DEPARTMENT OF MATHEMATICS
C0llege of Natural and Computational science

System of Non-linear Ordinary Differential Equations and Stability Analysis

BY
BRHANE KEBEDE MELESS

ADVISOR: **Dr. TESFA BISET**

A Thesis Submitted to the Department of Mathematics in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

Addis Ababa, Ethiopia
September, 2018

Approval

This thesis has been examined and approved as meeting the requirements for the partial fulfillment of Master of Science in Mathematics.

Examining board members

Name	Signature	Date
1. **Dr .Tesfa Biset**(Advisor)	___________	________
2. ___________________(Examiner)	___________	________
3. ___________________ (Examiner)	___________	________
4. ____________________ (Chairperson)	___________	________

DECLARATION

I, Brhane Kebede Meless, with student ID GSK/0095/06, here by declare that this thesis entitled "System of Non-linear Ordinary Differential Equations and Stability Analysis" has been compiled and organized by myself under the supervision of Dr. Tesfa Biset and that it has neverbeen submitted for completion of graduate qualification at any higher learning institution. Any work done by others hasbeen acknowledged and referenced accordingly.

Brhane Kebede Meless

Addis Ababa,Ethiopia

September, 2018

ACKNOWLEDGMENT

First, I would like to express my deepest gratitude to God for giving me patience; I am also grateful to my advisor Dr. Tesfa Biset for his helpful discussion, comments and providing the necessary materials in the preparation of this project.

Secondly I would like to extend my thank to my families specially sebli nati lali samri and all who encouraged me to complete my project and their heart full help while writing the paper. At last, but not the least I would like thanks to Department of mathematics, Addis Ababa University for giving the necessary materials throughout the preparation of this paper.

TABLE OF CONTENTS

Abstract.

The focus of this thesis is on the use of linearization techniques and linear differential equation theory to analyze nonlinear differential equations. Often, mathematical models of real-world phenomena are formulated in terms of systems of nonlinear differential equations, which may be difficult to solve explicitly. To overcome this barrier, we take a qualitative approach to the analysis of solutions to nonlinear systems by making phase portraits and using stability analysis. We demonstrate these techniques in the analysis of two systems of nonlinear differential equations. The first part of this paper gives a survey of standard linearization techniques in ordinary differential equation theory. The second part of the paper presents applications of these techniques to particular systems of nonlinear ordinary differential equation.

UNIT 1

INTRODUCTION AND BASIC DEFINITIONS

Many mathematical models of real-world phenomena are formulated in terms of systems of nonlinear differential equations, which may be difficult to solve explicitly. To overcome this barrier, we take a qualitative approach to the analysis of solutions to nonlinear systems by making phase portraits and using stability analysis. Given any system of first-order ordinary differential equations (possibly nonlinear), we want to create a qualitative characterization of the behavior of solutions depending on predefined initial conditions. We do this by first finding the equilibrium solutions, and then using stability analysis by making a phase portrait for the system that indicates the general behavior of solutions depending on their initial conditions. The stability analysis carried out at each equilibrium involves linearization techniques and solution methods for systems of linear ordinary differential equations, which further involve ideas from linear algebra, including computing eigenvalues and eigenvectors, among other topics. The theory behind these topics is crucial to our work.

Although most of the theory discussed in this thesis can be extended to arbitrary dimensions, for simplicity we will restrict our analysis in two dimensions. Though there are differences between the application of linearization techniques to ordinary differential equations and partial differential equations, the ideas of simplifying the problem at hand and analyzing eigenvalues to gain a general understanding of the solutions are shared. This can be observed, for instance, in separation of variable techniques applied to solve the heat equation and the wave equation.

1.1 Existence and Uniqueness of Solutions

Before analyzing systems of ordinary differential equations, we better first establish the existence of solution of an ordinary differential equation which is fundamental for further analyze. This is the purpose of the existence and uniqueness theory for systems of differential equations. In this section we discuss the issues of existence and uniqueness for Initial value problems corresponding to first order systems of ordinary differential equation. This discussion includes the case of scalar first order ordinary differential equation and also general scalar ordinary differential equation of higher order.

1.1.1 Existance of solution

Definition: Let$\Omega \in \mathbb{R}^n$be a domain and $I \subseteq \Re$ be an interval. Let$f$: $I \times \Omega \to \Re^n$ be a continuous function defined by $(x, Y) \mapsto f(x, Y)$ where Y= $(y_1, y_2, y_3, \ldots, y_n)$ and $x \in I$.

Let (x, Y) be arbitrary point in $I \times \Omega$. An initial value problem (IVP) for a first order system of n-ordinary differential equations is given by:

$$\begin{cases} Y' = f(x, Y), \\ Y(x_0) = Y_0 \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.1)$$

Definition(Solution of an IVP for systems of ordinary differential equation):- An n-tuple of functions$U = (u_1, u_2, u_3, \ldots, u_n) \in C^1[I_0]$ where$I_0 \subseteq I$ is a subinterval containing the point $x_0 \in$I is called a solution of IVP (1) if for every $x \in I_0$, the $n+1$ tuple $(x, u_1(x), u_3(x), \ldots, u_n(x)) \in I \times \Omega$

$$U'(\text{x}) = f(x, U(x)), \forall x \epsilon I_0 \text{ and } U(x_0) = Y_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.2)$$

1.1.2 Uniqueness of solution

Definition:(local uniqueness): An initial value problem is said to have local uniqueness property if for each $(x_0, Y_0) \in I \times \Omega$ and for any two solutions Y_1 and Y_2 of IVP(1) define on intervals I_1 and I_2 respectively, there exists an open interval $I_{\delta_{lr}}$ ⍰$:= (x_0 - \delta_l, x_0 + \delta_r)$ containing the point x_0 such that $Y_1(x) = Y_2(x)$ for all $x \in I_{\delta_{lr}}$.

Definition(Global uniqueness):- An initial value problem is said to have global uniqueness property if for each $(x_0, Y_0) \in I \times \Omega$ and for any two solutions Y_1 *and* Y_2 of IVP (1) defined on intervals I_1 *and* I_2 respectively, then equality $Y_1(x) = Y_2(x)$ holds for all $x \in I_1 \cap I_2$.

1.2. CRITICAL POINTS AND *PHASE PORTRAITS*

Autonomous systems of equations, when they are interpreted as describing the motion of a point in phase space, are particularly susceptible to some very beautiful techniques of local analysis. By performing a local analysis of the system near what are known as *critical points*, one can make remarkably accurate predictions about the global properties of the solution.

1.2.1 Phase-Space Interpretation of Autonomous Equations

Definition:-Differential equations which do not contain the independent variable explicitly are said to be autonomous.

Any differential equation is equivalent to an autonomous equation of one higher order; to remove the explicit reference to the independent variable x one simply solves for x in terms of y and its derivatives and then differentiates the resulting equation one.

It is convenient to study the approximate behavior of an autonomous equation of order n when it is in the form of a system of n coupled first-order equations. Also, by convention we will think of the independent variable of the system to be time t and the dependent variables$x_1, x_2, \dots, x_n$.

The general form of such a system is:

$$\begin{cases} \frac{d}{dt}x_1(t) = f_1(t, x_1, \dots, x_n) \\ \vdots \\ \frac{d}{dt}x_n(t) = f_n(t, x_1, \dots, x_n) \end{cases} \quad \dots\dots(1.3)$$

The solution of the system (1.3) is a curve or trajectory in an n-dimensional space called phase space. We will assume $f_1, f_2, \dots, f_n$are continuously differentiable with respect to each of their arguments. Thus, by the existence and uniqueness theorem of differential equations, any initial condition $x_1(0) = a_1, x_2(0) = a_2, \dots, x_n(0) = a_n$ gives rise to a unique trajectory through the point $(x_1, x_2, x_3, \dots, x_n)$.

To understand this uniqueness property geometrically, note that at every point on the trajectory $(x_1(t), x_2(t), x_3(t), \dots, x_n(t))$. The system (1.3) assigns a unique velocity vector

$(\frac{dx_1}{dt}(t), \frac{dx_2}{dt}(t), \dots, \frac{dx_n}{dt}(t))$ which is tangent to the trajectory at that point. It immediately follows that two trajectories cannot cross; otherwise, the tangent vector at the crossing point would not be unique.

1.2.2 Critical Points in Phase Space

If there are any solutions to the set of simultaneous algebraic equations:

$$\begin{cases} f_1(t, x_1, \dots, x_n) = 0 \\ \dots \\ f_n(t, x_1, \dots, x_n) = 0 \end{cases} \quad \text{……………………………………………………}(1.4)$$

Then there are special degenerate trajectories in phase space which are just points. (The velocity at these points is zero so the position vector does not move.) Such points are called *critical points***.**

Definition:-Phase plane is a visual display of certain characteristics of certain kinds of equations; a coordinate plane with axes being the values of the two state variables. It is a two dimensional case of the general n-dimensional phase space.

The parametric curves traced by the solutions are called *trajectories***.**

Definition:-For a system of ordinary differential equation phase portrait is a representative set of its solutions plotted as parametric curves (with t as its parameter) on the Cartesian plane tracing the path of each particular solution$(x, y) = (x(t), y(t)) - \infty < t < \infty$.

1.2.3 Two- Dimensional Phase Space

Two-dimensional phase space (the phase plane) is used to study a system of two coupled first-order equations. The phase plane is more complicated than the phase line, but, it is still possible to make elegant global analyses of systems of two coupled differential equations. First, we enumerate the possible global behaviors of a trajectory in a two-dimensional system:

1. The trajectory may approach a critical point as$t \to \infty$.

2. The trajectory may approach ∞ as$t \to \infty$.

3. The trajectory may remain motionless at a critical point for all t.

4. The trajectory may describe a closed orbit or cycle.

5. The trajectory may approach a closed orbit (by spiraling inward or outward the orbit) as $t \to \infty$.

We enumerate the possible local behaviors for trajectories near a critical point:
1. All trajectories may approach the critical point along curves which are asymptotically straight lines as $t \to \infty$. We call such a critical point a *stable node*.

