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Project on

Eigenpairs of Fredholm Integral Operators with Separable Kernel

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Abstract

An integral equation is an equation in which the unknown function appears in the integrand. There are different types of integral equations, among these Fredholm integral equation is an integral equation with constant limit of integration. Depending on the appearance of the unknown function we have three kinds of Fredholm integral equations, first kind, second kind and third kind.

In this project we discuss the techniques of solving Fredholm integral equations. Even though, it is difficult to find the solution of each kind, we consider the solution of the first kind in terms of a linear combination of eigenpairs of separable kernel. The solution of the second kind is also determined by using different decomposition methods.

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Introduction

An integral equation is an equation in which the unknown function $f(x)$ appears in the integrand. A standard integral equation in $f(x)$ is of the form,

$$a(x)f(x) = g(x) + \lambda \int_{m(x)}^{h(x)} K(x,y)f(y)dy$$

where $m(x)$ and $h(x)$ are the limits of integration, λ is a parameter and $K(x,y)$ is a kernel or the nucleus of the integral equation. The function $f(x)$ yet to be determined, indeed it appears inside and outside the integral sign as well. The functions $a(x), g(x)$ and $K(x,y)$ are presumably known functions. It is to be noted that the limits of integration $m(x)$ and $h(x)$ can be variables, constants, infinite or mixed, thereby rendering Volterra, Fredholm and singular integral equations respectively.

If $h(x) = x$ and $m(x) = a = constant$ then the resulting integral equation is Volterra type, while for $h(x) = b = constant$ and $m(x) = a = constant$ the corresponding class is Fredholm integral equation. If one of $h(x),\ m(x)$ or both are infinite the integral equation is a class of singular type. Many initial and boundary value problems associated with ODE and PDE can be transformed into problems of solving some approximate integral equations. All of the conditions specifying the initial boundary data for a differential equation can be condensed into a single integral equation. Solution techniques for the resulting integral equation include fixed point iteration which is friendly with numerical schemes.

Linear Fredholm integral equation is an equation that is linear in the unknown function f.

i.e.

$$\int_a^b k(x,y)\,[c_1 f_1(y) + c_2 f_2(y)]dy = c_1 \int_a^b K(x,y) f_1\,(y) + c_2 \int_a^b K(x,y) f_2(y)dy$$

Linear Fredholm integral equation appears in many forms. Three distinct kinds that depend on the appearance of the unknown function $f(x)$, are defined as follows,

a) Linear Fredholm Integral Equation of *first* kind (IEFK);the unknown function $f(x)$ occurs only under the integral sign in the form

$$g(x) = \int_a^b K(x,y)\, f(y) dy$$

The solution of this kind of integral equation is illustrated later in the second chapter it shows that the function f is expressed as a linear combination of eigenpairs.

b) Linear Fredholm integral equation of *second* kind; the unknown function $f(x)$ occurs inside and outside the integral sign in the form

$$f(x) = g(x) + \lambda \int_a^b K(x,y) f(y) dy$$

If $g(x) = 0$ then

$$f(x) = \lambda \int_a^b K(x,y) f(y) dy$$

This equation is called homogenous linear Fredholm integral equation of the *second* kind.

c) Linear Fredholm integral equation of *third* kind is given by

$$a(x) f(x) = g(x) + \lambda \int_a^b K(x,y) f(y) dy$$

If $a(x) \neq 0$, equation of the *third* kind can be reduced to equation of the *second* kind.

The concern will be on the determination of the solution f of the linear Fredholm integral equations.

Chapter 1

1. Fredholm Integral Equation

The standard form of linear Fredholm integral equation is,

$$a(x)f(x) = g(x) + \lambda \int_{a}^{b} K(x, y)f(y)dy \qquad x \in [a, b]$$

The function g, a and $k(x, y)$ are known functions, while f is yet to be determined; λ is a nonzero (real or complex) parameter.

I. In the Fredholm integral equation of the *second* kind, $a(x) = 1$;

$$f(x) = g(x) + \lambda \int_{a}^{b} K(x, y)f(y)dy$$

II. The homogeneous Fredholm integral equation of the *second kind* is a special case of (I) above. In this case, $g(x) = 0$;

$$f(x) = \lambda \int_{a}^{b} K(x, y)f(y)dy$$

III. In Fredholm integral equation of *first* kind (IFK), $a(x) = 0$.Thus

$$g(x) + \lambda \int_{a}^{b} K(x, y)f(y)dy = 0$$

Fredholm integral equation of *first* kind is ill-conditioned problem. Hardamord [1] postulated the following three properties

1) Existence of solution
2) Uniqueness of solution
3) Continuous dependence of the solution $f(x)$ on the data (x) . This property means that the small errors in $g(x)$ should cause small errors in the solution $f(x)$.

A problem is said to be a well-conditioned problem if it satisfies the above three properties. Problems that are not well-conditioned are called ill-conditioned problems. Even if the kernel $K(x, y)$ is smooth, the Fredholm integral equation of *first* kind is very often times ill-conditioned and the solution $f(x)$ is very sensitive to any change in the data $g(x)$.

1.1. From BVP to Fredholm integral equation

In this section we look at how a BVP can be converted to an equivalent Fredholm integral equation. Oftentimes the method of reducing a BVP to a Fredholm integral equation is complicated and rarely used. We demonstrate this method with example.

Consider the second-order ordinary differential equation,

$$y''(x) + p(x)y'(x) + q(x)y(x) = h(x) \qquad 1.1$$

with the boundary conditions $y(a) = \alpha$ and $y(b) = \beta$. Where α and β are given constants. Let us make a transformation,

$$y''(x) = f(x) \qquad 1.2$$

Integrating both sides of equation (1.2) over $[a, x]$ yield**s,**

$$y'(x) = y'(a) + \int_a^x f(t)dt \qquad 1.3$$

note that $y'(a)$ is not prescribed yet. Integrating both sides of equation (1.3) with respect to x from a to x and applying the given boundary condition at $y(a)$, we find

$$y(x) = y(a) + (x - a)y'(a) + \int_a^x \left(\int_a^x f(t)dt \right) dt$$

$$= \alpha + (x - a)y'(a) + \int_a^x \left(\int_a^x f(t)dt \right) dt \qquad 1.4$$

And using the boundary condition at $x = b$ yields

$$y(\mathrm{b}) = \beta = \alpha + (\mathrm{b} - \mathrm{a})y'(\mathrm{a}) + \int_{\mathrm{a}}^{\mathrm{b}} \left(\int_{\mathrm{a}}^{\mathrm{b}} f(\mathrm{t})\mathrm{d}t \right) \mathrm{dt}$$

And the unknown constant $y'(\mathrm{a})$ is determined as

$$y\,'(\mathrm{a}) = \frac{1}{\mathrm{b} - \mathrm{a}} \left(\beta - \alpha - \int_{\mathrm{a}}^{\mathrm{b}} \left(\int_{\mathrm{a}}^{\mathrm{b}} f(\mathrm{t})\mathrm{d}t \right) \mathrm{dt} \right) \qquad 1.5$$

