Coherently Driven Three-Level Laser with Parametric Amplifier

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Abstract

The squeezing and statistical properties of the light produced by a coherently driven degenerate as well as nondegenerate three-level laser with a parametric amplifier is studied applying c-number Langevin equations. Employing the solutions of these equations, we have determined the quadrature variance for the cavity and output modes, the squeezing spectrum for the output mode(s), the photon statistics of the cavity mode(s), and the photon number and count statistics of the output mode(s). It so turns out that the parametric amplifier increases the squeezing and the mean photon number significantly. Furthermore, it so happens that the coupling of the top and bottom levels enhances the degree of squeezing particularly when there are nearly equal number of atoms initially in the top and bottom levels. It is also found that one effect of the coupling of the top and bottom levels is to decrease the mean and the normally-ordered variance of the photon number. In addition, our calculation shows that the mean photon number of mode a is greater than that of mode b.

Contents

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Introduction

Light has played a special role in our attempt to understand nature quantum mechanically. Squeezing is one of the nonclassical features of light that has attracted a great deal of interest [1-50]. In squeezed light the noise in one quadrature is below the coherent-state level at the expense of enhanced fluctuations in the other quadrature, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation. Squeezed light has potential applications in low-noise optical communication and weak signal detection. Hence it is vital to find new optical devices or to combine the existing ones to generate highly squeezed and bright light.

It has been predicted that a three-level laser under certain conditions can produce squeezed light [1-22]. In a cascade three-level laser, three-level atoms in a cascade configuration are injected into a cavity coupled to a vacuum reservoir via a single-port mirror. The injected atoms may initially be prepared in a coherent superposition of the top and bottom levels and/or these levels may be coupled by strong coherent light after they are injected into the cavity. The superposition or the coupling of the top and bottom levels is responsible for the interesting nonclassical features of the generated light. When a three-level atom in a cascade configuration makes a transition from the top to the bottom level via the intermediate level, two photons are generated. If the two photons have the same frequency, the three-level atom is called degenerate otherwise it is called nondegenerate.

Some authors have studied the squeezing and statistical properties of the light produced by three-level lasers in which the crucial role is played by the superposition of the top and bottom levels [1-9]. Ansari [9] has studied the squeezing and the statistical properties of a degenerate three-level laser with the atoms initially prepared in coherent superposition of the top and bottom levels. He has predicted that the cavity mode of such a laser is in a squeezed state, when there are more atoms initially in the bottom level than in the top level. Moreover, with the aid of the Q function he has calculated the photon number distribution. He has shown that the probability of finding n photons decreases smoothly with increasing the number of photons. Furthermore, Lu and Zhu [8] have considered a nondegenerate three-level laser with the atoms initially prepared in coherent superposition of the top and bottom levels. They have predicted a maximum of 50% interacavity two-mode squeezing.

A three-level laser in which the top and bottom levels of the atoms injected into the cavity are coupled by a strong light has also been studied by different authors [9-15]. Ansari *et al* [10] have considered a degenerate three-level laser, with the atoms initially in the upper level and with the top and bottom levels of the atoms coupled by coherent light. They have shown that this system behaves like a parametric oscillator for sufficiently strong coherent light. They have also predicted that such a system can generate squeezed light over large range of the amplitude of the coherent light. Furthermore, Ansari [14] has studied a nondegenerate three-level laser in which the top and bottom levels of the atoms injected into the cavity are coupled by a strong light. He has predicted that under certain conditions this system behaves much like a nondegenerate parametric oscillator. In addition, some authors have studied the squeezing and statistical properties of the light produced by three-level lasers in which the injected atoms are initially prepared in a coherent superposition of the top and bottom levels and with these levels coupled by a strong coherent light after they are injected into the cavity [16-20].

On the other hand, it has been shown theoretically [21-39] and subsequently confirmed experimentally [40-42] that a parametric oscillator produces light with a maximum interacavity squeezing of 50% below the coherent-state level. Some authors [43-46] have considered a three-level laser whose cavity contains a parametric amplifier. Fesseha [43] has studied the squeezing and statistical properties of the light produced by a degenerate three-level laser whose cavity contains a degenerate parametric amplifier, and with the injected atoms prepared initially in coherent superposition of the top and bottom levels. He has shown that the effect of the parametric amplifier is to increase the interacavity squeezing by a maximum of 50%. He has pointed out that since the presence of the parametric amplifier also leads to a significant increase in the mean photon number, the system can produce a bright and highly squeezed light. Moreover, Alebachew and Fesseha [44] have considered a degenerate three-level laser whose cavity contains a degenerate parametric amplifier, with the top and bottom levels of the injected atoms coupled by the pump mode emerging from the parametric amplifier. With equal number of atoms initially in the top and bottom levels, they have found that this system can generate under certain conditions a highly squeezed light.

This PhD dissertation essentially has two parts. In the first part we wish to study the squeezing and statistical properties of the light produced by a degenerate three-level laser whose cavity contains a degenerate parametric amplifier and with the cavity mode driven by a strong coherent light and coupled to a vacuum reservoir. The three-level atoms injected into the cavity are initially prepared in a coherent superposition of the top and bottom levels. Moreover, the top and bottom levels of the three-level atoms are coupled by the pump mode emerging from the parametric amplifier. We first determine, using the pertinent master equation, c-number Langevin equations. Applying the solutions of the resulting c-number Langevin equations along with the properties of the noise forces, we calculate the quadrature variance of the cavity and output modes and the squeezing spectrum of the output mode. Furthermore, the same solutions are employed to obtain the antinormally ordered characteristic function, defined in the Heisenberg picture, which is used to determine the Q function. The resulting Q function is then applied to calculate the mean and the normally-ordered variance of the photon number, and the photon number distribution for the cavity mode. Moreover, using the expression for the l^{th} moment of the photon count in terms of the photon number distribution, we obtain the normally-ordered variance of the photon count for the output mode.

In the second part of this dissertation we seek to analyze the squeezing and statistical properties of of the two-mode light produced by a nondegenerate three-level laser whose cavity contains a nondegenerate parametric amplifier and with the cavity modes driven by a strong coherent light and coupled to a vacuum reservoir. The three-level atoms injected into the cavity are initially prepared in a coherent superposition of the top and bottom levels. Moreover, the top and bottom levels of the three-level atoms are coupled by the pump mode emerging from the parametric amplifier. Following a similar procedure as in the first part, we calculate the quadrature variance of the cavity and the output modes, the squeezing spectrum of the output modes, the mean and normally-ordered variance of the photon number sum and difference, the photon number distribution, and the normally-ordered photon count sum and difference for the output modes.

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Degenerate Three-Level Laser with Parametric Amplifier

In this chapter we consider a degenerate three-level laser whose cavity contains a degenerate parametric amplifier (DPA) and with the cavity mode driven by a strong coherent light and coupled to a vacuum reservoir. Moreover, the three-level atoms injected into the cavity are initially prepared in a coherent superposition of the top and bottom levels and with these levels coupled by the pump mode emerging from the parametric amplifier. We first derive

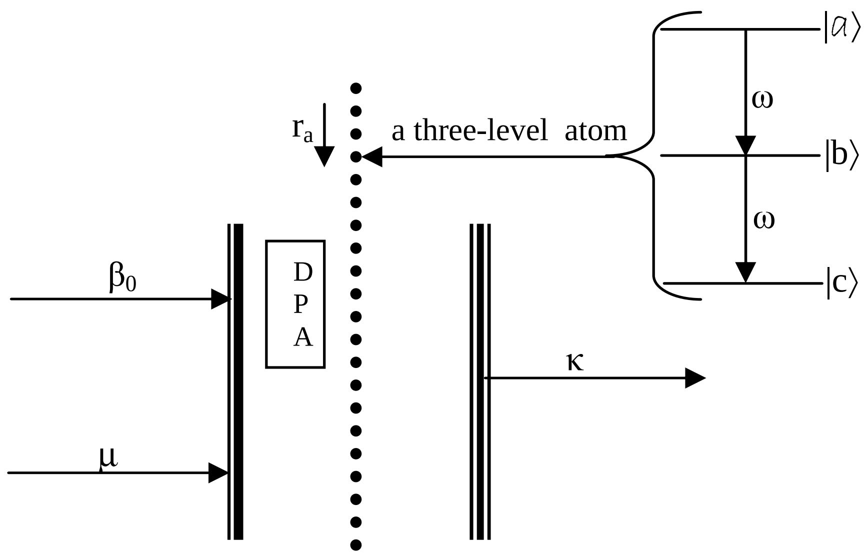

Fig. 2.1: A degenerate three-level laser with a parametric amplifier and strong coherent light.

the master equation (equation of evolution of the density operator) for the cavity mode and with the aid of this master equation, c-number Langevin equations are obtained. We determine, employing the solutions of the resulting c-number Langevin equations, the quadrature variance, the mean and the normally-ordered variance of the photon number, and the photon number distribution for the cavity mode. In addition, applying the same solutions and the input-output relation, we calculate the quadrature variance, the squeezing spectrum, and the normally-ordered variance of the photon count for the output mode.

2.1 c-number Langevin equations

We derive in the linear and adiabatic approximation schemes, the master equation for the cavity mode under consideration. Applying this master equation, we determine c-number Langevin equations associated with the normal ordering. Finally, we obtain the correlation properties of the noise forces and the solutions of the resulting c-number Langevin equations.

2.1.1 Master equation

Three-level atoms are injected at a constant rate r_a into a cavity and removed after a certain time τ. We denote the top, middle, and bottom levels of a three-level atom by $|a\rangle$, $|b\rangle$, and $|c\rangle$, respectively. We assume the transitions between levels $|a\rangle$ and $|b\rangle$ and between levels $|b\rangle$ and $|c\rangle$ to be dipole allowed, with direct transition between levels $|a\rangle$ and $|c\rangle$ to be dipole forbidden. We consider the case for which the cavity mode is at resonance with the two transitions $|a\rangle \rightarrow |b\rangle$ and $|b\rangle \rightarrow |c\rangle$ and with the top and bottom levels coupled by the pump mode emerging from the parametric amplifier. To this end, treating the pump mode classically, the coupling between the $|a\rangle$ and $|c\rangle$ levels can be described by the Hamiltonian

$$\hat{H}' = \frac{i\Omega}{2}\left(|c\rangle\langle a| - |a\rangle\langle c|\right), \tag{2.1}$$

in which

$$\Omega = 2\lambda\beta_0. \tag{2.2}$$

Here β_0, considered to be real and constant, is the amplitude of the pump mode and λ is the coupling constant between the pump mode and a three-level atom. In addition, the interaction of a three-level atom with the cavity mode can be described by the Hamiltonian

$$\hat{H}'' = ig\left[\hat{a}^\dagger\left(|b\rangle\langle a| + |c\rangle\langle b|\right) - \hat{a}\left(|a\rangle\langle b| + |b\rangle\langle c|\right)\right], \tag{2.3}$$

where g is the coupling constant between the cavity mode and a three-level atom, and $\hat{a}$ is the annihilation operator for the cavity mode. Thus upon combining (2.1) and (2.3) the interaction of a three-level atom with the cavity mode and the pump mode can be described by the Hamiltonian

$$\hat{H} = ig\left[\hat{a}^\dagger\left(|b\rangle\langle a| + |c\rangle\langle b|\right) - \hat{a}\left(|a\rangle\langle b| + |b\rangle\langle c|\right)\right] + \frac{i\Omega}{2}\left[|c\rangle\langle a| - |a\rangle\langle c|\right]. \tag{2.4}$$

We take the initial state of a three-level atom to be

$$|\psi_A(0)\rangle = c_a(0)|a\rangle + c_c(0)|c\rangle. \tag{2.5}$$

The initial density operator for the three-level atom can then be written as

$$\hat{\rho}_A(0) = \rho_{aa}^{(0)}|a\rangle\langle a| + \rho_{ac}^{(0)}|a\rangle\langle c| + \rho_{ca}^{(0)}|c\rangle\langle a| + \rho_{cc}^{(0)}|c\rangle\langle c|, \tag{2.6}$$

in which $\rho_{aa}^{(0)} = |c_a|^2$, $\rho_{ac}^{(0)} = c_a c_c^*$, $\rho_{ca}^{(0)} = c_c c_a^*$, $\rho_{cc}^{(0)} = |c_c|^2$. It proves to be convenient to introduce a new parameter η defined by [9]

$$\rho_{aa}^{(0)} = \frac{1-\eta}{2}. \tag{2.7}$$

Using the fact that

$$\rho_{aa}^{(0)} + \rho_{cc}^{(0)} = 1 \tag{2.8}$$

along with

$$|\rho_{ac}^{(0)}|^2 = \rho_{aa}^{(0)}\rho_{cc}^{(0)}, \tag{2.9}$$

one easily finds

$$\rho_{cc}^{(0)} = \frac{1+\eta}{2} \tag{2.10}$$

and

$$|\rho_{ac}^{(0)}| = \frac{1}{2}\sqrt{1-\eta^2}. \tag{2.11}$$

We note that the parameter η describes the initial preparation of a three-level atom. Upon setting

$$\rho_{ac}^{(0)} = |\rho_{ac}^{(0)}|e^{i\theta}, \tag{2.12}$$

expression (2.6) can be put in the form

$$\begin{aligned}\hat{\rho}_A(0) = {} & \frac{1-\eta}{2}|a\rangle\langle a| + \frac{1}{2}\sqrt{1-\eta^2}e^{i\theta}|a\rangle\langle c| \\ & + \frac{1}{2}\sqrt{1-\eta^2}e^{-i\theta}|c\rangle\langle a| + \frac{1+\eta}{2}|c\rangle\langle c|. \end{aligned} \tag{2.13}$$

Suppose $\hat{\rho}_{AR}(t, t_j)$ is the density operator for a single atom plus the cavity mode at time t, with the atom injected at time t_j, such that $(t-\tau) \leq t_j \leq t$. The density operator for all atoms in the cavity plus the cavity mode at time t can then be written as

$$\hat{\rho}_{AR}(t) = r_a \sum_j \hat{\rho}_{AR}(t, t_j)\Delta t_j, \tag{2.14}$$

where $r_a \Delta t_j$ represents the number of atoms injected into the cavity in a time Δt_j. Now converting the summation into integration in the limit $\Delta t_j \to 0$, we have

$$\hat{\rho}_{AR}(t) = r_a \int_{t-\tau}^{t} \hat{\rho}_{AR}(t, t_j) dt' \tag{2.15}$$

and differentiating with respect to t, and taking into account the Leibnitz' rule

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,x')dx' = \left(f(x,v)\frac{dv(x)}{dx} - f(x,u)\frac{du(x)}{dx} \right) + \int_{u(x)}^{v(x)} \frac{\partial}{\partial x} f(x,x')dx', \tag{2.16}$$

there follows

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a \left(\hat{\rho}_{AR}(t,t) - \hat{\rho}_{AR}(t, t-\tau) \right) + r_a \int_{t-\tau}^{t} \frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t')dt'. \tag{2.17}$$

We observe that $\hat{\rho}_{AR}(t,t)$ is the density operator for the cavity mode plus an atom injected at time t. This operator can be expressed as

$$\hat{\rho}_{AR}(t,t) = \hat{\rho}_A(t)\hat{\rho}(t), \tag{2.18}$$

with $\hat{\rho}(t)$ being the density operator for the cavity mode alone. We also note that $\hat{\rho}_{AR}(t, t-\tau)$ represents the density operator for an atom plus the cavity mode at time t, with the atom being removed from the cavity at this time. This operator can also be put in the form

$$\hat{\rho}_{AR}(t, t-\tau) = \hat{\rho}_A(t-\tau)\hat{\rho}(t). \tag{2.19}$$

Now in view of (2.18) and (2.19), one can write Eq. (2.17) as

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a \left(\hat{\rho}_A(t) - \hat{\rho}_A(t-\tau) \right) \hat{\rho}(t) + r_a \int_{t-\tau}^{t} \frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t')dt'. \tag{2.20}$$

In the absence of the damping of the cavity mode by vacuum reservoir, the density operator $\hat{\rho}_{AR}(t,t')$ evolves in time according to

$$\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t') = -i[\hat{H}, \hat{\rho}_{AR}(t,t')], \tag{2.21}$$

so that using this and taking into account (2.15), one can put Eq. (2.20) in the form

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a \left(\hat{\rho}_A(t) - \hat{\rho}_A(t-\tau) \right) \hat{\rho}(t) - i\big[\hat{H}, \hat{\rho}_{AR}(t)\big]. \tag{2.22}$$

Furthermore, tracing over the atomic variables and taking into account the damping of the cavity mode by the vacuum reservoir, we have

$$\frac{d\hat{\rho}}{dt} = -iTr_A[\hat{H}, \hat{\rho}_{AR}(t)] + \frac{1}{2}\kappa(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}), \tag{2.23}$$

where we have used the fact

$$Tr\hat{\rho}_A(t) = Tr\hat{\rho}_A(t-\tau) = 1. \tag{2.24}$$

The master equation associated with the interactions described by the Hamiltonian (2.4) can be put in the form

$$\frac{d\hat{\rho}}{dt} = g\big(\hat{a}^\dagger\hat{\rho}_{ab} - \hat{\rho}_{ab}\hat{a}^\dagger + \hat{a}^\dagger\hat{\rho}_{bc} - \hat{\rho}_{bc}\hat{a}^\dagger + \hat{\rho}_{ba}\hat{a} - \hat{a}\hat{\rho}_{ba} + \hat{\rho}_{cb}\hat{a} - \hat{a}\hat{\rho}_{cb}\big)$$
$$+ \frac{1}{2}\kappa\big(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\big), \tag{2.25}$$

in which the matrix element $\hat{\rho}_{\alpha\beta}$ is defined by

$$\hat{\rho}_{\alpha\beta} = \langle\alpha|\hat{\rho}_{AR}|\beta\rangle, \tag{2.26}$$

with $\alpha, \beta = a, b, c$.

On the other hand, we see from (2.22) that

$$\frac{d\hat{\rho}_{\alpha\beta}}{dt} = r_a\left(\langle\alpha|\hat{\rho}_A(0)|\beta\rangle - \langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle\right)\hat{\rho}(t) - i\left(\langle\alpha|\hat{H}\hat{\rho}_{AR}|\beta\rangle - \langle\alpha|\hat{\rho}_{AR}\hat{H}|\beta\rangle\right) - \gamma\hat{\rho}_{\alpha\beta}, \tag{2.27}$$

where the last term is included to account for the decay of the atoms due to spontaneous emission. Here γ, considered to be the same for all the three levels, is the atomic decay constant. We assume that the atoms are removed from the cavity after they have decayed to a level other than the middle or bottom level. We then see that

$$\langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle = 0 \tag{2.28}$$

and hence Eq. (2.27) reduces to

$$\frac{d\hat{\rho}_{\alpha\beta}}{dt} = r_a\langle\alpha|\hat{\rho}_A(0)|\beta\rangle\hat{\rho}(t) - i\left(\langle\alpha|\hat{H}\hat{\rho}_{AR}|\beta\rangle - \langle\alpha|\hat{\rho}_{AR}\hat{H}|\beta\rangle\right) - \gamma\hat{\rho}_{\alpha\beta}. \tag{2.29}$$

Applying this equation and taking into account (2.4) and (2.13), we obtain

$$\frac{d\hat{\rho}_{ab}}{dt} = g(\hat{\rho}_{aa}\hat{a} - \hat{\rho}_{ac}\hat{a}^\dagger - \hat{a}\hat{\rho}_{bb}) - \frac{\Omega}{2}\hat{\rho}_{cb} - \gamma\hat{\rho}_{ab}, \tag{2.30}$$

$$\frac{d\hat{\rho}_{bc}}{dt} = g(\hat{a}^\dagger\hat{\rho}_{ac} + \hat{\rho}_{bb}\hat{a} - \hat{a}\hat{\rho}_{cc}) + \frac{\Omega}{2}\hat{\rho}_{ba} - \gamma\hat{\rho}_{bc}, \tag{2.31}$$

$$\frac{d\hat{\rho}_{aa}}{dt} = \frac{r_a}{2}(1-\eta)\hat{\rho}(t) - g(\hat{\rho}_{ab}\hat{a}^\dagger + \hat{a}\hat{\rho}_{ba}) - \frac{\Omega}{2}(\hat{\rho}_{ca} + \hat{\rho}_{ac}) - \gamma\hat{\rho}_{aa}, \tag{2.32}$$

$$\frac{d\hat{\rho}_{bb}}{dt} = g(\hat{a}^\dagger\hat{\rho}_{ab} + \hat{\rho}_{ba}\hat{a} - \hat{\rho}_{bc}\hat{a}^\dagger - \hat{a}\hat{\rho}_{cb}) - \gamma\hat{\rho}_{bb}, \tag{2.33}$$

$$\frac{d\hat{\rho}_{ac}}{dt} = \frac{r_a}{2}\sqrt{1-\eta^2}e^{i\theta}\hat{\rho}(t) + g(\hat{\rho}_{ab}\hat{a} - \hat{a}\hat{\rho}_{bc}) - \frac{\Omega}{2}(\hat{\rho}_{cc} - \hat{\rho}_{aa}) - \gamma\hat{\rho}_{ac}, \tag{2.34}$$

$$\frac{d\hat{\rho}_{cc}}{dt} = \frac{r_a}{2}(1+\eta)\hat{\rho}(t) + g(\hat{a}^\dagger\hat{\rho}_{bc} + \hat{\rho}_{cb}\hat{a}) + \frac{\Omega}{2}(\hat{\rho}_{ac} + \hat{\rho}_{ca}) - \gamma\hat{\rho}_{cc}. \tag{2.35}$$

We confine ourselves to linear analysis and this can be achieved by dropping the g terms in Eqs. (2.32), (2.33), (2.34), and (2.35). We then see that

$$\frac{d\hat{\rho}_{aa}}{dt} = \frac{r_a}{2}(1-\eta)\hat{\rho}(t) - \frac{\Omega}{2}(\hat{\rho}_{ca} + \hat{\rho}_{ac}) - \gamma\hat{\rho}_{aa}, \tag{2.36}$$

$$\frac{d\hat{\rho}_{bb}}{dt} = -\gamma\hat{\rho}_{bb}, \tag{2.37}$$

$$\frac{d\hat{\rho}_{ac}}{dt} = \frac{r_a}{2}\sqrt{1-\eta^2}e^{i\theta}\hat{\rho}(t) - \frac{\Omega}{2}(\hat{\rho}_{cc} - \hat{\rho}_{aa}) - \gamma\hat{\rho}_{ac}, \tag{2.38}$$

$$\frac{d\hat{\rho}_{cc}}{dt} = \frac{r_a}{2}(1+\eta)\hat{\rho}(t) + \frac{\Omega}{2}(\hat{\rho}_{ac} + \hat{\rho}_{ca}) - \gamma\hat{\rho}_{cc}. \tag{2.39}$$

In addition, imposing the good cavity limit ($\kappa \ll \gamma$) in which the atomic variables reach steady state in a relatively short period of γ^{-1}, we can take the time derivatives of such variables to be zero, while keeping the zero order atomic and cavity mode variables at time t. This procedure may be referred as adiabatic approximation scheme. Thus applying the adiabatic approximation scheme, we get from Eqs. (2.36), (2.37), (2.38), and (2.39) that

$$\hat{\rho}_{aa} = \frac{r_a}{2\gamma}(1-\eta)\hat{\rho} - \frac{\Omega}{2\gamma}(\hat{\rho}_{ac} + \hat{\rho}_{ca}), \tag{2.40}$$

$$\hat{\rho}_{bb} = 0, \tag{2.41}$$

$$\hat{\rho}_{ac} = \frac{r_a}{2\gamma}\sqrt{1-\eta^2}e^{i\theta}\hat{\rho} - \frac{\Omega}{2\gamma}(\hat{\rho}_{cc} - \hat{\rho}_{aa}), \tag{2.42}$$

$$\hat{\rho}_{cc} = \frac{r_a}{2\gamma}\sqrt{1-\eta^2}e^{i\theta}\hat{\rho} + \frac{\Omega}{2\gamma}(\hat{\rho}_{ac} + \hat{\rho}_{ca}). \tag{2.43}$$

With the aid of Eq. (2.42) along with its complex conjugate, we easily find

$$\hat{\rho}_{ac} + \hat{\rho}_{ca} = \frac{r_a}{\gamma}(\sqrt{1-\eta^2}\cos\theta)\hat{\rho} - \frac{\Omega}{\gamma}(\hat{\rho}_{cc} - \hat{\rho}_{aa}). \tag{2.44}$$

On account of this result, Eqs. (2.40) and (2.43) take the forms

$$\hat{\rho}_{aa} = \frac{2\gamma r_a}{\gamma^2+\Omega^2}(1-\eta)\hat{\rho} - \frac{\Omega r_a}{2\gamma^2+\Omega^2}(\sqrt{1-\eta^2}\cos\theta)\hat{\rho} + \frac{\Omega^2}{2\gamma^2+\Omega^2}\hat{\rho}_{cc}, \tag{2.45}$$

$$\hat{\rho}_{cc} = \frac{\gamma r_a}{2\gamma^2+\Omega^2}(1+\eta)\hat{\rho} + \frac{\Omega r_a}{2\gamma^2+\Omega^2}(\sqrt{1-\eta^2}\cos\theta)\hat{\rho} + \frac{\Omega^2}{2\gamma^2+\Omega^2}\hat{\rho}_{aa}. \tag{2.46}$$

Substitution of (2.45) into (2.46) and (2.46) into (2.45) results in

$$\hat{\rho}_{aa} = \frac{r_a}{2\gamma^2}\hat{\rho} - \frac{\Omega r_a}{2\gamma^2+2\Omega^2}(\sqrt{1-\eta^2}\cos\theta)\hat{\rho} - \frac{r_a\gamma}{2\gamma^2+2\Omega^2}\eta\hat{\rho}, \tag{2.47}$$

$$\hat{\rho}_{cc} = \frac{r_a}{2\gamma}\hat{\rho} + \frac{\Omega r_a}{2\gamma^2+2\Omega^2}(\sqrt{1-\eta^2}\cos\theta)\hat{\rho} - \frac{r_a\gamma}{2\gamma^2+2\Omega^2}\eta\hat{\rho}. \tag{2.48}$$

Furthermore, application of (2.47) and (2.48) in Eq. (2.42) yields

$$\hat{\rho}_{ac} = \frac{r_a(2\gamma^2+\Omega^2)}{2\gamma(2\gamma^2+2\Omega^2)}\sqrt{1-\eta^2}e^{i\theta}\hat{\rho} - \frac{\Omega r_a}{2\gamma^2+2\Omega^2}\eta\hat{\rho} - \frac{r_a\Omega^2}{2\gamma(2\gamma^2+2\Omega^2)}\sqrt{1-\eta^2}e^{-i\theta}\hat{\rho}. \tag{2.49}$$

Now combination of (2.30), (2.41), (2.47), and (2.49) as well as (2.31), (2.42), (2.48), and (2.49) leads to

$$\frac{d\hat{\rho}_{ab}}{dt} = \frac{gr_a}{\gamma}\Big[\frac{\Omega\gamma}{2\gamma^2+2\Omega^2}(\eta)\hat{\rho}\hat{a}^\dagger + \frac{\Omega^2}{2(2\gamma^2+2\Omega^2)}\sqrt{1-\eta^2}e^{-i\theta}\hat{\rho}\hat{a}^\dagger - \frac{2\gamma^2+\Omega^2}{2(2\gamma^2+2\Omega^2)}\sqrt{1-\eta^2}e^{i\theta}\hat{\rho}\hat{a}^\dagger$$
$$+\frac{1}{2}\hat{\rho}\hat{a} - \frac{\Omega\gamma}{2\gamma^2+2\Omega^2}(\sqrt{1-\eta^2}\cos\theta)\hat{\rho}\hat{a} - \frac{\gamma^2}{2\gamma^2+2\Omega^2}\eta\hat{\rho}\hat{a}\Big] - \frac{\Omega}{2}\hat{\rho}_{cb} - \gamma\hat{\rho}_{ab}, \tag{2.50}$$

$$\frac{d\hat{\rho}_{bc}}{dt} = \frac{gr_a}{\gamma}\Big[\frac{2\gamma^2+\Omega^2}{2(2\gamma^2+2\Omega^2)}\sqrt{1-\eta^2}e^{i\theta}\hat{a}^\dagger\hat{\rho} - \frac{\Omega\gamma}{2\gamma^2+2\Omega^2}\eta\hat{a}^\dagger\hat{\rho} - \frac{\Omega^2}{2(2\gamma^2+2\Omega^2)}\sqrt{1-\eta^2}e^{-i\theta}\hat{a}^\dagger\hat{\rho}$$
$$-\frac{1}{2}\hat{a}\hat{\rho} - \frac{\Omega\gamma}{2\gamma^2+2\Omega^2}(\sqrt{1-\eta^2}\cos\theta)\hat{a}\hat{\rho} - \frac{\gamma^2}{2\gamma^2+2\Omega^2}(1+\eta)\hat{a}\hat{\rho}\Big] + \frac{\Omega}{2}\hat{\rho}_{ba} - \gamma\hat{\rho}_{bc}. \tag{2.51}$$

Using once more the adiabatic approximation scheme, we readily get

$$\hat{\rho}_{ab} = \frac{gr_a}{\gamma^2}\Big[\frac{\Omega\gamma}{2\gamma^2+2\Omega^2}\eta\hat{\rho}\hat{a}^\dagger + \frac{\Omega^2}{2(2\gamma^2+2\Omega^2)}\sqrt{1-\eta^2}e^{-i\theta}\hat{\rho}\hat{a}^\dagger - \frac{2\gamma^2+\Omega^2}{2(2\gamma^2+2\Omega^2)}\sqrt{1-\eta^2}e^{i\theta}\hat{\rho}\hat{a}^\dagger$$
$$+\frac{1}{2}\hat{\rho}\hat{a} - \frac{\Omega\gamma}{2\gamma^2+2\Omega^2}(\sqrt{1-\eta^2}\cos\theta)\hat{\rho}\hat{a} - \frac{\gamma^2}{2\gamma^2+2\Omega^2}\eta\hat{\rho}\hat{a}\Big] - \frac{\Omega}{2\gamma}\hat{\rho}_{cb}, \tag{2.52}$$

$$\hat{\rho}_{bc} = \frac{gr_a}{\gamma^2}\Big[\frac{2\gamma^2+\Omega^2}{2(2\gamma^2+2\Omega^2)}\sqrt{1-\eta^2}e^{i\theta}\hat{a}^\dagger\hat{\rho} - \frac{\Omega\gamma}{2\gamma^2+2\Omega^2}(\eta)\hat{a}^\dagger\hat{\rho} - \frac{\Omega^2}{2(2\gamma^2+2\Omega^2)}\sqrt{1-\eta^2}e^{-i\theta}\hat{a}^\dagger\hat{\rho}$$
$$-\frac{1}{2}\hat{a}\hat{\rho} - \frac{\Omega\gamma}{2\gamma^2+2\Omega^2}(\sqrt{1-\eta^2}\cos\theta)\hat{a}\hat{\rho} - \frac{\gamma^2}{2\gamma^2+2\Omega^2}\eta\hat{a}\hat{\rho}\Big] + \frac{\Omega}{2\gamma}\hat{\rho}_{ba}. \tag{2.53}$$

Moreover, substitution of the complex conjugate of (2.52) into (2.53) and the complex conjugate of (2.53) into (2.54) yields

$$\hat{\rho}_{ab} = \frac{A}{4Bg}\Bigg[\left(\frac{\Omega}{2\gamma}\eta + \frac{\Omega}{2\gamma}(1+\frac{\Omega^2}{\gamma^2}) - (1+\frac{\Omega^2}{4\gamma^2})\sqrt{1-\eta^2}e^{i\theta} + \frac{3\Omega^2}{4\gamma^2}\sqrt{1-\eta^2}e^{-i\theta}\right)\hat{\rho}\hat{a}^\dagger$$
$$+\left((\frac{\Omega^2}{2\gamma^2}-1)\eta + (1+\frac{\Omega^2}{\gamma^2}) + \frac{\Omega}{2\gamma}(\frac{\Omega^2}{2\gamma^2}-1)\sqrt{1-\eta^2}e^{i\theta} - (\frac{\Omega}{\gamma}+\frac{\Omega^3}{4\gamma^3})\sqrt{1-\eta^2}e^{i\theta}\right)\hat{\rho}\hat{a}\Bigg], \tag{2.54}$$

$$\hat{\rho}_{bc} = \frac{A}{4Bg}\left[\left(\frac{\Omega}{2\gamma}(\frac{\Omega^2}{\gamma^2}-1) - \frac{\Omega^3}{2\gamma^3}\eta + (1+\frac{\Omega^2}{4\gamma^2})\sqrt{1-\eta^2}e^{i\theta} - \frac{3\Omega^2}{4\gamma^2}\sqrt{1-\eta^2}e^{i\theta}\right)\hat{a}^\dagger\hat{\rho}\right.$$
$$\left.+\left(\frac{\Omega}{2\gamma}(\frac{\Omega^2}{2\gamma^2}-1)\sqrt{1-\eta^2}e^{i\theta} - (\frac{\Omega}{\gamma}+\frac{\Omega^3}{4\gamma^3})\sqrt{1-\eta^2}e^{-i\theta} - (1+\frac{\Omega^2}{\gamma^2}) + (\frac{3\Omega^2}{2\gamma^2}-1)\eta\right)\hat{a}\hat{\rho}\right], \tag{2.55}$$

where

$$B = (1+\frac{\Omega^2}{4\gamma^2})(1+\frac{\Omega^2}{\gamma}) \tag{2.56}$$

and

$$A = \frac{2r_a g^2}{\gamma^2} \tag{2.57}$$

is the linear gain coefficient. Finally, on account of (2.54), (2.55), and for $\theta = 0$, the master equation for the cavity mode of a degenerate three-level laser given by (2.25) takes the form [21]

$$\frac{d\hat{\rho}}{dt} = R\left(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\right) + S\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\right)$$
$$+ U\left(\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger + \hat{a}\hat{\rho}\hat{a} - \hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^{\dagger 2}\right) + V\left(\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger + \hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^2 - \hat{a}^{\dagger 2}\hat{\rho}\right), \tag{2.58}$$

in which

$$R = \frac{A}{4B}\left((\frac{\Omega^2}{2\gamma^2}-1)\eta - \frac{3\Omega}{2\gamma}\sqrt{1-\eta^2} + (1+\frac{\Omega^2}{\gamma^2})\right), \tag{2.59}$$

$$S = \frac{A}{4B}\left(\frac{2\kappa B}{A} + (1-\frac{\Omega^2}{2\gamma^2})\eta + \frac{3\Omega}{2\gamma}\sqrt{1-\eta^2} + (1+\frac{\Omega^2}{\gamma^2})\right), \tag{2.60}$$

$$U = \frac{A}{4B}\left(\frac{3\Omega}{2\gamma}\eta + (\frac{\Omega^2}{2\gamma^2}-1)\sqrt{1-\eta^2} + (\frac{\Omega}{2\gamma}+\frac{\Omega^3}{2\gamma^3})\right), \tag{2.61}$$

$$V = \frac{A}{4B}\left(\frac{3\Omega}{2\gamma}\eta + (\frac{\Omega^2}{2\gamma^2}-1)\sqrt{1-\eta^2} - (\frac{\Omega}{2\gamma}+\frac{\Omega^3}{2\gamma^3})\right). \tag{2.62}$$

On the other hand, a degenerate parametric amplifier with the pump mode treated classically is described by the Hamiltonian

$$\hat{H}_1 = \frac{i\varepsilon}{2}\left(\hat{a}^{\dagger 2} - \hat{a}^2\right), \tag{2.63}$$

in which $\varepsilon = 2\lambda'\beta_0$, with λ' being the coupling constant between the pump mode and the nonlinear crystal and β_0 being the amplitude of the pump mode. Moreover, the interaction of the cavity mode with a strong driving light can be described by the Hamiltonian

$$\hat{H}_2 = i\mu\left(\hat{a}^\dagger - \hat{a}\right), \tag{2.64}$$

where μ, considered to be real and constant, is proportional to the amplitude of the driving light. The master equation associated with the interactions described by the Hamiltonians (2.63) and (2.64) has the form [47]

$$\frac{d\hat{\rho}}{dt} = \frac{\varepsilon}{2}\left(\hat{\rho}\hat{a}^2 - \hat{a}^2\hat{\rho} + \hat{a}^{\dagger 2}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger 2}\right) + \mu\left(\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger + \hat{\rho}\hat{a} - \hat{a}\hat{\rho}\right). \tag{2.65}$$

Hence on account of Eqs. (2.58) and (2.65), the master equation for the cavity mode under consideration turns out to be

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= \mu\left(\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger + \hat{\rho}\hat{a} - \hat{a}\hat{\rho}\right) + \frac{\varepsilon}{2}\left(\hat{\rho}\hat{a}^2 - \hat{a}^2\hat{\rho} + \hat{a}^{\dagger 2}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger 2}\right)\\ &+ R\left(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\right) + S\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\right)\\ &+ U\left(\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger + \hat{a}\hat{\rho}\hat{a} - \hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^{\dagger 2}\right) + V\left(\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger + \hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^2 - \hat{a}^{\dagger 2}\hat{\rho}\right).\end{aligned} \tag{2.66}$$

2.1.2 c-number Langevin equations

Now employing the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{A}), \tag{2.67}$$

along with the master equation (2.66), one readily finds the following equations

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = (R-S)\langle\hat{a}(t)\rangle + (U-V+\varepsilon)\langle\hat{a}^\dagger(t)\rangle + \mu, \tag{2.68}$$

$$\frac{d}{dt}\langle\hat{a}^2(t)\rangle = 2(R-S)\langle\hat{a}^2(t)\rangle + 2(U-V+\varepsilon)\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle + 2\mu\langle\hat{a}(t)\rangle + (\varepsilon - 2V), \tag{2.69}$$

and

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle &= 2(R-S)\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle + (U-V+\varepsilon)(\langle\hat{a}^{\dagger 2}(t)\rangle + \langle\hat{a}^2(t)\rangle)\\ &+ \mu\big(\langle\hat{a}^\dagger(t)\rangle + \langle\hat{a}(t)\rangle\big) + 2R.\end{aligned} \tag{2.70}$$

We note that the operators in the above set of equations are in the normal-order. The c-number equations corresponding to these equations are

$$\frac{d}{dt}\langle\alpha(t)\rangle = -(S-R)\langle\alpha(t)\rangle + (U-V+\varepsilon)\langle\alpha^*(t)\rangle + \mu, \tag{2.71}$$

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -2(S-R)\langle\alpha^2(t)\rangle + 2(U-V+\varepsilon)\langle\alpha^*(t)\alpha(t)\rangle + 2\mu\langle\alpha(t)\rangle + (\varepsilon - 2V), \tag{2.72}$$

and

$$\begin{aligned}\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle &= -2(S-R)\langle\alpha^*(t)\alpha(t)\rangle + (U-V+\varepsilon)(\langle\alpha^{*2}(t)\rangle + \langle\alpha^2(t)\rangle)\\ &+ \mu\big(\langle\alpha^*(t)\rangle + \langle\alpha(t)\rangle\big) + 2R.\end{aligned} \tag{2.73}$$

We claim that the equation of evolution of $\alpha(t)$ (c-number Langevin equations) can be obtained from that of $\langle\alpha(t)\rangle$. This can be achieved by dropping the angular brackets in Eq. (2.71), and adding the noise forces $f_\alpha(t)$, so that

$$\frac{d}{dt}\alpha(t) = -(S-R)\alpha(t) + (U-V+\varepsilon)\alpha^*(t) + f(t) + \mu. \tag{2.74}$$

We next seek to determine the properties of the noise force $f(t)$. We observe that Eq. (2.71) and the expectation value of (2.74) will have the same form if

$$\langle f(t)\rangle = 0. \tag{2.75}$$

Using Eq. (2.74) together with the relation

$$\frac{d}{dt}\langle\alpha^2\rangle = 2\langle\alpha\frac{d\alpha}{dt}\rangle, \tag{2.76}$$

one readily gets

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -2(S-R)\langle\alpha^2(t)\rangle + 2(U-V+\varepsilon)\langle\alpha(t)\alpha^*(t)\rangle + 2\langle\alpha(t)f(t)\rangle + 2\mu\langle\alpha(t)\rangle. \tag{2.77}$$

Comparison of this with Eq. (2.72) indicates that

$$\langle\alpha(t)f(t)\rangle = \frac{1}{2}(\varepsilon - 2V). \tag{2.78}$$

A formal solution of Eq. (2.74) can be written as

$$\alpha(t) = \alpha(0)e^{-(S-R)t} + \int_0^t e^{-(S-R)(t-t')}\big[(U-V+\varepsilon)\alpha^*(t') + f(t') + \mu\big]dt'. \tag{2.79}$$

Multiplying this equation by $f(t)$ and taking the expectation value of the resulting expression, we have

$$\begin{aligned}\langle\alpha(t)f(t)\rangle &= \langle\alpha(0)f(t)\rangle e^{-(S-R)t} \\ &\quad + \int_0^t e^{-(S-R)(t-t')}[(U-V+\varepsilon)\langle\alpha^*(t')f(t)\rangle + \langle f(t')f(t)\rangle + \mu\langle f(t)\rangle]dt'.\end{aligned} \tag{2.80}$$

On account of Eq. (2.75) along with the fact that the noise force $f(t)$ at time t does not affect the system variables at earlier times, we find

$$\langle\alpha(t)f(t)\rangle = \int_0^t e^{-(S-R)(t-t')}\langle f(t')f(t)\rangle dt'. \tag{2.81}$$

In view of (2.78), we see that

$$\int_0^t e^{-(S-R)(t-t')}\langle f(t')f(t)\rangle dt' = \frac{1}{2}(\varepsilon - 2V). \tag{2.82}$$

Now on the basis of the relation

$$\int_0^t e^{-a(t-t')}\langle f(t')g(t)\rangle dt' = D, \tag{2.83}$$

we assert that [21]

$$\langle f(t)g(t)\rangle = 2D\delta(t-t'), \tag{2.84}$$

where a is a constant and D is a constant or some function of the time t. We then see that

$$\langle f(t')f(t)\rangle = (\varepsilon - 2V)\delta(t-t'). \tag{2.85}$$

Moreover, applying Eq. (2.74) and its complex conjugate along with the relation

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = \langle\alpha^*(t)\frac{d\alpha(t)}{dt}\rangle + \langle\alpha(t)\frac{d\alpha^*(t)}{dt}\rangle, \tag{2.86}$$

we get

$$\frac{d}{dt}\langle\alpha^*(t)\alpha\rangle(t) = -2(S-R)\langle\alpha^*(t)\alpha(t)\rangle + (U-V+\varepsilon)(\langle\alpha^{*2}(t)\rangle + \langle\alpha^2(t)\rangle)$$
$$+ \langle\alpha^*(t)f(t)\rangle + \langle\alpha(t)f^*(t)\rangle + \mu\big(\langle\alpha^*(t)\rangle + \langle\alpha(t)\rangle\big), \tag{2.87}$$

so that inspection of Eqs. (2.73) and (2.87) shows that

$$\langle\alpha^*(t)f(t)\rangle + \langle\alpha(t)f^*(t)\rangle = 2R. \tag{2.88}$$

In addition, employing (2.79) and its complex conjugate, we easily find

$$\langle\alpha^*(t)f(t)\rangle = \int_0^t e^{-(S-R)(t-t')}\langle f^*(t')f(t)\rangle dt', \tag{2.89}$$

$$\langle\alpha(t)f^*(t)\rangle = \int_0^t e^{-(S-R)(t-t')}\langle f(t')f^*(t)\rangle dt'. \tag{2.90}$$

Now taking into account (2.88), (2.89), (2.90), and assuming that

$$\langle f^*(t')f(t)\rangle = \langle f(t')f^*(t)\rangle, \tag{2.91}$$

we arrive at

$$\int_0^t e^{-(S-R)(t-t')}\langle f^*(t')f(t)\rangle dt' = \int_0^t e^{-(S-R)(t-t')}\langle f(t')f^*(t)\rangle dt' = R. \tag{2.92}$$

Therefore, with the help of (2.83) and (2.84), we obtain

$$\langle f^*(t')f(t)\rangle = \langle f(t')f^*(t)\rangle = 2R\delta(t-t'). \tag{2.93}$$

We note that (2.75), (2.85), and (2.93) describe the correlation properties of the noise force $f(t)$ associated with normal ordering.

