ON POINCARÉ-BENDIXSON THEOREM AND CLOSED ORBITS OF NONLINEAR DYNAMICAL SYSTEMS

By

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Table of Contents

Table of Contents **4**

List of Tables **6**

List of Figures **6**

Acknowledgements **7**

Abstract **8**

Introduction **9**

1 Closed Orbits and Limit Sets **10**

1.1 Closed Orbits 10

1.2 Limit Sets 22

2 Local Sections and Flow Boxes **27**

3 The Poincaré Map **31**

4 Monotone Sequences in Planar Dynamical Systems **37**

5 The Poincaré-Bendixson Theorem and Its Application **41**

5.1 The Poincaré-Bendixson Theorem 41

5.2 Applications of Poincaré-Bendixson Theorem 44

6 Conclusion **52**

Bibliography **53**

List of Figures

1.1 The solution curve of the shifted IVP. 11

1.2 A non-periodic open orbit. 17

1.3 A periodic solutions or closed orbits. 19

1.4 Phase portraits of an undamped pendulum 20

1.5 A dynamical system with a cylinder as its attracting set. 24

1.6 Homoclinic solutions in ω-limit set. 26

2.1 A local section $\mathcal{S}$ at X_0 and several representative vectors from the vector field along $\mathcal{S}$. 28

2.2 The flow box associated to $\mathcal{S}$. 28

2.3 Solutions crossing the local section $\mathcal{S}$. 29

3.1 The Poincaré or first return map. 32

4.1 Two solutions crossing a straight line. On the left,X_0, X_1, X_2 is monotone along the solution but not along the straight line. On the right, X_0, X_1, X_2 is monotone along both the solution and the line. 38

4.2 Solutions exit the region D through T. 39

4.3 The solution through X cannot cross $\mathcal{V}_1$ and $\mathcal{V}_2$ infinitely often. . . . 40

5.1 A solution spiraling toward a limit cycle. 45

5.2 The region A is positively invariant. 46

5.3 The region A is positively invariant. 47

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Abstract

In a certain sense, closed orbits are the only types of orbits that we can ever hope to understand completely throughout their evolution from the distant past (i.e. as $t \to -\infty$) to the distant future (i.e., as $t \to \infty$) since the entire course of their evolution is determined by knowledge over a finite time interval, i.e., the period. Like equilibrium points that are asymptotically stable, periodic solutions may also attract other solutions. Determining the long time behavior of closed orbits is much more difficult while in this project paper we do have a tool that resembles the linearization technique called the Poincaré map. Furthermore, by applying the Poincaré-Bendixson theorem the limiting behaviors of a planar flow is determined.

Introduction

The world, as we know it, is comprised of entities in space that evolve through time. The idea of modeling the motion of a physical system with mathematical equations probably dates back to Sir Isaac Newton. Today, mathematical analysis of dynamical systems places itself at the center of control theory and engineering, as well as, many sciences such as physics, chemistry, ecology, and economics.

In this paper we will investigate the long time behavior of periodic solutions or closed orbits, of dynamical systems, mainly, planar autonomous systems of the form

$$\dot{X} = F(X)$$

In a certain sense, closed orbits are the only types of orbits that we can ever hope to understand completely throughout their evolution from the distant past (i.e. as $t \to -\infty$) to the distant future (i.e., as $t \to \infty$) since the entire course of their evolution is determined by knowledge over a finite time interval, i.e., the period. Thus, it is of great interest to give conditions under which dynamical systems either have or do not have closed orbits, without doing simulation[5]. Like equilibrium points that are asymptotically stable, periodic solutions may also attract other solutions. That is, solutions may limit on periodic solutions just as they can approach equilibria. In the plane, the limiting behavior of solutions is essentially restricted to equilibria and closed orbits, although there are a few exceptional cases. We will investigate this phenomenon using the important tools Poincaré map and Poincaré-Bendixson theorem.

Chapter 1

Closed Orbits and Limit Sets

In this part we visit periodic solutions of differential equations and their limiting behavior. In particular, in section 1.1, the existence of periodic orbits of the planar autonomous systems resulting in the Green's theorem. Then this theorem is applied to determine the nonexistence of closed orbits in a simply connected domain. In section 1.2, the limiting behavior of solutions of systems of differential equations on a closed invariant set is characterized, yielding the result that a closed invariant set contains the α-limit and ω-limit sets in it.

1.1 Closed Orbits

We begin this section by defining some terms related to dynamical system.

Lemma 1.1.1. *(Shift-Invariance) Let* $t_0 \in \mathbb{R}, X_0 \in \mathcal{U},$ *where* $\mathcal{U}$ *is an open subset of* $\mathbb{R}^n$. *The function*

$$X : \mathcal{J}(t_0, X_0) \to \mathcal{U}$$

$$t \mapsto X(t) = X(t; t_0, X_0)$$

is a solution of the IVP

$$\begin{aligned} \dot{X} &= f(X(t)) \\ X(t_0) &= X_0 \end{aligned} \tag{1.1.1}$$

if and only if its shifted version

$$\sigma_{t_0}X : \mathcal{J}(0, X_0) \to \mathcal{U}$$
$$t \mapsto X(t+t_0)$$

is a solution of the shifted IVP

$$\begin{aligned} \dot{X} &= f(X(t)) \\ X(0) &= X_0 \end{aligned} \tag{1.1.2}$$

Here, $\mathcal{J}(t_0, X_0)$ is the maximal interval of existence.

The operator σ_{t_0} is called left-shift operator by t_0 as can be seen from figure 1.1. That is $\sigma_{t_0}X(t) = X(t+t_0)$.

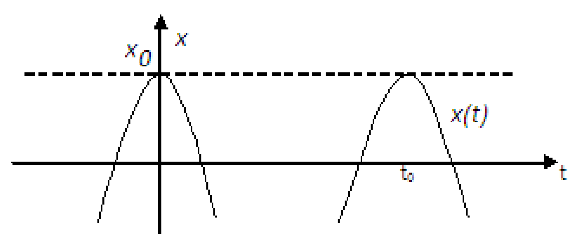

Figure 1.1: The solution curve of the shifted IVP.

Proof. ($\Rightarrow$) Suppose that X is a solution of the IVP (1.1.1). Set

$$Y(t) = X(t+t_0) \Rightarrow \dot{\sigma}_{t_0}X = \dot{Y} = \dot{X}(t+t_0) = f(X(t+t_0)) = f(Y)$$
$$Y(0) = X(0+t_0) = X(t_0) = X_0 = Y_0$$

Therefore, $Y(t)$ is a solution of the shifted IVP (1.1.2).

($\Leftarrow$) Suppose that X is a solution of the IVP (1.1.2). Set

$$Y(t) = X(t - t_0) \Rightarrow \dot{Y} = \dot{X}(t - t_0) = f(X(t - t_0)) = f(Y)$$
$$\Rightarrow \begin{cases} \dot{Y} = f(Y(t)) \\ Y(t_0) = X(0) = Y_0. \end{cases}$$

This completes the proof (since Y is a dummy variable).

□

This shows that for the autonomous system it suffices to consider situations at $t_0 = 0$, i.e; at the origin.

Let $\mathcal{J}(X_0) = \mathcal{J}(0, X_0)$ be maximal interval of existence,

$$\mathbf{D} = \{(t, X_0) : t \in \mathcal{J}(X_0), X_0 \in \mathcal{U}\},$$

and define a map

$$\phi : \mathbf{D} \to \mathcal{U}$$
$$(t, X_0) \mapsto \phi(t, X_0) = X(t; X_0)$$

solution of the shifted IVP (1.1.2) evaluated at time t,then ϕ has the following properties that can be justified by using lemma 1.1.1.

Properties of ϕ

(a) $\phi(0, X_0) = X_0, \forall X_0 \in \mathcal{U}$.

(b) $\phi(s, \phi(t, X_0)) = \phi(s + t, X_0), \forall (t, X_0) \in \mathbf{D}, (s, \phi(t, X_0)) \in \mathbf{D}$

(c) $\mathbf{D} \subset \mathbb{R} \times \mathcal{U}$ is open

(d) ϕ is continuous.

Notation $\phi(t, X) = \phi_t(X)$.

Definition 1.1.1. A mapping ϕ as above with properties (a)-(d) is called a *flow* or *dynamical system* on $\mathcal{U}$. If $\mathbf{D} = \mathbb{R} \times \mathcal{U}$, then ϕ is called a *global flow*. The set $\mathcal{U}$ is called the *phase space* of ϕ.

Let ϕ be a dynamical system on $\mathcal{U}$. Define a relation $\sim$ on $\mathcal{U}$ as follows, for $X, Y \in \mathcal{U}$ putting

$$X \sim Y \Leftrightarrow \exists t \in \mathcal{J}(X) : \phi(t, X) = Y.$$

Proposition 1.1.2. *The relation $\sim$ is an equivalence relation on $\mathcal{U}$.*

Proof. we have to show that the relation $\sim$ is reflexive, symmetry, and transitive.

(i) Reflexive,

Let $X \in \mathcal{U}$, Since

$$\phi(0, X) = X \text{ for } t = 0 \in \mathcal{J}(X) \text{ by property (a) },$$

$$\Rightarrow X \sim X.$$

Hence, $\sim$ is reflexive.