2. All trajectories may approach the critical point along spiral curves as $t \to \infty$. Such a critical point is called *a stable spiral point*. [It is also possible for trajectories to approach the critical point along curves which are neither spirals nor asymptotic to straight lines

3. All time-reversed trajectories [that is, x(t) with t decreasing] may move toward the critical point along paths which are asymptotically straight lines as$t \to -\infty$. Such a critical point is an *unstable node*. As t increases, all trajectories that start near an unstable node move away from the node along paths that are approximately straight lines, at least until the trajectory gets far from the node.

4. All time-reversed trajectories may move toward the critical point along spiral curves as $t \to \infty$. Such a critical point is called an unstable spiral point. As t increases, all trajectories move away from an unstable spiral point along trajectories that are, at least initially, spiral shaped.

5. All trajectories may form closed orbits about the critical point. Such a critical point is called a center.

Note that while nodes and saddle points occur in one-dimensional phase space, spiral points and centers cannot exist in less than two dimensions.

1.3 Linear Autonomous Systems

Since two dimensional linear autonomous systems can exhibit any of the critical point behaviors that we have described above it is appropriate to study linear systems before going on the non-linear systems.

A two-dimensional linear autonomous system $\begin{cases} \frac{dx_1}{dt} = ax_1 + bx_2 \\ \frac{dx_2}{dt} = cx_1 + dx_2 \end{cases}$ ……………………..…..……..(1.5)

Can be written in matrix form $X' = MX$ …………………………………………………..(1.6)

Where $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

If the eigenvalues $\lambda_1$andλ_2 of the matrix M are distinct and ξ_1 and ξ_2 are eigenvectors of M associated with the eigenvalues $\lambda_1$and λ_2 ,then the general solution to (1.5) has the form:

$$X(t) = c_1\xi_1 e^{\lambda_1 t} + c_2\xi_2 e^{\lambda_2 t} \text{...(1.7)}$$

Where c_1 andc_2 are constants of integration which are determined by the initial position X_0. The linear system (1.5) has a critical point at (0,0). Since $ax_1 + bx_2 = 0$ and $cx_1 + dx_2 = 0$

at $(x_1, x_2) = (0,0)$

It is possible to classify this critical point once $\lambda_1$and λ_2 are known.

λ_1 and λ_2 satisfy the eigenvalue condition;

$$\text{Det}(M - \lambda I) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = \lambda^2 - \lambda(a+d) + ad - bc \text{ ....(1.8)}$$

1. If $\lambda_1$and λ_2 are real and negative all the trajectories approach the origin as $t \to +\infty$ and (0,0) is a *stable node.*
2. If $\lambda_1$and λ_2 are real and positive then all the trajectories move away from (0,0) as $t \to +\infty$ and (0,0) is *unstable.*
3. If $\lambda_1$and λ_2 are real but $\lambda_1$is positive and λ_2 is negative then (0,0) is *a saddle point.*

Trajectories approach the origin in the direction of ξ_2 and move away from the origin in the directionξ_1. Eigenvalues $\lambda_1$and λ_2 may be complex. However, when the matrix is real, then $\lambda_1$and λ_2 must be complex conjugate pairs.

4. If $\lambda_1$and λ_2 are pure imaginary then the vector $X(t)$ represent a closed orbit for any $c_1$and c_2 and the critical point (0,0) is *centre*.
5. If $\lambda_1$and λ_2 are complex with non zero real part, then the critical point (0, 0) is *spiral point.*
    - When $Re(\lambda_{1,2}) < 0$, then $X(t) \to 0$ as $t \to \infty$ and (0,0) is a *stable spiral point*.
    - When $Re(\lambda_{1,2}) > 0$), then (0,0) is an *unstable spiral point*.

It is important to determine the direction of rotation of the trajectories in the vicinity of spiral point or center. To determine whether the rotation is clockwise or counterclockwise we simply let$x_2 = 0$, $x_1 > 0$ and see whether x_2' is negative or positive

Definition: Assume that $(0, 0)$ is an isolated critical point of the system. Let C be a path of (1.5); let $x_1 = f(t), x_2 = g(t)$ be a solution of (1.5) defining C parametrically. Let

$$D(t) = \sqrt{[f(t)]^2 + [g(t)]^2} \quad \ldots\ldots\ldots(1.9)$$

Denote the distance between the critical point (0, 0) and the point $R: (f(t), g(t))$ on C. the critical point $(0, 0)$ is called stable if for every $\epsilon > 0$, there exists a number $\delta > 0$ such that the following is true: every path C for which

$$D(t_0) < \delta \text{ for some value } t_0 \quad \ldots\ldots\ldots(1.10)$$

Is defined for all $t \geq t_0$ and is such that

$$D(t) < \epsilon \text{ for } t_0 \leq t < \infty \quad \ldots\ldots\ldots(1.11)$$

Let us try to analyse this definition, making use of **fig 1** as we do so. The critical point (0, 0) is said to be stable if, corresponding to every positive number ϵ, we can find another positive number δ. According to (1.9), the inequality $D(t_0) < \delta$ for some value t_0 in (1.10) means that the distance between the critical point (0, 0) and the point R on the path C must be less than δ at $t = t_0$. R lies within the circle K_1 of radius δ about $(0, 0)$ (**see fig 1),** Likewise the inequality $D(t) < \epsilon$ for $t_0 \leq t < \infty$ in (1.11) means that the distance between (0, 0) and R is less than ϵ for all $t \geq t_0$, and hence that for $t \geq t_0$, R lies within the circle K_2 of radius ϵ about (0, 0). If (0, 0) is stable, then every path C which is inside the circle K_1 of radius δ at $t = t_0$ will remain inside the circle $K_2$of radiusϵ.

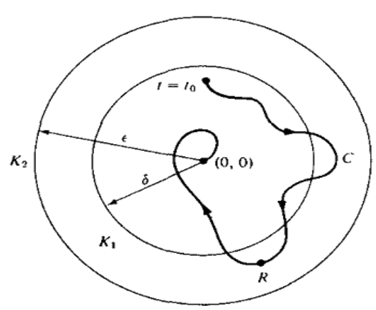


Figure 1

Definition: Assume that $(0,0)$ is an isolated critical point of the system). Let C be a path of (1.5); let $x = f(t), y = g(t)$ be a solution of (1.5) defining C parametrically. Let

$$D(t) = \sqrt{[f(t)]^2 + [g(t)]^2} \quad \text{...(1.9)}$$

Denote the distance between the critical point (0, 0) and the point R: $(f(t), g(t))$ on C. The critical point (0, 0) is called asymptotically stable if

1. It is stable and
2. There exists a number $\delta_0 > 0$ such that if

$$D(t_0) < \delta_0 \quad \text{..(1.12)}$$

For some valuet_0, then

$$\lim_{t\to\infty} f(t) = 0 \lim_{t\to\infty} g(t) = 0 \quad \text{...(1.13)}$$

To analyse this definition, note that condition (1) requires that (0, 0) must be stable. That is, every path C will stay as close to (0, 0) as we desire after it once gets sufficiently close. But asymptotic stability is a stronger condition than mere stability. In addition to stability, the condition (2) requires that every path that gets sufficiently close to (0, 0) ultimately approaches (0, 0) as $t \to \infty$.

Definition: A critical point is called unstable if it is not stable.

1.3.1 Critical point and paths of linear systems

We consider the linear system

$$\begin{cases} \frac{dx}{dt} = ax_1 + bx_2 \\ \frac{dy}{dt} = cx_1 + dx_2 \end{cases} \quad \text{...(1.14)}$$

Where a, b, c, and d are real constants. The origin (0, 0) is a critical point of (1.14).

We assume that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0 \quad \text{..(1.15)}$$

And hence (0, 0) is the only critical point of (1.14). then the solutions (1.14) is of the form

$$x = Ae^{\lambda t}, y = Be^{\lambda t} \quad \text{..(1.16)}$$

If (1.16) is to be the solution of (1.14), then λ must satisfy the quadratic equation

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0 \quad \text{...(1.17)}$$

called the characteristic equation of (1.14). Note that by condition (1.15) zero cannot be a root of the equation (1.17) in the problem under discussion.

Let λ_1 and λ_2 be the roots of the characteristic equation (1.17). We shall prove that the nature of the critical point (0, 0) of the system (1.14) depends upon the nature of roots λ_1 andλ_2. We would expect three possibilities according as λ_1 and λ_2 are real and distinct, real and equal, or conjugate complex. But actually the situation here is not quite so simple and we must consider the following five cases:

1. $\lambda_1$andλ_2 are real, unequal, and of the same sign.
2. λ_1 and λ_2 are real, unequal and of opposite sign
3. $\lambda_1$andλ_2 are real and equal.
4. λ_1 andλ_2 are conjugate complex but not pure imaginary.
5. λ_1 andλ_2 are pure imaginary.

Case 1:λ_1 and λ_2 are real, unequal, and of the same sign

Theorem: If the roots λ_1 and λ_2 of the characteristic equation (1.17) are real, unequal, and of the same sign, then the critical point (0, 0) of the linear system (1.17) is anode.

