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THE PROPAGATOR FORMULATION OF PHOTON STATISTICS AND SQUEEZED STATES IN PARAMETRIC AMPLIFICATION

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ABSTRACT

The photon statistics and squeezing properties of the signal light produced by the degenerate and the nondegenerate parametric amplifiers are analyzed employing the coherent state propagator. The propagator is evaluated using a method recently introduced by Fesseha [1]. The task of evaluating the propagator by means of this method essentially reduces to the problem of solving the pertinent Euler-Lagrange equations.

Contents

1. INTRODUCTION

A c-number formulation of the quantum dynamics of a physical system appears to be more convenient than the Heisenberg or the density operator formalism. In quantum optics the analysis of a system is very often carried out using the P-function or the Q-function. However, the task of finding the solution of the pertinent Fokker-Planck equation does not turn out to be so simple. In addition, for a system with a nonclassical feature, the diffusion coefficient in the Fokker-Planck equation becomes negative. As a result of this, it is not possible to obtain the solution of such equation.

On the other hand, we realize that the Q-function is expressible in terms of the propagator. In view of this, the problem of determining the Q-function reduces to the task of evaluating the propagator.

The main objective of this thesis is to analyze the statistical and squeezing properties of the signal light from the degenerate and the nondegenerate parametric amplifiers, employing the coherent state propagator. The propagator may be determined using path integral methods [2] or by directly solving the Schrödinger equation. However, we find it to be more convenient to evaluate propagator applying the method recently introduced by Fesseha [1].

This method has been developed by introducing the notion of a coherent state propagator action associated with an arbitrary quadratic Hamiltonian. This action is calculated by means of the Euler-Lagrange equation applying the definition

$A = A_0 + \int_0^T L_I \, dt$, where A_0 stands for the free action and where both the quadratic terms in the interaction Lagrangian L_I must be the product of the free and the full classical solutions, with a conjugated variable in one of these terms and its complex conjugate in the other first fixed, respectively, at the free and the full classical solutions. Once the action is evaluated for a quadratic Hamiltonian then the associated coherent state propagator can easily be determined using this method.

The mean and the variance of the photon number as well as the photon number and the photon count distributions are very often used to describe the statistical properties of light. According to the relation between the mean and the variance of the photon number, the photon statistics is usually classified as Poissonian, super-Poissonian or sub-Poissonian. In addition, based on the photon number distribution, one describes a light beam as coherent, chaotic or none of these.

Another interesting property of light is the fluctuations of its quadrature components. For light in a squeezed state the fluctuations in one quadrature component are below the vacuum level while the fluctuations in the other component are increased so that the uncertainty principle is not violated [3-5]. Squeezed light has recently been the object of considerable interest in view of its potential applications in high precision measurements and low-noise communications[6,7]. Squeezed states have been generated by several nonlinear optical processes such as degenerate parametric amplifier [8,9], second harmonic generation [10-12] and degenerate four wave mixing [13,14].

This thesis is devoted to the study of the photon statistics and the quadrature fluctuations of the signal light generated in parametric amplification, employing the new method of evaluating the propagator.

The organization of this thesis is as follows. In chapter two we derive an expression for the expectation value of an arbitrary operator in terms of the P-and the Q-functions. We also discuss various methods of evaluating the coherent state propagator. In chapter three we present a systematic derivation of the photon number and the photon count distributions. In addition, we obtain explicit expressions for the fluctuations of the quadrature components of light beam. In chapter four the statistical and squeezing properties of the signal light from the degenerate parametric amplifier are investigated. In chapter five, we study the statistical properties and quadrature fluctuations of the signal light from the nondegenerate parametric amplifier. Finally in chapter six we discuss briefly the main results of this thesis and make certain remarks of interest.

2. THE COHERENT STATE PROPAGATOR AND EXPECTATION VALUES

Coherent states are very useful in the analysis of quantum optical problems [15] since the coherent state representation has proved to be more suitable for such analysis. The coherent state is a minimum uncertainty state and is associated with the most classical state of light one can imagine in the framework of quantum theory [16]. A coherent state can be expressed as the linear superposition of the number states. The coherent states satisfy the completeness relation and the normalization condition but are not orthogonal.

The expectation value of an operator $A(a^{+},a)$ is expressible in terms of the density operator or in terms of a c-number equivalent of this operator for a certain ordering scheme. For example, the expectation value of $A(a^{+},a)$ can be expressed in terms of the Q-function which is the c-number equivalent of the density operator for the normal ordering. Since the Q-function is related to the coherent state propagator, the determination of this function, for a given system, reduces to the task of evaluating the propagator. The propagator may be determined by directly solving the Schrödinger equation or by using path integral methods or by employing the new method.

This chapter is structured as follows. In section one we discuss some important properties of the coherent states that will be frequently used in this work.

In section two we present a systematic derivation for the expectation value of an operator $A(a^{+},a)$ in terms of the Q- and P- functions. It is to be recalled that the P-function is the c-number equivalent of the density operator for the antinormal ordering.

In section three we derive the Schrödinger equation for the propagator. We also consider the integral representation of the propagator. Finally, we discuss the new method of evaluating the propagator.

