_r + \rho_m$, where $\rho_r \ \alpha \ a^{-4}$ and $\rho_m \ \alpha \ a^{-3}$ for radiation and matter respectively. The ratio of the radiation and matter densities scales as $(\rho_r/\rho_m) \ \alpha \ a^{-1}$ and falls as the universe expands[10]. Therefore, it is expected that the matter to dominate the radiation at sufficiently late times(assuming implicitly that the universe continues to expand indefinitely). On the other hand, the radiation will dominate at earlier times. In order to determine when the matter and radiation come to dominate, we need to solve the Friedmann equations for a universe dominated by matter and radiation independently and then we will answer the question when does the transition from radiation domination to matter domination occur.

3.2.1 For Matter Dominated Era with $k = \Omega_{v_0} = 0,\ k = 1, -1$, but $\Omega_{v_0} = 0$ and with $k = 0$, but $\Omega_{v_0} \neq 0$

The case $k = \Omega_{v_0} = 0$, Euclidean section: since $P = 0$ for matter dominated universe the two Friedmann equations can be rewritten as

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = 0, \tag{3.2.1}$$

$$\frac{\dot{a}^2}{a^2} = \frac{8}{3}\pi G\rho_m = \frac{8}{3}\pi G\rho_{m_0}\frac{a_0^3}{a^3}. \tag{3.2.2}$$

Where $a = a(t)$ but for the sake of simplicity we used a instead of $a(t)$. Taking equation(3.2.2) it is possible to express as

$$\dot{a} = \frac{da}{dt} = \left(\frac{8}{3}\pi G\rho_{m_0}a_0^3\right)^{1/2}\frac{a}{a^{3/2}},$$

$$\frac{da}{dt}a^{1/2} = \left(\frac{8}{3}\pi G\rho_{m_0}\right)^{1/2}a_0^{3/2},$$

$$a^{1/2}da = \left(\frac{8}{3}\pi G\rho_{m_0}a_0^3\right)^{1/2}dt.$$

Integration of this from $a_0 \longrightarrow a$ yields

$$\frac{2}{3}(a^{3/2} - a_0^{3/2}) = \left(\frac{8}{3}\pi G\rho_{m_0}a_0^3\right)^{1/2}t.$$

Where in the last line we used the assumption that $a_0 = 0$ at $t = 0$ when the distance between any two points was zero, this gives

$$\frac{2}{3}a^{3/2} = \left(\frac{8}{3}\pi G\rho_{m_0}\right)^{1/2}a_0^{3/2}t,$$

$$a = \frac{3}{2}\left(\frac{8}{3}\pi G\rho_{m_0}\right)^{1/3}a_0t^{2/3}.$$

But we know that, for $k = 0$

$$\rho_c = \rho_{m_0} = \frac{3H_0^2}{8\pi G} \Longrightarrow H_0^2 = \frac{8}{3}\pi G\rho_{m_0}. \tag{3.2.3}$$

Where ρ_c is the critical density and $H_0 = \dot{a}(t_0)/a_0$ is the Hubble constant. Making use of this in to the bracket to obtain

$$a = \frac{3}{2}\left(H_0^2\right)^{1/3} a_0 t^{2/3},$$

$$a = a_0\left(\frac{t}{t_0}\right)^{2/3}. \tag{3.2.4}$$

Where $t_0 = 2/3H_0 \approx 9.1 \times 10^9$ years. The constant a_0 is not determined and it has the dimension of length.

The case $k = 1$, but $\Omega_{v_0} = 0$, Closed section: For this case the Friedmann equations become

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2 + 1}{a^2} = 0, \tag{3.2.5}$$

$$\frac{\dot{a}^2 + 1}{a^2} = \frac{8\pi G\rho_m}{3} = \frac{8\pi G\rho_{m_0}}{3}\frac{a_0^3}{a^3}. \tag{3.2.6}$$

Here, it is better to introduce the decceleration parameter q that used to measure the accelerated expansion of the universe. Which is given by

$$q = -\frac{\ddot{a}a}{\dot{a}^2} \Longrightarrow \frac{\ddot{a}}{a} = -qH^2.$$

For our work $q_0 = -\ddot{a}a_0/\dot{a}^2$ and $H_0 = \dot{a}/a_0$ denote the present values. Using the defination given by the decceleration parameter and the Hubble constant in to the Friedmann equation(3.2.5)

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2 + 1}{a^2} = 0 \Longrightarrow \frac{1}{a^2} = -\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = (2q - 1)H^2.$$

In terms of q_0 and H_0, it is rewritten as

$$\frac{1}{a_0^2} = (2q_0 - 1)H_0^2. \tag{3.2.7}$$

And also equation(3.2.6) take the form

$$\left(H_0^2 + \frac{1}{a_0^2}\right) = \frac{8}{3}\pi G\rho_{m_0}\frac{a_0^3}{a^3}.$$

Substituting equation(3.2.7) in to the above equation and solving for ρ_{m_0} to obtain

$$\rho_{m_0} = \frac{3H_0^2 q_0}{4\pi G}. \tag{3.2.8}$$

Now using the relation given by Ω it is possible to write equation(3.2.8) as

$$\Omega = \frac{\rho}{\rho_c} \Longrightarrow \Omega_{m_0} = \frac{\rho_{m_0}}{\rho_{c0}}.$$

$$\rho_{m_0} = \Omega_{m_0}\rho_{c_0}.$$

Making use of equation(3.2.3) in to the above equation to get

$$\rho_{m_0} = \Omega_{m_0}\left(\frac{3H_0^2}{8\pi G}\right).$$

Comparing this equation with equation(3.2.8) to write

$$\Omega_{m_0} = 2q_0.$$

Since the left hand side of equation(3.2.7) is positive, it must be that $q_0 > \frac{1}{2}$ or $\Omega_{m_0} > 1$. Thus the closed model has density exceeding the so called critical density ρ_c. It is the value of the universal density that must be exceeded if the model is to describe a closed universe.

Now to eliminate a_0 and ρ_{m_0} from equation(3.2.6) we used equation(3.2.7) and equation(3.2.8). From equation(3.2.7) we have

$$a_0^2 = \frac{1}{(2q_0 - 1)H_0^2} \Longrightarrow a_0^3 = \frac{1}{(2q_0 - 1)^{3/2} H_0^3}.$$

Substituting this equation and equation(3.2.8) in to equation(3.2.6) gives

$$\frac{\dot{a}^2 + 1}{a^2} = \left(\frac{2q_0}{3a^3}\right)\left(\frac{1}{H_0(2q_0 - 1)^{3/2}}\right).$$

This implies that

$$\dot{a}^2 = \frac{1}{a}\left(\frac{2q_0}{H_0(2q_0 - 1)^{3/2}}\right) - 1 = \frac{\alpha}{a} - 1. \tag{3.2.9}$$

Where

$$\alpha = \frac{2q_0}{H_0(2q_0 - 1)^{3/2}} = \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}\frac{1}{H_0}.$$

This can be integrated as

$$\frac{da}{dt} = \left(\frac{\alpha}{a} - 1\right)^{1/2},$$

$$\int dt = \int \frac{\sqrt{a}}{\sqrt{\alpha - a}} da,$$

$$t = \int \frac{\sqrt{a}}{\sqrt{\alpha - a}} da.$$

Applying integration by substitution let

$$a = \alpha sin^2\frac{\theta}{2} = \frac{1}{2}\alpha(1 - cos\theta), \quad da = \alpha sin^2\frac{\theta}{2} d\theta.$$

Substituting this in to the above integral gives

$$t = \int \alpha sin^2\frac{\theta}{2} d\theta = \int \frac{1}{2}\alpha(1 - cos\theta) d\theta,$$

$$t = \int \frac{1}{2}\alpha(1 - cos\theta) d\theta = \frac{1}{2}\alpha(\theta - sin\theta). \tag{3.2.10}$$

In the same fashion to the Euclidean section where $k = 0$ here also we have taken $a = 0$ at $t = 0$, $(\theta = 0)$. Therefore it is possible to get $t = t_0$ by requiring $a = a_0$.