Proof: We first assume that λ_1 and λ_2 are both negative and take $\lambda_1 < \lambda_2 < 0$. Then the general equation of the system (1.14) may then be written

$$\begin{cases} x = c_1 A_1 e^{\lambda_1 t} + c_2 A_2 e^{\lambda_2 t} \\ y = c_1 B_1 e^{\lambda_1 t} + c_2 B_2 e^{\lambda_2 t} \end{cases} \quad \text{.....(1.18)}$$

Where A_1, B_1, A_2 and B_2 are definite constants and $A_1 B_2 \neq A_2 B_1$, and where c_1 and c_2 are arbitrary constants. Choosing $c_2 = 0$ we obtain the solutions

$$\begin{cases} x = c_1 A_1 e^{\lambda_1 t} \\ y = c_1 B_1 e^{\lambda_1 t} \end{cases} \quad \text{.....(1.19)}$$

Choosing $c_1 = 0$, we obtain the solutions

$$\begin{cases} x = c_2 A_2 e^{\lambda_2 t} \\ y = c_2 B_2 e^{\lambda_2 t} \end{cases} \quad \text{.....(1.20)}$$

For $c_1 > 0$, the solutions (1.19) represent a path consisting of "half" of the line $B_1 x = A_1 y$ of slope B_1/A_1. For any $c_1 < 0$, they represent a path consisting of the "other half" of the line. Since $\lambda_1 < 0$, both of these half line paths approach (0, 0) as $t \to \infty$. Also, since $\frac{y}{x} = \frac{B_1}{A_1}$, these two paths enter (0, 0) as $t \to \infty$.With slope $\frac{B_1}{A_1}$. In the same manner, for any $c_2 > 0$ the solutions (1.20) represent a path consisting of "half" of the line $B_2 x = A_2 y$; while for any $c_2 < 0$, they represent a path consisting of the "other half" of the line.these two half line paths approach (0, 0) as $t \to \infty$. Also, since $\frac{y}{x} = \frac{B_1}{A_1}$, these two half-line paths enter (0, 0) as $t \to \infty$ with slope $\frac{B_2}{A_2}$.

Thus the solutions (1.19)and (1.20) provide us with four half-line paths which all approach and enter (0, 0) as $t \to \infty$.

If $c_1 \neq 0$ and $c_2 \neq 0$, the general solution (1.18) represents nonrectilinear paths.Again,since $\lambda_1 < \lambda_2 < 0$ all of these paths approach (0, 0) as $t \to \infty$. Further, since

$$\frac{y}{x} = \frac{c_1 B_1 e^{\lambda_1 t} + c_2 B_2 e^{\lambda_2 t}}{c_1 A_1 e^{\lambda_1 t} + c_2 A_2 e^{\lambda_2 t}} = \frac{\left(\frac{c_1 B_1}{c_2}\right) e^{(\lambda_1 - \lambda_2)t} + B_2}{\left(\frac{c_1 A_1}{c_2}\right) e^{(\lambda_1 - \lambda_2)t} + A_2}$$

We have $\lim_{t\to\infty} \frac{y}{x} = \frac{B_2}{A_2}$ and so all of these paths enter (0, 0) with limiting slope $\frac{B_2}{A_2}$.

Thus all the paths (both rectilinear and nonrectilinear) enter (0, 0) as $t \to \infty$, and all except the two rectilinear ones defined by (1.19) enter with slope$\frac{B_2}{A_2}$. According to the definition the critial point (0, 0) is anode and it is asymptoticaly stable. A qualitative diagram of the paths appears in **fig 2**

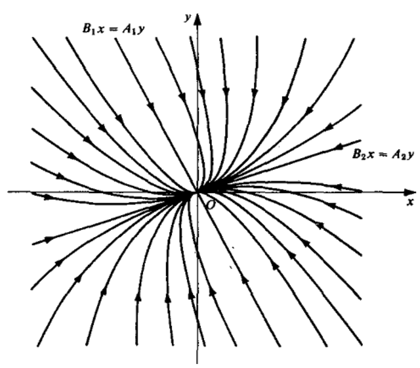


Figure 2

If now λ_1 andλ_2 are both positive and we take $\lambda_1 > \lambda_2 > 0$, we see that the general solution of (1.14) is still of the form (1.18) and particular solutions of the forms (1.19) and (1.20) still exist. The situation is the same as before, except all the paths approach and enter (0, 0) as $t \to -\infty$. The qualitative diagram of **fig 2** is unchanged, except that all the arrows now point in the opposite directions. The critical point (0, 0) is still a node, but in this case it is clear that it is unstable.

Case 2: λ_1 and λ_2 are real, unequal and of opposite sign

Theorem: If the roots λ_1 and λ_2 of the characteristic equation (1.17) are ral, unequal, and of opposite sign, then the critical point (0, 0) of the linear system (1.14) is a saddle point.

Proof: We assume that $\lambda_1 < 0 < \lambda_2$. The general solution of the system (1.14) may again be written in the form (1.18) and particular solutions of the forms (1.19) and (1.20) are again present.

For any $c_1 > 0$, the solutions (1.19) again represent a path consisting of "half" the line

$B_1x = A_1y$; while for any $c_1 < 0$, they again represent a path consisting of the "other half" of this line. Also, $\lambda_1 < 0$, both of these half-line paths still approach and enter (0, 0) as $t \to \infty$.

Also, for any$c_2 > 0$, the solutions (1.20) represent a path consisting of "half" the line

$B_2x = A_2y$; and for any $c_2 < 0$, the path which they represent consists of the "other half" of this line. But in this case, since$\lambda_2 > 0$, both of these half-line paths now approach and enter (0, 0)

as $t \to -\infty$.

Once again, if $c_1 \neq 0$ and$c_2 \neq 0$, the general solution (1.18) represents non-rectilinear paths. But here since$\lambda_1 < 0 < \lambda_2$, none of these paths can approach (0, 0) as $t \to +\infty$ or as $t \to -\infty$. Further, none of them pass through (0, 0) for any t_0 such that$-\infty < t_0 < \infty$, we see from (1.18) that each of these non-rectilinear paths becomes asymptotic to one of the half-line paths defined by (1.20). As $t \to -\infty$, each of them becomes asymptotic to one of the paths defined by (1.19).

Thus there are two half line paths which approach and enter (0, 0) as $t \to +\infty$ and two other half-line paths which approach and enter (0, 0) as $t \to -\infty$. All other paths are non-rectilinear paths which do not approach (0, 0) as $t \to +\infty$ or as $t \to -\infty$, but which become asymptotic to one or another of the four half-line paths as $t \to +\infty$ and as $t \to -\infty$. According to the definition, the critical point $(0, 0)$ is a saddle point and it is unstable. A qualitative diagram of the paths appears in fig 3.

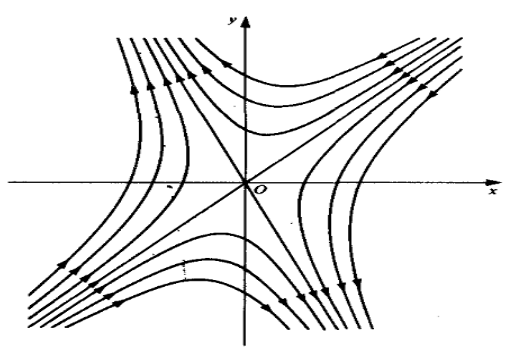


Figure 3

Case 3: λ_1 and λ_2 are real and equal.

Theorem: If the roots $\lambda_1$and λ_2 othe characteristic equation (1.17) are real and equal, then the critical point (0, 0) of the linear system is a node.

Proof: Let us first assume that$\lambda_1 = \lambda_2 = \lambda < 0$. We consider two subcases:

3(a) $a = d \neq 0, \quad b = c = 0.$

3(b) All other possibilities leading to a double root.

We consider first the special case 3(a). the characteristic equation (1.17) becomes

$$\lambda^2 - 2a\lambda + a^2 = 0$$

And hence $\lambda = a = d$. The system (14) becomes

$$\begin{cases} \dfrac{dx}{dt} = \lambda x \\ \dfrac{dy}{dt} = \lambda y \end{cases}$$

The general solution of this system is then

$$\begin{cases} x = c_1 e^{\lambda t} \\ y = c_2 e^{\lambda t} \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.21)$$

Where c_1 and c_2 are arbitrary constants. The paths defined by (1.21) for the various values of c_1 and c_2 are half lines of all possible slopes. Since$\lambda < 0$, we see that each of these half-lines approaches and enters $(0, 0)$ at $t \to +\infty$. That is, all the paths of the system enter $(0, 0)$as $t \to +\infty$. According to the definition, the critical point (0, 0) is a node and it is asymptotically stable. A qualitative diagram of the paths appears in fig 4.

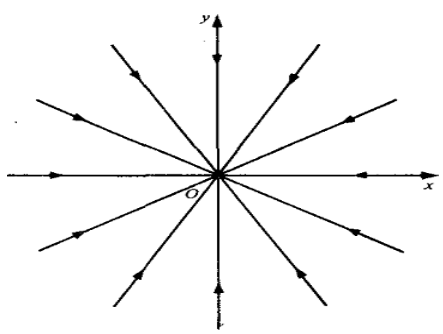


Figure 4

If $\lambda > 0$, the situation is the same except that the paths enter (0, 0) as $t \to -\infty$, the node (0, 0) is unstable, and the arrows in fig 4 are reversed.

We mention that this type of node is sometimes called a star-shaped node.