2.1 The Coherent States

A coherent state is defined as the eigenstate of the annihilation operator a such that [17]

$$a|\alpha\rangle = \alpha|\alpha\rangle \quad \text{and} \quad \langle\alpha|a^{+} = \langle\alpha|\alpha^{*} \tag{2.1}$$

Where α is a complex number. The boson operators a and a^{+} satisfy the commutation relation

$$[a, a^{+}] = 1. \tag{2.2}$$

Using the completeness relation for the number states

$$I = \sum_{n} |n\rangle\langle n| \ , \tag{2.3}$$

one can easily verify that the coherent state is expressible as

$$|\alpha\rangle = \exp(-\frac{1}{2}|\alpha|^{2}) \sum_{n} \frac{\alpha^{n}}{\sqrt{n!}} |n\rangle \ . \tag{2.4}$$

The coherent states are normalized

$$\langle\alpha|\alpha\rangle = 1 \tag{2.5}$$

but are not orthogonal

$$<\alpha|\beta> = \exp(-\frac{1}{2}|\alpha^2| - \frac{1}{2}|\beta^2| + \alpha^*\beta) \quad . \tag{2.6}$$

We note that

$$|<\alpha|\beta>|^2 = \exp(-|\alpha-\beta|^2), \tag{2.7}$$

so that the coherent states are approximately orthogonal if $|\alpha-\beta|$ is very large. The completeness relation for the coherent states has the form [18]

$$I = \int \frac{d^2\alpha}{\pi} |\alpha><\alpha| \tag{2.8}$$

where

$$d^2\alpha = d(\mathrm{Re}\alpha)\, d(\mathrm{Im}\alpha) \quad . \tag{2.9}$$

2.2 Expectation Value

The expectation value of an operator $A(a^+,a)$ for a system described by the density operator $\rho(a^+,a)$ can be written as

$$<A> = Tr(\rho A). \tag{2.10}$$

Now expanding $\rho(a^+,a)$ and $A(a^+,a)$ in a power series in the normal order and in the antinormal order, respectively, we have

$$<A> = \sum_{l,m} \sum_{n,k} C_{lm} C'_{nk} Tr(a^{+l} a^m a^n a^{+k}) \quad .$$

Using the cyclic property of the trace operation, we see that

$$\mathrm{Tr}(a^{+l}a^{m}a^{n}a^{+k}) = \mathrm{Tr}(a^{+k}a^{+l}a^{m}a^{n}) \quad .$$

Thus

$$\langle A\rangle = \sum_{l,m}\sum_{n,k} C_{lm}C'_{nk}Tr(a^{+k}a^{+l}a^{m}a^{n}) \quad . \tag{2.11}$$

With the aid of (2.8), expression (2.11) can be put in the form

$$\langle A\rangle = \int\frac{d^2\alpha}{\pi}\sum_{l,m}\sum_{n,k} C_{lm}C'_{nk}\langle\alpha\,|\,a^{+k}a^{+l}a^{m}a^{n}\,|\,\alpha\rangle$$

or

$$\langle A\rangle = \int\frac{d^2\alpha}{\pi}Q(\alpha^*,\alpha)\,A_a(\alpha^*,\alpha) \tag{2.12}$$

in which

$$Q(\alpha^*,\alpha) = \sum_{l,m} C_{lm}\alpha^{*l}\alpha^{m} \tag{2.13}$$

is the c-number equivalent of the density operator ρ for the normal ordering and

$$A_a(\alpha^*,\alpha) = \sum_{n,k} C'_{nk}\alpha^{*k}\alpha^{n} \tag{2.14}$$

is the c-number equivalent of the operator A for the antinormal ordering.

Alternatively, expanding $\rho(a^+,a)$ and $A(a^+,a)$ in a power series in the antinormal order and in the normal order, respectively, we have

$$\langle A\rangle = \sum_{l,m}\sum_{n,k} C_{lm}C'_{nk}Tr(a^{l}a^{+m}a^{+n}a^{k}) \quad .$$

Since

$$\mathrm{Tr}(a^{l}a^{+m}a^{+n}a^{k}) = \mathrm{Tr}(a^{+m}a^{+n}a^{k}a^{l}) \quad ,$$

we see that

$$\langle A\rangle = \sum_{l,m}\sum_{n,k} C_{lm}C'_{nk} Tr(a^{+m}a^{+n}a^{k}a^{l}) \quad .$$

Now following the above procedure, the expectation value of A can be expressed in the form

$$\langle A\rangle = \int \frac{d^2\alpha}{\pi} P(\alpha^*,\alpha) A_n(\alpha^*,\alpha) \tag{2.15}$$

where

$$P(\alpha^*,\alpha) = \sum_{l,m} C_{lm}\alpha^{*m}\alpha^{l}$$

is the c-number equivalent of the density operator ρ for the antinormal ordering and

$$A_n(\alpha^*,\alpha) = \sum_{n,k} C'_{nk}\alpha^{*n}\alpha^{k}$$

is the c-number equivalent of the operator A for the normal ordering.

In order to obtain the relation between the Q-function and the P-function, we expand the density operator $\rho(a^+,a)$ in a power series in the antinormal order. That is

$$\rho = \sum_{l,m} C_{lm}a^{l}a^{+m} \quad . \tag{2.16}$$

Now employing (2.8), we have

$$\begin{aligned}\rho &= \int \frac{d^2\alpha}{\pi}\sum_{l,m} C_{lm}a^{l}|\alpha\rangle\langle\alpha|a^{+m}\\ &= \int \frac{d^2\alpha}{\pi}\Big(\sum_{l,m} C_{lm}\alpha^{*m}\alpha^{l}\Big)|\alpha\rangle\langle\alpha|\\ &= \int \frac{d^2\alpha}{\pi} P(\alpha^*,\alpha)|\alpha\rangle\langle\alpha| \quad .\end{aligned} \tag{2.17}$$

Multiplying this from the left by $\langle\beta|$ and from the right by

$|\beta>$, we get

$$Q(\beta^*,\beta) = \int \frac{d^2\alpha}{\pi} P(\alpha^*,\alpha)\exp(-|\alpha-\beta|^2) \quad . \tag{2.18}$$

The density operator of a system describable by a quantum Hamiltonian $H(a^+,a)$ and represented by the state vector $|\Psi(t)>$ is defined as

$$\rho = |\Psi(t)><\Psi(t)| \tag{2.19}$$

where the state vector evolves in time according to the Schrödinger equation ($\hbar=1$)

$$i\frac{d}{dt}|\Psi(t)> = H(a^+,a)|\Psi(t)> \quad . \tag{2.20}$$

A formal solution of this equation, when the system is initially in a coherent state $|\alpha'>$, can be written as

$$|\Psi(t)> = U|\alpha'> \tag{2.21}$$

where

$$U = e^{-iHt} \tag{2.22}$$

is the evolution operator. It then follows from (2.19) and (2.21) that

$$\rho = U|\alpha'><\alpha'|U^+ \quad . \tag{2.23}$$

Multiplying this equation from the left by $<\alpha|$ and from the right by $|\alpha>$, we note that

$$Q(\alpha^*,\alpha) = K(\alpha,t|\alpha',0)K^*(\alpha,t|\alpha',0) \tag{2.24}$$

in which

$$Q(\alpha^*,\alpha) = \langle\alpha|\rho|\alpha\rangle \tag{2.25}$$

and

$$K(\alpha,t|\alpha',0) = \langle\alpha|U|\alpha'\rangle \tag{2.26}$$

is defined as the coherent state propagator of the system.