Now introducing a new variable defined by

$$\alpha_{\pm}(t) = \alpha^*(t) \pm \alpha(t), \tag{2.94}$$

it can be shown using (2.74) and its complex conjugate that

$$\frac{d}{dt}\alpha_{\pm}(t) = -\lambda_{\mp}\alpha_{\pm}(t) + (\varepsilon \pm \varepsilon) + f^*(t) \pm f(t), \tag{2.95}$$

where

$$\lambda_{\mp} = (S - R) \mp (U - V + \varepsilon). \tag{2.96}$$

The solution of Eq. (2.95) can be expressed as

$$\alpha_{\pm}(t) = \alpha_{\pm}(0)e^{-\lambda_{\mp}t} + \int_0^t e^{-\lambda_{\mp}(t-t')}\big[\mu \pm \mu + f^*(t') \pm f(t')\big]dt'. \tag{2.97}$$

In view of (2.94), there follows

$$\alpha(t) = B_+(t)\alpha(0) + B_-(t)\alpha^*(0) + E(t) + F(t), \tag{2.98}$$

in which

$$B_{\pm}(t) = \frac{1}{2}\big(e^{-\lambda_- t} \pm e^{-\lambda_+ t}\big), \tag{2.99}$$

$$E(t) = \frac{\varepsilon}{\lambda_-}(1 - e^{-\lambda_- t}), \tag{2.100}$$

and

$$F(t) = F_+(t) + F_-(t) \tag{2.101}$$

with

$$F_{\pm} = \frac{1}{2}\int_0^t e^{-\lambda_{\mp}(t-t')}\big(f(t') \pm f^*(t')\big)dt'. \tag{2.102}$$

2.2 Quadrature Squeezing

Applying the solutions of the c-number Langevin equations along with the correlation properties of the noise forces, we calculate the quadrature variance for the cavity mode. We also determine, employing the same solutions and the input-output relation, the quadrature variance and the squeezing spectrum for the output mode.

2.2.1 Quadrature variance of the cavity mode

The squeezing properties of a single mode-light are described by two quadrature operators

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a} \tag{2.103}$$

and

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}). \tag{2.104}$$

These operators satisfy the commutation relation

$$[\hat{a}_+, \hat{a}_-] = 2i. \tag{2.105}$$

A single-mode light is said to be in a squeezed state if either $\Delta a_+ < 1$ and $\Delta a_- > 1$ or $\Delta a_+ > 1$ and $\Delta a_- < 1$ such that $\Delta a_+ \Delta a_- \geq 1$. The quadrature variance is defined by

$$\Delta a_\pm^2(t) = \langle \hat{a}_\pm^2(t)\rangle - \langle \hat{a}_\pm(t)\rangle^2. \tag{2.106}$$

On account of (2.103) and (2.104), Eq. (2.106) can be written in the normal order as

$$\Delta a_\pm^2(t) = 1 \pm \langle \hat{a}^{\dagger 2}(t) + \hat{a}^2(t) \pm 2\hat{a}^\dagger(t)\hat{a}(t)\rangle \mp \langle \hat{a}^\dagger(t) \pm \hat{a}(t)\rangle^2. \tag{2.107}$$

This can be expressed in terms of c-number variables associated with the normal ordering as

$$\Delta a_\pm^2(t) = 1 \pm \langle \alpha_\pm(t), \alpha_\pm(t)\rangle, \tag{2.108}$$

in which $\langle \alpha_\pm(t), \alpha_\pm(t)\rangle = \langle \alpha_\pm^2(t)\rangle - \langle \alpha_\pm(t)\rangle^2$, with $\alpha_\pm(t)$ given by Eq. (2.94).

Assuming the cavity mode to be initially in a vacuum state and taking into account (2.97) along with Eq. (2.75), we obtain

$$\langle \alpha_\pm(t)\rangle = \frac{\mu \pm \mu}{\lambda_\pm}\big(1 - e^{-\lambda_\mp t}\big) \tag{2.109}$$

and at steady state

$$\langle \alpha_\pm(t)\rangle_{ss} = \frac{\mu \pm \mu}{\lambda_\mp}. \tag{2.110}$$

Furthermore, employing Eq. (2.95), we have

$$\frac{d}{dt}\langle \alpha_\pm^2(t)\rangle = -2\lambda_\mp \langle \alpha_\pm^2(t)\rangle + 2\big(\mu + \mu\big)\langle \alpha_\pm(t)\rangle + 2\big(\langle \alpha_\pm(t) f^*(t)\rangle \pm \langle \alpha_\pm(t) f(t)\rangle\big). \tag{2.111}$$

With the help of Eqs. (2.97) and (2.109), one can write

$$\begin{aligned}\frac{d}{dt}\langle \alpha_\pm^2(t)\rangle &= -2\lambda_\mp \langle \alpha_\pm^2(t)\rangle + 2\frac{(\mu \pm \mu)^2}{\lambda_\mp}\big(1 - e^{-\lambda_\mp t}\big) + 2\big(\langle \alpha_\pm(0) f^*(t)\rangle \pm \langle \alpha_\pm(0) f(t)\rangle\big)e^{-\lambda_\mp t}\\ &+ 2\bigg(\int_0^t e^{-\lambda_\mp (t-t')}\big[\mu\langle f^*(t)\rangle \pm \mu\langle f^*(t)\rangle + \langle f^*(t') f^*(t)\rangle \pm \langle f(t')^* f(t)\rangle\big]dt'\bigg)\\ &\pm 2\bigg(\int_0^t e^{-\lambda_\mp (t-t')}\big[\mu\langle f(t)\rangle \pm \mu\langle f(t)\rangle + \langle f^*(t') f(t)\rangle \pm \langle f(t') f(t)\rangle\big]dt'\bigg).\end{aligned} \tag{2.112}$$

On account of Eqs. (2.75), (2.85), and (2.93) along with the fact that the noise force $f(t)$ can not affect the system variables at the earlier times, we get

$$\frac{d}{dt}\langle\alpha_{\pm}^2(t)\rangle = -2\lambda_{\mp}\langle\alpha_{\pm}^2(t)\rangle + 2\frac{(\mu\pm\mu)^2}{\lambda_{\mp}}\big(1-e^{-\lambda_{\mp}t}\big) + 2\bigg((\varepsilon-2V\pm 2R)\int_0^t e^{-\lambda_{\mp}(t-t')}\delta(t-t')dt' \pm 2\big(2R\pm(\varepsilon-2V)\big)\int_0^t e^{-\lambda_{\pm}(t-t')}\delta(t-t')dt'\bigg), \tag{2.113}$$

so that upon carrying out the integration, we obtain

$$\frac{d}{dt}\langle\alpha_{\pm}^2(t)\rangle = -2\lambda_{\mp}\langle\alpha_{\pm}^2(t)\rangle + 2\frac{(\mu\pm\mu)^2}{\lambda_{\mp}}\big(1-e^{-\lambda_{\mp}t}\big) + 2\big(\varepsilon-2V\pm 2R\big). \tag{2.114}$$

The steady-state solution of this equation is found to be

$$\langle\alpha_{\pm}^2(t)\rangle_{ss} = \frac{(\varepsilon-2V\pm 2R)}{\lambda_{\mp}} + \frac{(\mu\pm\mu)^2}{\lambda_{\mp}^2}. \tag{2.115}$$

Finally, substitution of (2.110) and (2.115) into Eq. (2.108), the quadrature variance takes, at steady state, the form

$$\Delta a_{\pm ss}^2 = 1 \pm \frac{\varepsilon-2V\pm 2R}{\lambda_{\mp}}. \tag{2.116}$$

On the other hand, with the aid of Eqs. (2.59)-(2.62), expression (2.96) goes over into

$$\lambda_{\pm} = \frac{\kappa}{2} + \frac{\frac{3A\Omega}{\gamma}\sqrt{1-\eta^2} + A(2-\frac{\Omega}{\gamma})\eta \pm \frac{A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2})}{4(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})} \pm \varepsilon. \tag{2.117}$$

Therefore, application of Eqs. (2.59), (2.62), and (2.117) in (2.116) leads to

$$\Delta a_{\pm}^2 = \frac{2\kappa(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2}) \pm A(2-\frac{\Omega^2}{\gamma^2})\sqrt{1-\eta^2} + 2A(1+\frac{\Omega^2}{\gamma^2}) \mp \frac{3A\Omega}{\gamma}\eta}{(2\kappa\mp 4\varepsilon)(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2}) + \frac{3A\Omega}{\gamma}\sqrt{1-\eta^2} \mp \frac{A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2}) + A(2-\frac{\Omega^2}{\gamma^2})\eta}. \tag{2.118}$$

We see from Eq. (2.118) that the driving mode has no effect on the quadrature variance. Inspection of Eq. (2.117) shows that λ_+ is nonnegative while λ_- can be positive, negative, or zero. Rewriting (2.117) as

$$\lambda_- = G - \varepsilon, \tag{2.119}$$

with

$$G = \frac{\kappa}{2} + \frac{\frac{3A\Omega}{\gamma}\sqrt{1-\eta^2} + A(2-\frac{\Omega}{\gamma})\eta - \frac{A\Omega}{\gamma}(1+\frac{\beta^2}{\gamma^2})}{4(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})}. \tag{2.120}$$

We observe that the equation of evolution of $\alpha_-(t)$, described by (2.95), does not have a well-behaved solution for $\varepsilon > G$. We then identify $\varepsilon = G$ as the threshold condition.

We next proceed to analyze the quadrature variance of the light generated by the system operating below threshold. Using Eqs. (2.118) and (2.120) and writing a simple Matlab program, we have obtained for $A = 100$ and $\kappa = 0.8$ the values of η, $\frac{\Omega}{\gamma}$, ε, and Δa_-^2 for which Eq.

(2.95) has a solution. It so turns out that this equation has a solution for $-0.5 \leq \eta \leq 1$ and for $0 \leq \frac{\Omega}{\gamma} \leq 1.4$. We indicate in the table below the values of η, $\frac{\Omega}{\gamma}$, and ε corresponding to the two smallest values of the quadrature variance.

η	$\frac{\Omega}{\gamma}$	G	ε	Δa_-^2
0	0.1000	5.3135	5.3000	0.0731
0.1000	0	5.4000	5.3000	0.0608

Tab. 2.1: Values of Δa_-^2 for $A = 100$ and $\kappa = 0.8$.

We note that when there are equal number of atoms initially in the top and bottom levels ($\eta = 0$), the maximum interacavity squeezing is 93% below the coherent-state level for $\frac{\Omega}{\gamma} = 0.1$. We would like to point out that this result is in complete agreement with that obtained in Ref. [44]. On the other hand, when there are slightly more atoms initially in the bottom level than in the top level ($\eta = 0.1$), the maximum interacavity squeezing is found to be 94% below the coherent-state level for $\frac{\Omega}{\gamma} = 0$.

It is interesting to consider some special cases. We first inspect the case in which the nonlinear crystal is removed from the cavity, with the top and bottom levels of the atoms coupled by the pump mode. Thus upon setting $\varepsilon = 0$ (with $\beta_0 \neq 0$) in Eq. (2.118), we get

$$\Delta a_\pm^2 = \frac{2\kappa(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2}) \pm A(2-\frac{\Omega^2}{\gamma^2})\sqrt{1-\eta^2} + 2A(1+\frac{\Omega^2}{\gamma^2}) \mp \frac{3A\Omega}{\gamma}\eta}{2\kappa(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2}) + \frac{3A\Omega}{\gamma}\sqrt{1-\eta^2} \mp \frac{A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2}) + A(2-\frac{\Omega^2}{\gamma^2})\eta}. \tag{2.121}$$

It can be shown using (2.121) that for $\eta = 0$, $\frac{\Omega}{\gamma} = 0$, and any values of A and κ the light generated is not in a squeezed state. However, for $\eta = 0$, $A = 100$, and $\kappa = 0.8$ we readily get applying the same equation that the maximum squeezing to be 89% for $\frac{\Omega}{\gamma} = 0.1$. We therefore infer that the squeezing in this case is exclusively due to the coupling of the top and bottom levels. Moreover, for $\eta = 0.1$ and for the above values of A and κ, we find the maximum squeezing to be 88% for $\frac{\Omega}{\gamma} = 0$. This squeezing is due to the specific coherent superposition of the top and bottom levels.

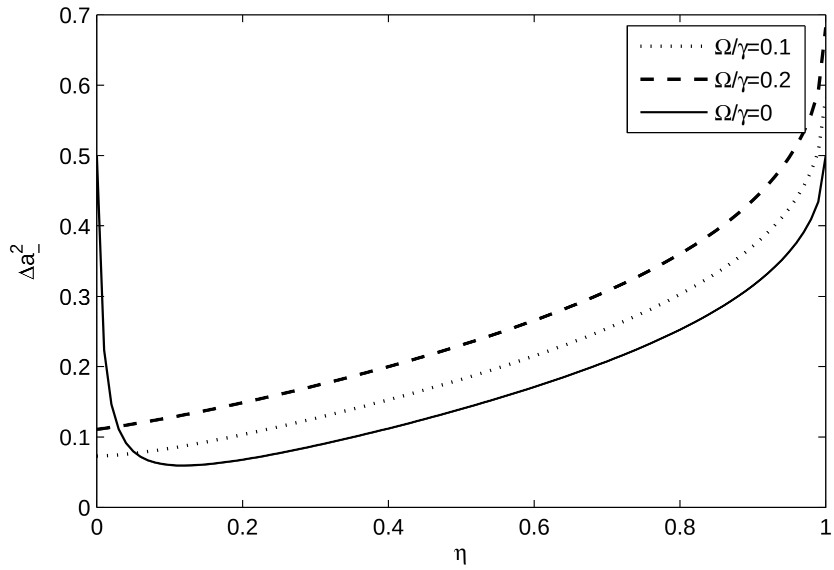

Fig. 2.2: Plots of the quadrature variance [Eq. (2.121)] versus η for $A = 100$, $\kappa = 0.8$, and different values of $\frac{\Omega}{\gamma}$.

In addition, inspection of the plots in Fig. 2.2 shows that for small values of the amplitude of the pump mode, the coupling of the top and bottom levels significantly enhances the intracavity squeezing particularly when there are nearly equal number of atoms initially in the top and bottom levels (around $\eta = 0$). Otherwise, it leads to the decrease in the intercavity squeezing. On the other hand, for a strong pump mode ($\Omega \gg \gamma$), Eq. (2.121) takes the form

$$\Delta a_{\pm}^2 = \frac{\frac{\kappa}{2} \pm \frac{A\gamma^2}{\Omega^2}\sqrt{1-\eta^2} + \frac{2A\gamma^2}{\Omega^2} \mp \frac{3A\gamma^3}{\Omega^3}\eta}{\frac{\kappa}{2} + \frac{3A\gamma^2}{\Omega^2}\sqrt{1-\eta^2} \mp \frac{A\gamma}{\Omega} - \frac{A\gamma^2}{\Omega^2}\eta}, \tag{2.122}$$

so that on dropping the terms $\frac{A\gamma^2}{\Omega^2}$ and $\frac{A\gamma^3}{\Omega^3}$, Eq. (2.122) reduces to

$$\Delta a_{\pm}^2 = \frac{1}{1 \mp \frac{2A\gamma}{\Omega\kappa}}. \tag{2.123}$$

This result indicates that a degenerate three-level laser pumped by a very strong light behaves like a degenerate parametric oscillator [10].

Furthermore, we consider the case in which the pump mode emerging from the nonlinear crystal does not couple the top and bottom levels. Hence upon setting $\frac{\Omega}{\gamma} = 0$ (with $\beta_0 \neq 0$), Eq.(2.118) reduces to [43]

$$\Delta a_{+}^2 = \frac{\kappa + A[1 + (1-\eta)^{1/2}]}{A\eta + \kappa - 2\varepsilon} \tag{2.124}$$

and

$$\Delta a_{-}^{2} = \frac{\kappa - A[1 + (1 - \eta)^{1/2}]}{A\eta + \kappa + 2\varepsilon}. \tag{2.125}$$

It is apparent that ε is the only parameter representing the parametric amplifier. And inspection of Eq. (2.125) shows that the effect of the parametric amplifier is to decrease the value of the quadrature variance Δa_{-}^{2}.

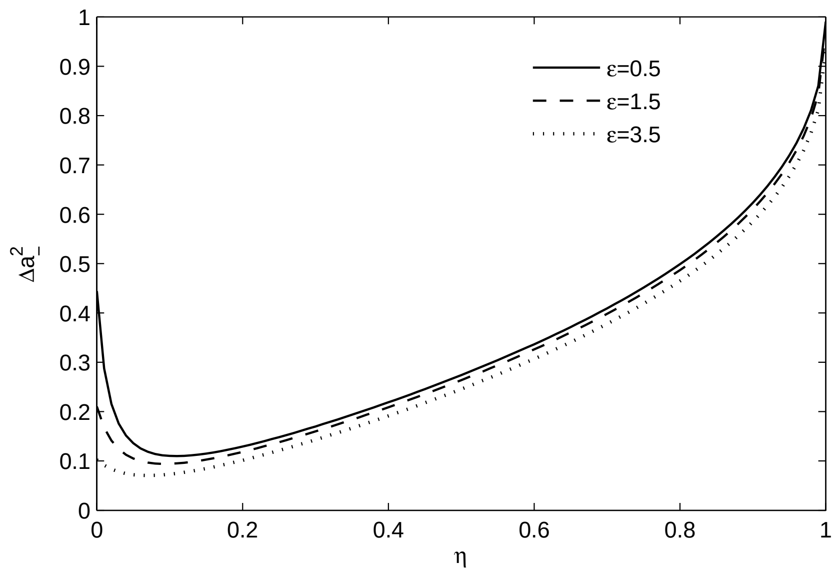

Fig. 2.3: Plots of the quadrature variance [Eq. (2.125)] versus η for $A = 100$, $\kappa = 0.8$, and different values of ε

The plots in Fig. 2.3 indicate that the presence of the nonlinear crystal leads to better squeezing. In addition, applying Eq. (2.125) with $A = 100$, $k = 0.8$, and $\eta = 0.1$ the maximum interacavity squeezing is 94% below the coherent-state level for $\varepsilon = 5.3$.

2.2.2 Quadrature variance of the output mode

We next seek to calculate the quadrature variance of the output mode. According to the derivation presented in Appendix A, the quadrature variance for the output mode is expressible as

$$\Delta a_{\pm out}^{2} = \kappa \Delta a_{\pm}^{2} + (1 - \kappa)\Delta a_{in\pm}^{2}, \tag{2.126}$$

where the first and the second terms represent the quadrature variance of the transmitted and reflected output mode. Taking into account Eq. (2.116) and the fact that the quadrature variance of the vacuum reservoir is unity, the quadrature variance for the output mode of the system under consideration takes at steady sate the form

$$\Delta a^2_{\pm out}(t) = 1 \pm \frac{\kappa(\varepsilon - 2V \pm 2R)}{\lambda_\mp}. \quad (2.127)$$

Finally, employing (2.59), (2.62), and (2.117) in Eq. (2.127), we obtain

$$\Delta a^2_{+out}(t) = \frac{(2\kappa - 4\varepsilon + 4\varepsilon\kappa)(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2}) + A(2\kappa - \frac{3\kappa\Omega}{\gamma} - \frac{\kappa\Omega^2}{\gamma^2} + \frac{3\Omega}{\gamma})(1-\eta^2)^{1/2}}{(2\kappa - 4\varepsilon)(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2}) + \frac{3A\Omega}{\gamma}(1-\eta^2)^{1/2} - \frac{A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2}) + A(2-\frac{\Omega}{\gamma})\eta}$$
$$+ \frac{A(\frac{\kappa\Omega^2}{\gamma^2} - \frac{3\kappa\Omega}{\gamma} - 2\kappa + 2 - \frac{\Omega^2}{\gamma^2})\eta + A(\frac{\kappa\Omega^3}{\gamma^3} + \frac{2\kappa\Omega^2}{\gamma^2} + \frac{\kappa\Omega}{\gamma} + 2\kappa - \frac{\Omega}{\gamma} - \frac{\Omega^3}{\gamma^3})}{(2\kappa - 4\varepsilon)(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2}) + \frac{3A\Omega}{\gamma}(1-\eta^2)^{1/2} - \frac{A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2}) + A(2-\frac{\Omega}{\gamma})\eta} \quad (2.128)$$

and

$$\Delta a^2_{-out}(t) = \frac{(2\kappa + 4\varepsilon - 4\varepsilon\kappa)(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2}) + A(\frac{\kappa\Omega^2}{\gamma^2} - 2\kappa - \frac{3\kappa\Omega}{\gamma} + \frac{3\Omega}{\gamma})(1-\eta^2)^{1/2}}{(2\kappa + 4\varepsilon)(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2}) + \frac{3A\Omega}{\gamma}(1-\eta^2)^{1/2} + \frac{A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2}) + A(2-\frac{\Omega}{\gamma})\eta}$$
$$+ \frac{A(\frac{\kappa\Omega^2}{\gamma^2} + \frac{3\kappa\Omega}{\gamma} - 2\kappa + 2 - \frac{\Omega^2}{\gamma^2})\eta + A(\frac{2\kappa\Omega^2}{\gamma^2} - \frac{\kappa\Omega^3}{\gamma^3} - \frac{\kappa\Omega}{\gamma} + 2\kappa + \frac{\Omega}{\gamma} + \frac{\Omega^3}{\gamma^3})}{(2\kappa + 4\varepsilon)(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2}) + \frac{3A\Omega}{\gamma}(1-\eta^2)^{1/2} + \frac{A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2}) + A(2-\frac{\Omega}{\gamma})\eta}. \quad (2.129)$$

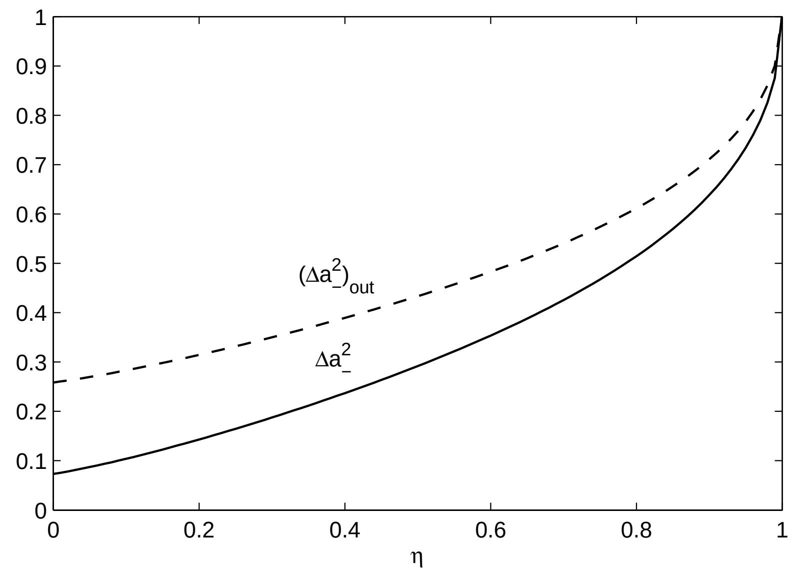

Fig. 2.4: Plots of the quadrature variance for the cavity mode [Eq. (2.118), solid curve] and for the output mode [Eq. (2.129), dashed curve] versus η for $\frac{\Omega}{\gamma} = 0.1$, $\varepsilon = 5.3$, $A = 100$, and $\kappa = 0.8$.

It can be seen from the plots in Fig. 2.4 that in general the cavity mode squeezing is greater than the output mode squeezing. Furthermore, applying Eq. (2.129) with $\eta = 0.1$, $\varepsilon = 5.3$, $A = 100$, and $\kappa = 0.8$, the maximum degree of squeezing for the output mode is found to be 75% (occurs at $\frac{\Omega}{\gamma} = 0$).

We next proceed to examine some special cases. We first consider the case in which the nonlinear crystal is removed from the cavity, with the top and bottom levels of the atoms coupled by the pump mode. Thus upon setting $\varepsilon = 0$ with $(\beta_0 \neq 0)$ in Eqs. (2.128) and (2.129) the output mode quadrature variance for this case takes the form

$$\Delta a^2_{+out}(t) = \frac{2\kappa(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2}) + A(2\kappa - \frac{3\kappa\Omega}{\gamma} - \frac{\kappa\Omega^2}{\gamma^2} + \frac{3\Omega}{\gamma})(1-\eta^2)^{1/2}}{2\kappa(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2}) + \frac{3A\Omega}{\gamma}(1-\eta^2)^{1/2} - \frac{A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2}) + A(2-\frac{\Omega}{\gamma})\eta}$$
$$+ \frac{A(\frac{\kappa\Omega^2}{\gamma^2} - \frac{3\kappa\Omega}{\gamma} - 2\kappa + 2 - \frac{\Omega^2}{\gamma^2})\eta + A(\frac{\kappa\Omega^3}{\gamma^3} + \frac{2\kappa\Omega^2}{\gamma^2} + \frac{\kappa\Omega}{\gamma} + 2\kappa - \frac{\Omega}{\gamma} - \frac{\Omega^3}{\gamma^3})}{2\kappa(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2}) + \frac{3A\Omega}{\gamma}(1-\eta^2)^{1/2} - \frac{A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2}) + A(2-\frac{\Omega}{\gamma})\eta} \quad (2.130)$$

and

$$\Delta a^2_{-out}(t) = \frac{2\kappa(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2}) + A(\frac{\kappa\Omega^2}{\gamma^2} - 2\kappa - \frac{3\kappa\Omega}{\gamma} + \frac{3\Omega}{\gamma})(1-\eta^2)^{1/2}}{2\kappa(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2}) + \frac{3A\Omega}{\gamma}(1-\eta^2)^{1/2} + \frac{A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2}) + A(2-\frac{\Omega}{\gamma})\eta}$$
$$+ \frac{A(\frac{\kappa\Omega^2}{\gamma^2} + \frac{3\kappa\Omega}{\gamma} - 2\kappa + 2 - \frac{\Omega^2}{\gamma^2})\eta + A(\frac{2\kappa\Omega^2}{\gamma^2} - \frac{\kappa\Omega^3}{\gamma^3} - \frac{\kappa\Omega}{\gamma} + 2\kappa + \frac{\Omega}{\gamma} + \frac{\Omega^3}{\gamma^3})}{2\kappa(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2}) + \frac{3A\Omega}{\gamma}(1-\eta^2)^{1/2} + \frac{A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2}) + A(2-\frac{\Omega}{\gamma})\eta}. \quad (2.131)$$

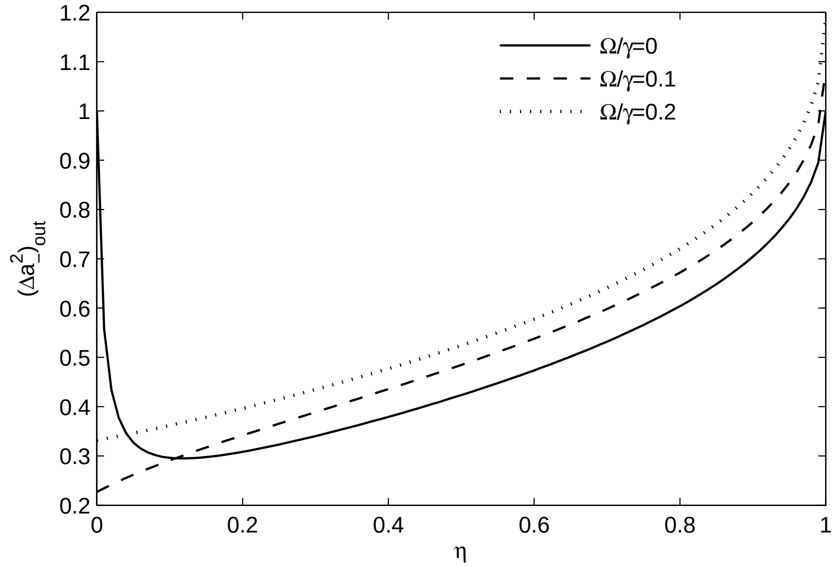

Fig. 2.5: Plots of the quadrature variance for the output mode [Eq. (2.131)] versus η for $A = 100$, $\kappa = 0.8$, and different values of $\frac{\Omega}{\gamma}$.

We easily see from the plots of Fig. 2.5 that the coupling of the top and bottom levels of the three-level atoms by the pump mode leads to a decrease in the degree of squeezing of the output mode, except when there are nearly equal number of atoms initially in the top and bottom levels. Moreover, using Eq. (2.131) with $\eta = 0.1$, $\kappa = 0.8$, and $A = 100$ the maximum degree of squeezing for the output mode is found to be 70% (occurs at $\frac{\Omega}{\gamma} = 0.02$). In addition, for the special case in which the pump mode emerging from the nonlinear crystal does not couple the top and bottom levels. Upon setting $\frac{\Omega}{\gamma} = 0$ (with $\beta_0 \neq 0$) Eqs. (2.128) and (2.129) become

$$\Delta a^2_{+out} = \frac{\kappa + 2\varepsilon\kappa - 2\varepsilon + A\kappa(1-\eta^2)^{1/2} + A(1-\kappa)\eta + A\kappa}{A\eta + \kappa - 2\varepsilon} \tag{2.132}$$

and

$$\Delta a^2_{-out} = \frac{\kappa - 2\varepsilon\kappa + 2\varepsilon - A\kappa(1-\eta^2)^{1/2} + A(1-\kappa)\eta + A\kappa}{A\eta + \kappa + 2\varepsilon}. \tag{2.133}$$

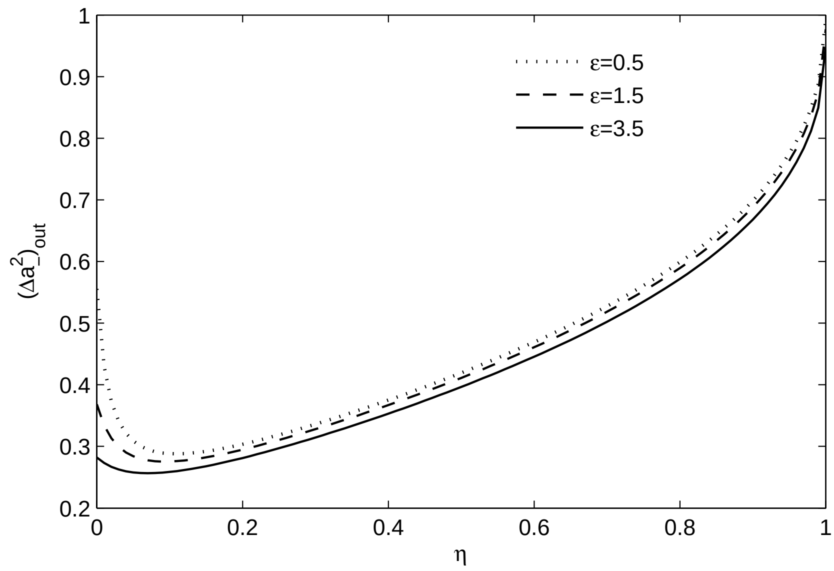

Fig. 2.6: Plots of the quadrature variance for the output mode [Eq. (2.133)] versus η for $A = 100$, $\kappa = 0.8$, and different values of ε.

We readily observe from the plots in Fig. 2.6 that the presence of the nonlinear crystal leads to better squeezing of the output mode. In addition, applying Eq. (2.133) with $\eta = 0.1$, $A = 100$, and $\kappa = 0.8$ the maximum output mode squeezing is found to be 75% below the coherent-state level for $\varepsilon = 5.3$.