Let us now consider case3(b). Here the characteristic equation has the double root $\lambda < 0$, but we exclude the special circumstance of case 3(a). the general solution of the system may in this case be written

$$\begin{cases} x = c_1 A e^{\lambda t} + c_2 (A_1 t + A_2) e^{\lambda t} \\ y = c_1 B e^{\lambda t} + c_2 (B_1 t + B_2) e^{\lambda t} \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.22)$$

Where the A's and B's are definite constants, c_1 and c_2 are arbitrary constants, and

$\frac{B_1}{A_1} = \frac{B}{A}$.chosing $c_2 = 0$ in (1.22) we obtain solutions

$$\begin{cases} x = c_1 A e^{\lambda t} \\ y = c_1 B e^{\lambda t} \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1.23)$$

Or $c_1 > 0$, the solutions (1.23) represent a path consisting of "half" of the line $Bx = Ay$ of slope $\frac{B}{A}$; for any $c_1 < 0$, they represent a oath which consists of the "other half" of the line. Since $\lambda < 0$, both of these half-line paths approach (0, 0) with slope B/A.

If $c_2 \neq 0$, the general solution (1.22) represents non-rectilinear paths. Since $\lambda < 0$, we see from (1.22) that

$$\lim_{t\to\infty} x_1 = 0\ , \qquad \lim_{t\to\infty} x_2 = 0$$

Thus the non-rectilinear paths defined by (1.22) all approach (0, 0) as $t \to \infty$.

Also

$$\frac{x_2}{x_1} = \frac{c_1 B e^{\lambda t} + c_2(B_1 t + B_2)e^{\lambda t}}{c_1 A e^{\lambda t} + c_2(A_1 t + A_2)e^{\lambda t}} = \frac{\left({}^{c_1 B}/_{c_2}\right) + B_2 + B_2 t}{\left({}^{c_1 A}/_{c_2}\right) + A_2 + A_2 t}$$

We see that $\lim_{t\to\infty} \frac{x_2}{x_1} = \frac{B}{A}$

Thus all the non rectilinear paths enter (0, 0) with limiting slope B/A.

Thus all the paths enter (0, 0) as $t \to \infty$ with slope B/A. Then the critical point $(0, 0)$ is node and it is asymptotically stable.

If $\lambda > 0$, the situation is again the same except that the paths enter (0, 0) as $t \to -\infty$, and node (0, 0) is unstable and arrows in **fig 4** are reversed.

Case 4:λ_1 and λ_2 are conjugate complex but not pure imaginary.

Theorem: if the roots λ_1 and λ_2 of the characteristic equation (1.17) are conjugate complex with real part non-zero(that is not pure imaginary) , then the critical point (0, 0) of the linear system (1.14) is a spiral point.

Proof: since λ_1 and λ_2 are complex conjugate with real part not zero, we may write these roots $\alpha \pm i\beta$, where α and β both real and unequal to zero.Then the general equation of the system may be written

$$x_1 = e^{\alpha t}[c_1(A_1 cos\beta t - A_2 sin\beta t) + c_2(A_2 cos\beta t + A_1 sin\beta t)]$$
$$x_2 = e^{\alpha t}[c_1(B_1 cos\beta t - B_2 sin\beta t) + c_2(B_2 cos\beta t + B_1 sin\beta t)] \quad \ldots\ldots\ldots\ldots(1.24)$$

where A_1, A_2, B_1 and B_2 are definite real constants and c_1 and c_2 are arbitrary constants.

Let us first assume that $\alpha < 0$ then from (1.24) we see that

$$\lim_{t\to\infty} x_1 = 0\ , \qquad \lim_{t\to\infty} x_2 = 0$$

And hence all paths defined by (1.24) approach $(0, 0)$ as $t \to \infty$. We can then write equation (1.24) in the form

$$x_1 = e^{\alpha t}(c_3 cos\beta t + c_4 sin\beta t)$$
$$x_2 = e^{\alpha t}(c_5 cos\beta t + c_6 sin\beta t) \quad \ldots\ldots\ldots\ldots(1.25)$$

where $c_3 = c_1A_1 + c_2A_2$, $c_4 = c_2A_1 + c_1A_2$,$c_5 = c_1B_1 + c_2B_2$ and $c_3 = c_2B_1 + c_1B_2$. Assuming $c_1$and $c_2$are real, the solutions (1.25) represent all paths in the real xy phase plane. We now put the solutions in the form

$$x_1 = K_1 e^{\alpha t}\cos(\beta t + \phi_1)$$
$$x_2 = K_2 e^{\alpha t}\cos(\beta t + \phi_2) \quad \ldots\ldots\ldots\ldots(1.26)$$

Where $K_1 = \sqrt{c_3^2 + c_4^2}, K_2 = \sqrt{c_5^2 + c_6^2}$ and ϕ_1 and ϕ_2 are defined by the equations

$$cos\phi_1 = \frac{c_3}{K_1}, \qquad cos\phi_1 = \frac{c_5}{K_2}$$

$$sin\phi_1 = -\frac{c_4}{K_1}, \qquad sin\phi_1 = -\frac{c_6}{K_2}$$

Let us now consider

$$\frac{x_2}{x_1} = \frac{K_2 e^{\alpha t}\cos(\beta t+\phi_2)}{K_1 e^{\alpha t}\cos(\beta t+\phi_1)} \quad \ldots\ldots\ldots\ldots(1.27)$$

Letting $K = \frac{K_2}{K_1}$and $\phi_3 = \phi_2 - \phi_1$, this becomes

$$= \frac{Kcos(\beta t + \phi_1 - \phi_3)}{\cos(\beta t + \phi_1)}$$

$$= K\left[\frac{cos(\beta t+\phi_1)cos\phi_3 + sin(\beta t+\phi_1)sin\phi_3}{\cos(\beta t+\phi_1)}\right] \quad \ldots\ldots\ldots\ldots(1.28)$$

$$= K[cos\phi_3 + sin\phi_3 \tan(\beta t + \phi_1)]$$

Provided that $\cos(\beta t + \phi_1) \neq 0$. As a result the periodicity of the trigonometric function we conclude from the expressions that $\lim_{t\to\infty}\frac{x_2}{x_1}$ does not exist and so that the paths do not enter (0, 0). Instead, the paths approach $(0, 0)$ in a spiral-like manner, winding around (0, 0) an infinite number of times as$t \to +\infty$. According to the definition, the critical point $(0, 0)$ is a spiral point and it is asymptotically stable.

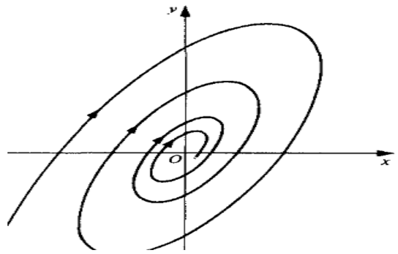


Figure 6

If$\alpha > 0$, the situation is the same except that the paths approach (0, 0) as $t \to -\infty$, the spiral point (0, 0) is unstable and the arrows are reversed.

Case 5 λ_1 and λ_2 are pure imaginary.

Theorem: If the roots $\lambda_1$and λ_2 of the characteristic equation (1.17) are pure imaginary, then the critical point (0, 0) of the linear system is a center.

Proof: Since $\lambda_1$and λ_2 are pure imaginary we may write as $\alpha \pm i\beta$, where $\alpha = 0$ but β is real and unequal to zero. Then the general solution of the system (1.14) is of the form (1.24), where $\alpha = 0$. In the notation (1.26) all real solutions may be written in the form

$$\begin{aligned} x_1 &= K_1\cos(\beta t + \phi_1) \\ x_2 &= K_2\cos(\beta t + \phi_2) \end{aligned} \qquad \text{..(29)}$$

WhereK_1, K_2,$\phi_1$and ϕ_2 are defined as before. The solutions (1.29) oscillate infinitely between -1 and 1 as $t \to +\infty$ and as$t \to -\infty$, the paths do not approach (0, 0) as $t \to +\infty$ or as $t \to -\infty$.

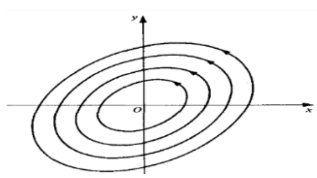


Figure 7

We summarize our result in the following table.

Nature of roots λ_1 and λ_2 of characteristic equation $\lambda^2-(a+d)\lambda+(ad-bc)=0$	Nature of critical point (0, 0) of the linear system $\frac{dx_1}{dt}=ax_1+bx_2$, $\frac{dx_2}{dt}=cx_1+dx_2$	Stability of critical point $(0,0)$
Real, unequal, and of same sign	Node	Asymptotically stable if roots are negative and unstable if roots are positive
Real, unequal, and of opposite sign	Saddle point	Unstable
Real, equal	Node	Asymptotically stable if roots are negative; unstable if roots are positive
Conjugate complex but not pure imaginary	Spiral point	Asymptotically stable if real part of roots is negative; unstable if real part of roots is positive
Pure imaginary	Centre	Stable but not asymptotically stable

UNIT 2

METHODS OF ANALYSIS

Our main goal when analyzing systems of ordinary differential equations is to gain an understanding of the behaviors of the solutions of the system. The natural approach for analyzing a system is to solve it explicitly and this system works well if the system is linear. So in this part, we will see the solution of linear system analytically and to solve nonlinear systems, linearize the system at its equilibrium (critical) point and gain a qualitative understanding of the solutions by analyzing the linearized system.

2.1 Solving explicitly for linear system

To demonstrate how to solve a linear system of ordinary differential equations and to introduce some general terminology about types of equilbria, we will work some examples of linear systems in two dimensions.

