GENERALIZED SOLUTION OF INHOMOGENEOUS DIFFERENTIAL EQUATIONS

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ABSTRACT

We study the method to find a generalized solution to the linear inhomogeneous differential equations of different orders. Here first, second and fourth order linear inhomogenous differential equations are considered.

We use generalized function for finding the generalized solutions of linear inhomogenous differential equations. Green's function plays a great role for solving such differential equations.

INTRODUCTION

In this paper, we study the method to find a generalized solution to the linear inhomogeneous differential equations of the form

$$L[y] = f(x) \ldots\ldots\ldots\ldots\ldots\ldots (*)$$

Of different orders by finding the general solution of the associated homogenous equation of (*) which is denoted by y_h and we find its Green's function which makes a particular solution to the non-homogeneous equation which is denoted by y_p. Thus the generalized solution of (*) becomes

$$y = y_h + y_p$$

The general solution of the n^{th} order linear inhomogeneous differential equation $L[y] = f(x)$ is given by

$$y = y_p + C_1y_1 + C_2y_2 + \ldots. C_ny_n$$

$$y_p + \sum_{k=1}^{n} ckyk$$

Where y_p is a particular solution, $\{y_1, y_2, \ldots\ldots, y_n\}$ is a set of linearly independent homogenous solutions, (y_h), and the C_k's are arbitrary constants.

Also we study the method to find a general solution to the linear inhomogenous differential equations with boundary conditions of the form.

$$L[y] = f(x),\ B_1(y) = B_2(y) = 0$$

The first chapter, focuses on the method to find a general solution of first order linear inhomegeneous differential equation. The second chapter focuses on the method to find the general solution of 2^{nd} order linear inhomogenous differential equations and the last (3^{rd}) chapter of this paper focuses on finding the general solution of 4^{th} order linear inhomogenous differential equations.

Chapter 1

1. First order Linear inhomogeneous Differential Equations.

Definition :- The first order linear DE is:

$a(x)y' + b(x)y + C(x) = 0$ ------------------- (1)

Where a(x), b(x), c(x) are continuous functions and $a(x)\neq 0$

(1) is called homogeneous if $c(x)\equiv 0$, and inhomogeneous other wise.

In any interval where a(x) does not vanish, the linear DE (1) can be reduced to the normal form as:

$dy/dx + p(x)y + q(x) = 0$ ---------------------------- (2)

With continuous coefficient functions $p = {}^{b}/_{a}$ and $q = {}^{c}/_{a}$.

Theorem 1: If f(x) is one solution of the inhomogeneous linear DE (2), and $g'(x) = p(x)$, then the general solution of (2) is $y = f(x) + ce^{-g(x})$, where c is an arbitrary constant.

Proof: First, we need to find the solution of the homogeneous case of (2), which is

$dy/dx + p(x)y = 0$

$\Rightarrow dy/dx = -p(x)y$

$\Rightarrow dy/y = -p(x)dx$ (variable separation)

$\Rightarrow \ell n \; |y| = -\int p(x)dx + c_1$, c_1 is arbitrary constant

$\Rightarrow y = C_o e^{-\int p(x)dx}$, where $C_0 = e^{c}{}_1$

Hence, the general solution of the homogeneous, y_h, is

$|Y_h| = C_o e^{-\int p(x)dx}$

$\Rightarrow y_h = Ce^{-\int p(x)dx}$, where C is an arbitrary constant.

$\Rightarrow y_h = Ce^{-g(x)}$, since $g'(x) = p(x)$.

Also the particular solution, y_p is

$y_p = f(x)$

Thus, the general solution of (2) is

$y = y_p + y_h$

$\Rightarrow y = f(x) + Ce^{-g(x)}$, where C is arbitrary constant.

1.1 ***Initial Value Problem***

The "initial value problem" for a first order DE $y' = F(x,y)$ consists in finding a solution $y = h(x)$ that satisfies an initial condition $y(a) = y_o$, where a and y_o are given constants. Thus, if we know a particular solution f(x) of an

inhomogeneous linear DE(2), then by setting $C = e^{g(x)}\,[y_o - f(a)]$ in theorem 1, we get a solution of the stated initial value problem.

Theorem: The first order inhomogeneous differential equation with homogeneous initial condition.

$L[y] = y' + p(x)y = f(x)$, for $x>a$, $y(a) = 0$

has the solution $y = \int_a^{\infty} G(x,\varepsilon) f(\varepsilon) d\varepsilon$

where $G(x,\varepsilon)$ satisfies the equation

$L[G(x, \varepsilon)] = \delta(x-\varepsilon)$, for $x>a$, $G(a, \varepsilon) = 0$

The Green function is $G(x, \varepsilon) = e^{\int_{\varepsilon}^{x} p(t)dt} H(x-\varepsilon)$

Proof: Applying the linear operator L to the integral, we have

$$L\left[\int_a^{\infty} G(x,\varepsilon) f(\varepsilon) d\varepsilon\right] = \int_a^{\infty} L[G(x,\varepsilon)] f(\varepsilon) d\varepsilon, \text{ since G is continuous}$$

$$= \int_a^{\infty} \delta(x-\varepsilon) f(\varepsilon) d\varepsilon$$

$$= f(x)$$

The integral also satisfies the initial condition:

$$B\left[\int_a^\infty G(x,\varepsilon)f(\varepsilon)d\varepsilon\right] = \int_a^\infty B[G(x,\varepsilon)]\, f(\varepsilon)d\varepsilon, \text{ since G is continuous}$$

$$= \int_a^\infty (0)f(\varepsilon)d\varepsilon$$

$$= 0$$

For $x\neq\varepsilon$, the Green function is simply a homogeneous solution of the differential equation.

However, at $x = \varepsilon$ we expect some singular behavior. $G'(x, \varepsilon)$ will have a dirac delta function type singularity.

That is, $G(x, \varepsilon)$ will have a jump discontinuity at $x = \varepsilon$.

We integrate the DE on the vanishing interval (ε^----ε^+) to determine this jump.

$$G' + p(x)\, G = \delta(x-\varepsilon)$$

$$G(\varepsilon^+, \varepsilon) - G(\varepsilon^-,\varepsilon) + \int_{\varepsilon-}^{\varepsilon+} p(x)G(x,\varepsilon)dx = 1$$

$$\Rightarrow G(\varepsilon^+,\varepsilon) - G(\varepsilon^-,\varepsilon) = 1$$

The homogeneous solution of the differential equation is

$$y_h = e^{-\int p(x)dx}$$

Since the Green function satisfies the homogeneous equation for $x\neq\varepsilon$, it will be a constant times this homogeneous solution for $x<\varepsilon$ and $x>\varepsilon$

$$G(x,\varepsilon) = \begin{cases} C_1 e^{-\int p(x)dx}, & a<x<\varepsilon \\ C_2 e^{-\int p(x)dx}, & x>\varepsilon \end{cases}$$

In order to satisfy the homogeneous initial condition $G(a, \varepsilon) = 0$, the Green function must vanish on the interval (a-----ε)

$$G(x, \varepsilon) = \begin{cases} 0, & a<x<\varepsilon \\ C\, e^{-\int p(x)dx}, & x>\varepsilon \end{cases}$$

$\Rightarrow G(\varepsilon^{+}, \varepsilon) = 1$

This determines the constant in the homogeneous solution for x>ε.

$$G(x, \varepsilon) = \begin{cases} 0, & 0<x<\varepsilon \\ e^{-\int_{\varepsilon}^{x} p(t)dt}, & x>\varepsilon \end{cases}$$

$$\Rightarrow G(x, \varepsilon) = e^{-\int_{\varepsilon}^{x} p(t)dt}\, H(x-\varepsilon)$$

Example:- Find the general solution of the DE

$$y' + 2y = 2x^2 + 4x + 7$$

With homogeneous initial condition $y_h(0) = 1$

Solution:- here p(x) = 2

Now from the above theorem, the homogeneous solution, y_h is

$$y_h = e^{-\int p(x)dx} = e^{-\int 2dx}\, e^{-\int 2dx}$$

$\Rightarrow yh = Ce^{-2x}$, C is arbitrary constant.

