



SOLUTION OF THE DIRAC EQUATION IN CURVED
SPACE TIME IN THE PRESENCE OF A STRONG
BACK GROUND MAGNETIC FIELD

By

Eshetu Mekonnen Abebe

A THESIS SUBMITTED TO
THE DEPARTMENT OF PHYSICS

PRESENTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF MASTER OF SCIENCE

ADDIS ABABA UNIVERSITY
ADDIS ABABA, ETHIOPIA
JUNE 2013

ADDIS ABABA UNIVERSITY
SCHOOL OF GRADUATE STUDIES

This is to certify that the thesis prepared by Eshetu Mekonnen Abebe, entitled “ **Solution of the Dirac equation in curved space time in the presence of a strong back ground magnetic field** ” and submitted in partial fulfillment of the requirements for the degree of Masters of science in complies with the regulations of the University and meets the accepted standards with respect to originality.

Approved by the Examination Committee:

Examiner: Prof. Singh P Signature: _____ Date: _____

Examiner: Prof. Chaubey A.K Signature: _____ Date: _____

Advisor: Dr. Legesse Wotro K Signature: _____ Date: _____

Abstract

In this thesis we propose a curved spacetime version of the Dirac equation. The equation has been developed mainly to try and account in a natural way for the solution of the Dirac equation in curved spacetime in an open cosmological universe with a partially horn topology in the presence of a time dependent magnetic field. Since the exact solution cannot be obtained explicitly for arbitrary time dependence of the field, we discuss the asymptotic behavior of the solutions with the help of the relativistic Hamilton-Jacobi equation.

Acknowledgements

First of all, I would like to thank the almighty God for letting me to accomplish this study.

Secondly, I would like to express my strong gratitude and respect to my advisor Dr. Legesse Wotro, for his invaluable advices, continuous support and friendly approach throughout this thesis.

Finally I would like to thank all my friends for their moral and material support. My special thanks goes to my wife Woynishet Worku who always encouraging me and helping me in many aspects.

Table of Contents

Abstract	iii
Acknowledgements	iv
Table of Contents	v
Introduction	1
Out line of the thesis	3
Notation	4
1 Introduction	5
1.1 Magnetic field and its origin in the universe	5
1.2 Atoms in very strong magnetic field	7
1.3 Magnetic vector potential(A)	7
1.4 Gauge choice and transformation	8
1.5 Convention for magnetic field and magnetic vector potential	8
1.5.1 Convention for magnetic field	8
1.5.2 Convention for magnetic vector potential	9
1.6 Pauli matrices	9
1.6.1 Commutation relations	10
1.7 Properties of the γ -matrices	10
1.8 Metric tensors	11
1.9 Homogeneity and isotropy	14
2 Relativistic Wave Equations in Curved Spacetime	15
2.1 Introduction	15
2.2 The Klein-Gordon equation	16
2.3 The Dirac equation	18
2.4 Spinors	19
2.5 Negative energy solutions and antimatter	20
2.6 Spinors for moving particles	21
2.7 The Dirac equation in curved space time	22
2.8 Solution to the curved spacetime Dirac equation	24

3	The Spin Connectin in Wely Space	26
3.1	Introduction	26
3.2	Basic Tetrad formalism	27
3.3	Parallel transfer and covariant differentiation	29
3.4	Spinors in curved space	32
3.5	Coordinate and Lorentz vectors in curved space	33
3.6	The Tetrad postulate	34
3.7	Derivation of Γ_μ	35
4	Solution of the Dirac Equation in Curved Spacetime in the Pres- ence of Strong Magnetic Field	38
4.1	Introduction	38
4.2	Solution of the Hamilton-Jacobi equation	39
4.3	Generally covariant Dirac equation in curved space-time	42
4.4	Solution of the Dirac equation	44
4.5	Relativistic Landau Energy Level in 4 dimensions	51
4.6	Spinors of the Dirac equation in the presence of strong magnetic field	52
5	Conclusion and Remarks	55
	References	56

Introduction

A successful relativistic wave equation was first obtained by the British physicist P.A.M Dirac in 1928, who made a fundamental contribution to the early development of both quantum mechanics and quantum electrodynamics. He formulated the Dirac equation which describes the behavior of fermions and which led to the prediction of the presence of anti matter. In the same way as the schrodinger equation can not be derived from the classical mechanics because it is essentially new physics, any relativistic wave equation can only be guessed by the proceusence against experiment. Dirac went off to search for an equation that was consistent with special theory of relativity and avoided the non-positive definiteness of induction, and its truth or other wise must be established by testing its conseq probability density, ρ . An equation which is linear in time derivative that is no more ρ less than zero, an equation that can reproduce the energy momentum relation and an equation which is relativistically covariant that is whose form is preserved under a lorentz boost are the demands on an equation describing a relativistic theory of quantum mechanics [1,2]. The Dirac equation is a relativistic quantum mechanical wave equation formulated by Paul Dirac, and provides a description of elementary spin-1/2 particles, such as electrons, consistence with the principle of both quantum mechanics and the theory of special relativity. The equation also demands the existence of anti-particles and actually predated their experimental discovery, making the discovery of the positron, the anti particle of the electron, which is one of the greatest triumphs of modern theoretical physics [3]. During the last years a large amount of observational data has been reported showing that our universe is almost isotropic and homogeneous [3]. The study

of the structure of the Cosmic Microwave Radiation leads us to conclude that the ratio of the total density to the critical density of the universe Ω_0 is likely to be close to one [13], favoring a spatially flat Robertson-Walker metric over other topologies [8]. It is well known that general relativity is a local metrical theory and therefore the corresponding Einstein field equations do not fix the global topology of spacetime and consequently the universe may have compact spatial sections with a nontrivial topology [4,5], then the observational data does not rule out the possibility that our universe possesses a hyperbolic topology [4,6,8]. In order to study quantum processes in curved space-times one has to fulfill a preliminary step which consists in having a description of the single-mode solution of the relativistic particles or perturbations in those background fields, i.e., exact solution of the relativistic scalar and spinor wave equations. Among different methods for solving relativistic wave equations in curved spaces and in curvilinear coordinates; the method of separation of variables is one of the most widely used [4,9]. In this thesis we solve the Dirac equations in the Friedman universe associated with the metric $a^{-2}(\eta)ds^2 = -d\eta^2 + dz^2 + e^{-2z}(dx^2 + dy^2)$ in the presence of a time dependent magnetic field. In order to solve the Dirac equation we apply the algebraic method of separation of variables. We compare the solutions with those obtained after solving the relativistic Hamilton Jacobi equation.

Out line of the thesis

The main purpose of this thesis is to find the solutions of the Dirac equation in curved spacetime in the presence of strong back-ground magnetic field. The first chapter of this thesis is devoted to the discussion of the concepts of magnetic field. In addition magnetic vector potential, gauge conditions, spin matrices, some astronomical objects with a very strong magnetic field as well as notations and conventions are discussed. The next chapter, is devoted to the discussion of relativistic wave equations in curved spacetime, the Dirac equation, the Dirac equation in curved spacetime, as well as the solutions to the curved Dirac equation for a free particle . The spin connection in wely space, basic Tetrad formalism, parallel transfer and covariant differentiation, spinors in curved space, coordinate and lorentz vectors in curved space, the Tetrad postulate, and derivation of Γ_μ is briefly discussed in chapter three. The solutions of the Dirac equation in curved spacetime for a particle in the presence of strong back ground magnetic field is briefly discussed in chapter four. At last conclusions and remarks are included.

Notation

4-vector: $(t, x, y, z) \rightarrow (x^0, x^1, x^2, x^3)$.

Indices convention:

- Roman letters (i, j, k, l, m, n) run from 1 to 3;
- Greek letters ($\alpha, \beta, \gamma, \delta, \mu, \nu, \eta, \lambda$) run from 0 to 3.

Einstein summation (summation over repeated indices):

$$v'^{\alpha} = \sum_{\beta=0}^3 \frac{\partial x'^{\alpha}}{\partial x^{\beta}} v^{\beta} \equiv \frac{\partial x'^{\alpha}}{\partial x^{\beta}} v^{\beta}.$$

Contravariant vector transforms as $A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta}$.

Covariant vector transforms as $A'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} A_{\beta}$.

Tensors: objects with multiple indices.

First rank (one index):

- Contravariant: $A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta}$.
- Covariant: $A'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} A_{\beta}$.

Second rank (two indices):

- Contravariant: $A'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} A^{\mu\nu}$.
- Covariant: $A'_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} A_{\mu\nu}$.
- Mixed: $A'^{\alpha}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} A^{\mu}_{\nu}$.

In this thesis we use natural units ($\hbar = c = 1$) and the signature of the Minkowski spacetime is assumed to be +, -, -, -.

Chapter 1

Introduction

1.1 Magnetic field and its origin in the universe

Magnetic fields are most frequently found in various length scales and various positions in our universe. The first ever magnetic field in the universe arose within 370,000 years of the big bang, a new analysis suggests. The work relies on standard physics, unlike some previous theories, and may shed light on how the very first star grew [14]. Relativity confined magnetic fields like those in the earth and sun are generated by the turbulent mixing of conducting fluids in their cores. But large scale fields tangled within galaxies and clusters of galaxies are harder to explain by fluid mixing alone. This is because most galaxies have rotated only a few dozen times since they formed. Due to this the origin of the magnetic fields in the galaxies and in the cluster of galaxies is unknown but most say in general one needs some initial, small magnetic field. Some researchers have tried to explain the origin of this so called seed field by invoking new physical mechanism-such as the coupling of electromagnetic fields with exotic particles or gravity in the first instants after the Big Bang [11]. There have been many models in this direction, most of which rely on new physics, and therefore not convincing. Now, researchers led by Kiyotomo chief of the national astronomical observatory of Japan in Tokyo have used standard physics to explain the seed field. They say the field began before the first atom formed, when the universe was a hot soup of protons, electrons and photons-a state that lasted for the first 370,000 after the big bang. Photons exert a pressure on electrons than protons and also

scatter off electrons more often. The researchers found that the difference in the movement of electrons and protons generated a rotating electric current, which produces a magnetic field. Magnetic fields in addition to gravity play a critical role in the formation process of various objects and their dynamical evolution in the universe. The presence of the magnetic field has an important consequence since the magnetic force can locally be much greater than the gravitational force [14]. When we come to the average magnitude of the magnetic field of some objects in our universe, the earth's magnetic field which can deflect compass needles is, 0.6 gauss. The magnetic field in a strong sun spot with dark magnetized area on the solar surface is 100 gauss. The strongest sustained that is steady magnetic field achieved so far in the laboratory which can be generated by huge electromagnets is 4.5×10^5 gauss. The strongest man-made field ever achieved which can be made by using focused explosive charges is 10^7 gauss. The strongest field ever detected on non neutron stars which can be found in a strongly magnetized compact white dwarf stars is 10^8 gauss. Typical surface polar magnetic fields of radio pulsars which are a familiar kind of spinning neutron stars is $\sim 10^{12}$ - 10^{13} gauss, Magnetars which are soft gamma repeaters, and anomalous X-ray pulsars is 10^{14} - 10^{16} gauss [14]. Physicists have not made fields stronger than 4.5×10^5 gauss in the lab because the magnetic stress of such fields exceed the tensile strength of terrestrial materials [15].

1.2 Atoms in very strong magnetic field

The strongest magnetic field that one can ever likely to encounter is 10^4 gauss if we have a magnetic resonance imaging (MRI) scan for medical diagnosis. Such fields pose no threat to ones health, hardly affecting the atoms in ones body. Fields in excess of 10^9 gauss however, would be instantly lethal. Such fields strongly distort atoms, compressing atomic electron clouds in to cigar shapes, thus rendering the chemistry of life impossible. In fields much stronger than 10^9 gauss, atoms are compressed in to thin needles. At 10^{14} gauss atomic needles have widths of one percent of their length and hundreds of times thinner than un magnetized atoms [14].

1.3 Magnetic vector potential(A)

The vector potential provides a mathematical way of defining a magnetic field in classical electromagnetism. It is not directly observable, only the field it describes may be measured. If the magnetic vector potential is time-dependent, it also contributes to the electric field.

Since the magnetic field is divergence free that is $\nabla \cdot B = 0$ (Gauss law for magnetism) this guaranties that A is fully specified by its divergence and curl, which is Helmholtzs theorem. Magnetic fields generated by steady and unsteady current satisfy $\nabla \cdot B = 0$. This immediately allows to write [11]

$$B = \nabla \times A, \tag{1.3.1}$$

since the divergence of a curl is automatically zero, that is

$$\nabla \cdot B = 0 = \nabla \cdot (\nabla \times A) = 0.$$

Eqn.(1.3.1) implies a magnetic vector potential is a three dimensional vector whose curl is the magnetic field. The vector potential A is used extensively when studying the lagrangian in classical mechanics and in quantum mechanics, such as the schrodinger equation for charged particles or the Dirac equation.

1.4 Gauge choice and transformation

It should be noted that the above definition eqn.(1.3.1) does not define magnetic vector potential uniquely because by definition, we can arbitrary add curl free components to the magnetic potential with out changing the observed magnetic field. Thus, there is a degree of freedom available when choosing A. This condition is known as gauge invariance. According to eqn.(1.3.1) the magnetic field is invariant under the transformation,

$$A \longrightarrow A - \nabla\phi. \quad (1.4.1)$$

In other words the vector potential is undetermined with in the gradient of a scalar field. The above equation, eqn.(1.4.1), is known as gauge transformation. The choice of a particular function ϕ is referred to us a choice of the gauge. We are free to fix the gauge to be what ever we like. The most sensible choice is the one which makes our equation as simple as possible. The usual gauge for A is such that

$$\nabla \cdot A = 0. \quad (1.4.2)$$

This particular choice is known as the coulomb gauge.

1.5 Convention for magnetic field and magnetic vector potential

1.5.1 Convention for magnetic field

To find the solutions of the Dirac equation in curved spacetime in the presence of strong back ground magnetic field, we consider the field to be directed along the z-axis so that the field always be

$$B = B\hat{z}, \quad (1.5.1)$$

where \hat{z} is the unit vector along the z-axis.

1.5.2 Convention for magnetic vector potential

From the discussion of gauge condition, the vector potential can be chosen in many equivalent ways, but the most sensible is the one which makes our equation simple. So through out this thesis we take,

$$A = A_x \hat{i}, \quad (1.5.2)$$

where \hat{i} is the unit vector along the x-axis.

1.6 Pauli matrices

The pauli matrices are a set of three 2×2 complex Hermitian and unitary matrices.

Usually designated by σ . They are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.6.1)$$

The names refers to Wolf gang Pauli, who suggested them.