Example1:- consider the system $\begin{cases} \frac{dx_1}{dt} = 7x_1 + 3x_2 \\ \frac{dx_2}{dt} = 3x_1 - x_2 \end{cases}$ ………………………….……………..*

the coefficient matrix of the system $A = \begin{pmatrix} 7 & 3 \\ 3 & -1 \end{pmatrix}$

To determine the eigenvalues of A solve the equation $\det(A - \lambda I) = 0$

The eigenvalues 0f A are $\lambda_1 = 8$ and$\lambda_2 = -2$ and eigenvectors are $\xi_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\xi_2 = \begin{pmatrix} \frac{-3}{5} \\ 1 \end{pmatrix}$

The solution of the system is $X(t) = c_1\xi_1 e^{\lambda_1 t} + c_2\xi_2 e^{\lambda_2 t} = \begin{pmatrix} c_1 e^{8t} + c_2 e^{-2t} \\ c_1 e^{8t} + c_2 e^{-2t} \end{pmatrix}$ where $c_1, c_2 \epsilon \mathbb{R}$ are arbitrary constants.

Solutions with initial conditions near the critical point do not stay close to the critical point, so we call this equilibrium unstable and furthermore we call this specific type (one positive eigenvalue and one negative eigenvalue) of point ***a saddle point.***

Example 2: Consider the system $\begin{cases} \frac{dx_1}{dt} = -\frac{1}{2}x_1 + x_2 \\ \frac{dx_1}{dt} = -x_1 - \frac{1}{2}x_2 \end{cases}$ ……………………………………..**

This system is of the form $X'(t) = MX$ with $M = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}$

The eigenvalues for this system are $\lambda_1 = -\frac{1}{2} + i$ and$\lambda_2 = -\frac{1}{2} - i$. Finding the corresponding eigenvectors $\xi_1 = \binom{1}{i}$ and $\xi_2 = \binom{1}{-i}$ yields two possible solutions $X^{(1)} = \binom{1}{i} e^{\left(-\frac{1}{2}+i\right)t}$ and $X^{(2)} = \binom{1}{-i} e^{\left(-\frac{1}{2}-i\right)t}$

However, in this case, the real and imaginary parts of $X^{(2)}$ are linear combinations of the real and imaginary parts of$X^{(1)}$, so we can express the whole solution with just $X^{(1)}$ in the form

$Re(X^{(1)}) + iIm(X^{(1)}$)or more simply $X^{(1)} = u(t) + iv(t)$:

$X^{(1)} = \begin{pmatrix} e^{\frac{-t}{2}} cost \\ -e^{\frac{-t}{2}} sint \end{pmatrix} + i \begin{pmatrix} e^{\frac{-t}{2}} sint \\ e^{\frac{-t}{2}} cost \end{pmatrix}$ where $u(t) = \begin{pmatrix} e^{\frac{-t}{2}} cost \\ -e^{\frac{-t}{2}} sint \end{pmatrix}$ and $v(t) = \begin{pmatrix} e^{\frac{-t}{2}} sint \\ e^{\frac{-t}{2}} cost \end{pmatrix}$.

As a result, any solution of (**) can be written as a linear combination of u and v.

While our phase portrait of the system will reveal its spiral nature, the solutions differ from the previous straight-line solutions because of the trigonometric factors that are involved. The direction of rotation can also be determined based ona_{ij}. In fact, with some elementary computations we can see that the solutions spiral in a clockwise direction when $a_{12} > a_{21}$, and in a counterclockwise direction when$a_{12} < a_{21}$. Thus, for this system, since$a_{12} = 1 > -1 = a_{21}$, the solutions will rotate clockwise about the critical point at the origin.

2.2 Linearization

Non-linear systems are in general much less amenable for the analytic and algebraic techniques, but we can use the method the mathematics of linear systems to understand the behavior of solution of non-linear systems near their equilibrium points.

- Linearization technique:- consider the autonomous system

$$\begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases} \quad \ldots\ldots\ldots\ldots(2.1)$$

Assume (x_0, y_0) is an equilibrium point. So we would like to find the closest linear system when (x,y) is close to (x_0, y_0).

In order to do that we need to approximate the function $f(x,y)$ and $g(x,y)$ when (x,y) is close to (x_0, y_0). This is a similar problem to approximating a real valued function by its tangent (around a point).

$$f(x,y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)[x - x_0] + \frac{\partial f}{\partial y}(x_0,y_0)[y - y_0]$$

$$g(x,y) \approx g(x_0,y_0) + \frac{\partial g}{\partial x}(x_0,y_0)[x - x_0] + \frac{\partial g}{\partial y}(x_0,y_0)[y - y_0]$$

When (x,y) is close to (x_0, y_0) then the non linear system may be approximated by

$$\begin{cases} \frac{dx}{dt} = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)[x - x_0] + \frac{\partial f}{\partial y}(x_0,y_0)[y - y_0] \\ \frac{dy}{dt} = g(x_0,y_0) + \frac{\partial g}{\partial x}(x_0,y_0)[x - x_0] + \frac{\partial g}{\partial y}(x_0,y_0)[y - y_0] \end{cases} \quad \ldots\ldots\ldots\ldots(2.2)$$

But since (x_0, y_0) is an equilibrium point, we have $f(x_0,y_0) = g(x_0,y_0) = 0$.

This implies

$$\begin{cases} \frac{dx}{dt} = \frac{\partial f}{\partial x}(x_0,y_0)[x - x_0] + \frac{\partial f}{\partial y}(x_0,y_0)[y - y_0] \\ \frac{dy}{dt} = \frac{\partial g}{\partial x}(x_0,y_0)[x - x_0] + \frac{\partial g}{\partial y}(x_0,y_0)[y - y_0] \end{cases} \quad \ldots\ldots\ldots\ldots(2.3)$$

This is a linear system with coefficient matrix $\begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{bmatrix}$

This matrix is called the Jacobian matrix of the system at the point(x_0, y_0).

It just needs to be calculated once for each nonlinear system. For each critical point of the system, all we need to do is to compute the coefficient matrix of the linearized system about the given critical point (x, y) =(x_0, y_0), and then use its eigenvalues to determine the (approximated) type and stability.

Example:- consider the non-linear system

$$\begin{cases} \dfrac{dx}{dt} = -2x + 2x^2 \\ \dfrac{dy}{dt} = -3x + y + 3x^2 \end{cases}$$

The critical points of the system are (0,0) and (1,0). To understand solutions that start near these points, we first compute the Jacobian matrix.

$f(x, y) = -2x + 2x^2$ and $g(x, y) = -3x + y + 3x^2$, then

$$\begin{pmatrix} f_x(x, y) & f_y(x, y) \\ g_x(x, y) & g_y(x, y) \end{pmatrix} = \begin{pmatrix} -2 + 4x & 0 \\ -3 + 6x & 1 \end{pmatrix}$$

At the two equilibrium points, we have

$\begin{pmatrix} f_x(0,0) & f_y(0,0) \\ g_x(0,0) & g_y(0,0) \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ -3 & 1 \end{pmatrix}$ and $\begin{pmatrix} f_x(1,0) & f_y(1,0) \\ g_x(1,0) & g_y(1,0) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$

Near (0,0) the phase portrait for the non-linear system should resemble that of the linearized system $\frac{dY}{dt} = \begin{pmatrix} -2 & 0 \\ -3 & 1 \end{pmatrix} Y$. The eigenvalues are -2 and 1. So the origin is saddle. We can compute that (0,1) is an eigenvector for eigenvalue 1 and (1,1) is an eigenvector for the eigenvalue -2.

At the other critical point (1,0) the linearized system is $\frac{dY}{dt} = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} Y$. Here the eigenvalues are $2 \; and \; 1, so \; the$ origin is a source Using the fact that (0, 1) is an eigenvector for the eigenvalue1 and (1,3) is an eigenvector for the eigenvalue2.

2.3 Stability of critical point

Definition: an equilibrium point $x = 0$ is said to be

a. Stable if for any positive scalar ε, there exists a positive scalar δ such that $\|x(t_0)\| < \delta$ implies $\|x(t)\| < \varepsilon$ for all $t \geq t_0$.
b. Asymptotically stable if it is stable and if in addition $x(t) \to 0$ as$t \to \infty$.
c. Unstable if it is not stable; that is there exists an $\varepsilon > 0$ such that for every $\delta > 0$, there exists an $x(t_0)$ with $\|x(t_0)\| < \delta$ and $\|x(t_1)\| \geq \varepsilon$ for some $t_1 > t_0$.

Definition:-An equilibrium point $x = 0$ is said to be bounded if there exists a constant M which may depend on t_0 and $x(t_0)$ such that $\|x(t)\| \leq M$ for all $t \geq t_0$.

Lemma 2.1: Let $\alpha > 0$ be a real number and let $m \geq 0$ be an integer, then there is a constant $C > 0$ (depending on α and$m$) such that $t^m e^{-\alpha t} \leq C, \forall t \geq 0$.

Proof:-In case $m = 0$, we can take $C = 1$.
In the other cases, the function $g(t) = t^m e^{-\alpha t}$ satisfies $g(0) = 0$, $g(t) \to 0$ as $t \to \infty$. Thus g is bounded on$[0, \infty)$. ■

***Lemma* 2.2**: Let λ be a complex number and let $m \geq 0$ be an integer. Suppose that $Re(\lambda) < \delta$, then there exist a constant C such that $|t^m e^{\lambda t}| \leq Ce^{\delta t}$.

Proof:-suppose $Re(\lambda) = \alpha < \delta$, then $\alpha - \delta$ is negative. Then by the above lemma there exist constant C such that$t^m e^{(\alpha-\delta)t} \leq C, \forall t \geq 0$. Multiplying by $e^{\delta t}$ yields $t^m e^{\alpha t} \leq Ce^{\delta t}$. Suppose $\lambda = \alpha + i\beta$, then$|t^m e^{\lambda t}| = t^m e^{\alpha t} \leq Ce^{\delta t}, \forall t \geq 0$. ■

***Theorem* 2.1**: Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the (distinct) eigenvalues of A. Suppose that $Re(\lambda_j) < \delta$ for $j = 1, 2, \dots, k$. Then there is a constant K such that

$\|e^{At}\| \leq Ke^{\delta t}, \forall t \geq 0$.