2.2.3 Squeezing spectrum of the output mode

The squeezing spectrum for a single-mode light can then be defined in the normal order as [47]

$$S^{out}_{\pm}(\omega) = 1 \pm 2Re\int_0^{\infty} \langle : \hat{a}^{out}_{\pm}(t), \hat{a}^{out}_{\pm}(t+\tau) : \rangle_{ss} e^{i(\omega-\omega_0)\tau} d\tau, \tag{2.134}$$

in which :: refers to normal-ordering. This can be expressed in terms of c-number variables associated with the normal ordering as

$$S^{out}_{\pm}(\omega) = 1 \pm 2Re\int_0^\infty \langle\alpha^{out}_{\pm}(t), \alpha^{out}_{\pm}(t+\tau)\rangle_{ss} e^{i(\omega-\omega_0)\tau} d\tau, \tag{2.135}$$

For a cavity mode coupled to a vacuum reservoir, the output and cavity variables can be related by

$$\alpha^{out}_{\pm} = \sqrt{\kappa}\alpha_{\pm}. \tag{2.136}$$

On account of Eq. (2.136) the squeezing spectrum takes the form

$$S^{out}_{\pm}(\omega) = 1 \pm 2\kappa Re\int_0^\infty \langle\alpha_{\pm}(t), \alpha_{\pm}(t+\tau)\rangle_{ss} e^{i(\omega-\omega_0)\tau} d\tau. \tag{2.137}$$

The solution of Eq. (2.95) can be written as

$$\alpha_{\pm}(t+\tau) = \alpha_{\pm}(t)e^{-\lambda_{\mp}\tau} + \int_0^\tau e^{-\lambda_{\mp}(\tau-\tau')}\big((\mu \pm \mu) + f^*(t+\tau') \pm f(t+\tau')\big)\, d\tau'. \tag{2.138}$$

Now multiplying Eq. (2.138) by $\alpha_{\pm}(t)$ and taking the the expectation value of the resulting expression along with the fact that the noise force at time $t+\tau'$ does not affect the system variables at the earlier time t, there follows

$$\langle\alpha_{\pm}(t)\alpha_{\pm}(t+\tau)\rangle_{ss} = \langle\alpha^2_{\pm}(t)\rangle_{ss} e^{-\lambda_{\mp}\tau} + (\mu \pm \mu)\langle\alpha_{\pm}(t)\rangle \int_0^\tau e^{-\lambda_{\mp}(\tau-\tau')} d\tau'. \tag{2.139}$$

With the aid of Eqs. (2.110) and (2.115), we have

$$\begin{aligned}\langle\alpha_{\pm}(t)\alpha_{\pm}(t+\tau)\rangle_{ss} = &\frac{(\mu \pm \mu)^2}{\lambda^2_{\mp}} e^{-\lambda_{\mp}\tau} + \frac{\varepsilon - 2V \pm 2R}{\lambda^2_{\mp}} e^{-\lambda_{\mp}\tau}\\ &+ \frac{(\mu \pm \mu)^2}{\lambda_{\mp}} \int_0^\tau e^{-\lambda_{\mp}(\tau-\tau')} d\tau'.\end{aligned} \tag{2.140}$$

Upon carrying out the integration, we obtain

$$\langle\alpha_{\pm}(t)\alpha_{\pm}(t+\tau)\rangle_{ss} = \frac{(\mu \pm \mu)^2}{\lambda^2_{\mp}} + \frac{\varepsilon - 2V \pm 2R}{\lambda^2_{\mp}} e^{-\lambda_{\mp}\tau}. \tag{2.141}$$

Furthermore, with the help of Eq. (2.138), one readily gets

$$\langle\alpha_{\pm}(t+\tau)\rangle_{ss} = \frac{\mu \pm \mu}{\lambda_{\mp}}. \tag{2.142}$$

Therefore, application of (2.110), (2.141), and (2.142) in Eq. (2.137) leads to

$$S^{out}_{\pm}(\omega) = 1 \pm 2\kappa Re\frac{\varepsilon - 2V \pm 2R}{\lambda^2_{\mp}} \int_0^\infty e^{(i(\omega-\omega_0)-\lambda_{\mp})\tau} d\tau, \tag{2.143}$$

so that upon performing the integration, we find

$$S^{out}_{\pm}(\omega) = 1 \pm \frac{2\kappa(\varepsilon - 2V \pm 2R)}{\lambda^2_{\mp} + (\omega-\omega_0)^2}. \tag{2.144}$$

Thus substitution of (2.59), (2.62), and (2.117) into Eq. (2.144) results in

$$S_+^{out}(\omega) = 1 + \frac{\kappa\left(2\varepsilon - \frac{A(\frac{\Omega^2}{\gamma^2}+\frac{3\Omega}{\gamma}-2)(1-\eta^2)^{1/2}-A(2+\frac{\Omega}{\gamma}+\frac{2\Omega^2}{\gamma^2}+\frac{\Omega^3}{\gamma^3})+A(\frac{3\Omega}{\gamma}-\frac{\Omega^2}{\gamma^2}+2)\eta}{2(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})}\right)}{\left(\frac{\kappa}{2}-\varepsilon+\frac{\frac{3A\Omega}{\gamma}(1-\eta^2)^{1/2}-\frac{A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2})+A(2-\frac{\Omega^2}{\gamma^2})\eta}{4(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})}\right)^2+(\omega-\omega_0)^2} \tag{2.145}$$

and

$$S_-^{out}(\omega) = 1 - \frac{\kappa\left(2\varepsilon - \frac{A(\frac{\Omega^2}{\gamma^2}-\frac{3\Omega}{\gamma}-2)(1-\eta^2)^{1/2}+A(2-\frac{\Omega}{\gamma}+\frac{2\Omega^2}{\gamma^2}-\frac{\Omega^3}{\gamma^3})+A(\frac{3\Omega}{\gamma}+\frac{\Omega^2}{\gamma^2}-2)\eta}{2(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})}\right)}{\left(\frac{\kappa}{2}+\varepsilon+\frac{\frac{3A\Omega}{\gamma}(1-\eta^2)^{1/2}+\frac{A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2})+A(2-\frac{\Omega^2}{\gamma^2})\eta}{4(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})}\right)^2+(\omega-\omega_0)^2}. \tag{2.146}$$

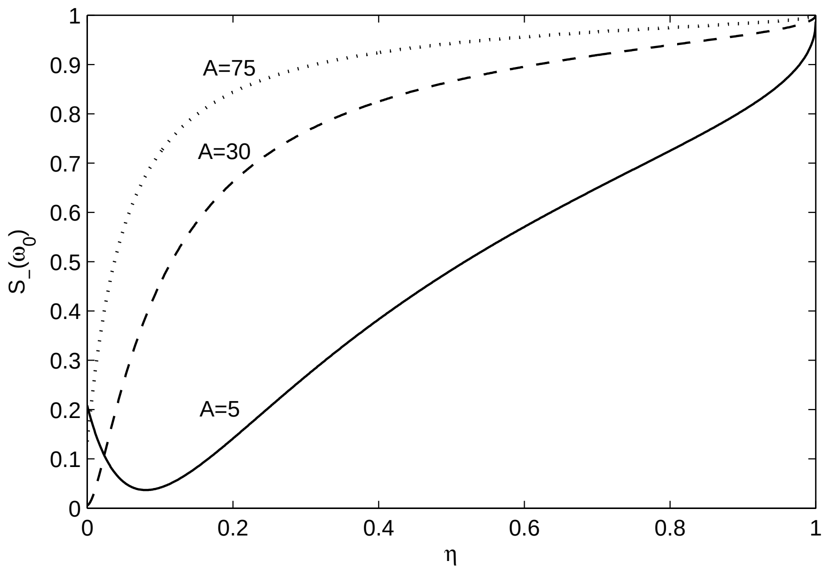

Fig. 2.7: Plots of the squeezing spectrum [Eq. (2.146)] versus η for $\frac{\Omega}{\gamma} = 0.01$, $\varepsilon = 0.1$, $\omega = \omega_0$, and for different values of the linear gain coefficient.

Fig. 2.7 shows that there appears to be perfect squeezing around $A = 30$ and for values of η very close to zero.

Next we wish to examine some special cases. We first consider the case in which the nonlinear crystal is removed from the cavity, with the top and bottom levels of the atoms coupled by the pump mode. Thus upon setting $\varepsilon = 0$ (with $\beta_0 \neq 0$) in Eqs. (2.145) and (2.146), we have

$$S_{+}^{out}(\omega) = 1 + \frac{\kappa\left(-\frac{A(\frac{\Omega^2}{\gamma^2}+\frac{3\Omega}{\gamma}-2)(1-\eta^2)^{1/2}-A(2+\frac{\Omega}{\gamma}+\frac{2\Omega^2}{\gamma^2}+\frac{\Omega^3}{\gamma^3})+A(\frac{3\Omega}{\gamma}-\frac{\Omega^2}{\gamma^2}+2)\eta}{2(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})}\right)}{\left(\frac{\kappa}{2}+\frac{\frac{3A\Omega}{\gamma}(1-\eta^2)^{1/2}-\frac{A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2})+A(2-\frac{\Omega^2}{\gamma^2})\eta}{4(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})}\right)^2+(\omega-\omega_0)^2} \tag{2.147}$$

and

$$S_{-}^{out}(\omega) = 1 - \frac{\kappa\left(-\frac{A(\frac{\Omega^2}{\gamma^2}-\frac{3\Omega}{\gamma}-2)(1-\eta^2)^{1/2}+A(2-\frac{\Omega}{\gamma}+\frac{2\Omega^2}{\gamma^2}-\frac{\Omega^3}{\gamma^3})+A(\frac{3\Omega}{\gamma}+\frac{\Omega^2}{\gamma^2}-2)\eta}{2(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})}\right)}{\left(\frac{\kappa}{2}+\frac{\frac{3A\Omega}{\gamma}(1-\eta^2)^{1/2}+\frac{A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2})+A(2-\frac{\Omega^2}{\gamma^2})\eta}{4(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})}\right)^2+(\omega-\omega_0)^2}. \tag{2.148}$$

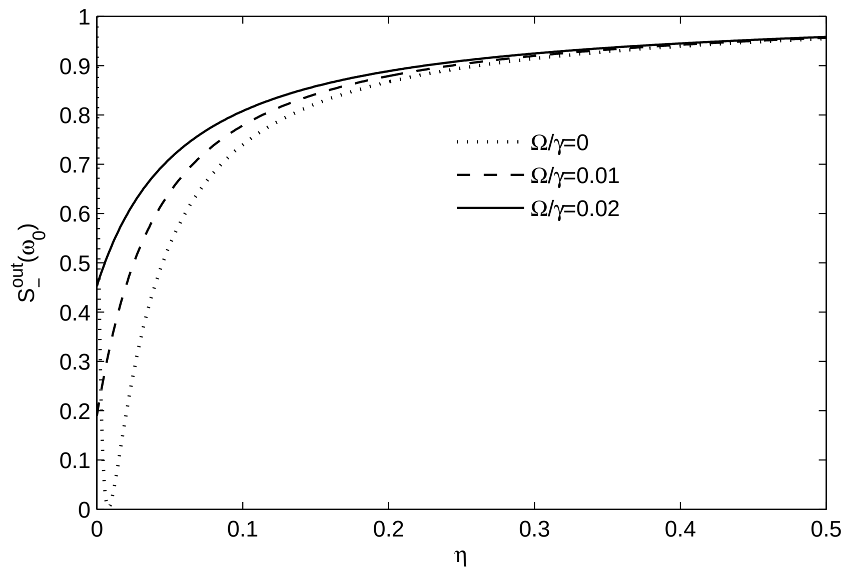

Fig. 2.8: Plots of the squeezing spectrum [Eq. (2.148)] versus η for $A = 100$, $\kappa = 0.8$, $\omega = \omega_0$, and different values of $\frac{\Omega}{\gamma}$.

We easily see from the plots of Fig. 2.8 that the coupling of the top and bottom levels of the three level atoms by the pump mode emerging from the nonlinear crystal increase the squeezing spectrum only when there are nearly equal number of atoms initially in the top and bottom levels ($\eta = 0$). Otherwise, it decreases the squeezing spectrum. Furthermore, employing Eqs. (2.148) with $\frac{\Omega}{\gamma} = 0$, $\kappa = 0.8$, $A = 100$, and $\omega = \omega_0$ we find that there is perfect squeezing at $\eta = 0.04$. In addition, we consider the special case in which the pump mode emerging from the nonlinear crystal does not couple the top and bottom levels of the three-level atoms. Thus upon setting $\frac{\Omega}{\gamma} = 0$ (with $\beta_0 \neq 0$) in Eqs. (2.145) and (2.146), we find

$$S_{+}^{out}(\omega) = 1 + \frac{\kappa\left(2\varepsilon + A(1-\eta^2)^{1/2} - A\eta + A\right)}{\left(\frac{\kappa}{2} - \varepsilon + \frac{A\eta}{2}\right)^2 + (\omega - \omega_0)^2} \tag{2.149}$$

and

$$S_{-}^{out}(\omega) = 1 - \frac{\kappa\left(2\varepsilon + A(1-\eta^2)^{1/2} + A\eta - A\right)}{\left(\frac{\kappa}{2} + \varepsilon + \frac{A\eta}{2}\right)^2 + (\omega - \omega_0)^2}. \tag{2.150}$$

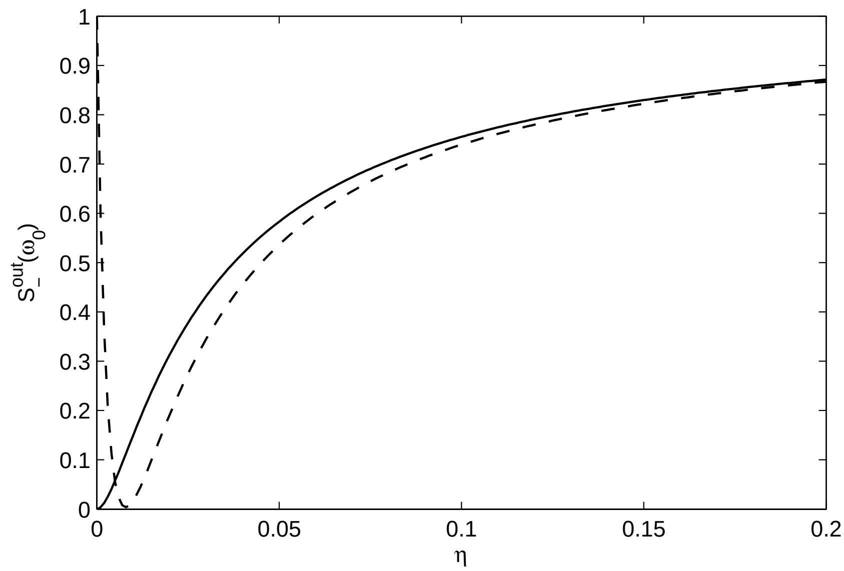

Fig. 2.9: Plots of the squeezing spectrum [Eq. (2.150)] verses η $A = 100$, $\kappa = 0.8$, $\omega = \omega_0$, and $\varepsilon = 0.4$ (solid curve) and $\varepsilon = 0$ (dashed curve).

We observe from Fig. 2.9 that the effect of the parametric amplifier is to increase the squeezing spectrum for $0 < \eta < 0.02$ and to decrease it for $0.02 < \eta < 0.32$. In addition, applying Eq. (2.150) with $A = 100$, $\kappa = 0.8$, and $\omega = \omega_0$ we find that there is perfect squeezing at $\eta = 0$ for $\varepsilon = 0.4$ and at $\eta = 0.04$ for $\varepsilon = 0$.

2.3 Photon number statistics of the cavity mode

We first determine, using the solutions of the c-number Langevin equations and the correlation properties of the noise forces, the antinormally ordered characteristic function defined in the Heisenberg picture for the cavity mode. With the aid of the resulting characteristic function, we obtain the Q function which is then used to calculate the mean and the normally-ordered variance of the photon number and the photon number distribution for the cavity mode. The Q function for a single-mode light is expressible as

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi^2} \int d^2 z \phi(z^*, z, t) exp(z^* \alpha - z \alpha^*), \tag{2.151}$$

with the characteristic function $\phi(z^*, z, t)$ defined in the Heisenberg picture by

$$\phi(z^*, z, t) = Tr\left(\hat{\rho(0)} e^{-z^* \hat{a}(t)} e^{z \hat{a}^\dagger(t)}\right). \tag{2.152}$$

Using the identity

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{B}} e^{\hat{A}} e^{[\hat{A}, \hat{B}]}, \tag{2.153}$$

the characteristic function can be put in the normal order as

$$\phi(z^*, z, t) = e^{-z^* z} \langle exp(z \hat{a}^\dagger - z^* \hat{a}) \rangle. \tag{2.154}$$

This can be expressed in terms of c-number variables associated with the normal ordering as

$$\phi(z^*, z, t) = e^{-z^* z} \langle exp(z \alpha^* - z^* \alpha) \rangle. \tag{2.155}$$

One can rewrite Eq. (2.98) as

$$\alpha(t) = \alpha'(t) + E(t), \tag{2.156}$$

where

$$\alpha'(t) = B_+(t) \alpha(0) + B_-(t) \alpha^*(t) + F(t). \tag{2.157}$$

Now using Eq. (2.156) and its complex conjugate, we have

$$\phi(z^*, z, t) = e^{-z^* z + (z - z^*) E} \left\langle exp(z \alpha'^* - z^* \alpha') \right\rangle. \tag{2.158}$$

Employing (2.157) along with Eq. (2.96), the equation of evolution for the expectation value of $\alpha'(t)$ can be expressed as

$$\frac{d}{dt} \langle \alpha'(t) \rangle = -(S - R) \langle \alpha'(t) \rangle + (U - V + \varepsilon) \langle \alpha'^*(t) \rangle. \tag{2.159}$$

Inspection of this equation indicate that $\alpha'(t)$ is a Gaussian variable [21]. Moreover, on account of (2.157) along with the assumption that the cavity mode is initially in a vacuum state, we easily show that $\langle \alpha'(t) \rangle = 0$. Hence $\alpha'(t)$ is a Gaussian variable with vanishing mean. One can then express (2.158) in the form [21]

$$\phi(z^*, z, t) = e^{-z z^* + (z - z^*) E} \times exp\left(\frac{1}{2} \left\langle [z^2 \alpha'^{*2} + z^{*2} \alpha'^2 - 2 z z^* \alpha' \alpha'^*] \right\rangle\right). \tag{2.160}$$

On the other hand, employing Eqs. (2.101), (2.102), and (2.157) along with the assumption that the cavity mode is initially in a vacuum state one readily finds

$$\langle\alpha'^2(t)\rangle = \frac{1}{4}\bigg[\int_0^t e^{-\lambda_-(2t-t'-t'')}\big\langle(f(t')+f(t')^*)(f(t'')+f(t'')^*)\big\rangle\,dt'dt''$$
$$+\int_0^t e^{-\lambda_+(2t-t'-t'')}\big\langle(f(t')-f(t')^*)(f(t'')-f(t'')^*)\big\rangle\,dt'dt''$$
$$+\int_0^t e^{-[(\lambda_-+\lambda_+)t+\lambda_-t'+\lambda_+t'']}\big\langle(f(t')+f(t')^*)(f(t'')-f(t'')^*)\big\rangle\,dt'dt''$$
$$+\int_0^t e^{-[(\lambda_-+\lambda_+)t+\lambda_+t'+\lambda_-t'']}\big\langle(f(t')-f(t')^*)(f(t'')+f(t'')^*)\big\rangle\,dt'dt''\bigg]. \tag{2.161}$$

With the aid of Eqs. (2.85) and (2.93), we get

$$\langle\alpha'^2(t)\rangle = \frac{2R-2V+\varepsilon}{2}\int_0^t e^{-\lambda_-(2t-t'-t'')}\delta(t'-t'')dt'dt''$$
$$-\frac{2R+2V-\varepsilon}{2}\int_0^t e^{-\lambda_+(2t-t'-t'')}\delta(t'-t'')dt'dt'', \tag{2.162}$$

so that upon carrying out the integration, we obtain

$$\langle\alpha'^2(t)\rangle = \frac{2R-2V+\varepsilon}{4\lambda_-}(1-e^{-2\lambda_-t}) - \frac{2R+2V-\varepsilon}{4\lambda_+}(1-e^{-2\lambda_+t}). \tag{2.163}$$

It can also be established following a similar procedure that

$$\langle\alpha'^*(t)\alpha'(t)\rangle = \frac{2R-2V+\varepsilon}{4\lambda_-}(1-e^{-2\lambda_-t}) + \frac{2R+2V-\varepsilon}{4\lambda_+}(1-e^{-2\lambda_+t}). \tag{2.164}$$

On account of Eq. (2.163) and its complex conjugate along with (2.164), there follows

$$\phi(z^*,z,t) = exp(-az^*z - \frac{b}{2}(z^2+z^{*2}) + (z-z^*)E), \tag{2.165}$$

in which

$$a = 1 + \frac{2R-2V+\varepsilon}{4\lambda_-}\left(1-e^{-2\lambda_-t}\right) + \frac{2R+2V-\varepsilon}{4\lambda_+}\left(1-e^{-2\lambda_+t}\right) \tag{2.166}$$

and

$$b = \frac{2R-2V+\varepsilon}{4\lambda_-}\left(1-e^{-2\lambda_-t}\right) - \frac{2R+2V-\varepsilon}{4\lambda_+}\left(1-e^{-2\lambda_+t}\right). \tag{2.167}$$

Furthermore, substitution of Eq. (2.165) into (2.151) leads to

$$Q(\alpha^*,\alpha.t) = \frac{1}{\pi^2}\int d^2z exp\left(-az^*z - \frac{b}{2}(z^2+z^{*2}) + (\alpha-E)z^* - (\alpha^*-E)z\right). \tag{2.168}$$

Upon carrying out the integration, the Q function is found to be

$$Q(\alpha^*,\alpha,t) = \frac{(c^2-d^2)^{1/2}}{\pi}exp\bigg[-c(\alpha^*\alpha-\alpha E-\alpha^*E+E^2)$$
$$-\frac{d}{2}(\alpha^2+\alpha^{*2}-2E\alpha-2E\alpha^*+2E^2)\bigg], \tag{2.169}$$

where

$$c = \frac{a}{a^2 - b^2} \tag{2.170}$$

and

$$d = \frac{b}{a^2 - b^2}. \tag{2.171}$$

2.3.1 Normally-ordered variance of the photon number

We next seek to calculate, applying the Q function we have found, the mean and variance of the photon number for the cavity mode.

Mean of the photon number

The interacavity photon number is represented by the operator

$$\hat{n} = \hat{a}^\dagger \hat{a}. \tag{2.172}$$

On the other hand, the expectation value of an operator function $\hat{A}(\hat{a}^\dagger, \hat{a})$ can be calculated using the Q function as [21, 22]

$$\langle \hat{A} \rangle = \int d^2\alpha Q(\alpha) A_a(\alpha), \tag{2.173}$$

in which $A_a(\alpha)$ is the c-number function corresponding to the operator function $\hat{A}$ in the antinormal order. Hence with the aid of Eq. (2.173) the mean photon number for the cavity mode can be expressed as

$$\bar{n} = \int d^2\alpha Q(\alpha)(\alpha^*\alpha - 1). \tag{2.174}$$

Now applying the Q function (2.169), we have

$$\bar{n} = \frac{(c^2 - d^2)^{1/2}}{\pi} e^{-(cE^2 + dE^2)} \int d^2\alpha \exp[-c\alpha^*\alpha + (cE + dE)(\alpha^* + \alpha) - \frac{d}{2}(\alpha^{*2} + \alpha^2)]\alpha^*\alpha - 1. \tag{2.175}$$

This can be put in the form

$$\bar{n} = -\frac{(c^2 - d^2)^{1/2}}{c} e^{-(cE^2 + dE^2)} \times \left\{ \frac{d}{dy} \frac{1}{\pi} \int d^2\alpha \exp[-yc\alpha^*\alpha + (cE + dE)(\alpha^* + \alpha) - \frac{d}{2}(\alpha^{*2} + \alpha^2)] \right\}_{y=1} - 1, \tag{2.176}$$

so that carrying out the integration, we get

$$\bar{n} = -\frac{(c^2 - d^2)^{1/2}}{c} e^{-(cE^2 + dE^2)} \left\{ \frac{d}{dy} \left(\frac{1}{y^2c^2 - d^2} \right)^{1/2} \exp\left(\frac{(cy - d)(cE + dE)^2}{y^2c^2 - d^2} \right) \right\}_{y=1} - 1. \tag{2.177}$$

Furthermore, upon performing the differentiation and applying the condition $y = 1$, we obtain

$$\bar{n} = \exp\left[-(cE^2 + dE^2) + \frac{(c-d)(cE+dE)^2}{c^2-d^2}\right] \times \left[\frac{c}{c^2-d^2} + \frac{\left[2c(c-d) - (c^2-d^2)\right](cE+dE)^2}{(c^2-d^2)^2}\right] - 1. \tag{2.178}$$

With the aid of Eqs. (2.170) and (2.171), one can readily show that

$$\exp\left[-(cE^2 + dE^2) + \frac{(c-d)(cE+dE)^2}{c^2-d^2}\right] = 1, \tag{2.179}$$

$$\frac{c}{c^2-d^2} = a, \tag{2.180}$$

and

$$\frac{\left[2c(c-d) - (c^2-d^2)\right](cE+dE)^2}{(c^2-d^2)^2} = E^2. \tag{2.181}$$

Thus substitution of (2.179), (2.180), and (2.181) into Eq. (2.178) results in

$$\bar{n} = E^2 + a - 1. \tag{2.182}$$

Therefore, on account of Eqs. (2.100) and (2.166), the mean photon number for the cavity mode turns out to be

$$\bar{n} = \frac{\mu^2}{\lambda_-^2}\left(1 + e^{-2\lambda_- t} - 2e^{-\lambda_- t}\right) + \frac{2R - 2V + \varepsilon}{4\lambda_-}\left(1 - e^{-2\lambda_- t}\right) + \frac{2R + 2V - \varepsilon}{4\lambda_+}\left(1 - e^{-2\lambda_+ t}\right). \tag{2.183}$$

At steady state the mean photon number goes over into

$$\bar{n}_{ss} = \frac{\mu^2}{\lambda_-^2} + \frac{2R - 2V + \varepsilon}{4\lambda_-} + \frac{2R + 2V - \varepsilon}{4\lambda_+}. \tag{2.184}$$

Finally, application of (2.59), (2.62), and (2.117) in Eq. (2.184) yields

$$\bar{n}_{ss} = \frac{16\mu^2(1+\frac{\Omega^2}{\gamma^2})^2(1+\frac{\Omega^2}{4\gamma^2})^2}{\Big((2\kappa-4\varepsilon)(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})+\frac{3A\Omega}{\gamma}(1-\eta^2)^{1/2}-\frac{A\Omega}{\gamma}(1+\Omega^2)+A(2-\frac{\Omega^2}{\gamma^2})\eta\Big)^2}$$
$$+\frac{2\varepsilon(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})-A(\frac{3\Omega}{2\gamma}+\frac{\Omega^2}{2\gamma^2}-1)(1-\eta^2)^{1/2}}{4(\kappa-2\varepsilon)(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})+\frac{6A\Omega}{\gamma}(1-\eta^2)^{1/2}-\frac{2A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2})+4A(1-\frac{\Omega^2}{2\gamma^2})\eta}$$
$$+\frac{-A(\frac{3\Omega}{2\gamma}-\frac{\Omega^2}{2\gamma^2}+1)\eta+A(\frac{\Omega}{2\gamma}+\frac{\Omega^3}{2\gamma^3}+\frac{\Omega^2}{\gamma^2}+1)}{4(\kappa-2\varepsilon)(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})+\frac{6A\Omega}{\gamma}(1-\eta^2)^{1/2}-\frac{2A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2})+4A(1-\frac{\Omega^2}{2\gamma^2})\eta}$$
$$-\frac{2\varepsilon(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})-A(\frac{\Omega^2}{2\gamma^2}-\frac{3\Omega}{2\gamma}-1)(1-\eta^2)^{1/2}}{4(\kappa+2\varepsilon)(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})+\frac{6A\Omega}{\gamma}(1-\eta^2)^{1/2}+\frac{2A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2})+2A(2-\frac{\Omega^2}{\gamma^2})\eta}$$
$$-\frac{-A(\frac{3\Omega}{2\gamma}+\frac{\Omega^2}{2\gamma^2}-1)\eta-A(\frac{\Omega^2}{\gamma^2}-\frac{\Omega^3}{2\gamma^3}-\frac{\Omega}{2\gamma}+1)}{4(\kappa+2\varepsilon)(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})+\frac{6A\Omega}{\gamma}(1-\eta^2)^{1/2}+\frac{2A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2})+2A(2-\frac{\Omega^2}{\gamma^2})\eta}. \quad (2.185)$$

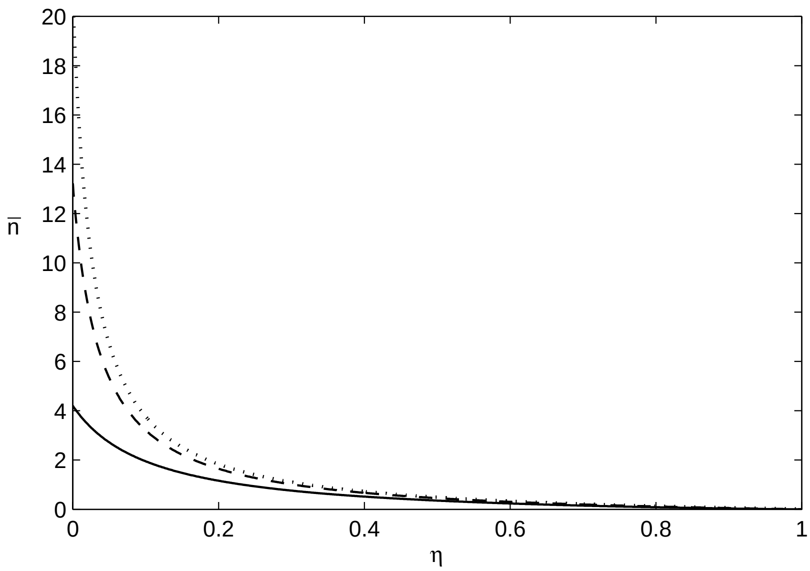

Fig. 2.10: The mean photon number [Eq. (2.185) with $\frac{\Omega}{\gamma} = 0.04$, $A = 100$, and $\kappa = 0.8$] versus η for $\varepsilon = \mu = 0$ (solid curve), for $\varepsilon = 3.5$ and $\mu = 0$ (dashed curve), and for $\varepsilon = 3.5$ and $\mu = 5$ (dotted curve).

The plots in Fig. 2.10 clearly indicate that the parametric amplifier and the driving light contribute significantly to the mean photon number. Moreover, we observe that in general the mean of the photon number decreases as η increases.

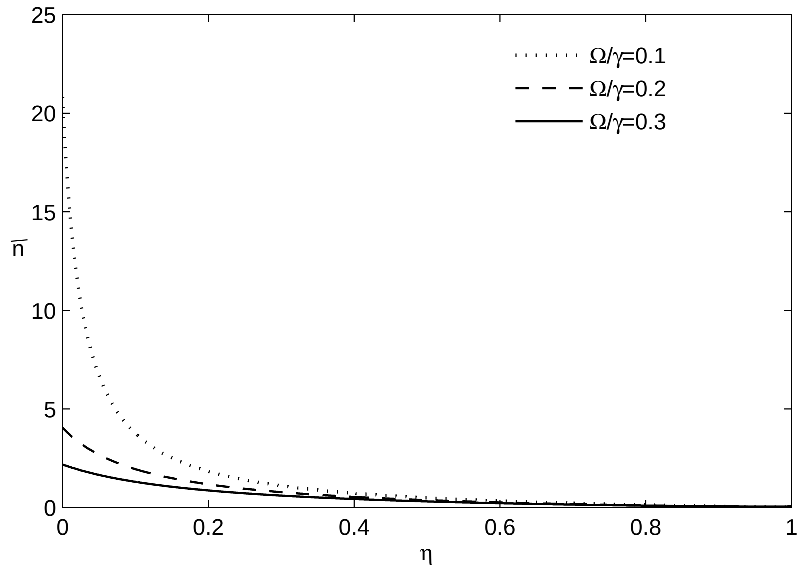

Fig. 2.11: The mean photon number [Eq. (2.185)] versus η for $\varepsilon = 3.5$, $\mu = 5$, $A = 100$, $\kappa = 0.8$, and different values of $\frac{\Omega}{\gamma}$.

The plots in Fig. 2.11 show that the mean photon number decreases with $\frac{\Omega}{\gamma}$. This must be due to the stimulated emission induced by the pump mode. The photons emitted this way are not included in the mean photon number for the cavity mode.

We next wish to examine some interesting special cases. First we consider the case in which the atoms are not injected into the cavity. Thus upon setting $A = 0$ in Eq. (2.185), we get [35]

$$\overline{n}_{ss} = \frac{4\mu^2}{(\kappa - 2\varepsilon)^2} + \frac{2\varepsilon^2}{\kappa^2 - 4\varepsilon^2}. \tag{2.186}$$

This represent the mean photon number of the cavity mode for a degenerate parametric oscillator driven by coherent light. Furthermore, we consider the case in which the parametric amplifier and the driving light are absent. Thus upon setting $\varepsilon = \mu = 0$ (with $\frac{\Omega}{\gamma} = 0$) in Eq. (2.185), we obtain [21]

$$\overline{n}_{ss} = \frac{A(1-\eta)}{2(A\eta + \kappa)}. \tag{2.187}$$

This represent the steady-state mean photon number for a degenerate three-level laser.

Normally-ordered variance of the photon number

We next proceed to calculate the normally-ordered variance of the photon number. The normally-ordered variance of the photon number is defined as [48]

$$:\Delta n^2 := \langle : \hat{n}^2 : \rangle - \langle \hat{n} \rangle^2, \tag{2.188}$$

Furthermore, using Eqs. (2.172), the normally-ordered variance of the photon number takes the form

$$: \Delta n^2 := \Delta n^2 - \bar{n}, \tag{2.189}$$

with Δn^2 and $\bar{n}$ being the variance and the mean of the photon number. The variance of the photon number for the cavity mode is expressible as

$$\Delta n^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2. \tag{2.190}$$

The variance of the photon number can also be put in the form

$$\Delta n^2 = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - \bar{n}^2 - 3\bar{n} - 2. \tag{2.191}$$

Employing (2.173) one can write

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \int d^2\alpha Q(\alpha) \alpha^2 \alpha^{*2}. \tag{2.192}$$

Now applying the Q function (2.169) in Eq. (2.192), we have

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{(c^2 - d^2)^{1/2}}{\pi} e^{-(cE^2 + dE^2)} \int d^2\alpha \exp[-c\alpha^*\alpha + (cE + dE)(\alpha^* + \alpha) - \frac{d}{2}(\alpha^{*2} + \alpha^2)] \alpha^{*2}\alpha^2 \tag{2.193}$$

or

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{(c^2 - d^2)^{1/2}}{c^2} e^{-(cE^2 + dE^2)} \times \left\{ \frac{d^2}{dy^2} \frac{1}{\pi} \int d^2\alpha \exp[-yc\alpha^*\alpha + (cE + dE)(\alpha^* + \alpha) - \frac{d}{2}(\alpha^{*2} + \alpha^2)] \right\}_{y=1}, \tag{2.194}$$

so that carrying out the integration, we get

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{(c^2 - d^2)^{1/2}}{c^2} e^{-(cE^2 + dE^2)} \left\{ \frac{d^2}{dy^2} \left(\frac{1}{y^2c^2 - d^2} \right)^{1/2} \exp\left(\frac{(cy - d)(cE + dE)^2}{y^2c^2 - d^2} \right) \right\}_{y=1}. \tag{2.195}$$

Furthermore, upon performing the differentiation and applying the condition $y = 1$, we obtain

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \exp\left[-(cE^2 + dE^2) + \frac{(c - d)(cE + dE)^2}{c^2 - d^2} \right] \left[\left(\frac{\left[2c(c - d) - (c^2 - d^2)\right](cE + dE)^2}{(c^2 - d^2)^2} \right)^2 + \frac{3c^2}{(c^2 - d^2)^2} - \frac{1}{c^2 - d^2} - \frac{6c\left[2c(c - d) - (c^2 - d^2)\right](cE + dE)^2}{(c^2 - d^2)(c^2 - d^2)^2} - \frac{2(c - d)(cE + dE)^2}{(c^2 - d^2)^2} \right]. \tag{2.196}$$

With the aid of Eqs. (2.170) and (2.171), one can easily show that

$$\frac{1}{c^2 - d^2} = a^2 - b^2 \tag{2.197}$$

and

$$\frac{2(c-d)(cE+dE)^2}{(c^2-d^2)^2} = 2E^2(a+b). \tag{2.198}$$

Thus substitution of (2.179), (2.180), (2.181), (2.197), and (2.198) into Eq. (2.196) leads to

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = E^4 + 4E^2 a - 2E^2 b + 2a^2 + b^2. \tag{2.199}$$

Application of (2.182) and (2.199) in Eq. (2.191) results in

$$\Delta n^2 = 2E^2 a + a^2 + b^2 - 2E^2 b - E^2 - a. \tag{2.200}$$

Therefore, with the aid of Eqs. (2.100), (2.166), and (2.167) the variance of the photon number turns out to be

$$\begin{aligned}\Delta n^2 = \bar{n} + \frac{\mu^2}{\lambda_-^2}\left(\frac{2R-2V+\varepsilon}{4\lambda_-}(1-e^{-2\lambda_- t})\right) \\ + 2\left(\frac{2R-2V+\varepsilon}{4\lambda_-}\right)^2\left(1-e^{-2\lambda_- t}\right)^2 + 2\left(\frac{2R+2V-\varepsilon}{4\lambda_+}\right)^2\left(1-e^{-2\lambda_+ t}\right)^2.\end{aligned} \tag{2.201}$$

At steady state the photon number variance goes over into

$$\Delta n_{ss}^2 = \bar{n}_{ss} + \frac{4\mu^2}{\lambda_-^2}\left(\frac{2R-2V+\varepsilon}{4\lambda_-}\right) + 2\left(\frac{2R-2V+\varepsilon}{4\lambda_-}\right)^2 + 2\left(\frac{2R+2V-\varepsilon}{4\lambda_+}\right)^2. \tag{2.202}$$

Finally, in view of this result the normally-ordered variance of the photon number (2.189) takes at steady state the form

$$:\Delta n_{ss}^2 := \frac{4\mu^2}{\lambda_-^2}\left(\frac{2R-2V+\varepsilon}{4\lambda_-}\right) + 2\left(\frac{2R-2V+\varepsilon}{4\lambda_-}\right)^2 + 2\left(\frac{2R+2V-\varepsilon}{4\lambda_+}\right)^2. \tag{2.203}$$

We see from Fig. 2.12 that the normally-ordered variance of the photon number is positive. This indicates that the photon number statistics is super-Poissonian. Furthermore, we note that one effect of the coupling of the top and bottom levels is to decrease the normally-ordered variance of the photon number. In addition, the same plots show that the normally-ordered variance of the photon number decreases with η.

2.3.2 Photon number distribution

We next seek to obtain, employing the Q function, the photon number distribution for the cavity mode of the system under consideration. The photon number distribution for a single-mode light is expressible in terms of the Q function as [21]

$$P(n,t) = \frac{\pi}{n!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}\left[Q(\alpha^*,\alpha,t)e^{\alpha^*\alpha}\right]_{\alpha^*=\alpha=0}. \tag{2.204}$$

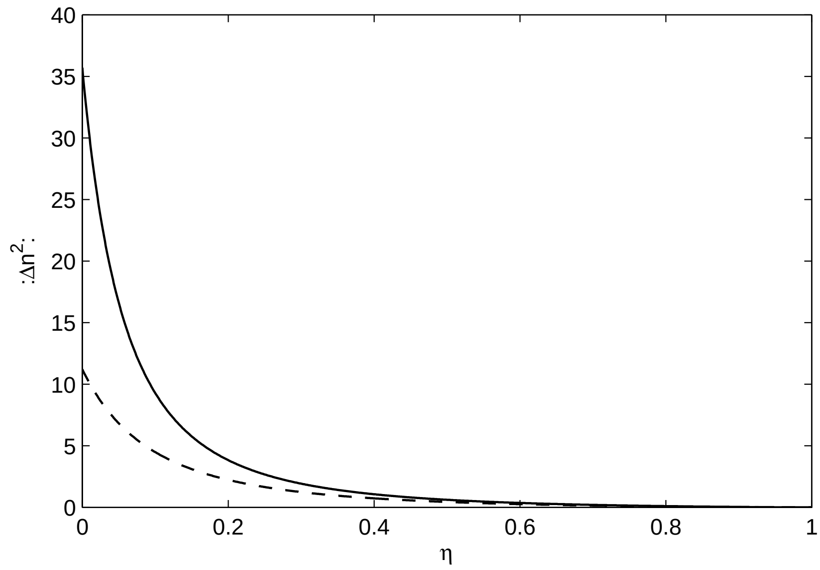

Fig. 2.12: Plots of the normally-ordered variance of the photon number [Eq. (2.203)] versus η for $\varepsilon = 3.5$, $\mu = 5$, $A = 100$, $\kappa = 0.8$, and $\frac{\Omega}{\gamma} = 0.2$ (solid curve) and $\frac{\Omega}{\gamma} = 0.3$ (dotted curve).