From equation(3.2.7) and the definition given for α, we can see that $a = a_0$ at $\theta = \theta_0$, where

$$\frac{1}{2}\alpha(1 - cos\theta_0) = \frac{1}{H_0}(2q_0 - 1)^{-1/2} = \frac{(2q_0 - 1)}{2q_0}\alpha.$$

That is

$$1 - cos\theta_0 = \frac{2q_0 - 1}{q_0} \Longrightarrow cos\theta_0 = \frac{1 - q_0}{q_0}, \quad \theta_0 = cos^{-1}\left(\frac{1 - q_0}{q_0}\right).$$

Using trigonometric identity, $sin^2\theta_0 + cos^2\theta_0 = 1 \Longrightarrow sin^2\theta_0 = 1 - cos^2\theta_0$

For $\theta = \theta_0$ together with the trigonometric identity equation(3.2.10) can be rewritten as

$$t_0 = \frac{q_0}{(2q_0 - 1)^{3/2}} \frac{1}{H_0} \left[cos^{-1}\left(\frac{1 - q_0}{q_0}\right) - \frac{\sqrt{2q_0 - 1}}{q_0}\right]. \tag{3.2.11}$$

For $(q_0 = 1)$ we get

$$t_0 = \left(\frac{\pi}{2} - 1\right) H_0^{-1}.$$

Note that a reaches a maximum value at $\theta = \pi$, when

$$a = a_{maxx} = \alpha = \frac{2q_0}{(2q_0 - 1)^{3/2}} \frac{1}{H_0}.$$

Thus for $q = 1$, the universe expands twice its present size. In closed models, therefore, expansion is followed by contraction and a decreases to zero. The value $a = 0$ is reached when $\theta = 2\pi$; that is, when

$$t = t_l = \pi\alpha = \frac{2\pi q_0}{(2q_0 - 1)^{3/2}} \frac{1}{H_0}.$$

The quantity t_l may be termed the life span of this universe. For $q_0 = 1,\ t_l = 2\pi H_0^{-1} = 2\pi T_0$, where $T_0 = H_0^{-1}$ which is reciprocal of time.

The case $k = -1$, but $\Omega_{v_0} = 0$, Open section: For such a case the Friedmann equations take the form

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2 - 1}{a^2} = 0, \tag{3.2.12}$$

$$\frac{\dot{a}^2 - 1}{a^2} = \frac{8\pi G\rho_{m_0}}{3} \frac{a_0^3}{a^3}. \tag{3.2.13}$$

In the similar fashion with that of the closed case for the present epoch(i.e. $a = a_0$) equation(3.2.12) can be written as

$$\frac{1}{a_0^2} = \frac{2\ddot{a}(t_0)}{a_0} + \left(\frac{\dot{a}(t_0)}{a_0}\right)^2.$$

For the sake of simplicity, here on wards it is better to use $\ddot{a}$ instead of $\ddot{a}(t_0)$ and also true for $\dot{a}$. Making use of the relation $\ddot{a}/a_0 = -q_0 H_0^2$, where $q_0 = -\ddot{a}a_0/\dot{a}^2$ and $H_0^2 = \dot{a}^2/a_0^2$ in to the above equation leads to write as

$$\frac{1}{a_0^2} = 2(-q_0 H_0^2) + H_0^2 = H_0^2(1 - 2q_0). \tag{3.2.14}$$

This implies that it is also possible to write in the form

$$a_0^3 = \frac{1}{H_0^3(1 - 2q_0)^{3/2}}. \tag{3.2.15}$$

Again from equation(3.2.13) we have

$$\rho_{m_0} = \frac{3}{8\pi G}\left(\frac{\dot{a}^2}{{a_0}^2} - \frac{1}{{a_0}^2}\right) = \frac{3q_0 H_0^2}{4\pi G}.$$

Since the LHS of equation(3.2.14) is depend on the value of q_0, q_0 may be lies between 0 and 1/2, that means

$$0 \leq q_0 < 1/2 \ , \quad 0 \leq \Omega_0 < 1 \ , \quad \Omega_0 = 2q_0.$$

Substitution of the value of ρ_{m_0} and equation(3.2.15) in to equation(3.2.13) gives

$$\dot{a}^2 = \left(\frac{\beta}{a} + 1\right). \tag{3.2.16}$$

Where,

$$\beta = \frac{1}{H_0} \frac{2q_0}{(1-2q_0)^{3/2}} = \frac{\Omega_0}{(1-\Omega_0)^{3/2}} \frac{1}{H_0}.$$

The solution for the differential equation(3.2.16) is evaluated integration by substitution as seen below. From(3.2.16) we have

$$\frac{da}{dt} = \left(\frac{\beta + a}{a}\right)^{1/2} \ , \quad \int dt = \int \frac{\sqrt{a}}{\sqrt{\beta + a}} da.$$

Now let

$$a = \left(\frac{1}{2}\right) \beta(cosh\varphi - 1) \ , \quad da = \left(\frac{1}{2}\right) \beta sinh\varphi d\varphi.$$

Putting these values in to the integral to get

$$t = \left(\frac{1}{2}\right) \beta(sinh\varphi - \varphi).$$

But from equation(3.2.14) and the value given for β we can see that $a = a_0$ at $\varphi = \varphi_0$, so that

$$\left(\frac{1}{2}\right) \beta(cosh\varphi_0 - 1) = \frac{1}{H_0}(1-2q_0)^{-1/2} = \left(\frac{1-2q_0}{2q_0}\right) \beta.$$

After some mathematics, this gives that

$$cosh\varphi_0 = \frac{1-q_0}{q_0} \ , \quad \varphi_0 = (cosh)^{-1} \left(\frac{1-q_0}{q_0}\right) \Longrightarrow sinh\varphi_0 = \frac{\sqrt{1-2q_0}}{q_0}.$$

If we set $t = 0$ at $a = 0$, as in the two preceding cases the present value of t is given by

$$t_0 = \left(\frac{1}{2}\right) \beta(sinh\varphi_0 - \varphi_0).$$

Now by substituting the values of β, φ_0 and $sinh\varphi_0$ in to the above equation t_0 can be written as

$$t_0 = \frac{1}{H_0}\frac{q_0}{(1-2q_0)^{3/2}}\left[\frac{\sqrt{1-2q_0}}{q_0} - (cosh)^{-1}\left(\frac{1-q_0}{q_0}\right)\right]. \tag{3.2.17}$$

As we can see from the above equation such models of the universe continue to expand for ever.

The ordinary matter through gravity attracts other matter, causing the expansion to slow down. If the density of the universe exceeds a certain threshold known as the critical density, this gravitational attraction is strong enough to stop and latter reverse the expansion of the universe, causing it eventually to re-collapse in what is known as the "Big Crunch". On the other hand, if the average density of the universe falls short of the critical density, the universe expands forever, and after a certain point the expansion proceeds much as if the universe were empty.

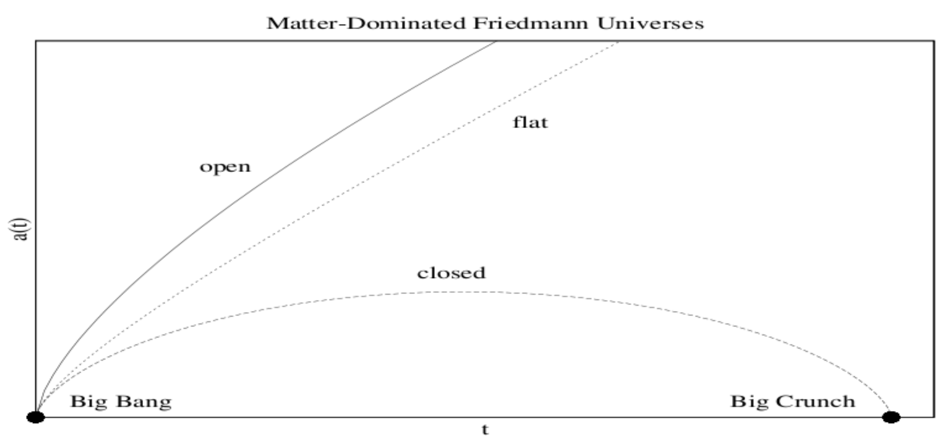

Figure 3.1: Evolution of the scale factor $a(t)$ for the flat($k = 0$), closed($k = 1$) and open($k = -1$) matter-dominated Friedmann Universes[4][15][16].