Proof: let $P(\lambda)$ be the characterstic polynomial of$A$. The roots of $P(\lambda)$ are the same as the eigenvalues of$A$. By our algorism for constructinge^{At}, we have$e^{At} = \sum_{j=0}^{n-1} r_{j+1}(t)A^j$, where r_j is a solution of $P(D)_r = 0$. Then by the last $lemma$2.2there exist a constant C_j such that

$|r_j(t)| \leq C_j e^{\delta t}$ for positive t.

Then $\|e^{At}\| = \left\|\sum_{j=0}^{n-1} r_{j+1}(t)A^j\right\| \leq \left[\sum_{j=0}^{n-1} C_{j+1}\|A\|^j\right]e^{\delta t}$.

Hence $\|e^{At}\| \leq Ke^{\delta t}$, where $K = \left[\sum_{j=0}^{n-1} C_{j+1}\|A\|^j\right]$. ■

***Lemma* 2.3**: Let $P(\lambda)$ be a polynomial of degree n. Suppose that every root of $P(\lambda)$ has

a non-positive real part and that the roots with real part zero are simple. Then if y is a solution of $P(D)_Y = 0$, there is a constant C such that $|y(t)| \leq C \forall t \geq 0$.

Proof: If λ is a root with negative real part, it contributes functions of the form $t^m e^{\lambda k}$ to the set of fundamental solutions. But, we know that these functions goes to zero as t goes to infinity, and so is surely bounded on the right half axis. If $\lambda = i\beta$ is a simple pure imaginary root, it contribute one function $e^{i\beta t}$ to the fundamental set of functions and this function is bounded. Thus we get a fundamental set of solutions which are all bounded on $[0, \infty)$. Hence there exist a constant C such that

$|y(t)| \leq C, \forall t \geq 0$. ■

Finally, we have the following theorem, which follows readily from the last lemma and an argument similar to the proof of Theorem (2.1)

***Theorem* 2.2**: Let A be an $n \times n$ matrix and suppose that all of the eigenvalues A have real part less than or equal to zero, and that the eigenvalues with zero real part are simple (as roots of the characteristic polynomial). Then, there is a constant K such that $\|e^{At}\| \leq K, \forall t \geq 0$.

Consider the linear homogenous system

$$x'(t) = Ax(t) \quad \text{...} (2.4)$$

Where A is a constant $n \times n$ matrix. The origin is an equilibrium point for this system.

***Theorem* 2.3**: Let A be an $n \times n$ matrix and let the spectrem of A (the eigenvalues of A) be denoted by $\delta(A)$ and consider the linear system of (2.4) .

1. If $Re(\delta(A)) \leq 0$ and all the eigenvalues of A with real part zero are simple, then 0 is a stable fixed point for (4).

2. If $Re\big(\delta(A)\big) < 0$, then 0 is asymptotically stable for(1).
3. If there is an eigenvalues of A with positive real part, then 0 is unstable.

Proof: Suppose first that $Re(\delta(A)) \leq 0$ and all imaginary eigenvalues are simple. Then by lemma (2.2) we can find K such that $\|e^{At}\| \leq K, \forall t \geq 0$. Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{K}$. Then if x_0 is an initial condition with $|0 - x_0| = |x_0| < \delta$, then $|0 - x(t,0,x_0)| = |e^{At}x_0| \leq \|e^{At}\||x_0| \leq K(\delta) = K\left(\frac{\varepsilon}{K}\right) = \varepsilon$. This shows that the zero solution is stable.

Now suppose that $Re\big(\delta(A)\big) < 0$. Then the zero solution is stable by the first part of the proof. We can choose a real number $\omega < 0$ such that $Re\big(\lambda_j\big) < \omega$ for all eigenvalues λ_j of A. Then by $lemma$(2.2), there exist a constant K such that $\|e^{At}\| \leq Ke^{\omega t}, \forall t \geq 0$. But then for any initial condition x_0,

$|x(t,0,x_0)| = |e^{At}x_0| \leq K|x_0|e^{\omega t}, \forall t \geq 0$. Since ω is negative $e^{\omega t} \to 0$ as $t \to \infty$ for any initial condition x_0.

For the last part of the proof, consider first the complex case. Suppose that we have eigenvalue $\lambda = \alpha + i\beta$ with $\alpha > 0$. Let u be an eigenvector of A belonging to the eigenvalue λ. The solution of the system with initial condition u is $e^{At}u = e^{\lambda t}u$. Let $\varepsilon > 0$ be given if we let $\rho = \frac{\varepsilon}{2|u|}$, then $|\rho u| = \varepsilon/2$. On the other hand the solution $x(t)$ of the system with initial condition ρu is$x(t) = e^{At}\rho u = e^{\lambda t}\rho u$.

UNIT 3

APPLICATION

3.1 Competing species

In this section and the next we explore the application of phase plane analysis to some problems in population dynamics. These problems involve two interacting populations.

Suppose that in some closed environment there are two similar species competing for a limited food supply; for example, two species of fish in a pond that do not prey on each other, but do compete for the available food. Let x and y be the populations of the two species at time t. We assume that the population of each of the species, in the absence of the other, is governed by a logistic equation. Thus

$$\begin{cases} \dfrac{dx}{dt} = x(\epsilon_1 - \delta_1 x) \quad \ldots\ldots\ldots (3.1a) \\ \dfrac{dy}{dt} = y(\epsilon_2 - \delta_2 y) \quad \ldots\ldots\ldots (3.1b) \end{cases}$$

Where ϵ_1 and ϵ_2 are the growth rates of the two populations, and ϵ_1/δ_1 and ϵ_2/δ_2 are their saturation levels. However, when both species are present, each will impinge on the available food supply for the other. In effect, they reduce the growth rates and saturation populations of each other. The simplest expression for reducing the growth rate of species x due to the presence of species y is to replace the growth rate factor $\epsilon_1 - \delta_1 x$ by $\epsilon_1 - \delta_1 x - \alpha_1 y$ where α_1 is a measure of the degree to which species y interferes with species x. Similarly, in Eq. (1b) we replace $\epsilon_2 - \delta_2 y$ by $\epsilon_2 - \delta_2 y - \alpha_2 x$. Thus we have the system of equations

$$\begin{cases} \frac{dx}{dt} = x(\epsilon_1 - \delta_1 x - \alpha_1 y) \\ \frac{dy}{dt} = y(\epsilon_2 - \delta_2 y - \alpha_2 x) \end{cases} \quad \ldots\ldots\ldots (3.2)$$

The values of the positive constants $\epsilon_1, \delta_1, \alpha_1, \epsilon_2, \delta_2$ and α_2 depend on the particular species under consideration and in general must be determined from observations. We are interested in solutions of Eqs. (3.2) for which x and y are nonnegative.

Example:-Discuss the qualitative behavior of solutions of the system

$$\begin{cases} \frac{dx}{dt} = x(1-x-y) \\ \frac{dy}{dt} = y(0.75-y-0.5x) \end{cases} \quad \text{...(3.3).}$$

We find the critical points by solving the system of algebraic equations

$$x(1-x-y) = 0 \; and \; y(0.75-y-0.5x) = 0 \quad \text{.................................(3.4)}$$

There are four points that satisfy eqs (3.4), namely, (0, 0), (0, 0.75), (1, 0), and (0.5, 0.5); they correspond to equilibrium solutions of the system (3.3). The first three of these points involve the extinction of one or both species; only the last corresponds to the long-term survival of both species. Other solutions are represented as curves or trajectories in the xy-plane that describe the evolution of the populations in time.

The system (3.3) is almost linear in the neighborhood of each critical point. There are two ways to obtain the linear system near a critical point(x_0, y_0). First, we can use the substitution $x = x_0 + u,\; y = y_0 + v$ in Eqs. (3.3), retaining only the terms that are linear in $u \; and \; v$.

Alternatively we can use

$$\left(\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{.......................................(3.5)}$$

Where, for the system (3.3)

$$f(x, y) = x(1-x-y) \text{ and } g(x, y) = y(0.75-y-0.5x) \quad \text{.................(3.6)}$$

Thus equation (3.5) becomes

$$\left(\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} 1-2x_0-y_0 & -x_0 \\ -0.5y_0 & 0.75-2y_0-0.5x_0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{..........................(3.7)}$$

- $x = 0, y = 0$, This critical point corresponds to a state in which both species die as a result of their competition. By setting $x_0 = y_0 = 0$ in equation (3.7) we see that near the origin, the corresponding linear system is

$$\left(\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0.75 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.8)$$

The eigenvalues and eigenvectors of the system (3.8) are

$$\lambda_1 = 1, \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \lambda_2 = 0.75, \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.9)$$

So the general solution of the system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{0.75t} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.10)$$

Thus the origin is an unstable node of both the linear system (3.8) and the nonlinear system(3.3). In the neighborhood of the origin all trajectories are tangent to the y-axis except for one trajectory that lies along the x-axis.

- $x = 1, y = 0$. This corresponds to a state in which species x survives the competition, but species y does not. The corresponding linear system is

$$\left(\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} -1 & -1 \\ 0 & 0.25 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.11)$$

Its eigenvalues and eigenvectors are

$$\lambda_1 = -1, \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \lambda_2 = 0.25, \xi^{(2)} = \begin{pmatrix} 4 \\ -5 \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.12)$$

$and\ its\ general$ Solution is

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 4 \\ -5 \end{pmatrix} e^{0.25t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.13)$$

Since the eigenvalues have opposite signs, the point (1, 0) is a saddle point, and hence is an unstable equilibrium point of the linear system (3.11) and of the nonlinear system (3.3). The behavior of the trajectories near (1, 0) can be seen from Eq. (3.13).If $c_2 = 0$, then there is one pair of trajectories that approaches the critical point along the x-axis. All other trajectories depart from the neighborhood of (1,0).