Since $y_h(0) = 1$

$\Rightarrow C = 1$

Thus, $y_h = e^{-2x}$

Thus, the Greens function G(x, ε) is

$$G(x, \varepsilon) = \begin{cases} 0, & 0<x<\varepsilon \\ e^{-2(x-\varepsilon)}, & \varepsilon<x \end{cases}$$

$\Rightarrow G(x, \varepsilon) = e^{-2(x-\varepsilon)} \,.\, H(x-\varepsilon)$, $G(0, \varepsilon) = e^{2\varepsilon} H(-\varepsilon) = 0 \Rightarrow \varepsilon = 1$, for ε>0

Thus, the general solution to $y' + 2y = x^2 + 4x + 7$ is

$$y(x) = \int_0^\infty G(x,\varepsilon) f(\varepsilon) d\varepsilon = \int_0^\infty e^{-2(x-\varepsilon)} f(\varepsilon) d\varepsilon$$

$$= e^{-2x} + x^2 + x + 3$$

Chapter Two

2. Second Order Linear Inhomogeneous differential equations.

Def:- The second order linear DE is

$p_0(x) d^2y/_{dx}{}^2 + p_1(x)^{dy}/_{dx} + p_2(x)y = p_3(x)$ ---------------- (1)

Where $p_j(x)$ [j = 0,1,2,3] are continuous functions and $P_0(x) \neq 0$

(1) is called homogeneous if $P_3(x) \equiv 0$, otherwise it is inhomogeneous.

Dividing through (1) by the leading coefficient $p_0(x)$, one obtains the normal form:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

where $P = P_1/P_0$, $q = P_2/P_0$, $r = p_3/p_0$

2.1 ***Variation of parameters***

Consider the second order case of $L[y] = f(x)$ and suppose $\{y_1, y_2\}$ is a fundamental set. Then $C_1y_1(x) + C_2y_2(x)$ is the general solution of $L[y] = 0$. A method due to Lagrange for solving $L[y] = f(x)$ is based on the idea of seeking a solution as:

$y_p(x) = C_1(x)y_1(x) + C_2(x)y_2(x)$

Then

$y'_p = C_1y'_1 + C_2y'_2 + C'_1y_1 + C'_2y_2$

To simplify the algebra, we impose the auxiliary condition

$C'_1y' + C'_2y_2 = 0$

Then

$$y''_p = C_1y''_1 + C_2y''_2 + C'_1y'_1 + C'_2y'_2$$

If we substitute into L[y] = f, we want

$$C_1(x)\,(y''_1+a_1 y'_1+a_o y_1) + C_2(x)\,(y''_2+a_1 y'_2+a_o y_2) + C'_1y'_1+C'_2y'_2 = f(x)$$

Note that the two-parenthesized expressions are zero because y_1 and y_2 are solutions of the homogeneous equation. Thus we need to solve

$$C'_1y_1 + C'_2y_2 = 0$$

$$C'_1y'_1 + C'_2y'_2 = f$$

By Cramer's Rule

$$C'_1(x) = \frac{-y_2(x)f(x)}{W(y_1,y_2)(x)}, \qquad C'_2(x) = \frac{y_1(x)f(x)}{W(y_1,y_2)(x)}$$

Thus a particular solution is given as

$$y_p(x) = -y_1(x)\int_{x_0}^{x}\frac{y_2(t)f(t)dt}{w(t)} + y_2(x)\int_{x_0}^{x}\frac{y_1(t)f(t)dt}{w(t)}$$

$$= \int_{x_0}^{x}\frac{y_1(x)y_2(t)-y_1(t)y_2(x)}{w(y_1,y_2)(t)}f(t)dt$$

$$= \int_{x_0}^{x} g(s,t)f(t)dt$$

g (x,t) is called the Fundamental solution.

Example:- Consider the problem

$y''+y = 1$ ----------------------- (1)

Find the general solution, using variation parameter.

Solution: - First we consider the homogenous part

$y'' + y = 0$ ---------------------- (2)

So we can take $y_1 = \cos x$, $y_2 = \sin x$ be the solutions of (2)

Thus the Wronskian is

$$W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

Thus by the variation of parameters formula:

$$y_p(x) = -y_1(x) \int_{x_0}^{x} \frac{y_2(t) f(t) dt}{w(t)} + y_2(x) \int_{x_0}^{x} \frac{y_1(t) f(x) dt}{w(x)}$$

$$= -\text{Cosx} \int_{x_0}^{x} \frac{S \text{int}\, dt + \sin}{1} x \int_{x_0}^{x} \frac{\cos t.1 dt}{1}$$

$$= -\text{Cosx} \int_{x_0}^{x} S \text{int}\, dt + \sin \int_{x_0}^{x} \cos t dt$$

$= \cos^2 x + \sin^2 x$

$= \underline{\underline{1}}$

2.2 ***Method of undetermined coefficients***

We consider here only linear DE'S with constant coefficients

That is $y'' + ay' + by = f(x)$

Where a,b are constants and f(x) is a polynomial function, exponential function, sine and cosine and sum and product of these functions.

The following table contains a guide for generating a particular solution when one applies the method of undetermined coefficients.

In particular consider.

Terms in f(x)	Choice for $y_p(x)$	Should not be the root of the characteristic equation
Polynomial $P_m(x) = a_o x^m + --- + a_m$	$a_o x^m = ---- + a_m$	0
$Ae^{\alpha x}$	$Ae^{\alpha x}$	α
A $\cos\beta x$ or A $\sin\beta x$	A $\cos \beta x$ + B $\sin\beta x$	β

Example 1. Find the particular solution y_p of

$y'' + 4y = 8x^2$ -------------------- (*)

Solution: $y_p = Ax^2+Bx+C$ (As defined from the previous table)

Now $y'_p = 2Ax+B$, $y''_p = 2A$

Substituting these into (*) we get

$y''_p + 4y_p = 8x^2$

$\Rightarrow 2A + 4(ax^2+Bx+C) = 8x^2$

$\Rightarrow A = 2, B = 0, C = -1$

$\therefore\ y_p = 2x^2-1$

Example 2. If $y'' - y' - 2y = 10\cos x$, ... (**)

Then solve the particular solution y_p.

Solution: The roots are

$m_1 = 2$, $m_2 = -1$ and $\beta = 1$ is not the root of the characteristic equation

Thus, $y_p = A \cos x + B \sin x$

$\Rightarrow y'_p = -A \sin x + B \cos x,\ y''_p = -a \cos x - B \sin x$

Substituting thus into (**) we get

$- A \cos x - B \sin x -(-A \sin x + B \cos x) -2 (A \cos x + B \sin x) = 10 \cos x$

$\Rightarrow (-3A-B) \cos x + 9A-3B) \sin x = 10 \cos x$

$\Rightarrow A = -3, B = -1$

$\therefore y_p = -3 \cos x - \sin x$

2.3 ***Boundary Value- Problem***

Initial value problems for second order liner differential equations are those in which we specify for a solution $y = y(x)$ the values $y(a)$ and $y'(a)$ at a single point $x = a$. We have that under a reasonable hypothesis such problems always have a unique solution.

If instead we try to specify the solution values $y(a)$ and $y(b)$ at two different points $x = a$ and $x = b$, which is a boundary value problem (BVP), then variety of outcomes appear if we ask for existence of a solution.

That is, there may be a unique solution, there may be infinitely many solutions or there may be no solution at all.

Theorem: Let the linear homogeneous differential equation

$y'' + p(x) y' + q(x)y = f(x)$ $p, q\in C^o[a,b]$ and Let α and β be

arbitrary numbers. Then the differential equation has a unique solution

satisfying both $y(a) = \alpha$ and $y(b) = \beta$ if and only either one or the other of the following equivalent statements hold

(a) The only solution satisfying $y(a) = 0$ is $y\equiv 0$

(b) For two independent solutions y_1 and y_2 we have

$$\begin{vmatrix} y_1(a) & y_2(a) \\ y_1(b) & y_2(b) \end{vmatrix} \neq 0$$

i.e. $y_1(a)\, y_2(b) - y_1(b)\, y_2(a)\neq 0$

Proof: Let the general solution of the differential equation is given by

$y(x) = C_1y_1(x) + C_2y(x)$

The boundary conditions $y(a) = \alpha$, $y(b) = \beta$

Thus, substituting we get

$C_1y_1(a)+C_2y_2(b) = \alpha$

$C_1y_1(b)+C_2y_2(b) = \beta$ --------------------- (2.3)

Using Cramer's rule it is easy to show that there equations (2.3) have a unique solution for C1 and C2 if and only if

$y_1(a)y_2(b) - y_2(a)y_1(b) \neq 0$

But the condition that the equations

$C_1y_1(a)+C_2y_2(a) = 0$

$C_1y_1(b)+C_2y_2(b) = 0$ have only the trivial solution. i.e. $C_1 = C_2 = 0$

Now we consider BVP for the non-homogenous equation with continuous coefficients p(x), q(x) and f(x).