The case $k = 0$, but $\Omega_{v_0} \neq 0$: For this case the Friedmann equation(3.1.7) take the form

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\Omega_{m_0}\left(\frac{a_0}{a}\right)^3 + \Omega_{v_0}\right].$$

In this case since the universe constitutes matter(Ω_m) and cosmological constant(Ω_v), We have only $\Omega_{m_0} + \Omega_{v_0} = 1 \Longrightarrow \Omega_{v_0} = 1 - \Omega_{m_0}$. And using the definition $H = \dot{a}/a$ in to the above equation gives

$$\frac{H^2}{H_0^2} = \left[\Omega_{m_0}\left(\frac{a_0}{a}\right)^3 + (1 - \Omega_{m_0})\right].$$

But $(a_0/a)^3 = a^{-3}$ by taking $a_0 = 1$ at the present epoch(to day) and also $H = H_0$ (for the present universe). Having these information the above equation can be written as

$$1 = \Omega_{m_0} a^{-3} + 1 - \Omega_{m_0}.$$

For $\Omega_{m_o} < 1$, $\Omega_{v_0} = \Omega_v > 0$ and from the above relation, the matter density and cosmological constant are equal at the expansion factor

$$a_{mv} = \left(\frac{\Omega_{m_0}}{\Omega_{v_0}}\right)^{1/3}. \tag{3.2.18}$$

To determine the cosmic scale factor and age of the universe with dark energy, let us rewrite the Friedmann equation by requiring $a_0 = 1$(for today) as

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[(1 - \Omega_{v_0})\frac{1}{a^3} + \Omega_{v_0}\right].$$

Solving for $\dot{a}$ gives

$$\dot{a} = \frac{da}{dt} = H_0\sqrt{\frac{1 - \Omega_{v_0}}{a} + \Omega_{v_0} a^2}.$$

This can be rewritten as

$$\begin{aligned} H_0 t_0 &= \int_0^1 \frac{da}{\sqrt{\frac{1-\Omega_{v_0}}{a} + \Omega_{v_0} a^2}} \\ &= \int_0^1 \frac{a^{1/2} da}{\sqrt{(1 - \Omega_{v_0}) + \Omega_{v_0} a^3}} \\ &= \frac{2}{3\sqrt{\Omega_{v_0}}} ln\left[2\left(\sqrt{\Omega_{v_0} a^3} + \sqrt{\Omega_{v_0}(a^3 - 1) + 1}\right)\right]_0^1 \\ &= \frac{2}{3\sqrt{\Omega_{v_0}}} ln\left[1 + \sqrt{\Omega_{v_0}} - \sqrt{1 - \Omega_{v_0}}\right] \\ &= \frac{2}{3\sqrt{\Omega_{v_0}}} ln\left[\frac{1 + \sqrt{\Omega_{v_0}}}{\sqrt{1 - \Omega_{v_0}}}\right]. \end{aligned}$$

Now from the relation $\dot{a}/a = H = H_0$ at the present epoch, we have

$$\int \frac{da}{a} = \int_0^{t_0} H dt = \int_0^{t_0} H_0 dt \implies ln(a) = H_0 t_0.$$

Substitution of the value $H_0 t_0$ in to the above result and solving for a gives that

$$a = \left[\frac{1+\sqrt{\Omega_{v_0}}}{\sqrt{1-\Omega_{v_0}}}\right] e^{\frac{2}{3\sqrt{\Omega_{v_0}}}}.$$

And the age of the universe with dark energy is

$$t_0 = \frac{2}{3H_0\sqrt{\Omega_{v_0}}} ln \left[\frac{1+\sqrt{\Omega_{v_0}}}{\sqrt{1-\Omega_{v_0}}}\right].$$

As $\Omega_{v_0} \longrightarrow 1,\ t_0 \longrightarrow \infty$, so some matter is needed to keep the age of the universe finite.

3.2.2 For Radiation Dominated Era with $k = \Omega_{v_0} = 0,\ k = 1, -1$, but $\Omega_{v_0} = 0$ and with $k = 0$, but $\Omega_{v_0} \neq 0$

The case $k = \Omega_{v_0} = 0$, flat universe: The general Friedmann equation(3.1.7) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho_{r_0}}{3} \left(\frac{a_0}{a}\right)^4.$$

This implies that

$$\frac{da}{dt} = \frac{1}{a}\sqrt{\frac{8\pi G \rho_{r_0} a_0^4}{3}} \implies \int a da = \int \sqrt{\frac{8\pi G \rho_{r_0} a_0^4}{3}} dt \implies \frac{a^2}{2} + C = \sqrt{\frac{8\pi G \rho_{r_0} a_0^4}{3}}\ t.$$

Again, at the Big Bang, $t = 0$, $a = 0$, so $C = 0$, and $a_0 = 1$. Also $\rho_{r_0} = \rho_c$. Therefore,

$$a = 2\left(\frac{2\pi G \rho_{r_0}}{3}\right)^{1/4} t^{1/2} = 2\left(\frac{2\pi G \rho_c}{3}\right)^{1/4} t^{1/2} = 2\left(\frac{2\pi G}{3}\frac{3H_0^2}{8\pi G}\right)^{1/4} t^{1/2}.$$

This can be rewritten as

$$a = 2\left(\frac{H_0}{2}\right)^{1/2} t^{1/2} = a_0\left(\frac{t}{t_0}\right)^{1/2}.$$

Where $t_0 = a_0^2/2H_0$ and $a = a(t)$.

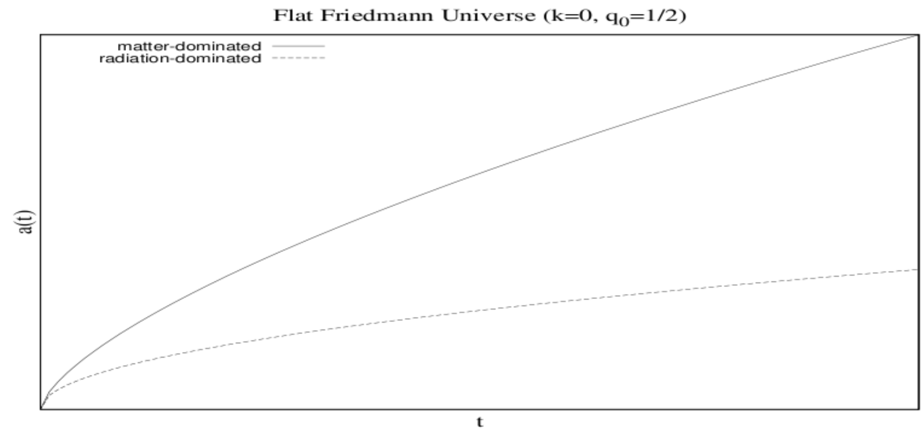

Figure 3.2: Evolution of the scale factor $a(t)$ for the flat($k=0$) Friedmann Universe[4]. .

The case $k = 1$, but $\Omega_{v_0} = 0$, closed universe: For this case the general Friedmann equation take the form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho_{r_0}}{3}\left(\frac{a_0}{a}\right)^4 - \frac{1}{a^2}.$$

Implies that

$$\frac{da}{dt} = \sqrt{\frac{8\pi G\rho_{r_0}a_0^4}{3a^2} - 1} \Longrightarrow \int dt = \int \frac{da}{\sqrt{\frac{8\pi G\rho_{r_0}a_0^4}{3a^2} - 1}}.$$

Now let as define the conformal time $d\eta \equiv dt/a \Longrightarrow ad\eta = dt$ and making use of $a_0 = 1$ for the present epoch, the above integral can be written as

$$\int d\eta = \int \frac{da}{\sqrt{\frac{8\pi G\rho_{r_0}}{3} - a^2}} = \int_0^a \frac{da}{\sqrt{\frac{8\pi G\rho_{r_0}}{3} - a^2}}.$$

Using the relation given by $\rho_0 = 3H_0^2 q_0/4\pi G$ and $H_0^2 = 1/(2q_0 - 1)$ at the present epoch($a_0 = 1$), it is possible to define the quantity A as $A = \frac{8\pi G\rho_{r_0}}{3} = \frac{2q_0}{2q_0-1}$. Again, rewrite the integral above in terms of η and A so as to have

$$\eta - \eta_0 = \int_0^a \frac{da}{\sqrt{A - a^2}} = sin^{-1}\left(\frac{a}{\sqrt{A}}\right).$$

The requirement $\eta = 0$ at $a = 0$ sets $\eta_0 = 0$, so we have

$$a = \sqrt{A}sin(\eta),$$

$$t - t_0 = -\sqrt{A}cos(\eta).$$

The requirement $\eta = 0$ at $t = 0$ sets $t_0 = \sqrt{A}$ so we finally have

$$a = \sqrt{\frac{2q_0}{2q_0 - 1}} sin(\eta),$$

$$t = \sqrt{\frac{2q_0}{2q_0 - 1}} (1 - cos\eta).$$

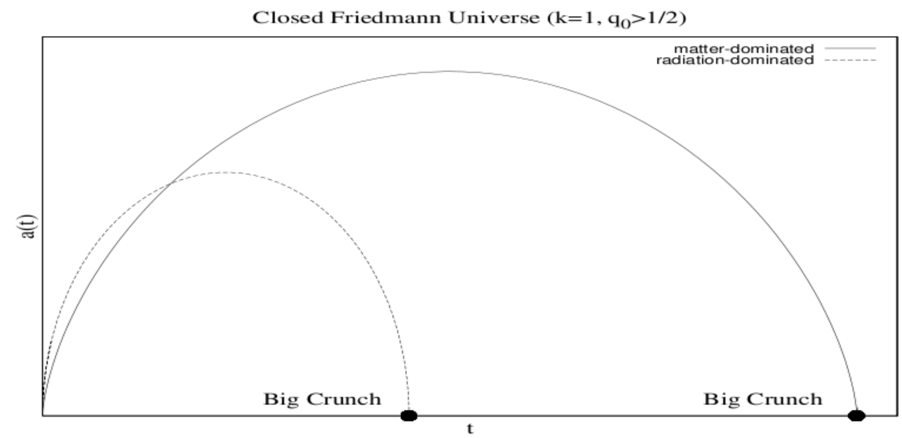

Figure 3.3: Evolution of the scale factor $a(t)$ for the closed($k = 1$) Friedmann Universe[15].