Hence the solution (1.4) can be rewritten as

$$y(\mathrm{x}) = \alpha + \frac{(\mathrm{x} - \mathrm{a})}{\mathrm{b} - \mathrm{a}} \left(\beta - \alpha - \int_{\mathrm{a}}^{\mathrm{b}} \left(\int_{\mathrm{a}}^{\mathrm{b}} f(\mathrm{t})\mathrm{d}t \right) \mathrm{dt} \right) + \int_{\mathrm{a}}^{\mathrm{x}} \left(\int_{\mathrm{a}}^{\mathrm{x}} f(\mathrm{t})\mathrm{d}t \right) \mathrm{dt} \qquad 1.6$$

Therefore, equation (1.1) can be written as

$$f(\mathrm{x}) = h(\mathrm{x}) - \mathrm{p}(\mathrm{x}) \left(y'(\mathrm{a}) + \int_{\mathrm{a}}^{\mathrm{x}} f(t)\mathrm{d}t \right)$$

$$- \mathrm{q}(\mathrm{x}) \left(\alpha + (\mathrm{x} - \mathrm{a})y'(\mathrm{a}) + \int_{\mathrm{a}}^{\mathrm{x}} \left(\int_{\mathrm{a}}^{\mathrm{x}} f(\mathrm{t})\mathrm{d}t \right) \mathrm{dt} \right) \qquad 1.7$$

where $f(\mathrm{x}) = y''(\mathrm{x})$ and so $y(\mathrm{x})$ can be determined, in principle, from equation (1.6). This is a complicated procedure to determine the solution of BVP by equivalent Fredholm integral equation.

Special case:

If $a = 0$ and $b = 1$, that means $0 \leq x \leq 1$, then from (1.4), we have;

$$y(x) = \alpha + xy'(0) + \int_{0}^{x} \left(\int_{0}^{\mathrm{x}} f(\mathrm{t})\mathrm{d}t \right) \mathrm{dt}$$

$$= \alpha + xy'(0) + \int_{0}^{x} (x - t)f(t)dt$$

And hence the unknown constant $y'(0)$ can be determined using (1.5), we have;

$$y'(0) = (\beta - \alpha) - \int_0^1 (1-t)f(t)dt$$

$$= (\beta - \alpha) - \int_0^x (1-t)f(t)dt - \int_x^1 (1-t)f(t)dt$$

And thus from equation (1.7) we have,

$$f(x) = h(x) - p(x)\left(y'(0) + \int_0^x f(t)dt\right) - q(x)\left(\alpha + xy'(0) + \int_0^x (x-t)f(t)dt\right)$$

$$= \mathrm{h(x)} - (\beta - \alpha)\big(\mathrm{p(x)} + \mathrm{xq(x)}\big) - \alpha\mathrm{q(x)} + \int_0^1 (x-t)K(x,t)f(t)dt$$

$$= \mathrm{g(x)} + \int_0^1 (x-t)K(x,t)f(t)dt \qquad 1.8$$

where $\quad \mathrm{g(x)} = \mathrm{h(x)} - (\beta - \alpha)(\mathrm{p(x)} + \mathrm{xq(x)}) - \alpha\mathrm{q(x)}$

$$\mathrm{K(x,t)} = \begin{cases} \big(\mathrm{p(x)} + \mathrm{tq(x)}\big)(1 - \mathrm{x}) & 0 \le \mathrm{t} \le \mathrm{x} \\ \big(\mathrm{p(x)} + \mathrm{xq(x)}\big)(1 - \mathrm{t}) & \mathrm{x} \le \mathrm{t} \le 1 \end{cases} \qquad 1.9$$

Equation (1.8) is the Fredholm integral equation of the *second* kind.

1.2. From Fredholm Integral Equation to BVP

In this section we will see the technique used to convert Fredholm integral equation to an equivalent BVP.

We first consider the Fredholm integral equation given by

$$f(x) = g(x) + \int_a^b K(x,y)f(y)dy \qquad 1.10$$

where $g(x)$ is a given function, and the kernel $K(x,y)$ is given by

$$K(x,y) = \begin{cases} y(1-x)q(x), 0 \le y \le x \\ x(1-y)q(x), x \le y \le 1 \end{cases} \quad 1.11$$

For simplicity reasons, we may consider $q(x) = \lambda$, where λ is constant. Equation (1.10) can be written as

$$f(x) = g(x) + \lambda \int_0^x y(1-x)f(y)dy + \lambda \int_x^1 x(1-y)f(y)dy \quad 1.12$$

Each term of the last two terms at the right side of (1.12) is a product of two functions of x. Differentiating both sides of (1.12) with respect to x, using the product rule of differentiation and using *Leibnitz* rule we obtain

$$f'(x) = g'(x) + \lambda \int_0^x y(1-x)f(y)dy + \lambda \int_x^1 (1-y)f(y)dy$$

$$= g'(x) - \lambda \int_0^x yf(y)dy + \lambda \int_x^1 (1-y)f(y)dy$$

We differentiate both sides of the above equation again with respect to x to find that

$$f''(x) = g''(x) - \lambda x f(x) - \lambda(1-x)f(x) \quad 1.13$$

That gives the ordinary differential equations:

$$f''(x) + \lambda f(x) = g''(x) \quad 1.14$$

The related boundary conditions can be obtained by substituting $x = 0$ and $x = 1$ in (1.12) to find that

$$f(0) = g(0), \quad f(1) = g(1) \quad 1.15$$

Combining (1.14) and (1.15) gives the BVP equivalent to the Fredholm integral equation. Recall that $y''(x) = f(x)$. Moreover, if $q(x)$ is not a constant, we can produce in a manner similar to the discussion presented above to obtain the boundary value problem.

Chapter 2

2. Fredholm Integral Equation with Separable Kernel

2.1. Linear Fredholm Integral Equation of First Kind (IEFK) with Separable Kernel

Consider an integral operator L_k of the form

$$L_k f(x) = \int_a^b K(x,y) f(y) dy \,, x \in [a,b] \qquad 2.1$$

This operator is helpful because a linear Fredholm integral equation of the *first* kind can be written as,

$$g(x) = L_k f(x) = \int_a^b K(x,y) f(y) dy \qquad 2.2$$

here, the kernel $K(x,y)$ and the function $g(x)$ are prescribed on the square

$a \le x, y \le b$, and g on the interval $a \le x \le b$. The function f is yet to be determined on $[a,b]$.

A kernel of the form is separable if it can be expressed as,

$$K(x,y) = \sum_{j=1}^{n} a_j(x) b_j(y)$$

observe that the set of functions a_j and b_j are linearly independent sets.