$y''+P(x)\,y'=q(x)y = f(x)$ ------------------------ (i)

The key is first to find a solution satisfying the special homogeneous boundary conditions.

$y(a) = 0, y(b) = 0$

It turns out that this can always be done provided that a and b are not conjugate points for the differential equation. The term conjugate point is used to denote pair of boundary points for which there fails to exist a unique solution.

Using variation of parameters, let us find first the particular solution of (i)

Let y_1 and y_2 be two linearly independent solution to the associated homogeneous equation. Form $y(x) = y_1(x)u_1(x)+y_2(x)u_2(x)$ where u_1 and u_2 are to be determined so that y(x) is a solution of the non-homogenous equation. Note that if u_1 and u_2 are constants, y(x) would be a solution of the associated homogenous equation.

Compute y' and y'' and substitute into (i)

Then rearrange terms as follows:

$(y_1''+Py'_1+qy_1)u_1+(y_2''+p\,y'_2+qy_2)u_2+(y_1u'_1+y_2u'_2)' +P(y_1u'_1+y_2u'_2)+$

$(y'_1u'_1+ y'_2u'_2) = f$

The first two collections of terms are zero, because y_1 and y_2 are homogenous solutions. What remains of the equation will be satisfied if we choose u_1 and u_2 so that the two equations.

$y_1u'_1+y_2u'_2 = 0$

$y'_1u'_1+ y'_2u'_2 = f$ ------------------------- (ii)

hold for all x on the interval in question. These little system of equations can be solved for u_1 and u_2 by elimination, with the result

$$u'_1(x) = \frac{-y_2(x)f(x)}{y_1(x)y'_2(x) - y_2(x)y'_1(x)}$$

$$u'_2(x) = \frac{y_1(x)f(x)}{y_1(x)y'_2(x) - y_2(x)y'_1(x)}$$

The expression in the denominators is the same in both formulas and can be written as

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix} = y_1(x)\,y'_2 x) - y''_2(x)\,y'_1(x)$$

Called the wronskian of y_1 and y_2

To complete the solution, integrate the formulas for u_1 and u_2 to find u_1 and u_2 and then combine with y_1, y_2 to get particular solution.

$y_p(x) = y_1(x)u_1(x)+y_2(x)u_2(x)$

$$y_p(x) = y_1(x)\int -\frac{y2(x)f(x)}{w(x)}dx + y2(x)\int \frac{y1(x)f(x)dx}{w(x)}$$

By choosing the constants of integration properly in the two integrals we can get the most general solution of the non-homogeneous differential equation.

In particular we can specify these constants by using integrals with limits of integration as follows:

$y_p(x) = -y_1(x) \int_b^x \frac{y_2(t) f(t) dt}{w(t)} + y_2(x) \int_a^x \frac{y_1(t) f(t) dt}{w(t)}$

$\Rightarrow y_p(x) = \int_a^x \frac{y_2(x) y_1(t) f(t) dt}{w(t)} + \int_x^b \frac{y_1(x) y_2(t) f(t) dt}{w(t)}$ + ------------- (iii)

Now suppose that the linearly independent solutions y_1 and y_2 have been chosen so that $y_1(a) = 0$ and $y_2(b) = 0$. It then follows by setting x = a and

x = b respectively, that $y_p(a) = 0$ and $y_p(b) = 0$ thus we have found a particular solution satisfying the preceding boundary condition.

We denote by G the function of two variables defined for $a \le x \le b$, $a \le t \le b$ by

$$G(x,t) = \begin{cases} \dfrac{y_2(x) y_1(t)}{w(t)}, & a \le t \le x \\ \dfrac{y_1(x) y_2(t)}{w(t)}, & x \le t \le b \end{cases}$$

Where $y_1(a) = y_2(b) = 0$ and $w(t) = y_1(t) y'_2(t) - y_2(t) y'_1(t)$ is the Wronskian of $y_1(t)$ and $y_2(t)$.
Then G is called a Green's function for the BVP and by equation (iii) we can write

$y_p(x) = \int_a^b G(x,t) f(t) dt$ for the solution of

$y'' + P(x) y' + q(x) y = f(x)$, $y(a) = y(b) = 0$

Example: Let $y'' - {}^2/_x y' + {}^2/_{x^2} y = x\cos x$, $x > 0$

Find a particular solution to the non-homogeneous

Solution: Homogeneous solutions are

$y_1(x) = x$ and $y_2(x) = x^2$

Thus, their wronskian is

$w(x) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2 \neq 0$, since $x > 0$

Hence $y_p(x) = x \int \frac{-x^2.x\cos x}{x^2} dx + x^2 \int \frac{x.x\cos x}{x^2} dx$

$\Rightarrow y_p(x) = -x \cos x$

To solve the non-homogeneous equation

$$y'' + p(x)y' + q(x)y = f(x)$$

with general non-homogeneous boundary conditions $y(a) = \alpha$, $y(b) = \beta$, where α and β are both not zero, we continue to assume that a and b are not conjugate points of the associated homogeneous equation. Then first solve the equation with homogeneous boundary conditions $y(a) = 0$, $y(b) = 0$ by the method discussed above say it y_p.

Now the general solution of the non-homogeneous equation is

$y(x) = C_1y_1(x) + C_2y_2(x) + y(x)$, where y_1 and y_2 are linearly independent homogeneous solutions and C_1 and C_2 are constants

Since

$y_p(a) = 0$, $y_p(b) = 0$, all we have to do to satisfy

$y(a) = \alpha$, $y(b) = \beta$ is choose C_1 and C_2 so that

$$C_1y_1(a) = C_2y_2(a) = \alpha$$

$$C_1y_1(b) + C_2y_2(b) = \beta$$

This can be done only if a and b are not conjugate points

Example: The differential equation $y'' + y = 1$ ---------- (*)

has linearly independent homogeneous solutions cosx and sinx.

To satisfy boundary conditions of the form $y(0) = \alpha$, $y(^{\pi}/_{2}) = \beta$, we can check that the points $x = 0$ and $x = {}^{\pi}/_{2}$ are not conjugate by observing

$$\begin{vmatrix} \cos 0 & \cos {}^{\pi}/_{2} \\ \sin 0 & \sin {}^{\pi}/_{2} \end{vmatrix} = 1$$

Rather than constructing a Green[s function we can observe by inspection that y(x) = 1 is a particular solution to(*).

So

$y(x) = C_1 \cos x + C_2 \sin x + 1$ is the most general solution of the differential equation.

To satisfy $y(0) = \alpha$, $y(^{\pi}/_2) = \beta$, we solve the equation

$$\begin{cases} C_1 \cos 0 + C_2 \sin 0 + 1 = \alpha \\ C_1 \cos{}^{\pi}/_2 + C_2 \sin{}^{\pi}/_2 + 1 = \beta \end{cases}$$

for C_1 and C_2 to get $C_1 = \alpha - 1$, $C_2 = \beta - 1$

Thus $y(x) = (\alpha-1) \cos x + (\beta-1) \sin x +1$ is the unique solution satisfying the boundary conditions.

On the other hand, the points $x = 0$, $x = \pi$ are conjugate for $y'+y = 0$, because $\sin x$ is a non-trivial solution so that is 0 at both points. Thus can attempt to solve,

$$\begin{cases} C_1 \cos 0 + C_2 \sin 0 + 1 = \alpha \\ C_1 \cos\pi + C_2 \sin\pi +1 = \beta \end{cases}$$

Leads to $C_1 = \alpha -1$ and $-C_1 = \beta-1$. Thus there is no solution to the BVP with boundary conditions $y(0) = \alpha$, $y(\pi) = \beta$ unless $\alpha - 1 = -\beta+1 \Rightarrow \alpha + \beta = 2$

In this case, there are infinitely many solutions all of the form

$y(x) = (\alpha - 1) \cos x + C_2 \sin x+1$ Where C_2 is arbitrary.

Theorem: The second order inhomogeneous differential equation with homogeneous boundary conditions.