We shall next proceed to show that the P-function and the Q-function are normalized. To this end, we recall that

$$\rho = \int \frac{d^2\alpha}{\pi} P(\alpha^*,\alpha)\,|\alpha\rangle\langle\alpha| \quad .$$

Since

$$\mathrm{Tr}(\rho) = 1 \tag{2.27}$$

we notice that

$$\int \frac{d^2\alpha}{\pi} P(\alpha^*,\alpha) = 1 \quad . \tag{2.28}$$

Moreover, on account of (2.25) we have

$$\begin{aligned} \int \frac{d^2\alpha}{\pi} Q(\alpha^*,\alpha) &= Tr\left(\int \frac{d^2\alpha}{\pi}\,|\alpha\rangle\langle\alpha|\,\rho\right) \\ &= Tr(I\rho) \\ &= 1 \quad . \end{aligned} \tag{2.29}$$

2.3 The Coherent State Propagator

A. The Equation of Evolution

The coherent state propagator of a system is the matrix element of the time evolution operator between the initial and the final states :

$$K(\alpha,t|\alpha',0) = \langle\alpha|U|\alpha'\rangle \ .$$

Differentiating both sides of this equation with respect to time, and noting that the state vectors are independent of time, we have

$$\frac{\partial K}{\partial t}(\alpha,t|\alpha',0) = \langle\alpha|\frac{\partial U}{\partial t}|\alpha'\rangle$$

or

$$i\frac{\partial K}{\partial t}(\alpha,t|\alpha',0) = \langle\alpha|HU|\alpha'\rangle. \tag{2.30}$$

Now expanding $H(a^{+},a)$ in a power series in the normal order, we have

$$i\frac{\partial K}{\partial t}(\alpha,t|\alpha',0) = \sum_{l,m} C_{lm}\langle\alpha|a^{+l}a^{m}U|\alpha'\rangle \ . \tag{2.31}$$

so that using (2.8) expression (2.31) can be put in the form

$$i\frac{\partial K}{\partial t}(\alpha,t|\alpha',0) = \int\frac{d^{2}\beta}{\pi}\sum_{l,m} C_{lm}\alpha^{*l}\beta^{m}\langle\alpha|\beta\rangle\langle\beta|U|\alpha'\rangle \ .$$

Since

$$\beta\langle\alpha|\beta\rangle = \left(\frac{1}{2}\alpha + \frac{\partial}{\partial\alpha^{*}}\right)\langle\alpha|\beta\rangle \quad , \tag{2.32}$$

we find that

$$i\frac{\partial K}{\partial t}(\alpha,t|\alpha',0) = \int\frac{d^{2}\beta}{\pi}\sum_{l,m} C_{lm}\alpha^{*l}\left(\frac{1}{2}\alpha + \frac{\partial}{\partial\alpha^{*}}\right)^{m}\langle\alpha|\beta\rangle\langle\beta|U|\alpha'\rangle$$

$$= \sum_{l,m} C_{lm}\alpha^{*l}\left(\frac{1}{2}\alpha + \frac{\partial}{\partial\alpha^{*}}\right)^{m}\langle\alpha|\left(\int\frac{d^{2}\beta}{\pi}|\beta\rangle\langle\beta|\right)U|\alpha'\rangle$$

or

$$i\frac{\partial K}{\partial t}(\alpha,t|\alpha',0) = H(\alpha^*,\frac{1}{2}\alpha+\frac{\partial}{\partial\alpha^*})\ K(\alpha,t\ \alpha',0) \quad . \tag{2.33}$$

where

$$H(\alpha^*,\frac{1}{2}\alpha+\frac{\partial}{\partial\alpha^*}) = \sum_{l,m} C_{lm}\alpha^{*l}\,(\frac{1}{2}\alpha+\frac{\partial}{\partial\alpha^*})^m \tag{2.34}$$

is the c- number equivalent of $H(a^+,a)$ for the normal ordering. This is the Schrödinger equation for the propagator subject to the initial condition [2]

$$K(\alpha,t|\alpha',0)|_{t=0} = \langle\alpha|\alpha'\rangle \ . \tag{2.35}$$

B. Path Integral Representation

We now wish to construct a path integral representation for the coherent state propagator

$$K(\alpha'',T|\alpha',0) = \langle\alpha''|e^{-iHT}|\alpha'\rangle \ . \tag{2.36}$$

One can divide the time T into N equal intervals, so that

$$T = N\epsilon \ . \tag{2.37}$$

Hence expression (2.36) can be rewritten as

$$\langle\alpha''|e^{-iHT}|\alpha'\rangle = \langle\alpha''|e^{-iHN\epsilon}|\alpha'\rangle$$

$$= \langle\alpha''|e^{-iH\epsilon}.\ e^{-iH\epsilon}\ldots\ e^{-iH\epsilon}|\alpha'\rangle.$$

Inserting N-1 identity operators defined in terms of the coherent states, the propagator can be expressed in the form

$$K(\alpha'', T|\alpha', 0) = \int \prod_{j=1}^{N-1} \frac{d^2\alpha_j}{\pi} \prod_{j=0}^{N-1} \langle\alpha_{j+1}|e^{-i\epsilon H}|\alpha_j\rangle \tag{2.38}$$

in which

$$\alpha_0 = \alpha' \quad \text{and} \quad \alpha_N = \alpha''. \tag{2.39}$$

For small ϵ, one can approximately write that

$$\langle\alpha_{j+1}|e^{-i\epsilon H}|\alpha_j\rangle = \langle\alpha_{j+1}|(1-i\epsilon H(a^+,a))|\alpha_j\rangle. \tag{2.40}$$

