ADDIS ABABA UNIVERSITY

COLLEGE OF NATURAL SCIENCES

DEPARTMENT OF MATHEMATICS

A GRADUATE SEMINAR REPORT

ON

GENERALIZED SOLUTIONS OF BOUNDARY VALUE PROBLEMS WITH JUMP DISCONTINUITY

(SUBMITTED IN PARTIAL FULFILLMENT OF THE M. Sc. DEGREE IN MATHEMATICS)

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Declaration

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other similar title to me.

Abdissa Shiferraw

Signature ________________Date____________________

Permission Letter

This is to certify that this project is compiled by **Mr. Abdissa Shiferraw** in the Department of Mathematics, Addis Ababa University, under my supervision.

I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

Dr. Tadesse Abdi

Signature _________________Date______________

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ii) Abstract

We study on how to find a generalized solution of boundary value problems of different orders with jump discontinuity. Here first, second order linear inhomogeneous differential equations are considered and then we extend it to the n^{th} order. Green's function plays a great role for solving such differential equations.

Classical theory is based on solving first- or higher-order derivatives with jumps on both sides of the boundaries and then attempting to satisfy the boundary conditions. Our aim is to develop the vector analysis of functions with jump discontinuities across surfaces and boundaries that cannot be solved by classical techniques. With the help of this we can solve many unsolved problems in the potential, scattering and wave propagation theories. Furthermore, problems whose solutions are already known can be solved by this method in a very simple fashion.

We can also show that there is a new solution of linear homogeneous systems of differential equations with singular in coefficients, in the space of generalized functions other than the classical solutions.

iii) Introduction

In this seminar, there are some concepts that should be pointed out so that the reader will smoothly go through. The first important point is the theory of distribution. One of the most useful aspects of the theory of distribution is that jump discontinuous functions can be handled as easily as continuous functions.

Secondly, we will see the generalized solution of linear inhomogeneous ordinary differential equations of generalized functions and linear homogeneous systems of differential equations with singular coefficients whose new solutions are in the space of generalized functions. We really gain no new insight into the solutions of the classical problems in ordinary differential equations by using the theory of distributions. However, the theory does enhance our knowledge if discontinuities are present in these equations or if we want to find the fundamental solutions.

iV) **List of symbols**

D	the space of test functions;
S	the space rapidly decreasing functions;
$D' = C_0^\infty$	the space of distributions;
S '	the space of tempered distributions;
$suppf$	Support of f ;
δ	Dirac delta function;
H	Heaviside function;
C^∞	Infinitely differentiable function;
$\subset\subset$	Strictly compact support;
L	linear ordinary differential operator;
L*	the formal adjoint of differential operator L;
C^0	*the space* of *continuous functions*;
y_h	hom*ogeneous solution*;
y_p	*particular solution*;
$[m/2]$	*the largest integer* $\geq m/2$;
$\langle f, \varphi \rangle$	*the distribution* f *measured by test function* φ;
BVP	*boundary value problem*;
IVP	*initial value problem*;
BC	*boundary condition* ;
IC	*initial condition*;
$[\ ]_{\xi_j}$	Jump in the quantity across the point ξ_j

Chapter 1

Basic Concepts in Distribution Theory

Basically, distributions are continuous linear functionals which were originally introduced by Schwartz and is the most popular and direct method of studying distributions. i.e., distributions are generalizations of ordinary functions which are defined $a.e$ in R^n and (Lebesgue) integrable in each finite interval [a, b]. For instance, Weiesrstrass function is not ordinary function since it is continuous nowhere differentiable.

1.1 The space of Test functions $\mathcal{D}$

Definition: Let $\Omega \subset \mathbb{R}^n$ be open. Included in the set of basic functions $\mathcal{D}(\Omega)$ are all functions which are finite (or have compact support) and infinitely differentiable function in Ω.That is,

$$\mathcal{D}(\Omega) = \{\varphi : \ \varphi \in C^{\infty}(\Omega) \ and \ \varphi \in C_0(\Omega)\},$$
$$= \{\varphi : \ \varphi \in C_0^{\infty}(\Omega)\}.$$

Example: $\varphi(x) = \begin{cases} e^{-\frac{1}{1-x^2}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$ is in $\mathcal{D}$ since $\varphi(x) \in C^{\infty}(\mathbb{R}^n)$ $and\ supp\ (\varphi) = [-1,1]$.

Convergence in $\mathcal{D}$: A given sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ in $\mathcal{D}(\Omega)$ converges to φ in $\mathcal{D}(\Omega)$ if the following two conditions are satisfied:

(i) $\exists$ a compact *set* $K \subset\subset \Omega$, such that $supp(\varphi_n) \subset K, \ \forall n,$

(ii) $\forall \alpha \in \mathbb{N}_0^n$ we have $\|D^{\alpha}\varphi_n - D^{\alpha}\varphi\|_{\infty} \longrightarrow 0.$

A linear set $\mathcal{D}(\Omega)$ equipped with such convergence is called the *space of test/basic functions* $\mathcal{D}$.

1.2 The space of Distributions $\mathcal{D}'$

Definition: A distribution (generalized function) $f \in \mathcal{D}'$ on a non-empty open set $\Omega \subset \mathbb{R}^n$ is any continuous linear functional on the space of test/basic functions $\mathcal{D}$. By this definition of distribution, we mean that

- A distribution f is a functional on $\mathcal{D}$, that is, with each $\varphi \in \mathcal{D}$ there is associated (complex) number $\langle f, \varphi \rangle$.

- A distribution $f \in \mathcal{D}'$ is a linear functional on $\mathcal{D}$, that is, if $\varphi, \psi \in \mathcal{D}$ and $\lambda, \mu \in \mathbb{C}$, then $\langle f, \lambda\varphi + \mu\psi \rangle = \lambda\langle f, \varphi \rangle + \mu\langle f, \psi \rangle$.
- A distribution $f \in \mathcal{D}'$ is a continuous functional on $\mathcal{D}$, that is, if $\varphi_n \longrightarrow \varphi$ in $\mathcal{D}(\Omega)$ as $n \longrightarrow \infty$, then $\langle f, \varphi_n \rangle \longrightarrow \langle f, \varphi \rangle$ as $n \longrightarrow \infty$.

One of the most important examples is the so called Dirac delta function δ.

Definition: Distributions generated by locally summable function $f \in L_1^{loc}(\Omega)$ via the formula:

$$\langle f, \varphi \rangle = \int f(x)\varphi(x)dx, \quad \varphi \in D(\Omega)$$

are called *regular* distributions and all others are called *singular* distributions.

Example: Dirac delta function δ is a singular distribution.

Convergence in $\mathcal{D}'$: A sequence $f_n \in \mathcal{D}'(\Omega)$ converges to f in $D'(\Omega)$, denoted by $f_n \longrightarrow f$ as $n \longrightarrow \infty$ if $\langle f_n, \varphi \rangle \longrightarrow \langle f, \varphi \rangle$ $\forall \varphi \in \mathcal{D}(\Omega)$.This convergence is termed as *weak convergence.*

1.2.1 The Dirac Delta Function and Delta Sequences

A) The Dirac Delta Function

In physical problems one often encounters idealized concepts such as a force concentrated at a point ξ or an impulsive force that acts instantaneously. Such forces are described by distribution called the Dirac delta $\delta(x-\xi)$:

$$\delta(x-\zeta) = \begin{cases} +\infty, & x = \xi, \\ 0, & x \neq \xi. \end{cases}$$

B) The Delta Sequences

Here we consider a sequence whose limit is the delta function.

Example: $s_m(x) = (\sin mx)/\pi x$, $m = 1, 2, \ldots$

It is clear that for a fixed m as |x| becomes large, $s_m(x)$ becomes small. Since

$\int_{-\infty}^{\infty} \frac{\sin x}{\pi x} dx = \frac{2}{\pi}\left(\int_0^{\infty} \frac{\sin x}{x} dx \right) \frac{2}{\pi}(\frac{\pi}{2}) = 1$ *for* changing x to mx, we obtain $\int_{-\infty}^{\infty} \frac{\sin mx}{\pi x} dx = 1, \ \forall m.$

Definition: A sequence s_m is called a delta-convergent sequence if

$$\lim_{m\to\infty}\int_{-\infty}^{\infty} s_m(x)\varphi(x)dx=\varphi(0)$$

for a ll f unctions $\varphi(x)$ sufficiently s mooth $-\infty<x<\infty$. Thus w e can say t hat f or a d elta convergent sequence $\lim_{m\to\infty} s_m(x)=\delta(x)$. In this case we have taken the unit charge located at $x=0$. If it is located at $x=\xi$, then the preceding formulas becomes

$$\lim_{m\to\infty}\int_{-\infty}^{\infty} s_m(x-\xi)\varphi(x)dx=\varphi(x-\xi) \text{ and } \lim_{m\to\infty} s_m(x-\xi)=\delta(x-\xi).$$

Thus for any basic function $\varphi\in D$, we have Dirac delta distribution in R^n is

$$\langle\delta(x-\xi),\varphi(x)\rangle=\varphi(\xi) \text{ for a fixed point } \xi \text{ in } R^n.$$

1.2.2 The Heaviside Function

The Heaviside function denoted $H(x)$ which has a jump discontinuity at x = 0 is defined by

$$H(x)=\begin{cases}0, & x<0,\\ 1, & x>0.\end{cases}$$

If t he j ump di scontinuous i n t he Heaviside f unction is a t a poi nt $x=\xi$, then it is w ritten $H(x-\xi)$:

$$H(x-\xi)=\begin{cases}0, & x<\xi,\\ 1, & x>\xi.\end{cases}$$

The function $H(x)$ will prove very useful in the study of the generalized functions, especially in the discussion of functions with jump discontinuities. For instance, let $F(x)$ be a function that is continuous everywhere except for the point $x=\xi$, at which point has a jump discontinuity:

$$\mathrm{F(x)}=\begin{cases}F_1(x), & x<\xi,\\ F_2(x), & x>\xi.\end{cases}$$

Then it can be written $F(x) = F_1(x)H(\xi - x) + F_2(x)H(x - \xi)$. *It is shown in Fig*. 1.2.

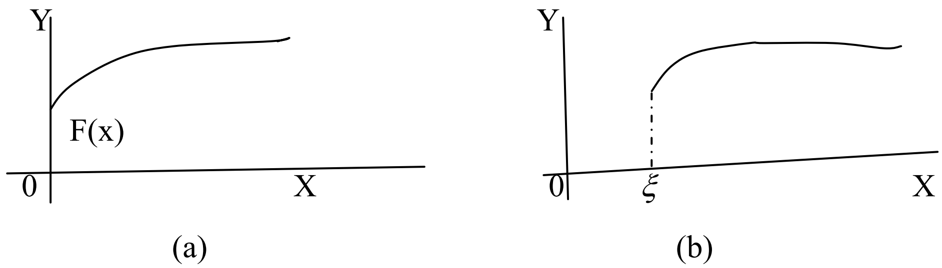

Fig. 1.2 (a) F(x); (b) $H(x-\xi)F(x-\xi)$

The Heaviside distribution in R^n is

$$\langle H_R, \varphi\rangle = \int_R \varphi(x)dx, \text{ where } H_R(x) = \begin{cases} 1, & x \in R, \\ 0, & x \notin R. \end{cases}$$

The *support* of $s(x)$ is the smallest closed set outside of which *it* vanishes. i.e., $s(x) = \overline{\{x \in R : s(x) \neq 0\}}$. For instance, for Dirac delta and Heaviside function, $supp\,(\delta(x)) = 0$, $supp\,(H(x)) =$ halve axis x > 0.

Singular support of $s(x)$ is closed set of points where $s(x)$ is not a smooth function. In view this , singular support of $\delta(x)$, $H(x)$ and |x| are x = 0.

A continuous function f has compact support if its support is compact set. A subset $M \subseteq X$ (X is topological space) is called a compact support if every open covering X has a finite cover M.

1.3 Product of Distributions

In general, it is difficult to define the product of two generalized functions. However, we can always assign a meaning to the product of a generalized function $t(x)$ and an infinitely differentiable function $\alpha(x)$ by setting

$$\langle \alpha t, \varphi\rangle = \langle t, \alpha\varphi\rangle, \qquad \alpha\varphi \in D.$$

In particular, $t = \delta$ *and for every* $\varphi \in D, \alpha \in C^\infty$,

$$\langle \alpha\delta, \varphi\rangle = \langle \delta, \alpha\varphi\rangle = \alpha(x)\varphi(x)|_{x=0} = \alpha(0)\varphi(0) = \alpha(0)\langle \delta, \varphi\rangle = \langle \alpha(0)\delta, \varphi\rangle.$$

$$\therefore\ \alpha(x)\delta(x) = \alpha(0)\delta(x).$$

Theorem 1.1: Let a function $f(x) \in C^n(\mathbb{R})$, then

$$f(x)\delta^{(n)}(x) = (-1)^n \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} f^{(n-k)}(0)\delta^{(k)}(x).$$

Proof : *For every* $\varphi \in D$,

$$\langle f(x)\delta^{(n)}(x), \varphi(x)\rangle = \int_{-\infty}^{\infty} [f(x)\varphi(x)]\delta^{(n)} dx,$$

$$= f(x)\varphi(x)\delta^{(n-1)}(x)\,|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} [f(x)\varphi(x)]'\,\delta^{(n-1)}(x)dx,$$

$$= -\int_{-\infty}^{\infty} [f(x)\varphi(x)]'\,\delta^{(n-1)}(x)dx.$$

After a succession of similar integrations by parts, we obtain

$\langle f(x)\delta^{(n)}(x), \varphi(x)\rangle = \int_{-\infty}^{\infty} [f(x)\varphi(x)]\delta^{(n)} dx = (-1)^n \int_{-\infty}^{\infty} [f(x)\varphi(x)]^n \delta(x)dx.$ *Substitution of the formula* $[f(x)\varphi(x)]^{(n)} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} f^{(n-k)}(x)\varphi^{(k)}(x)$ *in the preceding relation yields*

$$\langle f(x)\delta^{(n)}(x), \varphi(x)\rangle = \int_{-\infty}^{\infty} [f(x)\varphi(x)]\delta^{(n)} dx = (-1)^n \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} f^{(n-k)}(0)\varphi^{(k)}(0),$$

$$= (-1)^n \langle \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} f^{(n-k)}(0)\delta^{(k)}(x), \varphi(x)\rangle.$$

Corollary 1.1: $x^n \delta^{(m)}(x) = \begin{cases} 0, & n > m, \\ (-1)^n \dfrac{m!}{n!(m-n)!}\delta^{(m-n)}(x), & n \le m. \end{cases}$

Proof : *For every* $\varphi \in D$,

$$\langle x^n \delta^{(m)}(x), \varphi(x)\rangle = \int_{-\infty}^{\infty} [x^n \varphi(x)]\delta^{(m)}(x)dx,$$

$$= x^n \varphi(x)\delta^{(m-1)}(x)\,|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} [x^n\varphi(x)]'\,\delta^{(m-1)}(x)dx, \quad (\text{integration by part})$$

$$= (-1)^m \int_{-\infty}^{\infty} [x^n\varphi(x)]^m \delta(x)dx = (-1)^m \sum \binom{m}{n} (x^n)^m \varphi^{(m-n)}(x)\,|_{x=0}, \ (\textit{by theorem } 1.1)$$

$$= \begin{cases} 0, & n > m, \\ (-1)^m \dbinom{m}{n} \varphi^{(m-n)}(0), & n \le m. \end{cases}$$

$$= (-1)^m \binom{m}{n} \langle \delta^{(m-n)}, \varphi \rangle.$$

$$\therefore\ x^n \delta^{(m)}(x) = \begin{cases} 0, & n > m, \\ (-1)^n \dfrac{m!}{n!(m-n)!} \delta^{(m-n)}(x), & n \le m. \end{cases}$$

1.4 Generalized Derivatives of Distributions

Definition: A (generalized) derivative $D^\alpha f$ of a distribution $f \in \mathcal{D}'$ is defined by

$$\langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle \ \forall \varphi \in \mathcal{D}, \qquad \alpha \in \mathbb{N}_{0,}^n \quad |\alpha| = \sum_{i=1}^{n} \alpha_i.$$

Claim: $D^\alpha f \in \mathcal{D}'$

Clearly $D^\alpha f$ is a functional since $f \in \mathcal{D}'$.