(ii) Symmetry,

Suppose $X, Y \in \mathcal{U}$ and $X \sim Y$, then by the definition,

$$\begin{aligned} X \sim Y &\Leftrightarrow \phi(t, X) = Y \text{ for some } t \in \mathcal{J}(X) \\ &\Rightarrow \phi(-t, \phi(t, X)) = \phi(-t, Y) \text{ for some } -t \in \mathcal{J}(X) \\ &\Rightarrow \phi(0, X) = \phi(-t, Y) \\ &\Rightarrow \phi(-t, Y) = X. \\ &\Rightarrow Y \sim X. \end{aligned}$$

Hence, $\sim$ is symmetry.

(iii) Transitivity,

Suppose for $X, Y, Z \in \mathcal{U}$, $X \sim Y$ and $Y \sim Z$, then by the definition,

$$\begin{aligned} X \sim Y \text{ and } Y \sim Z &\Leftrightarrow \phi(t, X) = Y \text{ and } \phi(s, Y) = Z \text{ for } t, s \in \mathcal{J}(X) \\ &\Rightarrow \phi(s, \phi(t, X)) = Z \\ &\Rightarrow \phi(s+t, X) = Z, s+t \in \mathcal{J}(X) \text{ by property (b).} \\ &\Rightarrow X \sim Z. \end{aligned}$$

Hence, $\sim$ is transitive.

Therefore, $\sim$ is an equivalence relation. $\square$

Definition 1.1.2. The equivalence classes of $\mathcal{U}$ with respect to the relation $\sim$ are called *orbits of* ϕ, denoted by [X],i.e;

$$[X] = \{Y \in \mathcal{U} : X \sim Y\}.$$

Observe that $[X]$ is the image of the map

$$\begin{aligned} \eta_X : \mathcal{J}(X) &\to \mathcal{U} \\ t &\mapsto \phi(t, X) \end{aligned}$$

which is often called the *flow line* through X.

Lemma 1.1.3. *(Closed Subgroup of* $\mathbb{R}$*)*

Let $\phi : \mathbb{R} \times \mathcal{U} \to \mathcal{U}$ *be a global flow on* $\mathcal{U}$ *for any* $X \in \mathcal{U}$, *define*

$$\mathcal{S}_X = \{t \in \mathbb{R} : \phi(t, X) = X\},$$

then $\mathcal{S}_X$ *is a closed subgroup of* $\mathbb{R}$.

Proof. (i). $X \in \mathcal{U}, \phi(0, X) = X$ by property (a)

$$\Rightarrow 0 \in \mathcal{S}_X$$

$$\Rightarrow \mathcal{S}_X \neq \emptyset.$$

(ii). $s, t \in \mathcal{S}_X \Rightarrow s + t \in \mathcal{S}_X$, since

$$\phi(t, X) = X = \phi(s, X)$$

$$\Rightarrow \phi(s, \phi(t, X)) = \phi(s + t, \phi(0, X)) \text{ by property (b)}$$

$$= \phi(s + t, X) = X.$$

$$t \in \mathcal{S}_X \Rightarrow -t \in \mathcal{S}_X$$

$$\phi(t, X) = X \Rightarrow \phi(-t, X) = \phi(-t, \phi(t, X)) = \phi(0, X) = X$$

It remains to show that $\mathcal{S}_X$ is closed. The map

$$\eta_X : \mathcal{J}(X) \to \mathcal{U}$$

$$t \mapsto \phi(t, X)$$

is continuous.

Since $\eta_X^{-1}(\{X\}) = \mathcal{S}_X$ and $\{X\}$ is closed

it follows that $\mathcal{S}_X$ is closed. □

Define a map

$$\beta : \mathbb{R}/\mathcal{S}_X \to [X]$$

$$[t] \mapsto \phi(t, X).$$

<u>Claim!</u> β is a continuous bijection.

Proof. (i). $[t_1] = [t_2] \Rightarrow t_1 - t_2 \in \mathcal{S}_X$

$$\Rightarrow \phi(t_1 - t_2, X) = X$$

$$\Rightarrow \phi(t_2, X) = \phi(t_2, \phi(t_1 - t_2, X)) = \phi(t_1, X).$$

Therefore, β is well-defined.

(ii). $\phi(t_1, X) = \phi(t_2, X) \Rightarrow \phi(t_1 - t_2, X) = \phi(-t_2, \phi(t_1, X)) = \phi(-t_2, \phi(t_2, X)) = \phi(0, X) = X$

$$\Rightarrow t_1 - t_2 \in \mathcal{S}_X$$

$$\Rightarrow [t_1] = [t_2]$$

Therefore, β is injective.

(iii). β is surjective, since

$$\forall Y \in [X], \exists t_1 \in [t] \ni | \ \phi(t_1, X) = X$$

(iv). β is continuous (follows from continuity of ϕ). $\square$

<u>Fact!</u>(From Algebra)

There are three types of closed subgroups of $\mathbb{R}$

(i). $\{0\}$, (ii). $\mathbb{R}$, and

(iii). $\mathbb{Z}_{t_0} = \{nt_0 : n \in \mathbb{Z}\}, t_0 \in \mathbb{R}$ and $t_0 > 0$

For a global flow there are three types of orbits. Namely:

(i). $\mathbb{R}/\{0\} \cong \mathbb{R}$, (ii). $\mathbb{R}/\mathbb{R} \cong \{0\}$ and

(iii). $\mathbb{R}/\mathbb{Z}_{t_0} \cong \mathcal{S}_1 = \{(x, y) : x^2 + y^2 = 1\}$.

Orbits of the type (iii) are periodic with period t_0 and are often called *cycles* or *closed orbits*. Orbits of the type (i) are non-periodic *open orbits*. Orbits of the type (ii) consists of singleton(only one element) and are called *fixed point(stationary point/critical point/equilibrium point/rest point).* In what follows, we see some examples for the sake of demonstration.

Example 1.1.1. *$\phi(t,x) = e^t x$ is a global flow on $\mathcal{U} = \mathbb{R}$.*

For $x > 0$;

$$y \in [x] \Leftrightarrow y = \phi(t,x) = e^t x > 0 \text{ since } e^t > 0,$$

$$\Rightarrow [x] = [1]$$

As can be seen from figure 1.2(a) which is a non-periodic open orbit.

For $x = 0$;

$$y \in [x] = [0] \Leftrightarrow y = \phi(t,x) = e^t x = e^t.0 = 0$$

$$\Rightarrow [x] = \{0\}$$

this is also a non-periodic open orbit see figure 1.2(b).

For $x < 0$;

$$y \in [x] \Leftrightarrow y = \phi(t,x) = e^t x < 0 \text{ since } e^t > 0, x < 0$$

$$\Rightarrow [x] = [-1]$$

which is a non-periodic open orbit see figure 1.2(c).

Therefore, there are exactly three distinct orbits namely $[-1]$, $[0]$, and $[1]$.

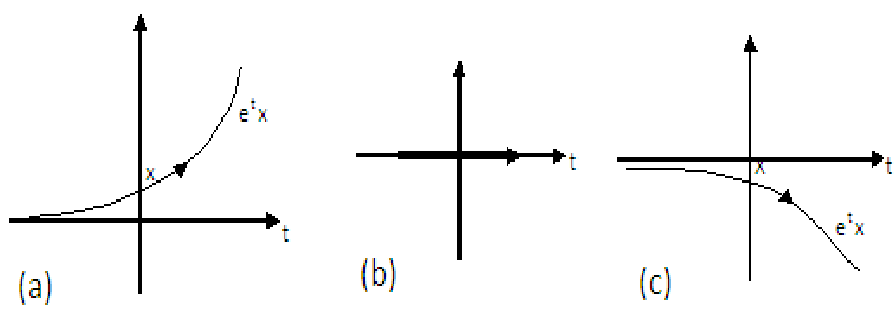

Figure 1.2: A non-periodic open orbit.

Example 1.1.2. $\phi\left(t, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 \cos t + x_2 \sin t \\ -x_1 \sin t + x_2 \cos t \end{pmatrix}$ *is a global flow on* $\mathcal{U} = \mathbb{R}^2$.

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right] \Leftrightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right] = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$$

which is an equilibrium point.

$$\text{For } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right] \Leftrightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \cos t + x_2 \sin t \\ -x_1 \sin t + x_2 \cos t \end{pmatrix}$$

$$\Leftrightarrow y_1^2 + y_2^2 = x_1^2 + x_2^2$$

$$\Leftrightarrow \| \widetilde{y} \|^2 = \| \widetilde{x} \|^2 = k > 0$$

$$\Rightarrow x_1^2 + x_2^2 = c^2, c \in \mathbb{R}, c > 0 \text{ where } c^2 = k.$$

$$\Rightarrow \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right] = \left\{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} : \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \cos t + x_2 \sin t \\ -x_1 \sin t + x_2 \cos t \end{pmatrix} \text{ and } \| \widetilde{y} \| = \| \widetilde{x} \| \right\}$$

which is a family of concentric circles called closed orbits. There are infinitely many closed orbits and exactly one equilibrium point see figure 1.3.