Since for normally ordered Hamiltonian $H(a^+,a)$

$$\langle\alpha_{j+1}|H(a^+,a)|\alpha_j\rangle = H(\alpha_{j+1}^*,\alpha_j)\langle\alpha_{j+1}|\alpha_j\rangle, \tag{2.41}$$

we see that

$$\langle\alpha_{j+1}|e^{-i\epsilon H}|\alpha_j\rangle = (1-i\epsilon H(\alpha_{j+1}^*,\alpha_j))\langle\alpha_{j+1}|\alpha_j\rangle$$

or

$$\langle\alpha_{j+1}|e^{-i\epsilon H}|\alpha_j\rangle = \langle\alpha_{j+1}|\alpha_j\rangle \exp(-i\epsilon H(\alpha_{j+1}^*,\alpha_j)). \tag{2.42}$$

Using this result in (2.38) the propagator can be rewritten as

$$K(\alpha'', T|\alpha', 0) = \lim_{\epsilon=0} \int \prod_{j=1}^{N-1} \frac{d^2\alpha_j}{\pi} \prod_{j=0}^{N-1} \langle\alpha_{j+1}|\alpha_j\rangle \exp\left(-i\epsilon H(\alpha_{j+1}^*,\alpha_j)\right) .$$

Noting that

$$\langle\alpha_{j+1} \ \alpha_j\rangle = \exp\left(\alpha^*_{j+1}\alpha_j - \frac{1}{2}\alpha^*_{j+1}\alpha_{j+1} - \frac{1}{2}\alpha^*_j\alpha_j\right) ,$$

we have

$$K(\alpha'', T|\alpha', 0)$$

$$= \lim_{\epsilon\to 0}\int\prod_{j=1}^{N-1}\frac{d^2\alpha_j}{\pi}\prod_{j=0}^{N-1}\exp\left[\alpha^*_{j+1}\alpha_j - \frac{1}{2}\alpha^*_{j+1}\alpha_{j+1} - \frac{1}{2}\alpha^*_j\alpha_j - i\epsilon H(\alpha^*_{j+1}, \alpha_j)\right]$$

$$= \lim_{\epsilon\to 0}\int\prod_{j=1}^{N-1}\frac{d^2\alpha_j}{\pi}\exp\sum_{j=0}^{N-1}\left[\alpha^*_{j+1}\alpha_j - \frac{1}{2}\alpha^*_{j+1}\alpha_{j+1} - \frac{1}{2}\alpha^*_j\alpha_j - i\epsilon H(\alpha^*_{j+1}, \alpha_j)\right]. \quad (2.43$$

Furthermore, applying the relations

$$\frac{1}{2}\alpha^*_{j+1}\alpha_j - \frac{1}{2}\alpha^*_j\alpha_j = \frac{1}{2}\epsilon\alpha_j \frac{(\alpha^*_{j+1} - \alpha^*_j)}{\epsilon} \quad (2.44a$$

and

$$\frac{1}{2}\alpha^*_{j+1}\alpha_j - \frac{1}{2}\alpha^*_{j+1}\alpha_{j+1} = -\frac{1}{2}\epsilon\alpha^*_{j+1} \frac{(\alpha_{j+1} - \alpha_j)}{\epsilon} \quad (2.44b)$$

we find that

$$\lim_{\epsilon\to 0}\sum_{j=0}^{N-1}\left(\alpha^*_{j+1}\alpha_j - \frac{1}{2}\alpha^*_{j+1}\alpha_{j+1} - \frac{1}{2}\alpha^*_j\alpha_j - i\epsilon H(\alpha^*_{j+1}, \alpha_j)\right)$$

$$= \lim_{\epsilon\to 0}\sum_{j=0}^{N-1}\left(\frac{\epsilon}{2}\alpha_j\frac{(\alpha^*_{j+1} - \alpha^*_j)}{\epsilon} - \frac{\epsilon}{2}\alpha^*_{j+1}\frac{(\alpha_{j+1} - \alpha_j)}{\epsilon} - i\epsilon H(\alpha^*_{j+1}, \alpha_j)\right)$$

$$= \int\left(\frac{1}{2}\dot{\alpha}^*\alpha - \frac{1}{2}\alpha^*\dot{\alpha} - iH(\alpha^*, \alpha)\right)dt . \quad (2.45)$$

Upon taking the limit $\epsilon\to 0$ one can put the coherent state propagator in the form

$$K(\alpha'', T|\alpha', 0) = \int D[\alpha(t)]\exp[\int_0^T L dt] \quad (2.46)$$

where

$$L = \frac{1}{2}\dot{\alpha}^{*}\alpha - \frac{1}{2}\alpha^{*}\dot{\alpha} - iH(\alpha^{*},\alpha) \tag{2.47}$$

is the Lagrangian associated with $H(\alpha^{*},\alpha)$ which is the c-number equivalent of $H(a^{+},a)$ for the normal ordering.

C. The New Method of Evaluation

According to the new method [1] introduced recently by Fesseha, the coherent state propagator for a single-mode quadratic Hamiltonian $H(a^{+},a)$ can be expressed in the form

$$K(\alpha'', T|\alpha', 0) = \left[\frac{\partial^2 A e^{i\omega T}}{\partial\alpha'\,\partial\alpha''^{*}}\right]^{1/2} e^{A} \tag{2.48}$$

where A is the coherent state propagator action defined by

$$A = A_0 + \int_0^T L_I\,dt \tag{2.49}$$

in which

$$A_0 = -\frac{1}{2}|\alpha''|^2 - \frac{1}{2}|\alpha'|^2 + \alpha'\alpha''^{*}e^{-i\omega T} \tag{2.50}$$

stands for the free action and L_I is the interaction Lagrangian. The quadratic terms in the interaction Lagrangian must be the product of the free and the full classical solutions, with a conjugated variable in one of these terms and its complex conjugate in the other first fixed, respectively, at the free and the full classical solutions (for details see Ref. [1]).