Linearity: Let $\varphi, \psi \in \mathcal{D}$ and $\lambda, \mu \in \mathbb{C}$, then

$$\begin{aligned} \langle D^\alpha f, \lambda\varphi + \mu\psi \rangle &= (-1)^{|\alpha|} \langle f, D^\alpha(\lambda\varphi + \mu\psi) \rangle, \\ &= (-1)^{|\alpha|} \langle f, \lambda D^\alpha \varphi + \mu D^\alpha \psi \rangle, \\ &= \lambda (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle + \mu (-1)^{|\alpha|} \langle f, D^\alpha \psi \rangle, \\ &= \lambda \langle D^\alpha f, \varphi \rangle + \mu \langle D^\alpha f, \psi \rangle. \end{aligned}$$

Hence $D^\alpha f$ is a linear functional on $\mathcal{D}$.

Continuity: Let $\varphi_n \longrightarrow \varphi$ as $n \longrightarrow \infty$ in $\mathcal{D}$, then

$$\langle D^\alpha f, \varphi_n \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi_n \rangle \longrightarrow (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle = \langle D^\alpha f, \varphi \rangle.$$

This implies that $\langle D^\alpha f, \varphi_n \rangle \longrightarrow \langle D^\alpha f, \varphi \rangle$ as $n \longrightarrow \infty$. Hence $D^\alpha f$ is continuous.

$$\therefore\ D^\alpha f \in \mathcal{D}'.$$

In particular, if $f = \delta$, then $\langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle \delta, D^\alpha \varphi \rangle,\ \varphi \in \mathcal{D},$

$$= (-1)^{|\alpha|} D^\alpha \varphi(0).$$

Chapter 2

Distributional Derivatives of Functions with Jump Discontinuities

Let $F(x)$ be a function of single variable x that has a jump discontinuity at x = ξ_1 of magnitude a_1 *i.e*, $[F]_{\xi_1} = a_1$ but that everywhere else has a continuous derivative. Let the derivative in the interval x < ξ_1 and x > ξ_1, be denoted F'(x). This derivative is undefined at x = ξ_1. However, with the help of generalized functions, the distributional derivative F'(x) in R^1 is obtained by setting

$$f(x) = F(x) - a_1 H(x - \xi_1) \tag{2.1}$$

where H is the Heaviside function. The function $f(x)$ is continuous at x = ξ_1. Its derivative coincides with that of F(x) on both sides of ξ_1. Accordingly, we differentiate both sides of (2.1) and obtain

$$F'(x) = \overline{F}'(x) - a_1\delta(x - \xi_1) \tag{2.2}$$

or

$$\overline{F}'(x) = F'(x) + a_1\delta(x - \xi_1) \tag{2.3}$$

Equation (2.2) is easily generalized to a function F(x) that has jumps of magnitude $a_1, a_2, ..., a_k$ at $\xi_1, \xi_2, ..., \xi_k$ *respectively*. The result is

$$\overline{F}'(x) = F'(x) + \sum_{j=1}^{k} a_j\delta(x - \xi_j). \tag{2.4}$$

Let us now consider a function F(x) that admits derivatives up to the second order on both sides of the point ξ_1, that has a jump discontinuity of strength a_1 and whose derivative has a jump discontinuity of strength b_1 at this point. To obtain F''(x), we substitute F'(x) for F(x) in (2.2) and we get

$$F'' = \overline{F}'' - a_1\delta'(x - \xi_1) - b_1\delta(x - \xi_1)$$

or

$$\overline{F}'' = F'' + a_1\delta'(x - \xi_1) + b_1\delta(x - \xi_1) \tag{2.5}$$

Theorem 1.2: Let a function $f(x) \in C^n(\square)$, then

$$\left(f(x)H(x-\xi)\right)^{(n)} = f^{(n)}(x)H(x-\xi) + f^{(n-1)}(\xi)\delta(x-\xi) + f^{(n-2)}(\xi)\delta'(x-\xi) +$$

$$\ldots + f(\xi)\delta^{(n-1)}(x-\xi).$$

$\mathrm{Pr}\,oof$: *For every* □ $\varphi \in D$,

□□□□ $$\langle (f(x)H(x-\xi))^n, \varphi(x)\rangle = (-1)^n \langle f(x)H(x-\xi), \varphi^n(x)\rangle,$$

$$= (-1)^n \langle H(x-\xi), f(x)\varphi^n(x)\rangle,$$

$$= (-1)^n \int_{-\infty}^{x} f(x)\varphi^n(x)\,d\xi, \text{ (int}\textit{egration by part)}$$

$$= \langle f^{(n)}(x)H(x-\xi) + f^{(n-1)}(\xi)\delta(x-\xi) + f^{(n-2)}(\xi)\delta'(x-\xi) + \cdots + f(\xi)\delta^{(n-1)}(x-\xi), \varphi\rangle.$$

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a) $D^\alpha H(x)$ b) $D^\alpha |x|$ c) $D^\alpha \operatorname{sgn} x$ d) $x\delta^m$

Solution: a) For every $\varphi \in D$, $\langle D^\alpha H(x)\ ,\varphi\rangle = \langle D^{\alpha-1}D^1 H(x), \varphi(x)\ \rangle$,

$$= (-1)^{|\alpha-1|}\langle D^1 H(x), D^{\alpha-1}\varphi(x)\rangle,$$

$$= (-1)^{|\alpha-1|}\langle \delta, D^{\alpha-1}\varphi(x)\rangle,$$

$$= \langle \delta^{\alpha-1}, \varphi\rangle.$$

Thus $D^\alpha H(x) = \delta^{(\alpha-1)}(x), \alpha \geq 1.$

b) $\langle D^\alpha |x|, \varphi\rangle = (-1)^{|\alpha|}\langle |x|, D^\alpha \varphi\rangle,$

$$= (-1)^{|\alpha|}\left[\int_0^\infty x D^\alpha \varphi(x)\,dx + \int_{-\infty}^{0} - x D^\alpha \varphi(x)\,dx\right],$$

$$=2(-1)^{|\alpha|}\int_0^\infty xD^\alpha\varphi(x)dx,$$

$$=2(-1)^{|\alpha|}xD^{\alpha-1}\varphi(x)\,|_0^\infty -2(-1)^{|\alpha|}\int_0^\infty D^{\alpha-1}\varphi(x)dx, \text{(Integration by part)}$$

$$=2(-1)^{|\alpha|}D^{\alpha-2}\varphi(0),$$

$$=2(-1)^{|\alpha|}\langle\delta, D^{\alpha-2}\varphi\rangle,$$

$$=2(-1)^{|\alpha|}(-1)^{|\alpha-2|}\langle\delta^{\alpha-2},\varphi\rangle,$$

$$=\langle 2\delta^{\alpha-2},\varphi\rangle, \quad \alpha\geq 2.$$

Hence $D^\alpha |x| = 2\delta^{(\alpha-2)}, \quad \alpha \geq 2.$

c) $\langle D^\alpha \operatorname{sgn} x,\varphi\rangle = (-1)^{|\alpha|}\langle \operatorname{sgn} x, D^\alpha\varphi\rangle,$

$$= (-1)^{|\alpha|}\left(\int_0^\infty D^\alpha\varphi(x)dx + \int_{-\infty}^0 - D^\alpha\varphi(x)dx\right),$$

$$=2(-1)^{|\alpha|}\int_0^\infty D^\alpha\varphi(x)dx, \qquad \text{(integration by part)}$$

$$=-2(-1)^{|\alpha|}D^{\alpha-1}\varphi(0),$$

$$=-2(-1)^{|\alpha|}\langle\delta, D^{\alpha-1}\varphi\rangle,$$

$$=\langle 2\delta^{\alpha-1},\varphi\rangle, \ \ \alpha\geq 1.$$

$\therefore D^\alpha \operatorname{sgn} x = 2\delta^{\alpha-1}, \alpha \geq 1.$

Hence, *inparticular* $|x|' = \operatorname{sgn} x.$

d) $\langle x\delta^{(m)},\varphi\rangle = \langle\delta^{(m)}, x\varphi\rangle,$

$$= (-1)^{|m|}\langle\delta,(x\varphi)^m\rangle,$$

$$= (-1)^{|m|}\langle\delta, m\varphi^{m-1} + x\varphi^m\rangle,$$

$$=(-1)^{|m|}\left(\langle\delta, m\varphi^{m-1}\rangle + \langle\delta, x\varphi^m\rangle\right), \ \text{since } \langle\delta, x\varphi^m\rangle = 0,$$

$$=\langle -m\delta^{m-1},\varphi\rangle.$$

$\therefore \ x\delta^{(m)} = -m\delta^{m-1}, m\geq 1.$

Corollary 1.2: The derivative of the product of a distribution t and a function $f \in C^1$ is

$$\frac{d}{dx}(ft) = \frac{df}{dx}t + f\frac{dt}{dx}.$$

Proof: For every $\varphi \in D$,

$$\langle \frac{d}{dx}(ft), \varphi \rangle = -\langle ft, \frac{d}{dx}\varphi \rangle,$$

$$= -\langle t, f\frac{d}{dx}\varphi \rangle,$$

$$= -\langle t, \frac{d}{dx}(f\varphi) - \frac{df}{dx}\varphi \rangle,$$

$$= \langle \frac{dt}{dx}, f\varphi \rangle + \langle \frac{df}{dx}t, \varphi \rangle$$

$$= \langle f\frac{dt}{dx} + \frac{df}{dx}t, \varphi \rangle.$$

The higher order derivatives of the product can be defined in similar manner and we have Leibnitz' formula

$$D^n(ft) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} f^{(n-k)} t^k$$

2.1 The Space of rapidly decreasing functions $\mathcal{S}$

Definition: The space $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ of all functions of the class $C^\infty(\mathbb{R}^n)$ that decrease together with all their derivatives, as $|x| \longrightarrow \infty$, faster than any power of $|x|^{-1}$ is called the space of rapidly decreasing functions also called Schwartz space. For instance, $\varphi(x) = e^{-|x|^2} \in S$.

Convergence in $\mathcal{S}$: The sequence of functions $\varphi_1, \varphi_2, \ldots$, belonging to $\mathcal{S}$ converges to a function φ in $\mathcal{S}$, denoted by $\varphi_n \longrightarrow \varphi$ as $n \longrightarrow \infty$ in $\mathcal{S}$, if for all α *and* β :

$$x^\beta D^\alpha \varphi_n(x) \Rightarrow x^\beta D^\alpha \varphi(x), \text{as } n \longrightarrow \infty.$$

2.2 The Space of Tempered Distributions $\mathcal{S}'$

Definition: A generalized function of slow growth is any continuous linear functional on the space $\mathcal{S}$ of test functions. We denote by $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ the set of all generalized functions of slow growth.

***Convergence in $\mathcal{S}'$*:** a sequence of generalized functions $f_1, f_2, \ldots$ taken from $\mathcal{S}'$ convergences to the generalized function $f \in \mathcal{S}'$, denoted by $f_k \longrightarrow f$, as $k \longrightarrow \infty$ in $\mathcal{S}'$, if for any $\varphi \in \mathcal{S}$, $\langle f_k, \varphi\rangle \longrightarrow \langle f, \varphi\rangle$, $k \longrightarrow \infty$. This convergence is called *a weak convergence of a sequence of functional.*

The linear set $\mathcal{S}'(\Omega)$ equipped with convergence is termed the spaces S' of generalized functions of slow growth or tempered distributions.

Note! $D \subset S \subset S' \subset D'$.

2.3 The Fourier Transform of rapidly decreasing functions S

Since the test functions from $\mathcal{S}$ are locally summable, the operation of Fourier Transform $\mathcal{F}(\varphi)$ and inverse Fourier Transform $\mathcal{F}^{-1}(\varphi)$ are defined on them.

Definition: If $\varphi \in \mathcal{S}(\mathbb{R}^{\boldsymbol{n}})$, we define the Fourier Transform and its inverse as follows:

$$(\mathcal{F}\varphi)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{\boldsymbol{n}}} \varphi(x) e^{-i\langle x,\xi\rangle}\, dx \quad and \quad (\mathcal{F}^{-1}\varphi)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{\boldsymbol{n}}} \varphi(x)\, e^{i\langle x,\xi\rangle}\, dx.$$

Proposition 1.1: Two important formulae can be proved here,

$$i)\ [\mathcal{F}\varphi](\xi) = (-i)^{|\alpha|}\mathcal{F}[x^\alpha\varphi](\xi)$$

$$ii)\ \mathcal{F}[D^\alpha\varphi](\xi) = (i\xi)^\alpha[\mathcal{F}\varphi](\xi)$$

Proof: $i)\ D^\alpha[\mathcal{F}\varphi](\xi) = D_\xi^\alpha\left[(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^{\boldsymbol{n}}}\varphi(x)\, e^{-i\langle x,\xi\rangle}\, dx\right],$

$$= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^{\boldsymbol{n}}}\varphi(x) D_\xi^\alpha e^{-i\langle x,\xi\rangle} dx,$$

$$= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^{\boldsymbol{n}}}\varphi(x)(-ix_1)^{\alpha_1}(-ix_2)^{\alpha_2}\ldots(-ix_n)^{\alpha_n} e^{-i\langle x,\xi\rangle}\, dx,$$

$$= (-i)^{|\alpha|}(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^{\boldsymbol{n}}}\varphi(x)\, x^\alpha e^{-i\langle x,\xi\rangle}\, dx,$$

$$= (-i)^{|\alpha|}\mathcal{F}[x^\alpha\varphi](\xi).$$

$$ii)\ \mathcal{F}[D^\alpha\varphi](\xi) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^{\boldsymbol{n}}} e^{-i\langle x,\xi\rangle} D^\alpha\varphi(x) dx,$$

$$= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^{\boldsymbol{n}}} \left(-1\right)^{|\alpha|}\varphi(x) D^\alpha e^{-i\langle x,\xi\rangle} dx,$$

$$= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \left(-1\right)^{|\alpha|} \varphi(x)(-i\xi)^{\alpha} e^{-i\langle x,\xi\rangle} dx,$$

$$= \left(-1\right)^{|\alpha|} (-i\xi)^{\alpha} (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(x) e^{-i\langle x,\xi\rangle} dx,$$

$$= (i\xi)^{\alpha} [\mathcal{F}\varphi](\xi).$$

2.4 Fourier Transform of Tempered Distributions $\mathcal{S}'(\mathbb{R}^n)$

Definition: If $f \in \mathcal{S}'(\mathbb{R}^n)$, then $\langle \mathcal{F}(f), \varphi \rangle = \langle f, \mathcal{F}(\varphi) \rangle$, $\forall \varphi \in \mathcal{S}$.

In particular if $f = \delta$, then

$$\langle \mathcal{F}(\delta), \varphi \rangle = \langle \delta, \mathcal{F}(\varphi) \rangle = \mathcal{F}\varphi(0) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(x) e^{-i\langle x,0\rangle} dx,$$

$$= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(x) dx,$$

$$= (2\pi)^{-\frac{n}{2}} \langle 1, \varphi \rangle.$$

$$= \langle (2\pi)^{-\frac{n}{2}}, \varphi \rangle.$$

$$\therefore \mathcal{F}(\delta) = (2\pi)^{-\frac{n}{2}}, \varphi \in \mathcal{S}.$$

Similarly, if $f = 1$, then

$$\langle \mathcal{F}(1), \varphi \rangle = \langle 1, \mathcal{F}(\varphi) \rangle = \int_{\mathbb{R}^n} \mathcal{F}\varphi(x) dx,$$

$$= (2\pi)^{\frac{n}{2}} \mathcal{F}^{-1}\mathcal{F}\varphi(0),$$

$$= (2\pi)^{\frac{n}{2}} \varphi(0),$$

$$= (2\pi)^{\frac{n}{2}} \langle \delta, \varphi \rangle,$$

$$= \langle (2\pi)^{\frac{n}{2}} \delta, \varphi \rangle.$$

Hence $\mathcal{F}(1) = (2\pi)^{\frac{n}{2}} \delta$.