Consider the autonomous system

$$\dot{X} = F(X). \tag{1.1.3}$$

With $F \in C^1(\mathcal{U})$ where $\mathcal{U}$ is an open subset of $\mathbb{R}^n$. Clearly, the system (1.1.3) defines a dynamical system $\phi(t, X)$ on $\mathcal{U}$. For $X \in \mathcal{U}$, the function $\phi(t, X) : \mathbb{R} \to \mathcal{U}$ defines a solution curve, trajectory, or orbit of (1.1.3) through the point X_0 in $\mathcal{U}$. If we identify

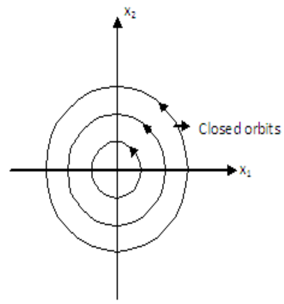

Figure 1.3: A periodic solutions or closed orbits.

the function $\phi(t, X)$ with its graph, we can think of a trajectory through the point $X_0 \in \mathcal{U}$ as a motion along the curve

$$\Gamma = \{X \in \mathcal{U} \mid X = \phi(t, X_0), t \in \mathbb{R}\}$$

Definition 1.1.3. A closed orbit of a dynamical system is the trace of the trajectory of a non-trivial (i.e., not a point) periodic solution. Thus $\Gamma \subset \mathbb{R}^2$ is a closed orbit if Γ is not an equilibrium point and there exists a time $\tau < \infty$ such that

$$\forall X \in \Gamma, \phi(n\tau, X) = X \forall n \in \mathbb{Z}.$$

By a closed orbit is meant a trajectory in the phase space that is periodic with a finite period. Thus, a collection of trajectories connecting a succession of saddle points is not referred to as a closed orbit; consider for example the trajectories connecting the saddles in Figure 1.4.

These trajectories are, roughly speaking, infinite periodic closed orbits. In the event, we will refer to such trajectories as saddle connections, and not as closed orbits.

Definition 1.1.4. A *cycle or closed(periodic) orbit* of (1.1.3) is any periodic solution curve of (1.1.3) which is not an equilibrium point of (1.1.3). A periodic orbit Γ is

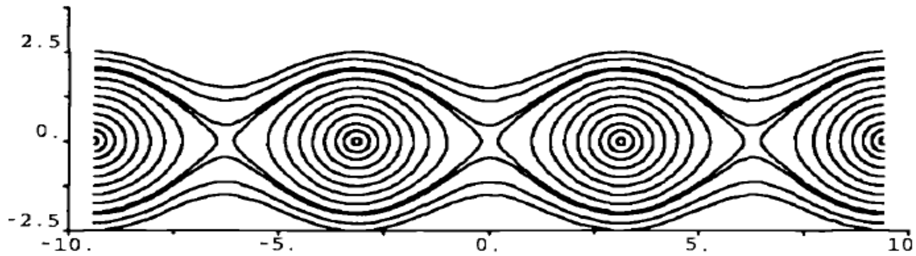

Figure 1.4: Phase portraits of an undamped pendulum

called *stable* if

$$\forall \epsilon > 0, \exists \mathcal{U} \ni \ \forall X \in \mathcal{U} \wedge t \geq 0,$$

$$d(\phi(t, X), \Gamma) < \epsilon.$$

Where $\mathcal{U}$ is open neigborhood of Γ. A periodic orbit Γ is called *unstable* if it is not stable; and Γ is called *asymptotically stable* if it is stable and if for all points X in some neighborhood $\mathcal{U}$ of Γ

$$\lim_{t \to \infty} d(\phi(t, X), \Gamma) = 0.$$

Example 1.1.3. *Consider a simple example of a system with a closed orbit:*

$$\begin{aligned} \dot{x} &= \quad y + \alpha x(\beta^2 - x^2 - y^2) \\ \dot{y} &= \ -x + \alpha y(\beta^2 - x^2 - y^2) \end{aligned} \tag{1.1.4}$$

The vector field is radially symmetric so that by defining polar coordinates as $r = \sqrt{x^2 + y^2}$, and $\theta = \arctan(y/x)$, we get

$$\begin{aligned} \dot{r} &= \alpha r(\beta^2 - r^2) \\ \dot{\theta} &= -1 \end{aligned} \tag{1.1.5}$$

The resultant system has a closed orbit which is a perfect circle at $r = \beta$. It also has an unstable source type equilibrium point at the origin since in its linearization there is

$$\begin{bmatrix} \alpha\beta^2 & 1 \\ -1 & \alpha\beta^2 \end{bmatrix}.$$

which has unstable eigenvalues at $\alpha\beta^2 \pm j$. In fact, the equations can be integrated to give $r(t) = \beta(1 + (\frac{\beta}{r(0)} - 1)e^{-2\beta^2\alpha t})^{-\frac{1}{2}}$ for $r(0) \neq 0$ and $\theta(t) = \theta(0) - t$. Thus all nonzero initial conditions converge to $r = \beta$. This example is somewhat contrived, since in general it is not easy to find closed orbits for nonlinear systems, even by detailed simulation. Thus, it is of great interest to give conditions under which planar dynamical systems either have or do not have closed orbits, without doing simulation.

We will now state a theorem on the nonexistence of closed orbits of the planar system

$$\begin{aligned} \dot{x} &= f(x, y), \\ \dot{y} &= g(x, y). \end{aligned} \tag{1.1.6}$$

Theorem 1.1.4. *Let $f(x, y)$ and $g(x, y)$ have continuous first partial derivatives in a simply connected domain $\boldsymbol{D}$ in $\mathbb{R}^2$. If $f_x + g_y$ has fixed sign in $\boldsymbol{D}$, then the system(1.1.6) has no closed orbit in $\boldsymbol{D}$[8].*

Proof. Let **C** be a closed curve in **D**. Then by Green's theorem, we have

$$\int_{\mathbf{C}} f(x, y)dy - g(x, y)dx = \int\int_{\mathbf{R}} (f_x + g_y)dxdy, \tag{1.1.7}$$

Where **R** is the region bounded by **C**. But if **C** is represented parametrically by $x = x(t)$, $y = y(t)$, then

$$\int_{\mathbf{C}} f(x, y)dy - g(x, y)dx = \int_0^{\tau} (f\frac{dy}{dt} - g\frac{dx}{dt})dt,$$

where τ is the period of **C**. Now using (1.1.6) we obtain

$$\int_{\mathbf{C}} f(x, y)dy - g(x, y)dx = \int_0^{\tau} (fg - gf)dt = 0.$$

Thus from (1.1.7) we have

$$\int\int_{\mathbf{R}} (f_x + g_y) dxdy = 0.$$

This relation holds true only if $f_x + g_y$ changes sign. This is a contradiction, and hence **C** is not a closed orbit in **D**. □

We will see such nonexistence and existence theorems and corollaries concerning closed orbits in chapter 5.

1.2 Limit Sets

We begin by describing the limiting behavior of solutions of systems of differential equations.

Definition 1.2.1. A point $p \in \mathcal{U}$ is an *ω-limit point* of the trajectory $\phi(t, X)$ of the system (1.1.3) if there is a sequence $t_n \to \infty$ such that

$$lim_{n\to\infty}\phi(t_n, X) = p$$

Similarly, if there is a sequence $t_n \to -\infty$ such that

$$lim_{n\to\infty}\phi(t_n, X) = q,$$

and the point $q \in \mathcal{U}$, then the point q is called an *α-limit point* of the trajectory $\phi(t, X)$ of (1.1.3). The set of all ω-limit points of a trajectory Γ is called the *ω-limit set* of Γ and it is denoted by $\omega(\Gamma)$. The set of all α-limit points of a trajectory Γ is called the *α-limit set* of Γ and it is denoted by $\alpha(\Gamma)$. The set of all limit points of Γ, $\alpha(\Gamma) \cup \omega(\Gamma)$ is called the *limit set of* Γ.

That is, the solution curve through X accumulates on the point p(respectively on q) as time moves forward(respectively as time moves backward). The set of all ω-limit points of the solution through X is the ω-limit set of X and is denoted by $\omega(X)$. The α-limit points and the α-limit set $\alpha(X)$ are defined by replacing $t_n \to \infty$ with $t_n \to -\infty$ in the above definition. By a limit set we mean a set of the form $\omega(X)$ or $\alpha(X)$.

Here are some examples of limit sets. If X^* is an asymptotically stable equilibrium, it is the ω-limit set of every point in its basin of attraction. Any equilibrium is its own α- and ω-limit set. A periodic solution is the α-limit and ω-limit set of every point on it. Such a solution may also be the ω-limit set of many other points.

Example 1.2.1. *Consider the system*

$$\begin{aligned} \dot{x} &= -y + x(1 - x^2 - y^2) \\ \dot{y} &= x + y(1 - x^2 - y^2) \\ \dot{z} &= \alpha \end{aligned} \tag{1.2.1}$$

It has the z-axis and the cylinder $x^2 + y^2 = 1$ as invariant sets. The cylinder is an attracting set as can be seen from Figure 1.5. where $\alpha > 0$. If in example 1.2.1, we identify the points (x, y, 0) and (x, y, 2π) in the planes $z = 0$ and $z = 2\pi$. we get a flow in $\mathbb{R}^3$ with a two-dimensional invariant torus $\mathbf{T}^2$ as an attracting set. The z-axis gets mapped onto an unstable cycle Γ and if a is an irrational multiple of it then the torus $\mathbf{T}^2$ is an attractor and it is the ω-limit set of every trajectory except the cycle Γ.

In three dimensions there are extremely complicated examples of limit sets, which are not very easy to describe. In the plane, however, limit sets are fairly simple. In fact, it is typical in that one can show that a closed and bounded limit set other than a closed orbit or equilibrium point is made up of equilibria and solutions joining them.

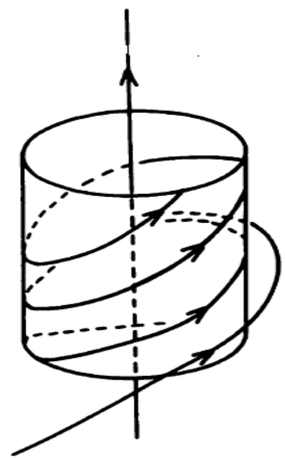

Figure 1.5: A dynamical system with a cylinder as its attracting set.