1.6.1 Commutation relations

The pauli matrices obey the following commutation and anti commutation relations as:

$$[\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k \quad (1.6.2)$$

and

$$\{\sigma_i, \sigma_j\} = 2\sigma_{ij}, \quad (1.6.3)$$

where ϵ_{ijk} is the levi-Civita symbol, in which

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is an even permutation} \\ -1 & \text{if } ijk \text{ is an odd permutation} \\ 0 & \text{if any index is repeated} \end{cases}, \quad (1.6.4)$$

and σ_{ij} is the kroneker delta

$$\sigma_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}. \quad (1.6.5)$$

1.7 Properties of the γ -matrices

All physically relevant information about the γ -matrices are given in the anti-commutation relation

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}. \quad (1.7.1)$$

For $\mu \neq \nu$ follows

$$\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu. \quad (1.7.2)$$

For $\mu = \nu = 0$:

$$(\gamma^0)^2 = 1. \quad (1.7.3)$$

For $\mu = \nu = k$, $k = 1, 2, 3$

$$(\gamma^k)^2 = -1. \quad (1.7.4)$$

1.8 Metric tensors

Flat Euclidian space. Our common sense has taught us to think in terms of a flat space metric (Euclidian), where parallel lines never cross and angles in a triangle always sum up to 180° , thus strongly reinforcing our Newtonian notion of absolute space. In this formulation, the invariant line element in Cartesian coordinates of space (x^1, x^2, x^3) is:

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \quad (1.8.1)$$

and space is assumed to be flat. Another way to write this is

$$ds^2 = \delta_{ij} dx^i dx^j, \quad (1.8.2)$$

where δ_{ij} is the Kronecker delta function ($\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ otherwise). Therefore, the Euclidian flat space metric tensor for Cartesian coordinates is given by:

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (1.8.3)$$

Invariant line element in an arbitrary coordinate system in flat space can be written in terms of Cartesian coordinates (change of variables) as:

$$ds^2 = \delta_{ij} dx^i dx^j = \delta_{ij} \frac{\partial x^i}{\partial x'^k} \frac{\partial x^j}{\partial x'^l} dx'^k dx'^l = p_{kl} dx'^k dx'^l, \quad (1.8.4)$$

where p_{kl} is the space metric of the new coordinate system.

Since the indices of the metric tensor enter the eqn.(1.8.4) in an identical fashion, the metric tensor is always symmetric. Furthermore, isotropy and homogeneity (as assumed in the flat Euclidian space) implies that the metric tensor in such a space will necessarily be diagonal.

Flat Minkowski spacetime. We can now generalize this to 4-vectors in flat spacetime (x^0, x^1, x^2, x^3) :

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta, \quad (1.8.5)$$

where $\eta_{\alpha\beta}$ is the Minkowski (flat) spacetime metric tensor

$$\eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (1.8.6)$$

Again, isotropy and homogeneity of spacetime leads to a diagonal metric tensor.

Curved spacetime. For a general (possibly curved) covariant spacetime metric tensor $g_{\alpha\beta}$, the invariant line element is given by

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta. \quad (1.8.7)$$

Of course this is in general far too complicated to consider. We will try to reduce the number of different components to the minimum possible and to isolate the spacetime dependence into a small number of factors. Formally this is done within differential geometry by postulating symmetries and then use the corresponding Killing vectors to impose constraints on the metric. Instead, we will just try to guess the correct result. An actual derivation can be found e.g. in *Gravitation and Cosmology* by Steven Weinberg. The fundamental, cosmological principle says that no point in the universe is preferred [8]. Thus the universe needs to be isotropic (i.e. the same in all directions, rotationally symmetric) and homogeneous (the same at all points, symmetric under translations) [8]. If we impose these conditions on spacetime, we are led to a steady-state universe which looks the same at all times. One can show that it is unique, and that it is the exponentially expanding de Sitter space. Thus we can only impose homogeneity and isotropy on the spatial part of the metric. A theorem of differential geometry tells us that the only 3-spaces which fulfil this requirement are those with (spatially) constant curvature $K(t)$. For any value of K the spaces are uniquely determined (up to isometries), and there are three classes: open ($K < 0$), flat ($K = 0$) and closed ($K > 0$) spaces. The line element is then in spherical coordinates (t, R, θ, φ) .

$$ds^2 = dt^2 - \left(\frac{dR^2}{1 - KR^2} + R^2 d\Omega \right). \quad (1.8.8)$$

Both R and K are functions of time (but not of space because of the homogeneity condition). $d\Omega = d\theta^2 + \sin^2\theta d\varphi^2$ is the two-dimensional angular volume element. We see as well that the metric is reduced to the Minkowski case with signature $(+, -, -, -)$ if $K = 0$ and if R is independent of time. t is the physical (proper) time and R the physical distance. We can rewrite the metric by extracting the time dependence of R into a scale factor $a(t)$ by introducing the comoving distance r , $dR(t) \equiv a(t)dr$. Then we are left with

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega \right). \quad (1.8.9)$$

If we set up two observers without relative motion at a comoving distance r_0 , then they will remain at this distance (hence the name), but their physical distance R will change over time. The curvature parameter K is related to κ by $\kappa = a^2 K$. Conventionally, the scale factor is normalised so that κ takes the values $-1, 0$ or 1 . Another common normalisation, especially for $\kappa = 0$, is to set the scale factor today to unity, $a(t_0) = a_0 = 1$.

It is possible and often useful to rewrite the metric (1.8.9) by transforming $r \rightarrow S_\kappa(r)$ where

$$S_\kappa(r) = \begin{cases} \sin r & \text{if } \kappa = 1 \\ r & \text{if } \kappa = 0 \\ \sinh r & \text{if } \kappa = -1 \end{cases} \quad (1.8.10)$$

The resulting metric (with the comoving distance called again r) is

$$ds^2 = dt^2 - a^2(t)(dr^2 + S_\kappa(r)^2 d\Omega). \quad (1.8.11)$$

The two expressions (1.8.9) and (1.8.11) are equivalent and we can use whichever is simpler in a given situation. We can also introduce the time analogue to the comoving distance, called conformal time η and defined by $dt \equiv a(t)d\eta$. Written in these coordinates, the line element (1.8.9) is

$$ds^2 = a(\eta)^2 (d\eta^2 - \left\{ \frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega \right\}) \quad (1.8.12)$$

and the equivalent expression for eqn.(1.8.11) is

$$ds^2 = a^2(\eta)(d\eta^2 - dr^2 - S_\kappa(r)d\Omega) \quad (1.8.13)$$

1.9 Homogeneity and isotropy

The evidence that the Universe becomes smooth on large scales supports the use of the cosmological principle. It is therefore believed that our large-scale Universe possesses two important properties, homogeneity and isotropy. Homogeneity is the statement that the Universe looks the same at each point, while isotropy states that the Universe looks the same in all directions. These do not automatically imply one another. For example, a Universe with a uniform magnetic field is homogeneous, as all points are the same, but it fails to be isotropic because directions along the field lines can be distinguished from those perpendicular to them. Alternatively, a spherically-symmetric distribution, viewed from its central point, is isotropic but not necessarily homogeneous. However, if we require that a distribution is isotropic about every point, then that does enforce homogeneity as well [8]. The Friedman universe is a cosmological model where the universe is assumed to be homogeneous, isotropic, and filled with ideal fluid. The assumption of homogeneity and isotropicity of the universe is often referred to as the cosmological principle.

Chapter 2

Relativistic Wave Equations in Curved Spacetime

2.1 Introduction

The Dirac Equation is a relativistic quantum mechanical wave equation invented by Paul Dirac in 1928 originally designed to overcome the criticism of the Klein-Gordon Equation [1]. The Klein-Gordon equation gave negative probabilities and this is considered to be physically meaningless. Despite this fact, this equation accounts well for Bosons, that is spin zero particles. This criticism leveled against the Klein-Gordon equation, motivated Dirac to successfully seek an equation devoid of negative probabilities. The Dirac Equation is consistent with Quantum Mechanics (QM) and fully consistent with the Special Theory of Relativity (STR). This equation accounts in a natural way for the nature of particle spin as a relativistic phenomenon and amongst its prophetic achievements was its successful prediction of the existence of anti-particles [2]. In its bare form, the Dirac Equation provided us with an impressive and accurate description of the Electron hence it being referred in most of the literature as the Dirac Equation for the Electron [2].

2.2 The Klein-Gordon equation

From elementary quantum mechanics we know the schrödinger equation

$$i\frac{\partial\psi}{\partial t} = \left[-\frac{1}{2m_0}\nabla^2 + v(x)\right]\psi(x, t) \quad (2.2.1)$$

corresponds to the nonrelativistic energy in operator form,

$$\hat{E} = \frac{\hat{p}^2}{2m_0} + v(x), \quad (2.2.2)$$

where

$$\hat{E} = i\frac{\partial}{\partial t}, \quad \hat{p} = -i\nabla \quad (2.2.3)$$

are the operators of energy and momentum, respectively. In order to get relativistic wave equation we start by considering free particles with the relativistic relation

$$p^\mu p_\mu = E^2 - p \cdot p = m_0^2. \quad (2.2.4)$$

We now replace the four-momentum p^μ by the four-momentum operator

$$\begin{aligned} \hat{p}^\mu &= i\frac{\partial}{\partial x_\mu} = i\left\{\frac{\partial}{\partial(t)}, \frac{-\partial}{\partial x}, \frac{-\partial}{\partial y}, \frac{-\partial}{\partial z}\right\} \\ &= i\left\{\frac{\partial}{\partial(t)}, -\nabla\right\} = \{\hat{p}_0, \hat{p}\}. \end{aligned} \quad (2.2.5)$$

Thus the Klein-Gordon equation for free particles,

$$\hat{p}^\mu \hat{p}_\mu \psi = m_0^2 \psi. \quad (2.2.6)$$

Here m_0 is the rest mass of the particle.

Free solutions are of the form

$$\begin{aligned} \psi &= \exp(-ip_\mu x^\mu) = \exp[-i(p_0 x^0 - p \cdot x)] \\ &= \exp[i(p \cdot x - p_0 x^0)] \\ &= \exp[i(p \cdot x - Et)]. \end{aligned} \quad (2.2.7)$$

Indeed, insertion of eqn.(2.2.7) into eqn.(2.2.6) leads to the condition

$$p^\mu p_\mu \psi = m_0^2 \psi \rightarrow p^\mu p_\mu \exp(-ip_\mu x^\mu) = m_0^2 \exp(-ip_\mu x^\mu)$$

$$\rightarrow p^\mu p_\mu = m_0^2 \quad \text{or} \quad E^2 - p \cdot p = m_0^2,$$

which results in

$$E = \pm \sqrt{m_0^2 + p^2}. \quad (2.2.8)$$

Thus, there exist solutions both for positive $E = +\sqrt{m_0^2 + p^2}$ as well as for negative $E = -\sqrt{m_0^2 + p^2}$ energies respectively. The solution yielding negative energy are physically connected with antiparticles.

2.3 The Dirac equation

Suppose we have a particle of rest mass m_0 and momentum p and energy E , Albert Einstein, from his 1905 special relativity paper, derived the basic equation [6]

$$E^2 = p^2 + m_0^2, \quad (2.3.1)$$

which later formed the basis of the Klein-Gordon theory upon which the Dirac theory was founded. Using the already established canonical quantisation procedures Klein and Gordon proposed the Klein-Gordon equation

$$\square\psi = (m_0^2)\psi, \quad (2.3.2)$$

which describes a spin-0 quantum mechanical scalar particle whose wave-function is ψ and where

$$\square = \nabla^2 - \frac{\partial^2}{\partial t^2}. \quad (2.3.3)$$

This equation allows for negative probabilities and as already stated, Dirac was not satisfied with the Klein-Gordon theory. He noted that the Klein-Gordon equation is second order differential equation and his suspicion was that the origin of the negative probability solutions may have something to do with this very fact. He sought an equation linear in both the time and spatial derivatives that would upon squaring reproduce the Klein-Gordon equation. The equation he found was [6]

$$[i\gamma^\mu\partial_\mu - m_0]\psi = 0, \quad (2.3.4)$$

Where

$$\gamma^0 = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ \sigma^i & \mathbf{0} \end{pmatrix}, \quad (2.3.5)$$

are the 4×4 Dirac gamma matrices (\mathbf{I} and $\mathbf{0}$ are the 2×2 identity and null matrices respectively) and ψ is the four component Dirac wave-function. Equation (2.3.4) is the original Dirac equation.

2.4 Spinors

The Dirac equation describes the behaviour of spin-1/2 fermions in relativistic quantum field theory. For a free fermion the wavefunction is the product of a plane wave and a **Dirac spinor**, $u(p^\mu)$:

$$\psi(x^\mu) = u(p^\mu)e^{-ip \cdot x}. \quad (2.4.1)$$

Substituting the fermion wavefunction, ψ , into the Dirac equation:

$$(\gamma^\mu p_\mu - m)u(p) = 0. \quad (2.4.2)$$

For a particle at rest, $\mathbf{p}=0$, we find the following equations;

$$(i\gamma^0 \frac{\partial}{\partial t} - m)u = (\gamma^0 E - m)u = 0$$

$$\hat{E}u = \begin{pmatrix} mI & 0 \\ 0 & -mI \end{pmatrix} u. \quad (2.4.3)$$

The solutions are four eigen spinors:

$$u^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad u^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad (2.4.4)$$

and the associated wave functions of the fermion is:

$$\psi^1 = e^{-imt}u^1, \quad \psi^2 = e^{-imt}u^2, \quad \psi^3 = e^{+imt}u^3, \quad \psi^4 = e^{+imt}u^4. \quad (2.4.5)$$

Note that the spinors are 1×4 column matrices, and that there are four possible states. The spinors are, however, not four-vectors: the four components do not represent t, x, y, z. The four components are a surprise: we would expect only two spin states for a spin-1/2 fermion! Note also the change of sign in the exponents of the plane waves in the states ψ^3 and ψ^4 . The four solutions in equations (2.4.4) and (2.4.5) describe two different spin states (\uparrow and \downarrow) with $E = m$, and two spin states with $E = -m$.

2.5 Negative energy solutions and antimatter

To describe the negative energy states, Dirac postulated that an electron in a positive energy state is produced from the vacuum accompanied by a hole with negative energy. The hole corresponds to a physical antiparticle, the positron, with charge $+e$. Another interpretation (Feynman-Stückelberg) is that the $E = -m$ solutions can either describe a negative energy particle which propagates backwards in time, or a positive energy antiparticle propagating forward in time:

$$e^{-i[(-E)(-t)-(-\mathbf{p})\cdot(-\mathbf{x})]} = e^{-i[Et-\mathbf{p}\cdot\mathbf{x}]} \quad (2.5.1)$$

2.6 Spinors for moving particles

For a moving particle, $\mathbf{p} \neq 0$ the Dirac equation becomes

$$(\gamma^\mu p_\mu - m)(u_A u_B) = \begin{pmatrix} E - m & -\vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -E - m \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0, \quad (2.6.1)$$

where u_A and u_B denote that 1×2 upper and lower components of u respectively.

The equations for u_A and u_B are coupled:

$$u_A = \frac{\vec{\sigma} \cdot \vec{p}}{E - m} u_B \quad u_B = \frac{\vec{\sigma} \cdot \vec{p}}{E + m} u_A. \quad (2.6.2)$$

The solutions are obtained by successively setting:

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (2.6.3)$$

to give;

$$u^1 = \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}, \quad u^2 = \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}, \quad u^3 = \begin{pmatrix} \frac{-p_z}{-E+m} \\ \frac{p_x - ip_y}{-E+m} \\ 1 \\ 0 \end{pmatrix}, \quad u^4 = \begin{pmatrix} \frac{-p_x + ip_y}{-E+m} \\ \frac{p_z}{-E+m} \\ 0 \\ 1 \end{pmatrix}, \quad (2.6.4)$$

where

$$u^p = \begin{pmatrix} u_A \\ u_B \end{pmatrix}. \quad (2.6.5)$$

The u^1 and u^2 solution describe an electron of energy $E = +\sqrt{m^2 + \vec{p}^2}$, and momentum \vec{p} . The u^3 and u^4 of eqn(2.6.4) describe a positron of energy $E = -\sqrt{m^2 + \vec{p}^2}$, and momentum \vec{p} .