Now using the Q function (2.169), the photon number distribution for the cavity mode under consideration can be written in the form

$$P(n,t) = \frac{(c^2-d^2)^{1/2}}{n!}e^{-(c+d)E^2} \times \frac{\partial^2}{\partial\alpha^{*n}\partial\alpha^n} exp\left[(1-c)\alpha^*\alpha + (cE+dE)\alpha + (cE+dE)\alpha^* - \frac{d}{2}\alpha^2 - \frac{d}{2}\alpha^{*2}\right]_{\alpha^*=\alpha=0}. \tag{2.205}$$

Furthermore, upon expanding the exponential functions in power series, we find

$$P(n,t) = \frac{(c^2-d^2)^{1/2}}{n!}e^{-(c+d)E^2}\sum_{ijklm}\frac{(-1)^{l+m}(1-c)^i(cE+dE)^{j+k}(d)^{l+m}}{2^{l+m}i!j!k!l!m!} \times \frac{\partial^2}{\partial\alpha^{*n}\partial\alpha^n}\left[\alpha^{*i+k+2m}\alpha^{i+j+2l}\right]_{\alpha^*=\alpha=0}. \tag{2.206}$$

In addition, performing the differentiation and using the condition $\alpha^* = \alpha = 0$, we get

$$P(n,t) = \frac{(c^2-d^2)^{1/2}}{n!}e^{-(c+d)E^2}\sum_{ijklm}\frac{(-1)^{l+m}(1-c)^i(cE+dE)^{j+k}(d)^{l+m}}{2^{l+m}i!j!k!l!m!} \times \frac{(i+j+2l)!}{(i+j+2l-n)!}\frac{(i+k+2m)!}{(i+k+2m-n)!}\delta_{i+j+2l,n}\delta_{i+k+2m,n}. \tag{2.207}$$

Applying the properties of the Kronecker delta, we have

$$j = n - i - 2l \tag{2.208}$$

and

$$k = n - i - 2m. \tag{2.209}$$

Finally, with the aid of these results, the photon number distribution is found to be of the form

$$P(n,t) = (c^2 - d^2)^{1/2} e^{-(c+d)E^2} \sum_{ilm} \frac{n!(-1)^{l+m}(1-c)^i (cE + dE)^{2(n-i-l-m)} d^{l+m}}{2^{l+m} i! l! m! (n-i-2l)!(n-i-2m)!}. \tag{2.210}$$

It is interesting to consider a special case in which the coherent driving light is absent ($\varepsilon = 0$). Thus upon setting $\varepsilon = 0$ in Eq. (2.100), we get

$$E = 0. \tag{2.211}$$

In view of this we see that

$$j = k = 0 \tag{2.212}$$

and

$$l = m = (n - i)/2. \tag{2.213}$$

Hence on account of Eqs. (2.211) and (2.213) along with the fact that a factorial is defined for nonnegative integers, the photon number distribution can be put in the form

$$P(n,t) = (c^2 - d^2)^{1/2} \sum_{l=0}^{[n]} \frac{n!(1-c)^{n-2l} d^{2l}}{2^{2l}(n-2l)! l!^2}, \tag{2.214}$$

where $[n] = n/2$ for even n and $[n] = (n-1)/2$ for odd n.

As can be seen from Fig. 2.13 the steady-state photon number distribution decreases with the photon number. Though the photons are generated in pairs in this quantum optical system, there is a finite probability to find odd number of photons inside the cavity. This is due to the damping of the cavity mode. In addition, the probability of finding even number of photons is in general greater than the probability of finding odd number of photons. One can also obtain using (2.214) that the probability of finding n photons, with $n \leq 17$, is smaller for the light generated by the three-level laser with the parametric amplifier. And the opposite of this holds for $n \geq 18$.

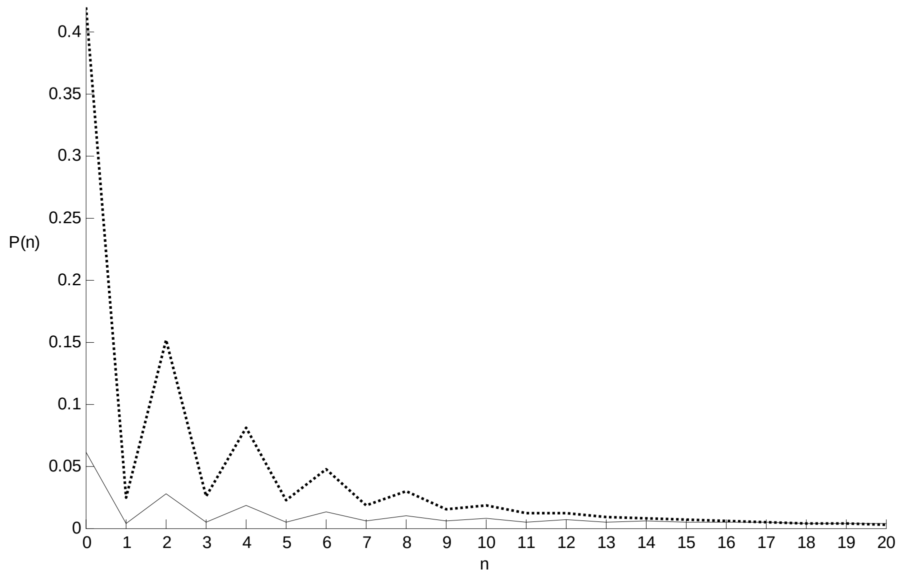

Fig. 2.13: Plots of the photon number distribution [Eq. (2.214)] at steady state versus the photon number for $\frac{\Omega}{\gamma} = 0$, $\eta = 0.1$, $A = 100$, $\kappa = 0.8$, and $\varepsilon = 5.3$ (solid curve) and $\varepsilon = 0$ (dotted curve).

2.4 Photon number and count statistics of the output mode

We first express the mean and the normally-ordered variance of the photon number for the output mode in terms of the mean and the normally-ordered variance of the photon number for the cavity mode. Then using the expression for the l^{th} moment of the photon count in terms of the photon number distribution along with the resulting mean and normally-ordered variance of the photon number for the output mode, we calculate the mean and normally-ordered variance of the photon count for the output mode.

2.4.1 Normally-ordered variance of the photon number

The mean photon number of the output mode is defined as

$$\bar{n}_{out} = \langle \alpha^*_{out}(t)\alpha_{out}(t)\rangle. \tag{2.215}$$

For a cavity mode coupled to a vacuum reservoir, the output and cavity variables can be related by

$$\alpha(t) = \sqrt{\kappa}\alpha_{out}(t), \tag{2.216}$$

so that in view of this, the mean photon number for the output mode takes the form

$$\bar{n}_{out} = \kappa\bar{n}, \tag{2.217}$$

with $\bar{n}$ being the mean photon number for the cavity mode. Thus with the aid of Eq. (2.185), the mean photon number for the output mode can be put at steady state in the form

$$\begin{aligned}\bar{n}_{out} = & \frac{16\kappa\mu^2(1+\frac{\Omega^2}{\gamma^2})^2(1+\frac{\Omega^2}{4\gamma^2})^2}{\Big((2\kappa-4\varepsilon)(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})+\frac{3A\Omega}{\gamma}(1-\eta^2)^{1/2}-\frac{A\Omega}{\gamma}(1+\Omega^2)+A(2-\frac{\Omega^2}{\gamma^2})\eta\Big)^2}\\ &+\frac{2\kappa\varepsilon(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})-A(\frac{3\Omega}{2\gamma}+\frac{\Omega^2}{2\gamma^2}-1)(1-\eta^2)^{1/2}}{4(\kappa-2\varepsilon)(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})+\frac{6A\Omega}{\gamma}(1-\eta^2)^{1/2}-\frac{2A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2})+4A(1-\frac{\Omega^2}{2\gamma^2})\eta}\\ &+\frac{-\kappa A(\frac{3\Omega}{2\gamma}-\frac{\Omega^2}{2\gamma^2}+1)\eta+A(\frac{\Omega}{2\gamma}+\frac{\Omega^3}{2\gamma^3}+\frac{\Omega^2}{\gamma^2}+1)}{4(\kappa-2\varepsilon)(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})+\frac{6A\Omega}{\gamma}(1-\eta^2)^{1/2}-\frac{2A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2})+4A(1-\frac{\Omega^2}{2\gamma^2})\eta}\\ &-\frac{2\kappa\varepsilon(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})-A(\frac{\Omega^2}{2\gamma^2}-\frac{3\Omega}{2\gamma}-1)(1-\eta^2)^{1/2}}{4(\kappa+2\varepsilon)(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})+\frac{6A\Omega}{\gamma}(1-\eta^2)^{1/2}+\frac{2A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2})+2A(2-\frac{\Omega^2}{\gamma^2})\eta}\\ &-\frac{-\kappa A(\frac{3\Omega}{2\gamma}+\frac{\Omega^2}{2\gamma^2}-1)\eta-A(\frac{\Omega^2}{\gamma^2}-\frac{\Omega^3}{2\gamma^3}-\frac{\Omega}{2\gamma}+1)}{4(\kappa+2\varepsilon)(1+\frac{\Omega^2}{\gamma^2})(1+\frac{\Omega^2}{4\gamma^2})+\frac{6A\Omega}{\gamma}(1-\eta^2)^{1/2}+\frac{2A\Omega}{\gamma}(1+\frac{\Omega^2}{\gamma^2})+2A(2-\frac{\Omega^2}{\gamma^2})\eta}. \end{aligned} \tag{2.218}$$

We next proceed to determine the normally-ordered variance of the photon number for the output mode. The normally-ordered variance of the photon number for the output mode can be defined as

$$:\Delta n_{out}^2 := \langle : \hat{n}_{out}^2 : \rangle - \langle \hat{n}_{out}\rangle^2. \tag{2.219}$$

With the aid of the input-output relation, we obtain for a cavity mode coupled to a vacuum reservoir

$$:\Delta n_{out}^2 := \kappa^2 : \Delta n^2 : . \tag{2.220}$$

in which $:\Delta n^2:$ is the normally-ordered variance of the photon number for the cavity mode given by (2.203). Therefore, on account of Eq. (2.203) the normally-ordered variance of the photon number for the output mode is found to be

$$:\Delta n_{out}^2 := \frac{4\mu^2\kappa^2}{\lambda_-^2}\left(\frac{2R-2V+\varepsilon}{4\lambda_-}\right)+2\kappa^2\left(\frac{2R-2V+\varepsilon}{4\lambda_-}\right)^2+2\kappa^2\left(\frac{2R+2V-\varepsilon}{4\lambda_+}\right)^2. \tag{2.221}$$

We note that κ is always positive. Thus one easily observes that the photon number statistics for the output mode is similar to that for the cavity mode. Moreover, in view of Eqs. (2.217) and (2.220) along with the fact that κ is less than one, we easily see that the mean and the normally-ordered variance of the photon number for the output mode are less than that for the cavity mode.

2.4.2 Normally-ordered variance of the photon count

We next wish to calculate, employing the l^{th} moment of the photon count along with the mean and the normally-ordered variance of the photon number, the mean and the normally-ordered variance of the photon count for the output mode. To this end, we recall that the l^{th} moment of the photon count is expressible as [21, 48]

$$\overline{m^l} = \sum_{n_{out}=0}^{\infty} P(n_{out}) \left((\lambda - 1)\frac{d}{d\lambda} \right)^l (1 - u\lambda)^{n_{out}} \Big|_{\lambda=0}, \tag{2.222}$$

where $P(n_{out})$ is the photon number distribution for the output mode and u represents the probability for the detection of a single photon. Thus using Eq. (2.222) the mean photon count for the output mode can be written as

$$\overline{m} = \sum_{n_{out}=0}^{\infty} P(n_{out}) \left((\lambda - 1)\frac{d}{d\lambda} \right) (1 - u\lambda)^{n_{out}} \Big|_{\lambda=0}. \tag{2.223}$$

Upon carrying out the differentiation and employing the condition $\lambda = 0$, we get

$$\overline{m} = u \sum_{n_{out}=0}^{\infty} P(n_{out}) n_{out}, \tag{2.224}$$

from which follows

$$\overline{m} = u\overline{n}_{out}. \tag{2.225}$$

Moreover, on account of Eq. (2.222) the second moment of the photon count is given by

$$\overline{m^2} = \sum_{n_{out}=0}^{\infty} P(n_{out}) \left((\lambda - 1)\frac{d}{d\lambda} \right)^2 (1 - u\lambda)^{n_{out}} \Big|_{\lambda=0}. \tag{2.226}$$

Upon performing the differentiation and applying the condition $\lambda = 0$, we find

$$\overline{m^2} = u^2 \sum_{n_{out}=0}^{\infty} P(n_{out})(n_{out}^2 - n_{out}) + u \sum_{n_{out}=0}^{\infty} P(n_{out}) n_{out}, \tag{2.227}$$

so that

$$\overline{m^2} = u^2(\overline{n^2}_{out} - \overline{n}_{out}) + u\overline{n}_{out}. \tag{2.228}$$

The normally-ordered second moment of the photon count can be written as

$$: \overline{m^2} := \overline{m^2} - \overline{m}. \tag{2.229}$$

Combining this equation with (2.225) and (2.228), we obtain

$$: \overline{m^2} := u^2 : \overline{n_{out}^2} : \tag{2.230}$$

The normally-ordered variance of the photon count is defined by

$$: \Delta m^2 :=: \overline{m^2} : - \overline{m}^2. \tag{2.231}$$

Hence substitution of (2.225) and (2.230) into Eq. (2.231) results in

$$: \Delta m^2 := u^2 : \Delta n_{out}^2 : . \tag{2.232}$$

On account of Eq. (2.220), the normally-ordered variance of the photon count (2.232) takes the form

$$: \Delta m^2 := \kappa^2 u^2 : \Delta n^2 :, \tag{2.233}$$

where $: \Delta n^2 :$ for the cavity mode under consideration is given by (2.203). Therefore, in view of Eq. (2.203) the normally-ordered variance of the photon count turns over at steady state into

$$: \Delta n_{out}^2 := \frac{4\mu^2 u^2 \kappa^2}{\lambda_-^2}\left(\frac{2R - 2V + \varepsilon}{4\lambda_-}\right) + 2\kappa^2 u^2 \left(\frac{2R - 2V + \varepsilon}{4\lambda_-}\right)^2 + 2\kappa^2 u^2 \left(\frac{2R + 2V - \varepsilon}{4\lambda_+}\right)^2 . \tag{2.234}$$

Since k and u are always positive, one can easily assert that the photon number count statistics for the output mode is similar to the photon number statistics for the output mode as well as cavity mode. Moreover, inspection of Eqs. (2.225) and (2.233) along with the fact κ and u are less than one, we observe that the mean and the normally-ordered variance of the photon count for the output mode is less than the mean and the normally-ordered variance of the photon number for the output and cavity modes.

3

Nondegenerate Three-Level Laser with Parametric Amplifier

We consider a nondegenerate three-level laser whose cavity contains a nondegenerate parametric amplifier (NOPA) and with the cavity modes driven by a strong coherent light and coupled to a vacuum reservoir. The three-level atoms injected into the cavity are initially prepared in a coherent superposition of the top and bottom levels. Moreover, the top and bottom levels of the three-level atoms are coupled by the pump mode emerging from the parametric amplifier.

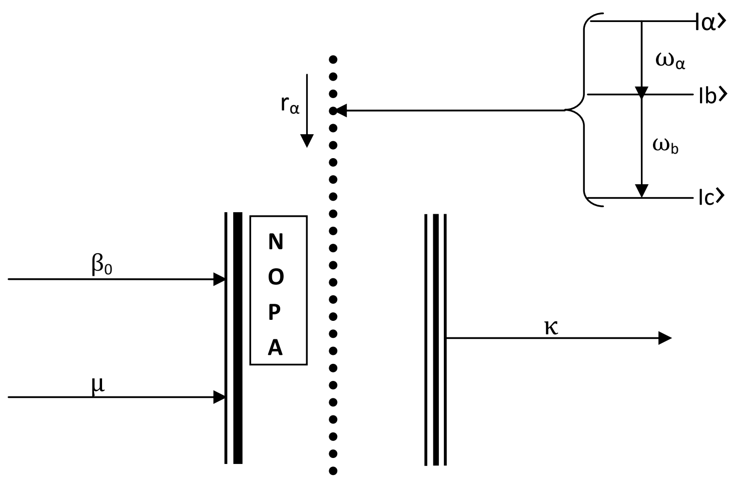

Fig. 3.1: A nondegenerate three-level laser with a parametric amplifier and strong coherent light.

We first derive the master equation (equation of evolution of the density operator) for the cavity modes and with the aid of this master equation, c-number Langevin equations associated with the normal ordering are obtained. Employing the solutions of the resulting c-number

Langevin equations, we determine the quadrature variance, the mean and the normally-ordered variance of the photon number sum and difference, and the photon number distribution of the cavity modes. In addition, applying the same solutions and the input-output relation, we calculate the quadrature variance, the squeezing spectrum, and the mean and the normally-ordered variance of the photon count sum and difference for the output modes.

3.1 c-number Langevin equations

We derive in the linear and adiabatic approximations schemes, the master equation for the cavity modes under consideration. Applying this master equation, we find the c-number Langevin equations associated with the normal ordering. Finally, we obtain the correlation properties of the noise forces.

3.1.1 Master equation

In a nondegenerate three-level laser, nondegenerate three-level atoms in a cascade configuration and initially prepared in coherent superposition of the top and bottom levels are injected at a constant rate r_a into a cavity coupled to a vacuum reservoir via a single port mirror and removed after a certain time τ. We denote the top, intermediate, and bottom levels of a three-level atom by $|a\rangle$, $|b\rangle$, and $|c\rangle$, respectively. We assume that the transitions between levels $|a\rangle$ and $|b\rangle$ and between $|b\rangle$ and $|c\rangle$ to be dipole allowed, with direct transition between levels $|a\rangle$ and $|c\rangle$ to be dipole forbidden. Further we assume the cavity mode a to be at resonance with transition $|a\rangle \rightarrow |b\rangle$ and the cavity mode b to be at resonance with the transition $|b\rangle \rightarrow |c\rangle$, with the top and bottom levels coupled by the pump mode emerging from the parametric amplifier. To this end, treating the pump mode classically, the coupling between the $|a\rangle$ and $|c\rangle$ levels can be described by the Hamiltonian

$$\hat{H}' = \frac{i\Omega}{2}(|c\rangle\langle a| - |a\rangle\langle c|), \tag{3.1}$$

in which

$$\Omega = 2\lambda\beta_0. \tag{3.2}$$

Here β_0, considered to be real and constant, is proportional to the amplitude of the pump mode and λ is the coupling constant between the pump mode and a three-level atom. In addition, the interaction of a three-level atom with the cavity modes can be described by the Hamiltonian

$$\hat{H}'' = ig\big[\hat{a}^\dagger|b\rangle\langle a| - \hat{a}|a\rangle\langle b| + \hat{b}^\dagger|c\rangle\langle b| - \hat{b}|b\rangle\langle c|\big], \tag{3.3}$$

where g is the coupling constant between the cavity modes and a three-level atom, and $\hat{a}$ and $\hat{b}$ are the annihilation operators for the cavity modes. Thus upon combining (3.1) and (3.3) the interaction of a three-level atom with the cavity modes and the pump mode can be described by the Hamiltonian

$$\hat{H} = ig\big[\hat{a}^\dagger|b\rangle\langle a| - \hat{a}|a\rangle\langle b| + \hat{b}^\dagger|c\rangle\langle b| - \hat{b}|b\rangle\langle c|\big] + \frac{i\Omega}{2}\left[|c\rangle\langle a| - |a\rangle\langle c|\right]. \tag{3.4}$$

With the aid of Eqs. (2.13) and (3.4), and following a similar procedure we have used for the degenerate case and taking into account the damping of the cavity modes by the vacuum reservoir, the master equation associated with the interactions described by the Hamiltonian (3.4) can be put in the form

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= g\left[\hat{a}^\dagger\hat{\rho}_{ab} - \hat{\rho}_{ab}\hat{a}^\dagger + \hat{\rho}_{ba}\hat{a} - \hat{a}\hat{\rho}_{ba} + \hat{b}^\dagger\hat{\rho}_{bc} - \hat{\rho}_{bc}\hat{b}^\dagger + \hat{\rho}_{cb}\hat{b} - \hat{b}\hat{\rho}_{cb}\right] \\ &+ \frac{1}{2}\kappa\left[2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\right] + \frac{1}{2}\kappa\left[2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\hat{\rho}\right].\end{aligned} \tag{3.5}$$

Applying Eq. (2.29) and taking into account (3.4) and (2.13), the equations of evolution of the matrix elements in Eq. (3.5) are found to be

$$\frac{d\hat{\rho}_{ab}}{dt} = g\left[\hat{\rho}_{aa}\hat{a} - \hat{a}\hat{\rho}_{bb} - \hat{\rho}_{ac}\hat{b}^\dagger\right] - \frac{\Omega}{2}\hat{\rho}_{cb} - \gamma\hat{\rho}_{ab}, \tag{3.6}$$

$$\frac{d\hat{\rho}_{bc}}{dt} = g\left[\hat{a}^\dagger\hat{\rho}_{ac} + \hat{\rho}_{bb}\hat{b} - \hat{b}\hat{\rho}_{cc}\right] + \frac{\Omega}{2}\hat{\rho}_{ba} - \gamma\hat{\rho}_{bc}, \tag{3.7}$$

$$\frac{d\hat{\rho}_{aa}}{dt} = \frac{r_a}{2}(1-\eta)\hat{\rho}(t) - g\left[\hat{\rho}_{ab}\hat{a}^\dagger + \hat{a}\hat{\rho}_{ba}\right] - \frac{\Omega}{2}\left(\hat{\rho}_{ca} + \hat{\rho}_{ac}\right) - \gamma\hat{\rho}_{aa}, \tag{3.8}$$

$$\frac{d\hat{\rho}_{bb}}{dt} = g\left[\hat{a}^\dagger\hat{\rho}_{ab} + \hat{\rho}_{ba}\hat{a} - \hat{\rho}_{bc}\hat{b}^\dagger - \hat{b}\hat{\rho}_{cb}\right] - \gamma\hat{\rho}_{bb}, \tag{3.9}$$

$$\frac{d\hat{\rho}_{ac}}{dt} = \frac{r_a}{2}\sqrt{1-\eta^2}e^{i\theta}\hat{\rho}(t) + g\left[\hat{\rho}_{ab}\hat{b} - \hat{a}\hat{\rho}_{bc}\right] - \frac{\Omega}{2}\left(\hat{\rho}_{cc} - \hat{\rho}_{aa}\right) - \gamma\hat{\rho}_{ac}, \tag{3.10}$$

$$\frac{d\hat{\rho}_{cc}}{dt} = \frac{r_a}{2}(1+\eta)\hat{\rho}(t) + g\left[\hat{b}^\dagger\hat{\rho}_{bc} + \hat{\rho}_{cb}\hat{b}\right] + \frac{\Omega}{2}\left(\hat{\rho}_{ac} + \hat{\rho}_{ca}\right) - \gamma\hat{\rho}_{cc}. \tag{3.11}$$

In the linear and adiabatic approximation schemes, we find

$$\hat{\rho}_{aa} = \frac{r_a}{2\gamma}(1-\eta)\hat{\rho} - \frac{\Omega}{2\gamma}(\hat{\rho}_{ac} + \hat{\rho}_{ca}), \tag{3.12}$$

$$\hat{\rho}_{bb} = 0, \tag{3.13}$$

$$\hat{\rho}_{ac} = \frac{r_a}{2\gamma}\sqrt{1-\eta^2}e^{i\theta}\hat{\rho} - \frac{\Omega}{2\gamma}(\hat{\rho}_{cc} - \hat{\rho}_{aa}), \tag{3.14}$$

$$\hat{\rho}_{cc} = \frac{r_a}{2\gamma}(1+\eta)\hat{\rho} + \frac{\Omega}{2\gamma}(\hat{\rho}_{ac} + \hat{\rho}_{ca}). \tag{3.15}$$

With the aid of Eq.(3.14) along with its complex conjugate, we get

$$\hat{\rho}_{ac} + \hat{\rho}_{ca} = \frac{r_a}{\gamma}(\sqrt{1-\eta^2})\cos\theta\hat{\rho} - \frac{\Omega}{\gamma}(\hat{\rho}_{cc} - \hat{\rho}_{aa}). \tag{3.16}$$

On account of this, Eqs. (3.12) and (3.15) take the forms

$$\hat{\rho}_{aa} = \frac{2\gamma r_a}{\gamma^2+\Omega^2}(1-\eta)\hat{\rho} - \frac{\Omega r_a}{2\gamma^2+\Omega^2}\sqrt{1-\eta^2}\cos\theta\hat{\rho} + \frac{\Omega^2}{2\gamma^2+\Omega^2}\hat{\rho}_{cc}, \tag{3.17}$$

$$\hat{\rho}_{cc} = \frac{\gamma r_a}{2\gamma^2+\Omega^2}(1+\eta)\hat{\rho} + \frac{\Omega r_a}{2\gamma^2+\Omega^2}\sqrt{1-\eta^2}\cos\theta\hat{\rho} + \frac{\Omega^2}{2\gamma^2+\Omega^2}\hat{\rho}_{aa}. \tag{3.18}$$

Substitution of (3.18) into (3.17) as well as (3.17) into (3.18) results in

$$\hat{\rho}_{aa} = -\frac{r_a\gamma}{2\gamma^2+2\Omega^2}\eta\hat{\rho} - \frac{\Omega r_a}{2\gamma^2+2\Omega^2}\sqrt{1-\eta^2}\cos\theta\hat{\rho} + \frac{r_a}{2\gamma}\hat{\rho}, \tag{3.19}$$

$$\hat{\rho}_{cc} = \frac{r_a\gamma}{2\gamma^2+2\Omega^2}\eta\hat{\rho} + \frac{\Omega r_a}{2\gamma^2+2\Omega^2}\sqrt{1-\eta^2}\cos\theta\hat{\rho} + \frac{r_a}{2\gamma}\hat{\rho}. \tag{3.20}$$

Furthermore, application of (3.19) and (3.20) in Eq. (3.14) yields

$$\hat{\rho}_{ac} = \frac{r_a(2\gamma^2+\Omega^2)}{2\gamma(2\gamma^2+2\Omega^2)}\sqrt{1-\eta^2}e^{i\theta}\hat{\rho} - \frac{r_a\Omega}{2\gamma^2+2\Omega^2}\eta\hat{\rho} - \frac{r_a\Omega^2}{2\gamma(2\gamma^2+2\Omega^2)}\sqrt{1-\eta^2}e^{-i\theta}\hat{\rho}. \tag{3.21}$$

Now combination of (3.6), (3.13), (3.19), and (3.20) as well as (3.7), (3.13), (3.20), and (3.21) leads to

$$\begin{aligned}\frac{d\hat{\rho}_{ab}}{dt} = \frac{gr_a}{\gamma}\Bigg[&\frac{\Omega\gamma}{2\gamma^2+2\Omega^2}\eta\hat{\rho}\hat{b}^\dagger + \frac{\Omega^2}{4\gamma^2+4\Omega^2}\sqrt{1-\eta^2}e^{-i\theta}\hat{\rho}\hat{b}^\dagger - \frac{2\gamma^2+\Omega^2}{4\gamma^2+4\Omega^2}\sqrt{1-\eta^2}e^{i\theta}\hat{\rho}\hat{b}^\dagger \\ &- \frac{\gamma^2}{2\gamma^2+2\Omega^2}\eta\hat{\rho}\hat{a} + \frac{1}{2}\hat{\rho}\hat{a} - \frac{\Omega\gamma}{2\gamma^2+2\Omega^2}\sqrt{1-\eta^2}\cos\theta\hat{\rho}\hat{a}\Bigg] - \frac{\Omega}{2}\hat{\rho}_{cb} - \gamma\hat{\rho}_{ab},\end{aligned} \tag{3.22}$$

$$\begin{aligned}\frac{d\hat{\rho}_{bc}}{dt} = \frac{gr_a}{\gamma}\Bigg[&-\frac{\gamma^2}{2\gamma^2+2\Omega^2}\eta\hat{b}\hat{\rho} - \frac{1}{2}\hat{b}\hat{\rho} - \frac{\Omega\gamma}{2\gamma^2+2\Omega^2}\sqrt{1-\eta^2}\cos\theta\hat{b}\hat{\rho} - \frac{\Omega\gamma}{2\gamma^2+2\Omega^2}\eta\hat{a}^\dagger\hat{\rho} \\ &+ \frac{2\gamma^2+\Omega^2}{4\gamma^2+4\Omega^2}\sqrt{1-\eta^2}e^{i\theta}\hat{a}^\dagger\hat{\rho} - \frac{\Omega^2}{4\gamma^2+4\Omega^2}\sqrt{1-\eta^2}e^{-i\theta}\hat{a}^\dagger\hat{\rho}\Bigg] + \frac{\Omega}{2}\hat{\rho}_{ba} - \gamma\hat{\rho}_{bc}.\end{aligned} \tag{3.23}$$

Using once more the adiabatic approximation scheme, we easily find

$$\begin{aligned}\hat{\rho}_{ab} = \frac{gr_a}{\gamma^2}\Bigg[&\frac{\Omega\gamma}{2\gamma^2+2\Omega^2}\eta\hat{\rho}\hat{b}^\dagger + \frac{\Omega^2}{4\gamma^2+4\Omega^2}\sqrt{1-\eta^2}e^{-i\theta}\hat{\rho}\hat{b}^\dagger - \frac{2\gamma^2+\Omega^2}{4\gamma^2+4\Omega^2}\sqrt{1-\eta^2}e^{i\theta}\hat{\rho}\hat{b}^\dagger \\ &- \frac{\gamma^2}{2\gamma^2+2\Omega^2}\eta\hat{\rho}\hat{a} + \frac{1}{2}\hat{\rho}\hat{a} - \frac{\Omega\gamma}{2\gamma^2+2\Omega^2}\sqrt{1-\eta^2}\cos\theta\hat{\rho}\hat{a}\Bigg] - \frac{\Omega}{2}\hat{\rho}_{cb},\end{aligned} \tag{3.24}$$

$$\hat{\rho}_{bc} = \frac{gr_a}{\gamma^2}\bigg[-\frac{\gamma^2}{2\gamma^2+2\Omega^2}\eta\hat{b}\hat{\rho} - \frac{1}{2}\hat{b}\hat{\rho} - \frac{\Omega\gamma}{2\gamma^2+2\Omega^2}\sqrt{1-\eta^2}\cos\theta\hat{b}\hat{\rho} - \frac{\Omega\gamma}{2\gamma^2+2\Omega^2}\eta\hat{a}^\dagger\hat{\rho} + \frac{2\gamma^2+\Omega^2}{4\gamma^2+4\Omega^2}\sqrt{1-\eta^2}e^{i\theta}\hat{a}^\dagger\hat{\rho} - \frac{\Omega^2}{4\gamma^2+4\Omega^2}\sqrt{1-\eta^2}e^{-i\theta}\hat{a}^\dagger\hat{\rho}\bigg] + \frac{\Omega}{2}\hat{\rho}_{ba}. \tag{3.25}$$

Moreover, substitution of the complex conjugate of (3.25) into (3.24) as well as the complex conjugate of (3.24) into (3.25), we get

$$\hat{\rho}_{ab} = \frac{A}{4Bg}\bigg([\frac{3\Omega}{\gamma}\eta + \frac{\Omega}{2\gamma}(1+\frac{\Omega^2}{\gamma^2}) - (2+\frac{\Omega^2}{2\gamma^2})\frac{\sqrt{1-\eta^2}}{2}e^{i\theta} + \frac{3\Omega^2}{2\gamma^2}\frac{\sqrt{1-\eta^2}}{2}e^{i\theta}]\hat{\rho}\hat{b}^\dagger + [(\frac{\Omega^2}{2\gamma^2}-1)\eta + (\frac{\Omega^2}{\gamma^2}+1) + (\frac{\Omega^3}{2\gamma^3}-\frac{\Omega}{\gamma})\frac{\sqrt{1-\eta^2}}{2}e^{i\theta} - (2\frac{\Omega}{\gamma}+\frac{\Omega^3}{2\gamma^3})\frac{\sqrt{1-\eta^2}}{2}e^{-i\theta}]\hat{\rho}\hat{a}\bigg) \tag{3.26}$$

$$\hat{\rho}_{bc} = \frac{A}{4Bg}\bigg([-\frac{3\Omega}{2\gamma}\eta + \frac{\Omega}{2\gamma}(1+\frac{\Omega^2}{\gamma^2}) + (2+\frac{\Omega^2}{2\gamma^2})\frac{\sqrt{1-\eta^2}}{2}e^{i\theta} - \frac{3\Omega^2}{2\gamma^2}\frac{\sqrt{1-\eta^2}}{2}e^{-i\theta}]\hat{a}^\dagger\hat{\rho} + [(\frac{\Omega^2}{2\gamma^2}-1)\eta - (\frac{\Omega^2}{\gamma^2}+1) + (\frac{\Omega^3}{2\gamma^3}-\frac{\Omega}{\gamma})\frac{\sqrt{1-\eta^2}}{2}e^{i\theta} - \frac{\Omega}{\gamma}(2+\frac{\Omega^2}{2\gamma^2})\frac{\sqrt{1-\eta^2}}{2}e^{-i\theta}]\hat{b}\hat{\rho}\bigg), \tag{3.27}$$

where

$$B = (1+\frac{\Omega^2}{4\gamma^2})(1+\frac{\Omega^2}{\gamma^2}) \tag{3.28}$$

and

$$A = \frac{2r_a g^2}{\gamma^2} \tag{3.29}$$

is the linear gain coefficient. Finally, on account of Eqs. (3.26), (3.27), and their complex conjugates the master equation for the cavity modes of a nondegenerate three-level laser given by (3.5) takes, for $\theta = 0$, the form

$$\frac{d\hat{\rho}}{dt} = R\Big(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}\Big) + S\Big(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}\Big) + U\Big(\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger + \hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{a}^\dagger - \hat{a}\hat{b}\hat{\rho}\Big) + V\Big(\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger + \hat{b}\hat{\rho}\hat{a} - \hat{b}^\dagger\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{b}\Big) + \frac{\kappa}{2}\Big(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\Big) + \frac{\kappa}{2}\Big(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}\Big), \tag{3.30}$$

in which

$$R = \frac{A}{4B}\bigg[-\frac{3\Omega}{2\gamma}(1-\eta^2)^{1/2} + (\frac{\Omega^2}{2\gamma^2}-1)\eta + (1+\frac{\Omega^2}{\gamma^2})\bigg], \tag{3.31}$$

$$S = \frac{A}{4B}\bigg[\frac{3\Omega}{2\gamma}(1-\eta^2)^{1/2} + (1-\frac{\Omega^2}{2\gamma^2})\eta + (1+\frac{\Omega^2}{\gamma^2})\bigg], \tag{3.32}$$

$$U = \frac{A}{4B}\bigg[(\frac{\Omega^2}{2\gamma^2}-1)(1-\eta^2)^{1/2} + \frac{3\Omega}{2\gamma}\eta + \frac{\Omega}{2\gamma}(1+\frac{\Omega^2}{\gamma^2})\bigg], \tag{3.33}$$

$$V = \frac{A}{4B}\Big[(\frac{\Omega^2}{2\gamma^2} - 1)(1-\eta^2)^{1/2} + \frac{3\Omega}{2\gamma}\eta - \frac{\Omega}{2\gamma}(1+\frac{\Omega^2}{\gamma^2})\Big]. \tag{3.34}$$

On the other hand, a nondegenerate parametric amplifier with the pump mode treated classically is described by the Hamiltonian

$$H_1 = i\varepsilon\big(\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b}\big), \tag{3.35}$$

in which $\varepsilon = \lambda'\beta_0$, with λ' being the coupling constant and β_0 proportional to the amplitude of the pump mode. In addition, the interaction of the cavity modes with the strong driving light can be described by the Hamiltonian

$$H_2 = i\mu\Big(\hat{a}^\dagger - \hat{a} + \hat{b}^\dagger - \hat{b}\Big), \tag{3.36}$$

where μ, considered to be real and constant, is proportional to the amplitude of the driving light. The master equation associated with the interactions described by the Hamiltonians (3.35) and (3.36) has the form

$$\frac{d\hat{\rho}}{dt} = \varepsilon\left(\hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{a}\hat{b}\hat{\rho} + \hat{\rho}\hat{a}\hat{b} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\right) + \mu\Big(\hat{a}^\dagger\hat{\rho} + \hat{b}^\dagger\hat{\rho} - \hat{a}\hat{\rho} - \hat{b}\hat{\rho} - \hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{b}^\dagger + \hat{\rho}\hat{a} + \hat{\rho}\hat{b}\Big). \tag{3.37}$$

Hence on combining Eqs. (3.30) and (3.37) the master equation for the cavity modes of the optical system under consideration turns out to be

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= \mu\Big(\hat{a}^\dagger\hat{\rho} + \hat{b}^\dagger\hat{\rho} - \hat{a}\hat{\rho} - \hat{b}\hat{\rho} - \hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{b}^\dagger + \hat{\rho}\hat{a} + \hat{\rho}\hat{b}\Big) + \varepsilon\Big(\hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{a}\hat{b}\hat{\rho} + \hat{\rho}\hat{a}\hat{b} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\Big)\\ &+ R\Big(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}\Big) + S\Big(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}\Big) + U\Big(\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger - \hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{a}^\dagger - \hat{a}\hat{b}\hat{\rho}\Big)\\ &+ V\Big(\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger + \hat{b}\hat{\rho}\hat{a} - \hat{b}^\dagger\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{b}\Big) + \frac{\kappa}{2}\Big(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\Big) + \frac{\kappa}{2}\Big(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}\Big).\end{aligned} \tag{3.38}$$

3.1.2 c-number Langevin equations

Now employing the relation $\frac{d}{dt}\langle\hat{A}\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{A})$ along with the master equation (3.38), one readily obtains the following equations