2.1.1. Properties of IEFK with Separable Kernel

For the moment we consider f to be a given function and define

$$g(x) = \int_a^b K(x,y) f(y) dy = \sum_{j=1}^{n} \int_a^b a_j(x) b_j(y) f(y) dy$$

$$= \sum_{j=1}^{n} a_j(x) \int_a^b b_j(y) f(y) dy$$

$$g(x) = \sum_{j=1}^{n} c_j \, a_j(x) \qquad 2.3$$

where

$$c_{j=} \int_a^b b_j(y) f(y) dy$$

Equation (2.3) shows that the function g that can be represented by this integral operator must be of a particular form, i.e. g must be a linear combination of $a_j(x)'$s.

Next, we note that there are many functions ψ that are orthogonal to the set of functions b_j's,

$$\int_a^b b_j\,(y)\psi(y)dy = 0, \qquad j = 1, \ldots ..n$$

This is a reflection of the fact that the set b_j is finite. Define $f_1 = f + \psi$ then

$$\int_a^b K(x,y) f_1(y) dy = \sum_{j=1}^{n} \int_a^b a_j\,(x) b_j(y)\{f(y) + \psi(y)\} dy$$

$$= \sum_{j=1}^{n} \int_a^b a_j\,(x) b_j(y) f(y) dy + \sum_{j=1}^{n} \int_a^b a_j\,(x) b_j(y) \psi(y) dy$$

$$= \sum_{j=1}^{n} \int_a^b a_j\,(x) b_j(y) f(y) dy = g(x)$$

Similarly

$$\sum_{j=1}^{n} \int_a^b a_j\,(x) b_j(y) \psi(y) dy = 0$$

Thus there exist many functions f_1 that give rise, through the integral operator, to the same function g. This implies that uniqueness is totally lacking.

Now let us reconsider equation (2.3), this time supposing g to be known while f is to be determined. The previous demonstration leads us to two observations.

a. g must be a linear combination of the a_j's or the problem is not solvable

b. If the problem is solvable, it has infinitely many solutions.

Although those facts now seem obvious, they are often overlooked. Now assume that g is of the proper form,

$$g(x) = \sum_{j=1}^{n} g_j \, a_j(x) \qquad 2.4$$

Let

$$f(y) = \sum_{i=1}^{n} f_i \, b_i(y) \qquad 2.5$$

where f_i are constants to be determined, if possible. We next compute equation (2.3)

$$g(x) = \sum_{j=1}^{n} g_j \, a_j(x) = \int_a^b K(x,y) f(y) dy$$

$$= \sum_{j=1}^{n} a_j\,(x) \int_a^b \{\sum_{i=1}^{n} f_i \, b_i(y) b_j(y)\} dy$$

$$= \sum_{j=1}^{n} a_j\,(x) \sum_{i=1}^{n} f_i \,\{\int_a^b b_i(y) b_j(y)\} dy$$

$$= \sum_{j=1}^{n} a_j\,(x) \sum\nolimits_{i=1}^{n} b_{ji} \, f_i$$

where

$$b_{ji} = \int_a^b b_i(y) b_j(y) dy$$

Since a_j's have been chosen from a linearly independent set and

$$\sum_{j=}^{n}\{g_j - \sum\nolimits_{i=1}^{n} b_{ji}f_i\} \;\; a_j(x) = 0$$

We have

$$g_j = \sum\nolimits_{i=1}^{n} b_{ji}f_i$$

Or in matrix form

$$\text{g} = B\text{f, where } B = (b_{ji})_{j,i=1}^{n}$$

It may be shown that the matrix B is non singular. Thus the vector f is uniquely determined, and we have a unique f of the form equation (2.5). This is not the unique solution of the IEFK. As noted earlier, the IEFK does not have a unique solution. We have established the fact that it has a solution for any g of the proper form.

2.1.2. Eigenpairs of IEFK with Separable Kernel

The preceding problem has been reduced to one in matrix theory, and eigenvalues and eigenvector considerations are important in that theory. It is helpful to study their analogues for integral operators. If we write the IEFK as,

$$\mu g(x) = \int_a^b K(x,y)\phi(y)\,dy \qquad 2.6$$

where the eigenvalue $\mu = \frac{1}{\lambda}$ and $\phi(y)$ is the corresponding eigenfunction.

Let

$$\phi(y) = \sum_{p=1}^{n} \phi_p b_p\,(y)$$

where $K(x,y)$ is separable. In view of the observation in equation (2.4) any function g satisfying equation (2.6) must have the form

$$\mu g(x) = \mu \sum_{j=1}^{n} \phi_j a_j (x)$$
$$= \int_a^b K(x, y)\phi(y)dy$$
$$= \sum_{j=1}^{n} a_j (x) \int_a^b b_j (y) \sum_{p=1}^{n} \phi_p a_p (y)dy \quad 2.7$$
$$= \sum_{j=1}^{n} a_j (x) \sum_{p=1}^{n} \phi_p \int_a^b b_j (y)b_p(y)dy$$

Let

$$\int_a^b b_j (y)b_p(y)dy = \gamma_{jp}$$

Since a_j's have been chosen from a linearly independent set and

$$\sum_{j=1}^{n}(\mu\phi_j - \sum_{p=1}^{n} \gamma_{jp}\phi_p)a_j(x) = 0$$

we have

$$\mu\phi_j = \sum_{p=1}^{n} \gamma_{jp}\phi_p$$

Or in matrix form

$$\mu\phi = \Gamma\phi \quad 2.8$$

where

$$\Gamma = \gamma_{jp}$$

The matrix system in equation (2.8) has n eigenvalues and n eigenvectors. Their exact properties may be discussed intermes of the structure of Γ, which in turn inherits its structure from $L_{k.}$. If it should happen that $a_j(x) = b_j(x)$, j=1,2,…….n, then $K(x, y) = K(y, x)$. Thus K is a symmetric kernel and Γ is a symmetric matrix.

Chapter 3

3. Analytical Solution of IEFK

In the previous sections we basically showed one important thing that is not always feasible to find a well-behaved solution to an IEFK equation, in general. This does not mean that the problem is completely unsolvable. We show in this section the basics about how to analytically find a solution to an IEFK.

First, we have to find a solution in the simple case, that is, when the kernel is symmetric function. This case is then used in the general case, when nonsymmetrical kernel is given, by means of turning any kernel into a symmetrical one. Finally, we get a general expression for a solution of any IEFK equation.

3.1. Symmetric Kernel

In the case where kernel $\mathrm{K}(\mathrm{x}, y)$ is a symmetric function, i.e., $\mathrm{K}(x, y) = \mathrm{K}(y, x)$, the operator $\mathrm{L_k}$ has some especially attractive properties, such as having eigenfunctions and eigenvalues. These functions can be thought as an analogy to the eigenvectors of matrices as equation (3.1) claims.

In the next line we consider the properties of eigenpairs of symmetrical kernel belonging to an IEFK. Furthermore, we obtain representations of kernel K in terms of a series of eigenfunctions. We first consider the equation for the eigenfunction

$$\mu\, g(x) = \int_{\mathrm{a}}^{\mathrm{b}} \mathrm{K}(x, y)\phi(y)\, \mathrm{d}y \qquad 3.1$$

For simplicity $L_k\phi = \lambda\phi$, where ϕ is eigenfunctions and λ is eigenvalues.