The Poincaré- Bendixson theorem we will discuss later states that if a closed and bounded limit set in the plane contains no equilibria, then it must be a closed orbit.

Definition 1.2.2. A set $\mathbf{P}$ is called *invariant* if for each $X \in \mathbf{P}$,$\phi(t, X)$ is defined and in $\mathbf{P}$ for all $t \in \mathbb{R}$. The set $\mathbf{P}$ is *positively*(respectively *negatively*) *invariant* if for each $X \in \mathbf{P}$, $\phi(t, X)$ is defined and in $\mathbf{P}$ for all $t \geq 0 (t \leq 0)$. Finally, an entire solution of a system is a set of the form $\{\phi(t, X) \mid t \in \mathbb{R}\}$.

Proposition 1.2.1.

1. *If X and Z lie on the same solution curve, then $\omega(X) = \omega(Z)$ and $\alpha(X) = \alpha(Z)$;*
2. *If D is a closed,positively invariant set and $Z \subset D$,then $\omega(Z) \subset D$, and similarly, for negatively invariant sets and α-limits;*
3. *A closed invariant set, in particular, a limit set, contains the α-limit and ω-limit sets of every point in it.*

Proof. (1). Let $Y \in \omega(X)$, and $\phi(s, X) = Z$.

suppose that $\lim_{n\to\infty} \phi(t_n, X) = Y$, then we have

$$\phi(t_n - s, Z) = \phi(t_n, X).$$

$$\Rightarrow \lim_{n\to\infty} \phi(t_n - s, Z) = \lim_{n\to\infty} \phi(t_n, X) = Y.$$

Hence $Y \in \omega(Z)$ as well.

(2) Let $\lim_{n\to\infty} \phi(t_n, X) = Y \in \omega(Z)$ as $t_n \to \infty$,

then we have $t_n \geq 0$ for sufficiently large n so that

$$\phi(t_n, Z) \in D.$$

Hence, $Y \in D$ since D is a closed set.

Finally, part (3) follows immediately from part (2). □

Definition 1.2.3. Heteroclinic Orbit: An orbit is said to be heteroclinic to the two invariant sets Λ_1 and Λ_2 if it approaches Λ_1 asymptotically under time evolution as time goes to $-\infty$ and approaches Λ_2 asymptotically under time evolution as time goes to $+\infty$. If the invariant sets have stable and unstable manifolds then a heteroclinic orbit lies in the intersection of the unstable manifold of Λ_1 and the stable manifold of Λ_2.

Definition 1.2.4. Homoclinic Orbit: An orbit is said to be homoclinic to an invariant set if it approaches the invariant set asymptotically under time evolution as time goes to $\pm\infty$. If the invariant set has stable and unstable manifolds then a homoclinic orbit lies in the intersection of the stable and unstable manifolds[5].

Example 1.2.2. *Another example of an ω-limit set that is neither a closed orbit nor* *an equilibrium is provided by a homoclinic solution. Consider the system*

$$\dot{x} = y$$
$$\dot{y} = \mu y + x - x^2 + xy. \tag{1.2.2}$$

There are two equilibrium points at (0 , 0) and (1 , 0). By linearization, we can see that (0 , 0) is always a saddle, while (1 , 0) is an unstable focus for $-1 < \mu < 1$. Let us limit our analysis to the range $-1 < \mu < 1$. Figure 1.6 shows the phase portrait for four different values of μ. The phase portraits for $\mu = -0.95$ and -0.88 are typical for $\mu < \mu_c \approx -0.8645$, while that for $\mu = -0.8$ is typical for $\mu < \mu_c$. For $\mu < \mu_c$ there is a stable limit cycle that encircles the unstable focus. As μ increases towards μ_c, the limit cycle swells and finally touches the saddle at $\mu = \mu_c$, creating a trajectory that starts and ends at the saddle; such trajectory is called homoclinic orbit. For $\mu > \mu_c$ the limit cycle disappears. Note that this bifurcation occurs without any changes to the equilibrium points (0,0) and (1,0)(see the detail [7]).

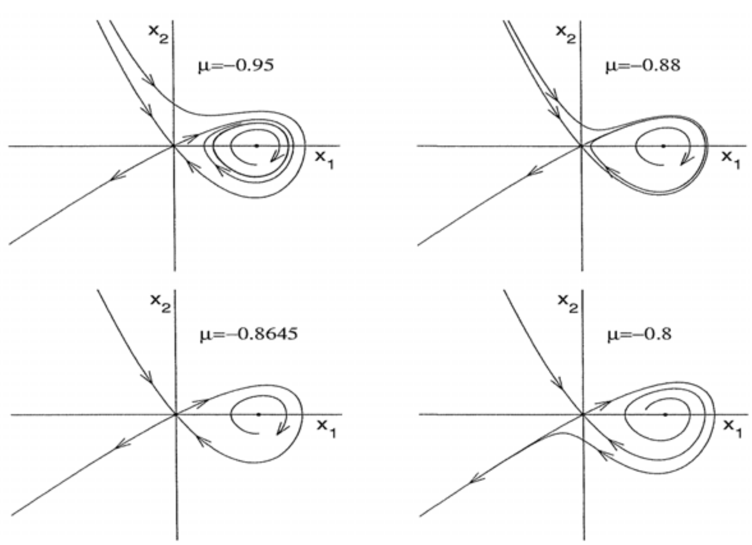

Figure 1.6: Homoclinic solutions in ω-limit set.

Chapter 2

Local Sections and Flow Boxes

In this chapter we describe the local behavior of the flow associated to

$$\dot{X} = F(X)$$

near a given point X_0, which is not an equilibrium point. Our goal is to construct first a local section at X_0, and then a flow box neighborhood of X_0. In this flow box, solutions of the system behave particularly simply.

Suppose $F(X_0) \neq 0$. The transverse line at X_0, denoted by $\ell(X_0)$, is the straight line through X_0, which is perpendicular to the vector $F(X_0)$ based at X_0. We parametrize $\ell(X_0)$ as follows. Let V_0 be a unit vector based at X_0 and perpendicular to $F(X_0)$. Then define $h : \mathbb{R} \to \ell(X_0)$ by $h(u) = X_0 + uV_0$. Since $F(X)$ is continuous, the vector field is not tangent to $\ell(X_0)$, at least in some open interval in $\ell(X_0)$ surrounding X_0. We call such an open subinterval containing X_0 a *local section* at X_0. At each point of a local section $\mathcal{S}$, the vector field points "away from" $\mathcal{S}$, so solutions must cut across a local section. In particular $F(X) \neq 0$ for $X \in \mathcal{S}$. See Figure 2.1.

Our first use of a local section at X_0 will be to construct an associated flow box in a neighborhood of X_0. A flow box gives a complete description of the behavior of the flow in a neighborhood of a nonequilibrium point by means of a special set of

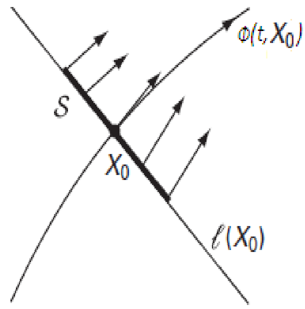

Figure 2.1: A local section $\mathcal{S}$ at X_0 and several representative vectors from the vector field along $\mathcal{S}$.

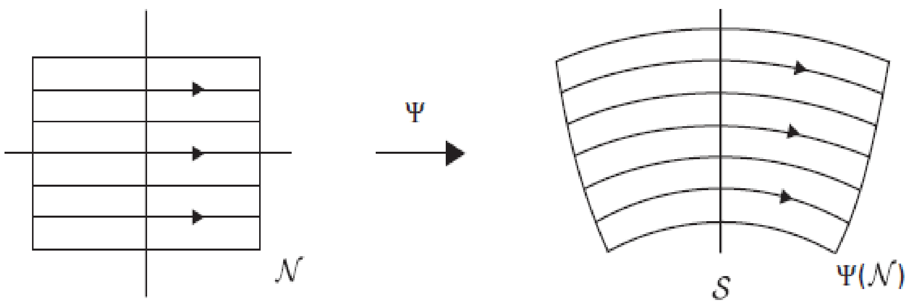

Figure 2.2: The flow box associated to $\mathcal{S}$.

coordinates. An intuitive description of the flow in a flow box is simple: points move in parallel straight lines at constant speed.

Given a local section $\mathcal{S}$ at X_0, we may construct a map Ψ from a neighborhood $\mathcal{N}$ of the origin in $\mathbb{R}^2$ to a neighborhood of X_0 as follows. Given $(s, u) \in \mathbb{R}^2$, we define

$$\Psi(s, u) = \phi(s, h(u)),$$

where h is the parameterization of the transverse line described above. Note that Ψ maps the vertical line (0,u) in $\mathcal{N}$ to the local section $\mathcal{S}$; Ψ also maps horizontal lines in $\mathcal{N}$ to pieces of solution curves of the system. Provided that we choose $\mathcal{N}$ sufficiently small, the map Ψ is then one to one on $\mathcal{N}$. Also note that $D\Psi$ takes the constant vector field (1,0) in $\mathcal{N}$ to vector field $F(X)$. Using the language of planar systems, Ψ is a local conjugacy between the flow of this constant vector field and the flow of the nonlinear system.

We usually take $\mathcal{N}$ in the form $\{(s,u)||s|<\sigma\}$ where $\sigma>0$. In this case we sometimes write $\mathcal{V}_\sigma=\Psi(\mathcal{N})$ and call $\mathcal{V}_\sigma$ the flow box at (or about) X_0. See Figure 2.2. An important property of a flow box is that if $X\in\mathcal{V}_\sigma$, then $\phi(t,X)\in\mathcal{S}$ for a unique $t\in(-\sigma,\sigma)$.