$L[y] = y''+P(x)y'+q(x)y= f(x)$, for $a<x<b$, $B_1[y] = B_2[y] = 0$, has the solution.

$$y = \int_a^b G(x,\varepsilon) f(\varepsilon) d\varepsilon$$

Where $G(x, \varepsilon)$ satisfies the equation

$L[G(x, \varepsilon)] = \delta(x-\varepsilon)$, for $a<x<b$, $\beta_1[G(x, \varepsilon)] = \beta_2[G(x, \varepsilon)] = 0$

$G(x, \varepsilon)$ is contenstions and $G'(x, \varepsilon)$ has a jump discontinuity of height 1 at

$x = \varepsilon$

Proof: [essentially same procedure as 1st order case]

Example: Solve the boundary value problem

$y'' = f(x)$ with boundary conditions $y(0) = y(1) = 0$

Solution: $y_1 = 1$ and $y_2 = x$ are a pair of linearly independent solutions for

$y'' = 0$

Only $y \equiv 0$ (trivial) solution satisfies the homogeneous boundary conditions.

Hence, there is a unique solution to this problem.

The Green function satisfies

$G''(x, \varepsilon) = \delta(x-\varepsilon, G(0, \varepsilon) = G(1, \varepsilon)= 0$

$$\Rightarrow G(x, \varepsilon) = \begin{cases} C_1+C_2x \text{ , for } x<\varepsilon \\ d_1+d_2x \text{ , for } x>\varepsilon \end{cases}$$

Applying the two boundary conditions, we get

$C_1 = 0$ and $d_1 = -d_2$

$$\Rightarrow G(x, \varepsilon) = \begin{cases} cx \text{ , for } x<\varepsilon \\ d(x-1) \text{ , for } x>\varepsilon \end{cases}$$

Since Green's function is continuous, we have

$C\varepsilon = d(\varepsilon-1) \Rightarrow d = c\varepsilon/\varepsilon-1$

From the jump condition

$$\frac{d}{dx}\frac{c\varepsilon(x-1)}{\varepsilon-1}\Bigg|_{x=\varepsilon} - \frac{d}{dx}cx\Bigg|_{x=\varepsilon} = 1$$

$\Rightarrow$ cε/ε-1 - c = 1

$\Rightarrow$ c = ε - 1

$\Rightarrow$ d = ε

$$\text{Thus } G(x, \varepsilon) = \begin{cases} (\varepsilon-1)x, & \text{for } x<\varepsilon \\ \varepsilon(x-1), & \text{for } x>\varepsilon \end{cases}$$

Thus, the general solution to $y'' = f(x)$ is

$$y(x) = \int_0^1 G(x,\varepsilon) f(\varepsilon) d\varepsilon$$

$$= (x-1)\int_0^x \varepsilon f(\varepsilon) d\varepsilon + x\int_x^1 (\varepsilon-1) f(\varepsilon) d\varepsilon$$

2.4 ***Variable Coefficients***

Let us consider the differential equation

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = \sum_{n=0}^{N} \beta n \delta^{(n)}(x-\varepsilon)$$

Where $\delta^{(n)}{}_{(x)}$ is the n^{th} derivative of $\delta(x)$

To fix the ideas, we shall take N = 1 So that we have only to solve

$L[y] = a(x)y'' + b(x)y' + c(x)y = \beta_o\delta(x-\varepsilon) + \beta_1\delta^1(x-\varepsilon)$ -------------- (1)

Since this is non-homogeneous equation, its solution is expressed as

$y(x) = y_h(x) + y_p(x)$ -- (2)

Where $y_h(x)$ is the solution of the homogeneous equation L[y] = 0 and $y_p(x)$ is the particular solution.

The part $y_h(x)$ is the same as we study in the classical analysis: we therefore attend only to the particular solution, which we assume to be

$y_p(x) = G(x)H(x-\varepsilon)$ --- (3)

Where H(x) is the Heaviside function and G(x) is the unknown function.

Now:

$y'_p(x) = G'(x)H(x-\varepsilon) + G(\varepsilon)\delta(x-\varepsilon)$ ------------------------------ (4a)

$y''_p(x) = G'(x)H(x-\varepsilon) + G'(\varepsilon)\delta(x-\varepsilon) + G(\varepsilon)\delta'(x-\varepsilon)$ ---------- (4b)

Thus

$b(x)y'_p(x) = b(x)G'(x)H(x-\varepsilon) + b(\varepsilon)G(\varepsilon)\delta(x-\varepsilon)$ -------------- (5a)

$a(x)y''_p(x) = a(x)G''(x)H(x-\varepsilon) + a(x)G'(\varepsilon)\delta(x-\varepsilon) + a(x)G(\varepsilon)\delta'(x-\varepsilon)$

$= a(x)G''(x)H(x-\varepsilon) + a(\varepsilon)G'(\varepsilon)\delta(x-\varepsilon) + G(\varepsilon)[-a'(\varepsilon)\delta(x-\varepsilon) +$

$a(\varepsilon)\delta'(x-\varepsilon)]$

$= a(x)G''(x)H(x-\varepsilon) + [a(\varepsilon)G'(\varepsilon) - a'(\varepsilon)G(\varepsilon)]\delta(x-\varepsilon) + a(\varepsilon)G(\varepsilon)\delta'(x-\varepsilon)$ ---- (5b)

Substituting (5) in (1), we get

$a(x)G''(x)H(x-\varepsilon) + [a(\varepsilon)G'(\varepsilon) - a'(\varepsilon)G(\varepsilon)]\delta(x-\varepsilon) + a(\varepsilon)G(\varepsilon)\delta'(x-\varepsilon) + b(x)G'(x)H(x-\varepsilon)+ b(\varepsilon)G(\varepsilon)\delta(x-\varepsilon) + c(x)G(x)H(x-\varepsilon)$

$= \beta_o\delta(x-\varepsilon) + \beta_1\delta'(x-\varepsilon)$

Or

$(LG(x))H(x-\varepsilon) + [a(\varepsilon)G'(\varepsilon) - a'(\varepsilon)G(\varepsilon)+b(\varepsilon)G(\varepsilon)]\, \delta(x-\varepsilon) + a(\varepsilon)G(\varepsilon)\delta'(x-\varepsilon)$

$= \beta_o\delta(x-\varepsilon) + \beta_1\delta'(x-\varepsilon)$ --- (6)

Equating derivatives of $\delta(x)$ of equal order on both sides of this equation, we have

$LG = a(x)G''(x) + b(x)G'(x) + c(x)G(x) = 0,$ --------------------------------------- (7a)

$a(\varepsilon)G'(\varepsilon+ (b(\varepsilon) - a'(\varepsilon))G(\varepsilon) = \beta_o$ -- (7b)

$a(\varepsilon)G(\varepsilon) = \beta_1$ -- (7c)

System (7) is the initial value problem for G(x) and can be solved by the classical method, there by solving the original problem completely.

For the special case when $\beta_1 = 0$, we have the fundamental solution $E(x-\varepsilon)$.

$E(x-\varepsilon) = G(x)H(x-\varepsilon)$ -- (8)

Example: Solve the following harmonic oscillator

$y'' + K^2 y = \delta(x-\varepsilon)$ --- (9)

Solution: comparing (9) with (1), we find that

$a(x) = 1,\ b(x) = 0,\ c(x) = K^2$

$\beta_o = 1,\ \beta_1 = 0$

Thus the system (7) reduces to

$G''(x) + K^2 G(x) = 0$ ---------------------------------- (10a)

$G'(\varepsilon) = 1$ -------------------------------------- (10b)

$G(\varepsilon) = 0$ --(10c)

The general solution of (10a) is

$G(x) = A \sin(kx+B)$ ------------------------ (11)

Then application of the initial conditions (10b) and (10c) yields

$G(\varepsilon) = A \sin(k\varepsilon+B) = 0$, so that $B = -k\varepsilon$ and

$AK \cos(k\varepsilon+B) = Ak \cos 0 = 1$

Thus $A = {}^1/_k$

Substituting these values in (11), we obtain

$G(x) = {}^1/_k \sin k(x-\varepsilon)$

And the general solution of (9) and the fundamental solution are

$$y(x) = \alpha \sin(kx+B) + [{}^{1}/_{k} \sin k(x-\varepsilon)]H(x-\varepsilon)$$

and $E(x-\varepsilon) = [{}^{1}/_{k} \sin k(x-\varepsilon)]\, H(x-\varepsilon)$ respectively.