The following facts are known in the case $K(x, y)$ is symmetrically well-behaved and the quadratic bounded conditions hold.

1) Equation (3.1) has at least one solution $\phi \neq 0$ when the value of $\mu \neq 0$.

2) If eigenfunctions ϕ_1 and ϕ_2 belong to different eigenvalues μ_1 and μ_2 then ϕ_1 and ϕ_2 are orthogonal, that is

$$\langle \phi_1, \phi_2 \rangle = \int_a^b \phi_1(x)\phi_2(x)\, dy = 0$$

3) Two different eigenfunctions may belong to the same eigenvalues. However, a particular non- zero eigenvalue can be only associated with a finite number of linearly independent eigenfunctions, which may be orthonormalized.
4) Equation (3.1) has only a finite number of nonzero eigenvalues if and only if $K(x, y)$ is separable.
5) If there are infinitely many non-zero eigenvalues μ_i then $\lim_{i\to\infty} \mu_i = 0$.It is customary to order the eigenvalues so that $|\mu_i| \geq |\mu_{i+1}|$.

The existence of eigenfunctions immediately suggests that we may be able to expand an arbitrary function in terms of these eigenfunctions. If for some f we can write,

$$g = L_k f$$

then we can expand g intermes of eigenfunctions ϕ as

$$g(x) = \sum_{n=1}^{\infty} a_n \phi_n(y) \qquad 3.2$$

$$a_n = \int_a^b g(x)\phi_n(x)\, dx$$

The kernel K can also be represented as

$$K(x, y) = \sum_{n=1}^{\infty} a_n(y)\phi_n(x) \qquad 3.3$$

$$a_n(y) = \int_a^b K(x, y)\, \phi_n(x) dx = \int_a^b K(y, x) h\, \phi_n(x)\, dx$$

Since $K(x, y)$ is symmetric

$$a_n(y) = \mu_n \phi_n(y) \qquad 3.4$$

This implies that

$$K(x,y) = \sum_{n=1}^{\infty} \mu_n \phi_n(y)\, \phi_n(x) \qquad 3.5$$

That is, there is identification between the kernel K and its eigenpairs.

3.2. Non-Symmetric Kernel

When dropping the assumption of symmetry, relatively little of the preceding can be saved.

1) An integral operator may have no eigenvalues at all.
2) It is not always possible to assure that nonzero eigenvalues exists.

Therefore, as matrices arising numerical computations, such manipulations may introduce entirely spurious eigenvalues. More problems, despite nonsymmetrical operators may have eigenvalues and eigenfunctions, may be complex.

Let us see the proper solution for nonsymmetrical kernel turning it into a symmetrical kernel. Because function g is known, we may apply to it any reasonable operator T that we choose and obtain a known function h,

$$g = L_k f$$

This implies that

$$Tg = TL_k f = h \qquad 3.6$$

Thus, the IEFK becomes $h = (TL_k)f$, where TL_k the new operator h is the data function and f is yet to be determined. A useful operator to solve the problem is the transposed operator

$$T = L_k^*$$

It is defined as follows

$$L_k^* f(y) = \int_a^b K(x,y)\, f(x) dx, y \in [a,b] \qquad 3.7$$

It is similar to L_k, but the role of the variable x has been interchanged with the variable y. Obeserve that if the kernel is symmetric,

$$L_k^* = L_k$$

That means

$$L_k f(x) = \int_a^b K(x,y)\, f(y) dy = \int_a^b K(y,x)\, f(y) dy$$

since $K(x, y)$ is symmetric

$$L_k\, f(x) = L_k^*\, f(x)$$

Therefore

$$L_k^* = L_k$$

Hence, from equation (3.6)

$$h = L_k^* L_k f$$

$$h(z) = \int_a^b K(x,z) g(x) dx = \int_a^b K(x,z) \int_a^b K(x,y)\, f(y) dy dx$$

$$= \int_a^b f(y)\, [\int_a^b K(x,z)\, K(x,y) dx] dy$$

This implies that

$$L_k^* L_k f(z) = \int_a^b [\int_a^b K(x,z)\, K(x,y) dx] f(y) dy \qquad 3.8$$

With kernel

$$K^2(z,y) = \int_a^b K(x,z) K(x\,,y) dx$$

This is to say that $L_k^* L_k$ is the integral operator with kernel

$$K^2(z,y) = K^2(y,z)$$

Therefore $L_k^* L_k$ is a symmetrical operator and has all the properties listed in section (3.1). It is also interesting to define the analogous operator $L_k L_k^*$.

Since

$$g = \mathrm{L_k} f \qquad 3.9$$

multiplying equation (3.9) by T we get $\mathrm{T}g = \mathrm{TL_k} f$. Then interchanging the position of T we have $g\mathrm{T} = \mathrm{L_k T} f = h$ for $T = L_k^*$, $\mathrm{h} = (\mathrm{L_k L_k^*}) f$

$$\mathrm{h(z)} = \int_a^{\mathrm{b}} \mathrm{K(z,} y) g(y)\, \mathrm{d}y$$

Using

$$g(\mathrm{y}) = \mathrm{L_k^*} f(y) = \int_a^{\mathrm{b}} \mathrm{K}(x, y) f(x) \mathrm{d}x$$

and Fubini theorem, we get

$$\mathrm{h(z)} = \int_a^{\mathrm{b}} \mathrm{k(z,} y)(\int_a^{\mathrm{b}} \mathrm{K}(x, y) f(x) \mathrm{d}x)\, \mathrm{d}y$$

$$h(z) = \int_a^b \int_a^b K(z, y) K(x, y) f(x) dx dy$$

$$= \int_a^{\mathrm{b}} f(x)\, \mathrm{d}x \int_a^{\mathrm{b}} \mathrm{k(z,} y)\, \mathrm{K}(x, y) \mathrm{d}y$$

This implies that

$$(\mathrm{L_k L_k^*}) f(\mathrm{z}) = \int_a^{\mathrm{b}} [\int_a^{\mathrm{b}} \mathrm{K(z,} y)\, \mathrm{K}(x, \mathrm{y}) \mathrm{d}y] f(x) \mathrm{dy} \qquad 3.10$$

which is to say that $\mathrm{L_k L_k^*}$ is the integral operator with kernel

$$\mathrm{K^2(z,} x) = \int_a^{\mathrm{b}} \mathrm{k(z,} y)\, \mathrm{K}(x, y) \mathrm{d}y$$

From this explicit representation we readily find that

$$\mathrm{K^2(z,} x) = \mathrm{K^2}(x, \mathrm{z})$$

Therefore $\mathrm{L_k L_k^*}$ is symmetric operator.