We note that the space of test functions D is not mapped into itself by the Fourier transform (since the Fourier transform of a function with compact support is an analytic function, and consequently is either not of compact support or zero).Thus it is not possible for $\varphi \in D'$ *so that*

$$\langle F(f), \varphi \rangle = \langle f, F(\varphi) \rangle.$$

Proposition 1.2: Two important formulae can be also proved

$i)\ D^{\alpha}[\mathcal{F}f](\xi) = \mathcal{F}[(-ix)^{\alpha} f](\xi)$

$ii)\ \mathcal{F}[D^{\alpha} f](\xi) = (i\xi)^{\alpha}[\mathcal{F}f](\xi)$

Proof: $i)\ \langle D^{\alpha}[\mathcal{F}f], \varphi\rangle = (-1)^{|\alpha|}\langle \mathcal{F}f, D^{\alpha}\varphi\rangle\ ,\ f \in \mathcal{S}',\ \varphi \in \mathcal{S}$

$$= (-1)^{|\alpha|}\langle f, \mathcal{F}(D^{\alpha}\varphi)\rangle = (-1)^{|\alpha|}\langle f, (ix)^{\alpha}(\mathcal{F}\varphi)\rangle,$$

$$= (-1)^{|\alpha|}\langle (ix)^{\alpha} f, \mathcal{F}\varphi\rangle = \langle (-ix)^{\alpha} f, \mathcal{F}\varphi\rangle = \langle \mathcal{F}[(-ix)^{\alpha} f], \varphi\rangle,$$

$\therefore D^{\alpha}[\mathcal{F}f](\xi) = \mathcal{F}[(-ix)^{\alpha} f](\xi).$

$ii)\ \langle \mathcal{F}[D^{\alpha} f], \varphi\rangle = \langle D^{\alpha} f, \mathcal{F}\varphi\rangle\ ,\ f \in \mathcal{S}',\ \varphi \in \mathcal{S}$

$$= (-1)^{|\alpha|}\langle f, D^{\alpha}(\mathcal{F}\varphi)\rangle = (-1)^{|\alpha|}\langle f, (-i)^{|\alpha|}\mathcal{F}(\xi^{\alpha}\varphi)\rangle,$$

$$= i^{|\alpha|}\langle f, \mathcal{F}(\xi^{\alpha}\varphi)\rangle = i^{|\alpha|}\langle \mathcal{F}(f), \xi^{\alpha}\varphi\rangle,$$

$$= i^{|\alpha|}\langle \xi^{\alpha}\mathcal{F}(f), \varphi\rangle = \langle (i\xi)^{\alpha}\mathcal{F}(f), \varphi\rangle,$$

$\therefore \mathcal{F}[D^{\alpha} f](\xi) = (i\xi)^{\alpha}[\mathcal{F}f](\xi).$

2.5 The Convolution of Generalized Functions

Definion: *Let* $f \in L_1^{loc}(R^n),\ g \in L_1^{loc}(R^n)$ *and* $\int f(y)g(x-y)dy \in L_1^{loc}(R^n)$. *Then the*

convolution of the functions f and g is

$$(f * g)(x) = \int f(y)g(x-y)dy,$$

or

$$(g * f)(x) = \int g(y)f(x-y)dy. \tag{2.5.1}$$

Function (2.5.1) i s l ocally i ntegrable i n R^n and t herefore d efines a (regular) g eneralized function, acting on the test functions $\varphi \in D(R^n)$, according to the rule:

$$\langle (f * g)(x), \varphi(x)\rangle = \int (f * g)(x)\varphi(x)dx,$$

$$= \int \varphi(x)\int f(y)g(x-y)dydx,$$

$$= \int f(y)\int g(x-y)\varphi(x)dxdy,$$

$$= \int f(y)\int g(\xi)\varphi(y+\xi)d\xi dy,\ set\ \xi = x-y,$$

$$= \int f(y)[\int g(x)\varphi(x+y)dx]dy.$$

(by virtue of Fubini's theorem), that is,

$$\langle (f*g)(x),\varphi(x)\rangle = \int f(y)g(x)\varphi(x+y)dxdy,\ \varphi\in D(R^n).$$

(Condition fo r the E xistence of a C onvolution) Let f b e an ar bitrary an d g a generalized function with compact support. Then the convolution f * g exists in D' and appears in the form:

$$\langle f*g,\varphi\rangle = \langle f(x)\otimes g(y),\eta(y)\varphi(x+y)\rangle,\ \ \varphi\in D$$

where η is a ny t est function equal t o 1 i n t he n eighborhood of t he s upport of g . F or this th e convolution is continuous with respect to f and g separately:

(1) If $f_k\to f$ as $k\to\infty$ in D', then $f_k*g\to f*g$ as $k\to\infty$ in D'.

(2) If $g_k\to g$ as $k\to\infty$ and for a certain R, supp $g_k\subset U_R$, then $f*g_k\to f*g$ as $k\to\infty$ in D'.

What happens with Y oung's I nequality if $\frac{1}{p}+\frac{1}{q}-1=\frac{1}{r}$ is not sat isfied? If $p,q,r\in[1,\infty)$ where $\frac{1}{p}+\frac{1}{q}-1=\frac{1}{r}$, then $f\in L_p,\ g\in L_q$:

$$f*g\in L_r \text{ and } \left|f*g\right|_{L_r}\le \|f\|_{L_p}\|g\|_{L_q}.$$

Remark: T he c onvolution of a ny g eneralized f unction s w ith t he Dirac d elta f unction always

exists and equal to s .i.e.,

$$\begin{aligned}\langle s*\delta,\varphi\rangle &= \langle s(x)\otimes\delta(y),\eta(y)\varphi(x+y)\rangle,\ \ \textit{for any } \varphi\in D\\ &= \langle s(x),\langle\delta(y),\eta(y)\varphi(x+y)\rangle\rangle,\\ &= \langle s(x),\varphi(y)\rangle,\end{aligned}$$

$$\therefore s*\delta=\delta*s=s.$$

Chapter 3

Generalized Solutions of Linear Ordinary Differential Equations

3.1 Ordinary Differential Operators

A linear differential equations of operator $L=\sum_{m=0}^{n} a_m(x)\frac{d^m}{dx^m}$, is defined by,

$$Lt=\left(a_n(x)\frac{d^n}{dx^n}+a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}}+\ldots+a_1(x)\frac{d}{dx}+a_0\right)t=\sum_{m=0}^{n} a_m(x)\frac{d^m t}{dx^m} \tag{3.1.1}$$

and its formal adjoin denoted L^*,

$$L^*\varphi=\sum_{m=0}^{n}(-1)^m\frac{d^m(a_m(x)\varphi)}{dx^m},$$

where the coefficients $a_m(x)\in C^\infty$, t is a distribution in $\mathcal{D}'$ and φ is a test function in D. These operators are related as distributional sense by the equation

$$\langle Lt,\varphi\rangle=\langle\sum_{m=0}^{n} a_m(x)\frac{d^m t}{dx^m},\varphi\rangle=\langle t,\sum_{m=0}^{n}(-1)^m\frac{d^m\left(a_m(x)\varphi\right)}{dx^m}\rangle=\langle t,L^*\varphi\rangle.$$

This means that the action of Lt on φ is equivalent to the action of t on the test function

$$\psi=L^*\varphi.$$

Consider the differential equation,

$$Lt=f(x), \tag{3.1.2}$$

where f is an arbitrary known distribution. A distribution t is a solution of differential equation (3.1.2) if for every test function φ we have

$$\langle Lt,\varphi\rangle=\langle f,\varphi\rangle \quad or \quad \langle t,\varphi\rangle=\langle f,L^*\varphi\rangle \tag{3.1.3}$$

In searching for a solution t of differential equation (3.1.2) we may have the following situations:

(1) A solution t is said to be *classical solution* if $t \in C^{\infty}$, so that the operations in (3.1.2) can be performed in the classical sense and the resulting equation is an identity.

(2) A solution t is said to be *weak solution* if it is not sufficiently smooth, so that the operation in (3.1.2) cannot be performed, but it satisfies (3.1.3) as a distribution.

(3) A solution t is said to be *distributional solution* if it is a singular distribution and satisfies (3.1.3).

3.2 Homogeneous Ordinary Differential Equations

Consider the simplest homogeneous differential equation,

$$dt/dx = 0. \qquad (3.2.1)$$

We know that the only classical solution to this equation is $t = c$, constant.

Lemma 1.1: Any test function $\psi(x)$ can be represented as the derivative of another test function $\varphi(x)$ if and only if $\int_{-\infty}^{\infty} \psi(x)dx = 0.$

Proof:

$(\Rightarrow)$ Suppose that $\psi(x) = \varphi'(x)$ for $\varphi(x) \in D$. Then we have

$$\int_{-\infty}^{\infty} \psi(x)dx = \int_{-\infty}^{\infty} \varphi'(x)dx = \varphi(x)|_{-\infty}^{\infty} = 0.$$

$(\Leftarrow)$ Suppose that $\varphi(x) = \int_{-\infty}^{\infty} \psi(u)du \in C^{\infty}$, and $\psi(x)$ and $\varphi'(x)$ vanish outside the same interval.

Thus $\psi(x) = \varphi'(x)$.

Theorem 1.3: The only generalized solution to (3.2.1) is $t = c$.

Proof: From equation (3.2.1) we have $\langle t', \varphi\rangle = -\langle t, \varphi'\rangle = 0$. Thus $\langle t, \psi\rangle = 0$ because of Lemma 1.1, $\psi(x) = \varphi'(x)$. Let us now take a fixed test function φ_0 that is normalized such that $\int_{-\infty}^{\infty} \varphi_0(x)dx = 1$, and write as decomposition

$$\varphi(x) = \varphi_0(x)\int_{-\infty}^{\infty} \varphi(u)du + \varphi(x) - \varphi_0(x)\int_{-\infty}^{\infty} \varphi(u)du, \;\; \varphi \in D.$$

If we set $\psi(x)=\varphi(x)-\varphi_0(x)\int_{-\infty}^{\infty}\varphi(u)du,$ and $\int_{-\infty}^{\infty}\psi(x)dx=\int_{-\infty}^{\infty}\varphi(x)dx-\int_{-\infty}^{\infty}\varphi_0(x)dx\int_{-\infty}^{\infty}\varphi(u)du=0.$

Then $\langle t,\varphi\rangle=\langle t,\varphi_0\rangle\int_{-\infty}^{\infty}\varphi(u)du+\langle t,\psi\rangle=\langle t,\varphi_0\rangle\int_{-\infty}^{\infty}\varphi(u)du=c\int_{-\infty}^{\infty}\varphi(u)du,\ \ constant\ c=\langle t,\varphi_0\rangle,$

$$=\langle c,\varphi\rangle.$$

Hence $t=c$ is the only generalized solution.

Corollary 1.3: If two generalized functions s and t have the same derivative, then s = t + c.

Proof: Suppose that $s'=t'.$ Then $\langle s',\varphi\rangle=\langle t',\varphi\rangle$ or $\langle s-t,\varphi'\rangle=0.$ Thus $\langle s-t,\psi\rangle=0$ because of Lemma 1.1, $\psi(x)=\varphi'(x).$ By using the decomposition

$\varphi(x)=\varphi_0(x)\int_{-\infty}^{\infty}\varphi_0(u)du+\varphi(x)-\varphi(x)\int_{-\infty}^{\infty}\varphi_0(u)du,$ and if we set $\psi(x)=\varphi(x)-\varphi_0(x)\int_{-\infty}^{\infty}\varphi(u)du,$ we

obtain
$$\int_{-\infty}^{\infty}\psi(x)dx=\int_{-\infty}^{\infty}\varphi(x)dx-\int_{-\infty}^{\infty}\varphi_0(x)dx\int_{-\infty}^{\infty}\varphi(u)du=0.$$

Then
$$\langle s-t,\varphi\rangle=\langle s-t,\varphi_0\rangle\int_{-\infty}^{\infty}\varphi(u)du+\langle s-t,\psi\rangle,$$

$$=\langle s-t,\varphi_0\rangle\int_{-\infty}^{\infty}\varphi(u)du,$$

$$=c\int_{-\infty}^{\infty}\varphi(u)du,\ \text{for constant}\ c=\langle s-t,\varphi_0\rangle,$$

$$=\langle c,\varphi\rangle.$$

So we obtain $s=t+c.$

Corollary 1.4: Let $T=(t_1,t_2,...,t_n)$ be an n-dimensional vector distribution. Then the solution of the differential equation $\frac{dT}{dx}=0$ is $T=C,(C=c_1,c_2,...,c_n).$

Proof: F rom equation $\frac{dT}{dx}=0,$ we ha ve $\langle T',\varphi\rangle=-\langle T,\varphi'\rangle=0.$ Thus $\langle T,\psi\rangle=0$ because o f Lemma 1.1, $\psi(x)=\varphi'(x).$ By using the decomposition

$$\varphi(x)=\varphi_0(x)\int_{-\infty}^{\infty}\varphi(u)du+\varphi(x)-\varphi_0(x)\int_{-\infty}^{\infty}\varphi(u)du,$$

and set $$\psi(x)=\varphi(x)-\varphi_0(x)\int_{-\infty}^{\infty}\varphi(u)du,$$

we obtain $$\int_{-\infty}^{\infty}\psi(x)dx=\int_{-\infty}^{\infty}\varphi(x)dx-\int_{-\infty}^{\infty}\varphi_0(x)dx\int_{-\infty}^{\infty}\varphi(u)du=0.$$

Then
$$\begin{aligned}\langle T,\varphi\rangle&=\langle T,\varphi_0\rangle\int_{-\infty}^{\infty}\varphi(u)du+\langle T,\psi\rangle,\\&=\langle T,\varphi_0\rangle\int_{-\infty}^{\infty}\varphi(u)du,\\&=C\int_{-\infty}^{\infty}\varphi(u)du,\text{ for constant } C=\langle T,\varphi_0\rangle,\\&=\langle C,\varphi\rangle.\end{aligned}$$

So the solution is T = C.

The idea of corollary 2.2 can be extended to the n[th]-order ordinary differential equation

$$Lt=a_nt^{(n)}+a_{n-1}t^{(n-1)}+...+a_1t'+a_0t=0,$$

where $a_j\in C^{\infty}\forall j=0,1,...,n;\ a_n\neq 0.$ *The solution in this case has also the same as the classical solution.* Indeed, this equation can be transformed to a system of linear first-order ordinary differential equations, as will be explained in Section 4.3. The solution in this case has also the same form as the classical solution.

3.3 Inhomogeneous Differential Equations

The simplest inhomogeneous ordinary differential equation is $dt/dx=f$, where $f\in C^0$.

A distribution t is a solution of this equation if $\langle t',\varphi\rangle=\langle f,\varphi\rangle$ *or* $\langle t,\varphi'\rangle=\langle f,-\varphi\rangle$, for every $\varphi\in D$. To solve this equation we call to the decomposition:

$$\varphi(x)=\varphi_0(x)\int_{-\infty}^{\infty}\varphi\ (u)du+\varphi(x)-\varphi_0(x)\int_{-\infty}^{\infty}\varphi\ (u)du.$$

Then we have $$\langle t,\varphi\rangle=\langle t,\varphi_0\rangle\int_{-\infty}^{\infty}\varphi(u)du+\langle t,\psi\rangle,\ \langle t,\varphi\rangle=\langle t,\varphi_0\rangle\int_{-\infty}^{\infty}\varphi(u)du+\langle t,\psi\rangle,$$

where $\psi(x)=\varphi(x)-\varphi(x)\int_{-\infty}^{\infty}\varphi_0(u)du.$ Let $\varphi_1(x)=\int_{-\infty}^{\infty}\varphi(u)du-\int_{-\infty}^{\infty}\varphi_0(v)dv\int_{-\infty}^{\infty}\varphi(u)du.$

since $\psi=\varphi'_1,$ we can use $\psi(x)=\varphi'(x),\ \varphi_1(x)=\varphi(x).$ *Thus* $\langle t,\varphi'\rangle=\langle f,-\varphi\rangle,$

$$\Leftrightarrow\langle t,\psi\rangle=\langle f,-\varphi_1\rangle=\langle t_0,\varphi\rangle.\ \textit{Since}\ \langle t,\psi\rangle\ \textit{is known},\ \textit{we have}$$

$$\langle t,\varphi\rangle=\langle t,\varphi_0\rangle\int_{-\infty}^{\infty}\varphi(u)du+\langle t,\psi\rangle=\langle c,\varphi\rangle+\langle t_0,\varphi\rangle.$$

So we obtain

$$t=c+t_0,$$

where c *is* hom*ogenous solution and* t_0 *is the particular solution.*

Corollary 1.5: The solution of a vector equation $\frac{dT}{dx}=F,$ where $T=(t_1,t_2,...,t_n)$ and $F=(f_1,f_2,...,f_n),$ is $T=C+T_0,$ where C is a constant vector distribution and T_0 *is the particular solution.*

Proof: By taking composition $\varphi(x)=\varphi_0(x)\int_{-\infty}^{\infty}\varphi(u)du+\varphi(x)-\varphi_0(x)\int_{-\infty}^{\infty}\varphi(u)du.$ Then we have

$$\langle T,\varphi\rangle=\langle T,\varphi_0\rangle\int_{-\infty}^{\infty}\varphi(u)du+\langle T,\psi\rangle,\text{where }\psi(x)=\varphi(x)-\varphi(x)\int_{-\infty}^{\infty}\varphi_0(u)du.$$

Let $\varphi_1(x)=\int_{-\infty}^{\infty}\varphi(u)du-\int_{-\infty}^{\infty}\varphi_0(v)dv\int_{-\infty}^{\infty}\varphi(u)du.$ since $\psi=\varphi'_1,$ we can use $\psi(x)=\varphi'(x),$

$\varphi_1(x)=\varphi(x).$ *Thus* $\langle T,\varphi'\rangle=\langle f,-\varphi\rangle\Leftrightarrow\langle T,\psi\rangle=\langle f,-\varphi_1\rangle=\langle T_0,\varphi\rangle$ sin*ce* $\langle T,\psi\rangle$ *is known.*

Accordingly, we have $\langle T,\varphi\rangle=\langle T,\varphi_0\rangle\int_{-\infty}^{\infty}\varphi(u)du+\langle T,\psi\rangle=\langle C,\varphi\rangle+\langle T_0,\varphi\rangle,$ *so we obtain*

$$T=C+T_0.$$

The generalized solution of the inhomogeneous ordinary differential equation

$$a_nt^{(n)}+a_{n-1}t^{(n-1)}+...+a_1t'+a_0t=f, \tag{3.3.1}$$

where $a_j\in C^{\infty}\forall j=0,1,...,n,$ t is a scalar distribution, $a_n\neq 0,$ and $f\in C^0$ (or the generalized solution of its equivalent system of the first order equations), is identical to the classical solution.