Moreover, according to this method the coherent state propagator for a two-mode quadratic Hamiltonian $H(a^+,a,b^+,b)$ can be put in the form

$$K(\alpha'',\beta'',T|\alpha',\beta',0) = \left[\frac{\partial^2 A e^{i\omega_a T}}{\partial\alpha'\,\partial\alpha''^*}\cdot\frac{\partial^2 A e^{i\omega_b T}}{\partial\beta'\,\partial\beta''^*}\right]^{1/2} e^{A} \,. \tag{2.51}$$

In this case, the free action A_0 is given by

$$A_0 = -\frac{1}{2}|\alpha''|^2 - \frac{1}{2}|\alpha'|^2 + \alpha'\alpha''^* e^{-i\omega_a T} - \frac{1}{2}|\beta''|^2 - \frac{1}{2}|\beta'|^2 + \beta'\beta''^* e^{-i\omega_b T} \quad (2.52$$

and the interaction Lagrangian is determined following the same procedure as that for the single mode Hamiltonian.

3. PHOTON STATISTICS AND SQUEEZED STATES

The statistical properties of a light beam is usually described in terms of the mean and the variance of the photon number as well as the photon number and the photon count distributions. The relation between the mean and the variance of the photon number is used to classify the photon statistics as Poissonian, super-Poissonian and sub-Poissonian. Unlike the other distributions, sub-Poissonian statistics represents a nonclassical feature of light. Such features of light cannot be described in terms of light waves.

Another interesting property of light is the fluctuations of the pertinent quadrature components. A light beam is said to be in a squeezed state if the fluctuations of one of the quadrature components is reduced below the vacuum level at the expense of enhanced fluctuations in the canonically conjugate quadrature [19,20]. A squeezed state like sub-Poissonian statistics represents a nonclassical feature of light.

Much work, both theoretical and experimental, has been done on squeezed states of light [21]. A variety of nonlinear optical systems are predicted to produce squeezed light. A squeezed light has attractive applications in precision measurements and low-noise communications [6,22].

This chapter is organized as follows. In section one we derive an expression for the photon number distribution P_n in terms of the Q-function.

In section two we obtain an expression for the photon count distribution $P_m(T)$ in terms of the Q-function.

In section three we discuss the fluctuations of the quadrature components of a light beam. The notion of a squeezed state of light is then defined in terms of the quadrature fluctuations.

3.1 The Photon Number Distribution

We now proceed to obtain the photon number distribution of a light beam described by the density operator ρ. The mean photon number is expressible as

$$\langle n\rangle = Tr(\rho n) \quad . \tag{3.1}$$

Employing (2.3), we can write

$$\begin{aligned}\langle n\rangle &= \sum_n Tr(|n\rangle\langle n|\rho n) \\ &= \sum_n \langle n|\rho n|n\rangle \\ &= \sum_n nP_n\end{aligned} \tag{3.2}$$

Where

$$P_n = \langle n|\rho|n\rangle \tag{3.3}$$

represents the photon number distribution. The quantity P_n represents the probability that the number of photons in a given light beam is n.

Now applying the completeness relation for the coherent states twice, expression (3.3) can be rewritten in the form

$$P_n = \int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}\langle n|\alpha\rangle\langle\alpha|\rho|\beta\rangle\langle\beta|n\rangle \ , \tag{3.4}$$

so that making use of (2.4) and (2.25), we get

$$P_n = \int \frac{d^2\alpha}{\pi} \frac{d^2\beta}{\pi} e^{-\alpha^*\alpha - \beta^*\beta + \alpha^*\beta} Q(\alpha^*, \beta) \frac{\alpha^n \beta^{*n}}{n!} \quad . \tag{3.5}$$

We recall that

$$\int \frac{d^2\alpha}{\pi} f(\alpha^*, \alpha) = \int \frac{dx\,dy}{\pi} f(x, y) \tag{3.6}$$

and in invoking the transformation $(x,y) \to (\alpha^*,\alpha)$, we have

$$\int \frac{dx\,dy}{\pi} f(x, y) = \int \frac{d\alpha^* d\alpha}{\pi} \left| J\left(\frac{x, y}{\alpha^*, \alpha}\right) \right| f(\alpha^*, \alpha) \tag{3.7}$$

where

$$J\left(\frac{x, y}{\alpha^*, \alpha}\right) = \begin{vmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial x}{\partial \alpha^*} \\ \frac{\partial y}{\partial \alpha} & \frac{\partial y}{\partial \alpha^*} \end{vmatrix} = \frac{i}{2} \tag{3.8}$$

is the Jacobian of the transformation. In view of (3.6), (3.7) and (3.8), the photon count distribution is expressible as

$$P_n = \left(\frac{1}{2\pi}\right)^2 \int d\alpha^* d\alpha\, d\beta^* d\beta\; e^{-\alpha^*\alpha - \beta^*\beta + \alpha^*\beta} Q(\alpha^*, \beta) \frac{\alpha^n \beta^{*n}}{n!} \quad , \tag{3.9}$$

so that application of the fact that

$$\beta^{*n} = (-1)^n \frac{\partial^n}{\partial \beta^n} e^{-\beta\beta^*} \quad \textit{and} \quad \alpha^n = (-1)^n \frac{\partial^n}{\partial \alpha^{*n}} e^{-\alpha^*\alpha} \tag{3.10}$$

leads to

$$P_n = \frac{1}{n!}\left(\frac{1}{2\pi}\right)^2 \int d\alpha^* d\alpha\, d\beta^* d\beta\, Q(\alpha^*, \beta)\; e^{\alpha^*\beta} \frac{\partial^n}{\partial \beta^n} e^{-\beta^*\beta} \frac{\partial^n}{\partial \alpha^{*n}} e^{-\alpha^*\alpha} \quad . \tag{3.11}$$