Note that

$$\int_a^{\mathrm{b}} (\int_a^{\mathrm{b}} \mathrm{k(z,} y)\, \mathrm{K}(x, y) \mathrm{d}y) f(x) \mathrm{d}x \neq \int_a^{\mathrm{b}} (\int_a^{\mathrm{b}} \mathrm{K}(x, \mathrm{z}) \mathrm{K}(x, y) \mathrm{d}x f(y)\, \mathrm{d}y$$

Then$L_k L_k^* \neq L_k^* L_k$. The eigenvalues of $L_k L_k^*$ and $L_k^* L_k$ are positive and identical but the eigenfunctions of the two operators differ in general [2].

For the positive eigenvalues μ_i of the integral operator $L_k^* L_k$ we define the set of singular values $\delta_i = +\sqrt{\mu_i}$.

If v_i are the eigenfunctions of the operator $L_k^* L_k$, thus

$$L_k^* L_k v_i = \delta_i^2 v_i \qquad 3.11$$

Now we define the next function

$$u_i = \frac{1}{\delta_i} L_k v_i \qquad 3.12$$

Applying the transpose operator L_k^* to equation (3.12) we get

$$L_k^* u_i = L_k^* \frac{1}{\delta_i} L_k v_i = \frac{1}{\delta_i} L_k^* L_k v_i = \delta_i v_i \qquad 3.13$$

Note that equation (3.12) and (3.13) are dual relations. If we apply L_k to equation (3.13), and substituting in (3.12) we get

$$L_k L_k^* u_i = \delta_i^2 u_i \qquad 3.14$$

Thus the function u_i is the eigenfunction of the operator $L_k L_k^*$. On the other hand, we may start with the eigenfunction u_i of $L_k L_k^*$ and reverse all the step of the argument to show that the function v_i are the eigen function of $L_k^* L_k$. We demonstrate this for the positive μ_i of the integral operator $L_k L_k^*$ and define the set of singular values δ_i of the operator $L_k L_k^*$.

If $u_i$is the eigenfunctions of the operator $L_k L_k^*$ thus

$$L_k L_k^* u_i = \delta_i^2 u_i \qquad 3.15$$

We define the next function

$$v_i = \frac{1}{\delta_i} L_k^* u_i \qquad 3.16$$

Applying the operator L_k to equation (3.16) we get

$$L_k v_i = \frac{1}{\delta_i} L_k L_k^* u_i = \delta_i u_i \qquad 3.17$$

if we apply the transpose operator L_k^* to equation (3.17) then we get

$$L_k^* L_k v_i = \delta_i L_k L_k^* u_i$$

from equation (3.16)

$$L_k^* u_i = \delta_i v_i$$

then

$$L_k^* L_k v_i = \delta_i^2 v_i \qquad 3.18$$

Thus the function v_i is the eigenfunction of the operator $L_k^* L_k$.

Summarizing to this point, we have associated with the original operator L_k two sets of functions v_i and u_i are eigenfunctions of $L_k^* L_k$ and $L_k L_k^*$ respectively. The following expressions encompass their properties

$$L_k^* u_i = \delta_i v_i \qquad L_k L_k^* u_i = \delta_i^2 u_i$$

$$L_k v_i = \delta_i u_i \qquad L_k^* L_k v_i = \delta_i^2 v_i$$

Functions v_i and u_i are called singular functions belonging to kernel K, and the values δ_i are singular values.

The following equation is also true and relates both operators by means of the scalar product of the two functions.

$$\langle L_k f_1, f_2 \rangle = \int_a^b (L_k f_1)\, f_2(y) dy = \int_a^b [\int_a^b K(y,t)\, f_1(t) dt] f_2(y) dy$$

$$= \int_a^b f_1(t) dt \int_a^b K(y,t)\, f_2(y) dy$$

$$= \int_a^b f_1(t) \mathrm{L}_\mathrm{k}^* f_2(y) \mathrm{dt} = \langle f_1, \mathrm{L}_\mathrm{k}^* f_2 \rangle$$

thus we have

$$\langle \mathrm{L}_\mathrm{k} f_1, f_2 \rangle = \langle f_1, \mathrm{L}_\mathrm{k}^* f_2 \rangle \qquad 3.19$$

Let as now consider a function g given by $g = L_k f$ for some f, suppose that g can be expanded in a series of u_i. That means $g = \mathrm{L}_\mathrm{k} f$ if and only if

$$g = \sum_{\mathrm{i}=1}^{\infty} \langle g, u_i \rangle\, u_i = \sum_{\mathrm{i}=1}^{\infty} \langle \mathrm{L}_\mathrm{k} f, u_i \rangle u_i = \sum_{\mathrm{i}=1}^{\infty} \langle f, \mathrm{L}_\mathrm{k}^* u_i \rangle\, u_i$$

from equation (3.16)

$$g = \sum_{\mathrm{i}=1}^{\infty} \langle f, \delta_\mathrm{i} v_i \rangle\, u_i = \sum_{\mathrm{i}=1}^{\infty} \delta_\mathrm{i} \langle f, v_i \rangle\, u_i \qquad 3.20$$

Furthermore, it should be notice that (3.20) is a generalization of the expansion in the case of symmetrical kernel. This result suggests a possible expansion for an arbitrary kernel $K(x, y)$ as a series of singular functions u_i and $\mathrm{v_i}$

$$\mathrm{K}(x, \mathrm{y}) = \sum_{\mathrm{i}=1}^{\infty} \langle \mathrm{K}(x, \mathrm{y}), v_i(\mathrm{y}) \rangle\, v_i(\mathrm{y})$$

but by definition of eigenfunction and inner product we have

$$\langle \mathrm{K}(x, \mathrm{y}), v_i(\mathrm{y}) \rangle = \int_a^b \mathrm{K}(\mathrm{x}, \mathrm{y})\, v_i(\mathrm{y})\, \mathrm{dy} = \mathrm{L}_\mathrm{k} v_i(x) = \delta_\mathrm{i} u_i(x)$$

$$K(x, y) = \sum_{i=1}^{\infty} \delta_i u_i(x)\, v_i(y) \qquad 3.21$$

That is the structure of the kernel $K(x, y)$ is defined by the singular functions and the singular values. Both the data and solution functions belong to the spaces spanned by the singular functions, which get determined by the kernel respectively.