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -(\frac{\kappa}{2} - R)\langle\hat{a}(t)\rangle + (U+\varepsilon)\langle\hat{b}^\dagger(t)\rangle + \mu, \tag{3.39}$$

$$\frac{d}{dt}\langle\hat{b}(t)\rangle = -(\frac{\kappa}{2} + S)\langle\hat{b}(t)\rangle - (V-\varepsilon)\langle\hat{a}^\dagger(t)\rangle + \mu, \tag{3.40}$$

$$\frac{d}{dt}\langle\hat{a}^2(t)\rangle = -(\kappa - 2R)\langle\hat{a}^2(t)\rangle + 2(U+\varepsilon)\langle\hat{b}^\dagger(t)\hat{a}(t)\rangle + 2\mu\langle\hat{a}(t)\rangle, \tag{3.41}$$

$$\frac{d}{dt}\langle\hat{a}(t)\hat{b}(t)\rangle = -(\kappa + S - R)\langle\hat{a}(t)\hat{b}(t)\rangle + (U+\varepsilon)\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle - (V-\varepsilon)\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle \\ + \mu(\langle\hat{a}(t)\rangle + \langle\hat{b}(t)\rangle) - (V-\varepsilon), \tag{3.42}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle = -(\kappa - 2R)\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle + (U+\varepsilon)\Big(\langle\hat{a}^\dagger(t)\hat{b}^\dagger(t)\rangle + \langle\hat{a}(t)\hat{b}(t)\rangle\Big) \\ + \mu(\langle\hat{a}^\dagger(t)\rangle + \langle\hat{a}(t)\rangle) + 2R, \tag{3.43}$$

$$\frac{d}{dt}\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle = -(\kappa + 2S)\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle - (V-\varepsilon)\Big(\langle\hat{a}^\dagger(t)\hat{b}^\dagger(t)\rangle + \langle\hat{a}(t)\hat{b}(t)\rangle\Big) \\ + \mu(\langle\hat{b}^\dagger(t)\rangle + \langle\hat{b}(t)\rangle). \tag{3.44}$$

We note that the operators in the above set of equations are in the normal-order. Thus the c-number equations associated with this ordering are

$$\frac{d}{dt}\langle\alpha(t)\rangle = -(\frac{\kappa}{2} - R)\langle\alpha(t)\rangle + (U+\varepsilon)\langle\beta^*(t)\rangle + \mu, \tag{3.45}$$

$$\frac{d}{dt}\langle\beta(t)\rangle = -(\frac{\kappa}{2} + S)\langle\beta(t)\rangle - (V-\varepsilon)\langle\alpha^*(t)\rangle + \mu, \tag{3.46}$$

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -(\kappa - 2R)\langle\alpha^2(t)\rangle + 2(U+\varepsilon)\langle\alpha(t)\beta^*(t)\rangle + 2\mu\langle\alpha(t)\rangle, \tag{3.47}$$

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = -(\kappa + S - R)\langle\alpha(t)\beta(t)\rangle + (U+\varepsilon)\langle\beta^*(t)\beta(t)\rangle - (V-\varepsilon)\langle\alpha^*(t)\alpha(t)\rangle \\ + \mu(\langle\alpha(t)\rangle + \langle\beta(t)\rangle) - (V-\varepsilon), \tag{3.48}$$

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = -(\kappa - 2R)\langle\alpha^*(t)\alpha(t)\rangle + (U+\varepsilon)\Big(\langle\alpha^*(t)\beta^*(t)\rangle + \langle\alpha(t)\beta(t)\rangle\Big) \\ + \mu(\langle\alpha^*(t)\rangle + \langle\alpha(t)\rangle) + 2R, \tag{3.49}$$

$$\frac{d}{dt}\langle\beta^*(t)\beta(t)\rangle = -(\kappa + 2S)\langle\beta^*(t)\beta(t)\rangle - (V-\varepsilon)\Big(\langle\alpha^*(t)\beta^*(t)\rangle + \langle\alpha(t)\beta(t)\rangle\Big) \\ + \mu(\langle\beta^*(t)\rangle + \langle\beta(t)\rangle). \tag{3.50}$$

We claim that the equation of evolution of $\alpha(t)$ and $\beta(t)$ (c-number Langevin equations) can be obtained from that of $\langle\alpha(t)\rangle$ and $\langle\beta(t)\rangle$. This can be achieved by dropping the angular brackets in Eqs. (3.45) and (3.46) and adding the noise forces $f_\alpha(t)$ and $f_\beta(t)$, so that

$$\frac{d}{dt}\alpha(t) = -(\frac{\kappa}{2} - R)\alpha(t) + (U+\varepsilon)\beta^*(t) + f_\alpha(t) + \mu \tag{3.51}$$

and

$$\frac{d}{dt}\beta(t) = -(\frac{\kappa}{2} + S)\beta(t) - (V - \varepsilon)\alpha^*(t) + f_\beta(t) + \mu. \tag{3.52}$$

We next seek to determine the properties of the noise forces $f(t)$ and $f_\beta(t)$. We note that Eq. (3.45) and the expectation value of (3.51) as well as Eq. (3.46) and the expectation value of (3.52) will have the same forms if

$$\langle f_\alpha(t)\rangle = \langle f_\beta(t)\rangle = 0. \tag{3.53}$$

Now employing Eqs. (3.51) and (3.52) together with the relation

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = \left\langle \alpha(t)\frac{d\beta(t)}{dt}\right\rangle + \left\langle \frac{d\alpha(t)}{dt}\beta(t)\right\rangle, \tag{3.54}$$

one easily gets

$$\begin{aligned}\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = &-(\kappa + S - R)\langle\alpha(t)\beta(t)\rangle + (U + \varepsilon)\langle\beta^*(t)\beta(t)\rangle - (V - \varepsilon)\langle\alpha^*(t)\alpha(t)\rangle \\ &+ \mu(\langle\alpha(t)\rangle + \langle\beta(t)\rangle) + \langle\alpha(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha(t)\rangle.\end{aligned} \tag{3.55}$$

Comparison of Eqs. (3.55) and (3.48) indicates that

$$\langle\alpha(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha(t)\rangle = -(V - \varepsilon). \tag{3.56}$$

The solutions of Eqs. (3.51) and (3.52) can be put in the form

$$\alpha(t) = \alpha(0)e^{-(\kappa/2-R)t} + \int_0^t e^{-(\kappa/2-R)(t-t')}\left[\mu + f_\alpha(t') + (U + \varepsilon)\beta^*(t')\right]dt' \tag{3.57}$$

and

$$\beta(t) = \beta(0)e^{-(\kappa/2+S)t} + \int_0^t e^{-(\kappa/2+S)(t-t')}\left[\mu + f_\beta(t') - (V - \varepsilon)\alpha^*(t')\right]dt'. \tag{3.58}$$

Upon multiplying Eq. (3.57) on the right by $f_\beta(t)$ and taking the expectation value of the resulting expression, we find

$$\begin{aligned}\langle\alpha(t)f_\beta(t)\rangle = \langle\alpha(0)f_\beta(t)\rangle e^{-(\kappa/2-R)t} + \int_0^t e^{-(\kappa/2-R)(t-t')}\Big[&\mu\langle f_\beta(t)\rangle \\ + \langle f_\alpha(t')f_\beta(t)\rangle + (U + \varepsilon)\langle\beta^*(t')f_\beta(t)\rangle&\Big]dt'.\end{aligned} \tag{3.59}$$

Assuming the noise force f_β at time t can not affect the system variables at the earlier times together with Eq. (3.53), we have

$$\langle\alpha(0)f_\beta(t)\rangle = \langle\alpha(0)\rangle\langle f_\beta(t)\rangle = 0, \tag{3.60}$$

$$\langle\beta(t')f_\beta(t)\rangle = \langle\beta(t')\rangle\langle f_\beta(t)\rangle = 0. \tag{3.61}$$

Hence on account of Eqs. (3.53), (3.60), and (3.61), we obtain

$$\langle\alpha(t)f_\beta(t)\rangle = \int_0^t e^{-(\kappa/2-R)(t-t')}\langle f_\alpha(t')f_\beta(t)\rangle dt'. \tag{3.62}$$

It can also be verified in a similar fashion that

$$\langle\beta(t)f_\alpha(t)\rangle = \int_0^t e^{-(\kappa/2+S)(t-t')}\langle f_\beta(t')f_\alpha(t)\rangle dt'. \tag{3.63}$$

Therefore, in view of Eqs. (3.56), (3.62), (3.63), and the assumption that

$$\langle f_\alpha(t')f_\beta(t)\rangle = \langle f_\beta(t')f_\alpha(t)\rangle, \tag{3.64}$$

we arrive at

$$\int_0^t e^{-(\kappa/2-R)(t-t')}\langle f_\alpha(t')f_\beta(t)\rangle dt' = \int_0^t e^{-(\kappa/2+S)(t-t')}\langle f_\beta(t')f_\alpha(t)\rangle dt' = -\frac{(V-\varepsilon)}{2}. \tag{3.65}$$

Now on the basis of the relations (2.83) and (2.84) , one readily gets

$$\langle f_\alpha(t')f_\beta(t)\rangle = \langle f_\beta(t')f_\alpha(t)\rangle = -(V-\varepsilon)\delta(t-t'). \tag{3.66}$$

Furthermore, applying Eq. (3.51) along with the relation

$$\frac{d}{dt}\langle\alpha(t)\alpha(t)\rangle = 2\left\langle\frac{d\alpha(t)}{dt}\alpha(t)\right\rangle, \tag{3.67}$$

we obtain

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -(\kappa-2R)\langle\alpha^2(t)\rangle + 2(U+\varepsilon)\langle\alpha(t)\beta^*(t)\rangle + 2\mu\langle\alpha(t)\rangle + 2\langle\alpha(t)f_\alpha(t)\rangle. \tag{3.68}$$

Comparison of Eqs. (3.68) and (3.47) shows that

$$\langle\alpha(t)f_\alpha(t)\rangle = 0. \tag{3.69}$$

Using Eq. (3.57) together with (3.69), we find

$$\int_0^t e^{-(\kappa/2-R)(t-t')}\langle f_\alpha(t)f_\alpha(t')\rangle dt' = 0, \tag{3.70}$$

from which follows

$$\langle f_\alpha(t)f_\alpha(t')\rangle = 0. \tag{3.71}$$

It can also be established in a similar manner that

$$\langle f_\beta(t)f_\beta(t')\rangle = \langle f^*_\alpha(t)f_\beta(t')\rangle = 0. \tag{3.72}$$

On the other hand, using Eq. (3.51) and its complex conjugate along with the relation

$$\frac{d}{dt}\left\langle \alpha^*(t)\alpha(t)\right\rangle = \left\langle \alpha^*(t)\frac{d\alpha(t)}{dt}\right\rangle + \left\langle \alpha(t)\frac{d\alpha^*(t)}{dt}\right\rangle, \tag{3.73}$$

we find

$$\begin{aligned}\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle &= -(\kappa-2R)\langle\alpha^*(t)\alpha(t)\rangle + (U+\varepsilon)\Big(\langle\alpha^*(t)\beta^*(t)\rangle + \langle\alpha(t)\beta(t)\rangle\Big)\\ &+ \langle\alpha(t)f^*_\alpha(t)\rangle + \langle\alpha^*(t)f_\alpha(t)\rangle + \mu(\langle\alpha^*(t)\rangle + \langle\alpha(t)\rangle).\end{aligned} \tag{3.74}$$

Inspection of Eqs. (3.74) and (3.49) indicates that

$$\langle\alpha(t)f^*_\alpha(t)\rangle + \langle\alpha^*(t)f_\alpha(t)\rangle = 2R. \tag{3.75}$$

With the aid of Eq. (3.57) and its complex conjugate, we get

$$\langle\alpha(t)f^*_\alpha(t)\rangle = \int_0^t e^{-(\kappa/2-R)(t-t')}\langle f^*_\alpha(t)f_\alpha(t')\rangle dt' \tag{3.76}$$

and

$$\langle\alpha^*(t)f_\alpha(t)\rangle = \int_0^t e^{-(\kappa/2-R)(t-t')}\langle f_\alpha(t)f^*_\alpha(t')\rangle dt'. \tag{3.77}$$

On account of Eqs. (3.75), (3.76), (3.77), and the assumption that

$$\langle f^*_\alpha(t)f_\alpha(t')\rangle = \langle f_\alpha(t)f^*_\alpha(t')\rangle, \tag{3.78}$$

there follows

$$\int_0^t e^{-(\kappa/2-R)(t-t')}\langle f^*_\alpha(t)f_\alpha(t')\rangle dt' = \int_0^t e^{-(\kappa/2-R)(t-t')}\langle f_\alpha(t)f^*_\alpha(t')\rangle dt' = R. \tag{3.79}$$

With the help of Eqs. (2.83) and (2.84), we have

$$\langle f^*_\alpha(t)f_\alpha(t')\rangle = \langle f_\alpha(t)f^*_\alpha(t')\rangle = 2R\delta(t-t'). \tag{3.80}$$

In addition, employing Eq. (3.52) and its complex conjugate together with the relation

$$\frac{d}{dt}\left\langle \beta^*(t)\beta(t)\right\rangle = \left\langle \beta^*(t)\frac{d\beta(t)}{dt}\right\rangle + \left\langle \beta(t)\frac{d\beta^*(t)}{dt}\right\rangle, \tag{3.81}$$

we readily get

$$\begin{aligned}\frac{d}{dt}\langle\beta^*(t)\beta(t)\rangle &= -(\kappa+2S)\langle\beta^*(t)\beta(t)\rangle - (V-\varepsilon)\Big(\langle\alpha^*(t)\beta^*(t)\rangle + \langle\alpha(t)\beta(t)\rangle\Big)\\ &+ \mu(\langle\beta^*(t)\rangle + \langle\beta(t)\rangle) + \langle\beta(t)f^*_\beta(t)\rangle + \langle\beta^*(t)f_\beta(t)\rangle.\end{aligned} \tag{3.82}$$

Comparison of Eqs. (3.82) and (3.50) shows that

$$\langle\beta(t)f^*_\beta(t)\rangle + \langle\beta^*(t)f_\beta(t)\rangle = 0. \tag{3.83}$$

On account of Eqs. (3.58) and (3.83), we obtain

$$\int_0^t e^{-(\kappa/2+S)(t-t')}\langle f_\beta^*(t)f_\beta(t')\rangle dt' + \int_0^t e^{-(\kappa/2+S)(t-t')}\langle f_\beta(t)f_\beta^*(t')\rangle dt' = 0. \tag{3.84}$$

Assuming that

$$\langle f_\beta^*(t)f_\beta(t')\rangle = \langle f_\beta(t)f_\beta^*(t')\rangle, \tag{3.85}$$

we find

$$\langle f_\beta^*(t)f_\beta(t')\rangle = \langle f_\beta(t)f_\beta^*(t')\rangle = 0. \tag{3.86}$$

We note that Eqs. (3.53), (3.66), (3.71), (3.72), (3.80), and (3.86) describe the correlation properties of the noise forces $f_\alpha(t)$ and $f_\beta(t)$ associated with the normal ordering.

3.2 Quadrature squeezing

Applying the solutions of the c-number Langevin equations we have obtained in Appendix B, we calculate the quadrature variance for the cavity modes. We also determine, employing the same solutions and the input-output relation, the quadrature variance and the squeezing spectrum for the output modes.

3.2.1 Quadrature variance of the cavity modes

We next proceed to determine the quadrature variance for the cavity radiation produced by the system under consideration. The squeezing properties of a two-mode light are described by two quadrature operators

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c} \tag{3.87}$$

and

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}), \tag{3.88}$$

where

$$\hat{c} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{b}). \tag{3.89}$$

It can easily be shown that

$$[\hat{c}, \hat{c}^\dagger] = 1 \tag{3.90}$$

and

$$[\hat{c}_+, \hat{c}_-] = 2i. \tag{3.91}$$

A two-mode light is said to be in a squeezed state if either $\Delta c_+ < 1$ and $\Delta c_- > 1$ or $\Delta c_+ > 1$ and $\Delta c_- < 1$ such that $\Delta c_+ \Delta c_- \geq 1$. The variance of the quadrature operators are defined by

$$\Delta c_{\pm}^2(t) = \langle \hat{c}_{\pm}^2(t)\rangle - \langle \hat{c}_{\pm}(t)\rangle^2. \tag{3.92}$$

Applying Eqs. (3.87) and (3.88), one can write (3.92) in the normal order as

$$\Delta c_{\pm}^2(t) = 1 \pm \langle \hat{c}^{\dagger 2}(t) + \hat{c}^2(t) \pm 2\hat{c}^{\dagger}(t)\hat{c}(t)\rangle \mp \langle \hat{c}^{\dagger}(t) \pm \hat{c}(t)\rangle^2. \tag{3.93}$$

This can be expressed in terms of c-number variables associated with the normal ordering as

$$\Delta c_{\pm}^2(t) = 1 \pm \langle [\gamma^*(t) \pm \gamma(t)]^2\rangle \mp \langle \gamma^*(t) \pm \gamma(t)\rangle^2, \tag{3.94}$$

where $\gamma(t)$ is the c-number variable corresponding to the operator $\hat{c}(t)$. The c- number equation corresponding to Eq. (3.89) can be written as

$$\gamma(t) = \frac{1}{\sqrt{2}}(\alpha(t) + \beta(t)) \tag{3.95}$$

and application of (3.95) to Eq. (3.94) leads to

$$\begin{aligned}\Delta c_{\pm}^2(t) = 1 \pm \Big[&\frac{1}{2}\big(\langle \alpha^2(t)\rangle + \langle \alpha^{*2}(t)\rangle + \langle \beta^2(t)\rangle + \langle \beta^{*2}(t)\rangle + 2\langle \alpha(t)\beta(t)\rangle + 2\langle \alpha^*(t)\beta^*(t)\rangle\big)\\ &\pm \big(\langle \alpha^*(t)\alpha(t)\rangle + \langle \beta^*(t)\beta(t)\rangle + \langle \alpha^*(t)\beta(t)\rangle + \langle \beta^*(t)\alpha(t)\rangle\big)\Big]\\ &\mp \frac{1}{2}\left\langle (\alpha^*(t) + \beta^*(t)) \pm (\alpha(t) + \beta(t))\right\rangle^2. \end{aligned}\tag{3.96}$$

With the aid of Eqs. (B34) and (B35) along with the assumption that the cavity modes are initially in a two-mode vacuum state, we find

$$\langle \alpha(t)\rangle = E_+(t) \tag{3.97}$$

and

$$\langle \beta(t)\rangle = E_-(t). \tag{3.98}$$

Taking into account the fact that the noise forces $f_\alpha(t')$ and $f_\beta(t')$ do not affect $\alpha(0)$ and $\beta(0)$, one readily obtains applying Eqs. (B34), (B35), (3.97), and (3.98) that

$$\begin{aligned}\Delta c_{\pm}^2(t) = 1 &+ \langle F_+^*(t)F_+(t)\rangle + \langle F_-^*(t)F_-(t)\rangle + \langle F_+^*(t)F_-(t)\rangle + \langle F_-^*(t)F_+(t)\rangle\\ &\pm \frac{1}{2}\Big[\langle F_+^2(t)\rangle + \langle F_+^{*2}(t)\rangle + \langle F_-^2(t)\rangle + \langle F_-^{*2}(t)\rangle + 2\langle F_+(t)F_-(t)\rangle + 2\langle F_+^*(t)F_-^*(t)\rangle\Big]\\ &\pm \Big[\frac{1}{2}\big(E_+^2(t) + E_+^{*2}(t) + E_-^2(t) + E_-^{*2}(t) + 2E_+(t)E_-^*(t) + 2E_+^*(t)E_-(t)\big)\\ &\pm \big(E_+(t)E_+^*(t) + E_-(t)E_-^*(t) + E_+^*(t)E_-^*(t) + E_-(t)E_+\big)\Big]\\ &\mp \frac{1}{2}\big[\big(E_+^*(t) + E_-(t)\big) \pm \big(E_+(t) + E_-^*(t)\big)\big]^2. \end{aligned}\tag{3.99}$$

It then follows that

$$\Delta c_{\pm}^2(t) = 1 + \langle F_+^*(t)F_+(t)\rangle + \langle F_-^*(t)F_-(t)\rangle + \langle F_+^*(t)F_-(t)\rangle + \langle F_-^*(t)F_+(t)\rangle$$
$$\pm \frac{1}{2}\Big[\langle F_+^2(t)\rangle + \langle F_+^{*2}(t)\rangle + \langle F_-^2(t)\rangle + \langle F_-^{*2}(t)\rangle + 2\langle F_+(t)F_-(t)\rangle + 2\langle F_+^*(t)F_-^*(t)\rangle\Big]. \tag{3.100}$$

Now employing Eq. (B38), we have

$$\begin{aligned}\langle F_+^2(t)\rangle = \frac{1}{4}\bigg(\int_0^t &\big[(1+p)e^{-\lambda_2(t-t')} + (1-p)e^{-\lambda_1(t-t')}\big]\\ &\times \big[(1+p)e^{-\lambda_2(t-t'')} + (1-p)e^{-\lambda_1(t-t'')}\big]\langle f_\alpha(t')f_\alpha(t'')\rangle dt'dt''\bigg)\\ &+ \frac{q_+^2}{4}\int_0^t \big[e^{-\lambda_1(t-t')} - e^{-\lambda_2(t-t')}\big]\big[e^{-\lambda_1(t-t'')} - e^{-\lambda_2(t-t'')}\big]\langle f_\beta^*(t')f_\beta^*(t'')\rangle dt'dt''\\ &+ \frac{q_+}{4}\bigg(\int_0^t \big[(1+p)e^{-\lambda_2(t-t')} + (1-p)e^{-\lambda_1(t-t')}\big]\\ &\times \big[e^{-\lambda_1(t-t'')} - e^{-\lambda_2(t-t'')}\big]\langle f_\alpha(t')f_\beta^*(t'')\rangle dt'dt''\bigg)\\ &+ \frac{q_+}{\lambda^2}\bigg(\int_0^t \big[e^{-\lambda_1(t-t')} - e^{-\lambda_2(t-t')}\big]\\ &\times \big[(1+p)e^{-\lambda_2(t-t'')} + (1-p)e^{-\lambda_1(t-t'')}\big]\langle f_\beta^*(t')f_\alpha(t'')\rangle dt'dt''\bigg).\end{aligned} \tag{3.101}$$

In view of Eqs. (3.71) and (3.72), we see that

$$\langle F_+^2(t)\rangle = 0. \tag{3.102}$$

One can also establish in a similar manner that

$$\langle F_-^2(t)\rangle = \langle F_+(t)F_-^*(t)\rangle = 0. \tag{3.103}$$

Hence introducing Eqs. (3.102), (3.103), and their complex conjugates into (3.100), one finds

$$\Delta c_{\pm}^2(t) = 1 + \langle F_+^*(t)F_+(t)\rangle + \langle F_-^*(t)F_-(t)\rangle \pm \big[\langle F_+(t)F_-(t)\rangle + \langle F_+^*(t)F_-^*(t)\rangle\big]. \tag{3.104}$$

On the other hand, using Eq. (B38) and its complex conjugate one can write

$$\begin{aligned}\langle F_+(t)F_+^*(t)\rangle &= \frac{1}{4}\bigg(\int_0^t \big[(1+p)e^{-\lambda_2(t-t')} + (1-p)e^{-\lambda_1(t-t')}\big]\\ &\times \big[(1+p)e^{-\lambda_2(t-t'')} + (1-p)e^{-\lambda_1(t-t'')}\big]\langle f_\alpha(t')f_\alpha^*(t'')\rangle dt'dt''\bigg)\\ &+ \frac{q_+^2}{4}\int_0^t \big[e^{-\lambda_1(t-t')} - e^{-\lambda_2(t-t')}\big]\big[e^{-\lambda_1(t-t'')} - e^{-\lambda_2(t-t'')}\big]\langle f_\beta^*(t')f_\beta(t'')\rangle dt'dt''\\ &+ \frac{q_+}{4}\bigg(\int_0^t \big[(1+p)e^{-\lambda_2(t-t')} + (1-p)e^{-\lambda_1(t-t')}\big]\big[e^{-\lambda_1(t-t'')} - e^{-\lambda_2(t-t'')}\big]\langle f_\alpha(t')f_\beta(t'')\rangle dt'dt''\bigg)\\ &+ \frac{q_+}{4}\bigg(\int_0^t \big[e^{-\lambda_1(t-t')} - e^{-\lambda_2(t-t')}\big]\big[(1+p)e^{-\lambda_2(t-t'')} + (1-p)e^{-\lambda_1(t-t'')}\big]\langle f_\beta^*(t')f_\alpha^*(t'')\rangle dt'dt''\bigg).\end{aligned} \tag{3.105}$$

With the aid of (3.66), (3.80), and (3.86), we get

$$
\begin{aligned}
\langle F_+(t)F_+^*(t)\rangle = {} & \frac{R}{2}\bigg(\int_0^t \big[(1+p)e^{-\lambda_2(t-t')} + (1-p)e^{-\lambda_1(t-t')}\big] \\
& \times \big[(1+p)e^{-\lambda_2(t-t'')} + (1-p)e^{-\lambda_1(t-t'')}\big]\delta(t'-t'')dt'dt''\bigg) \\
& + \frac{(V-\varepsilon)q_+}{4}\bigg(\int_0^t \big[(1+p)e^{-\lambda_2(t-t')} + (1-p)e^{-\lambda_1(t-t')}\big] \\
& \times \big[e^{-\lambda_1(t-t'')} - e^{-\lambda_2(t-t'')}\big]\delta(t'-t'')dt'dt''\bigg) \\
& - \frac{(V-\varepsilon)q_+}{4}\bigg(\int_0^t \big[e^{-\lambda_1(t-t')} - e^{-\lambda_2(t-t')}\big] \\
& \times \big[(1+p)e^{-\lambda_2(t-t'')} + (1-p)e^{-\lambda_1(t-t'')}\big]\langle f_\beta^*(t')f_\alpha^*(t'')\rangle dt'dt''\bigg).
\end{aligned} \tag{3.106}
$$

Upon carrying out the integration, we obtain

$$
\begin{aligned}
\langle F_+(t)F_+^*(t)\rangle = {} & \frac{R(1-p)^2 - (V-\varepsilon)q_+(1-p)}{4\lambda_1}(1-e^{-2\lambda_1 t}) \\
& + \frac{R(1+p)^2 - (V-\varepsilon)q_+(1+p)}{4\lambda_2}(1-e^{-2\lambda_2 t}) \\
& + \frac{R(1-p^2) - (V-\varepsilon)q_+ p}{\lambda_1+\lambda_2}(1-e^{-(\lambda_1+\lambda_2)t}).
\end{aligned} \tag{3.107}
$$

Furthermore, employing (B39) and its complex conjugate, we have

$$
\begin{aligned}
\langle F_-(t)F_-^*(t)\rangle = {} & \frac{q_-}{4}\int_0^t \big[e^{-\lambda_2(t-t')} - e^{-\lambda_1(t-t')}\big]\big[e^{-\lambda_2(t-t'')} - e^{-\lambda_1(t-t'')}\big]\langle f_\alpha^*(t')f_\alpha(t'')\rangle dt'dt'' \\
& + \frac{1}{4}\bigg(\int_0^t \big[(1+p)e^{-\lambda_1(t-t')} + (1-p)e^{-\lambda_2(t-t')}\big] \\
& \times \big[(1+p)e^{-\lambda_1(t-t'')} + (1-p)e^{-\lambda_2(t-t'')}\big]\langle f_\beta(t')f_\beta^*(t'')\rangle dt'dt''\bigg) \\
& + \frac{q_-}{4}\bigg(\int_0^t \big[e^{-\lambda_2(t-t')} - e^{-\lambda_1(t-t')}\big] \\
& \times \big[(1+P)e^{-\lambda_1(t-t'')} + (1-p)e^{-\lambda_2(t-t'')}\big]\langle f_\alpha^*(t')f_\beta^*(t'')\rangle dt'dt''\bigg) \\
& + \frac{q_-}{4}\bigg(\int_0^t \big[(1+P)e^{-\lambda_1(t-t')} + (1-p)e^{-\lambda_2(t-t')}\big] \\
& \times \big[e^{-\lambda_2(t-t'')} - e^{-\lambda_1(t-t'')}\big]\langle f_\beta(t')f_\alpha(t'')\rangle dt'dt''\bigg).
\end{aligned} \tag{3.108}
$$

Applying Eqs. (3.66), (3.80), and (3.86), we find

$$\begin{aligned}\langle F_-(t)F_-^*(t)\rangle &= \frac{Rq_-}{2}\int_0^t \left[e^{-\lambda_2(t-t')} - e^{-\lambda_1(t-t')}\right]\left[e^{-\lambda_2(t-t'')} - e^{-\lambda_1(t-t'')}\right]\delta(t'-t'')dt'dt'' \\ &- \frac{(V-\varepsilon)q_-}{4}\bigg(\int_0^t \left[e^{-\lambda_2(t-t')} - e^{-\lambda_1(t-t')}\right] \\ &\times \left[(1+P)e^{-\lambda_1(t-t'')} + (1-p)e^{-\lambda_2(t-t'')}\right]\delta(t'-t'')dt'dt''\bigg) \\ &- \frac{(V-\varepsilon)q_-}{4}\bigg(\int_0^t \left[(1+P)e^{-\lambda_1(t-t')} + (1-p)e^{-\lambda_2(t-t')}\right] \\ &\times \left[e^{-\lambda_2(t-t'')} - e^{-\lambda_1(t-t'')}\right]\delta(t'-t'')dt'dt''\bigg),\end{aligned} \tag{3.109}$$

so that performing the integration there follows

$$\begin{aligned}\langle F_-(t)F_-^*(t)\rangle &= \frac{Rq_-^2 - (V-\varepsilon)q_-(1+p)}{4\lambda_1}(1-e^{-2\lambda_1}) \\ &+ \frac{Rq_-^2 + (V-\varepsilon)q_-(1-p)}{4\lambda_2}(1-e^{-2\lambda_2}) \\ &- \frac{Rq_-^2 - (V-\varepsilon)q_-p}{\lambda_1+\lambda_2}(1-e^{-(\lambda_1+\lambda_2)t}).\end{aligned} \tag{3.110}$$

Moreover, using Eqs. (B38) and (B39), we get

$$\begin{aligned}\langle F_+(t)F_-(t)\rangle &= \frac{q_-}{4}\bigg(\int_0^t \left[(1+p)e^{-\lambda_2(t-t')} + (1-p)e^{-\lambda_1(t-t')}\right] \\ &\times \left[e^{-\lambda_2(t-t'')} - e^{-\lambda_1(t-t'')}\right]\langle f_\alpha(t')f_\alpha^*(t'')\rangle dt'dt''\bigg) \\ &+ \frac{q_+}{4}\bigg(\int_0^t \left[e^{-\lambda_1(t-t')} - e^{-\lambda_2(t-t'')}\right] \\ &\times \left[(1+p)e^{-\lambda_1(t-t'')} + (1-p)e^{-\lambda_2(t-t'')}\right]\langle f_\beta^*(t')f_\beta(t'')\rangle dt'dt''\bigg) \\ &+ \frac{1}{4}\bigg(\int_0^t \left[(1+p)e^{-\lambda_2(t-t')} + (1-p)e^{-\lambda_1(t-t')}\right] \\ &\times \left[(1+p)e^{-\lambda_1(t-t'')} + (1-p)e^{-\lambda_2(t-t'')}\right]\langle f_\alpha(t')f_\beta(t'')\rangle dt'dt''\bigg) \\ &+ \frac{q_+q_-}{4}\int_0^t \left[e^{-\lambda_1(t-t')} - e^{-\lambda_2(t-t'')}\right]\left[e^{-\lambda_2(t-t'')} - e^{-\lambda_1(t-t'')}\right]\langle f_\beta^*(t')f_\alpha^*(t'')\rangle dt'dt''.\end{aligned} \tag{3.111}$$

In view of Eqs. (3.66), (3.80), and (3.86), we see that

$$\begin{aligned}
\langle F_+(t)F_-(t)\rangle &= \frac{Rq_-}{2}\bigg(\int_0^t \big[(1+p)e^{-\lambda_2(t-t')} + (1-p)e^{-\lambda_1(t-t')}\big] \\
&\times \big[e^{-\lambda_2(t-t'')} - e^{-\lambda_1(t-t'')}\big]\delta(t'-t'')dt'dt''\bigg) \\
&- \frac{(V-\varepsilon)}{4}\bigg(\int_0^t \big[(1+p)e^{-\lambda_2(t-t')} + (1-p)e^{-\lambda_1(t-t')}\big] \\
&\times \big[(1+p)e^{-\lambda_1(t-t'')} + (1-p)e^{-\lambda_2(t-t'')}\big]\delta(t'-t'')\bigg)dt'dt'' \\
&- \frac{(V-\varepsilon)q_+q_-}{4}\int_0^t \big[e^{-\lambda_1(t-t')} - e^{-\lambda_2(t-t'')}\big]\big[e^{-\lambda_2(t-t'')} - e^{-\lambda_1(t-t'')}\big]\delta(t'-t'')dt'dt''
\end{aligned} \tag{3.112}$$

and carrying out the integration, we obtain

$$\begin{aligned}
\langle F_+(t)F_-(t)\rangle &= \frac{2Rq_-(1-p) - (V-\varepsilon)[1-p^2+q_+q_-]}{8\lambda_1}(1-e^{-2\lambda_1}) \\
&- \frac{2Rq_-(1+p) + (V-\varepsilon)[1-p^2+q_+q_-]}{8\lambda_2}(1-e^{-2\lambda_2}) \\
&+ \frac{2Rq_-p - (V-\varepsilon)[1+p^2-q_+q_-]}{2(\lambda_1+\lambda_2)}(1-e^{-(\lambda_1+\lambda_2)t}).
\end{aligned} \tag{3.113}$$

Finally, on account of Eqs. (3.107), (3.110), (3.113), and the complex conjugate of (3.113), the quadrature variance (3.104) takes the form

$$\begin{aligned}
\Delta c_\pm^2 &= 1 + \bigg(R[(1-p)^2 + q_-^2 \pm 2q_-(1-p)] - (V-\varepsilon)[q_+(1-p) + q_-(1+p) \\
&\pm (1-p^2+q_+q_-)]\bigg)\frac{(1-e^{-2\lambda_1 t})}{4\lambda_1} + \bigg(R[(1+p)^2 + q_-^2 \mp 2q_-(1+p)] \\
&+ (V-\varepsilon)[q_+(1+p) + q_-(1-p) \mp (1-p^2+q_+q_-)]\bigg)\frac{(1-e^{-2\lambda_2 t})}{4\lambda_2} \\
&+ \bigg(R[1-p^2-q_-^2 \pm 2q_-p] - (V-\varepsilon)[q_+p - q_-p \pm (1+p^2-q_+q_-)]\bigg)\frac{(1-e^{-(\lambda_1+\lambda_2)t})}{\lambda_1+\lambda_2}.
\end{aligned} \tag{3.114}$$

At steady state the quadrature variance (3.114) goes over into

$$\begin{aligned}
\Delta c_\pm^2 &= 1 + \big[R\left(p^2 + q_-^2 \mp 2q_-p\right) + (V-\varepsilon)\left(p(q_+-q_-) \pm (p^2-q_+q_-)\right)\big]\frac{(\lambda_1+\lambda_2)^2 - 4\lambda_1\lambda_2}{4\lambda_1\lambda_2(\lambda_1+\lambda_2)} \\
&+ [R \mp (V-\varepsilon)]\frac{(\lambda_1+\lambda_2)^2 + 4\lambda_1\lambda_2}{4\lambda_1\lambda_2(\lambda_1+\lambda_2)} + [2R(p\mp q_-) + (V-\varepsilon)(q_+ + q_-)]\frac{(\lambda_1-\lambda_2)}{4\lambda_1\lambda_2}.
\end{aligned} \tag{3.115}$$

We see from Eq. (3.115) that the driving light has no effect on the quadrature variance. Now inspection of Eqs. (B13) and (B14) shows that λ_1 is nonnegative while λ_2 can be positive, negative, or zero. We observe that the equations of evolutions of $\alpha(t)$ and $\beta(t)$, described by

(B1) and (B2), do not have well-behaved solutions for $\lambda_2 < 0$. We then identify, using Eq. (B14)

$$\varepsilon = \frac{(V-U) + \sqrt{(V-U)^2 + 4UV + (S-R+\kappa)^2 - (R+S)^2}}{2} \tag{3.116}$$

as the threshold condition.

We next proceed to analyze the quadrature variance of the light generated by the system operating below threshold. Using Eqs. (B14), (3.115), and (3.116) and writing a simple Matlab program, we have obtained for $A = 100$ and $\kappa = 0.8$ the values of η, $\frac{\Omega}{\gamma}$, ε, and Δc_-^2 for which Eqs. (B1) and (B2) have solutions. It so turns out that these equations have solutions for $0 \leq \eta \leq 1$ and for $0 \leq \frac{\Omega}{\gamma} \leq 1$. We indicate in the table below the values of η, $\frac{\Omega}{\gamma}$, and ε corresponding to the two smallest values of the quadrature variance.

η	$\frac{\Omega}{\gamma}$	ε	Δc_-^2
0	0.0100	0.4000	0.3789
0.1000	0	0.7000	0.2923

Tab. 3.1: Values of Δc_-^2 for $A = 100$ and $\kappa = 0.8$.

We note that when there are equal number of atoms initially in the top and bottom levels ($\eta = 0$), the maximum interacavity squeezing is 62% below the coherent-state level for $\frac{\Omega}{\gamma} = 0.01$. On the other hand, when there are slightly more atoms initially in the bottom level than in the top level ($\eta = 0.1$), the maximum interacavity squeezing is found to be 70% below the coherent-state level for $\frac{\Omega}{\gamma} = 0$.