We shall discuss in Section 4.3 the equation obtained from (3.3.1) by substituting $\delta(x)$ for the right side.

For the sake of demonstration, we consider the following problems.

1. Find the general solution of the equation $x^m dt/dx = 0,\ \mathrm{m} \geq 1.$

Solution: Indeed, we assume that $t(x) = c_1 + c_2 H(x) + c_3\delta(x) + c_4\delta'(x) + ... + c_{m+1}\delta^{m-2}(x)$, so that $t'(x) = c_2\delta(x) + c_3\delta'(x) + ... + c_{m+1}\delta^{m-1}(x)$.

Thus $\langle x^m t'(x), \varphi\rangle = \langle c_2 x^m \delta(x), \varphi\rangle + \langle c_3 x^m \delta'(x), \varphi\rangle + ... + \langle c_{m+1} x^m \delta^{(m-1)}(x), \varphi(x)\rangle,$

$$= c_2\langle \delta(x), x^m\varphi\rangle - c_3\langle \delta, (x^m\varphi)'\rangle + ... + (-1)^{|m-1|} c_{m+1}\langle \delta(x), x^m\varphi(x))^{(m-1)}\rangle,$$

$$= c_2 x^m \varphi(x)|_{x=0} - c_3 (x^m\varphi(x))'|_{x=0} + ... + (-1)^{|m-1|} c_{m+1} x^m \varphi(x))^{(m-1)}|_{x=0},$$

$$= 0 = \langle 0, \varphi\rangle.$$

and we have $x^m t'(x) = 0$ as required. Hence,

$$t(x) = c_1 + c_2 H(x) + c_3\delta(x) + c_4\delta'(x) + ... + c_{m+1}\delta^{m-2}(x)$$

is the general solution of the given equation.

For m = 1 the solution reduces to $t(x) = c_1 + c_2 H(x)$, which is a weak solution since the ordinary function H(x) is not differentiable at x=0.

For $m \geq 2$, $t(x) = c_1 + c_2 H(x) + c_3\delta(x) + c_4\delta'(x) + ... + c_{m+1}\delta^{m-2}(x)$ is the distributional solution.

2**.** We show that the general solution of $x\dfrac{dt}{dx} = 1$ is

$$t(x) = c_1 + c_2 H(x) + \ln|x|.$$

Solution: For every $\varphi \in D$,

$$\langle x\frac{d}{dx}\left(c_1 + c_2 H(x) + \ln|x|\right), \varphi\rangle = \langle x c_2 \delta(x) + x pf(\frac{1}{x}), \varphi\rangle,\quad pf(\frac{1}{x}) \text{ is } \textit{principal part } (\frac{1}{x})$$

$$= c_2\langle x\delta(x), \varphi\rangle + \langle x pf(\frac{1}{x}), \varphi\rangle,$$

$$= \lim_{\varepsilon\to 0}\left[\int_{-\infty}^{-\varepsilon} x\frac{\varphi(x)dx}{x} + \int_{\varepsilon}^{\infty} x\frac{\varphi(x)dx}{x}\right],$$

$$= \lim_{\varepsilon \to 0} \left[\int_{-\infty}^{-\varepsilon} \varphi(x)dx + \int_{\varepsilon}^{\infty} \varphi(x)dx \right],$$

$$= \int_{-\infty}^{\infty} \varphi(x)dx = \langle 1, \varphi \rangle.$$

Hence $x\frac{dt}{dx} = 1$. Thus $t(x) = c_1 + c_2 H(x) + \ln|x|$ is the general solution.

3. Show that the general solution of $x^2 \frac{dt}{dx} = pf(\frac{1}{x})$ is

$$t(x) = c_1 + c_2 H(x) + c_3 \delta(x) - \frac{1}{2} pf(\frac{1}{x^2}).$$

Solution: For every $\varphi \in D$,

$$\langle x^2 \frac{d}{dx}\left(c_1 + c_2 H(x) + c_3 \delta(x) - \frac{1}{2} pf(\frac{1}{x^2}) \right), \varphi \rangle = \langle x^2 c_2 \delta(x) + x^2 c_3 \delta'(x) + pf(\frac{1}{x}), \varphi \rangle,$$

$$= c_2 \langle x^2 \delta(x), \varphi \rangle + c_3 \langle x^2 \delta'(x), \varphi \rangle + \langle pf(\frac{1}{x}), \varphi \rangle,$$

$$= 0 + 0 + \langle pf(\frac{1}{x}), \varphi \rangle,$$

$$= \langle pf(\frac{1}{x}), \varphi \rangle.$$

So $t(x) = c_1 + c_2 H(x) + c_3 \delta(x) - \frac{1}{2} pf(\frac{1}{x^2})$ *is the general solution.* .

3.4 Fundamental Solutions and Green's Functions

Definition: The fundamental solution E for differential operator L is defined as $LE(x) = \delta(x)$, where L is defined in (3.1.1).

Let us start with the simple ordinary differential equation $\frac{dE}{dx} + aE = \delta(x)$. If $a = 0$, $E(x) = H(x) + C$ and if $a \neq 0$, its general solution is $E(x) = H(x)U(x) + Ce^{-ax}$, where U satisfies *IVP:* $\frac{dU}{dx} + aU = 0,\ U|_{x=0} = 1$. These arguments extend to a system of first-order differential equations

$$\frac{dE}{dx} + AE = \delta(x)I,$$

where the fundamental solution E is now an n x n matrix, A is a given n x n matrix whose entries are in C^{∞}, and I is the identity matrix. Its particular solution is $E(x) = H(x)U(x) + Ce^{-ax}$, where U satisfies *IVP*:

$$\frac{dE}{dx} + AE = 0, \; E\,|_{x=0} = I.$$

We have right fundamental solution $E(x) = H(x)U(x) + e^{-xA}C$ which satisfies the equation $\frac{dE}{dx} + AE = I\delta(x)$ and the left fundamental solution $E(x) = H(x)U(x) + Ce^{-xA}$ is the distribution that satisfies the equation

$$\frac{dE}{dx} + EA = \delta(x)I.$$

This solution helps us in solving the inhomogeneous equation for a distribution vector t,

$$\frac{dt}{dx} + At = T,$$

where T is a distribution vector with a compact support. Indeed, the particular solution is the convolution

$$t_p(x) = E(x) * T = \int_{-\infty}^{x} U(y)T(x-y)dy + \int_{-\infty}^{\infty} e^{-yA}T(x-y)dy,$$

while the homogenous solution is the same as the classical one.

Green's function provides an powerful alternative to the method of separation of variables to solve inhomogeneous *BVP*.

Example: The general solution of the *BVP* (in classical sense)

$$y'' = f(x), \; y(0) = 0 = y(1),$$

can be written in the form $y = \int_0^1 G(x,s)f(s)ds$, where $G(x,s)$ is the Green's function defined by

$$G(x,s) = \begin{cases} s(1-x), & 0 \le s \le x, \\ x(1-s), & x \le s \le 1. \end{cases}$$

Note that: Fundamental solution helps us in solving the inhomogeneous equation $dt/dx + At = T$, but Green's functions used to solve inhomogeneous *BVP*.

The next natural step would be to consider the general nth-order ordinary differential operator. We shall, of course, attend to it, but first we want to discuss second-order ordinary differential equations, because they are the cornerstone of mathematical physics and theoretical mechanics.

3.5 Second-Order Differential Equations with Constant Coefficients

Let us consider the simplest second differential equation with constant coefficient:

$$\frac{d^2E(x,\xi)}{dx^2} = \delta(x-\xi). \tag{3.5.1}$$

Since $H'(x) = \delta(x)$, integrate this equation both side with respect to x by fix ξ as constant, we obtain

$$\frac{dE(x,\xi)}{dx} = H(x-\xi) + \alpha(\xi),$$

where $\alpha(\xi)$ is an arbitrary function. Next, integrate this equation and we obtain

$$E(x,\xi) = \int_{-\infty}^{x} H(x-\xi)dx + x\alpha(\xi) + \beta(\xi),$$

$$= xH(x-\xi)\,|_{-\infty}^{x} - \int_{-\infty}^{x} x\delta(x)dx + x\alpha(\xi) + \beta(\xi), \quad \text{(Integration by part)}$$

$$= (x-\xi)H(x-\xi) + x\alpha(\xi) + \beta(\xi), \tag{3.5.2}$$

where $\beta(\xi)$ is another arbitrary function.

The solution $E(x,\xi) = (x-\xi)H(x-\xi) + x\alpha(\xi) + \beta(\xi)$ is a continuous and piece-wise differentiable function. Thus it is the solution of (3.5.1) and it helps us in solving the inhomogeneous equation

$$\frac{d^2t}{dx^2} = f(x), \tag{3.5.3}$$

where $f(x)$ is a distribution with compact support. Indeed, the solution of (3.5.3) is

$$t(x) = E * f, \tag{3.5.4}$$

Let us see if we can solve with the help of the above analysis an *IVP* or *BVP* in the classical theory of ordinary differential equations.

Let us consider *BVP:* $\frac{d^2u}{dx^2} = f(x), \quad 0 \le x \le 1, \quad u(0) = u(1) = 0,$ where $f(x)$ is a n i ntegrable function with compact support in $[0,1]$. Appealing to relations (3.5.2) and (3.5.4), we find that

$$U(x) = E * f = [(x-\xi)H(x-\xi) + x\alpha(\xi) + \beta(\xi)] * f,$$

$$= \int_{-\infty}^{\infty} (x-\xi)H(x-\xi) f(\xi)d\xi + x\int_{-\infty}^{\infty} \alpha(\xi) f(\xi)d\xi + \int_{-\infty}^{\infty} \beta(\xi) f(\xi)d\xi,$$

$$= \int_{-\infty}^{x} (x-\xi) f(\xi)d\xi + x\int_{-\infty}^{\infty} \alpha(\xi) f(\xi)d\xi + \int_{-\infty}^{\infty} \beta(\xi) f(\xi)d\xi.$$

From *BCs:* $u(0) = -\int_{-\infty}^{0} \xi f(\xi)d\xi + \int_{-\infty}^{\infty} \beta(\xi) f(\xi)d\xi = 0,$ (3.5.5)

$$u(1) = \int_{-\infty}^{1} (1-\xi) f(\xi)d\xi + \int_{-\infty}^{\infty} \alpha(\xi) f(\xi)d\xi + \int_{-\infty}^{\infty} \beta(\xi) f(\xi)d\xi = 0 \qquad (3.5.6)$$

From (3.5.5) *we find that* $\int_{-\infty}^{0} \xi f(\xi)d\xi = \int_{-\infty}^{\infty} \beta(\xi) f(\xi)d\xi \Leftrightarrow \beta(\xi) = \xi H(-\xi).$ *Then* (3.5.6) *yields*

$$\alpha(\xi) = \begin{cases} -1 + \xi H(\xi), & -\infty \le \xi \le 1, \\ 0, & \xi > 1. \end{cases}$$

Thus particular solution is

$$u(x) = \int_{0}^{x} (1-\xi) f(\xi)d\xi - x\int_{0}^{1} (1-\xi) f(\xi)d\xi$$

$$= \int_{0}^{1} (x-\xi)H(x-\xi) f(\xi)d\xi - \int_{0}^{1} x(1-\xi) f(\xi)d\xi,$$

$$= \int_{0}^{1} [(x-\xi)H(x-\xi) - x(1-\xi)] f(\xi)d\xi = \int_{0}^{x} G(x,\xi) f(\xi)d\xi,$$

where $$G(x,\xi) = (x-\xi)H(x-\xi) - x(1-\xi) = \begin{cases} -x(1-\xi), & x < \xi, \\ -\xi(1-x), & x > \xi. \end{cases}$$

The f unction $G(x,\xi)$ is c alled a G reen's f unction. I t i s t he s olution of differential e quation (3.5.1) and the same *BCs* as does $u(x)$, namely,

$$G(0,\xi) = G(1,\xi) = 0.$$

Chapter 4

Boundary Value Problems with Jump Discontinuity

Definition: The problem of finding a solution to a given differential equation in a given closed intervals I with the solution required to meet certain specified requirements on the boundary I of that intervals is called *BVP*.

We can go a step further and examine the case of inhomogeneous boundary values, that is $u(0) = a, u(1) = b.$ For this purpose we split the function u into two parts u_1 and u_2 such that $u = u_1 + u_2$. The function u_1 satisfies the same differential equation and has the same boundary values as the function u, that is, $u_1(0) = u_1(1) = 0.$ Accordingly, its value is

$$u_1 = \int_0^x (x-\xi) f(\xi) d\xi - x\int_0^1 (1-\xi) f(\xi) d\xi.$$

The function u_2 satisfies *BVP*: $\frac{d^2 u_2}{dx^2} = 0,\ u_2(0) = a,\ u_2(1) = b.$ We can look at the quantity a as the strength of the jump discontinuity at 0 and b as the strength of the jump discontinuity at 1 (so that the function rises at 0 and falls back at 1). Then we can appeal to (1.4.5) and obtain

$$\frac{\overline{d}^2 u_2}{dx^2} = a\delta'(x) - b\delta'(x-1),\ u_2(0) = 0,\ u_2(1) = 0.$$

Its solution u_2 is determined by integrating

$$\frac{d^2 u_2}{dx^2} = 0,\ and\ u_2 = x\alpha(\xi) + \beta(\xi).$$

Thus $u_2 = bx - a(x-1)$ and $u = \int_0^x (x-\xi) f(\xi) d\xi - x\int_0^1 (1-\xi) f(\xi) d\xi + bx - a(x-1).$

Let us now consider *IVP:*

$$\frac{d^2u}{dx^2} + a^2u = \delta(x),\ \ u\,|_{x=0^+} = 0,\ \ u'|_{x=0^+} = 1. \tag{4.1}$$

Here u is defined for $x \geq 0$. Accordingly, the solution is such that u is zero for $x < 0$ and satisfies the homogeneous part $\frac{d^2u}{dx^2} + a^2u = 0$ for $x > 0$. The solution that satisfies the conditions

$$u(0) = 0,\ u'(0) = 1,\ \text{ is }\ u = \frac{\sin(ax)}{a}.$$

Thus the solution of (4.1) is $u = \frac{1}{a}H(x)\sin(ax)$. It is the fundamental solution of the operator $d^2/dx^2 + a^2$. Let us use this u to construct the solution of *IVP:*

$$\frac{d^2u}{dx^2} + a^2u = f(x),\ \ u\,|_{x=0^+} = u_0,\ \ u\,|_{x=0^+} = u_1,$$

where $f \in C^0$ for $x \geq 0$. For this purpose we continue the functions u and f in the following way:

$$v(x) = \begin{cases} 0, & x < 0, \\ u(x), & x \geq 0; \end{cases}$$

$$g(x) = \begin{cases} 0, & x < 0, \\ f(x), & x \geq 0. \end{cases}$$

We then find from (1.4.4) and (1.4.5) that

$$(\bar{v})' = v' + u_0\delta(x)\,,$$

$$(\bar{v})'' = v'' + u_0\delta'(x) + u_1\delta(x).$$

Accordingly, the function v satisfies the differential equation

$$\frac{\bar{d}^2 v}{dx^2} + a^2v = g(x) + u_0\delta'(x) + u_1\delta(x),$$

whose solution is $v(x) = E * (g + u_0\delta' + u_1\delta),$

$$= E * g + u_0E' + u_1E,$$

$$= \int_{-\infty}^{\infty} g(y)E(x-y)dy + u_0[\frac{1}{a}H(x)\sin(ax)]' + u_1\frac{1}{a}H(x)\sin(ax),$$

$$= \frac{1}{a}\int_0^x f(y)\sin a(x-y)dy + u_0\cos(ax) + u_1\frac{\sin(ax)}{a}, \quad x \ge 0.$$

An Alternative Approach

We can obtain $E(x,\xi) = (x-\xi)H(x-\xi) + x\alpha(\xi) + \beta(\xi)$ and various other interesting results by an alternative approach. For this purpose let us first recall,

$$\frac{\overline{d}^2}{dx^2}(\frac{1}{2}|x-\xi|) = \delta(x-\xi).$$

Accordingly, the solution (Green's function) to *BVP:*

$$\frac{\overline{d}^2}{dx^2}G(x,\xi) = \delta(x-\xi), \; G(0,\xi) = G(1,\xi) = 0,$$

is
$$G(x,\xi) = \frac{1}{2}|x-\xi| + xA(\xi) + B(\xi).$$

Applying the *BCs*, we find that

$$G(x,\xi) = \frac{1}{2}|x-\xi| + x(\xi - \frac{1}{2}) - \frac{\xi}{2} = \begin{cases} -x(1-\xi) \;, x < \xi \\ -\xi(1-x) \;, x > \xi. \end{cases}$$

At this stage it is useful to introduce the symbols $x_<$ and $x_>$. They stand for the values

$$x_< = \min(x,\xi) = \begin{cases} x, & x \in [a,\xi], \\ \xi, & x \in [\xi,b], \end{cases} \qquad \text{and} \qquad x_> = \max(x,\xi) = \begin{cases} \xi, & x \in [a,\xi], \\ x, & x \in [\xi,b], \end{cases}$$

as shown in Fig. 1.1. The corresponding quantities $G_<$ and $G_>$ stand for the values of G in the $x_<$ and $x_>$ regions, respectively.