Example: Consider

$$(1-x^2)y''-2xy' = \delta(x-\varepsilon) \text{ -- } (12)$$

Solve its general solution

Solution: Here $a(x) = 1-x^2$, $b(x) = -2x$, $c(x) = 0$

$\beta_o = 1, \beta_1 = 0$

System (7) becomes

$$(1-x^2)G''(x) - 2xG'(x) = 0 \text{ -- } (13a)$$

$$(1-\varepsilon^2)G'(\varepsilon) = 1 \text{ -- } (13b)$$

$$(a-\varepsilon^2)\, G(\varepsilon) = 0 \text{ -- } (13c)$$

The general solution of (13a) is

$$G(x) = A\, \ell n \frac{1+x}{1-x} + B \text{ --- } (14)$$

Initial condition (13c) yields

$$A\, \ell n \frac{1+\varepsilon}{1-\varepsilon} + B = 0 \text{ -- } (15)$$

Since $G'(x) = A\, ({}^{1}/_{1tx} + {}^{1}/_{1-x}) = 2A/1-x^2$, from condition (13b) we find that

$(1-\varepsilon^2)\ {}^{2A}/_{1-\varepsilon}{}^{2} = 1 \ \ \Rightarrow A = ½$

Then (15) gives the value of B as

$$B = -½\, \lambda n\, 1+\varepsilon/1-\varepsilon = ½\, \lambda n\, 1^{-}\varepsilon/1+\varepsilon$$

Substituting these values of A and B in (14) we obtain

$$G(x) = \tfrac{1}{2}[\lambda n\ 1^{-}\varepsilon/1+\varepsilon\ .\ 1+x/1-x] \text{ --- } (16)$$

So the general solution of (12) is

$$y = \alpha\ \lambda n\ 1+x/1-x + \beta + 1/2\ (\lambda n\ 1-\varepsilon/1+\varepsilon\ .\ 1+x/1-x)H(x-\varepsilon) \text{ -------------- } (17)$$

The fundamental solution $E(x, \varepsilon)$ is

$$E(x-\varepsilon) = E(x, \varepsilon) = \tfrac{1}{2}\ (\lambda n\ 1-\varepsilon\ /1+\varepsilon\ .\ 1+x/1-x)H(x-\varepsilon)$$

Example: Let us extend the method of the previous examples to solve the ''

$$y'' + 3y' + 2y = \delta'''x) \text{ -- } (18)$$

Solution: It is clear that the terms $\delta(x)$ and $\delta'(x)$ will appear in the expression for $y(x)$. However, we need only assume that the coefficients of $\delta(x)$ and $\delta'(x)$ are constants. Accordingly, we assume

$$y(x) = G(x)H(x) + a\delta(x) + b\delta'(x) \text{ ------------------------------------- } (19)$$

Differentiating twice we obtain

$$y'(x) = G'(x)H(x) + G(0)\delta(x) + a\delta'(x) + b\delta''(x)$$

$$y''(x) = G''(x)H(x) + G'(0)\delta(x) + G(0)\delta'(x) - a\delta''(x) + b\delta'''(x)$$

Substitution of these in (18) yields the following system:

$$G''(x) + 3G'(x) + 2G(x) = 0,$$

$$G'(0) + 3G(0) + 2a = 0,$$

$$G(0) + 3a + 2b = 0,$$

$$a + 3b = 0,$$

$$b = 1$$

Solving this system we obtain

$$G(x) = -e^{-x} + 8e^{-2x},\ a = -3,\ b = 1,$$

So, from (19) the particular solution is

$$y(x) = (-e^{-x} + 8e^{-2x})H(x) - 3\delta(x) + \delta'(x)$$

2.5 CONSTANT COEFFICIENTS THIS SIMPLEST EQUATION OF THIS TYPE IS

$$d^2E/dx^2=\delta(x-\varepsilon) \qquad (1)$$

Since $H'X) =\delta(x)$, integration of (1) gives

$$dE(x, \varepsilon)/dx= H(x- \varepsilon)+\alpha(\varepsilon), \qquad (2)$$

Where $\alpha(\varepsilon)$ is an arbitrary function. Next, we integrate (2) and obtain

$$E(x, \varepsilon)=\int_{-\infty}^{x} H(x-\varepsilon)dx + x\propto(\varepsilon) + \beta(\varepsilon)$$

$$= (x- \varepsilon)H(x- \varepsilon)+x\alpha(\varepsilon)+ \beta(\varepsilon) \qquad (3)$$

Where $\beta(\varepsilon)$ is another arbitrary function. It is easily verified that (3) satisfies (1). Equation (3) shows that $E(x,\varepsilon)$ is a continuous and piecewise differentiable function.

This solution helps us in solving the inhomogeneous equation

$$d^2t/dx^2 = \tau(x), \qquad (4)$$

Where $\tau(x)$, is a distribution with compact support. Indeed,

$$\tau(x)=E * \tau \qquad (5)$$

As is easily verified by operating on both sides of (5) with d^2/dx^2.

Let us see if we can solve with the help of the above analysis an initial or boundary value problem in the classical theory of ordinary differential equations.

Theorem:- the general solution of the boundary value problem

$$d^2u/dx^2=f(x),\ o\le x \le 1,\ u(0)=u(1) = 0, \qquad (6)$$

Is $u(x)=\int_0^x G(x,\varepsilon)f(\varepsilon)d\varepsilon$

Where f(x) is an integrable function with compact support.

Proof:- appealing to relations (3) and (5), we find that

$$U(x) = \int_{-\infty}^{\infty}(x-\varepsilon)H(x-\varepsilon)f(\varepsilon)d\varepsilon + x\int_{-\infty}^{\infty}\propto(\varepsilon)f(\varepsilon)d\varepsilon + \int_{-\infty}^{\infty}\beta(\varepsilon)f(\varepsilon)d\varepsilon$$

$$= \int_{-\infty}^{x}(x-\varepsilon)f(\varepsilon)d\varepsilon + x\int_{-\infty}^{\infty}\propto(\varepsilon)f(\varepsilon)d\varepsilon + \int_{-\infty}^{\infty}\beta(\varepsilon)f(\varepsilon)d\varepsilon.$$

The boundary conditions (6) then give

$$0 = -\int_{-\infty}^{0}\varepsilon f(\varepsilon)d\varepsilon + \int_{-\infty}^{\infty}\beta(\varepsilon)f(\varepsilon)d\varepsilon. \qquad (7)$$

$$0 = \int_{-\infty}^{1}(1-\varepsilon)f(\varepsilon)d\varepsilon + \int_{-\infty}^{\infty} \propto(\varepsilon)f(\varepsilon)d\varepsilon + \int_{-\infty}^{\infty}\beta(\varepsilon)f(\varepsilon)d\varepsilon. \qquad (8)$$

From (7) we find that $\beta(\varepsilon) = \varepsilon H(-\varepsilon)$, then (8) yields

$$\propto(\varepsilon) = \begin{cases} -1+\varepsilon H(\varepsilon), & -\infty \le \varepsilon \le 1 \\ 0, & \varepsilon > 1. \end{cases}$$

Thus the required solution is

$$u(x) = \int_0^x (x-\varepsilon)f(\varepsilon)d\varepsilon - x\int_0^1(\mathbf{1}-\varepsilon)f(\varepsilon)d\varepsilon$$

$$= \int_0^1 (x-\varepsilon)H(x-\varepsilon)f(\varepsilon)d\varepsilon - \int_0^1 x(1-\varepsilon)f(\varepsilon)d\varepsilon$$

$$= \int_0^1 [(x-\varepsilon)H(x-\varepsilon) - x(1-\varepsilon)]f(\varepsilon)d\varepsilon \qquad (9)$$

$$= \int_0^x G(x,\varepsilon)f(\varepsilon)d\varepsilon, \qquad (10)$$

Where

$$G(x,\varepsilon) = (x-\varepsilon)H(x-\varepsilon) - x(1-\varepsilon)$$

$$= \begin{cases} -x(1-\varepsilon), & x < \varepsilon, \\ -\varepsilon(1-x), & x > \varepsilon. \end{cases} \qquad (11)$$

The function $G(x,\varepsilon)$ is called a Green's function. It satisfies differential equation (1) and the same boundary conditions as does u(x), namely.

$$G(0,\varepsilon) = G(1\ \varepsilon) = 0 \qquad (12)$$

We can go a step further and examine the case of inhomogeneous boundary values, that is u(0) = a, u(1) = b. For this purpose we split the function u into two parts u_1 and u_2 such that $u = u_1 + u_2$. The function u_1 satisfies the same differential equation and has the same boundary values as the function u in (6). Accordingly, its value, from (9), is

$$U_1 = \int_0^x (x-\varepsilon)f(\varepsilon)d\varepsilon - x\int_0^1(1-\varepsilon)f(\varepsilon)d\varepsilon.$$