Since

$$\frac{1}{2\pi} \int d\beta^* e^{-\beta^*\beta} = \delta(\beta) \tag{3.12a}$$

and

$$\frac{1}{2\pi}\int d\alpha\, e^{-\alpha^*\alpha} = \delta(\alpha^*) \tag{3.12b}$$

we have

$$P_n = \int d\alpha^* d\beta\, \varrho(\alpha^*, \beta)\, \frac{\partial^n \delta(\alpha^*)}{\partial \alpha^{*n}}\, \frac{\partial^n \delta(\beta)}{\partial \beta^n}\, . \tag{3.13}$$

Furthermore, on account that

$$\int dx \frac{\partial^n \delta(x)}{\partial x^n} f(x) = (-1)^n \frac{\partial^n f(x)}{\partial x^n}\Big|_{x=0}\, , \tag{3.14}$$

the photon count distribution can be put in the form

$$P_n = \frac{1}{n!}\, \frac{\partial^{2n}}{\partial \alpha^{*n} \partial \beta^n}\left[\varrho(\alpha^*, \beta)\, e^{\alpha^*\beta}\right]_{\alpha^* = \beta = 0}\, .$$

Finally, since α^* and α are independent variables, one can write that

$$P_n = \frac{1}{n!}\, \frac{\partial^{2n}}{\partial \alpha^{*n} \partial \alpha^n}\left[\varrho(\alpha^*, \alpha)\, e^{\alpha^*\alpha}\right]_{\alpha^* = \alpha = 0} \tag{3.15}$$

3.2 The Photon Count Distribution

The photon count distribution $P_m(T)$ represents the probability of detecting m photons in a time interval T. A quantum mechanical derivation of the photon count distribution was first presented by Kelly and Kleider [23]. The physical significance of photon count distribution is most transparent for photon counting experiments on a light beam whose photons are restricted to a single mode. For this case the photon count distribution can be written as [17]

$$P_m(T) = Tr\left[\rho\, :\frac{(\gamma a^+ a)^m}{m!} \exp(-\gamma a^+ a):\right], \tag{3.16}$$

where n orders the creation operator a^+ to the left of the annihilation operator a, and γ is the quantum efficiency of the detector. We easily see that

$$P_m(T) = \sum_l \frac{\gamma^m}{m!}\frac{(-\gamma)^l}{l!} Tr[\rho (a^+)^{m+l} a^{m+l}] . \tag{3.17}$$

Now using (2.8) twice, we have

$$P_m(T) = \sum_l \frac{\gamma^m}{m!}\frac{(-\gamma)^l}{l!} \int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi} \langle\alpha|\rho|\beta\rangle\langle\beta|(a^+)^{m+l}a^{m+l}|\alpha\rangle$$

$$= \sum_l \frac{\gamma^m}{m!}\frac{(-\gamma)^l}{l!} \int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi} Q(\alpha^*,\beta)\, \beta^{*m+l}\alpha^{m+l} |\langle\alpha|\beta\rangle|^2$$

or

$$P_m(T) = \frac{\gamma^m}{m!}\int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi} e^{-\gamma\alpha\beta^*} Q(\alpha^*,\beta)\, \beta^{*m}\alpha^m |\langle\alpha|\beta\rangle|^2 , \tag{3.18}$$

which represents the expression for the photon count distribution in terms of the Q-function.

3.3 Squeezed States

In order to describe the squeezing properties of a light beam of frequency ω, we define the pertinent annihilation operator a in terms of two Hermitian operators a_1 and a_2 by [24]

$$a = (a_1 + ia_2)\, e^{-i\omega t} . \tag{3.19}$$

One can easily see that quadrature components are expressible as

$$a_1 = \frac{1}{2}(ae^{i\omega t} + a^+e^{-i\omega t}) \tag{3.20a}$$

and

$$a_2 = \frac{1}{2i}(ae^{i\omega t} - a^+e^{-i\omega t}) \quad . \tag{3.20b}$$

We shall next obtain the commutation relation for the quadrature components. To this end, we note that

$$[a_1, a_2] = a_1a_2 - a_2a_1 \quad . \tag{3.21}$$

Substitution of (3.20) into (3.21) then gives

$$[a_1, a_2] = \frac{i}{2}[a, a^+]$$

or in view of (2.2), we find that

$$[a_1, a_2] = \frac{i}{2} \quad . \tag{3.22}$$

This represents the commutation relation for the quadrature components.

The variance of each quadrature component is give by

$$\Delta a_1^2 = \langle a_1^2 \rangle - \langle a_1 \rangle^2 \tag{3.23a}$$

and

$$\Delta a_2^2 = \langle a_2^2 \rangle - \langle a_2 \rangle^2 \quad . \tag{3.23b}$$

From (3.20a) it follows that

$$\langle a_1^2 \rangle = \frac{1}{4}\langle aa^+ \rangle + \frac{1}{4}\langle a^+a \rangle + \frac{1}{4}e^{2i\omega t}\langle a^2 \rangle + \frac{1}{4}e^{-2i\omega t}\langle a^{+2} \rangle \tag{3.24a}$$

and

$$\langle a_1 \rangle^2 = \frac{1}{4}e^{2i\omega t}\langle a \rangle^2 + \frac{1}{4}e^{-2i\omega t}\langle a^+ \rangle^2 + \frac{1}{2}\langle a \rangle\langle a^+ \rangle \quad . \tag{3.24b}$$

Now on combining (3.24a) and (3.24b), there emerges

$$\Delta a_1^2 = \frac{1}{4}\langle aa^+\rangle + \frac{1}{4}\langle a^+a\rangle + \frac{1}{4}e^{2i\omega t}\langle a^2\rangle + \frac{1}{4}e^{-2i\omega t}\langle a^{+2}\rangle$$
$$-\frac{1}{4}e^{2\omega t}\langle a\rangle^2 - \frac{1}{4}e^{-2i\omega t}\langle a^+\rangle^2 - \frac{1}{2}\langle a\rangle\langle a^+\rangle \ . \qquad (3.25)$$