The case $k = -1$, but $\Omega_{v_0} = 0$, open universe: For this case the general Friedmann equation take the form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho_{r_0}}{3}\left(\frac{a_0}{a}\right)^4 + \frac{1}{a^2}$$

This can be rewritten as

$$\frac{da}{dt} = \sqrt{\frac{8\pi G\rho_{r_0} a_0^4}{3a^2} + 1} \Longrightarrow \int dt = \int \frac{da}{\sqrt{\frac{8\pi G\rho_{r_0} a_0^4}{3a^2} + 1}}$$

Again, rewrite the integral above in terms of conformal time $d\eta = dt/a$ and quantity $A = \frac{8\pi G\rho_{r_0}}{3} = \frac{2q_0}{2q_0 - 1}$:

$$\eta - \eta_0 = \int_0^a \frac{da}{\sqrt{A + a^2}} = sinh^{-1}\left(\frac{a}{\sqrt{A}}\right)$$

Again, the requirement $\eta = 0$ at $a = 0$ sets $\eta_0 = 0$, so we have

$$a = \sqrt{A} sinh(\eta),$$

$$t - t_0 = -\sqrt{A} cosh(\eta).$$

The requirement $\eta = 0$ at $t = 0$ sets $t_0 = \sqrt{A}$, so we finally have

$$a = \sqrt{\frac{2q_0}{2q_0 - 1}} sinh(\eta),$$

$$t = \sqrt{\frac{2q_0}{1 - 2q_0}} (cosh\eta - 1).$$

Early times (small η limit): For small values of η, the trigonometric and hyperbolic functions can be expanded in Taylor series (keeping only the first two terms):

$$sin(\eta) = \eta - \frac{1}{6}\eta^3 \ , \quad cos(\eta) = 1 - \frac{1}{2}\eta^2.$$

$$sinh(\eta) = \eta + \frac{1}{6}\eta^3 \ , \quad cosh(\eta) = 1 + \frac{1}{2}\eta^2.$$

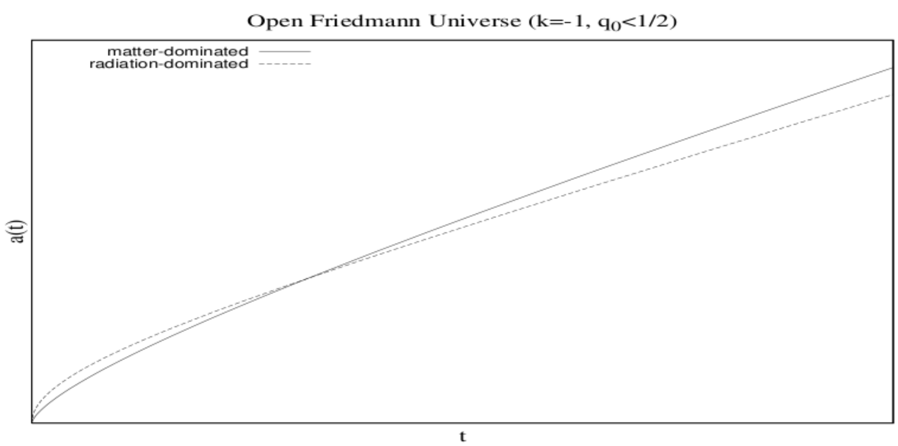

Figure 3.4: Evolution of the scale factor $a(t)$ for the open($k = -1$) Friedmann Universe[16].

The case $k = 0$, but $\Omega_{v_0} \neq 0$: For this case the Friedmann equation(3.1.7) become

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left[\Omega_{r_0} \left(\frac{a_0}{a}\right)^4 + \Omega_{v_0}\right].$$

Since the universe under consideration is containing only radiation and dark energy we have $\Omega_{r_0} + \Omega_{v_0} = 1 \Longrightarrow \Omega_{r_0} = 1 - \Omega_{v_0}$. So the above equation with $a_0 = 1$(for today) can be written as

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left[(1 - \Omega_{v_0}) \left(\frac{1}{a}\right)^4 + \Omega_{v_0}\right].$$

Solving for $\dot{a}$ gives

$$\dot{a} = \frac{da}{dt} = H_0\sqrt{\frac{1-\Omega_{v0}}{a^2} + \Omega_{v0}a^2}.$$

We can write this again in the form

$$H_0t_0 = \int_0^1 \frac{da}{\sqrt{\frac{1-\Omega_{v0}}{a^2} + \Omega_{v0}a^2}}$$

$$= \int_0^1 \frac{ada}{\sqrt{(1-\Omega_{v0}) + \Omega_{v0}a^4}}$$

$$= \frac{1}{2\sqrt{\Omega_{v0}}} ln\left[2\left(a^2\sqrt{\Omega_{v0}} + \sqrt{\Omega_{v0}(a^4-1)+1}\right)\right]_0^1$$

$$= \frac{1}{2\sqrt{\Omega_{v0}}} ln\left[1 + \sqrt{\Omega_{v0}} - \sqrt{1-\Omega_{v0}}\right]$$

$$= \frac{1}{2\sqrt{\Omega_{v0}}} ln\left[\frac{1+\sqrt{\Omega_{v0}}}{\sqrt{1-\Omega_{v0}}}\right].$$

Again applying $\dot{a}/a = H = H_0$ at the present epoch, we have

$$\int \frac{da}{a} = \int_0^{t_0} Hdt = \int_0^{t_0} H_0 dt \implies ln(a) = H_0t_0.$$

Substitution of the value H_0t_0 in to the above result and solving for a gives that

$$a = \left[\frac{1+\sqrt{\Omega_{v0}}}{\sqrt{1-\Omega_{v0}}}\right] e^{\frac{1}{2\sqrt{\Omega_{v0}}}}.$$

And the age of the universe with dark energy is

$$t_0 = \frac{1}{2H_0\sqrt{\Omega_{v0}}} ln\left[\frac{1+\sqrt{\Omega_{v0}}}{\sqrt{1-\Omega_{v0}}}\right].$$

Here also as $\Omega_{v_0} \longrightarrow 1,\ t_0 \longrightarrow \infty$, so some radiation is needed to keep the age of the universe finite.

3.2.3 For Both Matter and Radiation with $k = \Omega_{v_0} = 0$

In our universe, radiation-matter equality took place at a scale factor $a_{rm} = a_{eq} \equiv \Omega_{r_0}/\Omega_{m_0} \approx 2.8 \times 10^{-4}$. At scale factors $a \ll a_{rm}$, the universe is well described by a flat, radiation-only model. At scale factors $a \sim a_{rm}$, the universe is better described by a flat model containing both radiation and matter. The Friedmann equation around the time of radiation-matter equality can be written in the approximate form

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r_0}}{a^4} + \frac{\Omega_{m_0}}{a^3},$$

or in more general form for such a case the Friedmann equation can be written as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\rho_{m_0}\frac{a_0^3}{a^3} + \rho_{r_0}\frac{a_0^4}{a^4}\right). \tag{3.2.19}$$

Where we have normalized the densities in terms of present day values, as usual. To proceed, it is better to define two constants,

$$\sigma_m = \frac{8\pi G\rho_{m_0}a_0^3}{3}, \quad \sigma_r = \frac{8\pi G\rho_{r_0}a_0^4}{3}.$$

Taking the positive square root of the Friedmann equation(3.2.19) (since we are interested in expanding universe) this implies

$$a\frac{da}{dt} = [\sigma_m a + \sigma_r]^{1/2}. \tag{3.2.20}$$

Equation(3.2.20) can be solved by separation of variables and integrating by parts. This yields the solution in terms of $t(a)$, i.e. in terms of time as a function of scale factor. Unfortunately, the solution can not be inverted to give $a(t)$ and is not particularly illuminating. However, a parametric solution can be found by defining a new time dependent variable

$$\eta = \int \frac{dt}{a} \Longrightarrow d\eta = \frac{dt}{a} \Longrightarrow a\frac{d}{dt} = \frac{d}{d\eta}. \tag{3.2.21}$$