The function u_2 satisfies the boundary value problem.

$$d^2u_2/dx^2 = 0, \qquad u_2(0)=a,\ u_2(1)=b,$$

We can look at the quantity a as the strength of the jump discontinuity at 0 and b as the strength of the jump discontinuity at x=1 (so that the functions rises at 0 and falls back at x=1). Then we can appeal and obtain.

$$d^2u_2/dx^2 = a\delta'(x) - b\delta'(x-1), \qquad u2(0) = 0, u2(1) = 0,$$

its solution follows from (9):$u_2 = bx - a(x-1)$, Thus

$$u(x)= \int_0^x (x-\varepsilon) f(\varepsilon) d\varepsilon - x\int_0^1 (1-\varepsilon) f(\varepsilon) d\varepsilon + bx - a(x-1). \qquad (13)$$

let us now consider the initial value problem

$$d^2u/dx^2 + a^2u=\delta(x), \qquad u(x=0+)=0, u'(x=0+)=1 \qquad (14)$$

Here u is defined for $x\geq 0$. Accordingly, the solution is such that u is zero for $x<0$ and satisfies the differential equation $d^2u/dx^2+a^2u=0$ for $x>0$. The solution that satisfies the conditions $u(0)=0$, $u'(0)=1$, is $\sin(ax)/a$. Thus the solution of the initial value problem (14)is

$$U(x)=(1/a)H(x)\sin(ax). \qquad (15)$$

It is the fundamental solution E (also called the Causal solution) of the operator d^2/dx^2+a^2.

Let us use this fundamental solution to construct the solution of the initial value problem.

$$d^2u/dx^2+a^2u=f(x), \qquad u(x=0+)=u_o, u'(x=0+)=u_1, \qquad (16)$$

where f is a continuous function for $x\geq 0$. For this purpose we continue the functions u and f in the following way:

$$v(x)=\begin{cases} 0, & x<0, \\ u(x), & x\geq 0 \end{cases}$$

$$g(x) = \begin{cases} 0, & x<0, \\ f(x) & x\geq 0 \end{cases}$$

We then find that

$$V'(x) = v'+u_0\delta(x) \qquad v''(x) = v'' + u0\delta'(x) + u1\delta(x)$$

Accordingly, the function v satisfies the differential equation

$$d^2v/dx^2 +a2v=g(x)+u_0\delta'(x) + u1\delta(x), \qquad (17)$$

whose solution is

$$v(x)=E*(g+uo\delta'+u1\delta) = E*g + u0E' + u1E$$

$$= 1/a\int_0^x f(y)\sin a(x-y)dy + u0\cos(ax) + \frac{u1\sin(ax)}{a} \qquad (18)$$

An Alternative Approach

We can obtain (3) various other interesting results by an alternative approach. For this purpose let us first recall

$$\frac{d^2}{dx^2}\left[\frac{1/x-\varepsilon/}{2}\right] =\delta\ (x - \varepsilon) \qquad (19)$$

Accordingly, the solution (Green's function) to the boundary value problem

$$d^2\, G(x, \varepsilon) = \delta(x - \varepsilon) \qquad (20)$$

$$G(0, \varepsilon) = 0, \qquad G(1, \varepsilon) = 0 \qquad (21)$$

Is

$$G(x, \varepsilon) = \frac{1}{2} /x - \varepsilon / + xA(\varepsilon) + B(\varepsilon). \qquad (22)$$

Appling the boundary conditions (21), we find that

$$A(\varepsilon) = \varepsilon - \frac{1}{2}, \quad B(\varepsilon) = -\frac{1}{2}\varepsilon \qquad (23)$$

Substituting theses values in (22), we recover (11).
At this stage it is useful to introduce the symbols x< and x>. They stand for values.

$$X_< = \min(x, \varepsilon) = \begin{cases} x, & a \leq x < \varepsilon, \\ \varepsilon, & \varepsilon \leq x \leq b, \end{cases}$$

And

$$X_> = \min(x, \varepsilon) = \begin{cases} \varepsilon, & a \leq x < \varepsilon, \\ x, & \varepsilon \leq x \leq b, \end{cases}$$

As shown in Fig. 9.1 the corresponding quantities G< and G> stands for values of G in the values of G in the x< and x> regions, respectably.

In the present case, a = 0, b = 1, so that $G_< = x(\varepsilon - 1)$ and $G_> = \varepsilon(x - 1)$. The functions G< and G> satisfy differential equation (1) in the x< and x> regions respectably. The function G< satisfies the boundary condition G (0, ε) = 0 while G> satisfies the conduction G (1, ε) = 0. At x = ε these two solutions are equal. However, there is a jump in their derivates; that is,

$$[dG_>/dx - G_</dx]_{x=\varepsilon} = 1. \qquad (24)$$

G< G>

0 x< ε x> b

Fig. 9.1

The preceding remarks re valid for a general strum- liouvile problem.

$$d/dx\ (p(x)\, dG(x, \varepsilon)/dx) + q(x)G(x, \varepsilon) = \delta(x - \varepsilon), a \leq x,\ \varepsilon \leq b, \qquad (25)$$

$$G(a, \varepsilon) = 0, \qquad G(b, \varepsilon) = 0, \tag{26}$$

Where p and q re real-valued functions on $a \leq x \leq b$,p,p',q are continuous in the interval and p is positive. In this cases,

$$G(x, \varepsilon)/_{x=\varepsilon+} = G(x, \varepsilon)_{x=\varepsilon-} \tag{27}$$

An

$$\frac{dG(x, \varepsilon)}{dx}\Bigg|_{x=\varepsilon+} \frac{-dG(x, \varepsilon)}{dx}\Bigg|_{x=\varepsilon-} = \frac{1}{p(\varepsilon)} \tag{28}$$

Example 1. Let us find the green's function that satisfies the Sturm Liovile problem

$$\frac{d^2G(x, \varepsilon)}{dx^2} + G(x, \varepsilon) = \delta(x = \varepsilon), \qquad 0 \leq x \leq \pi, \tag{38}$$

$$G(0) - G'(0) = 0, \quad G(\pi) + G'(\pi) = 0. \tag{39}$$

To find the complete solution of this system, we fist find the solutions in the regions

x<($0 \leq x \leq \varepsilon$) and x> ($\varepsilon \leq X \leq 1$) regions.

In the x< region, the solution of the homogenous part of (38) is

$$G(x, \varepsilon) = \tilde{A} \sin x + B \cos x, \tag{40}$$

and

$$G(x, \varepsilon) = \tilde{A} \cos x - B \sin x. \tag{41}$$

In view of the boundary condition at x=0, obtain $\tilde{A} = B$ thus the solution in the x< region is

$$G(x, \varepsilon) = \tilde{A} (\sin x + \cos x) = \sqrt{2}\tilde{A}\, [(1/\sqrt{2}) \sin x + (1\sqrt{2}) \cos x].$$

$$= A \sin (x + {}^{-}i4). \qquad A = \sqrt{2}\,\tilde{A}. \tag{42}$$

In the x> region, we take the solution of homogenous part of equation (38) to be .

$$G(x, \varepsilon) = C_1 \sin x + D \cos x, \tag{43}$$

So that

$$G'(x, \varepsilon) = C_1 \cos x - D \sin x, \tag{44}$$

The boundary condition (39) yields $C_1 = -D$. Thus

$$G(x, \varepsilon) = C_1 \sin (x - \pi/4).\ C_1 = \sqrt{2} C_1 \tag{45}$$

Then condition of continuity is

$$A \sin (x + \pi/4)/x=\varepsilon = C \sin (x - \pi/4)/x = \varepsilon$$

Or

$$\frac{A}{\sin(\varepsilon - \pi/4)} = \frac{C}{\sin(\varepsilon + \pi/4)} = B = \text{Const.} \tag{46}$$

This leaves undermined only B, Which we find by applying the jump condition

$$dG/dx/_{x=\varepsilon+} - dG/dx/_{x=\varepsilon-} \quad =1$$

or

$$B\,[\sin(\varepsilon + \pi/4)\,\mathrm{Cos}(\varepsilon - \pi/4) - \sin(\varepsilon - \pi/4)\cos(\varepsilon + \pi/4)] = 1,$$