2.7 The Dirac equation in curved space time

We note that the Dirac equation is designed so that it is consistent with the STR, that is flat spacetime. What is its equivalent in curved spacetime and could this explain some of what the flat spacetime Dirac equation is incapable of accounting for? Taking it from the General Theory of Relativity (GTR) that matter curves spacetimes, it goes without say that we live in a curved spacetime since we are inhabitants of a massive object, the Earth and a Universe vastly populated by massive objects [1]. From this it is expected that some subtle connection to the fabric of curved spacetime will arise. Let us begin by looking at the Einstein equation(2.3.1) . We know that its equivalent in curved spacetime is given by

$$g_{\mu\nu}p^\mu p^\nu = m_0^2, \quad (2.7.1)$$

where $p^\mu=(E,P)$ is the usual four momentum and $g_{\mu\nu}$ is the metric of spacetime. In the case of flat spacetime, the metric is given by

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (2.7.2)$$

and thus there are no off-diagonal terms in the expression of the dot product $(E, -p)$. $(E, -p)$. In curved spacetime off-diagonal terms will emerge thus what we seek are the off-diagonal terms to come into existence for the curved spacetime Dirac equation. Taking the general case in which the spacetime is fully curved, that is, for all μ, ν we have $g_{\mu\nu} \neq 0$ the off-diagonal terms are no longer equal to zero. This should result in the following equation

$$E^2 - \delta^{\mu\nu}[p_\mu, p_\nu] - p^2 = m_0^2, \quad (2.7.3)$$

where the bracket $[,]$ is the usual commutator bracket and [1]

$$\sigma^{\mu\nu} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}. \quad (2.7.4)$$

This equation is the equivalent of eqn.(2.3.1) in curved spacetime and admits two solutions E_+, E_- where we set $E_+ > E_-$. It follows from the above that

$$E = \sum |p_i| \pm \sqrt{(2p^2 + 2\hat{\sigma}^{ij}p_i p_j + m_0^2)}, \quad (2.7.5)$$

where E_+ takes the case of the plus and E_- takes the case of the minus. The term $|p_i|$ has been made strictly positive because the time axis relative to the spatial x, y, z axis is assumed to configure itself in such a manner that the term $g_{0i}p^0 p^i > 0$. From this equation, it follows that for all p_i and $m_0 > 0$, the energy $E_+, E_- > 0$. Likewise, it follows that the Klein-Gorden equation in curved spacetime will be given by

$$\square - \hat{\sigma}^{\mu\nu}[\partial_\mu, \partial_\nu]\psi = (m_0)\psi. \quad (2.7.6)$$

If this is the curved spacetime Klein-Gordon equation, it follows that there must correspond an equivalent Dirac equation which upon "squaring" should reduce to this equation. following Dirac (1928) we write

$$[i\gamma^0\partial_0 + i\gamma^i\partial_i - m_0]\psi = 0 \quad (2.7.7)$$

and upon "squaring", that is multiplying the above from the left by

$[i\gamma^0\partial_0 + i\gamma^i\partial_i + m_0]$ we get

$$-(\bar{\gamma}^0)^2\partial_0^2\psi - (\bar{\gamma}^i)^2\partial_i^2\psi - \{\bar{\gamma}^0, \bar{\gamma}^i\}\partial_0\partial_i\psi - \{\bar{\gamma}^i, \bar{\gamma}^j\}_{i\neq j}\partial_i\partial_j\psi - (m_0^2)\psi = 0 \quad (2.7.8)$$

and this compeles the constraints

$$(\bar{\gamma}^0)^2 = I, \quad (\bar{\gamma}^i)^2 = -I, \quad \{\bar{\gamma}^0, \bar{\gamma}^i\} = 2I, \quad \{\bar{\gamma}^i, \bar{\gamma}^j\}_{i\neq j} = 2I, \quad (2.7.9)$$

where $\bar{\gamma}^\mu$ are what we shall call the Dirac bar matrices or simple the gamma-bar matrices. The solution to the above is given by

$$\bar{\gamma}^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \bar{\gamma}^i = \begin{pmatrix} I & \sigma^i \\ \sigma^i & -I \end{pmatrix}. \quad (2.7.10)$$

The bar-gamma matrices like Dirac matrices satisfy the relation

$$\{\bar{\gamma}^\mu, \bar{\gamma}^\nu\} = 2I^{\mu\nu}, \quad (2.7.11)$$

where

$$I^{\mu\nu} = \begin{pmatrix} I & I & I & I \\ I & -I & I & I \\ I & I & -I & I \\ I & I & I & -I \end{pmatrix}, \quad (2.7.12)$$

where I is nothing but the 4×4 identity matrix. It follows that the Dirac equation in curved space time will be given by

$$[i\bar{\gamma}^\mu \partial_\mu - m]\psi = 0. \quad (2.7.13)$$

2.8 Solution to the curved spacetime Dirac equation

It is most logical at this point to ask what kind of solutions does eqn.(2.7.13) admit? First we begin by considering the solutions for which the wave-function is independent of position, that is

$$\frac{\partial\psi}{\partial x^1} = \frac{\partial\psi}{\partial x^2} = \frac{\partial\psi}{\partial x^3} = 0, \quad (2.8.1)$$

which leads to

$$i\bar{\gamma}^0 \frac{\partial\psi}{\partial t} - m\psi = 0, \quad (2.8.2)$$

which is the same as the flat spacetime Dirac equation thus giving the same solutions. To see this let

$$\psi = \begin{pmatrix} \Phi \\ \chi \end{pmatrix}, \quad (2.8.3)$$

where

$$\Phi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \chi = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} \quad (2.8.4)$$

and substitute this into eqn.(2.8.2) and all this leads to two decoupled equations $i\bar{\gamma}^0 \frac{\partial\Phi}{\partial t} - m\Phi = 0$ and $i\bar{\gamma}^0 \frac{\partial\chi}{\partial t} - m\chi = 0$, the solutions of which are $\Phi = \Phi(0)\exp(+imt)$, $\chi = \chi(0)\exp(-imt)$ respectively. The solution Φ represents a particle with a positive energy while χ is for a particle with negative energy. Though this has not been

explicitly mentioned, one of the main objectives of searching for the curved space-time Dirac equation was to reed ourself of negative energy solutions. We note however that if the wave-function is dependent on position, at least for one of the position, x^i , no negative solutions will exist [1]. Therefore, if we are to disallow for the existence of negative energy, we are here forced to conclude that the wave-function of the particle must always be dependent on both time and at least one of the space coordinates! For free particle solutions, as usual we propose $\psi = u_p \exp(ip_\mu x^\mu)$ where

$$u_p = \begin{pmatrix} \Phi \\ \chi \end{pmatrix}. \quad (2.8.5)$$

Now inserting this into eqn.(2.7.13) leads to the solution

$$\chi = \frac{\vec{\sigma} \cdot \vec{p}}{E + \sum_i p_i + m} \Phi, \quad (2.8.6)$$

$$\Phi = \frac{\vec{\sigma} \cdot \vec{p}}{E + \sum_i p_i - m} \chi, \quad (2.8.7)$$

which in-turn leads to a solution of four particles with the first pair having energy E_+ and the second pair having energy E_- . The four particle solutions are

$$\chi_u = \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E + \sum_i p_i + m} \\ \frac{p_z + ip_y}{E + \sum_i p_i + m} \end{pmatrix}, \quad \chi_d = \begin{pmatrix} 0 \\ 1 \\ \frac{p_z - ip_y}{E + \sum_i p_i + m} \\ \frac{-p_z}{E + \sum_i p_i + m} \end{pmatrix}, \quad (2.8.8)$$

$$\Phi_u = \begin{pmatrix} \frac{-p_z}{E + \sum_i p_i + m} \\ \frac{-p_z - ip_y}{E + \sum_i p_i + m} \\ 1 \\ 0 \end{pmatrix}, \quad \Phi_d = \begin{pmatrix} \frac{-p_z + ip_y}{E + \sum_i p_i - m} \\ \frac{p_z}{E + \sum_i p_i - m} \\ 0 \\ 1 \end{pmatrix} \quad (2.8.9)$$

where the subscript u and d mean spin up and spin down respectively. These solutions are the same as the Dirac solution with the exception of the fact that a new term, namely $\sum_i p_i$ comes into being.

Chapter 3

The Spin Connection in Weyl Space

3.1 Introduction

In 1928 Dirac showed that spinors were fundamental to the quantum mechanical description of $\text{spin}\frac{1}{2}$ particles (electrons). However, the spacetime stage that Dirac's spinors operated in was still Lorentzian. Because spinors are neither scalars nor vectors [7], at that time it was unclear how spinors behaved in curved spaces. Weyl's paper provided a means for this description using tetrads (vierbeins) as the necessary link between Lorentzian space and curved Riemannian space.

Weyl's elucidation of spinor behavior in curved space and his development of the so-called spin connection $\omega_{b\lambda}^a$ and the associated spin vector $\omega_\lambda = \omega_{ab\lambda}\sigma^{ab}$ was noteworthy, but his primary purpose was to demonstrate the profound connection between quantum mechanical gauge invariance and the electromagnetic field. Weyl's 1929 paper served to complete his earlier (1918) theory in which Weyl attempted to derive electrodynamics from the geometrical structure of a generalized Riemannian manifold via a scale-invariant transformation of the metric tensor. This attempt failed, but the manifold he discovered (known as Weyl space), is still a subject of interest in theoretical physics.

Although Weyl's paper reflected upon his earlier effort, it is obvious from the 1929 paper that he had moved on, and consequently he did not address spinor descriptions in a curved Weyl space. In the following elementary discussion we pick up

on this topic and consider the modifications that such a space forces upon the spin connection and its associated algebra. In particular, we shall consider the issue of metricity for Lorentz and coordinate vector spaces and how Weyls geometry affects metricity in these spaces.

3.2 Basic Tetrad formalism

It is always possible to find a coordinate system in which the space is Lorentzian at a given point. Einstein demonstrated this with his famous thought experiment involving a passenger on an elevator. Unless the elevator is equipped with windows, the passenger cannot know whether she is in a stationary elevator in Earth's gravitational field or if her elevator is being uniformly accelerated somewhere out in space. If she stands in one place in the elevator, then her coordinates are sufficiently local to the extent that slight variations in the Earth's radial gravitational field cannot be detected. However, if her elevator is sufficiently large, she could move around and discover that the gravity field is convergent (that is, points to the center of the Earth) and gets either weaker or stronger as she climbs up and down the elevator walls. If a locally-flat coordinate system can be found even in a strong gravitational field, then there must be a way to express the Lorentzian metric $\eta_{\mu\nu}$ with the curved-space metric $g_{\mu\nu}(x)$. In four-dimensional spacetime, one uses quantities called tetrads e_μ^a (or vierbeins, which is German for four legs) to link the two metrics:

$$g_{\mu\nu} = e_\mu^a(x)e_\nu^b(x)\eta_{ab}. \quad (3.2.1)$$

A tetrad is a rather odd little fellow having one foot in flat space and the other in curved space. To distinguish the two with regard to tetrad notation, we will utilize Latin indices (a,b,c.etc.) for the lorentz index and Greek indices for the curved-space part. Thus, you can think of a tetrad as a tensor quantity whose curved-space part transforms just like a coordinate vector:

$$e_\mu^{\prime a} = \frac{\partial x^\lambda}{\partial x^{\prime\mu}} e_\lambda^a.$$

From eqn.(3.2.1) it is easy to see that we can also write

$$\eta_{ab} = e_a^\mu(x)e_b^\nu(x)g_{\mu\nu}(x). \text{ Provided we make the requirement}$$

that

$$e_a^\mu(x)e_\mu^b(x) = \sigma_a^b \quad \text{and} \quad e_a^\mu(x)e_\nu^a(x) = \sigma_\nu^\mu. \quad (3.2.2)$$

Because each tetrad index runs from 0 to 3, there are a total of 16 components in the tetrad.

3.3 Parallel transfer and covariant differentiation

Consider the change in a given contravariant vector field $\xi(x)$ from point to point in some manifold. Provided ξ^μ is not a constant field, the vector $\xi^\mu(x + dx)$ at an infinitesimally-near point will differ to first order from $\xi^\mu(x)$ according to [16]

$$\begin{aligned}\xi^\mu(x + dx) &= \xi^\mu(x) + \partial_\alpha \xi^\mu dx^\alpha, \text{ or} \\ d\xi^\mu &= \partial_\mu \xi^\mu dx^\alpha,\end{aligned}$$

Where $d\xi^\mu = \xi^\mu(x + dx) - \xi^\mu(x)$ is the total change in the vector.

The quantity $d\xi^\mu$ cannot be a tensor because, as it is obtained by taking the difference between two vectors at different points in space, it is not a coordinate-independent quantity. More importantly, the partial differential $\partial_\alpha \xi^\mu$ is not a tensor. This is something of a disaster, as there seems to be no way to define vector differentiation in a covariant sense. To see this more clearly, consider how ξ^μ is transformed by a change of coordinates to the system X' :

$$\xi'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} \xi^\nu(x). \quad (3.3.1)$$

The total change in $\xi'^\mu(x')$ at the neighboring point is therefore

$$d\xi'^\mu(x') = \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\alpha} \xi^\nu(x) dx^\alpha + \frac{\partial x'^\mu}{\partial x^\nu} d\xi^\nu(x). \quad (3.3.2)$$

Thus, $d\xi^x$ does not transform properly because of the second order differential term.

Clearly, we need a prescription for vector and tensor differentiation that obeys standard tensor transformation laws. To do this, we need to be able to compute the difference $\xi^\mu(x + dx) - \xi^\mu(x)$ at the same point. As odd as this sounds, it is in fact possible using the concept known as parallel transfer. The figure on the following page shows a vector $\xi^\mu(x)$ located at an arbitrary point x on a given curve λ . Treated as a vector field, it will have a slightly different orientation at the infinitesimally-near point $x + dx$ located elsewhere on the curve. The vector difference is given by $d\xi^\mu = \xi^\mu(x + dx) - \xi^\mu(x)$ which, as explained above, has

no intrinsic geometrical significance because the vectors are separated. On the other hand, if the vector ξ^μ represented a constant vector field (that is, $d\xi^x = 0$), then the separation clearly would not matter anymore, as the vectors at x and $x + dx$ would be (trivially) identical. Under the change of coordinates eqn.(3.3.1), however, the new vector ξ'^μ would clearly be a function of its coordinates, and eqn.(3.3.2) would become

$$d\xi'^\mu(x) = \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\alpha} \xi^\nu dx^\alpha \quad (3.3.3)$$

$$d\xi'^\mu(x) = \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\alpha} \frac{\partial x^\nu}{\partial x'^\lambda} \frac{\partial x^\alpha}{\partial x'^\beta} \xi'^\lambda dx'^\beta, \quad (3.3.4)$$

where in the last line we have transformed everything into the primed coordinate system. Eqn.(3.3.4), which represents a transformed constant vector field, can be more succinctly written as

$$d\xi'^\mu = \Gamma'_{\lambda\beta}{}^\mu(x') \xi'^\lambda dx'^\beta, \quad (3.3.5)$$

Where $\Gamma'_{\lambda\beta}{}^\mu = \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\alpha} \frac{\partial x^\nu}{\partial x'^\lambda} \frac{\partial x^\alpha}{\partial x'^\beta}$.