3.3. Solution of IEFK with General Kernel

In the previous section, the function f was supposed to be known in order to write function g as a series. Now we consider g is the known function, with f to be determined. It is clear that if there is a solution f to IEFK $g = L_k f$, then g must be of the form in equation (3.20). We write, assuming the u_i orthonormal,

$$g(\mathrm{x}) = \sum_{\mathrm{i}=1}^{\infty} g_{\mathrm{i}}\, u_i(x), \qquad g_{\mathrm{i}} = \langle g, u_i \rangle$$

The presence of the term $\langle f, \mathrm{v_i} \rangle$ in equation (3.20) suggests that we expand f in the series of $v_i's$ and we get,

$$f(\mathrm{y}) = \sum_{\mathrm{i}=1}^{\infty} f_{\mathrm{i}}\, v_i(\mathrm{y})$$

putting this expression in $g = \mathrm{L_k} f$ then gives

$$g = \sum_{\mathrm{i}=1}^{\infty} g_{\mathrm{i}}\, u_i(\mathrm{x}) = \sum_{\mathrm{i}=1}^{\infty} f_{\mathrm{i}}\, \mathrm{L_k} v_i(\mathrm{y}) = \sum_{\mathrm{i}=1}^{\infty} f_{\mathrm{i}}\, \delta_{\mathrm{i}} u_i(\mathrm{x})$$

since $\mathrm{L_k} v_i = \delta_{\mathrm{i}} u_i$ and $\langle f, v_i \rangle = f_{\mathrm{i}}$

Identifying the coefficients of these two series it is clear that,

$$g_{\mathrm{i}} = f_i \delta_{\mathrm{i}} \,,$$

Then it implies that

$$f_{\mathrm{i}} = \frac{g_{\mathrm{i}}}{\delta_{\mathrm{i}}} = \frac{\langle g, u_i \rangle}{\delta_{\mathrm{i}}}$$

hence, finally a solution for the IEFK equation is

$$f(\mathrm{y}) = \sum_{\mathrm{i}=1}^{\infty} \frac{\langle g, u_i \rangle}{\delta_{\mathrm{i}}} v_i(y) \qquad 3.22$$

Chapter 4

4. Linear Fredholm Integral Equation of Second Kind

4.1 Linear Fredholm Integral Equation of Second kind with Separable Kernel

Separable kernels $K(x, y)$ as

$$K(x,y) = \sum_{i=1}^{n} a_i(x) b_i(y) \qquad 4.1$$

Where the functions $a_1(x)$, $a_2(x)$, $a_3(x)\ldots$ $a_n(x)$ and $b_1(y)$, $b_2(y)$, $b_3(y)$ ...$b_n(y)$ are linearly independent. With such kernel, the linear Fredholm integral of the *second* kind,

$$f(x) = g(x) + \lambda \int K(x,y) f(y) dy \qquad 4.\ 2$$

becomes

$$f(x) = g(x) + \lambda \sum_{i=1}^{n} a_i(x) \int b_i(y) f(y) dy \qquad 4.3$$

It emerges that the technique of solving this equation is essentially depends on the choice of the parameter λ. On the definition of

$$C_i = \int b_i(y) g(y) dy \qquad 4.4$$

The quantities C_i are constants. By substituting equation (4.4) into equation (4.3) gives

$$f(x) = g(x) + \lambda \sum_{i=1}^{n} C_i a_i(x) \qquad 4.5$$

And the problem reduces to finding the quantitiesC_i. To this end, we put the value of $f(x)$ given by equation (4.5) in equation (4.3) and we get

$$g(x) + \lambda \sum_{i=1}^{n} C_i a_i(x) = g(x) + \lambda \sum_{i=1}^{n} a_i(x) \int b_i(y) \left(g(y) + \lambda \sum_{k=1}^{n} C_k a_k(y) \right) dy$$

$$\sum_{i=1}^{n} a_i(x) \left(C_i - \int b_i(y) \left(g(y) + \lambda \sum_{k=1}^{n} C_k a_k(y) \right) dy \right) = 0$$

but the functions $a_i(x)$ are linearly independent; therefore

$$C_i - \int b_i(y) \left(g(y) + \lambda \sum_{k=1}^{n} C_k a_k(y) \right) dy = 0 \qquad 4.6$$

using the simplified notation

$$\int b_i(y) g(y) dy = g_i \,, \qquad \int b_i(y) a_k(y) = a_{ik}$$

Where g_i and a_{ik} are known constants. Equation (4.6) becomes

$$C_i - \lambda \sum_{k=1}^{n} a_{ik} C_k = g_i \qquad 4.7$$

Where (4.7) is a system of n algebraic equations for the unknown C_i .

The determinant $D(\lambda)$ of the system is

$$D(\lambda) = \begin{vmatrix} 1 - \lambda a_{11} & -\lambda a_{12} & \cdots & -\lambda a_{1n} \\ -\lambda a_{21} & 1 - \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & & & \vdots \\ -\lambda a_{n1} & -\lambda a_{n2} & \cdots & 1 - a_{nn} \end{vmatrix}$$

For all values of λ in which $D(\lambda) \neq 0$, the algebraic system equation (4.7), and the Fredholm integral equation of *second* kind has unique solution. On the other hand, for all values of λ for which $D(\lambda)$ is equal to zero, the algebraic system (4.7), and with the Fredholm integral equation of *second* kind either insolvable or has an infinite number of solutions.

In the following sections we shall discuss the various methods of solutions of the Fredholm integral equation of the *second* kind.

4.2 Decomposition Method

4.2.1 Adomian Decomposition Method

Adomian decomposition method consists of decomposing the unknown function $f(\mathrm{x})$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$f(x) = \sum_{\mathrm{k}=0}^{\infty} f_{\mathrm{k}}(x) \qquad 4.8$$

Where the components of $f_{\mathrm{k}}(x\ ), \mathrm{k} \geq 0$ will be determined recurrently.

To establish the recurrence relation, we substitute equation (4.8) in to equation (4.2) to obtain

$$\sum_{\mathrm{k}=0}^{\infty} f_{\mathrm{k}}(x) = g(x) + \lambda \int_{a}^{\mathrm{b}} \mathrm{K}(x,y) \left(\sum_{\mathrm{n}=0}^{\infty} f_{\mathrm{k}}(y) \right) \mathrm{d}y \qquad 4.9$$

Or equivalently

$$f_0(x) + f_1(x) + f_3(x) \ldots = g(x) + \lambda \int_{a}^{\mathrm{b}} \mathrm{K}(x,y)[f_0(y) + f_1(y) + f_3(y) \ldots]\mathrm{d}y \qquad 4.10$$

The zeroth component $f_0(x)$ is identified by all terms that are not included under the integral sign. This means that the component $f_{\mathrm{k}}(x), \mathrm{k} \geq 0$ of the unknown function $f(x)$ is completely determined by setting the recurrence relation.

$$f_0(x) = g(x), f_{\mathrm{k}+1}(x) = \lambda \int_{a}^{\mathrm{b}} \mathrm{K}(x,y) f_{\mathrm{k}}\,(y)\mathrm{d}y, \mathrm{k} \geq 0 \qquad 4.11$$

Or equivalently

$$f_0(x) = g(x)$$

$$f_1(x) = \lambda \int_{a}^{\mathrm{b}} \mathrm{K}(x,y) f_0(y)\,\mathrm{d}y$$

$$f_2(x) = \lambda \int_{a}^{\mathrm{b}} \mathrm{K}(x,y)\, f_1(y)\mathrm{d}y \qquad 4.12$$

$$f_3(x) = \lambda \int_a^b \mathrm{K}(x,y)\, f_2(y) \mathrm{d}y$$

And so on for other components.