Which yields B= 1. Thus

$$G(x,\varepsilon) = \begin{cases} \mathrm{Sin}(\varepsilon - \pi/4)\,\sin(x + \pi/4) & x < \varepsilon. \\ \mathrm{Sin}(\varepsilon + \pi/4)\,\sin(x - \pi/4), & x > \varepsilon. \end{cases} \qquad (47)$$

Example 2 let us use the information of the previous example to solve the boundary value problem.

$$U''(x) + u(x) = f(x)\,, \qquad 0 \le x \le \pi \qquad (48)$$

$$f(x) = \begin{cases} f_1(x) = 0, & 0 \le x \le \pi/2 \\ f_2(x) = 2x/\pi & \pi/2 \le x \le \pi \end{cases}$$

$$u(0) - u'(O) = 0 \qquad u(\pi) + u'(\pi) = 0 \qquad (49)$$

Thus, the jump discontinuity is at the point $\varepsilon = \pi/2$. In view of he jump in the function f and the Green' function at x $=\pi/2$, we exact the solution fu and its derivative u to have jump discontinuities at $\pi/2$, let square brackets label the jump of a function at $x=\pi/2$ we thus set.

$$[u] - u(\tfrac{\pi}{2}+) - u(\tfrac{\pi}{2}-) = \alpha, \qquad [u'] = u'(\tfrac{\pi}{2}+) - u'(\tfrac{\pi}{2}-) = \beta$$

Now recall the values of the distributional derivatives, namely,

$$\acute{U}' = u' + x\delta(x - \pi/2), \quad \H{u}'' = \H{u}'' + \alpha\delta'(x-\pi/2) + \beta\delta(x - \pi/2)$$

Thus (48) can be written as $\H{u} + \H{u} = f(x) + \beta\delta(x - \pi/2) + \alpha\delta(x - \pi/2)$

So that

$$U(x) = \int_0^x G(x,\varepsilon)[f(t)] + \delta\beta(t - \tfrac{\pi}{2}) + \alpha\delta(t - \tfrac{\pi}{2})dr$$

$$= 2/\_\pi \int_{\pi/2}^{\pi} G(x,\varepsilon)[f(t)]\,dt + \beta G(x - \tfrac{\pi}{2}) - \alpha G(x, \tfrac{\pi}{2}) \text{ at}$$

$$= \frac{1}{\sqrt{2}} \sin(x - \tfrac{\pi}{4})(1 - \tfrac{4}{\pi}) + \beta G(x, \tfrac{\pi}{}) - \alpha\, aG(x, \tfrac{\pi}{2}).$$

Where we have used the solution for G from (47)

As in the previous example, we find the values of ű in the x< and x> regions.

In the x< region

$$u_<(x) = \int_0^x [G_<(x, \varepsilon) f(t)]\, dt + \beta G_<(x, \frac{\pi}{2}) - \alpha \frac{aG_<}{at}(x, \frac{\pi}{2})$$

Next, we substitute the required values from the previous example, namely,

$G_<(x, t)\ t{=}\pi/2 = [\sin(t - \pi/4) \sin(x + \pi/4)]\pi/2 = (\sqrt{2}/2) \sin(x + \pi/4)$

$aG_</at/t{=}\pi 2 = [\cos(t - \pi/4)] \sin(x + \pi/4)]\pi/2 = (\sqrt{2}/2)\sin(x+\pi/4)$,

And obtain

$$\H{U}_< = (1/\sqrt{2}) [\sin(x - \pi/4)(1 - 4/\pi) + \sin(x + \pi/4)(\beta - \alpha)] \qquad (50)$$

In the x> region,

$$u_> = \frac{1}{\sqrt{2}} \sin(x + \frac{\pi}{4})(1 - \frac{4}{\pi}) + \beta G_>(x \frac{\pi}{2}) - \alpha \frac{aG_>}{at}(x, \frac{\pi}{2}) \qquad (51)$$

Because $G_>(x,t) = \sin(t + \pi/4) \sin(x - \pi/4)$ we have

$$G_>(x,\frac{\pi}{2}) = \frac{\sqrt{2}}{2} \sin(x - \frac{\pi}{4}), \quad aG_>/at(x,\frac{\pi}{2}) = \frac{-\sqrt{2}}{2} \sin(x - \frac{\pi}{4})$$

Thus (51) yields

$$u_> = (1/\sqrt{2}) \sin(x - \pi/4)[\beta + \alpha + 1 - 4/\pi]. \qquad (52)$$

Combining (50) and (52), we have

$$u(x) = 1/\sqrt{2} \begin{cases} \sin\left(x - \frac{\pi}{4}\right)\left(1 - \frac{4}{\pi}\right) + \sin\left(x + \frac{\pi}{4}\right)(\beta - \alpha) & 0 \le x \le \pi/2 \\ \sin\left(x - \frac{\pi}{4}\right)\left(\beta + \alpha + 1 - \frac{4}{\pi}\right), & \frac{\pi}{2} < x \le \pi \end{cases}$$

Chapter 3

3. Fourth-order Linear In homogeneous differential Equation

Fourth-orders ordinary differential equations arise in various steady-state and vibration problems in applied Mechanics. We present the Green's function theory of such an equation in this section. Let us consider the differential equation.

$$L[y] = a(x)\frac{d^4 y}{dx^4} + b(x)\frac{d^3 y}{dx^3} + c(x)\frac{d^2 y}{dx^2} + d(x)\frac{dy}{dx} + e(x)y$$

$= \beta_o\delta(x-\varepsilon) + \beta_1\delta^{'}(x-\varepsilon) + \beta_2\delta^{''}(x-\varepsilon) + \beta_3\delta^{'''}(x-\varepsilon)$ ----------------------- (1)

Where we have stopped at the term $\delta^{'''}(x)$ for the sake of simplicity; the following analysis can be easily extended to include $\delta^{(n)}(x-\varepsilon)$ for $n \geq 4$

Again, the general solution y(x) is

$y(x) = y_h(x) + y_p(x)$ --(2)

Our aim is to find the particular solution, which we again assume to be

$y_p(x) = G(x)H(x-\varepsilon)$ -- (3)

Differentiating this relation four times we obtain

$y_p^{'}(x) = G^{'}(x)H(x-\varepsilon) + G(\varepsilon)\delta(x-\varepsilon)$ -------------------------------------(4a)

$y^{''}_p(x) = G^{'}(x)H(x-\varepsilon) + G^{'}(\varepsilon)\delta(x-\varepsilon) + G(\varepsilon)\delta^{'}(x-\varepsilon)$ --------------------(4b)

$y^{''}_p(x) = G^{'''}(x)H(x-\varepsilon) + G^{''}(™)\delta(x-\varepsilon) = G^{'}(\varepsilon)\delta^{'}(x-\varepsilon) + G(\varepsilon)\delta^{''}(x-\varepsilon)$ ---(4c)

$y^{iv}_p(x) = G^{iv}(x)H(x-\varepsilon) + G^{'''}(\varepsilon)\delta(x-\varepsilon) + G^{''}(\varepsilon)\delta^{'}(x-\varepsilon) + G^{'}(\varepsilon)\delta^{''}(x-\varepsilon) +$

$G(\varepsilon)\delta^{'''}(x-\varepsilon)$ -- (4d)

Now

$d(x)y'_p(x) = d(x)G^{'}(x)H(x-\varepsilon) + d(\varepsilon)g(\varepsilon)\delta(x-\varepsilon)$ -------------------------------- (5a)

$c(x)y''_p(x) = c(x)G^{'}(x)H(x-\varepsilon) + [c(\varepsilon)G^{'}(\varepsilon) - c^{'}(\varepsilon)G(\varepsilon)]\delta(x-\varepsilon) +$

$c(\varepsilon)G(\varepsilon)\delta^{'}(x-\varepsilon)$ ---(5b)

To find the fundamental solution of the operator L, we set

$\beta_o = 1$ and $\beta_1 = \beta_2 = \beta_3 = 0$

Example: Let us solve the differential equation

$$\frac{d^2}{dx^2}[E\lambda_o(1+mx)^3\frac{d^2y}{dx^2}] = \beta 1\delta'(x - \frac{1}{2}^{\lambda}) \text{ ----------------------------- (8)}$$

Which embodies the static problem of finding the deflection y of a linearly tapered beam that is free at x = 0, clamped at x = λ, and subjected to a couple β_1 at its mid-length (fig. 1).