- $x = 0, y = 0.75$ In this case species y survives but x does not. The analysis is similar to that for the point (1, 0). The corresponding linear system is

$$\left(\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} 0.25 & 0 \\ -0.375 & -0.75 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{...(3.14)}$$

The eigenvalues and eigenvectors are

$$\lambda_1 = 0.25, \xi^{(1)} = \begin{pmatrix} 8 \\ -3 \end{pmatrix}; \lambda_2 = -0.75, \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{.....................................(3.15)}$$

So the general Solution of (14) is

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 8 \\ -3 \end{pmatrix} e^{0.25t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-0.75t} \quad \text{...(3.16)}$$

Thus the point $(0, 0.75)$ is also a saddle point. All trajectories leave the neighborhood of this point except one pair that approaches along the y-axis.

- $x = 0.5, y = 0.5$ This critical point corresponds to a mixed equilibrium state, or coexistence, in the competition between the two species. The eigenvalues and eigenvectors of the corresponding linear system

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -0.5 & -0.5 \\ -0.25 & -0.5 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{...(3.17)}$$

Are

$$\lambda_1 = \frac{(-2+\sqrt{2})}{4} \cong -0.146, \ \ \xi^{(1)} = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} \text{and} \lambda_2 = \frac{(-2-\sqrt{2})}{4} \cong -0.854, \xi^{(2)} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \ \text{..... (3.18)}$$

Therefore the general solution of Eq. (17) is

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} e^{-0.146t} + c_2 \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{-0.854t} \quad \text{.....................................(3.19)}$$

Since both eigenvalues are negative, the critical point (0.5, 0.5) is an asymptotically stable node of the linear system (3.17) and of the nonlinear system (3.3). All trajectories approach the critical point as$t \to \infty$. One pair of trajectories approaches the critical

Example2:- Discuss the qualitative behavior of the solutions of the system

$$\begin{cases} \frac{dx}{dt} = x(3 - x - 2y) \\ \frac{dy}{dt} = y(1 - 2x - y) \end{cases} \quad \text{.......................................(3.20)}$$

Once again, there are four critical points, namely,(0,0),3,0),(0,2)and (1,1)

Corresponding to equilibrium solutions of the system (3.20). The mixed equilibrium solution (1,1) is a saddle point, and therefore unstable, while the points (3,0) and (0,2) are asymptotically stable.

- $x = 0, y = 0$, the linearized system is $\left(\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ ……………………(3.21)

Which is valid near the origin. The eigenvalues and eigenvectors of the system (3.21) are:

$$\lambda_1 = 3, \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \lambda_2 = 2, \ \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{....................................(3.22)}$$

So the general solution is $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}$ …………………………………(3.23)

Therefore the origin is an unstable node of the linear system (3.21) and also of the nonlinear system (3.20). All trajectories leave the origin tangent to the y-axis except for one trajectory that lies along the x-axis.

- $x = 3, y = 0$, the corresponding linear system is

$$\left(\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} -3 & -6 \\ 0 & -5 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \quad \text{..(3.24)}$$

The eigenvalues and eigenvectors of the system (3.24) are

$$\lambda_1 = -3, \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \lambda_2 = -5, \ \xi^{(2)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad \text{....................................(3.25)}$$

And its general solution is $\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-5t}$ ……………………..…..(3.26)

The point (3,0)is an asymptotically stable node of the linear system (3.24) and of the nonlinear system (3.20).

➢ $x = 0, y = 2$, the corresponding linear system is

$$\left(\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.27)$$

The eigenvalues and eigenvectors of the system are

$$\lambda_1 = -1,\ \xi^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix};\ \lambda_2 = -2,\ \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.28)$$

And its general solution is $\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}$ ……………………(3.29)

Thus the critical point (0, 2) is an asymptotically stable node of both the linear system (3.27) and the nonlinear system (3.20). All trajectories approach the critical point along the y-axis except for one trajectory that approaches along the line with slope -2.

➢ $x = 1, y = 1$, the corresponding linear system is

$$\left(\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.30)$$

The eigenvalues and eigenvectors of the system are

$$\lambda_1 = -1 + \sqrt{2},\ \xi^{(1)} = \begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \end{pmatrix};$$

$$\lambda_2 = -1 - \sqrt{2},\ \xi^{(2)} = \begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2} \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.31)$$

So the general solution is

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -0.707 \end{pmatrix} e^{(-1+\sqrt{2})t} + c_2 \begin{pmatrix} 1 \\ 0.707 \end{pmatrix} e^{(-1-\sqrt{2})t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.32)$$

Since the eigenvalues are of opposite sign, the critical point (1, 1) is a saddle point, and therefore is unstable.

Examples 1 and 2 shows that in some cases the competition between two species leads to an equilibrium state of coexistence, while in other cases the competition results in the eventual coexistence of one of the species.

Again at the general system (3.2), there are four cases to be considered, depending on the relative orientation of the lines

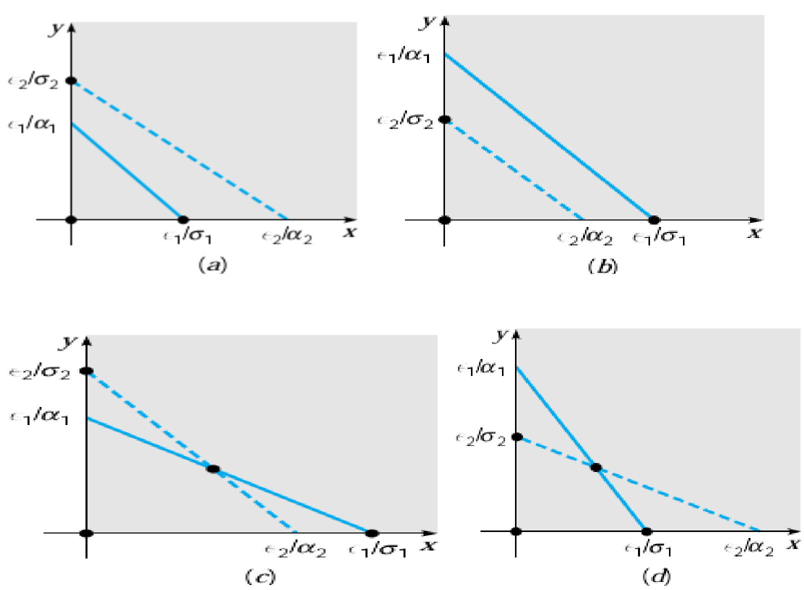
$\epsilon_1 - \delta_1 x - \alpha_1 y = 0$ and $\epsilon_2 - \delta_2 x - \alpha_2 y = 0$ ……………………………………………(3.33)

These lines are called the x *and* y *nullclines* respectively, because $x'$is zero on the first and y' is zero on the second. Let (X, Y) denote any critical in any one of the four cases.

$$\left(\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} \epsilon_1 - 2\delta_1 X - \alpha_1 Y & -\alpha_1 X \\ -\alpha_2 Y & \epsilon_2 - 2\delta_2 Y - \alpha_2 X \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \text{……….…………..(3.34)}$$

We now use Eq. (3.34) to determine the conditions under which the model described by $Eqs.$(3.2) permits the coexistence of the two species x and y. Of the four possible cases shown in $Figure$ 3.2 coexistence is possible only in cases (c) and (d). In these cases the nonzero values of X and Y are readily obtained by solving the algebraic equation (3.33), the result is

$$X = \frac{\epsilon_1\delta_2 - \epsilon_2\alpha_1}{\delta_1\delta_2 - \alpha_1\alpha_2},\ Y = \frac{\epsilon_2\delta_1 - \epsilon_1\alpha_2}{\delta_1\delta_2 - \alpha_1\alpha_2} \text{ ……………………….……………………(3.35)}$$



$Figure$3.2:- The various cases for the competing species system (3.2)

Since $\epsilon_1 - \delta_1 X - \alpha_1 Y = 0$ and $\epsilon_2 - \delta_2 Y - \alpha_2 X = 0$, equation (3.34) reduces to

$$\left(\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} -\delta_1 X & -\alpha_1 X \\ -\alpha_2 Y & -\delta_2 Y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \text{ ……………………………………….(3.36)}$$

The eigenvalues of the system (3.36) are found from the equation

$$\lambda^2 + (\delta_1 X + \delta_2 Y)\lambda + (\delta_1\delta_2 - \alpha_1\alpha_2)XY = 0 \text{ ………………….…….(3.37)}$$

Thus $\lambda_{1,2} = \frac{-(\delta_1 X + \delta_2 Y) \pm \sqrt{(\delta_1 X + \delta_2 Y)^2 - 4(\delta_1 \delta_2 - \alpha_1 \alpha_2)}}{2}$ ……………………………………..(3.38)

If $(\delta_1\delta_2 - \alpha_1\alpha_2) < 0$, then the radicand of Eq. (3.38) is positive and greater than $(\delta_1 X + \delta_2 Y)^2$. Thus the eigenvalues are real and of opposite sign. Consequently, the critical point$(X, Y)$is an (unstable) saddle point, and coexistence is not possible.