Note that $dt > 0 \Leftrightarrow d\eta > 0$ and consequently increasing t corresponds to increasing η. This implies that the variable η make a good time variable. Substituting equation(3.2.21) in to equation(3.2.20) and separating the variables gives

$$\int_0^a [\sigma_r + \sigma_m a]^{-1/2} da = \int_0^\eta d\eta. \tag{3.2.22}$$

Here we have chosen the limits of integration so that the scale factor must vanishes at some finite time in the past. Evaluating the integral(3.2.22) yields

$$\frac{2}{\sigma_m}[\sigma_r + \sigma_m a]^{1/2} - \frac{2\sqrt{\sigma_r}}{\sigma_m} = \eta.$$

And rearranging implies that

$$a(\eta) = \frac{\sigma_m \eta^2}{4} + \sqrt{\sigma_r}\ \eta. \tag{3.2.23}$$

The dependence of time(t) on η is now deduced by substituting equation(3.2.23) in to equation(3.2.21) that gives

$$\int_0^t dt = \int_0^\eta a(\eta) d\eta.$$

When we have chosen the limits so that the origin of the universe occurs at $t = 0$. We find that

$$t(\eta) = \frac{\sigma_m \eta^3}{12} + \frac{\sqrt{\sigma_r}\eta^2}{2}. \tag{3.2.24}$$

Equation(3.2.23) and equation(3.2.24) represents a parametric solution describing the expansion of a universe filled with matter and radiation with $k = \Omega_v = 0$. The asymptotic limits of the solution at early ($t \longrightarrow 0$) and late($t \longrightarrow \infty$) times can be deduced.
At early times, $t\ \alpha\ \eta^2 \longrightarrow 0$ and $a\ \alpha\ \eta\ \alpha\ t^{1/2}$. This is the behaviour given by equation

$$a = a_0 \left(\frac{t}{t_0}\right)^{1/2}, \qquad a \sim t^{1/2} \quad for \quad t \ll t_{eq}$$

This is for a universe containing just radiation and implies that the radiation dominates over the matter during the early history of the universe. At late times, $t\ \alpha\ \eta^3 \longrightarrow \infty$, and $a\ \alpha\ \eta^2\ \alpha\ t^{2/3}$. This is the expansion for a universe comprised of only pressure-less matter, that can be seen from equation(3.2.4) that is

$$a = a_0 \left(\frac{t}{t_0}\right)^{2/3}, \qquad a \sim t^{2/3} \quad for \quad t \gg t_{eq}$$

Thus at sufficiently late times, the matter will dominate over the radiation, where as the radiation dominates at very early times($a \longrightarrow 0$).

The key question to address at this stage, therefore is, when does the transition from radiation domination to matter domination occur? This epoch is referred to as the 'epoch of matter-radiation equality', and is denoted by t_{eq}. We may now employ the solutions (3.2.23) and (3.2.24) to eliminate t_{eq}. Matter-radiation equality occurs when the densities of the matter and radiation are equal. Thus, by equating the two terms on the RHS of the Friedmann equation(3.2.19), the scale factor at this time is given in-terms of its present day value by

$$a_{eq} = \frac{\rho_{r_0}}{\rho_{m_0}} a_0 = \frac{\sigma_r}{\sigma_m}. \tag{3.2.25}$$

Where $a_{eq} = a(t_{eq})$. Now to determine η_{eq} let us substitute equation(3.2.25) in to equation(3.2.23), to obtain

$$\eta_{eq} = 2(\sqrt{2} - 1)\frac{\sqrt{\sigma_r}}{\sigma_m}. \tag{3.2.26}$$

Then substituting equation(3.2.26) in to the solution (3.2.24) allows us to determine when matter-radiation equality occurs. We can find, after a little algebra, that is at

$$t_{eq} = \frac{2\sqrt{2}\left(\sqrt{2} - 1\right)}{3}\left(\frac{8\pi G\rho_{m_0}}{3}\right)^{-1/2}\left(\frac{\rho_{r_0}}{\rho_{m_0}}\right)^{3/2} = \frac{2\sqrt{2}\left(\sqrt{2} - 1\right)}{3} H_0^{-1},$$

it was the time when the transition from being radiation dominated to matter dominated occured(epoch of matter-radiation equality). At early times($t \ll t_{eq}$), the universe was dominated by radiation(radiation dominates over the matter) and after radiation-matter equality($t = t_{eq}$), at late times($t \gg t_{eq}$) matter dominates over the radiation and the universe became matter dominated.

Chapter 4

Evidences For The Accelerated Expansion of The Friedmann Universe

4.1 Redshift-Distance Relation

The best known way to trace the evolution of the universe observationally is to look in to the redshift-distance relation. The well-measured quantity of a far distant object is the redshift of light it emitted due to the expansion of the universe[12,17].

Another important observational quantity is the distance to the object from which red-shifted light is emitted. Now let us see the definition given for redshift first.

Redshift is the change in observed wave length of a spectral line relative to the wave length of a spectral line at emission[29,30]. In other word it means that a measure of the amount by which the energy of a photon is reduced due to the expansion of the universe[1].

To show the relation, first, let us see the mathematical definition given for redshift. We know that, light travels along the geodesic curves of a manifold, essentially locally straight lines, satisfying $d\tau = 0$. With this fact, let us recall the Friedmann-Robertson-Walker metric that was given by

$$d\tau^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2\right].$$

and look at the case where light travels along a radial curve with constant θ and ϕ (so, $d\theta = d\phi = 0$), from the point of emission at $t = t_e$, $r = r_e$ to our observation to day at $t = t_0$, $r = r_0 = 0$. We have from the above metric

$$0 = d\tau^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2}\right).$$

Which can integrate along the path from observation to emission as:

$$\int_{t_0}^{t_e} \frac{dt}{a} = \int_0^{r_e} \frac{dr}{\sqrt{1-kr^2}}.$$

Note that the dr integral is a fixed function of r_e and k. Because of this, the equality must still hold for photons emitted at a later time, $t_e + \delta t_e$ and observed at $t_0 + \delta t_0$. This implies that

$$\int_{t_0}^{t_e} \frac{dt}{a} = \int_{t_0+\delta t_0}^{t_e+\delta t_e} \frac{dt}{a}.$$

Split up the range of integration on both sides to eliminate the overlap between $t_0 + \delta t_0$ to t_e:

$$\int_{t_0}^{t_0+\delta t_0} \frac{dt}{a} + \int_{t_0+\delta t_0}^{t_e} \frac{dt}{a} = \int_{t_0+\delta t_0}^{t_e} \frac{dt}{a} + \int_{t_e}^{t_e+\delta t_e} \frac{dt}{a}.$$

The two inner terms are equal, so we are just left with

$$\int_{t_0}^{t_0+\delta t_0} \frac{dt}{a} = \int_{t_e}^{t_e+\delta t_e} \frac{dt}{a}.$$

Now assume that the δt are small and Taylor expand the integrands:

$$\frac{1}{a(t+\delta t)} = \frac{1}{a(t)}\left[1 - \delta t\left(\frac{\dot{a}}{a}\right) + O(\delta t^2)\right] \backsimeq \frac{1}{a(t)}.$$

Where the approximation holds if

$$\frac{\delta t}{(\dot{a}/a)^{-1}} \backsimeq \frac{photon\ wave\ length}{age\ of\ universe} << 1.$$

and we have used $H = \dot{a}/a \sim 1/t_0$ and the fact that we are considering successive peaks in the sinusoidal light wave. This inequality obviously holds for any reasonable times. Now, applying this to $t = t_0$ and $t = t_e$ yields

$$\frac{\delta t_0}{a(t_0)} = \frac{\delta t_e}{a(t_e)}.$$

This gives that

$$\frac{a(t_0)}{a(t_e)} = \frac{\delta t_0}{\delta t_e} = \frac{\lambda_0/c}{\lambda_e/c} = v_e/v_0 = a_0/a_e.$$

And if we define the red shift by z to rewrite the above relation as

$$z = \frac{v_e}{v_0} - 1 = \frac{v_e - v_0}{v_0} = \delta v/v = \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{\delta\lambda}{\lambda} = \frac{a_0 - a_e}{a_e}.$$

Where λ and v are the wave length and frequency of the light respectively. In an expanding universe, $a_0 > a_e$, so that z is positive, as observed[17,29]. Cosmological red shift is symmetric between receiver and emitter.