We next seek to examine some special cases. We first consider the case in which the nonlinear crystal is removed from the cavity, with the top and bottom levels of the atoms coupled by the pump mode. Thus upon setting $\varepsilon = 0$ (with $\beta_0 \neq 0$) in Eq. (3.115), we get

$$\Delta c_\pm^2 = 1 + \left[R\left(p'^2 + q_-'^2 \mp 2q_-'p'\right) + V\left(p'(q_+' - q_-') \pm (p'^2 - q_+'q_-')\right)\right] \frac{(\lambda_1' + \lambda_2')^2 - 4\lambda_1'\lambda_2'}{4\lambda_1'\lambda_2'(\lambda_1' + \lambda_2')}$$
$$+ \left[R \mp V\right] \frac{(\lambda_1' + \lambda_2')^2 + 4\lambda_1'\lambda_2'}{4\lambda_1'\lambda_2'(\lambda_1' + \lambda_2')} + \left[2R(p' \mp q_-') + V(q_+' + q_-')\right] \frac{(\lambda_1' - \lambda_2')}{4\lambda_1'\lambda_2'}, \tag{3.117}$$

where

$$p' = \frac{1 + \frac{\Omega^2}{\gamma^2}}{\left[(1 + \frac{\Omega^2}{\gamma^2})^2 + \left(\frac{\Omega}{2\gamma}(1 + \frac{\Omega^2}{\gamma^2})\right)^2 - (\frac{3\Omega}{2\gamma}\eta - (1 - \frac{\Omega^2}{2\gamma^2})\sqrt{1-\eta^2})^2\right]^{1/2}}, \tag{3.118}$$

$$q'_{\pm} = \frac{-\frac{\Omega}{2\gamma}(1+\frac{\Omega^2}{\gamma^2}) \mp [\frac{3\Omega}{2\gamma}\eta - (1-\frac{\Omega^2}{\gamma^2})\sqrt{1-\eta^2}]}{\left[(1+\frac{\Omega^2}{\gamma^2})^2 + \left(\frac{\Omega}{2\gamma}(1+\frac{\Omega^2}{\gamma^2})\right)^2 - (\frac{3\Omega}{2\gamma}\eta - (1-\frac{\Omega^2}{2\gamma^2})\sqrt{1-\eta^2})^2\right]^{1/2}}, \tag{3.119}$$

$$\lambda'_1 = \frac{(a_+ + a_-) + \sqrt{(a_+ - a_-)^2 - 4UV}}{2}, \tag{3.120}$$

and

$$\lambda'_2 = \frac{(a_+ + a_-) - \sqrt{(a_+ - a_-)^2 - 4UV}}{2}. \tag{3.121}$$

It can be shown using (3.117) that for $\eta = \frac{\Omega}{\gamma} = 0$ and any values of A and κ the light generated is not in a squeezed state. However, for $\eta = 0$, $A = 100$, and $\kappa = 0.8$, we readily get applying the same equation that the maximum squeezing to be 54% for $\frac{\Omega}{\gamma} = 0.02$. We therefore deduce that the squeezing in this case is solely due to coupling of the top and bottom levels. Moreover, for $\eta = 0.1$ and for the above values of A and κ, we find the maximum squeezing to be 64% for $\frac{\Omega}{\gamma} = 0$. The squeezing in this case is due to the specific superposition of the top and bottom levels.

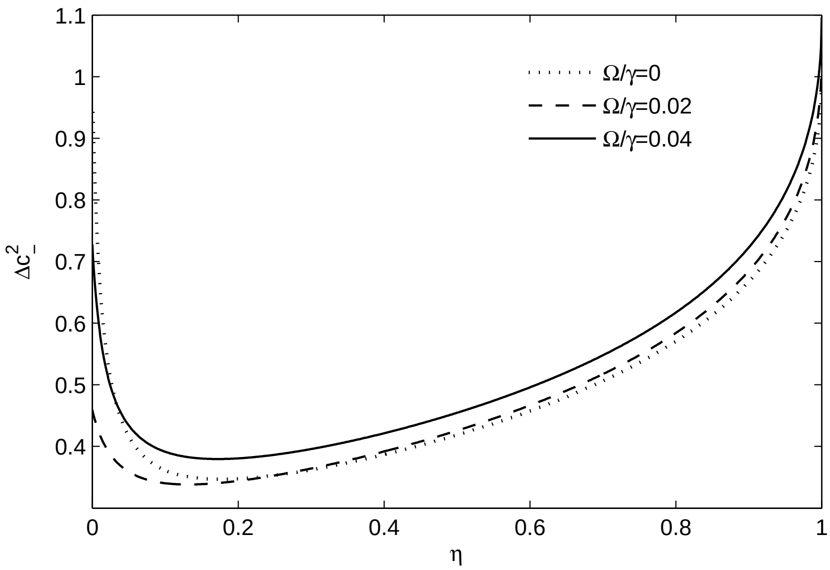

Fig. 3.2: Plots of the quadrature variance [Eq. (3.117)] versus η for $A = 100$, $\kappa = 0.8$, and different values of $\frac{\Omega}{\gamma}$.

In addition, one can also observe from the plots in Fig. 3.2 that for small values of the amplitude of the pump mode, the coupling of the top and bottom levels significantly enhances the intracavity squeezing particularly when there are nearly equal number of atoms initially in the top and bottom levels (around $\eta = 0$). Otherwise, it leads to the decrease in the intercavity squeezing. On the other hand, for a strong pump mode ($\Omega \gg \gamma$), we readily find

$$B = \frac{\Omega^4}{4\gamma^4}, \tag{3.122}$$

$$R = \frac{A\gamma^2}{2\Omega^2}(2+\eta), \tag{3.123}$$

$$V = -\frac{A\gamma}{2\Omega}, \tag{3.124}$$

$$p' = 0, \tag{3.125}$$

$$q'_{\pm} = -1, \tag{3.126}$$

$$\lambda'_1 = \frac{\kappa}{2} + \frac{A\gamma}{2\Omega}, \tag{3.127}$$

$$\lambda'_2 = \frac{\kappa}{2} - \frac{A\gamma}{2\Omega}, \tag{3.128}$$

Now applying the above results in Eq. (3.117), we get

$$\Delta c_{\pm}^2 = 1 + \left(\frac{A\gamma^2}{2\Omega^2}(2+\eta) \pm \frac{A\gamma}{2\Omega}\right)\frac{2\kappa^2\Omega^2 - A^2\gamma^2}{\kappa(\kappa^2\Omega^2 - A^2\gamma^2)}$$
$$\left(\frac{A\gamma^2}{2\Omega^2}(2+\eta) \pm \frac{A\gamma}{2\Omega}\right)\frac{A^2\gamma^2}{\kappa(\kappa^2\Omega^2 - A^2\gamma^2)} \pm \left(\frac{A\gamma^2}{2\Omega^2}(2+\eta) \pm \frac{A\gamma}{2\Omega}\right)\frac{2A\gamma\Omega}{\kappa^2\Omega^2 - A^2\gamma^2}. \tag{3.129}$$

It then follows that

$$\Delta c_{\pm}^2 = 1 + \left(\frac{A\gamma^2}{2\Omega^2}(2+\eta) \pm \frac{A\gamma}{2\Omega}\right)\left(\frac{2\kappa}{\kappa^2 - (\frac{A\gamma}{\Omega})^2} \pm \frac{2\frac{A\gamma}{\Omega}}{\kappa^2 - (\frac{A\gamma}{\Omega})^2}\right), \tag{3.130}$$

so that on dropping the term $\frac{A\gamma^2}{\Omega^2}$ there follows

$$\Delta c_{\pm}^2 = \frac{1}{1 \mp \frac{A\gamma}{\Omega\kappa}}. \tag{3.131}$$

This result indicates that a nondegenerate three-level laser driven by a strong light behaves like a nondegenerate parametric oscillator [14].

Furthermore, it is interesting to analyze the case in which the pump mode emerging from the nonlinear crystal does not couple the top and bottom levels. Hence upon setting $\frac{\Omega}{\gamma} = 0$ (with $\beta_0 \neq 0$), Eq.(3.115) reduces to

$$\begin{aligned}\Delta c_{\pm}^2 = 1 &+ \frac{4\varepsilon + A\sqrt{1-\eta^2}[8\kappa\varepsilon + A\eta(4\varepsilon - A\sqrt{1-\eta^2})]}{4(2\kappa + A\eta)(\kappa^2 + \kappa A\eta - 4\varepsilon^2)} \\ &+ \frac{A(1-\eta)(2\kappa + A\eta)[2\kappa + A(1+\eta)]}{4(2\kappa + A\eta)(\kappa^2 + \kappa A\eta - 4\varepsilon^2)} \\ &\pm \frac{2\kappa[2\kappa + A(1+\eta)](4\varepsilon + A\sqrt{1-\eta^2})}{4(2\kappa + A\eta)(\kappa^2 + \kappa A\eta - 4\varepsilon^2)}. \end{aligned} \tag{3.132}$$

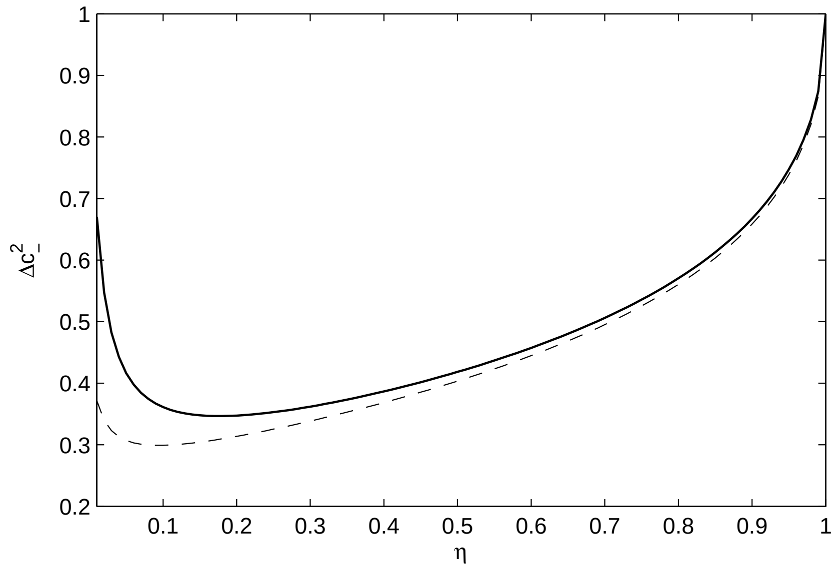

Fig. 3.3: Plots of the quadrature variance [Eq. (3.132)] versus η for $A = 100$, $\kappa = 0.8$, and $\varepsilon = 0.5$ (dashed curve) and $\varepsilon = 0$ (Solid curve).

Inspection of the plots in Fig. 3.3 shows that the presence of the nonlinear crystal leads to better squeezing. Furthermore, applying Eq. (3.132) with $A = 100$, $\kappa = 0.8$, and $\eta = 0.1$ the maximum interacavity squeezing is found to be 70% below the coherent-state level for $\varepsilon = 0.5$.

3.2.2 Quadrature variance of the output modes

We next proceed to calculate the quadrature variance of the output modes. Following a similar procedure as in Appendix A, one can express the quadrature variance for the output modes of the system under consideration as

$$\Delta c_{\pm out}^2 = \kappa \Delta c_{\pm}^2 + (1-\kappa)\Delta c_{in\pm}^2, \tag{3.133}$$

where the first and the second terms represent the quadrature variance of the transmitted and reflected modes. Taking into account Eq. (3.115) and the fact that the quadrature variance of the vacuum reservoir is unity, the quadrature variance for the output modes of the system under consideration takes at steady sate the form

$$\Delta c^2_{\pm out} = 1 + \kappa\left[R\left(p^2 + q_-^2 \mp 2q_-p\right) + (V-\varepsilon)\left(p(q_+ - q_-) \pm (p^2 - q_+q_-)\right)\right]\frac{(\lambda_1+\lambda_2)^2 - 4\lambda_1\lambda_2}{4\lambda_1\lambda_2(\lambda_1+\lambda_2)}$$
$$+\kappa\left[R \mp (V-\varepsilon)\right]\frac{(\lambda_1+\lambda_2)^2 + 4\lambda_1\lambda_2}{4\lambda_1\lambda_2(\lambda_1+\lambda_2)} + \kappa\left[2R(p \mp q_-) + (V-\varepsilon)(q_+ + q_-)\right]\frac{(\lambda_1-\lambda_2)}{4\lambda_1\lambda_2}. \tag{3.134}$$

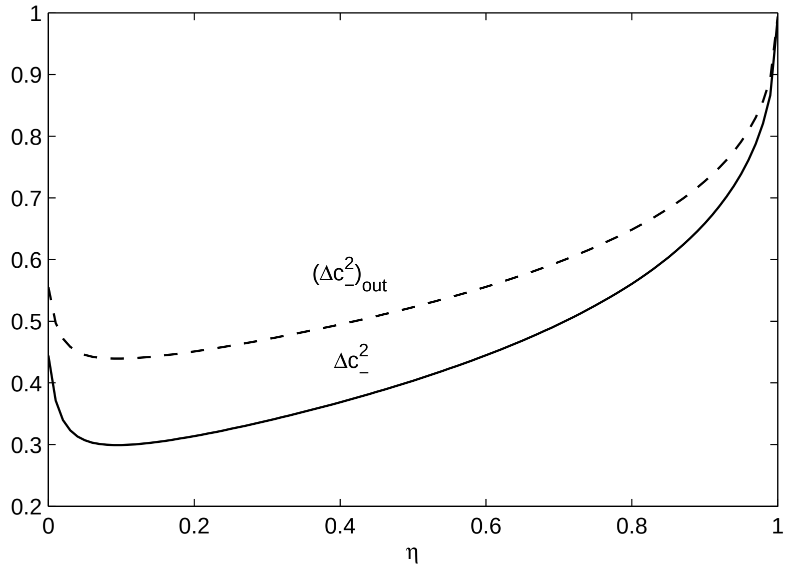

Fig. 3.4: Plots of the quadrature variance for the cavity modes [Eq. (3.115), solid cure] and for the output modes [Eq. (3.134), dashed curve] versus η for $\frac{\Omega}{\gamma} = 0.01$, $\varepsilon = 0.5$, $A = 100$, and $\kappa = 0.8$.

It can be seen from the plots in Fig. 3.4 that generally the degree of squeezing of the cavity modes is greater than the degree of squeezing of the output modes. Furthermore, applying Eq. (3.134) with $\eta = 0.1$, $\varepsilon = 0.5$, $A = 100$, and $\kappa = 0.8$, the maximum squeezing for the output modes is found to be 56% (occurs at $\frac{\Omega}{\gamma} = 0$).

We next wish to consider some special cases. We first take into consideration the case in which the nonlinear crystal is removed from the cavity, with the top and bottom levels of the atoms coupled by the pump mode. Thus upon setting $\varepsilon = 0$ (with $\beta_0 \neq 0$) in Eq. (3.134), we find

$$\Delta c^2_{\pm out} = 1 + \kappa\left[R\left(p'^2 + q'^2_- \mp 2p'_- p'\right) + (V-\varepsilon)\left(p'(q'_+ - q'_-) \pm (p'^2 - q'_+ q'_-)\right)\right] \frac{(\lambda'_1+\lambda'_2)^2 - 4\lambda'_1\lambda'_2}{4\lambda'_1\lambda'_2(\lambda'_1+\lambda'_2)}$$
$$+ \kappa\left[R \mp (V-\varepsilon)\right] \frac{(\lambda'_1+\lambda'_2)^2 + 4\lambda'_1\lambda'_2}{4\lambda'_1\lambda'_2(\lambda'_1+\lambda'_2)} + \kappa\left[2R(p' \mp q'_-) + (V-\varepsilon)(q'_+ + q'_-)\right] \frac{(\lambda'_1 - \lambda'_2)}{4\lambda'_1\lambda'_2}, \tag{3.135}$$

with p', $q'_\pm$, λ'_1, and λ'_2 given by Eqs. (3.118)-(3.121).

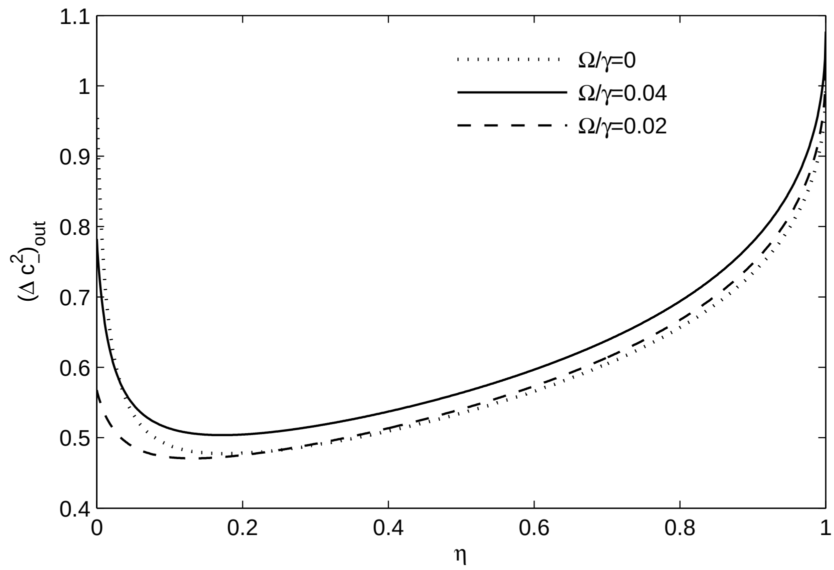

Fig. 3.5: Plots of the quadrature variance for the output modes [Eq. (3.135)] versus η for $A = 100$, $\kappa = 0.8$, and different values of $\frac{\Omega}{\gamma}$.

We can easily see from the plots in Fig. 3.5 that the coupling of the top and bottom levels of the three-level atoms by the pump mode leads to the decrease in the squeezing of the output modes, except when there are nearly equal number of atoms initially in the top and bottom levels. Furthermore, applying Eq. (3.135) with $\eta = 0.1$, $A = 100$, and $\kappa = 0.8$, the maximum squeezing for the output modes is found to be 53% (occurs at $\frac{\Omega}{\gamma} = 0.02$).

In addition, for the special case in which the pump mode emerging from the nonlinear crystal does not couple the top and bottom levels. Upon setting $\frac{\Omega}{\gamma} = 0$ (with $\beta_0 \neq 0$), Eq.

(3.134) reduces to

$$\Delta c^2_{\pm out} = 1 + \frac{\kappa(4\varepsilon + A\sqrt{1-\eta^2})[8\kappa\varepsilon + A\eta(4\varepsilon - A\sqrt{1-\eta^2})]}{4(2\kappa + A\eta)(\kappa^2 + \kappa A\eta - 4\varepsilon^2)}$$
$$+ \frac{\kappa A(1-\eta)(2\kappa + A\eta)[2\kappa + A(1+\eta)]}{4(2\kappa + A\eta)(\kappa^2 + \kappa A\eta - 4\varepsilon^2)}$$
$$\pm \frac{2\kappa^2[2\kappa + A(1+\eta)](4\varepsilon + A\sqrt{1-\eta^2})}{4(2\kappa + A\eta)(\kappa^2 + \kappa A\eta - 4\varepsilon^2)}. \quad (3.136)$$

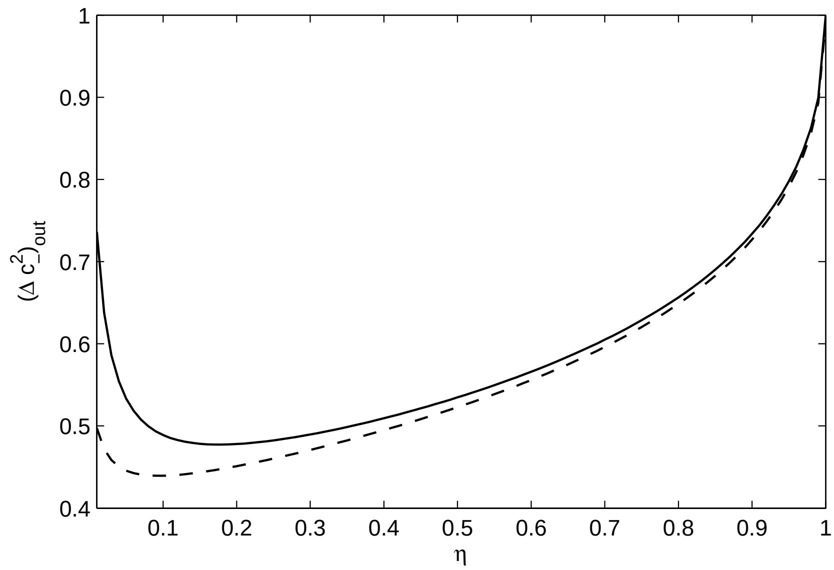

Fig. 3.6: Plots of the quadrature variance for the output modes [Eq. (3.136)] versus η for $A = 100$, $\kappa = 0.8$, and $\varepsilon = 0.5$ (dashed curve) and $\varepsilon = 0$ (Solid curve).

The plots in Fig. 3.6 shows that the presence of the nonlinear crystal leads to better squeezing of the output modes. Furthermore, applying Eq. (3.136) with $A = 100$, $\kappa = 0.8$, and $\eta = 0.1$ the maximum output modes squeezing is 56% below the coherent-state level for $\varepsilon = 0.5$.

3.2.3 Squeezing spectrum of the output modes

The squeezing spectrum of a two-mode light can be expressed as

$$S^{out}_{\pm}(\omega) = 1 \pm 2Re\int_0^{\infty} \langle : \hat{c}^{out}_{\pm}(t), \hat{c}^{out}_{\pm}(t+\tau) : \rangle_{ss} e^{i(\omega-\omega_0)\tau} d\tau. \quad (3.137)$$

This can be expressed in terms of c-number variables associated with the normal ordering as

$$S^{out}_{\pm}(\omega) = 1 \pm 2Re\int_0^{\infty} \langle \gamma^{out}_{\pm}(t), \gamma^{out}_{\pm}(t+\tau) \rangle_{ss} e^{i(\omega-\omega_0)\tau} d\tau, \quad (3.138)$$

in which

$$\gamma_{\pm}^{out}(t) = \gamma_{out}^{*}(t) \pm \gamma_{out}(t). \tag{3.139}$$

For a cavity mode coupled to vacuum reservoir the output and cavity variables can be related by

$$\gamma_{\pm}^{out}(t) = \sqrt{\kappa}\gamma_{\pm}(t). \tag{3.140}$$

On account of Eq. (3.140) the squeezing spectrum takes the form

$$S_{\pm}^{out}(\omega) = 1 \pm 2\kappa Re \int_{0}^{\infty} \langle \gamma_{\pm}(t), \gamma_{\pm}(t+\tau)\rangle_{ss} e^{i(\omega-\omega_0)\tau} d\tau. \tag{3.141}$$

With the aid of Eq. (3.95), one can readily show that

$$\gamma_{\pm}(t) = \frac{1}{\sqrt{2}} \left[\alpha^*(t) + \beta^*(t) \pm (\alpha(t) + \beta(t))\right] \tag{3.142}$$

and

$$\gamma_{\pm}(t+\tau) = \frac{1}{\sqrt{2}} \left[\alpha^*(t+\tau) + \beta^*(t+\tau) \pm (\alpha(t+\tau) + \beta(t+\tau))\right]. \tag{3.143}$$

We note that Eqs. (B34) and (B35) can be put in the form

$$\alpha(t+\tau) = A_{+}(t+\tau)\alpha(t) + B_{+}(t+\tau)\beta^*(t) + F_{+}(t+\tau) + E_{+}(t+\tau) \tag{3.144}$$

and

$$\beta(t+\tau) = A_{-}(t+\tau)\beta(t) + B_{-}(t+\tau)\alpha^*(t) + F_{-}(t+\tau) + E_{-}(t+\tau). \tag{3.145}$$

Now using Eqs. (3.142) and (3.143), we have

$$\begin{aligned}\langle \gamma_{\pm}(t)\gamma_{\pm}(t+\tau)\rangle_{ss} = \frac{1}{2}\Big[&\langle\alpha(t)\alpha(t+\tau)\rangle_{ss} + \langle\beta(t)\alpha(t+\tau)\rangle_{ss} + \langle\alpha(t)\beta(t+\tau)\rangle_{ss} \\ &+ \langle\beta(t)\beta(t+\tau)\rangle_{ss} \pm [\langle\alpha(t)\alpha^*(t+\tau)\rangle_{ss} + \langle\beta(t)\alpha^*(t+\tau)\rangle_{ss} \\ &+\langle\beta(t)\beta^*(t+\tau)\rangle_{ss} + \langle\alpha(t)\beta^*(t+\tau)\rangle_{ss}] + \langle\alpha^*(t)\alpha^*(t+\tau)\rangle_{ss} \\ &+ \langle\beta^*(t)\alpha^*(t+\tau)\rangle_{ss} + \langle\alpha^*(t)\beta^*(t+\tau)\rangle_{ss} + \langle\beta^*(t)\beta^*(t+\tau)\rangle_{ss} \\ &\pm [\langle\alpha^*(t)\alpha(t+\tau)\rangle_{ss} + \langle\beta^*(t)\alpha(t+\tau)\rangle_{ss} + \langle\beta^*(t)\beta(t+\tau)\rangle_{ss} \\ &+\langle\alpha^*(t)\beta(t+\tau)\rangle_{ss}]\Big].\end{aligned} \tag{3.146}$$

With the aid of Eqs. (B34), (B35), (3.144), (3.145) along with the fact that the noise force at $t+\tau$ do not affect the system variables at the earlier times, we get

$$\langle\alpha(t)\alpha(t+\tau)\rangle_{ss} = A_{+}(\tau)E_{+}^2(t) + B_{+}(\tau)E_{+}(t)E_{-}(t) + E_{+}(\tau)E_{+}(t), \tag{3.147}$$

$$\langle\beta(t)\alpha(t+\tau)\rangle_{ss} = A_+(\tau)\bigg(E_-(t)E_+(t) + \langle F_-(t)F_+(t)\rangle_{ss}\bigg) + B_+(\tau)\bigg(E_-^2(t) + \langle F_-(t)F_-^*(t)\rangle_{ss}\bigg) + E_+(\tau)E_-(t), \tag{3.148}$$

$$\langle\alpha(t)\beta(t+\tau)\rangle_{ss} = A_-(\tau)\bigg(E_-(t)E_+(t) + \langle F_-(t)F_+(t)\rangle_{ss}\bigg) + B_-(\tau)\bigg(E_+^2(t) + \langle F_+(t)F_+^*(t)\rangle_{ss}\bigg) + E_+(\tau)E_-(t), \tag{3.149}$$

$$\langle\beta(t)\beta(t+\tau)\rangle_{ss} = A_-(\tau)E_-^2(t) + B_-(\tau)E_-(t)E_+(t) + E_-(\tau)E_-(t), \tag{3.150}$$

$$\langle\alpha(t)\alpha^*(t+\tau)\rangle_{ss} = A_+(\tau)\bigg(E_+^2(t) + \langle F_+(t)F_+^*(t)\rangle_{ss}\bigg) + B_+(\tau)\bigg(E_+(t)E_-(t) + \langle F_+(t)F_-(t)\rangle_{ss}\bigg) + E_+(\tau)E_+(t), \tag{3.151}$$

$$\langle\beta(t)\alpha^*(t+\tau)\rangle_{ss} = A_+(\tau)E_-(t)E_+(t) + B_+(\tau)E_-^2(t) + E_+(\tau)E_-(t), \tag{3.152}$$

$$\langle\beta(t)\beta^*(t+\tau)\rangle_{ss} = A_-(\tau)\bigg(E_-^2(t) + \langle F_-(t)F_-^*(t)\rangle_{ss}\bigg) + B_-(\tau)\bigg(E_-(t)E_+(t) + \langle F_-(t)F_+(t)\rangle_{ss}\bigg) + E_-(\tau)E_-(t), \tag{3.153}$$

and

$$\langle\alpha(t)\beta^*(t+\tau)\rangle_{ss} = A_-(\tau)E_+(t)E_-(t) + B_-(\tau)E_+^2(t) + E_-(\tau)E_+(t). \tag{3.154}$$

Furthermore, employing Eqs. (B34), (B35), (3.144), and (3.145), we find

$$\begin{aligned}\langle\gamma_\pm(t)\rangle_{ss}\langle\gamma_\pm(t+\tau)\rangle_{ss} = \bigg[&A_+(\tau)E_+^2(t) + B_+(\tau)E_+(t)E_-(t) + A_-(\tau)E_-(t)E_+(t)\\ &+ B_-(\tau)E_+^2(t) + A_+(\tau)E_+(t)E_-(t) + B_+(\tau)E_-^2(t) + A_-(\tau)E_-^2(t)\\ &+ B_-(\tau)E_+(t)E_-(t) \pm [A_+(\tau)E_+^2(t) + B_+(\tau)E_+(t)E_-(t)\\ &+ A_-(\tau)E_-(t)E_+(t) + B_-(\tau)E_+^2(t) + A_+(\tau)E_+(t)E_-(t)\\ &+ B_+(\tau)E_-^2(t)] + A_-(\tau)E_-^2(t) + B_-(\tau)E_+(t)E_-(t))\\ &+ E_+(t)E_+(\tau) + E_-(t)E_-(\tau) + E_+(t)E_-(\tau) + E_-(t)E_+(\tau)\\ &\pm [E_+(t)E_+(\tau) + E_-(t)E_-(\tau) + E_+(t)E_-(\tau) + E_-(t)E_+(\tau)]\bigg].\end{aligned} \tag{3.155}$$

Hence applying Eqs. (3.147)-(3.154) and their complex conjugate in (3.146) and subtracting Eq. (3.155) from the resulting equation, we obtain

$$\begin{aligned}\langle\gamma_\pm(t),\gamma_\pm(\tau)\rangle_{ss} &= \langle\gamma_\pm(t)\gamma_\pm(t+\tau)\rangle_{ss} - \langle\gamma_\pm(t)\rangle_{ss}\langle\gamma_\pm(t+\tau)\rangle_{ss} = B_+(\tau)\langle F_-F_-^*\rangle_{ss}\\ &+ [A_+(\tau)+A_-(\tau)]\langle F_-(t)F_+(t)\rangle_{ss} + B_-(\tau)\langle F_+F_+^*\rangle_{ss}\\ &\pm (A_+(\tau)\langle F_+(t)F_+^*(t)\rangle_{ss} + A_-(\tau)\langle F_-(t)F_-^*(t)\rangle_{ss}\\ &+ [B_+(\tau)+B_-(\tau)]\langle F_+(t)F_-(t)\rangle_{ss}).\end{aligned} \tag{3.156}$$

On account of Eqs. (B36) and (B37), we have

$$\begin{aligned}\langle\gamma_\pm(t),\gamma_\pm(t+\tau)\rangle_{ss} &= \frac{1}{2}\bigg[(2\pm(q_++q_-))\langle F_-(t)F_+(t)\rangle_{ss} \pm [(1+p\pm q_+)\langle F_-(t)F_-^*(t)\rangle_{ss}\\ &+ (1-p\pm q_-)\langle F_+(t)F_+^*(t)\rangle_{ss}]\bigg]e^{-\lambda_1\tau}\\ &+ \frac{1}{2}\bigg[(2\mp(q_++q_-))\langle F_-(t)F_+(t)\rangle_{ss} \pm [(1-p\mp q_+)\langle F_-(t)F_-^*(t)\rangle_{ss}\\ &+ (1+p\mp q_-)\langle F_+(t)F_+^*(t)\rangle_{ss}]\bigg]e^{\lambda_2\tau}.\end{aligned} \tag{3.157}$$

Now substitution of (3.157) into Eq. (3.141) yields

$$\begin{aligned}S_\pm^{out}(\omega) &= 1\pm\kappa\bigg[(2\pm(q_++q_-))\langle F_-(t)F_+(t)\rangle_{ss} \pm [(1+p\pm q_+)\langle F_-(t)F_-^*(t)\rangle_{ss}\\ &+ (1-p\pm q_-)\langle F_+(t)F_+^*(t)\rangle_{ss}]\bigg] Re\int_0^\infty e^{-(\lambda_1-(\omega-\omega_0))\tau}d\tau\\ &\pm\kappa\bigg[(2\mp(q_++q_-))\langle F_-(t)F_+(t)\rangle_{ss} \pm [(1-p\mp q_+)\langle F_-(t)F_-^*(t)\rangle_{ss}\\ &+ (1+p\mp q_-)\langle F_+(t)F_+^*(t)\rangle_{ss}]\bigg] Re\int_0^\infty e^{-(\lambda_2-(\omega-\omega_0))\tau}d\tau.\end{aligned} \tag{3.158}$$

Finally, upon carrying out the integration, the squeezing spectrum is found to be

$$\begin{aligned}S_\pm^{out}(\omega) &= 1\pm\frac{\kappa\lambda_1}{\lambda_1^2+(\omega-\omega_0)^2}\bigg[(2\pm(q_++q_-))\langle F_-(t)F_+(t)\rangle_{ss}\\ &\pm [(1+p\pm q_+)\langle F_-(t)F_-^*(t)\rangle_{ss} + (1-p\pm q_-)\langle F_+(t)F_+^*(t)\rangle_{ss}]\bigg]\\ &\pm\frac{\kappa\lambda_2}{\lambda_2^2+(\omega-\omega_0)^2}\bigg[(2\mp(q_++q_-))\langle F_-(t)F_+(t)\rangle_{ss}\\ &\pm [(1-p\mp q_+)\langle F_-(t)F_-^*(t)\rangle_{ss} + (1+p\mp q_-)\langle F_+(t)F_+^*(t)\rangle_{ss}]\bigg].\end{aligned} \tag{3.159}$$

We see from Eq. (3.159) that the driving mode has no effect on the squeezing spectrum. Moreover, inspection of the plot in Fig. 3.7 indicates that almost perfect squeezing occurs for small values of η.

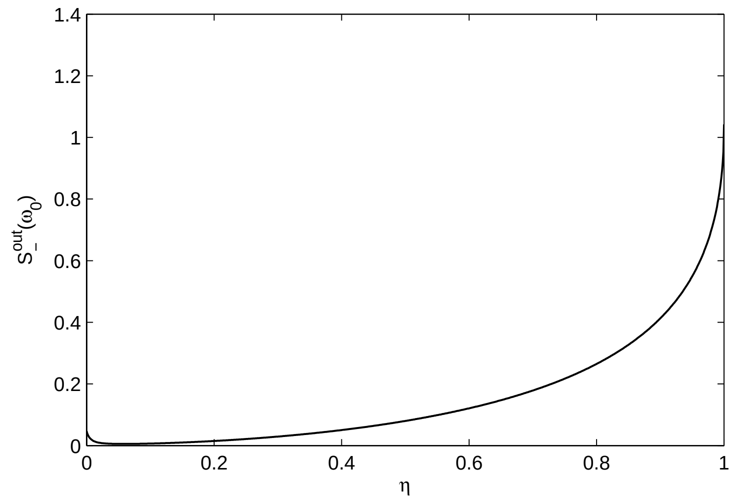

Fig. 3.7: Plots of the squeezing spectrum [Eq. (3.159)] versus η for $\omega = \omega_0$, $A = 100$, $\kappa = 0.8$, $\varepsilon = 0.3$, and $\frac{\Omega}{\gamma} = 0.01$.

We next proceed to analyze some special cases. For the case in which the nonlinear crystal is removed from the cavity, with the top and bottom levels of the atoms coupled by the pump mode. Thus upon setting $\varepsilon = 0$ (with $\beta_0 \neq 0$) in Eq. (3.159), we have plotted the graph of the resulting equation

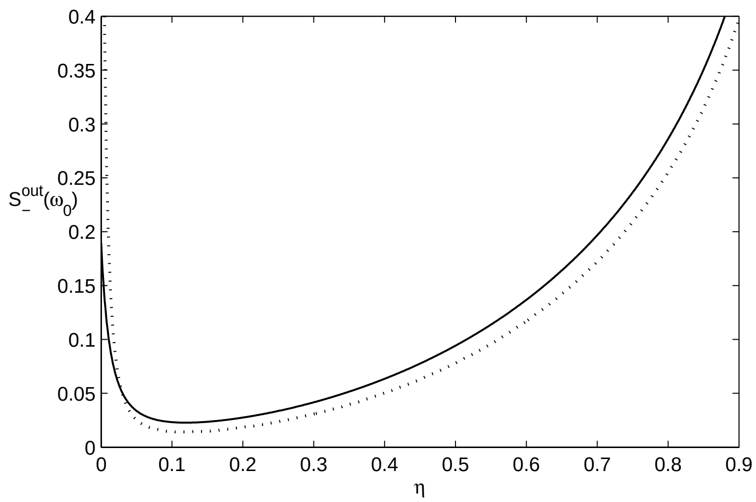

Fig. 3.8: Plots of the squeezing spectrum [Eq. (3.159)] versus η for $\omega = \omega_0$, $A = 100$, $\kappa = 0.8$, $\varepsilon = 0$, and $\frac{\Omega}{\gamma} = 0$ (dotted curve) and $\frac{\Omega}{\gamma} = 0.02$ (solid curve).

The plots in Fig. 3.8 show that for small values of the amplitude of the pump mode, the coupling of the top and bottom levels significantly enhances the squeezing spectrum particularly when there are nearly equal number of atoms initially in the top and bottom levels (around $\eta = 0$). Otherwise, it leads to the decrease in the squeezing spectrum. Furthermore, it is interesting to consider the special case in which the parametric amplifier is present but the pump mode emerging from the nonlinear crystal does not couple the top and bottom levels. Hence upon setting $\frac{\Omega}{\gamma} = 0$ (with $\beta_0 \neq 0$) in Eq.(3.159), we have plotted the graph of the resulting equation.

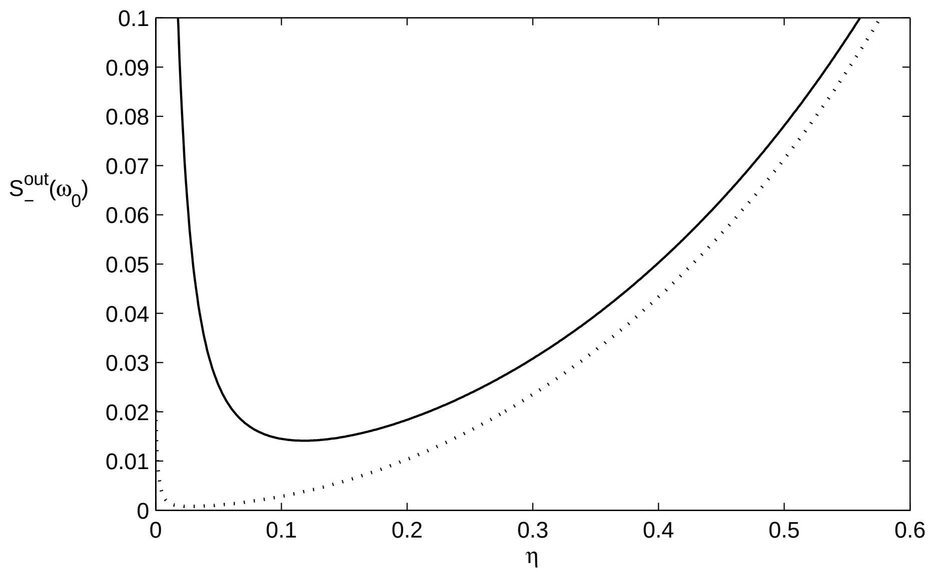

Fig. 3.9: Plots of the squeezing spectrum [Eq. (3.159)] versus η for $\omega = \omega_0$, $A = 100$, $\kappa = 0.8$, $\frac{\Omega}{\gamma} = 0$, and $\varepsilon = 0$ (solid curve) and $\varepsilon = 0.3$ (dotted curve).

We easily see from the plots in Fig. 3.9 that the presence of the parametric amplifier leads to the increase in the squeezing spectrum. On the other hand, we observe from the same plots that for $\omega = \omega_0$, $A = 100$, $\kappa = 0.8$, $\frac{\Omega}{\gamma} = 0$, and $\varepsilon = 0.3$ almost perfect squeezing occurs for small values of η.