In the present case, the functions $G_< = x(\xi-1)$ and $G_> = \xi(x-1)$ satisfy differential equation (3.5.1) in the $x_<$ and $x_>$ regions, respectively. The function $G_<$ satisfies the *BC:* $G(0,\xi) = 0$, while $G_>$ satisfies the *BC:* $G(1,\xi) = 0$. At $x = \xi$ these two solutions are equal. However, there is a jump in their derivatives; that is,

$$\left[\frac{\overline{d}G_>}{dx} - \frac{\overline{d}G_<}{dx}\right]_{x=\xi} = \left[\frac{\overline{d}(-\xi(1-x))}{dx} - \frac{\overline{d}(-x(1-\xi))}{dx}\right]_{x=\xi} = 1.$$

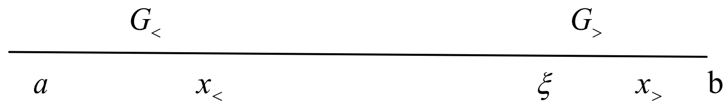

Fig. 1.1

The preceding remarks are valid for a general Sturm-Liouville problem,

$$\frac{d}{dx}\left[p(x)\frac{dG(x,\xi)}{dx}\right]+q(x)G(x,\xi)=\delta(x-\xi),\ a\le x,\ \ \xi\le b, \tag{4.2}$$

$$G(a,\xi)=G(b,\xi)=0,$$

where p and q are real-valued functions on $[a,b]$; $p,p',q\in C^0[a,b]$ and $p>0$. In this case the condition of continuity is $G(x,\xi)|_{x=\xi^-}=G(x,\xi)|_{x=\xi^+}$ *and* by first integration (4.2) as:

$$\int_{-\infty}^{\infty}\frac{d}{dx}\left[p(x)\frac{dG(x,\xi)}{dx}\right]dx+\int_{-\infty}^{\infty}q(x)G(x,\xi)dx=\int_{-\infty}^{\infty}\delta(x-\xi)dx,$$

$$\Leftrightarrow P(\xi)\left[\frac{dG(x,\xi)}{dx}|_{x=\xi^+}-\frac{dG(x,\xi)}{dx}|_{x=\xi^-}\right]+\int_{-\infty}^{\infty}q[G(x,\xi)|_{x=\xi^+}-G(x,\xi)|_{x=\xi^+}]dx=\int_{0}^{\infty}H'(x-\xi)dx,$$

$$\Leftrightarrow p(\xi)[\frac{dG(x,\xi)}{dx}|_{x=\xi+}-\frac{dG(x,\xi)}{dx}|_{x=\xi^-}]=1,$$

$$\Leftrightarrow\frac{dG(x,\xi)}{dx}|_{x=\xi+}-\frac{dG(x,\xi)}{dx}|_{x=\xi^-}=\frac{1}{p(\xi)}.$$

The following problem demonstrates application of this theory in real word problems.

In quantum mechanics there is a very interesting phenomenon in which a particle passes through a potential barrier that classical mechanics predicts is dense. Because this phenomenon implicitly involves motion, let us begin with the time-dependent Schrodinger wave equation:

$$\frac{1}{i}\frac{\partial}{\partial t}\psi(x,t)=\left[-\varepsilon^2\frac{\partial^2}{\partial x^2}+V(x)\right]\psi(x,t),$$

where ψ is the wave function,

V is the potential,

ε is a suitable parameter.

Aim: To illustrate the phenomenon of tunneling

We make two very simple choices. First, we assume that the time dependence of $\psi(x,t)$ is purely oscillatory, so that $\psi(x,t) = y(x)e^{iEt}$. Second, we choose $V(x) = \delta(x)$. Then the given wave equation reduces as $\frac{1}{i}\frac{\partial}{\partial t}\psi(x,t) = \left[-\varepsilon^2 \frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x,t)$

$$\Leftrightarrow \frac{1}{i} iEy(x)e^{iEt} = \left[-\varepsilon^2 \frac{d^2 y(x)}{dx^2} + \delta(x)y(x)\right]e^{iEt}$$

$$\Leftrightarrow \varepsilon^2 \frac{d^2 y(x)}{dx^2} + Ey(x) = y(x)\delta(x) = y(0)\delta(x).$$

Its solutions are: $y(x) = ae^{-\frac{ix\sqrt{E}}{\varepsilon}} + be^{\frac{ix\sqrt{E}}{\varepsilon}},\ x < 0,$ and $y(x) = ce^{-\frac{ix\sqrt{E}}{\varepsilon}} + de^{\frac{ix\sqrt{E}}{\varepsilon}},\ x > 0.$

The two constants a and d can be determined by physical considerations. Suppose we aim a monoenergetic unit-amplitude incident beam of particles moving toward x = 0 from the left, $a = 1$. Then the term $be^{ix\sqrt{E}/\varepsilon}$ is a reflected wave, moving left, for $x \le 0$. In the region x > 0 there will be only a transmitted wave, moving right. Accordingly, $d = 0$ and we have the following solutions:

$$y(x) = \begin{cases} e^{\frac{ix\sqrt{E}}{\varepsilon}} + be^{\frac{ix\sqrt{E}}{\varepsilon}} & ,x < 0, \\ ce^{\frac{-ix\sqrt{E}}{\varepsilon}} & ,x > 0. \end{cases}$$

The remaining two constants b and c are determined as:

1. y(x) is continuous at x = 0;
2. $[dy/dx]_{0+} - [dy/dx]_{0-} = y(0)/\varepsilon^2$.

Then we thereby obtain

$$b = \frac{2\varepsilon\sqrt{E}i - 1}{4\varepsilon^2 E + 1}, \qquad c = \frac{2\varepsilon\sqrt{E}(2\varepsilon\sqrt{E} + i)}{4\varepsilon^2 E + 1}.$$

The quantities $R = |b|^2 = \frac{1}{4\varepsilon^2 E + 1}$ and $T = |c|^2 = \frac{4\varepsilon^2 E}{4\varepsilon^2 + 1}$ are called the rejection and transmission coefficients, respectively.

In the language of quantum mechanics, R is the probability that an incident particle of energy E will be reflected, and T is the probability that the incident particle will be transmitted. Note that the total probability that a particle will be reflected or transmitted is unity between $T+R=1$. When $E\to\infty, R\to 0$ and $T\to 1$, and when $E\to 0, T\to 0$ and $R\to 1$. For these two limits the classical and the quantum-mechanical predictions agree.

With the help of the following example, we shall shade light on the use of theory in solving second order problems.

Example: Find the Green's function that satisfies the Sturm-Liouville problem with constant coefficient

$$\frac{\bar{d}^2 G(x,\xi)}{dx^2}+G(x,\xi)=\delta(x-\xi), \quad 0\le x\le \pi,$$

$$G(0)-G'(0)=0, \quad G(\pi)-G'(\pi)=0.$$

Solution:

To find the complete solution of this system, we first find the solutions in the regions $x_<$ and $x_>$.

In the $x_<$ region, the solution of the homogeneous part is $G(x,\xi)=\tilde{A}sinx+\tilde{B}cosx$ so that

$$G'(x,\xi)=\tilde{A}cosx-\tilde{B}sinx.$$

From the BC: $G(0)-G'(0)=0$, $G(\pi)-G'(\pi)=0$, we obtain $\tilde{A}=\tilde{B}$. Thus the solution in the $x_<$ region is

$$G(x,\xi)=\tilde{A}(sinx+cosx),$$

$$=A[1/\sqrt{2}\sin x+1/\sqrt{2}\cos x],$$

$$=A\sin(x+\pi/4), \quad A=\sqrt{2}\,\tilde{A}.$$

In the $x_>$ region, we take the solution of the homogeneous part is

$$G(x,\xi)=\tilde{C}sinx-\tilde{D}cosx$$

so that

$$G'(x,\xi)=\tilde{C}cosx+\tilde{D}sinx.$$

Again from the BCs, we obtain $\tilde{C}=-\tilde{D}$. Thus the solution in the $x_>$ region is

$$G(x,\xi)=\tilde{C}(sinx-cosx),$$

$$=C[1/\sqrt{2}\sin x-1/\sqrt{2}\cos x],$$

$$=C\sin(x-\pi/4), \quad C=\sqrt{2}\,\tilde{C}.$$

The condition of continuity is $A\sin(x+\pi/4)\,|_{x=\xi^+} = C\sin(x-\pi/4)\,|_{x=\xi^-}$

$$\frac{A}{\sin(\xi-\pi/4)} = \frac{C}{\sin(\xi+\pi/4)} = B = cons\tan t,$$

or

$$\Leftrightarrow A = B\sin(\xi-\pi/4), \quad C = B\sin(\xi+\pi/4).$$

By applying jump condition: $\frac{dG}{dx}|_{x=\xi^+} - \frac{dG}{dx}|_{x=\xi^-} = 1,$

$$\Leftrightarrow A\cos(\xi+\pi/4) \; - C\cos(\xi-\pi/4) = 1,$$

$$\Leftrightarrow B[\sin(\xi-\pi/4)\cos(\xi+\pi/4) - \sin(\xi+\pi/4)\cos(\xi-\pi/4)] = 1,$$

$$\Leftrightarrow B(\cos^2\xi + \sin^2\xi) = 1.$$

$So\ B = 1\ and\ . A = \sin(\xi-\pi/4)\ , C = \sin(\xi+\pi/4).$

Thus the Green's function is

$$G(x,\xi) = \begin{cases} \sin(\xi-\pi/4)\sin(x+\pi/4), & x < \xi, \\ \sin(\xi+\pi/4)\sin(x-\pi/4), & x > \xi. \end{cases}$$

4.1 Second-Order Differential Equations with Variable Coefficients

Let us consider the differential equation with variable coefficients

$$a(x)y'' + b(x)y' + c(x)y = \sum_{n=0}^{N} \beta_n \delta^{(n)}(x-\xi),$$

where $\delta^{(n)}(x)$ is the n^{th} derivative of $\delta(x)$. To fix the ideas, we shall take $N = 1$ so that we have only to solve

$$a(x)y'' + b(x)y' + c(x)y = \beta_0\delta(x-\xi) + \beta_1\delta'(x-\xi). \tag{4.1.1}$$

Since this is an inhomogeneous equation, its solution is expressed as

$$y(x) = y_h(x) + y_p(x),$$

where $y_h(x)$ is the solution of the homogeneous part and $y_p(x)$ is a particular solution. The part $y_h(x)$ is the same as we study in the classical analysis; we therefore attend only to the particular solution, which we assume to be

$$y_p(x) = G(x)H(x-\xi),$$

where $G(x)$ is a n unknow n f unction and $H(x)$ is the H eaviside f unction. T hen we a ppeal to Theorem 1.2, obtaining

$$y'_p(x) = G'(x)H(x-\xi) + G(\xi)\delta(x-\xi),$$

$$y''_p(x) = G''(x)H(x-\xi) + G'(\xi)\delta(x-\xi) + G(\xi)\delta'(x-\xi).$$

Thus

$$c(x)y_p(x) = c(x)G(x)H(x-\xi),$$

$$b(x)y'_p(x) = b(x)G'(x)H(x-\xi) + b(\xi)G(\xi)\delta(x-\xi),$$

$$a(x)y_p''(x) = a(x)G''(x)H(x-\xi) + a(\xi)G'(\xi)\delta(x-\xi) + a(x)G(\xi)\delta(x-\xi),$$

$$\begin{aligned} a(x)y''_p(x) &= a(x)G''(x)H(x-\xi) + a(\xi)G'(\xi)\delta(x-\xi) + G(\xi)[-a'(\xi)\delta(x-\xi) + a(\xi)\delta'(x-\xi)], \\ &= a(x)G''(x)H(x-\xi) + [a(\xi)G'(\xi) - a'(\xi)G(\xi)]\delta(x-\xi) + a(\xi)G(\xi)\delta'(x-\xi). \end{aligned}$$

By substituting into the given equation, we obtain

$$\begin{aligned} &(LG)H(x-\xi) + [a(\xi)G'(\xi) - a'(\xi)G(\xi) + b(\xi)G(\xi)]\delta(x-\xi) + a(\xi)G(\xi)\delta'(x-\xi) \\ &\qquad = \beta_0\delta(x-\xi) + \beta_1\delta'(x-\xi). \end{aligned}$$

Equating derivatives of $\delta(x)$ of equal orders on both sides of this equation, it reduced to

$$\begin{aligned} LG = a(x)G''(x) + b(x)G'(x) + c(x)G(x) = 0, \\ a(\xi)G'(\xi) + [-a'(\xi) + b(\xi)]G(\xi) = \beta_0, \\ a(\xi)G(\xi) = \beta_1. \end{aligned} \qquad (4.1.2)$$

This system is the *IVP* for $G(x)$ and can be solved by the classical method, thereby solving the original problem completely, i.e. $y(x) = y_h(x) + G(x)H(x-\xi)$. For the special case when $\beta_1 = 0$, we have the fundamental solution

$$E(x-\xi) = E(x,\xi) = G(x)H(x-\xi).$$

Example 1: Find the Green's function for the harmonic oscillator,

$$y'' + k^2 y = \delta(x-\xi).$$

Solution: $a(x)y'' + b(x)y' + c(x)y = \beta_0\delta(x-\xi) + \beta_1\delta'(x-\xi),$

where $a(x) = 1, b(x) = 0, c(x) = k^2, \beta_0 = 1, \beta_1 = 0.$

Thus by the system (2.7.2) it reduces to *IVP* for G(x):

$$G''(x)+k^2G(x)=0,$$
$$G'(\xi)=1,$$
$$G(\xi)=0.$$

Its general s olution is $G(x)=A\sin(kx+B)$. But $G'(x)=Ak\cos(kx+B)$, *where A and B are constants*. Then f rom *ICs* we obt ain $G(\xi)=A\sin(k\xi+B)=0 \Leftrightarrow B=-k\xi$ and $G'(\xi)=Ak\cos(k\xi-k\xi)=1 \Leftrightarrow A=1/k$. substituting these values into $G(x)$ and we get

$$G(x)=\frac{1}{k}\sin k(x-\xi).$$

Thus general solution of the given harmonic equation is

$$y(x)=\alpha\sin(kx+\beta)+[\frac{1}{k}\sin k(x-\xi)]H(x-\xi),$$

where α and β are constants, and the fundamental solution is

$$E(x,\xi)=[\frac{1}{k}\sin k(x-\xi)]H(x-\xi).$$

Example 2: Find the general solution and fundamental solution of the differential equation

$$(1-x^2)y''-2xy'=\delta(x-\xi).$$

Solution: $(1-x^2)y''-2xy'=\delta(x-\xi)$

Here $a(x)=1-x^2, b(x)=-2x, c(x)=0, \beta_0=1, \beta_1=0.$ Then by the system (4.1.2) it becomes to

$$(1-x^2)G''(x)-2xG'(x)=0,$$
$$(1-\xi^2)G'(\xi)=1,$$
$$(1-\xi^2)G(\xi)=0.$$

The general solution of this system is $G(x)=A\ln\frac{1+x}{1-x}+B.$ But $G'(x)=\frac{2A}{1-x^2}$. Then from *ICs* we obtain $(1-\xi^2)[A\ln\frac{1+\xi}{1-\xi}+B]=0 \Leftrightarrow B=-A\ln\frac{1+\xi}{1-\xi}$ and

$$(1-\xi^2)G'(\xi)=(1-\xi^2)\frac{2A}{1-\xi^2}=1 \Leftrightarrow A=\frac{1}{2}, B=-\frac{1}{2}\ln\frac{1+\xi}{1-\xi}.$$