Now we can see the relation between red shift and distance. Expansion of $a(t)$ in a Taylor series expansion can be written as

$$a(t) = a(t_0) + (t - t_0)\dot{a}(t_0) + \frac{1}{2}(t - t_0)^2\ddot{a}(t_0) + ...$$

In terms of H_0 and the decceleration parameter(q_0) this can be rewritten as

$$a(t) = a_0\left(1 + H_0(t - t_0) - \frac{1}{2}q_0H_0^2(t - t_0)^2 + ...\right).$$

This give us the red shift parameter z as a power series in time of flight, namely

$$\frac{1}{1+z} = \frac{a_1}{a_0} = 1 + (t_1 - t_0)H_0 - \frac{1}{2}q_0H_0^2(t_1 - t_0)^2 + ...$$

In other way it is possible to write as

$$z = (t_1 - t_0)H_0 + \left(1 + \frac{1}{2}q_0\right)H_0^2(t_0 - t_1)^2 + ...$$

for small $H_0(t_0 - t_1)$ this can be inverted

$$t_0 - t_1 = \frac{1}{H_0}\left[z - \left(1 + \frac{1}{2}q_0\right)z^2 + ...\right].$$

Now recall the integral given by

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_{r_1}^{0} \frac{dr}{\sqrt{1 - kr^2}}.$$

and to express $(t_0 - t_1)$ in terms of r_1,

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = r_1 + O(r_1^3).$$

While expanding $a(t)$ in the denominator we get

$$\begin{aligned}\int_{t_1}^{t_0} \frac{dt}{a(t)} &= \frac{1}{a_0}\int_{t_1}^{t_0} \frac{dt}{[1+(t-t_0)H_0+...]} \\ &= \frac{1}{a_0}\int_{t_1}^{t_0} dt[1+(t_0-t)H_0+...] \\ &= \frac{1}{a_0}\left[(t_0-t_1)+t_0(t_0-t_1)H_0-\frac{1}{2}(t_0^2-t_1^2)H_0+...\right] \\ &= \frac{1}{a_0}\left[(t_0-t_1)+\frac{1}{2}(t_0-t_1)^2H_0+...\right].\end{aligned}$$

Therefore, we obtain

$$r_1 = \frac{1}{a_0}\left[(t_0-t_1)+\frac{1}{2}(t_0-t_1)^2H_0+...\right].$$

Using the result for $t_0 - t_1$ given above we can rewrite r_1 as

$$r_1 = \frac{1}{a_0H_0}\left[z-\frac{1}{2}(1+q_0)z^2+...\right].$$

This clearly indicates to first order a linear dependence of the red-shift on the distance of the galaxy and identifies H_0(Hubble's law)[29,30]. Where a_0r_1 is the present distance to the galaxy, not the distance at the time the light was emitted. In other word the redshift of the light from galaxies is proportional to their distance. No cause of galaxy redshift other than a velocity away from the observer was considered plausible, so Hubble's result was taken to mean that, the farther away from us a galaxy is, the faster it moves away from us. Hence overall universe had to be expanding.

The effects that are due to the curvature of space-time will be important at length scales bigger than (or comparable with) the Hubble radius, defined as $d_{H(t)} \equiv (\dot{a}/a)^{-1}$. Referring back the Friedmann equation(3.1.7) to write as

$$\frac{\dot{a}^2+k}{a^2} = H_0^2\left[\Omega_r\left(\frac{a_0}{a}\right)^4+\Omega_m\left(\frac{a_0}{a}\right)^3+\Omega_v\right],$$

Where Ω_r, Ω_m Ω_v and Ω represent the density parameters for relativistic matter [with $P_r = (1/3)\rho_r$; $\rho_r \; \alpha \; a^{-4}$], non-relativistic matter [with $P_m = 0$; $\rho_m \; \alpha \; a^{-3}$], cosmological constants ($P_v = -\rho_v$; $\rho_v =$ constant), and total energy density($\Omega = \Omega_r + \Omega_m + \Omega_v$) respectively. For $k = 0$ and making use of the relation $(a_0/a) = 1 + z$, the above equation follows that

$$d_{H(z)} = H_0^{-1}\left[\Omega_r(1+z)^4 + \Omega_m(1+z)^3 + (1-\Omega)(1+z)^2 + \Omega_v\right]^{-1/2}.$$

This has the limiting forms

$$d_{H(z)} \cong \begin{cases} H_0^{-1}\Omega_r^{-1/2}(1+z)^{-2} & (z \gg z_{eq}) \\ H_0^{-1}\Omega_m^{-1/2}(1+z)^{-3/2} & (z_{eq} \gg z \gg z_{curv};\ \Omega_v = 0) \\ H_0^{-1}(1+z)^{-1}(1-\Omega)^{-1/2} & (z_{eq} \gg z;\ \Omega_m \simeq \Omega; \Omega_v = 0) \\ H_0^{-1}\Omega_m^{-1/2}[(1+z)^3 + \Omega_m^{-1} - 1]^{-1/2} & (z_{eq} \gg z;\ \Omega = \Omega_m + \Omega_v = 1) \end{cases}$$

during various epochs. Where $z_{eq} = \left(\frac{a_{eq}-a_0}{a_0}\right)$ and a_{eq} is the cosmic scale factor at which the matter energy density is equal to the radiation energy density($\Omega_{m_0} = \Omega_{r_0}$).

The physical length scale characterizing a region of size λ_0 today will evolve as $\lambda(z) = \lambda_0(1+z)^{-1}$ with redshift. Since d_H increases faster with redshift,[as $(1+z)^{-3/2}$ in the matter dominated phase and as $(1+z)^{-2}$ in the radiation dominated phase], $\lambda(z) > d_{H(z)}$ at sufficiently large redshifts. For a given λ_0 we can assign a particular redshift z_{enter} such that $\lambda(z_{enter}) = d_{H(z_{enter})}$. For $z > z_{enter}$, the proper wavelength is bigger than the Hubble radius, at this time general relativistic effects are important; while for $z < z_{enter}$, we have $\lambda < d_H$, in this case we can ignore the effects of general relativity. It is conventional to say that the scale λ_0 'enters the Hubble radius' at the epoch z_{enter}. The exact relation between λ_0 and z_{enter} differs in the case of radiation dominated phases since $d_H(z)$ has different scalings in these two cases. Using the preceding relation it is easy to verify that: (i) a scale $\lambda_{eq} \cong \left(\frac{H_0^{-1}}{\sqrt{2}}\left(\frac{\Omega_r^{1/2}}{\Omega_m}\right)\right) \cong 14(\Omega_m h^2)^{-1}Mpc$ enters the Hubble radius at $z = z_{eq}$; (ii) scales with $\lambda > \lambda_{eq}$ enter the Hubble radius in the matter dominated epoch with $z_{enter} \cong 900(\Omega_m h^2)^{-1}\left(\frac{\lambda_0}{100Mpc}\right)^{-2}$; (iii) scales with $\lambda < \lambda_{eq}$ enter the Hubble radius in the radiation dominated epoch with $z_{enter} \cong 4.55 \times 10^5\left(\frac{\lambda}{1Mpc}\right)^{-1}$. We can characterize

the wavelength λ_0 of the perturbation more meaningfully as: As the universe expands, the wavelength λ grows as $\lambda(t) = \lambda_0 \left[\frac{a(t)}{a_0}\right]$ and the density of non-relativistic matter decreases as $\rho(t) = \rho_0 \left[\frac{a_0}{a_t}\right]^3$. Hence, the mass $M(\lambda_0)$ contained inside a sphere of radius $(\lambda/2)$ remains constant as the universe expands:
$M = \frac{4\pi}{3}\rho(t)\left[\frac{\lambda(t)}{2}\right]^3 = \frac{4\pi}{3}\rho_0\left[\frac{\lambda_0}{2}\right]^3 = 1.45 \times 10^{11} M_\odot(\Omega_m h^2)\left(\frac{\lambda_0}{1Mpc}\right)^3$.
This relation shows that a co-moving scale $\lambda_0 \cong 1Mpc$ contains a typical galaxy mass; and $\lambda_0 \cong 10Mpc$ contains a typical cluster mass. From the equation given by $z_{enter} \cong 4.55 \times 10^5 \left(\frac{\lambda}{1Mpc}\right)^{-1}$, we see that all these scales enter the Hubble radius in a radiation dominated epoch.