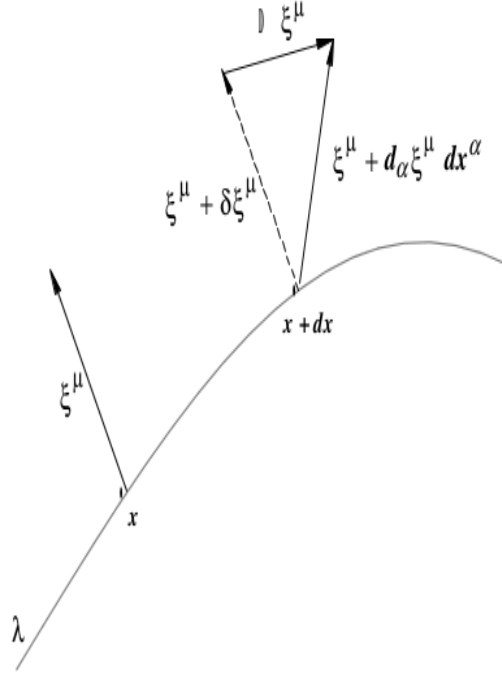
The (non-tensor) quantities $\Gamma'_{\lambda\beta}{}^\mu$ are called coefficients of affine connection because they affinely (linearly) relate the change in a vector with the vector itself and the transport distance dx . With the use of eqn.(3.3.1), it can easily be shown that the connections transform according to

$$\Gamma'_{\lambda\beta}{}^\mu = \frac{\partial x^\nu}{\partial x'^\lambda} \frac{\partial x^\sigma}{\partial x'^\beta} \frac{\partial x'^\mu}{\partial x^\beta} \Gamma_{\nu\sigma}{}^\beta(x) + \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\alpha} \frac{\partial x^\nu}{\partial x'^\lambda} \frac{\partial x^\alpha}{\partial x'^\beta}.$$

We now consider eqn.(3.3.5) to be the quantity which, when added to a vector field at x , effectively resurrects the original vector at the point $x + dx$. Taken as such, we will rename eqn.(3.3.5) as

$$\delta\xi^\mu = \Gamma_{\lambda\beta}^\mu \xi^\lambda dx^\beta. \quad (3.3.6)$$

Thus, $\delta\xi^\mu$ is a non-vector quantity which, when added to $\xi^\mu(x)$, produces a parallel copy of the vector at the neighboring point on the curve. The vector quantity $\xi^\mu(x) + \delta\xi^\mu(x)$ therefore represents the parallel- transferred vector which can now be compared with the vector $\xi^\mu(x) + \partial_\beta \xi^\mu$ in a truly covariant sense, since they



both occur at the same spacetime point. It can easily be shown that the difference

$$\partial_\beta \xi^\mu dx^\beta - \delta \xi^\mu = [\partial_\beta \xi^\mu - \Gamma_{\lambda\beta}^\mu \xi^\lambda] dx^\beta \quad (3.3.7)$$

is in fact a tensor. We define the covariant derivative of ξ^μ as

$$D_\beta \xi^\mu = \partial_\beta \xi^\mu - \Gamma_{\lambda\beta}^\mu \xi^\lambda \quad (3.3.8)$$

so that $\frac{D}{dx^\beta} = D_\beta$. The covariant derivative is of paramount importance in differential geometry and Einsteins theory of general relativity, where the coefficients of affine connection account for the presence of gravitational fields.

3.4 Spinors in curved space

Recall that Diracs equation in an electromagnetic field is

$$i\gamma^\mu(\partial_\mu - ieA_\mu)\psi - m\psi = 0, \quad (3.4.1)$$

where A_μ is the electromagnetic 4-potential in some given set of units. The presence of an electromagnetic field mandates the transformation of the simple partial differential operator from ∂_μ to $\partial_\mu - ieA_\mu$ (which is called the covariant derivative for spinor quantities). This change is also necessary if the Dirac-Maxwell Lagrangian is to be gauge invariant. One might reasonably expect that a similar form holds for the covariant derivative of a spinor in curved space. Indeed, Weyl had the insight to recognize this identification holds as a general principle, and he expressed the curved-space derivative of the Dirac spinor as

$$\begin{aligned} \partial_\mu\psi &\longrightarrow D_\mu\psi \\ \partial_\mu\psi &= (\partial_\mu + \Gamma_\mu)\psi, \end{aligned} \quad (3.4.2)$$

where $\Gamma_\mu(x)$ is some 4×4 matrix that makes the Dirac equation valid for curved space.

As for the transformation properties of $\psi(x)$ itself, consider the change in ψ that results from an infinitesimal pure displacement:

$$\psi(x + dx) = \psi(x) + \partial_\mu\psi(x)dx^\mu, \quad (3.4.3)$$

or

$$d\psi = \partial_\mu\psi(x)dx^\mu \quad (3.4.4)$$

where $d\psi = \psi(x + dx) - \psi(x)$. We can then demand that the total change in ψ in curved space under parallel transfer be

$$\psi(x + dx) = \psi(x) + \Gamma_\mu\psi(x)dx^\mu, \quad (3.4.5)$$

or

$$D\psi = \Gamma_\mu\psi(x)dx^\mu, \quad (3.4.6)$$

and

$$D\psi^\dagger = \psi^\dagger \Gamma_\mu^\dagger dx^\mu. \quad (3.4.7)$$

This is our starting point for the derivation of the field Γ_μ .

3.5 Coordinate and Lorentz vectors in curved space

In order to derive $\Gamma_\mu(x)$, we will utilize the equivalence of vectors expressed in what is called the coordinate form (or C-form) $V^\mu(x)$ and the lorentz form (L-form) $V^a(x)$, where

$$V^a = e_\mu^a V^\mu \quad (3.5.1)$$

and vice versa. The magnitude or length L of these vectors is the same for both forms:

$L^2 = \eta_{ab} V^a V^b = g_{\mu\nu}(x) V^\mu V^\nu$. The use of L-vectors presents an immediate problem: how do they transform under parallel displacement? In similar with eqn.(3.3.5) we assume the existence of an L-form connection term such that

$$D(\omega)V^a = \omega_{b\lambda}^a dx^\lambda, \quad (3.5.2)$$

where $\omega_{b\lambda}^a$ is called the spin connection. Although the spin connection and the metric connection $\Gamma_{\mu\nu}^\alpha$ can be viewed as different versions of the same quantity, we shall see that the spin connection has different symmetry properties with respect to its indices. The notion of covariant differentiation can also be specified for L-forms. We define the covariant derivative of the vector V^a as

$$D_\lambda(\omega)V^a = \partial_\lambda V^a - \omega_{b\lambda}^a V^b.$$

Further more, we will define the total covariant derivative of a "mixed" tensor as

$$D_\lambda(\omega + \Gamma)T_\beta^a = \partial_\lambda T_\beta^a + \Gamma_{\beta\lambda}^\mu T_\mu^a - \omega_{\beta\lambda}^a T_\beta^b.$$

Of particular interest is the total covariant derivative of the lorentz matrix η_{ab}

$$\begin{aligned} D_\lambda(\omega + \Gamma)\eta_{ab} &= D_\lambda(\omega)\eta_{ab} \\ &= \eta_{as}\omega_{b\lambda}^s + \eta_{sb}\omega_{a\lambda}^s \end{aligned}$$

$$= \omega_{ab\lambda} + \omega_{ba\lambda}.$$

If we make the reasonable demand that the Lorentz metric be constant under parallel transfer, its covariant derivative should vanish; the lower-index spin connection must then be antisymmetric in its first two indices:

$$\omega_{ab\lambda} = -\omega_{ba\lambda}. \quad (3.5.3)$$

3.6 The Tetrad postulate

Let us parallel-transfer the vector relation expressed in eqn.(3.5.1)

$$\begin{aligned} D(\omega + \Gamma)V^a &= V^\mu D(\omega + \Gamma)e_\mu^a + e_\mu^a D(\omega + \Gamma)V^\mu \\ \omega_{b\lambda}^a V^b dx^\lambda &= V^\mu \partial_\lambda e_\mu^a dx^\lambda + e_\mu^a \Gamma_{\alpha\lambda}^\mu V^\alpha dx^\lambda \end{aligned} \quad (3.6.1)$$

(Remember that the Tetrad is not a vector, so it transfers via the partial derivative.) Relabeling indices, we get

$$e_\mu^s \omega_{s\lambda}^a V^\mu dx^\lambda = V^\mu \partial_\lambda e_\mu^a dx^\lambda + e_\alpha^\mu \Gamma_{\mu\lambda}^\alpha V^\mu dx^\lambda \quad (3.6.2)$$

Dropping the common $V^\mu dx^\lambda$ term, we have

$$e_\mu^s \omega_{s\lambda}^a = \partial_\lambda e_\mu^a + e_\alpha^\mu \Gamma_{\mu\lambda}^\alpha. \quad (3.6.3)$$

However, this is just the statement that the total covariant derivative of the tetrad vanishes:

$$D_\lambda(\omega + \Gamma)e_\mu^a = 0. \quad (3.6.4)$$

This important result is known as the Tetrad postulate.

It will now be instructive to show from the Tetrad postulate the explicit relationship between the two connection $\Gamma_{\mu\lambda}^\alpha$ and $\omega_{s\lambda}^a$. Using eqn.(3.2.2) and eqn.(3.6.3), it is easy to see that

$$\Gamma_{\mu\nu}^\lambda = -e_a^\lambda \partial_\nu e_\mu^a + e_a^\lambda e_\mu^s \omega_{s\nu}^a. \quad (3.6.5)$$

This is a most interesting result-the connection separates in to two terms, one representing the chrisoffel term and another involving spin.

3.7 Derivation of Γ_μ

Consider the Dirac scalar quantity

$$I = \bar{\psi}\psi = \psi^\dagger\gamma^0\psi, \quad (3.7.1)$$

where γ^0 is the time-coordinate gamma matrix in the Dirac representation.

We assume that the Dirac matrices are expressible in either C-form or L-form, eventhough they are not vector quantities:

$$\gamma^\mu(x) = e_a^\mu\gamma^a. \quad (3.7.2)$$

In doing so, we must allow for the possibility that the C-form gamma matrices can be functions of the coordinates. Consequently, we will write γ^0 as $\gamma^0(x)$, despite the fact that we have no idea what these coordinate-dependent matrices might look like. Under parallel transfer of the scalar I, we then have

$$\begin{aligned} DI &= D\psi^\dagger(x)\gamma^0(x)\psi(x) + \psi^\dagger D\gamma^0(x)\psi(x) + \psi^\dagger(x)\gamma^0(x)D\psi(x) \\ &= \psi^\dagger\Gamma_\mu^\dagger\gamma^0\psi dx^\mu + \psi^\dagger\partial_\mu\gamma^0\psi dx^\mu + \psi^\dagger\gamma^0\Gamma_\mu\psi dx^\mu = 0. \end{aligned}$$

The terms ψ^\dagger and ψdx^μ bracket each of these quantities; dropping them, we thus have the requirement

$$\Gamma_\mu^\dagger\gamma^0 + \partial_\mu\gamma^0 + \gamma^0\Gamma_\mu = 0. \quad (3.7.3)$$

Now the quantity $\psi^-\gamma^\lambda\psi$ is a coordinate vector; let us call it $V^\lambda(x)$. We assume that this vector is parallel-transferred in the same manner as it is in general relativity, which is

$$\delta V^\lambda = \Gamma_{\alpha\mu}^\lambda(x)V^\alpha dx^\mu. \quad (3.7.4)$$

Similarly, we shall assume that $V^\alpha = \psi\gamma^\alpha\psi$ is an L-vector. Parallel transferring the identity $V^\lambda = \psi^-\gamma^\lambda\psi$ then gives

$$\begin{aligned} DV^\lambda &= D\psi^\dagger\gamma^0\gamma^\lambda\psi + \psi^\dagger D\gamma^0\gamma^\lambda\psi + \psi^\dagger\gamma^0 D\gamma^\lambda\psi + \psi^\dagger\gamma^0\gamma^\lambda D\psi \\ &= \psi^\dagger\Gamma_\mu^\dagger\gamma^0\gamma^\lambda\psi dx^\mu + \psi^\dagger(\partial_\mu\gamma^0)\gamma^\lambda\psi dx^\mu + \psi^\dagger\gamma^0\partial_\mu\gamma^\lambda\psi dx^\mu + \psi^\dagger\gamma^0\gamma^\lambda\Gamma_\mu\psi dx^\mu \\ &= \Gamma_{\alpha\mu}^\lambda V^\alpha dx^\mu \\ DV^\lambda &= \psi^\dagger\Gamma_{\alpha\mu}^\lambda\gamma^\alpha\psi dx^\mu \end{aligned} \quad (3.7.5)$$

Cancelling common terms, we have

$$\Gamma_\mu^\dagger \gamma^0 \gamma^\lambda + (\partial_\mu \gamma^0) \gamma^\lambda + \gamma^0 \partial_\mu \gamma^\lambda + \gamma^0 \gamma^\lambda \Gamma_\mu = \Gamma_{\alpha\mu}^\lambda \gamma^\alpha$$

Using eqn.(3.7.3), we can get rid of the $\Gamma_\mu^\dagger \gamma^0$ term and thus arrive at the elegant identity

$$D_\mu(\Gamma) \gamma^\lambda = \Gamma_\mu \gamma^\lambda - \gamma^\lambda \Gamma_\mu, \quad (3.7.6)$$

where $D_\mu(\Gamma) \gamma^\lambda = \partial_\mu \gamma^\lambda - \gamma^\alpha \Gamma_{\alpha\mu}^\lambda$. Note that the $\partial_\mu \gamma^0$ vanished when we inserted eqn.(3.7.3) in to eqn.(3.7.5).

Similarly, for the L-form $V^\alpha = \psi^- \gamma^\alpha \psi$, we have the differential quantity

$$\begin{aligned} DV^\alpha &= D\psi^\dagger \gamma^0 \gamma^\alpha \psi + \psi^\dagger D\gamma^0 \gamma^\alpha \psi + \psi^\dagger \gamma^0 \gamma^\alpha D\psi \\ &= \psi^\dagger \Gamma_\mu^\dagger \gamma^0 \gamma^\alpha \psi dx^\mu + \psi^\dagger (\partial_\mu \gamma^0) \gamma^\alpha \psi dx^\mu + \psi^\dagger \gamma^0 \gamma^\alpha \Gamma_\mu \psi dx^\mu \\ &= \omega_{b\mu}^a V^b dx^\mu = \psi^\dagger \omega_{b\mu} \gamma^b \psi dx^\mu \end{aligned}$$

(Note that $d\gamma^\alpha = 0$ because γ^α are the constant Dirac matrices.) Again, the spinor terms ψ^\dagger and ψdx^μ bracket the other quantities, and we're left with

$$\omega_{b\mu}^\alpha \gamma^b = \gamma^\alpha \Gamma_\mu - \Gamma_\mu \gamma^\alpha. \quad (3.7.7)$$

By using the tetrad identity $\gamma^b = e_\mu^b \gamma^\mu$, it is easy to show from eqn.(3.7.7) and eqn.(3.6.5) that

$$\omega_{b\mu}^a = e_b^\lambda D_\mu(\Gamma) e_\lambda^a. \quad (3.7.8)$$

Let us now consider the length of some L-vector V^α which is

$$L^2 = \eta_{ab} V^a V^b. \quad (3.7.9)$$

Under parallel transfer, the total change in the length is then

$$\begin{aligned} 2lDl &= \eta_{ab} V^a DV^b + \eta_{ab} V^b DV^a \\ &= \eta_{ab} V^a \omega_{\delta\mu}^b V^\delta dx^\mu + \eta_{ab} V^b \omega_{\delta\mu}^a V^\delta dx^\mu \\ &= (\omega_{ab\mu} + \omega_{ba\mu}) V^a V^b dx^\mu \end{aligned}$$

(Note that we have lowered the upper index on $\omega_{\delta\mu}^a$ with the L-metric). Since dl vanishes in Riemannian space, we see confirmation that the lower-indexed connection $\omega_{ab\mu}$ is antisymmetric with respect to its L-indices.