In view of (4.12), the components $f_0(x), f_1(x), f_2(x)$ ... are completely determined. As the result, the solution of $f(x)$ readily obtained in a series form by using the series assumption in equation (4.8). It is clearly seen that the decomposition method converted the integral equation in to an elegant determination of computable components. It was formally shown that if an exact solution exists for the problem, then obtained series converges very rapidly to that exact solution. The convergence concept of the decomposition series was thoroughly investigated by many reasrchars to confirm the rapid convergence of the resulting series. However, for concrete problems, where a closed form solution is not obtainable, a truncated number of terms are usually used for numerical purpose. The more components we use the higher accuracy we obtain.

Demonstration of this can be seen for Fredholm integral equation of the *second,*

$$f(x) = \mathrm{e}^x - x + x \int_0^1 y f(y) \mathrm{d}y \qquad 4.13$$

Adomian decomposition method assumes that the solution $f(x)$ has a series form given in (4.8). Substitute the decomposition series (4.8) in to both sides of (4.13) gives

$$\sum_{\mathrm{n}=0}^{\infty} f_{\mathrm{n}}(x) = \mathrm{e}^x - x + x \int_0^1 y \sum_{\mathrm{n}=0}^{\infty} f_{\mathrm{n}}(y)\, \mathrm{d}y \qquad 4.14$$

or equivalently

$$f_0(x) + f_1(x) + f_2(x) \ldots = \mathrm{e}^x - x + x \int_0^1 y(f_0(y) + f_1(y) + f_2(y) + \cdots) \mathrm{d}y \qquad 4.15$$

We identify the zeroth component by all terms that are not include under the integral sign.Therfore, we obtain the following recurrence relation

$$f_0(x) = \mathrm{e}^x - x$$

$$f_{k+1} = x\int_0^1 y f_k(y)dy, k \geq 0 \quad 4.16$$

Consequently, we obtain

$$f_0(x) = e^x - x$$

$$f_1(x) = x\int_0^1 y\, f_0(y)dy = x\int_0^1 y(e^y - y)dy = \frac{2x}{3}$$

$$f_2(x) = x\int_0^1 y\, f_1(y)dy = \int_0^1 y(\frac{2y}{3})dy = \frac{2x}{9} \quad 4.17$$

$$f_3(x) = x\int_0^1 f_2(y)dy = x\int_0^1 \frac{2y^2}{9}dy = \frac{2x}{27}$$

and so on. Using (4.8) gives the series solution

$$f(x) = e^x - \frac{2}{3}x\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots\right) \quad 4.18$$

Note that, the infinite geometric series at the right side has $a_1 = 1$and the ratio $r = \frac{1}{3}$ .The sum of infinite series for $|r| < 1$ is given by

$$\lim_{n\to\infty} S_n = \frac{a_1}{1-r}$$

$$S = \frac{a_1}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2} \quad 4.19$$

The series solution (4.18) converge the closed form solution

$$f(x) = e^x - x + \frac{2}{3}x \times \frac{3}{2} \ = e^x \quad 4.20$$

4.2.2. Modified Decomposition Method

As stated before, the Adomian decomposition method provides the solution in an infinite series of components. The components $f_{\mathrm{k}}, \mathrm{k} \geq 0$ are easily computed if the inhomogeneous term $g(x)$ in the Fredholm integral equation

$$f(x) = g(x) + \lambda \int_{\mathrm{a}}^{\mathrm{b}} \mathrm{K}(x, y) f(y) \mathrm{d}y$$

Consists of a polynomial of one or two terms. However, if the function $g(x)$ consists of a combination of two or more polynomials, trigonometric functions, hyperbolic functions, and others, the evaluation of the components $f_k, k \geq 0$ requires more work.

The standard Adomian decomposition method employs the recurrence relations

$$\begin{cases} f_0(x) = g(x) \\ f_{1+\mathrm{k}}(x) = \lambda \int_{a}^{\mathrm{b}} \mathrm{K}(x, y)\, f_{\mathrm{k}}(y) \mathrm{d}y, \mathrm{k} \geq 0 \end{cases} \quad 4.21$$

where the solution $f(x)$ is expressed by infinite sum of components defined by

$$f(x) = \sum_{\mathrm{k}=0}^{\infty} f_{\mathrm{k}}(x) \quad 4.22$$

In view of (4.21) the components $f_{\mathrm{k}}(x)$, $\mathrm{k} \geq 0$ is readily obtained.

The modified decomposition method presents a slight variation to the recurrence relation (4.21) to determine the components of $f(x)$ in an easier and faster manner. For many cases, the function $g(x)$ can be set as the sum of two partial functions, namely $g_1(x)$and $g_2(x)$.In other words, we can set

$$g(x) = g_1(x) + g_2(x) \quad 4.23$$

In view of (4.23), we introduce a qualitative change in the formation of the recurrence relation (4.12). The modified decomposition method identifies the zeroth component $f_0(x)$

by one part of $g(x)$, namely $g_1(x)$ or $g_2(x)$. The other part of $g(x)$ added to the components $f_1(x)$ that exist in the standard recurrence relation. The modified decomposition method admits the use of the modified recurrence relation

$$f_o(x) = g_1(x)$$

$$f_1(x) = g_2(x) + \lambda \int_a^b K(x,y) f_o(y) dy \qquad 4.24$$

$$f_{k+1}(x) = \lambda \int_a^b K(x,y)\, f_k(y) dy, k \geq 1$$

It is obvious that the difference between the standard recurrence relations (4.21) and the modified recurrence relation (4.24) rests only in the formation of the first two components $f_0(x)$ and $f_1(x)$ only. The other components $f_k(x), \mathrm{k} \geq 2$ remain the same in the two recurrence relations. Although this variation in the formation of $f_0(x)$ and $f_1(x)$ is slight, however it has been shown that it accelerates the convergence of the solution and minimizes the size of calculation.

The success of this modification depends only on the proper choice of $g_1(x)$ and $g_2(x)$, and this can be made through trials only. The modified decomposition method cannot be used if $g(x)$ consists of only one term.