The quantity E is young's modulus,

$\lambda = \lambda_o(1+mx^3)$ is the moment of inertia of the cross section, and λ_o and m are constants.

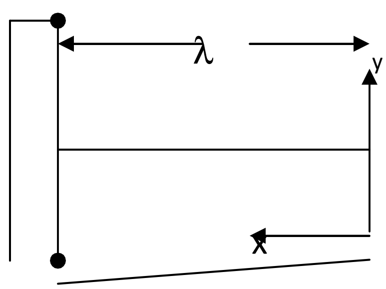

Fig. 1 The deflection of a tempered beam.

Solution: Written explicitly, (8) becomes

$$E\lambda_o(1+mx)^3y^{(iv)} + 6E\lambda_o m(1+mx)^2y''' + 6E\lambda_o m^2(1+mx)y'' = \beta_1\delta'(x-\tfrac{1}{2}\lambda) \text{ ---(9)}$$

Comparing (9) with (1), we have

$$a(x) = E\lambda_o(1+mx)^3,$$

$$b(x) = 6E\lambda_o m(1+mx)^3 \text{ --(10)}$$

$$c(x) = 6E\lambda_o m^2(1+mx),$$

$$d(x) = e(x) = 0$$

$$b(x)y'''(x) = b(x)G'''(x)H(x-\varepsilon)$$

$$+ [b(\varepsilon)G''(\varepsilon) - b'(\varepsilon)G'(\varepsilon) + b''(\varepsilon)G(\varepsilon)]\delta(x-\varepsilon)$$

$$+ [b(\varepsilon)G'(\varepsilon) - 2b'(\varepsilon)G(\varepsilon)\delta'(x-\varepsilon) + b(\varepsilon)G(\varepsilon)\delta''(x-\varepsilon) \text{ ------------------(5c)}$$

$$a(x)y^{iv}(x) = a(x)G^{iv}(x)H(x-\varepsilon) + [a(\varepsilon)G'''(\varepsilon) = a^1(\varepsilon)G''(\varepsilon) + a''(\varepsilon)G'(\varepsilon) - a'''(\varepsilon)G(\varepsilon)]\delta$$

$$(x-\varepsilon) + [a(\varepsilon)G''(\varepsilon) - 2a'(\varepsilon)G'(\varepsilon) + 3a''(\varepsilon)G(\varepsilon)]\delta'(x-\varepsilon) +$$

$$[a(\varepsilon)G'(\varepsilon) - 3a'(\varepsilon)G(\varepsilon)]\delta''(x-\varepsilon) + a(\varepsilon)G(\varepsilon)\delta'''(x-\varepsilon) \text{ ----------- (5d)}$$

The next step is to substitute these values in (1), so that

$$(LG)H(x-\varepsilon) + \{a(\varepsilon)G'''(\varepsilon) - [a'(\varepsilon) - b(\varepsilon)]G''(\varepsilon)$$

$$+ [a''(\varepsilon) - b'(\varepsilon) + c(\varepsilon)]G'(\varepsilon)$$

$$- [a'''(\varepsilon) - b'(\varepsilon) + c'(\varepsilon) - d(\varepsilon)]G(\varepsilon)\}\delta(x-\varepsilon)$$

$$+ \{a(\varepsilon)G''(\varepsilon) - [2a'(\varepsilon) - b(\varepsilon)]G'(\varepsilon)$$

$$+ [2a''(\varepsilon) - 2b'(\varepsilon) + c(\varepsilon)]G(\varepsilon)\}\delta'(x-\varepsilon)$$

$$+ \{a(\varepsilon)G'(\varepsilon) - [3a'(\varepsilon) - b(\varepsilon)]G(\varepsilon)\}\delta''(x-\varepsilon) + a(\varepsilon)G(\varepsilon)\delta'''(x-\varepsilon)$$

$$= \beta_o\delta(x-\varepsilon) + \beta_1\delta'(x-\varepsilon) + \beta_2\delta'(x-\varepsilon) + \beta_3\delta'''(x-\varepsilon) \text{ ----------------- (6)}$$

We equate coefficients of generalized functions on both sides of (6) to obtain the initial value problem:

$$LG = a(x)G^{iv}(x) + b(x)G'''(x) + c(x)G''(x) + d(x)G'(x) + e(x)G(x) = 0 \quad \text{---------- (7a)}$$

$$a(\varepsilon)G'''(\varepsilon) - [a'(\varepsilon) - b(\varepsilon)]G''(\varepsilon) + [a''(\varepsilon) - b'(\varepsilon) + c(\varepsilon)]G'(\varepsilon)$$

$$- [a'''(\varepsilon) - b''(\varepsilon) + c'(\varepsilon) - d(\varepsilon)]G(\varepsilon) = \beta_o \quad \text{------------------------------------- (7b)}$$

$$a(\varepsilon)G''(\varepsilon) - [2a'(\varepsilon) - b(\varepsilon)]G'(\varepsilon) + [3a''(\varepsilon) - 2b'(\varepsilon) + c(\varepsilon)]G(\varepsilon) = \beta_1 \quad \text{------- (7c)}$$

$$a(\varepsilon)G'(\varepsilon) - [3a'(\varepsilon) - b(\varepsilon)]G(\varepsilon) = \beta_2 \quad \text{---(7d)}$$

$$a(\varepsilon)G(\varepsilon) = \beta_3 \quad \text{-- (7e)}$$

To find the particular solution with start with (3) with $\varepsilon = 1/2\lambda$, so that (7a) in this case becomes

$$G^{iv}+6m/1+mx\ G''' + 6m^2/(1+mx)^2\ G'' = 0 \quad \text{------------------------------------ (11)}$$

Its solution is

$$G(x) = c_1 + c_2x + c_3(1+mx) + c_4\ \lambda n(1+mx) \quad \text{------------------------- (12)}$$

Where $c_i(i = 1,2,3,4)$ are constants to find these constants, we substitute (12) in the inital conditions (7b0 -(7c), to get

$$c_1 = \frac{-\beta_1\lambda}{4mE\lambda_o(1+½m\lambda)^2} - \frac{\beta_1}{2m^2E\lambda_o(1+½m\lambda)}$$

$$c_2 = \frac{\beta_1}{2mE\lambda_o(1+½m\lambda)^2}\ ,c_3 = \frac{\beta_1}{2m^2E\lambda_o}\ ,\ c_4 = 0 \quad \text{-------------(13)}$$

From (3), (12) and (13) we find that the particular solution of (9) is

$$y_p(x) = G(x)H(x-½\lambda)$$

$$= \frac{\beta_1}{E\lambda_o}\left[-\frac{1}{2m^2(1+\frac{1}{2}m\lambda)} - \frac{\lambda}{4m(1+\frac{1}{2}m\lambda)^2}\right.$$

$$\left. + \frac{x}{2m(1+\frac{1}{2}m\lambda)^2} + \frac{1}{2m^2(1+mx)}\right] H(x-{}^{\lambda}/_{2}) \text{ ----------------- } (14)$$

The general solution can now be readily obtained by adding to (14) the solution to the homogenous equation, namely,

$$y_h(x) = b_1 + b_2x + b_3(1+mx) + b_4\ \lambda n(1+mx) \text{ --------------------------------- } (15)$$

Where $b_i(i = 1,2,3,4)$ are constants

These constants are obtained with the help of the boundary conditions that the beam is free at x = 0 and clamped at x = λ, that is,

$$y(\lambda) = 0,\ y'(\lambda) = 0,\ y''(0) = 0,\ y'''(0) = 0 \text{ ------------------------- } (16)$$

Thus, the general solution is:

$$\frac{2mE\lambda_0}{\beta 1} y(x) = -\left[\frac{\lambda}{(1+m\lambda)^2} - \frac{\lambda}{2(1+\frac{1}{2}m\lambda)^2} - \frac{1}{m(1+m\lambda)^2} - \frac{1}{m(1+\frac{1}{2}m\lambda)^2}\right]$$

$$+\left[\frac{1}{(1+m\lambda)^2} - \frac{1}{(1+\frac{1}{2}m\lambda)^2}\right] X$$

$$+\left[-\frac{1}{m(1+\frac{1}{2}m\lambda)} \quad \frac{-\lambda}{2(1+\frac{1}{2}m\lambda)^2} + \frac{x}{(1+\frac{1}{2}m\lambda)^2} + \frac{1}{m(1+mx)}\right] H(x-\tfrac{1}{2}\lambda$$

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