On the other hand, if$(\delta_1\delta_2 - \alpha_1\alpha_2) > 0$, then the radicand is less than $(\delta_1 X + \delta_2 Y)^2$ Thus the eigenvalues are real, negative, and unequal, or complex with negative real part.the eigenvalues cannot be complex, then the critical point is asymptotically stable node and sustained coexistence is possible.This is illustrated by example 1, where $\delta_1 = 1, \delta_2 = 1, \alpha_1 = 1, \alpha_2 = 0.5$, and $\delta_1\delta_2 - \alpha_1\alpha_2 = 0.5$

Let us relate this result to Figures 3.2c and 3.3d. In Figure 3.2 c we have

$$\frac{\epsilon_1}{\delta_1} > \frac{\epsilon_2}{\alpha_2} \text{ or } \epsilon_1\alpha_2 > \delta_1\epsilon_2 \text{ and } \frac{\epsilon_2}{\delta_2} > \frac{\epsilon_1}{\alpha_1} \text{ or } \epsilon_2\alpha_1 > \delta_2\epsilon_1 \quad \text{…………………(3.39)}$$

These inequalities, coupled with the condition that X and Y given by $Eqs.$(3.35) be positive, yield the inequality$\delta_1\delta_2 < \alpha_1\alpha_2$. Hence in this case the critical point is a saddle point. On the other hand, in Figure 3.$3d$ we have

$$\frac{\epsilon_1}{\delta_1} < \frac{\epsilon_2}{\alpha_2} \text{ or } \epsilon_1\alpha_2 < \delta_1\epsilon_2 \text{ and } \frac{\epsilon_2}{\delta_2} < \frac{\epsilon_1}{\alpha_1} \text{ or } \epsilon_2\alpha_1 < \delta_2\epsilon_1 \quad \text{…… .(3.40)}$$

Now the condition that X and Y are positive yields $\delta_1\delta_2 > \alpha_1\alpha_2$. Hence the critical point is asymptotically stable. For this case we can also show that the other critical points

3.2 Lotka -Volterra Predator-Prey

Let us consider the predator-prey problem. Here, we study an ecological situation involving two species, one of which preys on the other (does not compete with it for food but preys on it), while the other lives on a different source of food. An example is foxes and rabbits in a closed forest; the foxes prey on the rabbits, the rabbits live on vegetation in the forest. Let $x(t)$ and $y(x)$ be the populations of prey and predator respectively, at time t.

Let us build a simple model of interaction and make the following assumptions:

1 .The prey grows without bound in the absence of the predator. Thus $\frac{dx}{dt} = ax, a > 0 \; for \; y = 0$

2 .The predator dies out in the absence of the prey.

Thus $\frac{dy}{dt} = -cy, c > 0 \; for \; x = 0$

3 .The increase in the number of predators' dependents wholly on the food supply (the prey) and the prey are consumed at a rate proportional to the number of encounters between predators and prey. For example, if the number of prey is doubled, the number of encounters is doubled. Encounters increase the number of predators and decrease the number of prey. A fixed proportion of prey is killed in each encounter, and the rate at which the population of the predator grows is enhanced by a factor proportional to the amount of prey consumed.

As a consequence, we have the following equations

$$\frac{dx}{dt} = ax - bxy = x(a - by) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3.41b)$$
$$\frac{dy}{dt} = -cy + dxy = y(-c + dx) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3.41a)$$

The constants $a, b, c \; and \; d$ and are positive; a and c are the growth rate of the prey and the death rate of the predator, respectively, b and d are the measures of the effect of the interaction between the two species. Equations $(1a and \; 1b)$ are known as the Lokta-Volterra equations. They were developed in papers by Lokta in 1925 and Volterra in 1926.

The equilibrium points of the equations $3.41a$ and $4.41b$) are the solutions of

$$x(a - by) = 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3.42a)$$
$$y(-c + dx) = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3.42b)$$

These solutions are $x = 0, y = 0,$ and $x = \frac{c}{d}, y = \frac{a}{b}$ ……………………………….(3.43)

We will examine the predator-prey model ($3.41a$ and $4.41b$) in the neighborhood of each critical point. The stability of the critical points tells us how the two species interact. The system is nonlinear, so we will linearize it to determine the stability of each equilibrium point. The Jacobian matrix is given by

$$J = \begin{bmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{bmatrix} = \begin{bmatrix} a - by & -bx \\ dy & -c + dx \end{bmatrix}$$

Where $f(x, y) = x(a - by), g(x, y) = y(-c + dx)$

- For $(0,0)$, $J = \begin{bmatrix} f_x(0,0) & f_y(0,0) \\ g_x(0,0) & g_y(0,0) \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}$

The linearized system is $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ with characteristic equation

$(a-\lambda)(-c-\lambda) = 0$. The eigenvalues are $\lambda_1 = a > 0$ and $\lambda_2 = -c < 0$. The critical point $(0,0)$ is unstable saddle point.This case is not important since the critical solution $x(t) \equiv 0$ and $y(t) \equiv 0$ corresponds to the extinction of both species.

The important case here is the coexistence of the two species; this depends on the stability of the nonzero critical point$\left(\frac{c}{d}, \frac{a}{b}\right)$. So let us check the stability of the critical point$\left(\frac{c}{d}, \frac{a}{b}\right)$. We have $J = \begin{bmatrix} a-by & -bx \\ dy & -c+dx \end{bmatrix}_{\left(\frac{c}{d}, \frac{a}{b}\right)} = \begin{bmatrix} 0 & \frac{-bc}{d} \\ \frac{ad}{b} & 0 \end{bmatrix}$

Using the substitution $u = x - \frac{c}{d}, v = y - \frac{a}{b}$ we obtain a corresponding critical point $(0,0)$ for $\left(\frac{c}{d}, \frac{a}{b}\right)$.

Hence the linearized system is $\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & \frac{-bc}{d} \\ \frac{ad}{b} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$. The characteristic equation is $\lambda^2 + ac = 0$ and the corresponding eigenvalues $\lambda_1, \lambda_2 = \pm i\sqrt{ac}$.

Since the roots of the characteristic equation are pure imaginary, the critical point is a stable center of the linear system. The trajectories of the linear system are closed curves corresponding to the solutions that are periodic in time. They do not approach or recede from the critical point. The trajectories can be shown in the following way:

Divide equation $(1b)$ by $(1a)$ we get $\frac{dy}{dx} = \frac{y(-c+dx)}{x(a-by)}$

By separating variables we obtain $\int \frac{a-by}{y} dy = \int \frac{-c+dx}{x} dx$

So we have $a \ln y - by = -c \ln x + dx + \ln C$, where C is constant of integration.

We cannot solve the equation explicitly for x in terms of y or for y in terms of x. The equation defines closed curve around the equilibrium point $\left(\frac{c}{d}, \frac{a}{b}\right)$. . This means that the critical point is a stable center.

Therefore, the critical solution $x(t) \equiv \frac{c}{d} \, and \; y(t) \equiv \frac{a}{b}$shows that both populations (the predator and the prey) coexist in the same environment without extinction.

CONCLUSION

The last section demonstrates the power of linearization techniques on phase-portrait.The systems of nonlinear ordinary differential equations discussed in the last section are difficult to solve. However, through the use of the techniques in this paper, we can simplify the problems and understand the general behaviors of the solutions. This process involves finding the equilibria of the nonlinear system, then linearizing the nonlinear system by taking the total derivative at the critical points. The next step was to use the linearized system to classify each equilibrium as stable or unstable based upon the eigenvalues of the coefficient matrix A.

It would be preferable to find explicit solutions to nonlinear systems of ordinary differential equations, but finding such solutions can be very difficult (if not impossible) in general. Using linearization and stability analysis to construct qualitative solutions in phase planes is a viable alternative to searching for explicit solutions, as demonstrated here. Understanding the types of critical points that exist for a given system allowed us to draw phase portraits and visually express the behaviors of the solutions. We then applied these techniques to real-world models and characterized how the parameters of each system influenced the behaviors.

Generally, this type of linear analysis of nonlinear problems is an accurate method of understanding the behaviors of solutions in the nonlinear system. Methods of linearization become very useful when dealing with complicated systems that could be difficult or even impossible to solve, so it is beneficial to have a firm understanding of how to linearize and analyze real-world modeling problems.

BIBLIOGRAPHY

[1] A.C. King, J. Billingham and S.R. Otto Differential Equations Linear, Nonlinear, Ordinary, Partial,Cambridge University Press 2003.

[2] Carl M. Bender Steven A. Orszag*Advanced Mathematical Methods for Scientists and EngineersI, Springer-Verlag New York, Inc. in 1999.*

[3] Dennis G. Zill A First Course in Differential Equation with Modeling Applications ninth edition, Brooks/Cole, Cengage Learning, 2005;2009.

[4] Dennis G. Zill, Michael R. Cullea. *Differential Equation with Boundary Value Problems.Third edition* Jones and Bartlett Learning, LLC, an Ascend Learning Company 2014.

[5] Dennis G. Zill Warren S. Wright *Advanced Engineering Mathematics Fifth EditionJones& Bartlett Learning, LLC, an Ascend Learning Company 2014.*

[6] Garrett Birkhoff,Gian-Carlo Rota*ordinary differential equations fourth edition,John Wiley 8c Sons, Inc1959, 1960, 1962, 1969, 1978, and 1989* .

[7] J. C. BurkillScdFRS.*The Theory of Ordinary Differential Equations;* New York: inter science Publishers Inc. 1962.

[8] Lawrence Perko; differential equations and dynamical systems third edition, Springer-Verlag, New York. Inc,1991, 1996, 2001.

[9] Peter O'Neil *Advanced Engineering Mathematics International Students edition Nelson, a division of Thomson Canada Limited 2007.*

[10]Shapley L. Ross *Differential Equation Third Edition, John Wiley & Sons, Inc.2004.*

[11] William E. Boyce, Rechared C. Diprima*Elementary Differential Equations and Boundary value problems seventh edition John Wiley & Sons, Inc*2001.