Now consider eqn.(3.7.7), where lower-index form is

$$\omega_{ab\mu}\gamma^b = \gamma_a\Gamma_\mu - \Gamma_\mu\gamma_a.$$

Left multiplying this by γ^a and rearrangeing, we get

$$4\gamma_\mu = \omega_{ba\mu}\gamma^a\gamma^b + \gamma_a\Gamma_\mu\gamma^a, \quad (3.7.10)$$

where we have used the fact that $\gamma^a\gamma_a=4$. Unfortunately, Γ_μ is also present on the rhs of this expression. In order to solve for it, lets pre- and post-multiply both sides by γ_a and γ^a , respectively. This gives

$$4\gamma_a\Gamma_\mu\gamma^a = \omega_{ba\mu}\gamma_c\gamma^a\gamma^b\gamma^c + \gamma_b\gamma_a\Gamma_\mu\gamma^a\gamma^b$$

where care has been taken in labeling the indices. Evaluation of the term $\omega_{ba\mu}\gamma_c\gamma^a\gamma^b\gamma^c$ requires a trick: we have to move the γ^c term over to its partner γ_c , where they can cancel each other (remember that $\gamma_c\gamma^c = 4$).

We do this in two steps, first by writing $\gamma^b\gamma^c = 2\eta^{bc} - \gamma^c\gamma^b$ and then

$$\gamma^a\gamma^c = 2\eta^{ac} - \gamma^c\gamma^a.$$

This gives

$$\begin{aligned} \omega_{ba\mu}\gamma_c\gamma^b\gamma^c &= (\omega_{ba\mu} + \omega_{ab\mu})(\gamma^a\gamma^b + \gamma^b\gamma^a) \\ &= 2\eta^{ab}(\omega_{ba\mu} + \omega_{ab\mu}) \\ &= 0. \end{aligned}$$

We now have

$$\begin{aligned} 4\gamma_a\Gamma_\mu\gamma^a &= \gamma_b\gamma_a\Gamma_\mu\gamma^a\gamma^b \text{ or} \\ \gamma_a\Gamma_\mu\gamma^a &= \frac{1}{4}\gamma_b\gamma_a\Gamma_\mu\gamma^a\gamma^b \end{aligned} \quad (3.7.11)$$

If we pre- and post-multiply eqn.(3.7.11) by γ_a and γ^a , we get

$$\gamma_a\Gamma_\mu\gamma^a = \frac{1}{16}\gamma_c\gamma_b\gamma_a\Gamma_\mu\gamma^a\gamma^b\gamma^c \quad (3.7.12)$$

After k such iterations, we have

$$\gamma_a\Gamma_\mu\gamma^a = \frac{1}{4^{k+1}}[- \dots - \gamma_d\gamma_c\gamma_b\gamma_a\Gamma_\mu\gamma^a\gamma^b\gamma^c\gamma^d - \dots] \quad (3.7.13)$$

Provided the term in brackets remains finite, we will have $\gamma_a\Gamma_\mu\gamma^a = 0$ as $k \rightarrow \infty$.

The sought-after identity for the spin vector is then

$$\Gamma_\mu = \frac{1}{4}\omega_{ba\mu}\gamma^a\gamma^b. \quad (3.7.14)$$

Chapter 4

Solution of the Dirac Equation in Curved Spacetime in the Presence of Strong Magnetic Field

4.1 Introduction

During the last years a large amount of observational data has been reported showing that our universe is almost isotropic and homogeneous. The study of the structure of the Cosmic Microwave Radiation leads us to conclude that the ratio of the total density to the critical density of the universe Ω_0 is likely to be close to one [4,5,9], favoring a spatially flat Robertson-Walker metric over other topologies. It is well known that general relativity is a local metrical theory and, therefore, the corresponding Einstein field equations do not fix the global topology of spacetime and, consequently, the universe may have compact spatial sections with a nontrivial topology. Then the observational data does not rule out the possibility that our universe possesses a hyperbolic topology.

The study of cosmological models with nonstandard topologies is not new and goes back to the works by Zelmanov [4,9], showing that upon different coordinate transformations, spatially closed or flat sections can be transformed into hyperbolic sections and vice versa. The line element associated with a spatially open Friedman universe has the form

$$ds^2 = a^2(\eta)[d\eta^2 - dr^2 - \sinh^2(d\theta^2 + \sin^2\theta)d\phi^2]. \quad (4.1.1)$$

Making the coordinate transformation

$$e^{-z} = \cosh r - \sinh r \cos \theta, \quad e^{-z} x = \sin \theta \cos \phi \sinh r, \quad e^{-z} y = \sin \theta \sin \phi \sinh r, \quad (4.1.2)$$

$$\frac{\partial}{\partial z} e^{-z} dz = \frac{\partial}{\partial r} (\cosh r - \sinh r \cos \theta) dr + \frac{\partial}{\partial \theta} (\cosh r - \sinh r \cos \theta) d\theta + \frac{\partial}{\partial \phi} (\cosh r - \sinh r \cos \theta) d\phi$$

$$\Rightarrow -dz = (\sinh r - \cosh r \cos \theta) dr + \sinh r \sin \theta d\theta$$

$$\frac{\partial}{\partial x} e^{-z} x = \frac{\partial}{\partial r} (\sin \theta \cos \phi \sinh r) dr + \frac{\partial}{\partial \theta} (\sin \theta \cos \phi \sinh r) d\theta + \frac{\partial}{\partial \phi} (\sin \theta \cos \phi \sinh r) d\phi$$

$$\Rightarrow e^{-z} dx = \sin \theta \cos \phi \cosh r dr + \cos \theta \cos \phi \sinh r d\theta - \sin \theta \sin \phi \sinh r d\phi \quad \frac{\partial}{\partial y} e^{-z} y =$$

$$\frac{\partial}{\partial r} (\sin \theta \sin \phi \sinh r) dr + \frac{\partial}{\partial \theta} (\sin \theta \sin \phi \sinh r) d\theta + \frac{\partial}{\partial \phi} (\sin \theta \sin \phi \sinh r) d\phi$$

$$\Rightarrow e^{-z} dy = \sin \theta \sin \phi \cosh r dr + \cos \theta \sin \phi \sinh r d\theta + \sin \theta \cos \phi \sinh r d\phi$$

the metric in eqn.(4.1.1) becomes

$$a^{-2}(\eta) ds^2 = d\eta^2 - dz^2 - e^{-2z}(dx^2 + dy^2). \quad (4.1.3)$$

In order to study quantum processes in curved spacetimes one has to fulfill a preliminary step which consists in having a description of the single-mode solution of the relativistic particles or perturbations in those background fields, i.e., exact solution of the relativistic scalar and spinor wave equations. We have different methods for solving relativistic wave equations in curved spaces ; among them, the method of separation of variables is one of the most widely used. It is the purpose of this present thesis to solve the Dirac equations in the Friedman universe associated with the metric (4.1.3) in the presence of magnetic field. In order to solve the Dirac equation we apply the algebraic method of separation of variables. We compare the solutions with those of obtained after solving the relativistic Hamilton Jacobi equation.

4.2 Solution of the Hamilton-Jacobi equation

The covariant generalization of the hamilton-Jacobi equation has the form [4,9]

$$g^{\alpha\beta} \left(\frac{\partial S}{\partial x^\alpha} - e A_\alpha \right) \left(\frac{\partial S}{\partial x^\beta} - e A_\beta \right) + M^2 = 0, \quad (4.2.1)$$

where $g^{\alpha\beta}$ is the contravariant metric, A_α is the vector potential and M is the mass of the particle. Let us introduce an electromagnetic field associated with vector potential

$$A^\mu = A_1(y)\delta_1^\mu, \quad (4.2.2)$$

Where the index $\mu = 0$ is associated with the evaluation parameter η and $\mu=1,2,3$ correspond to the space coordinates x,y,z respectively. Looking at the relativistic invariants [3,4,9]

$$\frac{1}{2}F^{\mu\nu}F_{\mu\nu} = \vec{B}^2 - \vec{E}^2 = \frac{e^{4z}}{\alpha(\eta)^4} \left(\frac{dA_1(y)}{dy} \right)^2, \quad (4.2.3)$$

$$F^{\mu\nu}F_{\mu\nu}^* = 0, \quad (4.2.4)$$

and taking into account that only F_{23} is different from zero, we notice that eqn.(4.2.2) corresponding to magnetic field directed long the z -axis, with strength

$$\vec{B} = \frac{e^{2z}}{\alpha(\eta)^2} \left| \frac{dA_1(y)}{dy} \right|, \quad (4.2.5)$$

whose value is inversely proportional to the expansion factor $\alpha(\eta)^2$

The line element eqn.(4.1.3) is a stackel space [4,9], and the Hamilton-Jacobi equation (4.2.1) is completely separable in (4.1.3) in the presence of the vector potential(4.2.2), therefore we can look for a solution in the form

$$S = K_x + S_y(y) + S_z(z) + S_\eta(\eta). \quad (4.2.6)$$

substituting eqn.(4.2.6) in to eqn.(4.2.1) we obtain

$$\frac{(k_x - A(y))^2}{e^{-2z}} + \frac{1}{e^{-2z}} \left(\frac{dS_y}{dy} \right)^2 + \left(\frac{dS_z}{dz} \right)^2 - \left(\frac{dS_\eta}{d\eta} \right)^2 - M^2 \alpha(\eta)^2 = 0. \quad (4.2.7)$$

Equation(4.2.7) reduces to the following system of differential equations:

$$\left(\frac{dS_z}{dz} \right)^2 + K_{xy}^2 e^{2z} = K_z^2, \quad (4.2.8)$$

$$\left(\frac{dS_\eta}{d\eta} \right)^2 + M^2 \alpha(\eta)^2 = K_z^2, \quad (4.2.9)$$

$$(K_x - A_1(y))^2 + \left(\frac{dS_y}{dy} \right)^2 = K_{xy}^2, \quad (4.2.10)$$

where K_{xy}^2 , and K_z^2 are separation constant.

In the absence of electromagnetic interaction, we have that $A_1(y) = 0$ and the solution of eqn.(4.2.10) takes the form

$$S_y = \pm \sqrt{K_{xy}^2 - K_x^2} y = \pm K_y y, \quad (4.2.11)$$

where we have introduced the constant k_y .

When the vector potential has the simple form $A_1(y) = A_1 y$ the magnetic field reads $\vec{B} = \frac{e^{2z}}{\alpha(\eta)^2} |A_1|$ and the function $S_y(y)$ is

$$S_y(y) = -\frac{K_x - A_1 y}{2A_1} \sqrt{K_{xy}^2 - (K_x - A_1 y)^2} + \frac{K_{xy}^2}{2A_1} \arctan \frac{A_1 y - K_x}{\sqrt{K_{xy}^2 - (K_x - A_1 y)^2}}. \quad (4.2.12)$$

The solution of eqn.(4.2.8) can be expressed in terms of elementary functions as follows:

$$S_z = \sqrt{K_z^2 - K_{xy} \exp(2z)} - K_z \tanh^{-1} \sqrt{\frac{K_z^2 - K_{xy} \exp(2z)}{K_z^2}}. \quad (4.2.13)$$

The solution of eqn.(4.2.9) can be written as

$$S_\eta = \pm \int \sqrt{K_z^2 - M^2 \alpha(\eta)^2} d\eta, \quad (4.2.14)$$

whose explicit form in terms of elementary function will depend on a particular choice of the expansion function $\alpha(\eta)$.

Since we have been able to solve the Hamilton-Jacobi equation in the Stückel space given by eqn.(4.2.1), we can construct the quasiclassical modes of the relativistic wave equations through the identification[9]

$$\Phi = e^{iS} = e^{\pm i \int \sqrt{k_z^2 - M^2 \alpha(\eta)^2} d\eta} e^{ik_x x} e^{iS_y} e^{iS_z}, \quad (4.2.15)$$

where S_z and S_y take the following values at the asymptotes

$$S_{z(\infty)} \rightarrow ik_{xy} e^z, \quad S_{z(-\infty)} \rightarrow k_z z, \quad (4.2.16)$$

$$S_{y(\infty)} \rightarrow \mp i \frac{(k_x - A_1 y)^2}{2A_1}. \quad (4.2.17)$$

When the electromagnetic interaction is not present we have that $S_y = \exp(ik_y y)$.

4.3 Generally covariant Dirac equation in curved space-time

To write the generally covariant Dirac equation in curved space-time with metric $g_{\mu\nu}$, one first introduces the spinor affine connections [3] $\Gamma_\mu = \frac{1}{4}\omega_\mu^{ab}\delta^{ab}$ where

$$\delta^{ab} = \frac{1}{2}(\gamma_a\gamma_b - \gamma_b\gamma_a). \quad (4.3.1)$$

Here γ_a are the Dirac-puli matrices with the following relation,

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}, \quad (4.3.2)$$

and

$$\gamma_0^\dagger = -\gamma_0, \quad \gamma_i^\dagger = \gamma_i, \quad (i = 1, 2, 3), \quad (4.3.3)$$

$$\gamma_0 = i\beta \quad \gamma_i = -i\beta\alpha_i, \quad (4.3.4)$$

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad (4.3.5)$$

Where I is the 2×2 identity matrix, $\eta^{ab} = \eta_{ab} = \text{diag}(-1, 1, 1, 1)$ is the minkoski metric tensor, and σ are the standard pauli mtrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (4.3.6)$$

ω_μ^{ab} is defined by the vanish of the generalied covariant derivatve of the tetrad (or vierbein) field e_μ^a .

$$D_\mu e_\nu^a = \partial_\mu e_\nu^a - \Gamma_{\mu\nu}^\lambda - \eta_{bc}\omega_\mu^{ab}e_\nu^c$$

$$D_\mu e_\nu^a = \nabla_\mu e_\nu^a - \eta_{bc}\omega_\mu^{ab}e_\nu^c = 0, \quad (4.3.7)$$

Where the tetrad field

$$g_{\mu\nu} = \eta_{ab}e_\mu^a e_\nu^b, \quad (4.3.8)$$

$$e_\mu^a e_b^\mu = \sigma_b^a, \quad (\mu, \nu, a, b = 0, 1, 2, 3), \quad (4.3.9)$$

μ and ν are the space-time indices lowered with the metric $g_{\mu\nu}$, and a and b are lorentz group indices lowered with η_{ab} .

One also needs to introduce generalized Dirac-pauli matrices $\bar{\gamma}_\mu = e_\mu^a \gamma_a$, which satisfy the equation

$$\bar{\gamma}_\mu \bar{\gamma}_\nu + \bar{\gamma}_\nu \bar{\gamma}_\mu = 2g_{\mu\nu}. \quad (4.3.10)$$

The covariant derivative acting on a spinor field ψ is then

$$D_\mu \bar{\psi} = \partial_\mu \bar{\psi} - \Gamma_\mu \bar{\psi}, \quad (4.3.11)$$

and the generally covariant form of the Dirac equation in pure gravitational field is

$$\bar{\gamma}^\mu D_\mu \bar{\psi} + m \bar{\psi} = 0, \quad (4.3.12)$$

where $\bar{\gamma}^\mu = g^{\mu\nu} \bar{\gamma}_\nu$. For an electron near the atomic nucleus one needs to consider the effect of the electromagnetic vector potential A_μ . So the covariant derivative acting on a spinor field should be rewritten as

$$D_\mu \bar{\psi} = (\partial_\mu - \Gamma_\mu - ieA_\mu) \bar{\psi}. \quad (4.3.13)$$

Then the generally covariant form of the Dirac equation in gravitational and electromagnetic fields is

$$\bar{\gamma}^\mu (\partial_\mu - \Gamma_\mu - ieA_\mu) \bar{\psi} + m \bar{\psi} = 0. \quad (4.3.14)$$

4.4 Solution of the Dirac equation

The Dirac equation is a system of coupled partial differential equations which is separable in a very restricted set of metrics. Among the spacetimes where the separability of the Dirac equation has been studied one can mention the Stackel spaces [4,9], which are those metrics where the Hamilton-Jacobi equation is separable. A systematic classification of the gravitational backgrounds where the Dirac equation is separable with the help of the algebraic method is used. The line element eqn.(4.1.3) belongs to this family and, consequently, one can apply the algebraic method of separation.