If $\mathcal{S}$ is a local section, the solution through a point Z_0 (perhaps far from $\mathcal{S}$) may reach $X_0\in\mathcal{S}$ at a certain time t_0; see Figure 2.3. We show that in a certain local sense,

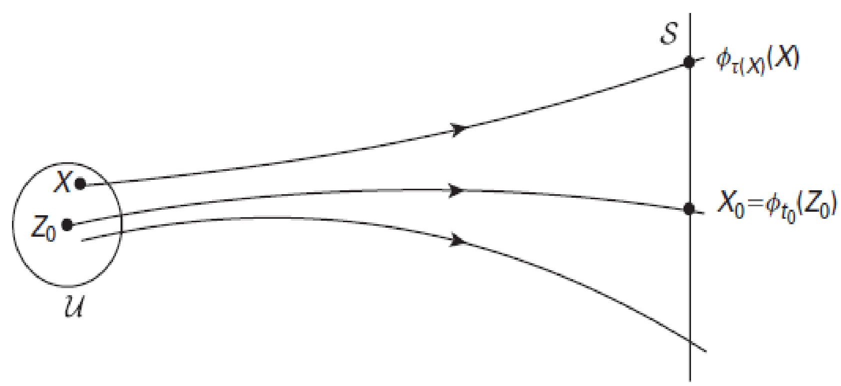

Figure 2.3: Solutions crossing the local section $\mathcal{S}$.

this "time of first arrival" at $\mathcal{S}$ is a continuous function of Z_0. More precisely:

Proposition 2.0.2. *Let $\mathcal{S}$ be a local section at X_0 and suppose $\phi(t_0,Z_0)=X_0$. Let $\mathcal{W}$ be a neighborhood of Z_0. Then there is an open set $\mathcal{U}\subset\mathcal{W}$ containing Z_0 and a differentiable function $\tau:\mathcal{U}\to\mathbb{R}$ such that $\tau(Z_0)=t_0$ and*

$$\phi(\tau(X),X)\in\mathcal{S}$$

for each $X\in\mathcal{U}$.

Proof. Let $F(X_0) = (\alpha, \beta) \neq (0,0)$.

For $Y = (y_1, y_2) \in \mathbb{R}^2$, defined $\eta : \mathbb{R}^2 \to \mathbb{R}$ by

$$\eta(Y) = Y.F(X_0) = \alpha y_1 + \beta y_2$$

$Y \in \ell(X_0) \Leftrightarrow Y = X_0 + V$, where $V.F(X_0) = 0$.

$\Rightarrow Y \in \ell(X_0) \Leftrightarrow \eta(Y) = Y.F(X_0) = X_0.F(X_0)$.

Now define $G : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ by

$$G(X,t) = \eta(\phi(t,X)) = \phi(t,X).F(X_0).$$

We have $G(Z_0, t_0) = X_0.F(X_0)$ since $\phi(t_0, Z_0) = X_0$. Furthermore,

$$\frac{\partial G}{\partial t}(Z_0, t_0) = \mid F(X_0) \mid^2 \neq 0.$$

We may thus apply the implicit function theorem to find a smooth function $\tau : \mathbb{R}^2 \to \mathbb{R}$ defined on a neighborhood $\mathcal{U}_1$ of (Z_0, t_0) such that $\tau(Z_0) = t_0$ and

$$G(X, \tau(X)) \equiv G(Z_0, t_0) = X_0.F(X_0).$$

Hence $\phi(\tau(X), X)$ belongs to the transverse line $\ell(X_0)$. If $\mathcal{U} \subset \mathcal{U}_1$ is a sufficiently small neighborhood of Z_0, then $\phi(\tau(X), X) \in \mathcal{S}$, as required. $\square$

Chapter 3

The Poincaré Map

Probably the most basic tool for studying the stability and bifurcations of periodic orbits is the Poincaré map or first return map, defined by Henri Poincaré in 1881;[2]. As in the case of equilibrium points, closed orbits may also be stable, asymptotically stable, or unstable. The definitions of these concepts for closed orbits are entirely analogous to those for equilibria(see the detail[1]). However, determining the stability of closed orbits is much more difficult than the corresponding problem for equilibria. While we do have a tool that resembles the linearization technique that is used to determine the stability of most equilibria, generally this tool is much more difficult to use in practice. Here is the tool.

Given a closed orbit Γ , there is an associated Poincaré map for Γ near a closed orbit, this map is defined as follows.

Definition 3.0.5. Choose $X_0 \in \Gamma$ and let $\mathcal{S}$ be a local section at X_0. We consider the *first return map* or the *Poincaré map* on $\mathcal{S}$. This is the function P that associates to $X \in \mathcal{S}$ the point $P(X) = \phi(t, X) \in \mathcal{S}$ where t is the smallest positive time for which $\phi(t, X) \in \mathcal{S}$. Now P may not be defined at all points on $\mathcal{S}$ as the solutions through certain points in $\mathcal{S}$ may never return to $\mathcal{S}$. But we certainly have $P(X_0) = X_0$, and the previous proposition guarantees that P is defined and continuously differentiable in a neighborhood of X_0.

See figure 3.1[2] Where U is a neighborhood of P.

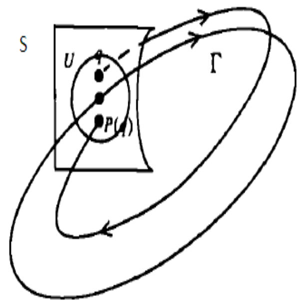

Figure 3.1: The Poincaré or first return map.

In the case of planar systems, a local section is a subset of a straight line through X_0, so we may regard this local section as a subset of $\mathbb{R}$ and take $X_0 = 0 \in \mathbb{R}$. Hence the Poincaré map is a real function taking 0 to 0. If $|P'(0)| < 1$, it follows that P assumes the form $P(x) = ax+$ higher order terms, where $|a| < 1$. Hence, for x near 0, P(x) is closer to 0 than x. This means that the solution through the corresponding point in $\mathcal{S}$ moves closer to Γ after one passage through the local section. Continuing, we see that each passage through $\mathcal{S}$ brings the solution closer to Γ, and so we see that Γ is asymptotically stable. We have:

Proposition 3.0.3. *Let $\dot{X} = F(X)$ be a planar system and suppose that X_0 lies on a closed orbit Γ . Let P be a Poincaré map defined on a neighborhood of X_0 in some local section. If $|P'(X_0)| < 1$, then Γ is asymptotically stable.*

Proof. Let b be any number between $| P'(X_0) |$ and 1, for example b could be chosen to be $\frac{(1+|P'(X_0)|)}{2}$. Since

$$\lim_{X \to X_0} \frac{| P(X) - P(X_0) |}{| X - X_0 |} = | P'(X_0) |$$

There is a neighborhood $\mathcal{N}_\epsilon(X_0)$ for some $\epsilon > 0$ so that

$$\frac{\mid P(X) - P(X_0) \mid}{\mid X - X_0 \mid} < b$$

for $X \in \mathcal{N}_\epsilon(X_0), X \neq X_0$

In other words, $P(X)$ is closer to X_0 than X is, by at least a factor of b (which is less than 1). This implies two things: First, if $X \in \mathcal{N}_\epsilon(X_0)$, then $P(X) \in \mathcal{N}_\epsilon(X_0)$; that means if X is with in ϵ neigborhood of X_0, then so is P(X), and by repeating the argument, so are $P^2(X), P^3(X)$, and so forth. Second, it follows that

$$\mid P^k(X) - X_0 \mid \leq a^k \mid X - X_0 \mid, \forall k \geq 1.$$

□

Example 3.0.3. *Consider the planar system*

$$\begin{aligned} \dot{x} &= x - y - x(x^2 + y^2) \\ \dot{y} &= x + y - y(x^2 + y^2). \end{aligned} \tag{3.0.1}$$

then determine the Poincaré map and the stability of the closed orbit.

solution: Take the local section $\mathcal{S}$ as

$$\mathcal{S} = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y = 0\}.$$

Transforming (3.0.1) to polar coordinates $r = (x^2+y^2)^{1/2}$, $\theta = \arctan(y/x)$, we obtain

$$\begin{aligned} \dot{r} &= r(1 - r^2), \\ \dot{\theta} &= 1. \end{aligned} \tag{3.0.2}$$

and the local section becomes

$$\mathcal{S} = \{(r, \theta) \in \mathbb{R}^+ \times S^1 \mid r > 0, \theta = 0\}.$$

It is easy to solve (3.0.2) to obtain the global flow

$$\phi_t(r_0,\theta_0) = ((1+(\frac{1}{r_0^2}-1)e^{-2t})^{-1/2}, t+\theta_0).$$

The time of flight τ for any point $q \in \mathcal{S}$ is simply $\tau = 2\pi$, and thus the Poincaré map is given by

$$P(r_0) = (1+(\frac{1}{r_0^2}-1)e^{-4\pi})^{-1/2} \tag{3.0.3}$$

[4]. Clearly, P has a fixed point at $r_0 = 1$, reflecting the circular closed orbit of radius 1 of (3.0.2). Here P is one-dimensional map and its linearization is given by

$$P'(1) = \frac{dP}{dr_0} \mid_{r_0=1} = \frac{-1}{2}(1+(\frac{1}{r_0^2}-1)e^{-4\pi})^{-3/2}.(\frac{-2e^{-4\pi}}{r_0^3}) \mid_{r_0=1} = e^{-4\pi} < 1. \tag{3.0.4}$$