Substituting these values into G(x), we obtain

$$G(x)=\frac{1}{2}\ln\frac{1+x}{1-x}-\frac{1}{2}\ln\frac{1+\xi}{1-\xi}=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\left(\frac{1-\xi}{1+\xi}\right).$$

So the general solution of the equation is

$$G(x)=\alpha\ln\frac{1+x}{1-x}+\beta+\frac{1}{2}[\ln\left(\frac{1+x}{1-x}\right)\left(\frac{1-\xi}{1+\xi}\right)]H(x-\xi)$$

and its fundamental solution is $E(x,\xi)=\frac{1}{2}[\ln\left(\frac{1+x}{1-x}\right)\left(\frac{1-\xi}{1+\xi}\right)]H(x-\xi).$

Example 3: Let us extend the method of the previous examples to solve the

$$y''+3y'+2y=\delta'''(x).$$

Solution: It is clear that the terms $\delta(x)$ and $\delta'(x)$ will appear in the expression *y(x)*. However, in v iew C orollary (1 .1) w e n eed o nly assu me t hat the co efficients o f $\delta(x)$ and $\delta'(x)$ are constants say a and b. Accordingly, we assume that the particular solution is

$$y_p(x)=G(x)H(x)+a\delta(x)+b\delta'(x).$$

Differentiating two times we obtain

$$y'_p(x)=G'(x)H(x)+G(0)\delta(x)+a\delta'(x)+b\delta''(x),$$

$$y''_p(x)=G''(x)H(x)+G'(0)\delta(x)+G(0)\delta'(x)+a\delta''(x)+b\delta'''(x).$$

Substitution of these into the equation $y''+3y'+2y=\delta'''(x)$:

$G''(x)H(x)+G'(0)\delta(x)+G(0)\delta'(x)+a\delta''(x)+b\delta'''(x)+$
$3[G'(x)H(x)+G(0)\delta(x)+a\delta'(x)+b\delta''(x)]+2[G(x)H(x)+a\delta(x)+b\delta'(x)]=\delta'''(x),$ it y ields the following system:

$$G''(x)+3G'(x)+2G(x)=0,$$

$$G'(0)+3G(0)+2a=0,$$

$$G(0)+3a+2b=0,$$

$$a+3b=0,$$

$$b=1.$$

Its general solution is $G(x) = Ae^{-x} + Be^{-2x}$. But $G'(x) = -Ae^{-x} - 2Be^{-2x}$. From ICs:

$$G(0) + 3a + 2b = 0 \Leftrightarrow A + B = 7 \ \text{since} \ \ a = -3, b = 1.$$

And $G'(0) + 3G(0) + 2a = 0 \Leftrightarrow -A - 2B + 3(A + B) = 6,$

$$\Leftrightarrow \begin{cases} A + B = 7, \\ 2A + B = 6. \end{cases}$$

Hence $A = -1, B = 8$ and $G(x) = -e^{-x} + 8e^{-2x}$. So the particular solution is

$$y_p(x) = [-e^{-x} + 8e^{-2x}]H(x) - 3\delta(x) + \delta'(x),$$

and the general solution is $y(x) = \alpha e^{-x} + \beta e^{-2x} + [-e^{-x} + 8e^{-2x}]H(x) - 3\delta(x) + \delta'(x)$.

Example 4: Find the fundamental solution $E(x)$ of the differential operator $d^2/dx^2 - a^2$ with the help of the Fourier transforms.

Solution: Let $E(x)$ be fundamental solution so that $\bar{d}^2 E/dx^2 - a^2 E = \delta(x)$. If we take the Fourier transform F on both sides of this equation, we obtain

$$F[(\bar{d}^2 E/dx^2 - a^2 E)] = F[\delta(x)],$$

$$\Leftrightarrow -u^2 F[E] - a^2 F[E] = 1,$$

$$\Leftrightarrow F[E] = \frac{-1}{a^2 + u^2},$$

$$\Leftrightarrow E(x) = F^{-1}[\frac{-1}{a^2 + u^2}],$$

$$= -\frac{1}{2a} e^{-a|x|}.$$

4.2 Eigenvalue Problems

Let u s now see the s pectral theory of t he f ollowing regular Sturm- Liouville pr oblem w ith general *BCs*. For this purpose we find the Green's function $G(x, \xi, \lambda)$ that satisfies the system:

$$(L + \lambda r)G = (pG')' + (q + \lambda r)G = \delta(x - \xi), \quad a \le x \le b, \tag{4.2.1}$$

$$B_a G = (\cos\alpha)G(a) - (\sin\alpha)G'(a) = 0,$$
$$B_b G = (\cos\beta)G(b) + (\sin\beta)G'(b) = 0,$$

where $p', q, r \in C^0[a,b]$, $p > 0$ and $r > 0$ in [a, b], λ is the eigenvalue, α and β are given real numbers such that $0 \le \alpha < \pi, 0 \le \beta < \pi$. The signs in the *BCs* have been chosen so that the eigenvalues decrease as α or β increases.

In the notation of Fig. 1.1, the system $\frac{d^2E}{dx^2} = \delta(x - \xi)$ can be split into two simpler systems:

i) In the $x_<$ region, $G(x, \xi, \lambda)$ is a solution of the homogeneous equation

$$(L + \lambda r)G = 0, \qquad B_a G = 0.$$

We can connect the solution of this system to the solution $\phi_\lambda(x)$ of the *IVP:*

$$L\phi_\lambda + \lambda r \phi_\lambda = 0,$$

$$\phi_\lambda(a) = \sin\alpha,$$

$$\phi'_\lambda(a) = \cos\alpha,$$

So that $B_a\phi_\lambda(a) \equiv 0$. Thus $G(x, \xi, \lambda)$ is a constant multiple of $\phi_\lambda(x)$.

ii) In the $x_>$ region, $G(x, \xi, \lambda)$ is a solution of the homogeneous equation $LG + \lambda rG = 0$, $B_b G = 0$, with the solution $\chi_\lambda(x)$ of the *IVP:*

$$L\chi_\lambda + \lambda r \chi_\lambda = 0,$$

$$\chi_\lambda(a) = \sin\beta,$$

$$\chi'_\lambda(a) = \cos\beta.$$

So that $B_b\chi_\lambda(b) \equiv 0$. Thus $G(x, \xi, \lambda)$ is a constant multiple of $\chi_\lambda(x)$. It follows that

$$G(x, \xi, \lambda) = C\phi_\lambda(x_<)\chi_\lambda(x_>).$$

G is continuous at $x = \xi$. To find the constant C, we appeal to the jump condition

$$C[\phi_\lambda(\xi)\chi'_\lambda(\xi) - \phi'_\lambda(\xi)\chi_\lambda(\xi)] = C\begin{vmatrix} \phi_\lambda & \chi_\lambda \\ \phi'_\lambda & \chi'_\lambda \end{vmatrix} = \frac{1}{p(\xi)}.$$

The quantity inside the brackets is the Wronskian $W_\xi(\phi_\lambda, \chi_\lambda)$ evaluated at $x = \xi$, *i.e.*,

$$CW_\xi(\phi_\lambda, \chi_\lambda) = \frac{1}{p(\xi)}.$$

Now we appeal to Abel's formula for the Wronskian, namely,

$$W(\phi_\lambda,\chi_\lambda)=\phi_\lambda(x)\chi'_\lambda(x)-\phi'_\lambda(x)\chi_\lambda(x)=\frac{\omega(\lambda)}{p(x)},$$

where $\omega(\lambda)$ is independent of x but may depend on λ.

Then

$$CW_\xi(\phi_\lambda,\chi_\lambda)=\frac{1}{p(\xi)}\Leftrightarrow C\frac{\omega(\lambda)}{p(\xi)}=\frac{1}{p(\xi)}.$$

So we find that $C=\dfrac{1}{\omega(\lambda)}$ and, we obtain the Green's function

$$G(x,\xi,\lambda)=\frac{1}{\omega(\lambda)}\phi_\lambda(x_<)\chi_\lambda(x_>).$$

It follows that the zeros of $\omega(\lambda)$ coincide with the eigenvalues of the given system. Indeed, let $\lambda=\mu$ be a zero of $\omega(\lambda)$, so that $\omega(\mu)=0$. From above Abel's formula it follows that the Wronskian of $\phi_\mu(x)$ and $\chi_\mu(x)$ is zero, and therefore these functions are linearly dependent. However, neither of these functions can vanish identically because of their initial values. Accordingly, $\phi_\mu(x)=k\chi_\mu(x)$, where $k\neq 0$ constant and both these functions satisfy the two *BCs* of (4.2.1). Thus, μ is an eigenvalue of the system, while $\phi_\mu(x)$ is the corresponding eigenfunction.

Furthermore, we shall soon find $G(x,\xi,\lambda)=\sum_{n=1}^{\infty}\dfrac{u_n(x)\overline{u_n}(x)}{\lambda_n-\lambda}$, that at an eigenvalue Green's function has a singularity and therefore $\omega(\lambda)$ must vanish. Thus, the zeros of $\omega(\lambda)$ coincide with the eigenvalue of system. Let us label these zeros

$$\lambda_1<\lambda_2<\dots\lambda_n<\dots,\quad \lambda_n\to\infty, \text{and set } \phi_{\lambda_n}(x)=k_n\chi_{\lambda_n}(x).$$

To normalize these eigenfunctions we appeal to the residue of G at $\lambda=\lambda_n$, which is

$$\frac{\phi_{\lambda_n}(x_<)\chi_{\lambda_n}(x_>)}{\omega'(\lambda_n)}=k_n\frac{\chi_{\lambda_n}(x_<)\chi_{\lambda_n}(x_>)}{\omega'(\lambda_n)},$$

$$=\frac{\phi_{\lambda_n}(x_<)\phi_{\lambda_n}(x_>)}{k_n\omega'(\lambda_n)}.$$

Note that $\phi_{\lambda_n}(x_<)\phi_{\lambda_n}(x_>) = \phi_{\lambda_n}(x)\phi_{\lambda_n}(\xi)$ and $\chi_{\lambda_n}(x_<)\chi_{\lambda_n}(x_>) = \chi_{\lambda_n}(x)\chi_{\lambda_n}(\xi)$ in both the regions $x_<$ and $x_>$. The required value of the residue of G at $\lambda = \lambda_n$, is

$$\frac{\phi_{\lambda_n}(x)\chi_{\lambda_n}(\xi)}{\omega'(\lambda_n)} = \frac{\phi_{\lambda_n}(x)\phi_{\lambda_n}(\xi)}{k_n\omega'(\lambda_n)} = u_n(x)\overline{u_n}(\xi),$$

where $u_n(x)$ are the orthonormalized eigenfunctions of the (4.1.1) and $\overline{u_n}(\xi)$ are their complex conjugates. From this relation it follows that

$$\frac{\phi_{\lambda_n}(x)\chi_{\lambda_n}(x)}{\omega'(\lambda_n)} = \frac{\phi_{\lambda_n}(x)\phi_{\lambda_n}(x)}{k_n\omega'(\lambda_n)} = u_n(x)\overline{u_n}(x) = u_n^2(x),$$

$$\Leftrightarrow u_n(x) = \pm\frac{\phi_{\lambda_n}(x)}{\sqrt{k_n\omega'(\lambda_n)}} = \pm\chi_{\lambda_n}\sqrt{\frac{k_n}{\omega'(\lambda_n)}}.$$

Next we expand Green's function in terms of the orthonormalized eigenfunctions $u_n(x)$, as

$$G(x,\xi,\lambda) = \sum_{n=1}^{\infty} g_n(\xi,\lambda)u_n(x), \text{ where } g_n = \langle G, u_n\rangle = \int_a^b r(x)G(x,\xi,\lambda)\overline{u_n}(x)dx.$$

To find g_n,

$$LG + \lambda rG = \delta(x-\xi),$$

$$\Leftrightarrow LG\overline{u_n} + \lambda rG\overline{u_n} = \delta(x-\xi)\overline{u_n},$$

$$\Leftrightarrow \int_a^b LG\overline{u_n}(x)dx + \lambda\int_a^b r(x)G\overline{u_n}(x)dx = \int_a^b \delta(x-\xi)\overline{u_n}(x)dx,$$

$$\Leftrightarrow \langle LG, u_n\rangle - \lambda g_n = \overline{u_n}(\xi), \qquad \textit{but } \lambda_n g_n = \langle LG, u_n\rangle.$$

$$\Leftrightarrow \lambda_n g_n - \lambda g_n = \overline{u_n},$$

$$\Leftrightarrow g_n(\xi,\lambda_n) = \frac{\overline{u_n}}{\lambda_n - \lambda}.$$

From $G(x,\xi,\lambda) = \sum_{n=1}^{\infty} g_n(\xi,\lambda)u_n(x)$, it then follows that an eigenvalue Green's function

$$G(x,\xi,\lambda) = \sum_{n=1}^{\infty}\frac{u_n(x)\overline{u_n}(x)}{\lambda_n - \lambda}.$$

There are two interesting results follow from this discussion:

i. We find that the solution $w(x,\lambda)$ of the inhomogeneous system

$$(pw')'+(q+\lambda r)w = f(x), \qquad a \le x \le b,$$

is
$$w(x,\lambda) = \int_a^b G(x,\xi,\lambda) f(\xi) d\xi,$$

$$= \int_a^b \left(\sum_{n=1}^{\infty} \frac{u_n(x)\overline{u_n}(\xi) f(\xi)}{\lambda_n - \lambda} \right) d\xi,$$

$$= \sum_{n=1}^{\infty} u_n(x) \frac{\int_a^b f(\xi)\overline{u_n}(\xi) d\xi}{(\lambda_n - \lambda)},$$

$$= \sum_{n=1}^{\infty} \langle f, \overline{u_n} \rangle u_n(x).$$

ii. When we expand $\delta(x)/r(x)$, w e obtain

$$\frac{\delta(x-\xi)}{r(x)} = \sum_{n=1}^{\infty} \left\langle \frac{\delta(x-\xi)}{r(y)}, \overline{u_n}(y) \right\rangle u_n(x),$$

$$= \sum_{n=1}^{\infty} \left(\frac{\delta(y-\xi)}{r(y)} r(y)\overline{u_n}(y) \right) u_n(x),$$

$$= \sum_{n=1}^{\infty} u_n(x)\overline{u_n}(\xi).$$

Example 5: Let us consider the system

$$(xu')'+\lambda xu = 0, \quad 0 \le x \le 1,$$

$$\lim_{x \to 0} xu' = 0, \qquad u(1) = 1.$$

The uni que s olution that s atisfies t his s ystem i s $u(x) = J_0(\sqrt{\lambda} x)$. From the *BCs* we h ave t he eigenvalues $J_0(\sqrt{\lambda}) = 0$. The c orresponding e igenfunctions a re $u_n(x) = \sqrt{2}\,\frac{J_0(\sqrt{\lambda_n} x)}{J_0(\sqrt{\lambda_n})}$ and i ts conjugate $\overline{u_n}(x) = \sqrt{2}\,\frac{J_0(\sqrt{\lambda_n}\xi)}{J_0(\sqrt{\lambda_n})}$. Then from relation it follows that

$$\frac{\delta(x-\xi)}{r(x)} = \sum_{n=1}^{\infty} u_n(x)\overline{u_n}(\xi) = \sum_{n=1}^{\infty} \frac{2J_0(\sqrt{\lambda_n} x) J_0(\sqrt{\lambda_n}\xi)}{\left(J_0(\sqrt{\lambda_n})\right)^2}.$$

4.3 Higher Order Differential Equations

Finally, we consider the fundamental solution of the nth-order differential operator (3.1.1) with coefficients $a_n(x) = 1$. This means that we have to solve the differential equation

$$\frac{\overline{d}^n E}{dx^n} + a_{n-1}(x)\frac{\overline{d}^{n-1} E}{dx^{n-1}} + ... + a_0(x)E = \delta(x). \tag{4.3.1}$$

To distinguish between the classical and distributional derivatives we shall put a bar over the above differential equation whenever there is an ambiguity.