3.3 Photon statistics of the cavity modes

We first determine, using the solutions of the c-number Langevin equations, the antinormally ordered characteristic function defined in the Heisenberg picture for the cavity modes. With the aid of the resulting characteristic function, we obtain the Q function which is then used to calculate the mean and the normally-ordered variance of the photon number sum and difference and the photon number distribution for the cavity modes. The Q function for a two-mode light is expressible as [21]

$$Q(\alpha,\beta,t)=\frac{1}{\pi^4}\int d^2zd^2\chi\phi(z,\chi,t)exp(z^*\alpha+\chi^*\beta-z\alpha^*-\chi\beta^*), \tag{3.160}$$

with the characteristic function $\phi(z,\chi,t)$ defined in Heisenberg picture by

$$\phi(z,\chi,t)=Tr\{\hat{\rho}(0)e^{-z^*\hat{a}(t)}e^{-\chi^*\hat{b}(t)}e^{z\hat{a}^\dagger(t)}e^{\chi\hat{b}^\dagger(t)}\}. \tag{3.161}$$

Employing the identity

$$e^{\hat{A}}e^{\hat{B}}=e^{\hat{B}}e^{\hat{A}}e^{[\hat{A},\hat{B}]}, \tag{3.162}$$

the characteristic function can be put in the normal order as

$$\phi(z,\chi,t)=exp\left[-(z^*z+\chi^*\chi)\right]\langle exp(z\hat{a}^\dagger-z^*\hat{a}+\chi\hat{b}^\dagger-\chi^*\hat{b})\rangle. \tag{3.163}$$

This can be expressed in terms of c-number variables associated with the normal ordering as

$$\phi(z,\chi,t)=exp\left[-(z^*z+\chi^*\chi)\right]\langle exp(z\alpha^*-z^*\alpha+\chi\beta^*-\chi^*\beta)\rangle. \tag{3.164}$$

One can rewrite Eqs. (B34) and (B35) as

$$\alpha(t)=\alpha'(t)+E_+(t) \tag{3.165}$$

and

$$\beta(t)=\beta'(t)+E_-(t), \tag{3.166}$$

in which

$$\alpha'(t)=A_+(t)\alpha(0)+B_+\beta^*(0)+F_+(t) \tag{3.167}$$

and

$$\beta'(t)=A_-(t)\beta(0)+B_-\alpha^*(0)+F_-^*(t). \tag{3.168}$$

Furthermore, using Eqs. (3.165) and (3.166) along with their complex conjugates, we have

$$\begin{aligned}\phi(z,\chi,t)=&\exp\left[-(z^*z+\chi^*\chi)+(z-z^*)E_++(\chi-\chi^*)E_-\right]\\&\times\left\langle\exp(z\alpha'^*-z^*\alpha'+\chi\beta'^*-\chi^*\beta')\right\rangle.\end{aligned} \tag{3.169}$$

In view of Eqs. (3.165) and (3.166) we see that $\alpha(t)=\alpha'(t)$ and $\beta(t)=\beta'(t)$ if $E_\pm(t)=0$. One can also easily see from Eq. (B40) that $E_\pm(t)=0$ provided that $\mu=0$. Thus we note that upon setting $\mu=0$ in Eqs. (3.45) and (3.46), the equations of evolution take the forms

$$\frac{d}{dt}\langle\alpha'(t)\rangle=-(\frac{\kappa}{2}-R)\langle\alpha'(t)\rangle+(U+\varepsilon)\langle\beta'^*(t)\rangle \tag{3.170}$$

and

$$\frac{d}{dt}\langle\beta'(t)\rangle = -(\frac{\kappa}{2}+S)\langle\beta'(t)\rangle - (V-\varepsilon)\langle\alpha'^{*}(t)\rangle. \tag{3.171}$$

Inspection of these equations indicate that $\alpha'(t)$ and $\beta'(t)$ are Gaussian variables [21]. In addition, on account of (3.167) and (3.168) along with the assumption that the cavity radiation is initially in a two-mode vacuum state, we easily see that $\langle\alpha'(t)\rangle = \langle\beta'(t)\rangle = 0$. Thus $\alpha'(t)$ and $\beta'(t)$ are Gaussian variables with vanishing means. One can then express (3.169) in the form [21]

$$\begin{aligned}\phi(z,\chi,t) &= \exp\left[-(z^{*}z+\chi^{*}\chi)+(z-z^{*})E_{+}(t)+(\chi-\chi^{*})E_{-}(t)\right]\\ &\times \exp\frac{1}{2}\left\langle[z\alpha'^{*}-z^{*}\alpha'+\chi\beta'^{*}-\chi^{*}\beta']^{2}\right\rangle\end{aligned} \tag{3.172}$$

or

$$\begin{aligned}\phi(z,\chi,t) &= \exp\left[-(z^{*}z+\chi^{*}\chi)+(z-z^{*})E_{+}(t)+(\chi-\chi^{*})E_{-}(t)\right]\\ &\times \exp\left[\frac{z^{2}}{2}\langle\alpha'^{*2}\rangle+\frac{z^{*2}}{2}\langle\alpha'^{2}\rangle+\frac{\chi^{2}}{2}\langle\beta'^{*2}\rangle+\frac{\chi^{*2}}{2}\langle\beta'^{2}\rangle-zz^{*}\langle\alpha'\alpha'^{*}\rangle\right.\\ &\left.-\chi\chi^{*}\langle\beta'\beta'^{*}\rangle+z\chi\langle\alpha'^{*}\beta'^{*}\rangle+z^{*}\chi^{*}\langle\alpha'\beta'\rangle-z^{*}\chi\langle\alpha'\beta'^{*}\rangle-z\chi^{*}\langle\alpha'^{*}\beta'\rangle\right].\end{aligned} \tag{3.173}$$

Moreover, applying Eqs. (3.167) and (3.168), we get

$$\begin{aligned}\phi(z,\chi,t) &= \exp\left[-(z^{*}z+\chi^{*}\chi)+(z-z^{*})E_{+}(t)+(\chi-\chi^{*})E_{-}(t)\right]\\ &\times \exp\left[\frac{z^{2}}{2}\langle F_{+}^{*2}(t)\rangle+\frac{z^{*2}}{2}\langle F_{+}^{2}(t)\rangle+\frac{\chi^{2}}{2}\langle F_{-}^{*2}(t)\rangle+\frac{\chi^{*2}}{2}\langle F_{-}^{2}(t)\rangle\right.\\ &-zz^{*}\langle F_{+}(t)F_{+}^{*}(t)\rangle-\chi\chi^{*}\langle F_{-}(t)F_{-}^{*}(t)\rangle+z\chi\langle F_{+}^{*}(t)F_{-}^{*}(t)\rangle\\ &\left.+z^{*}\chi^{*}\langle F_{+}(t)F_{-}(t)\rangle-z^{*}\chi\langle F_{+}(t)F_{-}^{*}(t)\rangle-z\chi^{*}\langle F_{+}^{*}(t)F_{-}(t)\rangle\right].\end{aligned} \tag{3.174}$$

On account of Eqs. (3.102) and (3.103), the characteristic function is found to be

$$\phi(z,\chi,t) = \exp\left[-az^{*}z-b\chi^{*}\chi+c(z\chi+z^{*}\chi^{*})+E_{+}(t)(z-z^{*})+E_{-}(t)(\chi-\chi^{*})\right], \tag{3.175}$$

where

$$a = 1+\langle F_{+}(t)F_{+}^{*}(t)\rangle, \tag{3.176}$$

$$b = 1+\langle F_{-}(t)F_{-}^{*}(t)\rangle, \tag{3.177}$$

and

$$c = \langle F_{+}(t)F_{-}(t)\rangle. \tag{3.178}$$

Now substitution of Eq. (3.175) into the expression for the Q function (3.160), there follows

$$Q = \frac{1}{\pi^4}\int d^2\alpha d^2\chi \exp(z^*\alpha + \chi^*\beta - z\alpha^* - \chi\beta^*)$$
$$\times \exp\left[-az^*z - b\chi^*\chi + c(z\chi + z^*\chi^*) + E_+(t)(z - z^*) + E_-(t)(\chi - \chi^*)\right], \tag{3.179}$$

so that upon carrying out the integration, the Q function turns out to be

$$Q = \frac{uv - w^2}{\pi^2} e^x \exp\left[-u\alpha^*\alpha - v\beta^*\beta + w(\alpha\beta + \alpha^*\beta^*) + y(\alpha^* + \alpha) + z(\beta^* + \beta)\right], \tag{3.180}$$

in which

$$u = \frac{b}{ab - c^2}, \tag{3.181}$$

$$v = \frac{a}{ab - c^2}, \tag{3.182}$$

$$w = \frac{c}{ab - c^2}, \tag{3.183}$$

$$x = 2E_+(t)E_-(t)w - E_+^2(t)u - E_-^2(t)v, \tag{3.184}$$

$$y = E_+(t)u - E_-(t)w, \tag{3.185}$$

and

$$z = E_-(t)v - E_+(t)w. \tag{3.186}$$

3.3.1 Normally-ordered variance of the photon number sum and difference

We next proceed to determine, employing the resulting Q function, the mean and the normally-ordered variance of the photon number sum and difference for the cavity radiation produced by the system under consideration.

Mean of the photon number sum and difference

The interacavity photon number for mode a and mode b can be represented by the operators

$$\hat{n}_a = \hat{a}^\dagger\hat{a}, \tag{3.187}$$

$$\hat{n}_b = \hat{b}^\dagger\hat{b}. \tag{3.188}$$

Using (2.173) the mean photon number for mode a and b can be expressed as

$$\bar{n}_a = \int d^2\alpha d^2\beta Q(\alpha,\beta)\alpha^*\alpha - 1 \tag{3.189}$$

and

$$\bar{n}_b = \int d^2\alpha d^2\beta Q(\alpha,\beta)\beta^*\beta - 1. \tag{3.190}$$

Now applying the Q function (3.180), we have

$$\begin{aligned}\bar{n}_a =& \frac{uv-w^2}{\pi^2}e^x \\ &\times \int d^2\alpha d^2\beta \exp\left(-u\alpha^*\alpha - v\beta^*\beta + w(\alpha\beta+\alpha^*\beta^*) + y(\alpha^*+\alpha) + z(\beta^*+\beta)\right)\alpha^*\alpha - 1.\end{aligned} \tag{3.191}$$

This can be put in the form

$$\begin{aligned}\bar{n}_a =& -\frac{uv-w^2}{U}e^x\frac{d}{dg}\Bigg[\frac{1}{\pi^2} \\ &\times \int d^2\alpha d^2\beta \exp\left(-gu\alpha^*\alpha - v\beta^*\beta + w(\alpha\beta+\alpha^*\beta^*) + y(\alpha^*+\alpha) + z(\beta^*+\beta)\right)\Bigg]_{g=1} - 1,\end{aligned} \tag{3.192}$$

so that upon carrying out the integration, we get

$$\bar{n}_a = -\frac{uv-w^2}{uv}\exp\left(x+\frac{z^2}{v}\right)\frac{d}{dg}\left[\frac{1}{gu-w^2/v}\exp\left(\frac{(y+wz/v)^2}{gu-w^2/v}\right)\right]_{g=1} - 1. \tag{3.193}$$

Furthermore, upon performing the differentiation and applying the condition $g=1$, we obtain

$$\bar{n}_a = \exp\left(x+\frac{z^2}{v}+\frac{(y+\frac{wz}{v})^2}{u-\frac{w^2}{v}}\right)\left[\frac{(y+\frac{wz}{v})^2}{(u-\frac{w^2}{v})^2}+\frac{1}{u-\frac{w^2}{V}}\right] - 1. \tag{3.194}$$

With the aid of Eqs. (3.181), (3.182), (3.183), (3.184), (3.185), and (3.186), one can readily show that

$$y + \frac{wz}{v} = \frac{E_+(t)}{a}, \tag{3.195}$$

$$u - \frac{w^2}{v} = \frac{1}{a}, \tag{3.196}$$

and

$$\exp\left(x+\frac{z^2}{v}+\frac{(y+\frac{wz}{v})^2}{U-\frac{W^2}{V}}\right) = 1. \tag{3.197}$$

Application of these results in (3.194) yields

$$\bar{n}_a = E_+^2 + a - 1. \tag{3.198}$$

One can also establish following a similar procedure that

$$\bar{n}_b = E_-^2 + b - 1. \tag{3.199}$$

Thus in view of (3.176) and (3.177), the mean photon number for mode a and mode b takes the forms

$$\bar{n}_a = E_+^2(t) + \langle F_+(t)F_+^*(t)\rangle \tag{3.200}$$

and

$$\bar{n}_b = E_-^2(t) + \langle F_-(t)F_-^*(t)\rangle. \tag{3.201}$$

Therefore, on account of Eqs. (B40), (3.107), and (3.110), the mean photon number for mode a and mode b at steady state turns over into

$$\begin{aligned}\bar{n}_{a_{ss}} = {}& \frac{\mu^2}{4}\bigg[\frac{p^2+q_+^2+2q_+-2p-2pq_++1}{\lambda_1^2} + \frac{p^2+q_+^2+2p-2q_+-2pq_++1}{\lambda_2^2} \\ &+ \frac{4pq_+-2p^2-2q_++2}{\lambda_1\lambda_2}\bigg] + \frac{R(1-p)^2-(V-\varepsilon)q_+(1-p)}{4\lambda_1} \\ &+ \frac{R(1+p)^2+(V+\varepsilon)q_+(1+p)}{4\lambda_2} + \frac{R(1-p^2)-(V-\varepsilon)q_+p}{\lambda_1+\lambda_2},\end{aligned} \tag{3.202}$$

$$\begin{aligned}\bar{n}_{b_{ss}} = {}& \frac{\mu^2}{4}\bigg[\frac{p^2+q_-^2+2q_-+2p+2pq_-+1}{\lambda_1^2} + \frac{p^2+q_-^2-2p-2q_-+2pq_-+1}{\lambda_2^2} \\ &+ \frac{2-4pq_--2p^2-2q_-}{\lambda_1\lambda_2}\bigg] + \frac{Rq_-^2-(V-\varepsilon)q_-(1+p)}{4\lambda_1} \\ &+ \frac{Rq_-^2+(V-\varepsilon)q_-(1-p)}{4\lambda_2} - \frac{Rq_-^2-(V-\varepsilon)q_-p}{\lambda_1+\lambda_2}.\end{aligned} \tag{3.203}$$

We easily see from Eqs. (3.202) and (3.203) that the driving coherent light increases the mean photon number of mode a and mode b significantly. In addition, employing these same equations one can readily find that the parametric amplifier also enhance the mean photon number of mode a and mode b considerably.

The intracavity photon number sum and difference is defined by

$$\hat{n}_\pm = \hat{n}_a \pm \hat{n}_b. \tag{3.204}$$

The mean of the photon number sum and difference can then be expressed as

$$\bar{n}_\pm = \bar{n}_a \pm \bar{n}_b. \tag{3.205}$$

Hence introducing (3.202) and (3.203) in Eq. (3.205), the mean of the photon number sum and difference takes at steady state the form

$$\begin{aligned}\bar{n}_{\pm_{ss}} =& \frac{\mu^2}{4}\Bigg[\frac{p^2+q_+^2+2q_+-2p-2pq_++1}{\lambda_1^2}+\frac{p^2+q_+^2+2p-2q_+-2pq_++1}{\lambda_2^2}\\ &+\frac{4pq_+-2p^2-2q_++2}{\lambda_1\lambda_2}\Bigg]+\frac{R(1-p)^2-(V-\varepsilon)q_+(1-p)}{4\lambda_1}\\ &+\frac{R(1+p)^2+(V+\varepsilon)q_+(1+p)}{4\lambda_2}+\frac{R(1-p^2)-(V-\varepsilon)q_+p}{\lambda_1+\lambda_2}\\ &\pm\Bigg[\frac{\mu^2}{4}\bigg(\frac{p^2+q_-^2+2q_-+2p+2pq_-+1}{\lambda_1^2}+\frac{p^2+q_-^2-2p-2q_-+2pq_-+1}{\lambda_2^2}\\ &+\frac{2-4pq_--2p^2-2q_-}{\lambda_1\lambda_2}\bigg)+\frac{Rq_-^2-(V-\varepsilon)q_-(1+p)}{4\lambda_1}\\ &+\frac{Rq_-^2+(V-\varepsilon)q_-(1-p)}{4\lambda_2}-\frac{Rq_-^2-(V-\varepsilon)q_-p}{\lambda_1+\lambda_2}\Bigg]. \end{aligned} \tag{3.206}$$

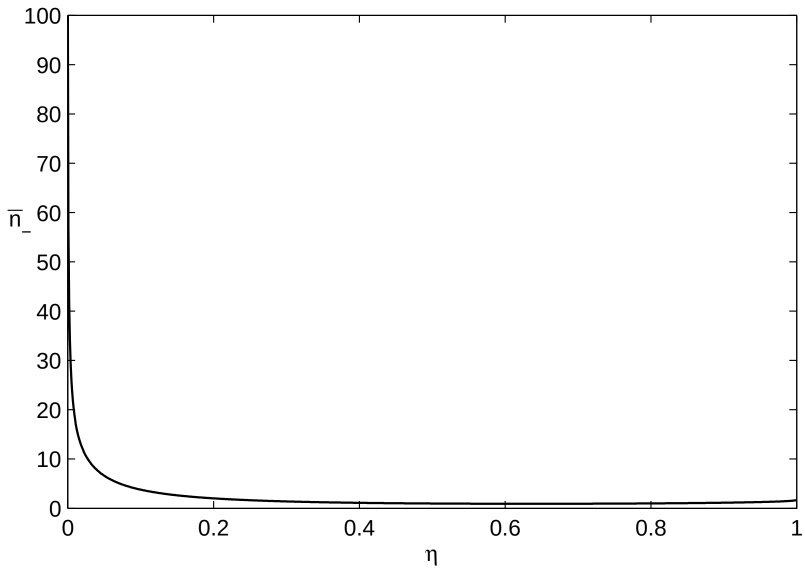

Fig. 3.10: A plot of the photon number difference for the cavity modes [Eq. (3.206)] versus η for $\frac{\Omega}{\gamma} = 0.01$, $\varepsilon = \mu = 0.5$, $A = 100$, and $\kappa = 0.8$.

The plot in Fig. 3.10 clearly indicates that the mean of the photon number difference is positive. This indicates that the mean photon number of mode a is greater than that of mode b. Moreover, we observe that in general the mean of the photon number difference decreases as η increases.

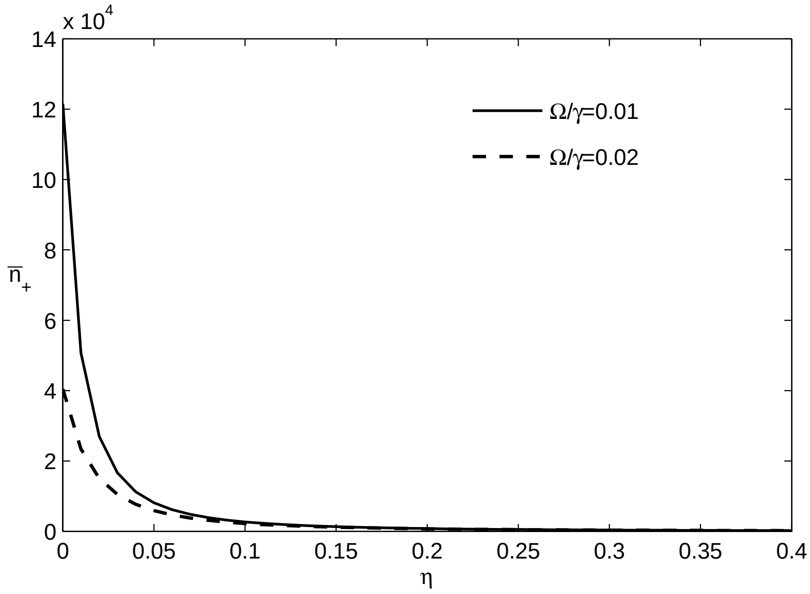

Fig. 3.11: Plots of the photon number sum [Eq. (3.206)] versus η for $\varepsilon = \mu = 0$, $A = 100$, $\kappa = 0.8$, and different values of $\frac{\Omega}{\gamma}$.

The plots in Fig. 3.11 show that the mean photon number decreases with $\frac{\Omega}{\gamma}$. This must be due to stimulated emission induced by the pump mode. The photons emitted this way do not contribute to the mean photon number of the cavity modes.

We next wish to analyze some interesting special cases. First we consider the case in which the atoms are not injected into the cavity and in the absence of the driving light. Thus upon setting $A = \mu = 0$ in Eq. (3.206), we get

$$\overline{n}_{\pm} = (1 \pm 1)\left[\frac{2\varepsilon^2}{\kappa^2 - 4\varepsilon^2}\right]. \tag{3.207}$$

This represent the mean of the photon number sum and difference of the cavity modes for a nondegenerate parametric oscillator. Inspection of Eq. (3.207) indicates that the mean of the photon number difference is zero. Furthermore, we consider the case in which the parametric amplifier and the driving light are absent. Thus upon setting $\varepsilon = \mu = 0$ (with $\frac{\Omega}{\gamma} = 0$) in Eq. (3.206), we obtain

$$\overline{n}_{\pm} = \frac{2A(1-\eta)(2\kappa + A\eta) + (1 \pm 1)A^2(1-\eta^2)}{4(\kappa + A\eta)(2\kappa + A\eta)}. \tag{3.208}$$

This is the mean of the photon number sum and difference for the cavity modes for a nondegenerate three-level laser. One can easily see from Eq. (3.208) that the mean of the photon number difference is always positive, this indicates that the mean photon number of mode a is greater than that of mode b. Hence inspection of Eqs. (3.207) and (3.208) shows that the mean of the photon number difference is due to the three-level laser. This must be due to the decay of some atoms from the intermediate level to levels other than the bottom level spontaneously.

Normally-ordered variance of the photon number sum and difference

Next we wish to determine the normally-ordered variance of the photon number sum and difference. To this end, the normally-ordered variance of the photon number sum and difference is defined as [48]

$$: \Delta n_{\pm}^2 := \langle : \hat{n}_{\pm}^2 : \rangle - \langle \hat{n}_{\pm} \rangle^2. \tag{3.209}$$

Furthermore, employing Eqs. (3.187) and (3.188), the normally-ordered variance of the photon number sum and difference takes the form

$$: \Delta n_{\pm}^2 := \Delta n_{\pm}^2 - \bar{n}_{+}, \tag{3.210}$$

with $\Delta n_{\pm}^2$ and $\bar{n}_{+}$ being the variance of the photon number sum and difference and the mean of the photon number sum. The variance of the photon number sum and difference is expressible as

$$\Delta n_{\pm}^2 = \langle \hat{n}_{\pm}^2 \rangle - \langle \hat{n}_{\pm} \rangle^2. \tag{3.211}$$

With the aid of Eq. (3.204) the variance of the photon number sum and difference can be put in the form

$$\Delta n_{\pm}^2 = \Delta n_a^2 + \Delta n_b^2 \pm 2\big(\langle \hat{n}_a \hat{n}_b \rangle - \langle \hat{n}_a \rangle \langle \hat{n}_b \rangle\big). \tag{3.212}$$

The photon number variance for mode a is expressible as

$$\Delta n_a^2 = \langle \hat{n}_a^2 \rangle - \langle \hat{n}_a \rangle^2. \tag{3.213}$$

This can be put in the form

$$\Delta n_a^2 = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - \bar{n}_a^2 - 3\bar{n}_a - 2. \tag{3.214}$$

Now employing Eq. (2.173), we have

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \int d^2\alpha d^2\beta Q(\alpha, \beta) \alpha^2 \beta^2. \tag{3.215}$$

Introducing the Q function (3.180) in Eq. (3.215), we get

$$\begin{aligned} \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle &= \frac{uv - w^2}{\pi^2} e^x \\ &\times \int d^2\alpha d^2\beta \alpha^2 \alpha^{*2} \exp[-u\alpha^*\alpha - v\beta^*\beta + w(\alpha\beta + \alpha^*\beta^*) + y(\alpha^* + \alpha) + z(\beta^* + \beta)]. \end{aligned} \tag{3.216}$$

This can be put in the form

$$\langle \hat{a}^2\hat{a}^{\dagger 2}\rangle = \frac{uv-w^2}{u^2}e^x\frac{d^2}{dg^2}$$
$$\times\left[\frac{1}{\pi^2}\int d^2\alpha d^2\beta\exp[-u\alpha^*\alpha - v\beta^*\beta + w(\alpha\beta+\alpha^*\beta^*) + y(\alpha^*+\alpha) + z(\beta^*+\beta)]\right]_{g=1}, \tag{3.217}$$

so that carrying out the integration, we obtain

$$\langle \hat{a}^2\hat{a}^{\dagger 2}\rangle = \frac{uv-w^2}{vu^2}e^{x+\frac{z^2}{v}}\frac{d^2}{dg^2}\left[\frac{1}{gu-\frac{w^2}{v}}\exp\left(\frac{(y+\frac{wz}{v})^2}{gu-\frac{w^2}{v}}\right)\right]_{g=1}. \tag{3.218}$$

Differentiating with respect to g and applying the condition $g=1$, there follows

$$\langle \hat{a}^2\hat{a}^{\dagger 2}\rangle = \exp[x+\frac{z^2}{v}+\frac{(y+\frac{wz}{v})^2}{gu-\frac{w^2}{v}}]\frac{uv-w^2}{u^2}$$
$$\times\left[\frac{3u^2(y+\frac{wz}{v})^2}{(u-\frac{w^2}{v})^4}+\frac{u^2(y+\frac{wz}{v})^4}{(u-\frac{w^2}{v})^5}+\frac{2u^2}{(u-\frac{w^2}{v})^3}+\frac{u^2(y+\frac{wz}{v})^2}{(u-\frac{w^2}{v})^4}\right]. \tag{3.219}$$

On account of Eqs. (3.195), (3.196), and (3.197), we find

$$\langle \hat{a}^2\hat{a}^{\dagger 2}\rangle = E_+^4 + 4E_+^2 a + 2a^2. \tag{3.220}$$

Furthermore, substitution of this result into Eq. (3.214) yields

$$\Delta n_a^2 = 2E_+^2 a + a^2 - E_+^2 - a. \tag{3.221}$$

Finally, employing Eq. (3.176), we get

$$\Delta n_a^2 = \bar{n}_a + 2E_+^2\langle F_+(t)F_+^*(t)\rangle + \langle F_+(t)F_+^*(t)\rangle^2. \tag{3.222}$$

Following a similar procedure one can also establish that

$$\Delta n_b^2 = \bar{n}_b + 2E_-^2\langle F_-(t)F_-^*(t)\rangle + \langle F_-(t)F_-^*(t)\rangle^2. \tag{3.223}$$

On the other hand, one can write

$$\langle \hat{n}_a\hat{n}_b\rangle = \langle \hat{a}\hat{a}^\dagger\hat{b}\hat{b}^\dagger\rangle - \bar{n}_a - \bar{n}_b - 1. \tag{3.224}$$

Now using Eq. (2.173) along with the Q function (3.180), we have

$$\langle \hat{a}\hat{a}^\dagger\hat{b}\hat{b}^\dagger\rangle = \frac{uv-w^2}{\pi^2}e^x\int d^2\alpha d^2\beta\alpha\alpha^*\beta\beta^*\exp[-u\alpha^*\alpha - v\beta^*\beta$$
$$+ w(\alpha\beta+\alpha^*\beta^*) + y(\alpha^*+\alpha) + z(\beta^*+\beta)]. \tag{3.225}$$

It then follows that

$$\langle \hat{a}\hat{a}^\dagger \hat{b}\hat{b}^\dagger \rangle = \frac{uv - w^2}{uv} e^x \frac{d^2}{dhdg} \left[\frac{1}{\pi^2} \int d^2\alpha d^2\beta \exp[-gu\alpha^*\alpha - hv\beta^*\beta + w(\alpha\beta + \alpha^*\beta^*) + y(\alpha^* + \alpha) + z(\beta^* + \beta)] \right]_{g=h=1}, \tag{3.226}$$

so that upon performing the integration, we obtain

$$\langle \hat{a}\hat{a}^\dagger \hat{b}\hat{b}^\dagger \rangle = \frac{uv - w^2}{vu} e^x \frac{d^2}{dhdg} \left[\frac{1}{hv(gu - \frac{w^2}{hv})} \exp\left(\frac{(y + \frac{wz}{hv})^2}{gu - \frac{w^2}{hv}} + \frac{z^2}{hv} \right) \right]_{h=g=1}. \tag{3.227}$$

Furthermore, differentiating with respect to g, there follows

$$\langle \hat{a}\hat{a}^\dagger \hat{b}\hat{b}^\dagger \rangle = \frac{uv - w^2}{vu} e^x \frac{d}{dh} - \left[\frac{huv}{[hv(gu - \frac{w^2}{hv})]^2} \exp\left(\frac{(y + \frac{wz}{hv})^2}{gu - \frac{w^2}{hv}} + \frac{z^2}{hv} \right) + \frac{u(y + \frac{wz}{hv})^2}{hv(gu - \frac{w^2}{hv})^3} \exp\left(\frac{(y + \frac{wz}{hv})^2}{gu - \frac{w^2}{hv}} + \frac{z^2}{hv} \right) \right]_{h=g=1} \tag{3.228}$$

and differentiating with respect to h and applying the conditions $g = h = 1$, we find

$$\begin{aligned} \langle \hat{a}\hat{a}^\dagger \hat{b}\hat{b}^\dagger \rangle &= \exp\left(\frac{(y + \frac{wz}{v})^2}{u - \frac{w^2}{v}} + \frac{z^2}{v} + x \right) \frac{uv - w^2}{uv} \\ &\times \left[\frac{uv}{v^2(u - \frac{w^2}{v})^2} \left(\frac{2(y + \frac{wz}{v})\frac{wz}{v}(u - \frac{w^2}{v}) + \frac{w^2}{v}(y + \frac{wz}{v})^2}{(u - \frac{w^2}{v})^2} + \frac{z^2}{v} \right) \right. \\ &+ \frac{2u(y + \frac{wz}{v})(u - \frac{w^2}{v})wz + u(y + \frac{wz}{v})^2[3w^2 + v(u - \frac{w^2}{v})]}{v^2(u - \frac{w^2}{v})^4} \\ &\left. + \frac{u(y + \frac{wz}{v})^2}{v(u - \frac{w^2}{v})^3} \left(\frac{2(y + \frac{wz}{v})\frac{wz}{v}(u - \frac{w^2}{v}) + \frac{w^2}{v}(y + \frac{wz}{v})^2}{(u - \frac{w^2}{v})^2} + \frac{z^2}{v} \right) - \frac{u(u - \frac{w^2}{v}) - 2u^2}{v(u - \frac{w^2}{v})^3} \right]. \end{aligned} \tag{3.229}$$

Employing Eqs. (3.195), (3.196), and (3.197), we get

$$\langle \hat{a}\hat{a}^\dagger \hat{b}\hat{b}^\dagger \rangle = \frac{a}{v^2}\left(2E_+wz + E_+^2w^2 + z^2\right) + \frac{a}{v^2}\left(2E_+wz + 3E_+^2w^2 + \frac{E_+^2 v}{a}\right) + \frac{E_+^2}{v^2}\left(2E_+wz + E_+^2w^2 + z^2\right) - \frac{a}{v}\left(1 - 2ua\right). \tag{3.230}$$

With the aid of Eqs. (3.181), (3.182), and (3.183), we have

$$\langle \hat{a}\hat{a}^\dagger \hat{b}\hat{b}^\dagger \rangle = E_-^2 a + 2E_+E_-c + E_+^2 b + E_+^2E_-^2 + ab + c^2. \tag{3.231}$$

Application of this result in Eq. (3.224) leads to

$$\langle \hat{n}_a \hat{n}_b \rangle = 2E_+E_-c + E_+^2 b + E_+^2E_-^2 + ab + c^2 - \bar{n}_a - \bar{n}_b - 1. \tag{3.232}$$

Moreover, using Eqs. (3.198) and (3.199), there follows

$$\langle \hat{n}_a \hat{n}_b \rangle = E_+^2 E_-^2 + E_-^2 a + E_+^2 b + ab - a - b + 1 + 2E_+ E_- c + c^2. \tag{3.233}$$

This can be put in the form

$$\langle \hat{n}_a \hat{n}_b \rangle = \bar{n}_a \bar{n}_b + 2E_+ E_- c + c^2. \tag{3.234}$$

In view of Eq. (3.178), we see that

$$\langle \hat{n}_a \hat{n}_b \rangle = \bar{n}_a \bar{n}_b + 2E_+ E_- \langle F_+(t) F_-(t) \rangle + \langle F_+(t) F_-(t) \rangle^2. \tag{3.235}$$

Hence upon substituting Eqs. (3.222), (3.223), and (3.235) into (3.212), the variance of the photon number sum and difference turns out to be

$$\begin{aligned} \Delta n_\pm^2(t) = \bar{n}_a + \bar{n}_b + 2E_+^2 \langle F_+ F_+^* \rangle + 2E_-^2 \langle F_- F_-^* \rangle + \langle F_+ F_+^* \rangle^2 + \langle F_- F_-^* \rangle^2 \\ \pm 2\bigg(2E_+ E_- \langle F_+ F_- \rangle + \langle F_+ F_- \rangle^2 \bigg). \end{aligned} \tag{3.236}$$

Therefore, on account of Eq. (3.236), the normally-ordered variance of the photon number sum and difference (3.210) takes the form

$$\begin{aligned} : \Delta n_\pm^2(t) : = 2E_+^2 \langle F_+ F_+^* \rangle + 2E_-^2 \langle F_- F_-^* \rangle + \langle F_+ F_+^* \rangle^2 + \langle F_- F_-^* \rangle^2 \\ \pm 2\bigg(2E_+ E_- \langle F_+ F_- \rangle + \langle F_+ F_- \rangle^2 \bigg). \end{aligned} \tag{3.237}$$

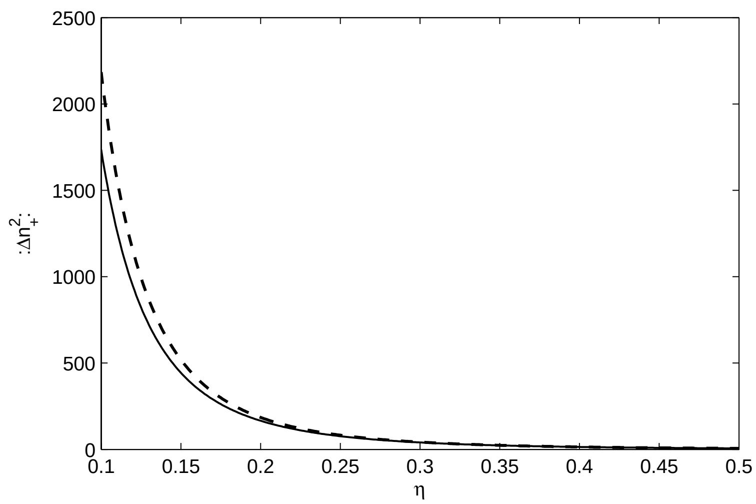

Fig. 3.12: Plots of the normally-ordered variance of the photon number sum [Eq. (3.237)] versus η for $\mu = 0.2$, $\varepsilon = 0.3$, $A = 100$, $\kappa = 0.8$, and $\frac{\Omega}{\gamma} = 0$ (dotted curve) and $\frac{\Omega}{\gamma} = 0.04$ (solid curve).

We see from Fig. 3.12 that the normally-ordered variance of the photon number sum is positive. This indicates that the photon number sum statistics is super-Poissonian. Furthermore, we note that one effect of the coupling of the top and bottom levels is to decrease the normally-ordered variance of the photon number sum. In addition, we observe that the normally-ordered variance of the photon sum decreases as η increases.

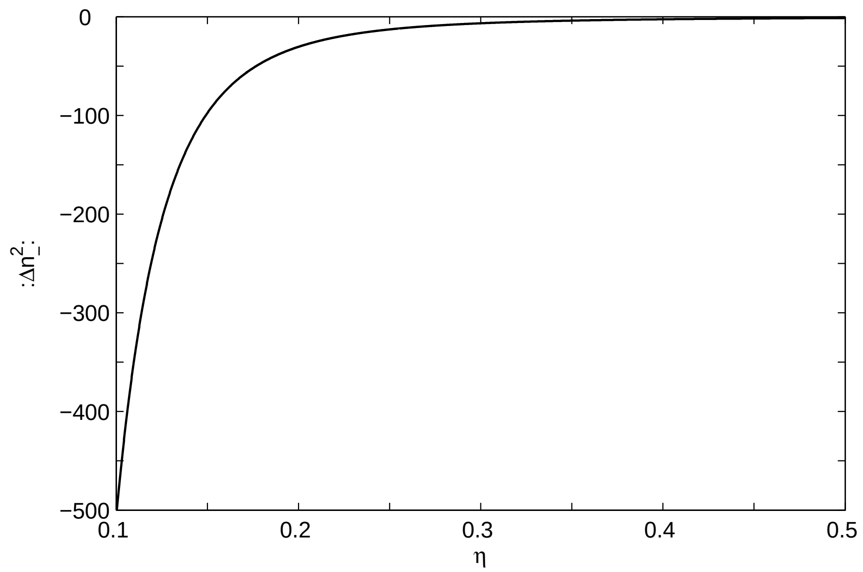

Fig. 3.13: Plots of the normally-ordered variance of the photon number difference [Eq. (3.237)] versus η for $\mu = 0.2$, $\varepsilon = 0.3$, $A = 100$, $\kappa = 0.8$, and $\frac{\Omega}{\gamma} = 0.04$.

We observe from Fig. 3.13 that the normally-ordered variance of the photon number difference is negative. This indicates that the photon number difference statistics is sub-Poissonian. Moreover, the plot shows that the normally-ordered variance of the photon number difference increases with η.