Thus $P = 1$ is a stable fixed point and Γ is a stable or attracting closed orbit. The Poincaré map is usually more useful when setting up a geometric model of a specific system, see [1].

we saw that the stability of a limit cycle Γ of a planar system is determined by the derivative of the Poincaré map, $P'(X_0)$, at a point $X_0 \in \Gamma$; in fact, if $\mid P'(X_0) \mid < 1$ then the limit cycle Γ is (asymptotically) stable. In this section we shall see that similar results, concerning the stability of periodic orbits, hold for higher dimensional systems of the form

$$\dot{X} = F(X) \tag{3.0.5}$$

with $F \in C^1(\mathcal{U})$ where $\mathcal{U}$ is an open subset of $\mathbb{R}^n$. Assume that (3.0.5) has a periodic orbit of period τ

$$\{\Gamma : X = \Gamma(t), 0 < t < \tau\},$$

contained in $\mathcal{U}$. In this case, the derivative of the Poincaré map, $\mathbf{DP}(X_0)$, at a point $X_0 \in \Gamma$ is an $(n-1)\times(n-1)$ matrix and we shall see that if $\parallel \mathbf{DP}(X_0) \parallel < 1$ then the periodic orbit Γ is asymptotically stable (see the detail on [2])

Example 3.0.4. *Consider the three-dimensional system*

$$\dot{r} = r(1 - r)$$
$$\dot{\theta} = 1 \tag{3.0.6}$$
$$\dot{z} = -z$$

Compute the Poincaré map along the closed orbit lying on the unit circle given by $r = 1$ *and show that this closed orbit is asymptotically stable.*

solution: Clearly $\Gamma(t) = (cost, sint, 0)^T$ is a closed orbit of the given system on the unit circle. First we have to solve the system.

$$\frac{dr}{dt} = r(1 - r)$$

$$\frac{dr}{r(1 - r)} = dt$$

Integrating both sides we obtain

$$r(t) = \frac{Ke^t}{1 + Ke^t}$$

where $K = e^c$ and c is a constant. Since $r(t = 0) = r_0$ we have

$$K = \frac{r_0}{1 - r_0}.$$

Hence

$$r(t) = \frac{r_0 e^t}{1 - r_0 + r_0 e^t}$$

Again,

$$\frac{d\theta}{dt} = 1$$

This implies that

$$d\theta = dt.$$

Integrating both sides we get

$$\theta(t) = t + c.$$

Since $\theta(0) = \theta_0$ we have

$$\theta(t) = t + \theta_0.$$

Similarly,

$$z(t) = z_0 e^{-t}.$$

Then the flow of the system becomes

$$\phi(t, (r_0, \theta_0, z_0)) = (\frac{r_0 e^t}{1 - r_0 + r_0 e^t}, t + \theta_0, z_0 e^{-t}).$$

For $\theta \in [0, 2\pi)$ we have $\theta(t) = t$. Hence $\tau = 2\pi$ is the period of the flow of the system. Therefore, the Poincaré map becomes

$$P(r_0, z_0) = (\frac{r_0 e^{2\pi}}{1 - r_0 + r_0 e^{2\pi}}, z_0 e^{-2\pi})^T.$$

Then

$$\mathbf{DP}(1, 0) = \begin{bmatrix} e^{-2\pi} & 0 \\ 0 & e^{-2\pi} \end{bmatrix}.$$

Hence

$$\parallel \mathbf{DP}(1, 0) \parallel = e^{-4\pi} < 1.$$

Therefore, $\Gamma(t) = (cost, sint, 0)^T$ is asymptotically stable.

Chapter 4

Monotone Sequences in Planar Dynamical Systems

Let $X_0, X_1, ... \in \mathbb{R}^2$ be a finite or infinite sequence of distinct points on the solution curve through X_0. We say that the sequence is monotone along the solution if $\phi(t_n, X_0) = X_n$ with $0 \leq t_1 < t_2 \cdots$.

Let $Y_0, Y_1, ...$ be a finite or infinite sequence of points on a line segment I in $\mathbb{R}^2$. We say that this sequence is monotone along I if Y_n is between Y_{n-1} and Y_{n+1} in the natural order along I for all $n \geq 1$.

A sequence of points may be on the intersection of a solution curve and a segment I; they may be monotone along the solution curve but not along the segment, or vice versa; see Figure 4.1. However, this is impossible if the segment is a local section in the plane.

Proposition 4.0.4. *Let $\mathcal{S}$ be a local section for a planar system of differential equations and let $Y_0, Y_1, Y_2, ...$ be a sequence of distinct points in $\mathcal{S}$ that lie on the same solution curve. If this sequence is monotone along the solution, then it is also monotone along $\mathcal{S}$.*

Proof. Consider three points $Y_0, Y_1, and Y_2$ in $\mathcal{S}$. Let Σ be the simple closed curve

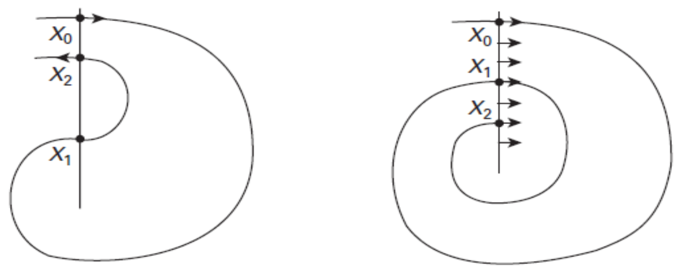

Figure 4.1: Two solutions crossing a straight line. On the left,X_0, X_1, X_2 is monotone along the solution but not along the straight line. On the right, X_0, X_1, X_2 is monotone along both the solution and the line.

made up of the part of the solution between Y_0 and Y_1 and the segment $T \subset \mathcal{S}$ between Y_0 and Y_1. Let D be the region bounded by Σ. We suppose that the solution through Y_1 leaves D at Y_1 (see Figure 4.2; if the solution enters D, the argument is similar). Hence the solution leaves D at every point in T since T is part of the local section.

It follows that $\mathbb{R}^2 - D$ is positively invariant. For no solution can enter D at a point of T; nor can it cross the solution connecting Y_0 and Y_1, by uniqueness of solutions.

Therefore, $\phi(t, Y_1) \in \mathbb{R}^2 - D$ for all $t > 0$.

In particular, $Y_2 \in \mathcal{S} - T$.

The set $\mathcal{S} - T = I_0 \cup I_1$ with Y_j an endpoint of I_j for $j = 0, 1$. One can draw an arc from a point $\phi(\epsilon, Y_1)$ (with $\epsilon > 0$ very small) to a point of I_1, without crossing Σ.

Therefore I_1 is outside D. Similarly I_0 is inside D.

It follows that $Y_2 \subset I_1$ since it must be outside D.

This shows that Y_1 is between Y_0 and Y_2 in $\mathcal{S}$, proving the proposition. $\square$

We now come to an important property of limit points.

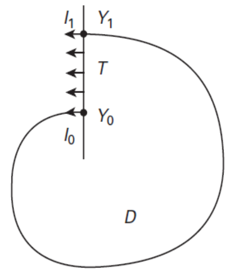

Figure 4.2: Solutions exit the region D through T.

Proposition 4.0.5. *For a planar system, suppose that $Y \in \omega(X)$. Then the solution through Y crosses any local section at no more than one point. The same is true if $Y \in \alpha(X)$.*

Proof. Assume the contrary that Y_1 and Y_2 are distinct points on the solution through Y and $\mathcal{S}$ is a local section containing Y_1 and Y_2.

Suppose $Y \in \omega(X)$ (the argument for $\alpha(X)$ is similar). Then $Y_k \in \omega(X)$ for $k = 1, 2$.

Let $\mathcal{V}_k$ be flow boxes at Y_k defined by some intervals $J_k \subset \mathcal{S}$; we assume that J_1 and J_2 are disjoint as depicted in Figure 4.3.

The solution through X enters each $\mathcal{V}_k$ infinitely often; hence it crosses J_k infinitely often.

Hence there is a sequence

$$a_1, b_1, a_2, b_2, a_3, b_3, ...,$$

which is monotone along the solution through X, with an $a_n \in J_1, b_n \in J_2$ for $n = 1, 2, ...$

But such a sequence cannot be monotone along $\mathcal{S}$ since J_1 and J_2 are disjoint, contradicting the previous proposition. $\square$

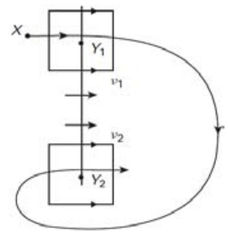

Figure 4.3: The solution through X cannot cross $\mathcal{V}_1$ and $\mathcal{V}_2$ infinitely often.