Let us set

$$\begin{aligned} u_1 &= E, \\ u'_1 &= E' = u_2, \\ u'_2 &= E'' = u_3, \\ u'_3 &= E''' = u_4, \\ &\vdots \quad \vdots \quad \vdots \\ u'_{n-1} &= E^{(n)} = u_n. \end{aligned}$$

Using these standard transformation $u_1 = E, \quad u_2 = u'_1, ..., \quad u_n = u'_{n-1}$, we can write (4.3.1) as a first-order system,

$$\frac{\overline{dU}}{dx} + AU = F, \tag{4.3.2}$$

where $U = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & & & & -1 \\ a_0 & a_1 & a_3 & \cdots & a_{n-1} \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \delta(x) \end{bmatrix}.$

There are two well-known results from the theory of ordinary differential equations:

1. The general solution of the equation $LU = dU/dx + A(x)U = 0,$ can be written $U(x) = K(x)C,$ where $A(x)$ is a given n x n matrix of differentiable functions, C is a constant

vector and $K(x)$ is a fundamental matrix which is nonsingular , $\det K(x) \neq 0$. From the matrix K we form the Green's function (matrix) for the *IVPs* associated with the operator L by setting

$$G(x,y) = K(x)[K(y)]^{-1}.$$

2. $G(x,y)$ is independent of the particular fundamental matrix K that we use. Moreover, if A is a constant, then $G(x,y)$ depends only $x-y; i.e., G(x,y) = g(x-y)$, where $g(x)$ satisfies the *IVP:*

$$dg/dx + Ag = 0, \; g(0) = I.$$

The Green's function $G(x,y)$ enables us to solve the inhomogeneous equation (4.3.2) where F is a vector whose components are continuous functions.

Indeed,

$$U(x) = \int_0^x G(x,y)F(y)dy + G(x,0), \tag{4.3.3}$$

$$U(0) = C,$$

where C is a constant vector, is the solution of (4.3.2).

We are interested only in evaluating the fundamental solution E. Accordingly, in view of transformation we take the first component of (4.3.3) and obtain

$$E(x) = \int_0^x G_{1n}(x,y)\delta(y)dy + \sum_{i=1}^{n} c_i G_{1i}(x,0),$$

$$= G_{1n}(x,0)H(x) + \sum_{i=1}^{n} c_i G_{1i}(x,0), \tag{4.3.4}$$

where G_{1i} and c_i are the components of G and C respectively.

Suppose that $u_1(x), u_2(x), ..., u_n(x)$ are n linearly independent solutions of the homogeneous equation associated with (4.3.1). Then

$$K(x) = \begin{bmatrix} u_1(x) & \cdots & u_n(x) \\ u'_1(x) & \cdots & u'_n(x) \\ \vdots & & \vdots \\ u_1^{(n-1)}(x) & \cdots & u_n^{(n-1)}(x) \end{bmatrix}$$

is a fundamental matrix of (4.3.2). Thus

$$G_{1n}(x,y)=\sum_{i=1}^{n}u_i(x)A_{ni}(y), \qquad (4.3.5)$$

where the quantity A_{ni} is the (n,i) element of the inverse of *K,* i.e.,

$$A_{ni}=(-1)^{n+i}\begin{vmatrix} u_1 & \cdots & u_{i-1} & u_{i+1} & \cdots & u_n \\ \vdots & & \vdots & \vdots & & \vdots \\ u_1^{(n-2)} & \cdots & u_i^{(n-2)} & u_{i+1}^{(n-2)} & \cdots & {}_n^{(n-2)} \end{vmatrix}\times\begin{vmatrix} u_1 & \cdots & u_n \\ \vdots & \vdots & \vdots \\ u_1^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix}^{-1}. \qquad (4.3.6)$$

From the relation (4.3.5) and (4.3.6) we obtain

$$G_{1n}(x,y)=\begin{vmatrix} u_1(y) & \cdots & u(y) \\ \vdots & & \vdots \\ u_1^{(n-2)}(y) & & u_n^{(n-2)}(y) \\ u_1(y) & \cdots & u_n(y) \end{vmatrix}\times\begin{vmatrix} u_1(y) & \cdots & u_n(y) \\ \vdots & & \vdots \\ u_1^{(n-1)}(y) & \cdots & u_n^{(n-1)}(y) \end{vmatrix}^{-1}.$$

Then equation (4.3.4) can be written as

$$E(x)=g(x)H(x)+\sum_{i=1}^{n}c_iu_i(x),$$

where g satisfies the *IVP:*

$$\frac{d^ng}{dx^n}+a_{n-1}\frac{d^{n-1}g}{dx^{n-1}}+\ldots+a_0g=0,$$

$$g(0)=g'(0)=\cdots=g^{(n-2)}(0)=0, \qquad g^{(n-1)}(0)=1.$$

If the polynomial $p(x)=x^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$, has roots $\lambda_1,\lambda_2,\ldots,\lambda_n$ that are simple, then it is easy to show that $g(x)=\sum_{i=1}^{n}e^{\lambda_ix}/p'(\lambda_i)$, and so the fundamental solution is

$$E(x)=H(x)\sum_{i=1}^{n}e^{\lambda_ix}/p'(\lambda_i)+\sum_{i=1}^{n}c_ie^{\lambda_ix}.$$

We can readily extend the analysis of this section to the case in which the right side of (4.3.1) contains terms such as $\delta',\delta'',\cdots,\delta^{(k)}$ the same lines as presented in the previous sections.

4.4 Ordinary Differential Equations with Singular Coefficients

Recall that the sturm-Liouville problem

$$(p(x)y'(x))'+[q(x)+\lambda r(x)]y(x)=0,\ x\in I$$

is said to be singular if atleast one of the following holds:

i) The interval I is infinite /unbounded

ii) P(x) or r(x) vanishes at one / both endpoints of I

iii) P(x) or r(x) is not smooth on I.

Example: i. Bessel DE: $x^2y''+xy'+(x^2-p^2)y=0, \quad 0\le x\le 1.$

ii. Hermite DE: $(e^{-x^2}y')'+\lambda e^{-x^2}y=0, \qquad -\infty<x<\infty.$

iii. Legendre DE: $((1-x^2)y')'+\lambda y=0, \qquad -1\le x\le 1.$

iV. Chebychev DE: $((1-x^2)^{1/2}y')'+\lambda(1-x^2)^{-1/2}y=0, \quad -1<x<1.$

Linear homogeneous systems of differential equations with infinitely smooth coefficients have no solutions in the space of generalized functions other than the classical solutions. However, for equations with singularities in the coefficients, there may appear new solutions in generalized functions.

Example1: Show that

$$y(x)=C[\delta''(x)+\frac{1}{4}\delta(x)],$$

where C is a constant, is a solution of the Bessel equation $x^2y''(x)+xy'(x)+(x^2-9)y(x)=0.$

Solution: We can show it by using Corollary 1.1.

Then $x^2y''(x)+xy'(x)+(x^2-9)y(x)$

$$=x^2[C(\delta''(x)+\frac{1}{4}\delta(x)]''+x[C(\delta''(x)+\frac{1}{4}\delta(x)]'+[x^2-9][C(\delta''(x)+\frac{1}{4}\delta(x)],$$

$$=C[x^2\delta^{IV}(x)+\frac{1}{4}x^2\delta''(x)+x\delta'''(x)+\frac{1}{4}x\delta'(x)+x^2\delta''(x)+\frac{1}{4}x^2\delta(x)-9\delta''(x)-\frac{9}{4}\delta(x)],$$

$$=C[\frac{4!}{2!}\delta''(x)+\frac{1}{4}\frac{2!}{0!}\delta(x)-\frac{3!}{2!}\delta''(x)-\frac{1}{4}\delta(x)+\frac{2!}{0!}\delta(x)+0-9\frac{2!}{2!}\delta''(x)-\frac{9}{4}\delta(x)],$$

$$=C[12\delta''(x)+\frac{1}{2}\delta(x)-3\delta''(x)-\frac{1}{4}\delta(x)+2\delta(x)-9\delta''(x)-\frac{9}{4}\delta(x)],$$

$$=0.$$

Hence y(x) satisfies the given differential equation.

Example 2: Verify that, in general

$$y(x) = C\sum_{k=0}^{[m/2]} \frac{(m-k)!}{4^k k!(m-2k)!}\delta^{(m-2k)}(x),$$

where the symbol $[m/2]$ stands for the greatest integer $\le m/2$, is a solution of the Bessel equation

$$x^2 y''(x) + xy'(x) + [x^2 - (m+1)^2]y(x) = 0.$$

Solution: If $m = 2,\ y(x) = C\sum_{k=0}^{[2/2]} \frac{(2-k)!}{4^k k!(2-2k)!}\delta^{(2-2k)}(x) = C[\delta''(x) + \frac{1}{4}\delta(x)]$

then it is the solution of $x^2 y''(x) + xy'(x) + (x^2 - 9)y(x) = 0$, *what is already shown on the above example. But if* $m > 2$,

$$x^2 y''(x) + xy'(x) + [x^2 - (m+1)^2]y(x)$$

$$= x^2\left(C\sum_{k=0}^{[m/2]} \frac{(m-k)!}{4^k k!(m-2k)!}\delta^{(m-2k)}\right)'' + x\left(C\sum_{k=0}^{[m/2]} \frac{(m-k)!}{4^k k!(m-2k)!}\delta^{(m-k)}\right)'$$

$$+ [x^2 - (m+1)^2]C\sum_{k=0}^{[m/2]} \frac{(m-k)!}{4^k k!(m-2k)!}\delta^{(m-2k)},$$

$$= \left(C\sum_{k=2}^{[m/2]} \frac{(m-k)!}{4^k k!(m-2k)!}x^2\delta^{(m-2k+2)}\right) + \left(C\sum_{k=1}^{[m/2]} \frac{(m-k)!}{4^k k!(m-2k)!}x\delta^{(m-2k+1)}\right)$$

$$+ C\sum_{k=0}^{[m/2]} \frac{(m-k)!}{4^k k!(m-2k)!}x^2\delta^{(m-2k)} - C(m+1)^2\sum_{k=0}^{[m/2]} \frac{(m-k)!}{4^k k!(m-2k)!}\delta^{(m-2k)},$$

$$= \left(C\sum_{k=2}^{[m/2]} \frac{(m-k)!(m-2k+2)!}{4^k k!(m-2k)^2!}\delta^{(m-2k)}\right) - \left(C\sum_{k=1}^{[m/2]} \frac{(m-k)!(m-2k+1)!}{4^k k!(m-2k)^2!}\delta^{(m-2k)}\right)$$

$$+ C\sum_{k=0}^{[m/2]} \frac{(m-k)!}{4^k k!(m-2k-2)!}\delta^{(m-2k-2)} - C(m+1)^2\sum_{k=0}^{[m/2]} \frac{(m-k)!}{4^k k!(m-2k)!}\delta^{(m-2k)},$$

$$= 0.$$

Let us now consider the confluent hypergeometric equation

$$xy'' + (b-x)y' - ay = 0.$$

It has the distributional solution

$$y(x) = c\frac{d^{a-1}}{dx^{a-1}}\left(\frac{d}{dx}-1\right)^{b-a-1}\delta(x),$$

$$where\ a\ and\ b\ are\ positive\ \text{int}egers\ such\ that\ b-a-1\ge 0.$$

Claim! A solution of the differential equation

$$xy''+ay'+bxy=0,$$

$$where\ \ b\neq 0\ and\ is\ a\ positive\ even\ \text{int}egers,\ is\ \ y(x)=c(d^2/dx^2+b)^{(a-2)/2}\delta(x).$$

Similarly, the distributional solution of the equation

$$xy''+(x+a+b)y'+ay=0,$$

where a and b are positive integers, is

$$y(x) = c\frac{d^{a-1}}{dx^{a-1}}\left(\frac{d}{dx}+1\right)^{b-1}\delta(x).$$

Justification :

$$y(x) = c\frac{d^{a-1}}{dx^{a-1}}\left(\frac{d}{dx}+1\right)^{b-1}\delta(x) = c\frac{d^{a-1}}{dx^{a-1}}\sum_{k=0}^{b-1}\binom{b-1}{k}\left(\frac{d}{dx}\right)^{b-1}1^{b-1-k}\delta,\ (binomial\ \text{exp}asion)$$

$$= c\sum_{k=0}^{b-1}\binom{b-1}{k}\left(\frac{d^{b+a-2}}{dx^{b+a-2}}\right)\delta.$$

$$xy''+(x+a+b)y'+ay \quad = cx\sum_{k=2}^{b-1}\binom{b-1}{k}\left(\frac{d^{b+a}}{dx^{b+a}}\right)\delta + cx\sum_{k=1}^{b-1}\binom{b-1}{k}\delta^{(b+a-1)} + (a+b)\sum_{k=1}^{b-1}\binom{b-1}{k}\delta^{(b+a-1)}$$

$$+ac\sum_{k=0}^{b-1}\binom{b-1}{k}\delta^{(b+a-2)},$$

$$= -c(b+a)\sum_{k=2}^{b-1}\binom{b-1}{k}\delta^{(b+a-1)} - c(b+a-1)\sum_{k=1}^{b-1}\binom{b-1}{k}\delta^{(b+a-2)} + (a+b)\sum_{k=1}^{b-1}\binom{b-1}{k}\delta^{(b+a-1)}$$

$$+ac\sum_{k=0}^{b-1}\binom{b-1}{k}\delta^{(b+a-2)} = 0.$$

These concepts can be extended to higher-order differential equations. For instance, the third-order differential equation

$$xy'''-(x+p)y''-(x-p-1)y'+(x-1)y=0,$$

has a distributional solution when p is a negative odd integer, $p\le -3$. This solution is

$$y(x)=c(d^2/dx^2-1)^{-(p+3)/2}\delta(x).$$

Distributional solutions to ordinary differential equations may also be found by deviations of arguments, as in the system

$$y'(x)=\sum_{l=0}^{N}A_l(x)y(\lambda_j x),$$

where the $A_l(x)$ are matrices and λ_j are numbers lying between - 1 and 1, while y is a column vector. Indeed, under certain assumptions it can be shown that equation $y'(x)=\sum_{l=0}^{N}A_l(x)y(\lambda_j x)$ has a distributional solution concentrated at $x=0$.

In what follows we visit Engineering problems the solution of which makes enormous use of this theory.

1. The governing differential equation that describes the response of a viscoelastic incomplete ring to a suddenly applied force at its free end is of the form

$$ay'''+by'=\alpha H(x)+\beta_0\delta(x)+\beta_1\delta'(x)+\beta_2\delta''(x),$$

where $a,b,\alpha,\beta_0,\beta_1,\beta_2$ are appropriate constants relating to the problem.
Solve this differential equation with $a=b=\alpha=\beta_0=\beta_1=\beta_2=1$.