The covariant generalization of the Dirac equation in curved spacetime is

$$\bar{\gamma}^\mu(\partial_\mu - \Gamma_\mu - ieA_\mu)\bar{\psi} + M\bar{\psi} = 0, \quad (4.4.1)$$

Where the curved Dirac matrices $\bar{\gamma}^\mu$ satisfy the anticommutation relation $\{\bar{\gamma}^\mu, \bar{\gamma}^\nu\} = 2g^{\mu\nu}$ and the spinor connection Γ_μ are

$$\Gamma_\mu = \frac{1}{4}g_{\lambda\alpha}\left[\left(\frac{\partial b_\nu^\beta}{\partial x^\mu}\right)a_\beta^\alpha - \Gamma_{\nu\mu}^\alpha\right]S^{\lambda\nu}, \quad (4.4.2)$$

Where

$$S^{\lambda\nu} = \frac{1}{2}(\bar{\gamma}^\lambda\bar{\gamma}^\nu - \bar{\gamma}^\nu\bar{\gamma}^\lambda), \quad (4.4.3)$$

and[12]

$$\Gamma_{\nu\mu}^\alpha = \frac{1}{2}g^{\delta\alpha}\left\{\frac{\partial g_{\mu\delta}}{\partial x^\nu} + \frac{\partial g_{\nu\delta}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\delta}\right\}. \quad (4.4.4)$$

The matrices $b_\nu^\beta, a_\beta^\alpha$ establish the connection between the Dirac matrices $\bar{\gamma}^\mu$ on a curved space-time and the flat Dirac matrices γ^μ as follows:

$$\bar{\gamma}_\mu = b_\mu^\alpha\gamma_\alpha, \bar{\gamma}^\mu = a_\beta^\mu\gamma^\beta. \quad (4.4.5)$$

From eqn.(3.2.1) it is easy to see that we can also write

$$\eta_{ab} = e_a^\mu e_b^\nu g_{\mu\nu}. \quad (4.4.6)$$

Since the line element eqn.(4.1.3) is associated with a diagonal metric, we can work in the diagonal tetrad gauge for $\bar{\gamma}^\mu$:

$$\bar{\gamma}^0 = \frac{\gamma^0}{a(\eta)}, \bar{\gamma}^1 = \frac{\gamma^1}{a(\eta)e^{-z}}, \bar{\gamma}^2 = \frac{\gamma^2}{a(\eta)e^{-z}}, \bar{\gamma}^3 = \frac{\gamma^0}{a(\eta)}. \quad (4.4.7)$$

substituting eqn.(4.4.7) into eqn.(4.4.2) and solving for the spinor connections

we have

$$\begin{aligned}
\Gamma_\mu &= \frac{1}{4}g_{\lambda\alpha}\left[\left(\frac{\partial b_\nu^\beta}{\partial x^\mu}\right)a_\beta^\alpha - \Gamma_{\nu\mu}^\alpha\right]\frac{1}{2}(\bar{\gamma}^\lambda\bar{\gamma}^\nu - \bar{\gamma}^\nu\bar{\gamma}^\lambda) \\
\Gamma_1 &= \frac{1}{4}g_{\lambda\alpha}\left[\left(\frac{\partial b_\nu^\beta}{\partial x^1}\right)a_\beta^\alpha - \Gamma_{\nu 1}^\alpha\right]\frac{1}{2}(\bar{\gamma}^\lambda\bar{\gamma}^\nu - \bar{\gamma}^\nu\bar{\gamma}^\lambda) \\
\Gamma_1 &= \frac{1}{4}g_{00}\left[\left(\frac{\partial b_\nu^\beta}{\partial x^1}\right)a_\beta^0 - \Gamma_{\nu 1}^0\right]\frac{1}{2}(\bar{\gamma}^0\bar{\gamma}^\nu - \bar{\gamma}^\nu\bar{\gamma}^0) + \frac{1}{4}g_{11}\left[\left(\frac{\partial b_\nu^\beta}{\partial x^1}\right)a_\beta^1 - \Gamma_{\nu 1}^1\right]\frac{1}{2}(\bar{\gamma}^1\bar{\gamma}^\nu - \bar{\gamma}^\nu\bar{\gamma}^1) + \\
&\quad \frac{1}{4}g_{22}\left[\left(\frac{\partial b_\nu^\beta}{\partial x^2}\right)a_\beta^2 - \Gamma_{\nu 1}^2\right]\frac{1}{2}(\bar{\gamma}^2\bar{\gamma}^\nu - \bar{\gamma}^\nu\bar{\gamma}^2) + \frac{1}{4}g_{33}\left[\left(\frac{\partial b_\nu^\beta}{\partial x^1}\right)a_\beta^3 - \Gamma_{\nu 1}^3\right]\frac{1}{2}(\bar{\gamma}^3\bar{\gamma}^\nu - \bar{\gamma}^\nu\bar{\gamma}^3) \\
\Rightarrow \Gamma_1 &= \frac{1}{4}g_{00}\left[\left(\frac{\partial b_0^0}{\partial x^1}\right)a_0^0 - \Gamma_{01}^0\right]\frac{1}{2}(\bar{\gamma}^0\bar{\gamma}^0 - \bar{\gamma}^0\bar{\gamma}^0) + \left[\left(\frac{\partial b_1^1}{\partial x^1}\right)a_1^0 - \Gamma_{11}^0\right]\frac{1}{2}(\bar{\gamma}^0\bar{\gamma}^1 - \bar{\gamma}^1\bar{\gamma}^0) + \\
&\quad \left[\left(\frac{\partial b_2^2}{\partial x^1}\right)a_2^0 - \Gamma_{21}^0\right]\frac{1}{2}(\bar{\gamma}^0\bar{\gamma}^2 - \bar{\gamma}^2\bar{\gamma}^0) + \left[\left(\frac{\partial b_3^3}{\partial x^1}\right)a_3^0 - \Gamma_{31}^0\right]\frac{1}{2}(\bar{\gamma}^0\bar{\gamma}^3 - \bar{\gamma}^3\bar{\gamma}^0) + \\
&\quad \frac{1}{4}g_{11}\left[\left(\frac{\partial b_0^0}{\partial x^1}\right)a_0^1\Gamma_{01}^1\right]\frac{1}{2}(\bar{\gamma}^1\bar{\gamma}^0 - \bar{\gamma}^0\bar{\gamma}^1) + \left[\left(\frac{\partial b_1^1}{\partial x^1}\right)a_1^1 - \Gamma_{11}^1\right]\frac{1}{2}(\bar{\gamma}^1\bar{\gamma}^1 - \bar{\gamma}^1\bar{\gamma}^1) + \\
&\quad \left[\left(\frac{\partial b_2^2}{\partial x^1}\right)a_2^1 - \Gamma_{21}^1\right]\frac{1}{2}(\bar{\gamma}^1\bar{\gamma}^2 - \bar{\gamma}^2\bar{\gamma}^1) + \left[\left(\frac{\partial b_3^3}{\partial x^1}\right)a_3^1 - \Gamma_{31}^1\right]\frac{1}{2}(\bar{\gamma}^1\bar{\gamma}^3 - \bar{\gamma}^3\bar{\gamma}^1) + \\
&\quad \frac{1}{4}g_{22}\left[\left(\frac{\partial b_0^0}{\partial x^1}\right)a_0^2 - \Gamma_{01}^2\right]\frac{1}{2}(\bar{\gamma}^2\bar{\gamma}^0 - \bar{\gamma}^0\bar{\gamma}^2) + \left[\left(\frac{\partial b_1^1}{\partial x^1}\right)a_1^2 - \Gamma_{11}^2\right]\frac{1}{2}(\bar{\gamma}^2\bar{\gamma}^1 - \bar{\gamma}^1\bar{\gamma}^2) + \\
&\quad \left[\left(\frac{\partial b_2^2}{\partial x^1}\right)a_2^2 - \Gamma_{21}^2\right]\frac{1}{2}(\bar{\gamma}^2\bar{\gamma}^2 - \bar{\gamma}^2\bar{\gamma}^2) + \left[\left(\frac{\partial b_3^3}{\partial x^1}\right)a_3^2 - \Gamma_{31}^2\right]\frac{1}{2}(\bar{\gamma}^2\bar{\gamma}^3 - \bar{\gamma}^3\bar{\gamma}^2) + \\
&\quad \frac{1}{4}g_{33}\left[\left(\frac{\partial b_0^0}{\partial x^1}\right)a_0^3 - \Gamma_{01}^3\right]\frac{1}{2}(\bar{\gamma}^3\bar{\gamma}^0 - \bar{\gamma}^0\bar{\gamma}^3) + \left[\left(\frac{\partial b_1^1}{\partial x^1}\right)a_1^3 - \Gamma_{11}^3\right]\frac{1}{2}(\bar{\gamma}^3\bar{\gamma}^1 - \bar{\gamma}^1\bar{\gamma}^3) + \\
&\quad \left[\left(\frac{\partial b_2^2}{\partial x^1}\right)a_2^3 - \Gamma_{21}^3\right]\frac{1}{2}(\bar{\gamma}^3\bar{\gamma}^2 - \bar{\gamma}^2\bar{\gamma}^3) + \left[\left(\frac{\partial b_3^3}{\partial x^1}\right)a_3^3 - \Gamma_{31}^3\right]\frac{1}{2}(\bar{\gamma}^3\bar{\gamma}^3 - \bar{\gamma}^3\bar{\gamma}^3)
\end{aligned}$$

$$\Gamma_1 = -\frac{1}{2}\frac{e^{-z}}{\alpha(\eta)}\{-\alpha(\eta)\gamma^1\gamma^3 + \frac{d\alpha(\eta)}{d\eta}\gamma^1\gamma^4\}, \quad (4.4.8)$$

In the same way, we find for Γ_2 , Γ_3 , and Γ_4 and their values are given by Eqn.(4.4.9), (4.4.10) and Eqn.(4.4.11) respectively.

$$\Gamma_2 = -\frac{1}{2}\frac{e^{-z}}{\alpha(\eta)}\{-\alpha(\eta)\gamma^2\gamma^3 + \frac{d\alpha(\eta)}{d\eta}\gamma^2\gamma^4\}, \quad (4.4.9)$$

$$\Gamma_3 = -\frac{1}{2}\frac{d\alpha(\eta)}{d\eta}\frac{1}{\alpha(\eta)}\gamma^3\gamma^4 \quad (4.4.10)$$

$$\Gamma_4 = 0. \quad (4.4.11)$$

Substituting eqn.(4.4.8) - eqn.(4.4.11) into eqn.(4.4.1)

$$\begin{aligned}
\Rightarrow \bar{\gamma}^0\left(\frac{\partial}{\partial\eta} - 0\right) + \bar{\gamma}^1\left(\frac{\partial}{\partial x} - \left\{\frac{1}{2}\frac{e^{-z}}{\alpha(\eta)}\alpha(\eta)\gamma^1\gamma^3 - \frac{1}{2}\frac{e^{-z}}{\alpha(\eta)}\frac{d\alpha(\eta)}{d\eta}\gamma^1\gamma^4\right\} - ieA_1(y)\right) + \bar{\gamma}^2\left(\frac{\partial}{\partial y} - \right. \\
\left. \left\{\frac{1}{2}\frac{e^{-z}}{\alpha(\eta)}\alpha(\eta)\gamma^2\gamma^3 - \left(\frac{1}{2}\frac{e^{-z}}{\alpha(\eta)}\frac{d\alpha(\eta)}{d\eta}\gamma^2\gamma^4\right)\right\} + \bar{\gamma}^3\left(\frac{\partial}{\partial z} + \frac{1}{2}\frac{d\alpha(\eta)}{d\eta}\frac{1}{\alpha(\eta)}\gamma^3\gamma^4\right)\bar{\psi} + M\bar{\psi}) = 0 \\
\Rightarrow [\gamma^0\left(\frac{\partial}{\partial\eta}\right) + \gamma^1e^z\left(\frac{\partial}{\partial x} - \left\{\frac{1}{2}e^{-z}\gamma^1\gamma^3 - \frac{1}{2}\frac{e^{-z}}{\alpha(\eta)}\frac{d\alpha(\eta)}{d\eta}\gamma^1\gamma^4\right\} - ieA_1(y)\right) + \gamma^2e^z\left(\frac{\partial}{\partial y} - \left\{\frac{1}{2}e^{-z}\gamma^2\gamma^3 - \right. \right. \\
\left. \left. \frac{1}{2}\frac{e^{-z}}{\alpha(\eta)}\frac{d\alpha(\eta)}{d\eta}\gamma^2\gamma^4\right\}\right) + \gamma^0\left(\frac{\partial}{\partial z} + \frac{1}{2}\frac{d\alpha(\eta)}{d\eta}\frac{1}{\alpha(\eta)}\gamma^2\gamma^0\right)]\bar{\psi} + M\alpha(\eta)\bar{\psi} = 0
\end{aligned}$$

we find that the Dirac equation takes the simple form

$$\left\{\gamma^0\frac{\partial}{\partial\eta} + \gamma^1e^z\left(\frac{\partial}{\partial x} - A_1(y)\right) + \gamma^2e^z\frac{\partial}{\partial y} + \gamma^3\frac{\partial}{\partial z} + M\alpha(\eta)\right\}\psi = 0, \quad (4.4.12)$$

where we have introduced the spinor ψ ,

$$\bar{\psi} = a(\eta)^{\frac{-3}{2}} e^z \psi. \quad (4.4.13)$$

Regarding Eqn.(4.4.12) we should mention that it does exhibit a nonfactorizable structure. In order to solve Eqn.(4.4.12) we apply the algebraic method of separation of variables. The method consists in rewriting the Dirac eqn.(4.4.12) as a sum of two first order differential operators \hat{K}_1, \hat{K}_2 satisfying the relation

$$[\hat{K}_1, \hat{K}_2] = 0, \{\hat{K}_1 + \hat{K}_2\}\Phi = 0, \quad (4.4.14)$$

with

$$\gamma^3 \gamma^0 \psi = \Phi, \quad (4.4.15)$$

and

$$\hat{K}_1(x, y)\Phi = \{\gamma^2 \frac{\partial}{\partial y} + \gamma^1 (\frac{\partial}{\partial x} - iA_1(y))\}\gamma^3 \gamma^0 \Phi = ik\Phi, \quad (4.4.16)$$

$$\hat{K}_2(z, \eta)\Phi = e^z \{\gamma^0 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z} + M\alpha(\eta)\}\gamma^3 \gamma^0 \Phi = -ik\Phi. \quad (4.4.17)$$