Proposition 4.0.6. *Show that a closed orbit of a planar system meets a local section in at most one point.*

Proof. Suppose that for a closed orbit Γ and some local section $\mathcal{S}$, Γ intersects $\mathcal{S}$ at two distinct points p and q. Since p and q are on a transversal $\mathcal{S}$, they are regular points, so we can choose $\mathcal{S}_\infty$ and $\mathcal{S}_\in$ of $\mathcal{S}$ containing, respectively, p and q, and for some $\epsilon > 0$, define flow boxes B_1 and B_2 by

$$B_i := \{\phi(t, X) \mid t \in [-\epsilon, \epsilon], X \in \mathcal{S}_i\}.$$

Now, the fact that $p, q \in \Gamma$ means that we can pick an increasing sequence of times $t_1, t_2, ...$ such that $\phi(t_j, X) \in B_1$ if j is odd and $\phi(t_j, X) \in B_2$ if j is even. In fact, because of the nature of the flow in B_1 and B_2, we can assume that $\phi(t_j, X) \in \mathcal{S}$ for each j. Although the sequence $\phi(t_1, X), \phi(t_2, X), ...$ is monotone on the trajectory Γ, it is not monotone on $\mathcal{S}$. This contradicts proposition 4.0.4. This completes the proof [3]. □

Chapter 5

The Poincaré-Bendixson Theorem and Its Application

5.1 The Poincaré-Bendixson Theorem

In this section we prove a celebrated result concerning planar systems:

Theorem 5.1.1. *(Poincaré-Bendixson) Suppose that Ω is a nonempty, closed and bounded limit set of a planar system of differential equations that contains no equilibrium point. Then Ω is a closed orbit.*

Proof. Assume that $\omega(X)$ is closed and bounded, $Y \in \omega(X)$, and let Γ be a closed orbit (the case of α-limit sets is similar). We show first that $Y \in \Gamma$ and later that $\Gamma = \omega(X)$.

Since $Y \in \omega(X)$, we know that $\emptyset \neq \omega(Y) \subset \omega(X)$, by proposition 1.2.1. Let $Z \in \omega(Y)$ and let $\mathcal{S}$ be a local section at Z. Let $\mathcal{V}$ be a flow box associated to $\mathcal{S}$. By proposition 4.0.5, the solution through Y meets $\mathcal{S}$ at exactly one point. On the other hand,

$$\exists t_n \to \infty \ni \lim_{n\to\infty} \phi(t_n, Y) = Z;$$

hence infinitely many $\phi(t_n, Y)$ belong to $\mathcal{V}$.

Therefore we can find $r, s \in \mathbb{R}$ such that $r > s$ and $\phi(r, Y), \phi(s, Y) \in \mathcal{S}$.
It follows that $\phi(r, Y) = \phi(s, Y)$; hence $\phi(r - s, Y) = Y$ and $r - s > 0$. Since $\omega(X)$ contains no equilibria, $Y \in \Gamma$.

It remains to prove that if $\Gamma \in \omega(X)$, then $\Gamma = \omega(X)$. For this, it is enough to show that

$$lim_{t\to\infty} d(\phi(t, X), \Gamma) = 0,$$

where $d(\phi(t, x), \Gamma)$ is the distance from $\phi(t, x)$ to the set Γ (that is, the distance from $\phi(t, x)$ to the nearest point of Γ). Let $\mathcal{S}$ be a local section at $Y \in \Gamma$. Let $\epsilon > 0$ and consider a flow box $\mathcal{V}_\epsilon$ associated to $\mathcal{S}$. Then there is a sequence $t_0 < t_1 < \cdots$ such that

1.$\phi(t_n, X) \in \mathcal{S}$;

2. $\phi(t_n, X) \to Y$;

3. $\phi(t, X) \notin \mathcal{S}$ for $t_{n-1} < t < t_n$, $n = 1, 2, ...$

Let $X_n = \phi(t_n, X)$. By proposition 4.0.4, X_n is a monotone sequence in $\mathcal{S}$ that converges to Y .

We claim that there exists an upper bound for the set of positive numbers $t_{n+1} - t_n$ for n sufficiently large. To see this, suppose $\phi(\tau, Y) = Y$ where $\tau > 0$. Then for X_n sufficiently near Y , $\phi(\tau, X_n) \in \mathcal{V}_\epsilon$ and hence

$$\phi(\tau + t, X_n) \in \mathcal{S}$$

for some $t \in [-\epsilon, \epsilon]$. Thus

$$t_{n+1} - t_n \leq \tau + \epsilon.$$

This provides the upper bound for $t_{n+1} - t_n$. Also, $t_{n+1} - t_n$ is clearly at least 2ϵ , so $t_n \to \infty$ as $n \to \infty$.

Let $\beta > 0$ be small. By continuity of solutions with respect to initial conditions, there exists $\delta > 0$ such that,

$$\text{if } |Z - Y| < \delta \text{ and } |t| \le \tau + \epsilon \text{ then } |\phi(t, Z) - \phi(t, Y)| < \beta$$

. That is, the distance from the solution $\phi(t, Z)$ to Γ is less than β for all t satisfying $|t| \le \tau + \epsilon$. Let n_0 be so large that $|X_n - Y| < \delta$ for all $n \ge n_0$. Then

$$|\phi(t, X_n) - \phi(t, Y)| < \beta$$

if $|t| \le \tau + \epsilon$ and $n \ge n_0$. Now let $t \ge t_{n_0}$. Let $n \ge n_0$ be such that $t_n \le t \le t_{n+1}$. Then

$$\begin{aligned} d(\phi(t, X), \Gamma) &\le |\phi(t, X) - \phi(t - t_n, Y)| \\ &= |\phi(t - t_n, X_n) - \phi(t - t_n, Y)| \quad < \beta \end{aligned} \tag{5.1.1}$$

since $|t - t_n| \le \tau + \epsilon$. This shows that the distance from $\phi(t, X)$ to Γ is less than β for all sufficiently large t . This completes the proof of the Poincaré-Bendixson theorem. □

Example 5.1.1. *Prove that the system of differential equation*

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + (1 - x^2 - 2y^2)y. \end{aligned} \tag{5.1.2}$$

has a nontrivial periodic solution.

solution: We try and find a bounded region **R** in the xy-plane, containing no equilibrium points of(5.1.2), and having the property that every solution x, y of (5.1.2) which starts in **R** at time $t = t_0$, remains there for all future time $t \geq t_0$. It can be shown that a simply connected region such as a square or disc will never work. Therefore, we try and take **R** to be an annulus surrounding the origin. To this end, compute

$$\frac{d}{dt}(\frac{x^2+y^2}{2}) = x\dot{x} + y\dot{y} = (1 - x^2 - 2y^2)y^2,$$

and observe that $1 - x^2 - 2y^2 > 0$ for $x^2 + y^2 < \frac{1}{2}$ and
$1 - x^2 - 2y^2 < 0$ for $x^2 + y^2 > 1$.
Hence, $x^2 + y^2$ is increasing along any solution x, y of (5.1.2) when $x^2 + y^2 < \frac{1}{2}$ and decreasing when $x^2 + y^2 > 1$.
This implies that any solution x, y of (5.1.2) which starts in the annulus $\frac{1}{2} < x^2+y^2 < 1$ at time $t = t_0$ will remain in this annulus for all future time $t \geq t_0$.
Now, this annulus contains no equilibrium points of (5.1.2).
Consequently, by Poincaré-Bendixson Theorem, there exists at least one periodic solution x, y of (5.1.2) lying entirely in this annulus, and then x is a nontrivial periodic solution of (5.1.2).

5.2 Applications of Poincaré-Bendixson Theorem

The Poincaré-Bendixson theorem essentially determines all of the possible limiting behaviors of a planar flow[1]. We give a number of corollaries of this important theorem in this section.

A limit cycle is a closed orbit Γ such that $\Gamma \subset \omega(X)$ or $\Gamma \subset \alpha(X)$ for some $X \notin \Gamma$. In the first case Γ is called an ω-limit cycle; in the second case, an α-limit cycle. We deal only with ω-limit sets in this section; the case of α-limit sets is handled by simply reversing time.

In the proof of the Poincaré-Bendixson theorem it was shown that limit cycles have the following property: If Γ is an ω-limit cycle, there exists $X \notin \Gamma$ such that

$$lim_{t\to\infty} d(\phi(t, X), \Gamma) = 0.$$

Geometrically this means that some solution spirals toward Γ as $t \to \infty$. See Figure 5.1. Not all closed orbits have this property. For example, in the case of a linear system with a center at the origin in $\mathbb{R}^2$, the closed orbits that surround the origin have no solutions approaching them, and so they are not limit cycles.

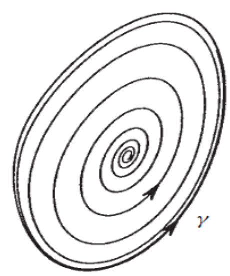

Figure 5.1: A solution spiraling toward a limit cycle.

Limit cycles possess a kind of (one-sided, at least) stability. Let Γ be an ω-limit cycle and suppose $\phi(t, X)$ spirals toward Γ as $t \to \infty$. Let $\mathcal{S}$ be a local section at $Z \in \Gamma$. Then there is an interval $T \subset \mathcal{S}$ disjoint from Γ, bounded by $\phi_(t_0, X)$ and $\phi(t_1, X)$ with $t_0 < t_1$, and not meeting the solution through X for $t_0 < t < t_1$. See Figure 5.2. The annular region A that is bounded on one side by Γ and on the other side by the union of T and the curve

$$\{\phi(t, X)|t_0 \leq t \leq t_1\}$$

is positively invariant, as is the set $B = A - \Gamma$. It is easy to see that $\phi(t, Y)$ spirals toward Γ for all $Y \in B$. Hence we have:

Corollary 5.2.1. *Let Γ be an ω-limit cycle. If $\Gamma = \omega(X)$ where $X \notin \Gamma$, then X has a neighborhood $\mathcal{O}$ such that $\Gamma = \omega(Y)$ for all $Y \in \mathcal{O}$. In other words, the set*

$$\{Y|\omega(Y) = \Gamma\} - \Gamma$$

is open.

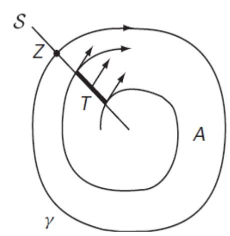

Figure 5.2: The region A is positively invariant.