4.2 The Peculiar Behavior of Angular Diameters

At very small distances the angular diameter would, of course, be inversely proportional to the distance, but for sources whose red shift is appreciable important relativistic effects come in to play. To determine these effects let us consider the galaxy G_1 to have a linear extent d, as shown in figure(4.1). Then it is possible to determine the angle that this length d subtend at our location. To determine the angle, we consider two neighboring null geodesics(representing light rays) from the two points A, B at the two extremities of G_1 directed towards our solar system. Without loss of generality we can choose our angular coordinates such that A has the coordinates θ_1, ϕ_1 while B has the coordinates $(\theta_1 + \Delta\theta_1,\ \phi_1)$. Although we have used homogeneity to take $r = 0$ at our location, we can also use isotropy to choose any particular direction as the polar axis $\theta = 0,\ \theta = \pi$. According to the Friedmann-Robertson-Walker metric(line element), the proper distance between A and B is obtained by putting $t = t_1 =$ constant, $r = r_1 =$constant, $\phi = \phi_1 =$constant(i.e. $dr = dt = d\phi = 0$), and $d\theta = \Delta\theta_1$ in to equation(1.6.6). Then we get

$$d\tau^2 = -r_1^2 a^2(t_1)(\Delta\theta_1)^2 = -d^2.$$

since in the rest frame of G_1 the space like separation $AB = d$. Thus

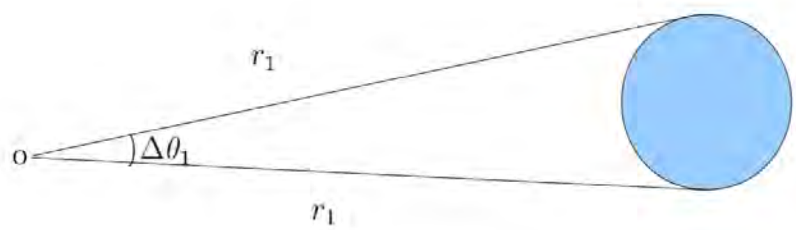

Figure 4.1: The angle subtended by galaxy G_1 at the observer O.

$$\Delta\theta_1 = \frac{d}{r_1 a(t_1)} = \frac{d(1+z)}{r_1 a(t_0)}.$$

Where $1 + z = a(t_0)/a(t_1)$ and t_1 is the time when the radiation is emitted. The result implies that as r_1 increases we are looking at more and more remote galaxies, which must therefore be seen at earlier and earlier epochs t_1. The $(d\theta,\ z)$ relation is of great interest. At large $z(z \gg 1)$, $d\theta$ is nearly proportional to z. In other words at large z the angular diameter actually increases with increasing red shift. However, $a(t_1)$ decreases as t_1 decreases, so it is not obvious that $r_1 a(t_1)$ should get progressively larger as we look at more and more remote galaxies. We see that as $a \Longrightarrow 0$ the angle increases to infinity and the object covers the whole horizon.

4.3 The Existence of Horizons

Now let us see how simply the function $a(t)$ describes the expansion of the universe. Here we take one galaxy to be at co-moving coordinates $(r, \theta, \varphi) = (0, 0, 0)$ and the other to be at $(r, 0, 0)$, at fixed time t. Thus, the portions of the line element depending on time and the angle θ and φ make no contribution. Therefore, from Friedmann-Robertson-Walker metric, it is possible to calculate the proper distance from us at a given cosmic time t as

$$D_p = a(t)\int_0^r \frac{dr}{\sqrt{1-kr^2}} = \begin{cases} a(t)sin^{-1}r \quad (k=1), \\ a(t)r \quad (k=0), \\ a(t)sinh^{-1}r \quad (k=-1). \end{cases} \tag{4.3.1}$$

Thus the distance is proportional to the cosmic scale factor $a(t)$ which changes with time. The proper velocity V_p is obtained by differentiating with respect to t, realizing that r remains constant because it is a co-moving coordinate. Using dots to denote differentiation with respect to time, we obtain

$$V_p = \dot{D}_p = \dot{a}(t)\int_0^r \frac{dr}{\sqrt{1-kr^2}} = \frac{\dot{a}(t)}{a(t)}D_p = H(t)D_p.$$

This tells us that at any given cosmic time t the speed of a galaxy relative to us is proportional to its proper distance from us. We see that in general the expansion rate changes with time.

4.3.1 The Particle(Object) Horizon

The cosmological particle(object) horizon separates co-moving distance(particles) we can currently see from those we cannot currently see. Now we demand to determine the coordinate r_{oh} of the most distant object(e.g. a galaxy) that we can see now. This coordinate is called the object horizon, or the particle horizon. Such an object must have emitted its light at the beginning of the universe t_{min}, on some models t_{min} is $-\infty$, on others t_{min} is zero. Therefore, from F-R-W metric we have

$$\int_{t_{min}}^{t_0} \frac{dt}{a(t)} = \int_0^{r_{0h}} \frac{dr}{\sqrt{1-kr^2}}.$$

Or referring back equation(4.3.1) we can write as the following

$$r_{oh} = \begin{cases} sin\left(\int_{t_{min}}^{t_0} \frac{dt}{a(t)}\right) \quad (k=1), \\ \int_{t_{min}}^{t_0} \frac{dt}{a(t)} \quad (k=0), \\ sinh\left(\int_{t_{min}}^{t_0} \frac{dt}{a(t)}\right) \quad (k=-1). \end{cases} \tag{4.3.2}$$

Objects beyond r_{oh} cannot now be seen by us. We know that no physical influence can travel faster that light, therefore the object horizon represents the greatest distance from which outside matter could have affected what is now happening at any given locality. In the early epochs of "Big Bang" model universe, the object horizons where very small; later, the horizons increased and mutual influence between masses become possible within ever larger co-moving volumes[11,13].

If we are interested to determine the object horizon by considering the steady state model, the cosmic scale factor for such a model is given by

$$a(t) = Aexp(H_0 t).$$

For flat space($k = 0$), so that the proper distance D corresponding to a co-moving coordinate r at the present time t_0 is

$$D = a(t_0)r.$$

This universe doesnot explode or implode, so it has an infinite past and future.

Referring back equation(4.3.2) the coordinate r_{oh} of the object horizon is

$$r_{oh} = \int_{-\infty}^{t_0} \frac{dt}{a(t)} = \frac{1}{A}\int_{-\infty}^{t_0} exp(-H_0 t)dt = \infty.$$

Therefore, $D = \infty$. This implies that there is no object horizon: all objects can in principle be seen by us.

4.3.2 The Event Horizon

The cosmological event horizon separates events we are able to see at some time, from events we will never be able to see. Similarly we like to determine the coordinate r_{eh} of the most distant event occurring now(that is, at cosmic time t_0). This coordinate is called the event horizon. The light from such an event must reach us before the universe ends, at t_{max}. Therefore from F-R-W metric we have

$$\int_{t_0}^{t_{max}} \frac{dt}{a(t)} = \int_0^{r_{eh}} \frac{dr}{\sqrt{1 - kr^2}}.$$

Or from equation(4.3.1) we have

$$r_{eh} = \begin{cases} sin\left(\int_{t_0}^{t_{max}} \frac{dt}{a(t)}\right) & (k=1), \\ \int_{t_0}^{t_{max}} \frac{dt}{a(t)} & (k=0), \\ sinh\left(\int_{t_0}^{t_{max}} \frac{dt}{a(t)}\right) & (k=-1). \end{cases} \tag{4.3.3}$$

Beyond r_{eh} will never be seen by us. The event horizon represents the greatest distance from which outside matter could eventually affect what will happen at any given locality[13,22]. For the steady state model using equation(4.3.3) the coordinate r_{eh} of the event horizon is

$$r_{eh} = \int_{t_0}^{\infty} \frac{dt}{a(t)} = \frac{1}{A}\int_{t_0}^{\infty} exp(-H_0 t)dt = \frac{exp(-H_0 t_0)}{H_0 A} = \frac{1}{H_0 a(t_0)}.$$

Therefore, $D = 1/H_0$. This implies that events occurring beyond this proper distance at the present cosmic time will never be seen by us.

Chapter 5

Summary and Conclusion

5.1 Summary

Space-time is a 4-dimensional manifold, with points referred to as events.

Events in Space-time are specified by four coordinates x^α with $\alpha = 0, 1, 2, 3$. Where the zero component of α corresponds to the time coordinate and the rest of three are the spacial coordinates x, y, z respectively.

The metric tensor($g_{\alpha\beta}$) in a gravitational field can be defined as the proper time interval between two events with a given infinitesimal coordinate separation.