It should be noticed that, using the pairwise scheme of separation, one has been able to reduce the problem of solving the Dirac equation to finding solutions of the decoupled system of eqn.(4.4.16) and eqn(4.4.17). A further problem arises when we try to separate variables in eqn(4.4.17). Here it is not possible to reduce the problem to a set of two commuting first order differential operators. In order to separate variables in eqn.(4.4.17), we rewrite it in the following form:

$$(\hat{L}_1 \gamma^3 \gamma^0 + \hat{L}_2)\Phi = 0, \quad (4.4.18)$$

Where \hat{L}_1 and \hat{L}_2 are two commuting differential operators given by the expressions

$$\hat{L}_1 = \gamma^0 \frac{\partial}{\partial \eta} + M\alpha(\eta), \quad (4.4.19)$$

$$\hat{L}_2 = \gamma^0 \frac{\partial}{\partial z} + ike^z. \quad (4.4.20)$$

In order to separate variables in eqn.(4.4.18), we introduce the auxiliary spinor Y ,

$$(\hat{L}_1 \gamma^3 \gamma^0 + \tilde{L}_2)Y = \Phi, \quad (4.4.21)$$

Where the differential operator \tilde{L}_2 is given by the expression

$$\tilde{L}_2 = \gamma^0 \frac{\partial}{\partial z} - ike^z. \quad (4.4.22)$$

Substituting eqn.(4.4.21) in to eqn.(4.4.18)

$$\begin{aligned} &\Rightarrow (\hat{L}_1 \gamma^3 \gamma^0 + \hat{L}_2)(\hat{L}_1 \gamma^3 \gamma^0 + \tilde{L}_2)Y=0 \\ &\Rightarrow [((\gamma^0 \frac{\partial}{\partial \eta} + M\alpha(\eta))\gamma^3 \gamma^0 + \gamma^0 \frac{\partial}{\partial z} + ike^z)((\gamma^0 \frac{\partial}{\partial \eta} + M\alpha(\eta))\gamma^3 \gamma^0 + \gamma^0 \frac{\partial}{\partial z} - ike^z)]Y = 0 \\ &\Rightarrow [(\gamma^0)^2 \frac{\partial^2}{\partial \eta^2} (\gamma^3)^2 (\gamma^0)^2 + \gamma^0 \frac{\partial}{\partial \eta} \gamma^3 \gamma^0 M(\eta) \gamma^3 \gamma^0 + \gamma^0 \frac{\partial}{\partial \eta} \gamma^3 (\gamma^0)^2 \frac{\partial}{\partial z} - \gamma^0 \frac{\partial}{\partial \eta} \gamma^3 \gamma^0 ike^z + \\ &\quad M^2 \alpha^2 (\gamma^3)^2 (\gamma^0)^2 + M\alpha(\eta) \gamma^3 (\gamma^0)^2 \frac{\partial}{\partial \eta} \gamma^3 \gamma^0 + M\alpha(\eta) \gamma^3 (\gamma^0)^2 \frac{\partial}{\partial z} - M\alpha(\eta) \gamma^3 \gamma^0 ike^z + \\ &\quad \gamma^0 \frac{\partial}{\partial z} \gamma^0 \frac{\partial}{\partial \eta} \gamma^3 \gamma^0 + \gamma^0 \frac{\partial}{\partial z} M\alpha(\eta) \gamma^3 \gamma^0 + (\gamma^0)^2 (\frac{\partial}{\partial z})^2 - \gamma^0 \frac{\partial}{\partial z} ike^z + ike^z \gamma^0 \frac{\partial}{\partial \eta} \gamma^3 \gamma^0 + \\ &\quad ike^z M\alpha(\eta) \gamma^3 \gamma^0 + ike^z \gamma^0 \frac{\partial}{\partial z} + k^2 e^{2z}]Y = 0 \\ &\Rightarrow (-\frac{\partial^2}{\partial \eta^2} - \gamma^0 \frac{\partial}{\partial \eta} M\alpha(\eta) + \gamma^0 \frac{\partial}{\partial \eta} \gamma^3 \frac{\partial}{\partial z} - \gamma^0 \frac{\partial}{\partial \eta} \gamma^3 \gamma^0 ike^z - M^2 \alpha^2(\eta) + M\alpha(\eta) \gamma^3 \frac{\partial}{\partial \eta} \gamma^3 \gamma^0 + \\ &\quad M\alpha(\eta) \gamma^3 \frac{\partial}{\partial z} - M\alpha(\eta) \gamma^3 \gamma^0 ike^z + \gamma^0 \frac{\partial}{\partial z} \gamma^0 \frac{\partial}{\partial \eta} \gamma^3 \gamma^0 \gamma^0 \frac{\partial}{\partial z} M\alpha(\eta) \gamma^3 \gamma^0 + \frac{\partial^2}{\partial z^2} - \gamma^0 ike^z + \\ &\quad ike^z \gamma^0 \frac{\partial}{\partial \eta} \gamma^3 \gamma^0 + ike^z M\alpha(\eta) \gamma^3 \gamma^0 + ike^z \gamma^0 \frac{\partial}{\partial z} + k^2 e^{2z})Y = 0 \\ &\Rightarrow (-\frac{\partial^2}{\partial \eta^2} - \gamma^0 M \frac{\partial}{\partial \eta}(\eta) - M^2 \alpha^2(\eta) + \frac{\partial^2}{\partial z^2} - i\gamma^0 ke^z + k^2 e^{2z})Y = 0 \end{aligned}$$

and we obtain that Y satisfies the following equation.

$$\{\hat{M}_1 + \hat{M}_2\}Y = 0, \quad (4.4.23)$$

with $[\hat{M}_1, \hat{M}_2]=0$ and

$$(\hat{M}_1 + \tilde{\lambda})Y = (-\frac{\partial^2}{\partial z^2} - i\gamma^0 ke^z + k^2 e^{2z} + \tilde{\lambda})Y = 0, \quad (4.4.24)$$

$$(\hat{M}_2 - \tilde{\lambda})Y = (-\frac{\partial^2}{\partial \eta^2} + \gamma^0 M \frac{d\alpha(\eta)}{d\eta} + M^2 \alpha^2(\eta) - \tilde{\lambda})Y = 0, \quad (4.4.25)$$

Where $\tilde{\lambda}$ is a separation constant. Introducing the new variable $u = 2ke^z$, we have that eqn.(4.4.24) can be written as,

$$(\frac{\partial^2}{\partial u^2} + \frac{i}{2u} \gamma^0 - \frac{1}{4} + (\frac{1}{4} - \lambda) \frac{1}{u^2})S = 0, \quad (4.4.26)$$

where

$$u^{-1/2}S = y. \quad (4.4.27)$$

Choosing the following representation of the Dirac matrices,

$$\gamma^0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}, \quad 1 \leq j \leq 3, \quad (4.4.28)$$

Substituting eqn.(4.4.28) in to eqn.(4.4.16)

$$\begin{aligned}
&\Rightarrow \{\gamma^2 \frac{\partial}{\partial y} + \gamma^1 (\frac{\partial}{\partial x} - iA_1(y))\} \gamma^3 \gamma^0 \Phi = ik\Phi \\
&\Rightarrow \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \frac{\partial}{\partial y} + \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} (\frac{\partial}{\partial x} - iA_1(y)) \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \Phi = ik\Phi \\
&\Rightarrow \begin{pmatrix} 0 & \sigma^2 \frac{\partial}{\partial y} \\ \sigma^2 \frac{\partial}{\partial y} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^1 (\frac{\partial}{\partial x} - iA_1(y)) \\ \sigma^1 (\frac{\partial}{\partial x} - iA_1(y)) & 0 \end{pmatrix} \begin{pmatrix} 0 & i\sigma^3 \\ -i\sigma^3 & 0 \end{pmatrix} \Phi = ik\Phi \\
&\Rightarrow \begin{pmatrix} 0 & \sigma^2 \frac{\partial}{\partial y} + \sigma^1 \frac{\partial}{\partial x} - iA_1(y) \\ \sigma^2 \frac{\partial}{\partial y} + \sigma^1 \frac{\partial}{\partial x} - iA_1(y) & 0 \end{pmatrix} \begin{pmatrix} 0 & i\sigma^3 \\ -i\sigma^3 & 0 \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = ik \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \\
&\Rightarrow \begin{pmatrix} -i\sigma^3 (\sigma^2 \frac{\partial}{\partial y} + \sigma^1 \frac{\partial}{\partial x} - iA_1(y)) & 0 \\ 0 & i\sigma^3 (\sigma^2 \frac{\partial}{\partial y} + \sigma^1 \frac{\partial}{\partial x} - iA_1(y)) \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = ik \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}
\end{aligned}$$

we readily obtain that the spinor ϕ has the following structure:

$$[\sigma_1 \frac{\partial}{\partial y} - i\sigma_2 (k_x - A_1(y))] \Phi_1 = ik\Phi_2, \quad (4.4.29)$$

$$[-\sigma_1 \frac{\partial}{\partial y} + i\sigma_2 (k_x - A_1(y))] \Phi_2 = ik\Phi_1, \quad (4.4.30)$$

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \phi(y) \\ F\sigma^3 \phi(y) \end{pmatrix} \exp(ik_x x), \quad (4.4.31)$$

where

$$\phi(y) = \begin{pmatrix} A(y) \\ B(y) \end{pmatrix}. \quad (4.4.32)$$

Using the representation (4.4.28) we obtain that the solution of eqn.(4.4.26) can be written in terms of Whittaker functions [10]

$$S_{1,2} = D_1 W_{-1/2, \sqrt{\lambda}(u)} + D_2 M_{-1/2, \sqrt{\lambda}(u)}, S_{3,4} = D_3 W_{1/2, \sqrt{\lambda}(u)} + D_4 M_{1/2, \sqrt{\lambda}(u)}, \quad (4.4.33)$$

where D_1, D_2, D_3, D_4 do not depend on the variable u . Looking at eqn.(4.4.25) and eqn.(4.4.26), we have that, for regular solutions at $u=0$, the spinor Y has the following structure [3].

$$Y = \begin{pmatrix} a(y)c_1(\eta)u^{-1/2}M_{+1/2, \sqrt{\lambda}(u)} \\ b(y)c_1(\eta)u^{-1/2}M_{+1/2, \sqrt{\lambda}(u)} \\ c(y)c_2(\eta)u^{-1/2}M_{-1/2, \sqrt{\lambda}(u)} \\ d(y)c_2(\eta)u^{-1/2}M_{-1/2, \sqrt{\lambda}(u)} \end{pmatrix} \exp(ik_x x). \quad (4.4.34)$$

Substituting eqn.(4.4.34) in to eqn.(4.4.18) and noticing that eqn.(4.4.25) is equivalent to the following system of equations,

$$\left(\frac{\partial}{\partial \eta} - iM\alpha(\eta)\right)c_1(\eta) = \sqrt{\tilde{\lambda}}c_2(\eta), \quad (4.4.35)$$

$$\left(\frac{\partial}{\partial \eta} + iM\alpha(\eta)\right)c_2(\eta) = \sqrt{\tilde{\lambda}}c_1(\eta), \quad (4.4.36)$$

we obtain that the spinor Φ has the following structure

$$\Phi = \begin{pmatrix} A(v)c_1(\eta)e^{-z/2}M_{-1/2}, \sqrt{\tilde{\lambda}}(2ke^z) \\ B(v)c_1(\eta)e^{-z/2}M_{-1/2}, \sqrt{\tilde{\lambda}}(2ke^z) \\ iA(v)c_2(\eta)e^{-z/2}M_{1/2}, \sqrt{\tilde{\lambda}}(2ke^z) \\ -iB(v)c_2(\eta)e^{-z/2}M_{1/2}, \sqrt{\tilde{\lambda}}(2ke^z) \end{pmatrix} \exp(ik_x x), \quad (4.4.37)$$

where $A(v)$ and $B(v)$ satisfy the system coupled system of equations

$$\left(\frac{d}{dy} - (k_x - A_1(y))\right)B = ikA, \quad (4.4.38)$$

$$\left(\frac{d}{dy} + (k_x - A_1(y))\right)A = ikB, \quad (4.4.39)$$

where

$$v = \frac{A_1 y - k_x}{\sqrt{A_1}}. \quad (4.4.40)$$

The corresponding solution of eqn.(4.4.14) in terms of the Whittaker functions $W_{k,\mu}(z)$ has the form [9]

$$\Phi = \begin{pmatrix} \sqrt{\tilde{\lambda}}A(v)c_1(\eta)e^{-z/2}W_{-1/2}, \sqrt{\tilde{\lambda}}(2ke^z) \\ -i\sqrt{\tilde{\lambda}}B(v)c_1(\eta)e^{-z/2}W_{-1/2}, \sqrt{\tilde{\lambda}}(2ke^z) \\ A(v)c_2(\eta)e^{-z/2}W_{1/2}, \sqrt{\tilde{\lambda}}(2ke^z) \\ B(v)c_2(\eta)e^{-z/2}W_{1/2}, \sqrt{\tilde{\lambda}}(2ke^z) \end{pmatrix} \exp(ik_x x). \quad (4.4.41)$$

Let us look for solutions of the system in eqn.(4.4.38) and eqn.(4.4.39) when the electromagnetic potential has the simple function dependence $A_1(y) = A_1 y$. In this case one can obtain exact solutions for $A(v)$ and $B(v)$ in terms of hypergeometric functions. After making the change of variable of eqn.(4.4.40) and using the recurrence relations

$$(b-1)M(a, b-1, z) = (b-1)M(a, b, z) + z \frac{dM(a, b, z)}{dz}, \quad (4.4.42)$$

$$\frac{1}{a} \frac{dM(a, b, z)}{dz} + M(a, b, z) = M(a + 1, b, z), \quad (4.4.43)$$

$$\frac{du(a, b, z)}{dz} - u(a, b, z) = -u(a, b + 1, z), \quad (4.4.44)$$

we find the general solution of the system of eqn.(4.4.38) and eqn(4.4.39) reads [3,4,9]

$$A = \sqrt{\frac{2A_1}{ik}} e^{-(1/2)v^2} (c_1 M(-\frac{k^2}{4A_1} + \frac{1}{2}, \frac{1}{2}, v^2) + c_2 u(-\frac{k^2}{4A_1} + \frac{1}{2}, \frac{1}{2}, v^2)), \quad (4.4.45)$$

$$B = e^{-(1/2)v^2} (c_1 M(-\frac{k^2}{4A_1} + \frac{1}{2}, \frac{3}{2}, v^2) - c_2 u(-\frac{k^2}{4A_1} + \frac{1}{2}, \frac{3}{2}, v^2)). \quad (4.4.46)$$

The exact solution of the system of equations (4.4.35) and (4.4.36) in the absence of electromagnetic interaction has the form

$$A = c_1 e^{i\sqrt{k^2 - k_x^2}y} + c_2 e^{-i\sqrt{k^2 - k_x^2}y}, \quad (4.4.47)$$

$$B = \frac{\sqrt{k^2 - k_x^2} - ik_x}{k} c_1 e^{i\sqrt{k^2 - k_x^2}y} - \frac{\sqrt{k^2 - k_x^2} + ik_x}{k} c_2 e^{-i\sqrt{k^2 - k_x^2}y}, \quad (4.4.48)$$

where C_1 and C_2 are arbitrary constants. The solutions of the Dirac eqn.(4.4.37) and eqn.(4.4.41) exhibit an asymptotic behavior which can be identified with the quasiclassical solutions of the HamiltonJacobi eqn(4.2.1). With the help of the asymptotic behavior, we find that the Dirac spinor Φ as $z \rightarrow -\infty$, and $y \rightarrow \infty$ takes the form [9]

$$\Phi_z \rightarrow -\infty = \begin{pmatrix} \frac{\sqrt{2A_1}}{ik} c_1(\eta) \\ -vc_1(v) \\ -\frac{\sqrt{2A_1}}{k} c_2(\eta) \\ ivc_2(v) \end{pmatrix} e^{-ke^z} e^{\sqrt{\lambda}z} e^{-v^2/2} v^{(k^2/2A_1)-1} \exp(ik_x x), \quad (4.4.49)$$

where the functions $c_1(\eta)$ and $c_2(\eta)$ satisfy the system of equations (4.4.35) and (4.4.36). For asymptotically large values of z we have that the spinor Φ takes the form.