Since

$$aa^+ = a^+a + 1$$

one finds that

$$\Delta a_1^2 = \frac{1}{4} + \frac{1}{2}\langle a^+a\rangle + \frac{1}{4}e^{2i\omega t}\langle a^2\rangle + \frac{1}{4}e^{-2i\omega t}\langle a^{+2}\rangle$$
$$-\frac{1}{4}e^{2\omega t}\langle a\rangle^2 - \frac{1}{4}e^{-2i\omega t}\langle a^+\rangle^2 - \frac{1}{2}\langle a\rangle\langle a^+\rangle \ . \qquad (3.26a)$$

Similarly

$$\Delta a_2^2 = \frac{1}{4} + \frac{1}{2}\langle a^+a\rangle - \frac{1}{4}e^{2i\omega t}\langle a^2\rangle - \frac{1}{4}e^{-2i\omega t}\langle a^{+2}\rangle$$
$$+\frac{1}{4}e^{2\omega t}\langle a\rangle^2 + \frac{1}{4}e^{-2i\omega t}\langle a^+\rangle^2 - \frac{1}{2}\langle a\rangle\langle a^+\rangle \ . \qquad (3.26b)$$

According to the Heisenberg uncertainty relation

$$\Delta a_1 \Delta a_2 \geq \frac{1}{2}\left|\langle [a_1, a_2]\rangle\right| \ . \qquad (3.27)$$

Thus combining this with (3.22) one gets

$$\Delta a_1 \Delta a_2 \geq \frac{1}{4} \ . \qquad (3.28)$$

For light in a coherent state the minimum uncertainty relation, with equal quadrature fluctuations, is satisfied. That is

$$\Delta a_1 = \frac{1}{2} \quad \mathit{and} \quad \Delta a_2 = \frac{1}{2} \ , \qquad (3.29)$$

so that [25]

$$\Delta a_1 \Delta a_2 = \frac{1}{4} \ . \qquad (3.30)$$

A light beam is said to be in a squeezed state if $\Delta a_1 < \frac{1}{2}$ and $\Delta a_2 > \frac{1}{2}$ or vise versa [26,27] such that the uncertainty relation $\Delta a_1 \, \Delta a_2 \geq \frac{1}{4}$ is not violated [28]. A squeezed state for which the minimum uncertainty relation holds is sometimes referred to as a squeezed coherent state.

4. THE DEGENERATE PARAMETRIC AMPLIFIER

There has been considerable interest in the statistical [28,29] and squeezing [30] properties of the light produced by a degenerate parametric amplifier. Takashi [31] was the first to indicate the light generated by this device to be in a squeezed state. In a degenerate parametric amplifier a strong pump photon of frequency 2ω interacts with a nonlinear crystal and is down converted into two signal photons at the subharmonic frequency ω [32].

The process of parametric amplification has been discussed by several authors [33-37] using different methods and approximation schemes. Some authors [38,39] have analysed this problem employing the so called parametric approximation.

This chapter is structured as follows. In section one we derive the coherent state propagator for the system adopting a certain approximation scheme [40]. Then employing the resulting propagator we study the statistical and squeezing properties of the signal mode in section two and in section three, respectively.

4.1 The Propagator of the System

The Hamiltonian for a degenerate parametric amplifier, in the absence of cavity damping, is given by

$$H = \omega a^+a + 2\omega b^+b + i\kappa(a^2b^+ - a^{+2}b) \tag{4.1}$$

where $a(a^+)$ and $b(b^+)$ are the annihilation (creation) operators for the signal and pump modes, respectively, and κ is a coupling constant between the two modes. The propagator associated with the Hamiltonian (4.1) cannot be determined exactly. We shall reduce this Hamiltonian to a quadratic form by replacing the pump mode with an approximate c-number function. To this end, we note that the Heisenberg equation of motion for the operators a and b are given by

$$i\frac{da}{dt} = [a,H] \tag{4.2a}$$

and

$$i\frac{db}{dt} = [b,H] \ . \tag{4.2b}$$

It then follows that

$$\dot{a} = -i\omega a - 2\kappa b a^{+} \tag{4.3a}$$

and

$$\dot{b} = -2i\omega b + \kappa a^{2} \ . \tag{4.3b}$$

As a first approximation, we shall replace the operator b in (4.3a) by the free classical solutions

$$\beta_{c_0} = \beta' e^{-2i\omega t} \tag{4.4}$$

with $\beta' = \beta(0)$. Consequently

$$\dot{a} = -i\omega a - 2\kappa\beta' e^{-2i\omega t} a^{+} \ . \tag{4.5}$$

Differentiating both sides of (4.5) with respect to time, one gets

$$\ddot{a} = -i\omega\dot{a} - 2\kappa\beta' e^{-2i\omega t}\dot{a}^{+} + 4i\omega\kappa\beta' e^{-2i\omega t} a^{+} \ . \tag{4.6}$$

Employing the adjoint of (4.5) in (4.6) and assuming β' to be real, we see that

$$\ddot{a} = -i\omega\dot{a} + 4\kappa^{2}\beta^{2} a + 2i\omega\kappa\beta' e^{-2i\omega t} a^{+} \ . \tag{4.7}$$

One can readily obtain from (4.5) the result

$$2i\omega\kappa\beta' e^{-2i\omega t} a^{+} = -i\omega\dot{a} + \omega^2 a \tag{4.8}$$

and substitution of this into (4.7) leads to

$$\ddot{a} + 2i\omega\dot{a} - (\omega^2+\lambda^2)\, a = 0 \tag{4.9}$$

where $\lambda = 2\kappa\beta'$. The solution of (4.9) can be written as

$$a(t) = (Ae^{\lambda t} + Be^{-\lambda t})\ e^{-i\omega t} \tag{4.10}$$

Where A and B are constants to be determined. Using the initial condition along with (4.5), we note that

$$a' = A + B\ . \tag{4.11a}$$

Moreover, from (4.5) and (4.10) we have

$$\dot{a}(0) = -i\omega a' - \lambda a'^{+} \tag{4.11b}$$

and

$$\dot{a}(0) = A(-i\omega + \lambda) + B(i\omega+\lambda)\ . \tag{4.11c}$$

It then follows from (4.11) that

$$A = \frac{a' - a'^{+}}{2} \tag{4.12a}$$

and

$$B = \frac{a' + a'^{+}}{2}\ , \tag{4.12b}$$

so that

$$a(t) = \left(\frac{a'-a'^{+}}{2} e^{\lambda t} + \frac{a'+a'^{+}}{2} e^{-\lambda t}\right) e^{-i\omega t}$$

or

$$a(t) = [a'\cosh\lambda t - a'^{+}\sinh\lambda t] e^{-i\omega t}\ . \tag{4.13}$$