Demonstration of this can be seen for Fredholm integral equation of *second* kind,

$$f(x) = 3x + \mathrm{e}^{4x} - \tfrac{1}{16}(17 + 3\mathrm{e}^4) + \int_0^1 y f(y) dy \qquad 4.25$$

We first decompose $g(x)$ given by

$$g(x) = 3x + \mathrm{e}^{4x} - \frac{1}{16}(17 + 3\mathrm{e}^4) \qquad 4.26$$

In to two

$$g_1(x) = 3x + \mathrm{e}^{4x}$$
$$g_2(x) = -\tfrac{1}{16}(17 + 3\mathrm{e}^4)$$

We next use the modified recurrence formula (4.24) to obtain

$$f_0(x) = g_1(x) = 3x + \mathrm{e}^{4x}$$

$$f_1(x) = g_2(x) + \lambda \int_a^b K(x,y) f_0(y) dy = -\frac{1}{16}(17 + 3\mathrm{e}^4) + \int_0^1 y(3y + \mathrm{e}^{4y}) dy = 0$$

$$f_{k+1}(x) = \int_0^1 K(x,y) f_k(y) dy = 0 \qquad k \geq 1$$

It obvious that each component of $f_{k,} k \geq 1$ is zero. This in turn gives the exact solution by

$$f(x) = 3x + \mathrm{e}^{4x}$$

4.2.3. Direct Computational Method

In this section, the direct computational method will be applied to solve the Fredholm integral equations. The method approaches Fredholm integral equations in a direct manner and gives the solution in an exact form and not in a sires form. It is important to point out that this method will be applied to the degenerate or separable kernels of the form

$$K(x,y) = \sum_{k=1}^{n} m_k(x) h_k(y) \qquad 4.27$$

The direct computation method can be applied as follows.

A. We first substitute (4.27) into the linear Fredholm integral equation of the form

$$f(x) = g(x) + \lambda \int_a^b K(x,y) f(y) dy \qquad 4.28$$

B. This substitution gives

$$f(x) = g(x) + \lambda m_1(x) \int_a^b h_1(y)f(y)dy + \lambda m_2(x) \int_a^b h_2(y)f(y)dy + \cdots$$

$$+ \lambda m_n(x) \int_a^b h_n(y)f(y)\, dy \qquad 4.29$$

C. Each integral at the right side depends only on the variable y with constant limit of integration for y.This means that each integral is equivalent to a constant. Based on this, equation (4.30) becomes

$$f(x) = g(x) + \lambda a_1 m_1(x) + \lambda a_2 m_2(x) + \cdots + \lambda a_n m_n(x) \qquad 4.30$$

where

$$a_i = \int_a^b h_i\,(y)f(y)dy \qquad 4.31$$

D. Substituting equation (4.30) in to equation (4.31) gives system n alegabric equations that can be solved to determine the constants; $a_i, i \leq n$**.** Using the obtained numerical values of a_i into (4.30) the solution $f(x)$of the Fredholm integral equation (4.28) is readily obtained.

Demonstration of this can be seen for Fredholm integral equation of *second* kind,

$$f(x) = 3x + 3x^2 + \frac{1}{2}\int_0^1 x^2 y f(y)dy \qquad 4.32$$

The kernel $K(x, y) = x^2 y$ is separable. Consequently we rewrite equation (4.32) as

$$f(x) = 3x + 3x^2 + \frac{1}{2}x^2 \int_0^1 y f(y)dy \qquad 4.33$$

The integral at the right side is equivalent to a constant because it depends only on functions of the variable y with constant limits of integration. Consequently equation (4.33) can be rewritten as

$$f(x) = 3x + 3x^2 + \frac{1}{2}x^2 a \qquad 4.34$$

where

$$a = \int_0^1 y f(y) dy \qquad 4.35$$

To determine a , we substitute equation (4.34) into (4.35) to obtain

$$a = \int_0^1 y\left(3y + 3y^2 + \frac{1}{2}ay^2\right)dy \qquad 4.36$$

By integration the right side of (4.36) yields

$$a = \frac{7}{4} + \frac{1}{8}a$$

that gives $a = 2$ by substituting the values of a into equation (4.34) leads to the exact solution

$$f(x) = 3x + 4x^2$$

Note:-the main goal for studying the homogeneous linear Fredholm integral equation of *second* kind is to find nontrivial solution because the trivial solution $f(x) = 0$ is a solution of this equation. Moreover, the Adomian decomposition method is not applicable here because it depends mainly on assigning a non-zero value for the zeroth component $f_0(x)$, and in this kind of equation $g(x) = 0$. Based on this the direct computation method will be employed here to handle this kind of equation.

As stated before, the direct computation method handles Fredholm integral equation in a direct manner and gives the solution an exact form but not in series form as Adomian method. It is important to point out that this method will be applied for the a separable kernel of the form

$$K(x,y) = \sum_k^n m_k(x)h_k(y) \qquad 4.37$$

The direct computational method can be applied as follows.

i. We first substitute (4.37) in to the homogeneous Fredholm integral equation the form

$$f(x) = \lambda \int_a^b K(x,y)\, f(y) dy \qquad 4.38$$

ii. This substitution leads to

$$f(x) = \lambda m_1(x) \int_a^b h_1(y) f(y) dy + \lambda m_2(x) \int_a^b h_2(y) f(y) dy + \ldots$$
$$+ \lambda m_n(x) \int_a^b h_n(y) f(y) dy \qquad 4.39$$

iii. Each integral at the right side depends only on the variable y with constant limit of integration. This means that each integral is equivalent to constant number. Based on this equation (4.39) becomes

$$u(x) = \lambda a_1 m_1(x) + \lambda a_2 m_2(x) + \cdots + \lambda a_n m_n(x) \qquad 4.40$$

where

$$a_i = \int_a^b h_i(y) f(y) dy \qquad 1 \le i \le n \qquad 4.41$$

iv. Substitution of equation (4.40) into equation (4.41) gives a system of n simultaneous algebraic equation that can be solved to determine the constants $a_i \;\; 1 \le i \le n$ using the obtained numerical values of a_i in equation (4.41), the solution $f(x)$ of the homogeneous Fredholm integral equation (4.39) follows immediately.

Demonstration of this can be seen for the homogeneous integral equation of *second* kind,

$$f(x) = \lambda \int_0^{\frac{\pi}{2}} cosx siny f(y) dy$$

This can be rewritten as

$$f(x) = a\lambda cosx \qquad 4.42$$

$$a = \int_0^{\frac{\pi}{2}} sinyf(y)dy \qquad 4.43$$

Substituting equation (4.42) into (4.43) we get,

$$a = \lambda a \int_0^{\frac{\pi}{2}} sinycosydy$$

this gives

$$a = \frac{1}{2} a\lambda$$

Recall that $a = 0$ gives a trivial solution. For $a \neq 0$ we find that the eigenvalues $\lambda = 2$.

This in turn gives the eigenfunction

$$f(x) = Acosx$$

Where A is a nonzero arbitrary constant with $A = 2a$.

Remark: The evaluation is completely dependent upon the structure of the kernel $\mathrm{K(x, y)}$ and sometimes it may happen that the value of a may contain more than one. The computational difficulties may arise in determining the constant a if the resulting algebraic equation is of third or higher.

5. Conclusion

This paper mainly deals with two issues.

First, a general expression for the solution of an IEFK in termes of eigenpairs of the kernel is put forward. This expression completely describes the nature of any solution to the recovery problem and exhibits the difficulties that may arise when dealing with that type of problem.

Second, Adomian decomposition, modified decomposition and direct methods had been exactly and successfully applied to find the solution of Fredholm integral equation of *second* kind.

Generally the concern was the determination of the solution $f(x)$ of the Fredholm integral equation of *first* kind and the *second* kind.

6. Reference

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