$$\Phi_z \rightarrow \infty = \begin{pmatrix} \sqrt{\lambda} \frac{\sqrt{2A_1}}{k} c_1(\eta) e^{-z} \\ i\sqrt{\lambda} v c_1(v) e^{-z} \\ -i2\sqrt{2A_1} c_2(\eta) \\ -2k v c_2(v) \end{pmatrix} e^{-ke^z} e^{-v^2/2} v^{k^2/2A_1} \exp(ik_x x). \quad (4.4.50)$$

Looking at the solution of the Hamilton-Jacobi equation we can identify eqn.(4.4.37) and (4.4.41) as the corresponding quasiclassical modes as $z \rightarrow -\infty$ and $z \rightarrow \infty$, respectively. An approximate expression for the time dependence of the spinor Φ can be obtained with the help of the WKB approximation. In this case we obtain

$$c_2(\eta) \sim \exp(i\omega(\eta)), \quad (4.4.51)$$

$$c_1(\eta) \sim -i \frac{c_{10}}{\omega(\eta) + M\alpha(\eta)} \exp(i\omega(\eta)), \quad (4.4.52)$$

where c_{10} is a normalization constant and $\omega(\eta) = \sqrt{iM \frac{d\alpha}{d\eta} + M^2\alpha^2 - \tilde{\lambda}}$. Looking at eqn.(4.4.51)- (4.4.52) and eqn.(4.4.49) we readily see that, for large values of η we obtain $c_1(\eta) \rightarrow -i \frac{c_{10}}{2M\alpha(\eta)\exp(i\omega(\eta))}$. Analytic solutions of the system of equations (4.4.51) and (4.4.52) can be obtained for some particular expansion parameter $\alpha(\eta)$

4.5 Relativistic Landau Energy Level in 4 dimensions

We now solve the Dirac equation to find the relativistic energy level of a charged fermion in the presence of external magnetic field in the 4 dimensional spacetime. As an approximation, we will ignore the effect of curvature on the energy levels of the fermions. Starting from the Dirac equation in flat spacetime

$$\gamma^\mu \partial_\mu \psi + M\psi = 0, \quad (4.5.1)$$

the energy solution $\psi(x) = u(p)e^{-ipx} = u(p)e^{-iEt+ipx}$ satisfies the equation $(\gamma^\mu p_\mu - m)u(p) = 0$. Let us consider a particle in an external magnetic field, the effect of the magnetic field can be taken into account by adding the field momentum, $p_\mu \rightarrow p_\mu - eA_\nu$. We will choose the magnetic field to point in the z direction and uniformly distributed over the entire x, y, z space. The equation of motion of the fermion in 4 dimensional space becomes

$$\{p_x + p_y - 2eB_x p_y + e^2 B^2 x^2 - eB\sigma_z\}\phi = (E^2 - M^2)\phi \quad (4.5.2)$$

The energy condition from the equation of motion is given by

$$E_n^2 = M^2 + p_z^2 + (2n - \nu + 1)2M\mu_\beta B. \quad (n = 0, 1, 2, \dots, \quad \nu = \pm 1) \quad (4.5.3)$$

If we let $j = n - \frac{\nu}{2}$, then we have

$$E_j = M^2 + p_z^2 + (j + \frac{1}{2})4M\mu_\beta B, \quad (4.5.4)$$

$$E_j = M^2 + p_n^2 + (j + \frac{1}{2})4M\mu_\beta B, \quad (p_n^2 = p_z^2) \quad (4.5.5)$$

From eqn.(4.5.4) and (4.5.5), energy is quantized in the x-y plane and contains certain degeneracy of states, i.e., there are several states with the same one-particle energy. The number of states g_j of a discrete energy level j is

$$g_j = g_s \int dp_x dp_y dx dy = g_s L_x L_y 2\pi \int_{p_j}^{p_{j+1}} p dp = g_s \pi L_x L_y (p_{j+1}^2 - p_j^2)$$

$$g_j = g_s \pi L_x L_y (4m\mu_\beta B). \quad (4.5.6)$$

Therefore $p_j^2 = (p_x^2 + p_y^2) = 4jM\mu_\beta B$, where $g_s (= 2s + 1)$ is a spin degeneracy independent of j. The degeneracy is proportional to the field and vanishes for $B \rightarrow 0$. The discrete energies from the degrees of freedom of the plane perpendicular to the magnetic field is called the Landau levels, characterizing the statistical properties of the fermionic system.

4.6 Spinors of the Dirac equation in the presence of strong magnetic field

As an approximation, we will ignore the effect of curvature on the fermions. Starting from the Dirac equation in curved spacetime

$$\bar{\gamma}^\mu (\partial_\mu - ieA_\mu) \bar{\psi} + M\bar{\psi} = 0, \quad (4.6.1)$$

where $\bar{\psi} = a(\eta)^{-\frac{3}{2}} e^z \psi$ and the bar gamma matrices are given by eqn.(4.4.7). Hence Eqn(4.6.1) can be rewritten as

$$\bar{\gamma}^\mu (\partial_\mu - ieA_\mu) a(\eta)^{-\frac{3}{2}} e^z \psi + M a(\eta)^{-\frac{3}{2}} e^z \psi = 0, \quad (4.6.2)$$

$$\Rightarrow [\bar{\gamma}^0(\partial_0 - ieA_0) + \bar{\gamma}^1(\partial_1 - ieA_1) + \bar{\gamma}^2(\partial_2 - ieA_2) + \bar{\gamma}^3(\partial_3 - ieA_3)]a(\eta)^{\frac{-3}{2}}e^z\psi + Ma(\eta)^{\frac{-3}{2}}e^z\psi = 0.$$

But from eqn.(1.5.2) we have only the x-component of vector potential. So the above equation can be reduced to

$$[\bar{\gamma}^0(\partial_0) + \bar{\gamma}^1(\partial_1 - ieA_1) + \bar{\gamma}^2(\partial_2) + \bar{\gamma}^3(\partial_3)]a(\eta)^{\frac{-3}{2}}e^z\psi + Ma(\eta)^{\frac{-3}{2}}e^z\psi = 0. \quad (4.6.3)$$

For free particle solutions, as usual we propose

$$\psi = u_p e^{ip_\mu x^\mu},$$

where $u_p = \begin{pmatrix} \Phi \\ \chi \end{pmatrix}$.

Substituting the value of ψ and using eqn.(4.4.7), eqn.(4.6.3) gives

$$[-i\gamma^0 E + i\gamma^1 e^z(p_1 - ieA_1(y)) + i\gamma^2 e^z p_2 + i\gamma^0 p_3]e^z u_p + e^z M u_p = 0. \quad (4.6.4)$$

Substituting eqn.(4.4.28) in to eqn.(4.6.4) we can arrive at

$$\begin{pmatrix} e^z(E + p_3 + M) & i\sigma_1 e^{2z}[p_1 - eA_1(y) + i\sigma_2 p_2] \\ i\sigma_1 e^{2z}[p_1 - eA_1(y) + i\sigma_2 p_2] & e^z(-E + p_3 + M) \end{pmatrix} \begin{pmatrix} \Phi \\ \chi \end{pmatrix} = 0 \quad (4.6.5)$$

$$e^z(E + p_3 + M)\Phi + i\sigma_1 e^{2z}[p_1 - eA_1(y) + i\sigma_2 p_2]\chi = 0 \quad (4.6.6)$$

$$i\sigma_1 e^{2z}[p_1 - eA_1(y) + i\sigma_2 p_2]\Phi + e^z(-E + p_3 + M)\chi \quad (4.6.7)$$

Solving for Φ and χ from eqn.(4.6.5),

$$\Phi = -\frac{i\sigma_1 e^z[p_1 - eA_1(y) + i\sigma_2 p_2]}{E + p_3 + M}\chi, \quad (4.6.8)$$

and

$$\chi = \frac{i\sigma_1 e^z[p_1 - eA_1(y) - \sigma_3 p_2]}{E - p_3 - M}\Phi. \quad (4.6.9)$$

substituting eqn.(4.6.9) to eqn.(4.6.6) we find

$$E^2 - p_3^2 - M^2 - 2p_3 M = -(i\sigma_1 e^z[p_1 - eA_1(y) + i\sigma_2 p_2])^2 \quad (4.6.10)$$

We can associate this energy eigen value with the relativistic form of the Landau energy levels, given by eqn.(4.5.3).

For the case of positive energy $E = E_n$, the lower component of the spinor, Φ from eqn.(4.6.8) is given by

$$\Phi = \frac{1}{E_n + p_3 + M} \begin{pmatrix} e^z p_2 & -ie^z(p_1 - eA_1(y)) \\ -ie^z(p_1 - eA_1(y)) & -e^z p_2 \end{pmatrix} \chi, \quad (4.6.11)$$

and for the case of negative energy $E = |-E_n|$ in to eqn.(4.6.9) we can arrive at

$$\chi = \frac{1}{|-E_n| + p_3 - M} \begin{pmatrix} -e^z p_2 & ie^z(p_1 - eA_1(y)) \\ ie^z(p_1 - eA_1(y)) & e^z p_2 \end{pmatrix} \Phi. \quad (4.6.12)$$

An appropriate solutions for Φ and for χ are to take

$$\Phi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ or } \Phi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ or } \chi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (4.6.13)$$

respectively.

substituting eqn.(4.6.13) in to eqn.(4.6.11) we get

$$\Phi = \begin{pmatrix} \frac{e^z p_2}{E_n + p_3 + M} \\ -\frac{e^z(p_1 - eA_1(y))}{E_n + p_3 + M} \end{pmatrix}, \text{ or } \Phi = \begin{pmatrix} -\frac{e^z(p_1 - eA_1(y))}{E_n + p_3 + M} \\ \frac{e^z p_2}{E_n + p_3 + M} \end{pmatrix}, \quad (4.6.14)$$

So that the full solution U_p becomes

$$U_p = \begin{pmatrix} \frac{e^z p_2}{E_n + p_3 + M} \\ -\frac{e^z(p_1 - eA_1(y))}{E_n + p_3 + M} \\ 1 \\ 0 \end{pmatrix}, \text{ or } U_p = \begin{pmatrix} -\frac{e^z(p_1 - eA_1(y))}{E_n + p_3 + M} \\ \frac{e^z p_2}{E_n + p_3 + M} \\ 0 \\ 1 \end{pmatrix}. \quad (4.6.15)$$

substituting eqn.(4.6.13) in to eqn.(4.6.12) we get

$$\chi = \begin{pmatrix} -\frac{e^z p_2}{|-E_n| + p_3 - M} \\ \frac{ie^z(p_1 - eA_1(y))}{|-E_n| + p_3 - M} \end{pmatrix}, \text{ or } \chi = \begin{pmatrix} -\frac{ie^z(p_1 - eA_1(y))}{|-E_n| + p_3 - M} \\ \frac{e^z p_2}{|-E_n| + p_3 - M} \end{pmatrix}. \quad (4.6.16)$$

By the same process as the positive energy, the negative energy spinors which are denoted by V_p can be done to be:

$$V_p = \begin{pmatrix} 1 \\ 0 \\ -\frac{e^z p_2}{|-E_n| + p_3 - M} \\ \frac{ie^z(p_1 - eA_1(y))}{|-E_n| + p_3 - M} \end{pmatrix}, \text{ or } V_p = \begin{pmatrix} 0 \\ 1 \\ -\frac{ie^z(p_1 - eA_1(y))}{|-E_n| + p_3 - M} \\ \frac{e^z p_2}{|-E_n| + p_3 - M} \end{pmatrix}. \quad (4.6.17)$$

We call the positive and negative energy solutions, U_p and V_p as positive and negative energy spinors respectively.

Chapter 5

Conclusion and Remarks

In this thesis, we introduced an algebra based on a matrix whose square is diagonal and involving the Dirac gamma matrices defined on curved space. This algebra is rich in sense that it allowed us to formulate the Dirac equation with the need for spin connections or vierbeins. As an illustration of our findings, we studied one relativistic particle in 4D space-time with static diagonal metric. By choosing an appropriate representation of the elements of our algebra, we obtained exact solutions for Dirac equations. The present work will shortly be extended and followed by a study of relativistic particles in 4D curved space in the presence of an electromagnetic field. We have solved the Dirac equations in an open cosmological universe with partially horn topology. The solutions of the relativistic wave equations are expressed in terms of special functions. In chapter 4 we have shown that the algebraic method of separation permits one a complete separation of variables of the Dirac equation in the line element associated with a horn topology. The identification of the quasiclassical modes with the help of the relativistic HamiltonJacobi equation shows that this method is a very useful tool in the study of quantum effects in curved spaces. As a final remark, we should mention that the introduction of nonstandard topologies in order to describe the large scale structure of the spacetime also opens new possibilities to discuss quantum effects in globally inhomogeneous and anisotropic backgrounds in the presence of non-trivial electromagnetic interactions.

References

- [1] G.G.Nyambuya, *New Curved Spacetime Dirac Equation on the Anomalous Gyromagnetic Ratio*, Foundations of Physics, Vol.38,July 2008.
- [2] Zhao Zhen-Hua, LIU Yu-Xiao, and LI Xi-Guo¹, Theor.Phys., Beijing, China, 2007.
- [3] Antonio C. Venetti, *Progress in Materials Science Research*, 2008.
- [4] Victor M. Villalba, *Separation of Variables and Exact Solution of the Klein – Gordon and Dirac Equations in an Open Universe*, Institut für Theoretische Physik, Universität Frankfurt.
- [5] A. D. Alhaidaria and A. Jellal, *Dirac and Klein–Gordon Equations in CurvedSpace*, Saudi Center for Theoretical Physics, Jeddah, Saudi Arabia.
- [6] Greiner, *Relativistic Quantum Mechanics Wave Equations*,3rd edition,
- [7] William O. Straub, *The Spin Connection In Weyl Space*, Pasadena, California.
- [8] Tom Theuns, *Physical Cosmology*, Springer-Verlege, Berlin, 1994.
- [9] Esteban Isasi Catal, *Journal of Mathematical Physics*, Vol. 43, Number 10 Oct. 2002.
- [10] N.N.Lebedev, *Special Functions and Their Applications*.
- [11] Shimeles Terefe, *MSc Thesis*, Addis Ababa University, 2009.
- [12] Steven Weinberg, *Gravitation and Cosmology*, Institut für theoretische physik, Universität Frankfurt.
- [13] Karolos Potamianos, *Relativistic Electron Theory the Dirac Equation*, Mathematical Physics Project.
- [14] <http://www.newscientists.com>
- [15] Malcom S. Longair, *High Energy Astro Physics*, 2nd edition , Cambrige Universiy Press 1997.
- [16] Norbert straumann, *General Relativity and Relativistic Astrophsics*, Berlin Heidelbers, Newyork, 1984.

Declaration

I hereby declare that this thesis is my original work and has not been presented for a degree in any other university. All sources of material used for the thesis have been duly acknowledged.

Name: Eshetu Mekonnen

Signature:

This thesis has been submitted for the examination with my approval as university advisor.

Name: Dr.Lgesse Wotro

Signature:

Addis Ababa University

Department of Physics

June, 2013