3.3.2 Photon number distribution

The joint probability to find n photons of mode a and m photons of mode b can be written in terms of the Q function as [21]

$$P(n,m,t) = \frac{\pi^2}{n!m!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^{n}}\frac{\partial^{2m}}{\partial\beta^{*m}\partial\beta^{m}}\left[Q(\alpha,\beta,t)\exp(\alpha^*\alpha+\beta^*\beta)\right]_{\alpha^*=\alpha=\beta^*=\beta=0}, \tag{3.238}$$

so that on account of Eq.(3.180), we have

$$P(n,m,t) = \frac{[uv - w^2]}{n!m!}e^x \frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}\frac{\partial^{2m}}{\partial\beta^{*m}\partial\beta^m}\exp\Big[(1-u)\alpha^*\alpha + (1-v)\beta^*\beta + w(\alpha\beta + \alpha^*\beta^*) + y(\alpha^* + \alpha) + z(\beta^* + \beta)\Big]_{\alpha^*=\alpha=\beta^*=\beta=0}. \tag{3.239}$$

Upon expanding in power series, we get

$$P(n,m,t) = \frac{[uv - w^2]}{n!m!}e^x \sum_{ijklpqrs}\frac{(1-u)^i(1-v)^j w^{k+l}y^{p+q}z^{r+s}}{i!j!k!l!p!q!r!s!} \times \frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}\frac{\partial^{2m}}{\partial\beta^{*m}\partial\beta^m}\Big[\alpha^{i+k+q}\alpha^{*i+l+p}\beta^{j+k+s}\beta^{*j+l+r}\Big]_{\alpha^*=\alpha=\beta^*=\beta=0}. \tag{3.240}$$

Then carrying out the differentiation and applying the condition $\alpha^* = \alpha = \beta^* = \beta = 0$, we obtain

$$P(n,m,t) = \frac{[uv - w^2]}{n!m!}e^x \sum_{ijklpqrs}\frac{(1-u)^i(1-v)^j w^{k+l}y^{p+q}z^{r+s}}{i!j!k!l!p!q!r!s!} \times \frac{(i+k+q)!}{(i+k+q-n)!}\frac{(i+l+p)!}{(i+l+p-n)!}\frac{(j+k+s)!}{(j+k+s-m)!}\frac{(j+l+r)!}{(j+l+r-m)!} \times \delta_{i+k+q,n}\delta_{i+l+p,n}\delta_{j+k+s,m}\delta_{j+l+r,m}. \tag{3.241}$$

With the aid of the properties of the Kronecker delta symbol, we have

$$P(n,m,t) = [uv - w^2]e^x n!m! \sum_{ijkl}\frac{(1-u)^i(1-v)^j w^{k+l}y^{2(n-i)-k-l}z^{2(m-j)-k-l}}{i!j!k!l!(n-l-i)!(n-k-i)!(m-l-j)!(m-k-j)!}. \tag{3.242}$$

It is interesting to consider the special case in which the driving coherent light is absent. Thus upon setting ($\mu = E_+ = E_- = 0$), we find

$$x = y = z = 0. \tag{3.243}$$

In view of this, we see that

$$k = l = (n-i) = (m-j) \tag{3.244}$$

and

$$j = i + m - n. \tag{3.245}$$

With the aid of these results the photon number distribution takes the form

$$P(n,m,t) = [uv - w^2]n!m! \sum_{i=n-m}^{n}\frac{(1-u)^i(1-v)^{i+m-n}w^{2(n-i)}}{i!(i+m-n)![(n-i)!]^2}. \tag{3.246}$$

From the fact that a factorial is defined for nonnegative integers, one can easily show that $n - m \leq i \leq n$. Thus we realize that the joint probability of observing n photons of mode a

and m photons of mode b inside the cavity with $m > n$ is zero. On the other hand, for $n = m$ the photon number distribution takes the form

$$P(n,n,t) = [uv - w^2]n!^2 \sum_{i=0}^{n} \frac{[(1-u)(1-v)]^i w^{2(n-i)}}{(i!)^2[(n-i)!]^2}. \tag{3.247}$$

From this result we note that there is a finite probability of finding equal number of photons in the two modes.

3.4 Photon number and count statistics of the output modes

We first seek to express the mean and the normally-ordered variance of the photon number sum and difference for the output modes in terms of the mean and the normally-ordered variance of the photon number sum and difference for the cavity modes. Then using the expression for the l^{th} moment of the photon count in terms of the photon number distribution along with the resulting mean and normally-ordered variance of the photon number sum and difference for the output modes, we calculate the mean and the normally-ordered variance of the photon count sum and difference for the output modes of the system under consideration.

3.4.1 Normally-ordered variance of the photon number sum and difference

The photon number sum and difference of the output modes is defined by

$$\hat{n}_{\pm}^{out} = \hat{n}_a^{out} \pm \hat{n}_b^{out}. \tag{3.248}$$

The mean photon number sum and difference of the output modes can then be given as

$$\bar{n}_{\pm}^{out} = \bar{n}_a^{out} \pm \bar{n}_b^{out}. \tag{3.249}$$

For a cavity mode coupled to a vacuum reservoir, one can write using the input-output relation $(A4)$ that

$$\hat{n}_a^{out} = \kappa \hat{n}_a \tag{3.250}$$

and

$$\hat{n}_b^{out} = \kappa \hat{n}_b. \tag{3.251}$$

In view of these results the mean photon number sum and difference for the output modes takes the form

$$\bar{n}_{\pm}^{out} = \kappa \bar{n}_{\pm}, \tag{3.252}$$

with $\bar{n}_{\pm}$ being the mean of the photon number sum and difference. Thus on account of Eq. (3.206), the mean of the photon number sum and difference for the output modes takes at steady state the form

$$\begin{aligned}\bar{n}^{out}_{\pm_{ss}} &= \frac{\mu^2\kappa}{4}\bigg[\frac{p^2+q_+^2+2q_+-2p-2pq_++1}{\lambda_1^2}+\frac{p^2+q_+^2+2p-2q_+-2pq_++1}{\lambda_2^2}\\ &+\frac{4pq_+-2p^2-2q_++2}{\lambda_1\lambda_2}\bigg]+\kappa\frac{R(1-p)^2-(V-\varepsilon)q_+(1-p)}{4\lambda_1}\\ &+\kappa\frac{R(1+p)^2+(V+\varepsilon)q_+(1+p)}{4\lambda_2}+\kappa\frac{R(1-p^2)-(V-\varepsilon)q_+p}{\lambda_1+\lambda_2}\\ &\pm\bigg[\frac{\mu^2\kappa}{4}\bigg(\frac{p^2+q_-^2+2q_-+2p+2pq_-+1}{\lambda_1^2}+\frac{p^2+q_-^2-2p-2q_-+2pq_-+1}{\lambda_2^2}\\ &+\frac{2-4pq_--2p^2-2q_-}{\lambda_1\lambda_2}\bigg)+\kappa\frac{Rq_-^2-(V-\varepsilon)q_-(1+p)}{4\lambda_1}\\ &+\kappa\frac{Rq_-^2+(V-\varepsilon)q_-(1-p)}{4\lambda_2}-\kappa\frac{Rq_-^2-(V-\varepsilon)q_-p}{\lambda_1+\lambda_2}\bigg].\end{aligned} \tag{3.253}$$

Next we want to express the normally-ordered variance of the photon number sum and difference for the output modes in terms of the normally-ordered variance of the photon number sum and difference for the cavity modes. The normally-ordered variance of the photon number sum and difference for the output modes can be written as

$$:\Delta n^2_{\pm out} := \langle : \hat{n}^2_{\pm} : \rangle_{out} - \langle \hat{n}_{\pm}\rangle^2_{out}. \tag{3.254}$$

This can also be rewritten employing Eq. (3.248) as

$$:\Delta n^2_{\pm out} := \big(:\Delta n_a^2:\big)_{out} + \big(:\Delta n_b^2:\big)_{out} \pm 2\left[\langle : \hat{n}_a\hat{n}_b : \rangle_{out} - \langle \hat{n}_a\rangle_{out}\langle \hat{n}_b\rangle_{out}\right]. \tag{3.255}$$

On account of (3.250) and (3.251) along with Eqs. (2.220), we find

$$:\Delta n^2_{\pm out} := \kappa^2\bigg[:\Delta n_a^2: + :\Delta n_b^2: \pm 2\left[\langle : \hat{n}_a\hat{n}_b : \rangle - \bar{n}_a\bar{n}_b\right]\bigg], \tag{3.256}$$

so that in view of Eq. (3.212), we see that

$$:\Delta n^2_{\pm out} := \kappa^2 :\Delta n^2_{\pm}:, \tag{3.257}$$

where $:\Delta n^2_{\pm}:$ is the normally-ordered variance of the photon number sum and difference for the cavity modes. With the aid of (3.237) the normally-ordered variance of the photon number for the output mode is found to be

$$\begin{aligned}:\Delta n^2_{\pm out}(t): &= 2\kappa^2E_+^2\langle F_+F_+^*\rangle + 2\kappa^2E_-^2\langle F_-F_-^*\rangle + \kappa^2\langle F_+F_+^*\rangle^2 + \kappa^2\langle F_-F_-^*\rangle^2\\ &\pm 2\kappa^2\Big(2E_+E_-\langle F_+F_-\rangle + \langle F_+F_-\rangle^2\Big).\end{aligned} \tag{3.258}$$

We note that κ is always positive. Thus one can easily see that the photon number sum and difference statistics for the output modes is similar to that for the cavity modes. Moreover, on account of Eqs. (3.252) and (3.257) along with the fact that κ is less than one, we observe that the mean and the normally-ordered variance of the photon number sum and difference for the output modes are less than that for the cavity modes.

3.4.2 Normally-ordered variance of the photon count sum and difference

We next seek to obtain, using the expression for the l^{th} moment of the photon count along with the resulting mean and normally-ordered variance of the photon number sum and difference for the output modes, the normally-ordered variance of the photon number sum and difference for the output mode. To this end, the l^{th} moment of the photon count sum and difference is expressible as

$$\overline{m^l_{\pm}} = \sum_{n,m=0}^{\infty} P_{out}(n,m)\left[(\lambda_1-1)\frac{d}{d\lambda_1} \pm (\lambda_2-1)\frac{d}{d\lambda_2}\right]^l (1-u\lambda_1)^n(1-u\lambda_2)^m\bigg|_{\lambda_1=\lambda_2=0}. \quad (3.259)$$

Thus using this relation, the mean of the photon count sum and difference can be written as

$$\overline{m}_{\pm} = \sum_{n,m=0}^{\infty} P_{out}(n,m)\left[(\lambda_1-1)\frac{d}{d\lambda_1} \pm (\lambda_2-1)\frac{d}{d\lambda_2}\right] (1-u\lambda_1)^n(1-u\lambda_2)^m\bigg|_{\lambda_1=\lambda_2=0}. \quad (3.260)$$

Upon carrying out the differentiation and applying the conditions $\lambda_1 = \lambda_2 = 0$, we obtain

$$\overline{m}_{\pm} = u\overline{n}^{out}_{\pm}. \quad (3.261)$$

In view of Eq. (3.252), we see that

$$\overline{m}_{\pm} = u\kappa\overline{n}_{\pm}. \quad (3.262)$$

Furthermore, with the help of Eq. (3.259) the second moment of the photon count sum and difference can be expressed as

$$\overline{m^2}_{\pm} = \sum_{n,m=0}^{\infty} P(n,m)_{out}\left[(\lambda_1-1)\frac{d}{d\lambda_1} \pm (\lambda_2-1)\frac{d}{d\lambda_2}\right]^2 (1-u\lambda_1)^n(1-u\lambda_2)^m\bigg|_{\lambda_1=\lambda_2=0}. \quad (3.263)$$

It then follows that

$$\overline{m^2}_{\pm} = \sum_{n=m=0}^{\infty} P(n,m)_{out}\left[(\lambda_1-1)^2\frac{d^2}{d\lambda_1^2} + (\lambda_1-1)\frac{d}{d\lambda_1}\right] + \left[(\lambda_2-1)^2\frac{d^2}{d\lambda_2^2} + (\lambda_2-1)\frac{d}{d\lambda_2}\right]$$
$$\pm\left[(\lambda_1-1)(\lambda_2-1)\frac{d^2}{d\lambda_1 d\lambda_2} + (\lambda_2-1)(\lambda_1-1)\frac{d^2}{d\lambda_2 d\lambda_1}\right](1-u\lambda)^n(1-u\lambda)^m\bigg|_{\lambda_1=\lambda_2=0}. \quad (3.264)$$

Upon performing the differentiation and applying the conditions $\lambda_1 = \lambda_2 = 0$, we find

$$\overline{m^2}_{\pm} = u^2\bigg(\langle\hat{n}_a^2\rangle_{out} - \langle\hat{n}_a\rangle_{out}\bigg) + u\langle\hat{n}_a\rangle_{out} + u^2\bigg(\langle\hat{n}_b^2\rangle_{out} - \langle\hat{n}_b\rangle_{out}\bigg) + u\langle\hat{n}_b\rangle_{out} \pm 2u^2\langle\hat{n}_a^{out}\hat{n}_b^{out}\rangle. \tag{3.265}$$

This can also be put in the form

$$\overline{m^2}_{\pm} = u^2\langle\hat{n}_{\pm}^2\rangle_{out} + u(1-u)\overline{n}_{+}^{out}, \tag{3.266}$$

with

$$\langle\hat{n}_{\pm}^2\rangle_{out} = \langle\hat{n}_a^2\rangle_{out} + \langle\hat{n}_b^2\rangle_{out} \pm 2\langle\hat{n}_a\hat{n}_b\rangle_{out}. \tag{3.267}$$

On the other hand, the normally-ordered second moment of the photon count sum and difference is expressible as

$$:\overline{m^2}_{\pm} := \overline{m^2}_{\pm} - \overline{m}_{+}. \tag{3.268}$$

Combination of this equation with (3.261) and (3.266) results in

$$:\overline{m^2}_{\pm} := u^2\langle : n_{\pm}^2 : \rangle_{out}, \tag{3.269}$$

in which

$$\langle : \hat{n}_{\pm}^2 : \rangle_{out} = \langle n_{\pm}^2\rangle_{out} - \overline{n}_{+out}. \tag{3.270}$$

The normally-ordered variance of the photon count sum and difference is defined by

$$:\Delta m_{\pm}^2 :=: \overline{m^2}_{\pm} : -\overline{m}_{\pm}^2. \tag{3.271}$$

Hence substitution of (3.261) and (3.269) into Eq. (3.271) leads to

$$:\Delta m_{\pm}^2 := u^2 : \Delta n_{\pm out}^2 : . \tag{3.272}$$

In view of Eq. (3.257), the normally-ordered variance of the photon number count sum and difference is found to be

$$:\Delta m_{\pm}^2 := u^2\kappa^2 : \Delta n_{\pm}^2 :, \tag{3.273}$$

with $: \Delta n_{\pm}^2 :$ being the normally-ordered variance of the photon number sum and difference for the cavity modes. Finally, with the help of Eq. (3.237) the normally-ordered variance of the photon number count sum and difference for the output modes of the system under consideration goes over into.

$$\begin{aligned} :\Delta m_{\pm}^2 : = {} & 2\kappa^2u^2E_{+}^2\langle F_{+}F_{+}^{*}\rangle + 2\kappa^2u^2E_{-}^2\langle F_{-}F_{-}^{*}\rangle + \kappa^2u^2\langle F_{+}F_{+}^{*}\rangle^2 + \kappa^2\langle F_{-}F_{-}^{*}\rangle^2 \\ & \pm 2\kappa^2u^2\bigg(2E_{+}E_{-}\langle F_{+}F_{-}\rangle + \langle F_{+}F_{-}\rangle^2\bigg). \end{aligned} \tag{3.274}$$

Since κ and u are always positive, one can easily assert that the photon count statistics for the output modes are similar to the photon number statistics for the output and cavity modes. Moreover, in view of Eqs. (3.262) and (3.273) along with the fact that κ and u are less than one, we also see that the mean and the normally-ordered variance of the photon count sum and difference for the output modes are less than the mean and the normally-ordered variance of the photon number sum and difference for the output as well as cavity modes.

4

Conclusion

In this dissertation we have seen the simplicity with which the squeezing and statistical properties of the light, generated by coherently driven degenerate as well as nondegenerate three-level lasers whose cavity contains a parametric amplifier, could be analyzed with the aid of c-number Langevin equations. Applying the solutions of these equations, we have calculated the quadrature variance for the cavity and output modes and the squeezing spectrum for the output mode(s) for both the degenerate and nondegenerate cases. Our results show that the light produced in both cases is in a squeezed state, with the degree of squeezing for the degenerate case being greater than that for the nondegenerate. We have also seen that for various initial superpositions of the three-level atoms, the presence of the parametric amplifier enhances the squeezing of the light generated by the systems under consideration, while the driving light has no effect on the squeezing. Furthermore, we have found that the pump mode which couples the top and bottom levels of the three-level atoms enhances the degree of squeezing particularly when there are nearly equal number of atoms initially in the top and bottom levels.

It so turns out that for $\eta = 0$, $A = 100$, and $\kappa = 0.8$ the maximum interacavity squeezing is 93% (for the degenerate case) and 62% (for the nondegenerate case) below the coherent-state level. This squeezing is exclusively due to the parametric amplifier and the coupling of the top and bottom levels. Furthermore, for $\eta = 0.1$ and the above values of A and κ the maximum interacavity squeezing is found to be 94% (for the degenerate case) and 70% (for the nondegenerate case) below the coherent-state level. This squeezing is due to the parametric amplifier and the superposition of the top and bottom levels. In addition, we have shown that the cavity mode(s) squeezing is greater than the output mode(s) squeezing by 19% (for the degenerate case) and by 14% (for the nondegenerate case).

On the other hand, for the degenerate case, we have determined via the Q function, the mean and variance of the photon number, the photon number distribution for the cavity mode, and the normally-ordered variance of the photon count for the output mode. From the results we have found, we note that the driving coherent light and the parametric amplifier increase the mean of the photon number significantly. We have seen that one effect of the coupling of the top and bottom levels is to decrease the mean of the photon number. This could be due to stimulated emission induced by the pump mode. The photons emitted this way do not contribute to the mean photon number of the cavity mode. Furthermore, we have also observed that the photon number as well as the photon count statistics are super-Poissonian. In addition, we found that there is a finite probability to find odd number of photons inside the cavity.

Moreover, for the nondegenerate case following a similar procedure as in the degenerate case, we have determined the mean and variance of the photon number sum and difference, the photon number distribution for the cavity modes, and the normally-ordered variance of the photon count sum and difference for the output modes. From the results we have obtained, we have seen that the driving coherent light and the parametric amplifier increase the mean of the photon number for mode a and mode b significantly. We have also found that the mean photon number of mode a is greater than that of mode b. This must be due to spontaneous decay of some atoms from the intermediate level to levels other than the bottom level. We have seen that another effect of the coupling of the top and bottom levels is to decrease the mean of the photon number sum. Furthermore, we have also observed that the photon number sum statistics is super-Poissonian while the photon number difference statistics is sub-Poissonian. In addition, we have noted that the joint probability of observing more photons of mode b than mode a inside the cavity is zero. And there is also a finite probability of finding equal number of photons in the two modes.

APPENDIX

Appendix A

Quadrature variance of the output mode for a cavity coupled to a squeezed vacuum

The squeezing properties of the output mode can be described by the quadrature operators

$$\hat{a}_{+}^{out} = \hat{a}_{out}^{\dagger} + \hat{a}_{out} \tag{A1}$$

and

$$\hat{a}_{-}^{out} = i(\hat{a}_{out}^{\dagger} - \hat{a}_{out}). \tag{A2}$$

The quadrature variance of the output mode is given by

$$\Delta a_{\pm out}^2(t) = \langle \hat{a}_{\pm out}^2(t) \rangle - \langle \hat{a}_{\pm out}(t) \rangle^2. \tag{A3}$$

We next proceed to obtain a general expression for the quadrature variance of the output mode in terms of the quadrature variance of the cavity mode as well as the input mode. Using the input-output relation

$$\hat{a}_{out} = \sqrt{\kappa}\hat{a} - \hat{a}_{in} \tag{A4}$$

along with Eqs. (2.103), (2.104), (A1), and (A2), we have

$$\hat{a}_{\pm out} = \sqrt{\kappa}\hat{a}_{\pm} - \hat{a}_{\pm in}. \tag{A5}$$

Thus on account of Eq. (A5), the quadrature variance (A3) takes the form

$$\Delta a_{\pm out}^2 = \kappa \Delta a_{\pm}^2 + \Delta a_{\pm in}^2 - \sqrt{\kappa}\bigg(\langle \hat{a}_{\pm}, \hat{a}_{\pm in} \rangle + \langle \hat{a}_{\pm in}, \hat{a}_{\pm} \rangle \bigg). \tag{A6}$$

The quantum Langevin equation for the operator $\hat{a}(t)$ has a general form

$$\frac{d}{dt}\hat{a}(t) = -A\hat{a}(t) + B\hat{a}^{\dagger}(t) + \hat{F}(t) + C, \tag{A7}$$

in which A, B, and C are constants and the noise operator $\hat{F}(t)$ has a vanishing mean. In addition, for a squeezed vacuum reservoir this operator satisfies the correlation functions

$$\langle \hat{F}(t)\hat{F}^{\dagger}(t') \rangle = \kappa(N+1)\delta(t-t'), \tag{A8}$$

$$\langle \hat{F}^\dagger(t)\hat{F}(t')\rangle = \kappa N\delta(t-t'), \tag{A9}$$

$$\langle \hat{F}(t)\hat{F}(t')\rangle = \langle \hat{F}^\dagger(t)\hat{F}^\dagger(t')\rangle = \kappa M\delta(t-t'), \tag{A10}$$

where for a squeezed vacuum reservoir

$$N = \sinh^2 r, \tag{A11}$$

$$M = \sinh r \cosh r, \tag{A12}$$

and r is the squeeze parameter. On account of Eqs. (2.103), (2.104), and (A7), the equation of evolution for the quadrature operators can be written as

$$\frac{d}{dt}\hat{a}_\pm(t) = -\lambda_\mp \hat{a}_\pm(t) + C \pm C + \hat{F}_\pm(t), \tag{A13}$$

in which

$$\lambda_\mp = A \mp B, \tag{A14}$$

$$\hat{F}_+(t) = \hat{F}^\dagger(t) + \hat{F}(t), \tag{A15}$$

and

$$\hat{F}_-(t) = i(\hat{F}^\dagger(t) - \hat{F}(t)). \tag{A16}$$

The formal solution of Eq. (A13) is expressible as

$$\hat{a}_\pm(t) = \hat{a}_\pm(0)e^{-\lambda_\mp t} + \int_0^t e^{-\lambda_\mp(t-t')}\left(C \pm C + \hat{F}_\pm(t')\right)dt'. \tag{A17}$$

The annihilation operator for the input mode is defined by

$$\hat{a}_{in}(t) = \frac{1}{\sqrt{\kappa}}\hat{F}(t). \tag{A18}$$

With the aid of Eqs. (A15), (A16), and (A17) along with the fact that the noise operator has a vanishing mean, we have

$$\langle \hat{a}_{\pm in}(t), \hat{a}_\pm(t)\rangle = \frac{1}{\sqrt{\kappa}}\langle \hat{F}_\pm(t)\hat{a}_\pm(t)\rangle. \tag{A19}$$

Furthermore, multiplying Eq. (A17) from the left by $\hat{F}(t)$ and taking the expectation value of the resulting expression, we get

$$\langle \hat{F}_\pm(t)\hat{a}_\pm(t)\rangle = \langle \hat{F}_\mp(t)\hat{a}_\pm(0)\rangle e^{-\lambda_\mp t} + \int_0^t e^{-\lambda_\mp(t-t')}\left(\langle \hat{F}_\pm(t)\rangle C \pm \langle \hat{F}_\pm(t)\rangle C + \langle \hat{F}_\pm(t)\hat{F}_\pm(t')\rangle\right)dt'. \tag{A20}$$

On account of the fact that the noise operator has a vanishing mean and does not affect the cavity mode operator at an earlier time, we find

$$\langle \hat{F}_\pm(t)\hat{a}_\pm(t)\rangle = \int_0^t e^{-\lambda_\mp(t-t')}\langle \hat{F}_\pm(t)\hat{F}_\pm(t')\rangle dt'. \tag{A21}$$

Moreover, with the aid of Eqs. (A15) and (A16), one readily gets

$$\langle \hat{F}_{\pm}(t)\hat{a}_{\pm}(t)\rangle = \int_0^t e^{-\lambda_{\mp}(t-t')}\left(\langle \hat{F}(t)\hat{F}^{\dagger}(t')\rangle + \langle \hat{F}^{\dagger}(t)\hat{F}^{\dagger}(t')\rangle \pm \langle \hat{F}(t)\hat{F}(t')\rangle \pm \langle \hat{F}^{\dagger}(t)\hat{F}^{\dagger}(t')\rangle\right) dt', \tag{A22}$$

so that with the aid of Eqs. (A8), (A9), and (A10), we obtain

$$\langle \hat{F}_{\pm}(t)\hat{a}_{\pm}(t)\rangle = \kappa(1 + 2N \pm 2M)\int_0^t e^{-\lambda_{\mp}(t-t')}\delta(t-t')dt'. \tag{A23}$$

Finally, upon carrying out the integration, we find

$$\langle \hat{F}_{\pm}(t)\hat{a}_{\pm}(t)\rangle = \frac{\kappa}{2}\Delta a^2_{\pm in}, \tag{A24}$$

where

$$\Delta a^2_{\pm in} = 1 + 2N \pm 2M \tag{A25}$$

being the quadrature variance of the input squeezed vacuum. Thus substitution of (A24) into Eq. (A19) leads to

$$\langle \hat{a}_{\pm in}(t), \hat{a}_{\pm}(t)\rangle = \frac{\sqrt{\kappa}}{2}\Delta a^2_{\pm in}. \tag{A26}$$

Following a similar procedure one can also obtain

$$\langle \hat{a}_{\pm}(t), \hat{a}_{\pm in}(t)\rangle = \frac{\sqrt{\kappa}}{2}\Delta a^2_{\pm in}. \tag{A27}$$

Therefore, application of (A26) and (A27) in Eq. (A6) yields

$$\Delta a^2_{\pm out} = \kappa\Delta a^2_{\pm} + (1-\kappa)\Delta a^2_{\pm in}, \tag{A28}$$

where the first and the second terms represent the quadrature variance of the transmitted and reflected modes.

Appendix B

Solutions of the c-number Langevin equations

One can rewrite Eq. (3.51) and the complex conjugate of Eq. (3.52) as

$$\frac{d}{dt}\alpha(t) = -a_+\alpha(t) - b_+\beta^*(t) + f_\alpha(t) + \mu \tag{B1}$$

and

$$\frac{d}{dt}\beta^*(t) = -a_-\beta^*(t) - b_-\alpha(t) + f^*_\beta(t) + \mu, \tag{B2}$$

in which

$$a_\pm = \frac{\kappa}{2} + \frac{A}{4B}\left[\frac{3\Omega}{2\gamma}\sqrt{1-\eta^2} + (1-\frac{\Omega^2}{2\gamma^2})\eta \mp (1+\frac{\Omega^2}{\gamma^2})\right] \tag{B3}$$

and

$$b_\pm = -\frac{A}{4B}\left[\frac{4B\varepsilon}{A} + \frac{\Omega}{2\gamma}(1+\frac{\Omega^2}{\gamma^2}) \pm \left(\frac{3\Omega}{2\gamma}\eta - (1-\frac{\Omega^2}{2\gamma^2})\sqrt{1-\eta^2}\right)\right]. \tag{B4}$$

To solve the coupled differential equations (B1) and (B2), we write the single-matrix equation

$$\frac{d}{dt}U(t) = -MU(t) + N(t), \tag{B5}$$

in which

$$U(t) = \begin{pmatrix} \alpha(t \\ \beta^*(t) \end{pmatrix}, \tag{B6}$$

$$M = \begin{pmatrix} a_+ & b_+ \\ b_- & a_- \end{pmatrix}, \tag{B7}$$

and

$$N(t) = \begin{pmatrix} f_\alpha(t) + \mu \\ f^*_\beta(t) + \mu \end{pmatrix}. \tag{B8}$$

In order to solve Eq. (B5), we need the eigenvalues and eigenvectors of M such that

$$MV_i = \lambda_i V_i, \tag{B9}$$

with $i = 1, 2$ and the eigenvectors

$$V_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \tag{B10}$$

subject to the normalization condition

$$x_i^2 + y_i^2 = 1. \tag{B11}$$

The eigenvalue equation (B9) has nontrivial solution, provided that

$$det(M - \lambda I) = 0, \tag{B12}$$

so that applying Eqs. (B7), the eigenvalues are found to be

$$\lambda_1 = \frac{(a_+ + a_-) + \sqrt{(a_+ - a_-)^2 + 4b_+b_-}}{2} \tag{B13}$$

and

$$\lambda_2 = \frac{(a_+ + a_-) - \sqrt{(a_+ - a_-)^2 + 4b_+b_-}}{2}. \tag{B14}$$

We next seek to obtain the eigenvectors of M. To this end, the eigenvector corresponding to λ_1 is expressible as

$$V_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \tag{B15}$$

Then employing Eqs. (B7) and (B9), we write the matrix equation

$$\begin{pmatrix} a_+ & b_+ \\ b_- & a_- \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}. \tag{B16}$$

Taking into account this equation and the normalization condition

$$x_1^2 + y_1^2 = 1, \tag{B17}$$

we get

$$V_1 = \frac{1}{\sqrt{b_+^2 + (\lambda_1 - a_+)^2}} \begin{pmatrix} b_+ \\ \lambda_1 - a_+ \end{pmatrix}. \tag{B18}$$

The eigenvector corresponding to λ_2 can also be established following a similar procedure that

$$V_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{b_+^2 + (\lambda_2 - a_+)^2}} \begin{pmatrix} b_+ \\ \lambda_2 - a_+ \end{pmatrix}. \tag{B19}$$

Finally, we construct a matrix V consisting of the eigenvectors of the matrix M as column matrices

$$V = \begin{pmatrix} V_1 & V_2 \end{pmatrix} = \begin{pmatrix} \frac{b_+}{\sqrt{b_+^2+(\lambda_1-a_+)^2}} & \frac{b_+}{\sqrt{b_+^2+(\lambda_2-a_+)^2}} \\ \frac{\lambda_1}{\sqrt{b_+^2+(\lambda_1-a_+)^2}} & \frac{\lambda_2}{\sqrt{b_+^2+(\lambda_2-a_+)^2}} \end{pmatrix}. \tag{B20}$$

We next proceed to determine the inverse of the matrix V. To this end, it can be readily verified that the characteristic equation

$$det(V - \lambda I) = 0 \tag{B21}$$

has explicit form

$$\lambda^2 - \left[\frac{b_+}{\sqrt{b_+^2 + (\lambda_1 - a_+)^2}} + \frac{\lambda_2 - a_+}{\sqrt{b_+^2 + (\lambda_2 - a_+)^2}}\right]\lambda - \frac{b_+(\lambda_1 - \lambda_2)}{\sqrt{b_+^2 + (\lambda_1 - a_+)^2}\sqrt{b_+^2 + (\lambda_2 - a_+)^2}} I = 0. \tag{B22}$$

Thus applying the Cayley-Hamilton theorem that a matrix satisfies its own characteristic equation, we have

$$V^2 - \left[\frac{b_+}{\sqrt{b_+^2 + (\lambda_1 - a_+)^2}} + \frac{\lambda_2 - a_+}{\sqrt{b_+^2 + (\lambda_2 - a_+)^2}}\right]V - \frac{b_+(\lambda_1 - \lambda_2)}{\sqrt{b_+^2 + (\lambda_1 - a_+)^2}\sqrt{b_+^2 + (\lambda_2 - a_+)^2}} I = 0. \tag{B23}$$

In view of this, we obtain

$$V^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} -\frac{\lambda_2 - a_+}{b_+}\sqrt{b_+^2 + (\lambda_1 - a_+)^2} & \sqrt{b_+^2 + (\lambda_1 - a_+)^2} \\ \frac{\lambda_1 - a_+}{b_+}\sqrt{b_+^2 + (\lambda_2 - a_+)^2} & -\sqrt{b_+^2 + (\lambda_2 - a_+)^2} \end{pmatrix}. \tag{B24}$$

Using the fact that $VV^{-1} = I$, Eq. (B5) can be rewritten as

$$\frac{d}{dt}U(t) = -VV^{-1}MVV^{-1}U(t) + N(t). \tag{B25}$$

Multiplying this equation by V^{-1} from the left, we get

$$\frac{d}{dt}\left(V^{-1}U(t)\right) = -DV^{-1}U(t) + V^{-1}N(t), \tag{B26}$$

where

$$D = V^{-1}MV = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{B27}$$

The formal solution of Eq. (B26) can be written as

$$V^{-1}U(t) = e^{-Dt}V^{-1}U(0) + \int_0^t e^{-D(t-t')}V^{-1}N(t')dt', \tag{B28}$$

from which follows

$$U(t) = Ve^{-Dt}V^{-1}U(0) + \int_0^t Ve^{-D(t-t')}V^{-1}N(t')dt'. \tag{B29}$$

In view of the fact that D is diagonal, we have

$$e^{-Dt} = \begin{pmatrix} e^{-\lambda_1 t} & 0 \\ 0 & e^{-\lambda_2 t} \end{pmatrix} \tag{B30}$$

and

$$e^{-D(t-t')} = \begin{pmatrix} e^{-\lambda_1(t-t')} & 0 \\ 0 & e^{-\lambda_2(t-t')} \end{pmatrix}. \tag{B31}$$

Therefore, on account of Eq. (B29) along with (B6), (B8), (B20), (B24), (B30), and (B31), we obtain

$$Ve^{-Dt}V^{-1}U(0) =$$
$$\frac{1}{\lambda_1-\lambda_2}\begin{pmatrix} \big[(\lambda_1-a_+)e^{-\lambda_2 t}-(\lambda_2-a_+)e^{-\lambda_1 t}\big]\alpha(0)+b_+\big[e^{-\lambda_1 t}-e^{-\lambda_2 t}\big]\beta^*(0) \\ \frac{(\lambda_1-a_+)(\lambda_2-a_+)}{b_+}\big[e^{-\lambda_2 t}-e^{-\lambda_1 t}\big]\alpha(0)+\big[(\lambda_1-a_+)e^{-\lambda_1 t}-(\lambda_2-a_+)e^{-\lambda_2 t}\big]\beta^*(0) \end{pmatrix} \tag{B32}$$

and

$$\int_0^t Ve^{-D(t-t')}V^{-1}N(t')dt' =$$
$$\frac{1}{\lambda_1-\lambda_2}\begin{pmatrix} \int_0^t\big[(\lambda_1-a_+)e^{-\lambda_2(t-t')}-(\lambda_2-a_+)e^{-\lambda_1(t-t')}\big]f_\alpha(t')dt' \\ \frac{(\lambda_1-a_+)(\lambda_2-a_+)}{b_+}\int_0^t\big[e^{-\lambda_2(t-t')}-e^{-\lambda_1(t-t')}\big]f_\alpha(t')dt' \end{pmatrix}$$
$$+\frac{1}{\lambda_1-\lambda_2}\begin{pmatrix} b_+\int_0^t\big[e^{-\lambda_1(t-t')}-e^{-\lambda_2(t-t')}\big]f^*_\beta(t')dt' \\ \int_0^t\big[(\lambda_1-a_+)e^{-\lambda_1(t-t')}-(\lambda_2-a_+)e^{-\lambda_2(t-t')}\big]f^*_\beta(t')dt' \end{pmatrix}$$
$$+\frac{\mu}{\lambda_1-\lambda_2}\begin{pmatrix} \frac{b_+-(\lambda_2-a_+)}{\lambda_1}\big(1-e^{-\lambda_1}\big)+\frac{(\lambda_1-a_+)-b_+}{\lambda_2}\big(1-e^{-\lambda_2 t}\big) \\ \frac{(\lambda_1-a_+)+b_-}{\lambda_1}\big(1-e^{-\lambda_1}\big)-\frac{(\lambda_2-a_+)+b_-}{\lambda_2}\big(1-e^{-\lambda_2 t}\big) \end{pmatrix}. \tag{B33}$$

Finally, application of (B32) and (B33) in Eq. (B29) results in

$$\alpha(t) = A_+(t)\alpha(0) + B_+(t)\beta^*(0) + F_+(t) + E_+(t) \tag{B34}$$

and

$$\beta(t) = A_-(t)\beta(0) + B_-(t)\alpha^*(0) + F_-(t) + E_-(t), \tag{B35}$$

where

$$A_\pm(t) = \frac{1}{2}\Big[(1\pm p)e^{-\lambda_2 t} + (1\mp p)e^{-\lambda_1 t}\Big], \tag{B36}$$

$$B_{\pm}(t) = \frac{q_{\pm}}{2}\Bigg[e^{-\lambda_1 t} - e^{-\lambda_2 t}\Bigg], \tag{B37}$$

$$F_{+}(t) = \frac{1}{2}\int_0^t \big[(1+p)e^{-\lambda_2(t-t')} + (1-p)e^{-\lambda_1(t-t')}\big] f_{\alpha}(t')dt'$$
$$+\frac{q_{+}}{2}\int_0^t \big[e^{-\lambda_1(t-t')} - e^{-\lambda_2(t-t')}\big] f_{\beta}^{*}(t')dt', \tag{B38}$$

$$F_{-}(t) = \frac{q_{-}}{2}\int_0^t \big[e^{-\lambda_2(t-t')} - e^{-\lambda_1(t-t')}\big] f_{\alpha}^{*}(t')dt'$$
$$+\frac{1}{2}\int_0^t \big[(1+p)e^{-\lambda_1(t-t')} + (1-p)e^{-\lambda_2(t-t')}\big] f_{\beta}(t')dt', \tag{B39}$$

and

$$E_{\pm}(t) = \frac{\mu}{2}\Bigg[\frac{1 \mp p + q_{\pm}}{\lambda_1}\big(1 - e^{-\lambda_1 t}\big) + \frac{1 \pm p - q_{\pm}}{\lambda_2}\big(1 - e^{-\lambda_2 t}\big)\Bigg], \tag{B40}$$

in which

$$\frac{1}{2}(1+p) = \frac{\lambda_1 - a_{+}}{Z}, \tag{B41}$$

$$\frac{1}{2}(p-1) = \frac{\lambda_2 - a_{+}}{Z}, \tag{B42}$$

and

$$\frac{q_{\pm}}{2} = \frac{b_{\pm}}{Z}. \tag{B43}$$

Employing Eqs. (B41)-(B43), one can readily show that

$$p = \frac{1 + \frac{\Omega^2}{\gamma^2}}{Z}, \tag{B44}$$

$$q_{\pm} = \frac{-[\frac{\Omega}{2\gamma}(1+\frac{\Omega^2}{\gamma^2}) + \frac{4B\varepsilon}{A}] \mp [\frac{3\Omega}{2\gamma}\eta - (1-\frac{\Omega^2}{2\gamma^2})\sqrt{1-\eta^2}]}{Z}, \tag{B45}$$

with

$$Z = \Bigg[(1+\frac{\Omega^2}{\gamma^2})^2 + (\frac{\Omega}{2\gamma}(1+\frac{\Omega^2}{\gamma^2}) + \frac{4B\varepsilon}{A})^2 - (\frac{3\Omega}{2\gamma}\eta - (1-\frac{\Omega^2}{2\gamma^2})\sqrt{1-\eta^2})^2\Bigg]^{1/2}. \tag{B46}$$

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DECLARATION

I hereby declare that this PhD dissertation is my original work and has not been presented for a degree in any other universities, and that all sources of material used for the dissertation have been duly acknowledged.

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