The relation developed between the metric tensor($g_{\alpha\beta}$) in a gravitational field and the affine connection ($\Gamma^\sigma_{\lambda\mu}$) is given by

$$\frac{1}{2} g^{v\sigma} \left(\frac{\partial g_{\mu v}}{\partial x^\lambda} + \frac{\partial g_{\lambda v}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^v} \right) = \Gamma^\sigma_{\lambda\mu}.$$

The Riemann-Christoffel curvature tensor is given by the relation

$$R^\lambda_{\mu v k} = \frac{\partial}{\partial x^k}(\Gamma^\lambda_{\mu v}) - \frac{\partial}{\partial x^v}(\Gamma^\lambda_{\mu k}) + \Gamma^\eta_{\mu v}\Gamma^\lambda_{k\eta} - \Gamma^\eta_{\mu k}\Gamma^\lambda_{v\eta}.$$

The Ricci tensor, the Ricci scaler and the Einstein's field tensor respectively are given by

$$R_{\mu k} = \frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^k} - \frac{\partial \Gamma^\lambda_{\mu k}}{\partial x^\lambda} + \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{k\eta} - \Gamma^\eta_{\mu k}\Gamma^\lambda_{\lambda\eta},$$

$$R \equiv g^{\mu k} R_{\mu k} = R^\mu_\mu,$$

$$G_{\mu k} \equiv R_{\mu k} - \frac{1}{2} g_{\mu k} R.$$

Energy-Momentum tensor of the source is

$$T^i_k = diag(\rho(t), -P(t), -P(t), -P(t)).$$

Einstein gravitational field equation with weak field approximation is calculated as

$$R_{\mu v} - \frac{1}{2} g_{\mu v} R = -8\pi G T_{\mu v}.$$

Maximally symmetric spaces are spaces with constant curvature and it is also homogeneous and isotropic. The metric in a maximally symmetric space III(Friedmann-Robertson-Walker metric) is given by

$$d\tau^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2\right].$$

With the above metric tensor, the Ricci tensor and the Ricci scaler respectively can be rewritten as

$$R_{ik} = \left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2 + 2k}{a^2}\right) g_{ik} \ ,$$

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right).$$

Friedmann universe is homogeneous and isotropic in its largest scale and it constitutes Radiation, Matter(dark+baryonic) and Dark energy.

The Friedmann equations that describe the dynamics of the universe can be written as

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8}{3}\pi G\rho = H_0^2\left[\Omega_{r_0}\left(\frac{a_0}{a}\right)^4 + \Omega_{m_0}\left(\frac{a_0}{a}\right)^3 + \Omega_{v_0}\right],$$

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} = -8\pi G P.$$

The time dependent solutions of the Friedmann equations are:

For matter dominated era: The cosmic scale factor and age of the universe for **the case** $k = \Omega_{v_0} = 0$**:** are

$$a = a_0\left(\frac{t}{t_0}\right)^{2/3}, \qquad t_0 = 2/3H_0.$$

the case $k = 1$ but $\Omega_{v_0} = 0$: are

$$a = \frac{1}{2}\alpha(1 - cos\theta), \qquad t = \frac{1}{2}\alpha(\theta - sin\theta).$$

the case $k = -1$ but $\Omega_{v_0} = 0$: are

$$a = \left(\frac{1}{2}\right)\beta(cosh\varphi - 1), \qquad t = \left(\frac{1}{2}\right)\beta(sinh\varphi - \varphi).$$

the case $k = 0$ but $\Omega_{v_0} \neq 0$: are

$$a = \left[\frac{1 + \sqrt{\Omega_{v_0}}}{\sqrt{1 - \Omega_{v_0}}}\right] e^{\frac{2}{3\sqrt{\Omega_{v_0}}}}, \qquad t_0 = \frac{2}{3H_0\sqrt{\Omega_{v_0}}} ln\left[\frac{1 + \sqrt{\Omega_{v_0}}}{\sqrt{1 - \Omega_{v_0}}}\right].$$

If the density of the universe exceeds a certain threshold known as the critical density, the gravitational attraction is strong enough to stop and latter reverse the expansion of the universe, causing it eventually to re-collapse in what is known as the "Big Crunch". If the average density of the universe falls short of the critical density,the universe expands forever, and after a certain point the expansion proceeds much as if the universe were empty.

For radiation dominated era: The cosmic scale factor and age of the universe for **the case $k = \Omega_{v_0} = 0$:** are

$$a = a_0\left(\frac{t}{t_0}\right)^{1/2}, \qquad t_0 = 2a_0^2/H_0.$$

the case $k = 1$ but $\Omega_{v_0} = 0$: are

$$a = \sqrt{\frac{2q_0}{2q_0 - 1}} sin(\eta), \qquad t = \sqrt{\frac{2q_0}{2q_0 - 1}}(1 - cos\eta).$$

the case $k = -1$ but $\Omega_{v_0} = 0$: are

$$a = \sqrt{\frac{2q_0}{2q_0 - 1}} sinh(\eta), \qquad t = \sqrt{\frac{2q_0}{1 - 2q_0}}(cosh\eta - 1).$$

the case $k = 0$ but $\Omega_{v_0} \neq 0$: are

$$a = \left[\frac{1 + \sqrt{\Omega_{v_0}}}{\sqrt{1 - \Omega_{v_0}}}\right] e^{\frac{1}{2\sqrt{\Omega_{v_0}}}}, \qquad t_0 = \frac{1}{2H_0\sqrt{\Omega_{v_0}}} ln\left[\frac{1 + \sqrt{\Omega_{v_0}}}{\sqrt{1 - \Omega_{v_0}}}\right].$$

For both matter and radiation with $k = \Omega_{v_0} = 0$: The cosmic scale factor, age of the universe and t_{eq}(epoch of matter-radiation equality) respectively are

$$a(\eta) = \frac{\sigma_m \eta^2}{4} + \sqrt{\sigma_r}\,\eta, \quad t(\eta) = \frac{\sigma_m \eta^3}{12} + \frac{\sqrt{\sigma_r}\eta^2}{2}, \quad t_{eq} = \frac{2\sqrt{2}(\sqrt{2}-1)}{3}\left(\frac{8\pi G\rho_{m_0}}{3}\right)^{-1/2}\left(\frac{\rho_{r_0}}{\rho_{m_0}}\right)^{3/2}.$$

Since $\rho_r \;\alpha\; a^{-4}$, $\rho_m \;\alpha\; a^{-3}$ and $\rho_v =$ constant, one can see that radiation will always dominate at early times and the vacuum (if non-zero) will always dominate at late times. The ratio of the radiation and matter densities scales as $(\rho_r/\rho_m) \;\alpha\; a^{-1}$ and falls as the universe expands.

Matter-radiation equality occurs when the densities of the matter and radiation are equal.

Observable quantities that are considered as an evidences for the expansion of the Friedmann universe are; redshift-distance relation, the peculiar behavior of angular size and the existence of horizons.

Red Shift is the change in observed wave length of a spectral line relative to the wave length of a spectral line at emission:

$$z = \frac{v_e}{v_0} - 1 = \frac{v_e - v_0}{v_0} = \delta v/v = \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{\delta\lambda}{\lambda} = \frac{a_0 - a_e}{a_e}.$$

Cosmological red shift is symmetric between receiver and emitter.

The redshift of the light from galaxies is proportional to their distance.

Hubble's Law was taken to mean that, the farther away from us a galaxy is, the faster it moves away from us.

For a galaxy G_1 to have a linear extent d the angle that this length d subtend at our location is

$$\Delta\theta_1 = \frac{d}{r_1 a(t_1)} = \frac{d(1+z)}{r_1 a(t_0)}.$$

The cosmological particle(object) horizon separates co-moving distance(particles) we can currently see from those we cannot currently see.

The cosmological event horizon separates events we are able to see at some time, from events we will never be able to see.

5.2 Conclusion

The effects of various densities in the universe were clear from the results that we obtained in the preceding chapters. From the results, the relationships between each of the components of the universe, its density, the scale factor and age of the universe can be determined. The age of the universe has different relationships among each component suggesting that a variation in one parameter would not have the same effect as another parameter. Many of the results that occurred from solving the Friedmann equations describes the dynamics of the universe. The dependencies upon the value of $a(t)$ for each density parameter describe how each component of matter effects the evolution of the universe. The a^{-4} term describes that the radiation density affects the evolution at small values of t, or the early universe. The radiation at this point causes rapid expansion. As the effect of the radiation decreases on the universe, the influence of the matter with the a^{-3} dependency increases. At larger values of t, the dark energy becomes dominant causing an accelerating expansion. Because at the current time both the matter density and dark energy densities are playing the largest role in the evolution and accelerated expansion of the universe, they also have much more of a role in age of the universe. As the Friedmann equation is solved backwards in time, the large expansion caused by the two components will also have an important role in how quickly the universe will contract back into a singularity. The results verify this hypothesis as the changes in the matter density and the dark energy density can create a wide range of ages for the universe while the radiation density has a much more limited range in which it can effect the age of the universe. Finally, as we have seen, the redshift-distance linear relationship(Hubble's Law), the redshift-angular diameter relation and also the existence of horizons shows the accelerated expansion of the Friedmann universe.

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Declaration

This thesis is my own work, has not been presented for a degree in any other university and that all the sources of material used for the thesis have been dully acknowledged.

Name: **Asnakew Bewketu**

Signature:– – – – – – – – – – – – –

Place and time of submission: Addis Ababa University, March 2012

This thesis has been submitted for examination with my approval as university advisor

Name: **Dr Legesse Wotro**

Signature:– – – – – – – – – – – –