Solution: First we have to derive the third–order DE:

$$a(x)y'''+b(x)y''+c(x)y'+d(x)y=\beta_0\delta(x)+\beta_1\delta'(x)+\beta_2\delta''(x).$$

Suppose that $y_p=G(x)H(x)$. Differentiate this particular solution three times we obtain

$$y'_p=G'(x)H(x)+G(0)\delta(x),$$
$$y''_p=G''(x)H(x)+G'(0)\delta(x)+G(0)\delta'(x),$$
$$y'''_p=G'''(x)H(x)+G''(0)\delta(x)+G'(0)\delta'(x)+G(0)\delta''(x),$$

Thus

$$d(x)y_p=d(x)G(x)H(x),$$
$$c(x)y'_p=c(x)G'(x)H(x)+c(0)G(0)\delta(x),$$
$$b(x)y''_p=b(x)G''(x)H(x)+[b(0)G'(0)-b'(0)G(0)]\delta(x)+b(0)G(0)\delta'(x),$$
$$a(x)y'''_p=a(x)G'''(x)H(x)+[a(0)G''(0)-a'(0)G'(0)+a''(0)G(0)]\delta(x)$$
$$+[a(0)G'(0)-2a'(0)G(0)]\delta'(x)+a(0)G(0)\delta''(x).$$

Substitute these into the given equation it yields *IVP* for G(x):

$$a(x)G'''+b(x)G''+c(x)G'+d(x)y=0,$$
$$a(0)G''(0)-a'(0)G'(0)+a''(0)G(0)+b(0)G'(0)-b'(0)G(0)+c(0)G(0)=\beta_0,$$
$$a(0)G'(0)-2a'(0)G(0)+b(0)G(0)=\beta_1,$$
$$a(0)G(0)=\beta_2.$$

Now we solve $y'''+y'=H(x)+\delta(x)+\delta'(x)+\delta''(x)$,
where $a(x)=1, b(x)=0, c(x)=1, d(x)=0,\ \alpha=\beta_0=\beta_1=\beta_2=1.$
Then by above system we can find that

$$G'''(x) + G'(x) = 1,$$
$$G''(0) + G(0) = 1,$$
$$G'(0) = 1,$$
$$G(0) = 1.$$

Its general solution is $G(x) = c_1 + c_2 \cos x + c_3 \sin x + x.$ From *ICs:*

$$G(0) = c_1 + c_2 = 1,$$
$$G'(0) = c_3 + 1 = 1 \Leftrightarrow c_3 = 0,$$
$$G''(0) + G(0) = -c_2 + c_1 + c_2 = 1 \Leftrightarrow c_1 = 1, c_2 = 0.$$

Thus $G(x) = 1 + x$ and $y(x) = c_1 + c_2 \cos x + c_3 \sin x + x + (1+x)H(x)$.

2. Solve the differential equation

$$\frac{d^2}{dx^2}[EI_0(1+mx)^3 \frac{d^2 y}{dx^2}] = \beta_1 \delta'(x - \frac{1}{2}l),$$

which embodies the static problem of finding the deflection y of a linearly tapered beam that is free at $x = 0$, clamped at $x = l$, and subjected to a couple β_1 at its midlength (see Fig. 1.3),

where E is Young's modulus,

$I = EI_0(1+mx)^3$ is the moment of inertia of the cross section,

I_0 and m are constants.

Solution: First we have to derive the fourth-order ordinary differential equations. Let us consider the differential equation $a(x)\frac{d^4 y}{dx^4} + b(x)\frac{d^3 y}{dx^3} + c(x)\frac{d^2 y}{dx^2} + d(x)\frac{dy}{dx} + e(x)y$

$$= \beta_0 \delta(x-\xi) + \beta_1 \delta'(x-\xi) + \beta_2 \delta^{(2)}(x-\xi) + \beta_3 \delta^{(3)}(x-\xi),$$

where we have stopped at the term $\delta'''(x)$ for the sake of simplicity.

Assume that $y_p(x) = G(x)H(x-\xi)$. Differentiating this relation four times we obtain

$$y'_p(x) = G'(x)H(x-\xi) + G(\xi)\delta(x-\xi),$$

$$y''_p(x) = G''(x)H(x-\xi) + G'(\xi)\delta(x-\xi) + G(\xi)\delta'(x-\xi),$$

$$y'''_p(x) = G'''(x)H(x-\xi) + G''(\xi)\delta(x-\xi) + G'(\xi)\delta'(x-\xi) + G(\xi)\delta''(x-\xi),$$

$$y^{IV}{}_p(x) = G^{IV}(x)H(x-\xi) + G'''(\xi)\delta(x-\xi) + G''(\xi)\delta'(x-\xi) + G'(\xi)\delta''(x-\xi) + G(\xi)\delta'''(x-\xi).$$

Thus

$$e(x)y_p(x)=e(x)G(x)H(x-\xi),$$

$$d(x)y'_p(x)=d(x)G'(x)H(x-\xi)+d(\xi)G(\xi)\delta(x-\xi),$$

$$c(x)y''_p(x)=c(x)G''(x)H(x-\xi)+[c(\xi)G'(\xi)-c'(\xi)G(\xi)]\delta(x-\xi)+c(\xi)G(\xi)\delta'(x-\xi),$$

$$b(x)y'''_p(x)=b(x)G'''(x)H(x-\xi)+[b(\xi)G''(\xi)-b'(\xi)G'(\xi)+b''(\xi)G(\xi)]\delta(x-\xi)$$
$$+[b(\xi)G'(\xi)-2b'(\xi)G(\xi)]\delta'(x-\xi)+b(\xi)G(\xi)\delta''(x-\xi),$$

$$a(x)y_p^{iV}(x)=a(x)G^{iV}(x)H(x-\xi)+[a(\xi)G'''(\xi)-a'(\xi)G'(\xi)+a''(\xi)G'(\xi)-a'''(\xi)G(\xi)]\delta(x-\xi)$$
$$+[a(\xi)G''(\xi)-2a'(\xi)G'(\xi)+3a''(\xi)G(\xi)]\delta'(x-\xi)$$
$$+[a(\xi)G'(\xi)-3a'(\xi)G(\xi)]\delta''(x-\xi)+a(\xi)G(\xi)\delta'''(x-\xi).$$

The next step is to substitute these values into given equation, so that

$$(LG)H(x-\xi)+\{a(\xi)G'''(\xi)-[a'(\xi)-b(\xi)]G''(\xi)+[a''(\xi)-b'(\xi)+c(\xi)]G'(\xi)$$
$$-[a'''(\xi)-b''(\xi)+c'(\xi)-d(\xi)]G(\xi)\}\delta(x-\xi)+\{a(\xi)G''(\xi)$$
$$-[2a'(\xi)-b(\xi)]G'(\xi)+[3a''(\xi)-2b'(\xi)+c(\xi)]G(\xi)\}\delta'(x-\xi)$$
$$+\{a(\xi)G'(\xi)-[3a'(\xi)-b(\xi)]G(\xi)\}\delta''(x-\xi)+a(\xi)G(\xi)\delta'''(x-\xi)$$
$$=\beta_0\delta(x-\xi)+\beta_1\delta'(x-\xi)+\beta_2\delta^{(2)}(x-\xi)+\beta_3\delta^{(3)}(x-\xi).$$

We equate the coefficients of generalized functions on both sides of this system to obtain the *IVP* for $G(x)$:

$$LG(x)=a(x)G^{IV}(x)+b(x)G'''(x)+c(x)G''(x)+d(x)G'(x)+e(x)G(x)=0,$$
$$a(\xi)G'''(\xi)-[a'(\xi)-b(\xi)]G''(\xi)+[a''(\xi)-b'(\xi)+c(\xi)]G'(\xi)$$
$$-[a'''(\xi)-b''(\xi)+c'(\xi)-d(\xi)]G(\xi)=\beta_0,$$
$$a(\xi)G''(\xi)-[2a'(\xi)-b(\xi)]G'(\xi)+[3a''(\xi)-2b'(\xi)+c(\xi)]G(\xi)=\beta_1,$$
$$a(\xi)G'(\xi)-[3a'(\xi)-b(\xi)]G(\xi)=\beta_2,$$
$$a(\xi)G(\xi)=\beta_3.$$

Now it is written explicitly, $\frac{d^2}{dx^2}[EI_0(1+mx)^3\frac{d^2y}{dx^2}]=\beta_1\delta'(x-\frac{1}{2}l),$ becomes

$$\frac{d^2}{dx^2}[EI_0(1+mx)^3\frac{d^2y}{dx^2}]=EI_0\frac{d}{dx}[3m(1+mx)^2\frac{d^2y}{dx^2}+(1+mx)^3\frac{d^3y}{dx^3}],$$
$$=EI_0[6m^2(1+mx)\frac{d^2y}{dx^2}+3m(1+mx)^2\frac{d^3y}{dx^3}+3m(1+mx)^2\frac{d^3y}{dx^3}+(1+mx)^3\frac{d^4y}{dx^4}],$$

$$= \beta_1 \delta'(x - \frac{1}{2}l).$$

$$\Leftrightarrow EI_0(1+mx)^3 y^{IV} + 6EI_0 m(1+mx)^2 y''' + 6EI_0 m^2(1+mx)y'' = \beta_1 \delta'(x - \frac{1}{2}l).$$

Here $a(x) = EI_0(1+mx)^3, b(x) = 6EI_0 m(1+mx)^2, c(x) = 6EI_0 m^2(1+mx),$

$$d(x) = e(x) = 0, \beta_0 = \beta_2 = \beta_3.$$

Assume that the particular solution *is* $y_p(x) = G(x)H(x-\xi)$ *with* $\xi = \frac{1}{2}l.$

$a'(x) = 3EI_0 m(1+mx)^2,$

$a''(x) = 6EI_0 m^2(1+mx),$

$a'''(x) = 6EI_0 m^3.$

$b'(x) = 12EI_0 m^2(1+mx),$

$b''(x) = 12EI_0 m^3,$

$c'(x) = 6EI_0 m^3.$

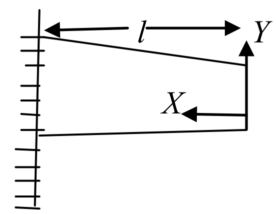

Fig .1.3.The deflection of *a* tempered beam

Then by this system it becomes to

$$G^{IV}(x) + \frac{6m}{1+mx}G'''(x) + \frac{6m^2}{(1+mx)^2}G''(x) = 0,$$

$$EI_0[(1+m\,l/2)^3 G''(l/2) - 6m(1+m\,l/2)(1+m\,l/2 - m)G(l/2)] = \beta_1,$$

$$(1+m\,l/2)^2 - [9m(1+m\,l/2) - 6m^2]G(l/2) = 0,$$

$$G(l/2) = 0.$$

Its solution is $G(x) = c_1 + c_2 x + c_3(1+mx) + c_4 \ln(1+mx).$ From *ICs:*

$$G(l/2) = c_1 + c_2\, l/2 + c_3(1+m\,l/2) + c_4 \ln(1+m\,l/2) = 0,$$

$$G'(l/2) = c_2 + c_3 m + c_4 \frac{m}{1+m\,l/2},\ G''(l/2) = c_4 \frac{m^2}{(1+m\,l/2)^2},\ G'''(l/2) = 2c_4 \frac{m^3}{(1+m\,l/2)^2}.$$

By substitution: $2c_4 \dfrac{m^3}{\left(1+m\, l/2\right)^2} - 3c_4 \dfrac{m^2}{\left(1+m\, l/2\right)^2} = 0 \Leftrightarrow c_4 = 0.$

$$-EI_0\left(1+m\, l/2\right)[6m(1+m\, l/2 - m][c_1 + c_2\, l/2 + c_3(1+m\, l/2)] = \beta_1,$$

$$\left(1+m\, l/2\right)^2[c_2 + c_3] - m\left(1+m\, l/2\right)[9-6m][c_1 + c_2\, l/2 + c_3(1+m\, l/2)] = 0,$$

$$\Leftrightarrow c_1 = -\frac{\beta_1 l}{4mEI_0\left(1+m\, l/2\right)^2} - \frac{\beta_1}{2m^2 EI_0\left(1+m\, l/2\right)},$$

$$c_2 = -\frac{\beta_1}{2mEI_0\left(1+m\, l/2\right)^2}, \qquad c_3 = -\frac{\beta_1 l}{2m^2 EI_0}.$$

Hence $y_p(x) = \dfrac{\beta_1}{EI_0}\left[\dfrac{-1}{2m^2(1+m\, l/2)} - \dfrac{l}{4m(1+m\, l/2)^2} + \dfrac{x}{2m(1+m\, l/2)^2} + \dfrac{1}{2m^2(1+mx)}\right]H(x - l/2).$

Thus the general solution $y(x) = b_1 + b_2 x + b_3(1+mx) + b_4 \ln(1+mx) + y_p(x),$

where b_1, b_2, b_3, b_4 are constants. These constants are obtained with the help of the *BCs* that the beam is free at $x = 0$ and clamped at $x = l$, that is,

$$y(l) = 0, \quad y'(l) = 0, \quad y''(0) = 0, \quad y'''(0) = 0.$$

$$y'(x) = b_2 + mb_3 + b_4 \frac{m}{1+mx} + y_p'(x), \; y''(x) = -b_4 \frac{m^2}{(1+mx)^2} + y_p''(x),$$

$$y'''(x) = 2b_4 \frac{m^3}{(1+mx)^3} + y_p'''(x), \; y^{IV}(x) = -6b_4 \frac{m^4}{(1+mx)^4} + y^{IV}{}_p(x).$$

From *BCs* it yields

$$y(l) = b_1 + b_2 l + b_3(1+ml) + b_4 \ln(1+ml) + y_p(l) = 0,$$

$$y'(l) = b_2 + b_3 ml + b_4 \frac{m}{1+ml} + y'_p(l) = 0,$$

$$y''(0) = b_4 m^2 + y''_p(0) = 0,$$

$$y'''(0) = 2b_4 m^3 + y'''_p(0) = 0,$$

$$y^{IV}(0) = -6b_4 m^4 + y^{IV}{}_p(0) = 0.$$

Thus the solution is

$$y(x) = -\frac{\beta_1}{2mEI_0}[\frac{l}{(1+ml)^2} - \frac{l}{2(1+m\,l/2)^2} - \frac{1}{m(1+ml)} - \frac{1}{m(1+m\,l/2)} + (\frac{1}{(1+ml)^2} - \frac{1}{(1+mx)^2})x$$

$$+(\frac{-1}{m(1+m\,l/2)} - \frac{l}{2(1+m\,l/2)} + \frac{x}{(1+m\,l/2)^2} + \frac{1}{m(1+mx)}]H(x - l/2).$$

3. Show that the infinite series $y(x) = \sum_{n=0}^{\infty} \frac{2^{n+1}\delta^{(n)}(x)}{n!(n+1)!}$ formally satisfies the first-order ordinary differential equation

$$x^2\, dy/dx - 2y = 0.$$

It is interesting to observe that x = 0 is an essential singularity for this differential equation.

Solution:

$$\begin{aligned}
x^2 y' - 2y &= x^2\left(\sum_{n=0}^{\infty} \frac{2^{n+1}\delta^{(n)}(x)}{n!(n+1)!}\right)' - 2\sum_{n=0}^{\infty} \frac{2^{n+1}\delta^{(n)}(x)}{n!(n+1)!}, \\
&= x^2\sum_{n=1}^{\infty} \frac{2^{n+1}\delta^{(n+1)}(x)}{n!(n+1)!} - 2\sum_{n=0}^{\infty} \frac{2^{n+2}\delta^{(n)}(x)}{n!(n+1)!}, \\
&= \sum_{n=1}^{\infty} \frac{2^{n+1}x^2\delta^{(n+1)}(x)}{n!(n+1)!} - \sum_{n=0}^{\infty} \frac{2^{n+2}\delta^{(n)}(x)}{n!(n+1)!}, \\
&= \sum_{n=1}^{\infty} \frac{2^{n+1}(-1)^2(n+1)!\delta^{(n-1)}(x)}{n!(n+1)!(n-1)!} - \sum_{n=0}^{\infty} \frac{2^{n+2}\delta^{(n)}(x)}{n!(n+1)!}, \\
&= \sum_{n=1}^{\infty} \frac{2^{n+1}\delta^{(n-1)}(x)}{n!(n-1)!} - \sum_{n=0}^{\infty} \frac{2^{n+2}\delta^{(n)}(x)}{n!(n+1)!}, \\
&= \sum_{n=0}^{\infty} \frac{2^{n+2}x\delta^{(n)}(x)}{n!(n+1)!} - \sum_{n=0}^{\infty} \frac{2^{n+2}\delta^{(n)}(x)}{n!(n+1)!}, \\
&= 0.
\end{aligned}$$

Hence y satisfies the given differential equation.

V) Conclusions

The space of *test/basic* function is a linear set of test function together with convergence. Any continuous functional on the space test function is called a *generalized* function. The Dirac delta and Heaviside function are functions with jump continuities.

Vector analysis functions with jump discontinuities across surface and boundaries help us to solve many unsolved problems in potential, scattering and wave propagation theories that can not solved by classical methods. Fundamental solution and Green's function are the great tools for solving boundary value problems with jump discontinuity.

There is a new solution of linear homogeneous systems of differential equations with singular coefficients in the space of generalized functions other than classical solution.

Vi) References

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