Proof. Let Γ be an ω-limit cycle and suppose $\lim_{t\to\infty} \phi(t, X) = \Gamma$. Let $\mathcal{S}$ be a local section at $Z \in \Gamma$. We can then find times t_0 and t_1 with $t_0 < t_1$ such that

$$\phi(t_0, X), \phi(t_1, X) \in \mathcal{S},$$

and we may assume that $\phi(t, X) \notin \mathcal{S}$ for $t_0 < t < t_1$. Then define A to be the area bounded by the segment $\overline{\phi(t_0, X)\phi(t_1, X)}$, the curve segment $\{\phi(t, X)\}_{t\in(t_0,t_1)}$, and the limiting trajectory Γ. Then the set A is positively invariant. Now take any small open box $B \subset A$ on the same side of the segment $\overline{\phi(t_0, X)\phi(t_1, X)}$ as given by the 'direction of the flow'. This may be chosen such that $\phi(t, Y) \in B$ for Y in a small neighborhood of X and $t > t_0$ sufficiently close to t_0 or t_1. Then we claim that for any $P \in B$, we have

$$\Gamma = \omega(P).$$

This gives the corollary, since then also $\Gamma = \omega(Y)$ for Y as in the immediately preceding. To see the claim, we note that the trajectory starting at $P \in B$ winds is nested between two branches of the spiral $\{\phi(t,x)\}$, and turns in the same sense. Since $\lim_{t\to\infty} \phi(t,X) = \Gamma$, the same must be true of the trajectory starting at P, which gives the claim $\Gamma = \omega(P)$. □

Proposition 5.2.2. *Let A be an annular region in $\mathbb{R}^2$. Let F be a planar vector field that points inward along the two boundary curves of A. Suppose also that every radial segment of A is local section. See Figure 5.3. Prove there is a periodic solution in A.*

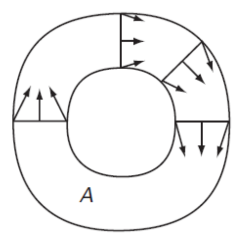

Figure 5.3: The region A is positively invariant.

Proof. Let $\mathcal{S}$ be a radial segment and Suppose $Z \in \mathcal{S}$. Consider the map $\mathcal{S} \to \mathcal{S}$ given by

$$Z \mapsto \phi(\tau(z), Z).$$

Then by the proof of Poincaré-Bendixson theorem,

$$\phi(\tau(z), Z) \in \mathcal{S}.$$

For a smallest $t = \tau(z) > 0$. Hence Z must lie on a closed orbit. Therefore, there is a periodic solution in A. □

Proposition 5.2.3. *Let X be a recurrent point of a planar system, that is, there is a sequence $t_n \to \pm\infty$ such that*

$$\phi(t_n, X) \to X.$$

Prove that either X is an equilibrium or X lies on a closed orbit.

Proof. Let us start the proof by defining recurrent trajectory or orbit. Let Γ be a trajectory of system (1.1.3). Then Γ is recurrent if $\Gamma \subset \alpha(\Gamma)$ or $\Gamma \subset \omega(\Gamma)$[2].

Suppose X is a recurrent point of the planar system, then by definition $X \subset \alpha(X)$ or $X \subset \omega(X)$. If X contains no equilibrium point, then X lies on a closed orbit. For otherwise, X is an equilibrium point. Hence X is either an equilibrium or X lies on a closed orbit. □

Corollary 5.2.4. *A compact set K that is positively or negatively invariant contains either a limit cycle or an equilibrium point.*

Proof. Suppose K be compact and positively invariant set and Γ be a closed orbit. If $X \in K$ such that $X \notin \Gamma$,then by the compactness of K we have

$$\omega(X) \neq \emptyset.$$

Hence $\omega(X)$ must also lie in K. If K contains no equilibrium point, then by Poincaré-Bendixson theorem for any $X \in K$,

$$\omega(X) = \Gamma.$$

Hence K contains a limit cycle. Otherwise for any $X \in K$,

$$\omega(X) \in K.$$

Hence K contains an equilibrium point since any equilibrium point is its own ω-limit set. This completes the proof. □

The next result exploits the spiraling property of limit cycles.

Corollary 5.2.5. *Let* Γ *be a closed orbit and let* $\mathcal{U}$ *be the open region in the interior of* Γ *. Then* $\mathcal{U}$ *contains either an equilibrium point or a limit cycle.*

Proof. Let D be the compact set $\mathcal{U} \cup \Gamma$. Then D is invariant since no solution in $\mathcal{U}$ can cross Γ . If $\mathcal{U}$ contains no limit cycle and no equilibrium, then, for any $X \in \mathcal{U}$,

$$\omega(X) = \alpha(X) = \Gamma$$

by the proof of Poincaré-Bendixson theorem. If $\mathcal{S}$ is a local section at a point $Z \in \Gamma$, there are sequences $t_n \to \infty$, $s_n \to -\infty$ such that $\phi(t_n, X), \phi(s_n, X) \in \mathcal{S}$ and both $\phi(t_n, X)$ and $\phi(s_n, X)$ tend to Z as $n \to \infty$. But this leads to a contradiction of the proposition 4.0.4 on monotone sequences. □

Actually this last result can be considerably sharpened:

Corollary 5.2.6. *Let* Γ *be a closed orbit that forms the boundary of an open set U. Then U contains an equilibrium point.*

Proof. Suppose U contains no equilibrium point. Consider first the case that there are only finitely many closed orbits in U. We may choose the closed orbit that bounds the region with smallest area. There are then no closed orbits or equilibrium points inside this region, and this contradicts corollary 5.2.5. □

Now suppose that there are infinitely many closed orbits in U. If $X_n \to X$ in U and each X_n lies on a closed orbit, then X must lie on a closed orbit. Otherwise, the solution through X would spiral toward a limit cycle since there are no equilibria in U. By corollary 5.2.1, so would the solution through some nearby X_n, which is impossible.

Let $\nu \geq 0$ be the greatest lower bound of the areas of regions enclosed by closed orbits in U. Let $\{\Gamma_n\}$ be a sequence of closed orbits enclosing regions of areas ν_n

such that $\lim_{n\to\infty} \nu_n = \nu$. Let $X_n \in \Gamma_n$. Since $\Gamma \cup U$ is compact, we may assume that $X_n \to X \in U$. Then if U contains no equilibrium, X lies on a closed orbit β bounding a region of area ν. The usual section argument shows that as $n \to \infty$, Γ_n gets arbitrarily close to β and hence the area $\nu_n - \nu$ of the region between Γ_n and β goes to 0. Then the argument above shows that there can be no closed orbits or equilibrium points inside Γ, and this provides a contradiction to corollary 5.2.5.

The following result uses the spiraling properties of limit cycles in a subtle way.

Corollary 5.2.7. *Let H be a first integral of a planar system. If H is not constant on any open set, then there are no limit cycles.*

Proof. Suppose there is a limit cycle Γ; let $c \in \mathbb{R}$ be the constant value of H on Γ. If $X(t)$ is a solution that spirals toward Γ, then $H(X(t)) \equiv c$ by continuity of H. In corollary 5.2.1 we found an open set whose solutions spiral toward Γ; thus H is constant on an open set. □

Finally, the following result is implicit in our development of the theory of Liapunov functions (see the detail[1])

Corollary 5.2.8. *If L is a strict Liapunov function for a planar system, then there are no limit cycles.*

Proof. Suppose $L : \mathbb{R}^2 \to \mathbb{R}$ is a strict Liapunov function for a planar system, that means,

(a) $L(X^*) = 0$

(b) $L(X) > 0$ if $X \neq X^*$; and

(c) $L' < 0$ in $\mathbb{R}^2 - X^*$, where X^* is an equilibrium point.

Assume that $\phi(t, X)$ is a periodic orbit with $\phi(t_0, X) \neq X^*$ for $t_0 = 0$. It follows that $\exists T > 0; \forall n \in \mathbb{N}$ such that

$$\phi(nT, X) = \phi(t_0, X) \neq X^*. \tag{5.2.1}$$

From (a),(b),(c) it follows that X^* is an asymptotically stable fixed - point, and therefore,

$$\lim_{t\to\infty} \phi(t, X) = X^*. \tag{5.2.2}$$

From (5.2.1) and (5.2.2)

$$\phi(t_0, X) = \lim_{n\to\infty} \phi(nT, X) = \lim_{t\to\infty} \phi(t, X) = X^*.$$

This implies

$$\phi(t_0, X) = X^*.$$

This contradicts the assumption that

$$\phi(t_0, X) \neq X^*.$$

Therefore,there is no closed orbit. Hence there are no limit cycles. $\square$

Chapter 6

Conclusion

In this project we addressed the following natural question: Like equilibrium points that are asymptotically stable,does closed orbits also attract other solutions? and why we need to compute Poincaré map? We relaxed these assumption mainly in planar nonlinear systems of differential equations by using the Poincaré map and the poincaré-Bendixson theorem.

The Poincaré map is the most basic tool for studying the stability of closed orbits. It is usually more useful when setting up a geometric model of a specific system, particularly,of a planar system.To determine the poincaré map, we actually first found formulas for all of the solutions starting at $(X, 0)^T$. But it is usually very difficult to compute the exact form of a Poincaré map or even its derivative along a closed orbit.

The Poincaré-Bendixson theorem tells that every nonequilibrium solution of any bounded solution of a planar dynamical system spirals towards either to a fixed point or a closed orbit. And it classifies the asymptotic behavior of closed orbits in smooth planar ordinary differential equations.

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