We now easily see that for the normal ordering

$$a^2 = [a'^2\cosh^2\lambda t - a'^{+}\sinh^2\lambda t - (2a'^{+}a'+1)\sinh\lambda t\,\cosh\lambda t]e^{-2i\omega t}. \qquad (4.14)$$

Finally, substitution of this into (4.36) results in

$$\dot{b} = -2i\omega b + \kappa[a'^2\cosh^2\lambda t + (a'^{+})^2\sinh^2\lambda t$$
$$-(2a'^{+}a'+1)\sinh\lambda t\,\cosh\lambda t]\,e^{-2i\omega t}. \qquad (4.15)$$

We shall be interested in the case for which the signal mode is initially in the vacuum state. Hence the c-number equivalent of (4.15) for the normal ordering and for $\alpha' = 0$ is

$$\dot{\beta} = -2i\omega\beta - \kappa\sinh\lambda t\,\cosh\lambda t\,e^{-2i\omega t} \qquad (4.16)$$

In order to obtain the solution of this equation, we let

$$\beta(t) = B(t)\,e^{-2i\omega t}. \qquad (4.17a)$$

Then we note that

$$\dot{\beta} = -2i\omega B e^{-2i\omega t} + \dot{B}e^{-2i\omega t}. \qquad (4.17b)$$

On account of this result, (4.16) takes the form

$$\dot{B} = -\kappa\sinh\lambda t\,\cosh\lambda t \qquad (4.18)$$

so that

$$B(t) = -\kappa\int\sinh\lambda t\,\cosh\lambda t + C \qquad (4.19a)$$

or

$$\beta(t) = \left[C - \frac{\kappa}{2\lambda}\sinh^2\lambda t\right]e^{-2i\omega t} \qquad (4.19b)$$

where C is a constant. Application of the initial condition

$$\beta(0) = \beta' \quad \text{along with} \quad \lambda = 2\kappa\beta'$$

leads to

$$\beta(t) = \beta'\left[1 - \frac{1}{4\beta'^2}\sinh^2(2\kappa\beta' t)\right]e^{-2i\omega t}. \qquad (4.20)$$

This represents approximately the time evaluation of the amplitude of the pump mode. This approximation scheme is valid for all interaction times $t < T$ where

$$T = \frac{1}{2\kappa\beta'} \sinh^{-1}(2\beta') \tag{4.21}$$

and $\beta(T) = 0$.

We now replace the operator b in the interaction part of (4.1) by this approximate c-number function. We then base our analysis of the degenerate parametric amplifier on the Hamiltonian

$$H = \omega a^{+}a + i\kappa f(t)[e^{2i\omega t}a^{2} - e^{-2i\omega t}a^{+2}] \tag{4.22}$$

where

$$f(t) = \beta'\left[1 - \frac{1}{4\beta'^{2}} \sinh^{2}(2\kappa\beta' t)\right] . \tag{4.23}$$

The approximation scheme adopted here is quite justifiable for a sufficiently large number of pump mode photons at the initial time.

We now proceed to obtain the propagator associated with the Hamiltonian (4.22) using the new method. To this end, we note that the corresponding Lagrangian is given by

$$L = \frac{1}{2}\dot{\alpha}^{*}\alpha - \frac{1}{2}\alpha^{*}\dot{\alpha} - i\omega\alpha^{*}\alpha + \kappa f(t)[e^{2i\omega t}\alpha^{2} - e^{-2i\omega t}\alpha^{*2}]. \tag{4.24}$$

The Euler-Lagrange equations that follow from (4.24) are

$$\dot{\alpha} = -i\omega\alpha - 2\kappa f(t)\, e^{-2i\omega t}\alpha^{*} \tag{4.25a}$$

and

$$\dot{\alpha}^{*} = i\omega\alpha^{*} - 2\kappa f(t)\, e^{2i\omega t}\alpha \ . \tag{4.25b}$$

If we let

$$\alpha(t) = B(t)\, e^{-i\omega t} \tag{4.26}$$

then we have

$$\dot{\alpha} = -i\omega\alpha + \dot{B}e^{-i\omega t} \,. \tag{4.27}$$

Combination of (4.27) and the complex conjugate of (4.26) with (4.25a) yields

$$\dot{B} = -2\kappa f(t) B^{*} \,. \tag{4.28a}$$

Similarly

$$\dot{B}^{*} = -2\kappa f(t) B \,. \tag{4.28b}$$

Now from (4.28) one readily obtains

$$\ddot{B} - \frac{\dot{f}(t)}{f(t)}\dot{B} - 4\kappa^2 f^2(t) B \tag{4.29a}$$

and

$$\ddot{B}^{*} - \frac{\dot{f}(t)}{f(t)}\dot{B}^{*} - 4\kappa^2 f^2(t) B^{*} \,. \tag{4.29b}$$

In order to obtain the solution of (4.29), we make a change of the independent variable B(t) → B(Z). Then

$$\frac{dB}{dt} = \frac{dB}{dZ}\frac{dZ}{dt} \tag{4.30a}$$

and

$$\frac{d^2B}{dt^2} = \frac{d^2B}{dZ^2}\left(\frac{dz}{dt}\right)^2 + \frac{dB}{dZ}\frac{d^2Z}{dt^2} \,, \tag{4.30b}$$

where Z is a real function of time. With the aid of (4.30), expression (4.29a) can be rewritten as

$$\frac{d^2B}{dZ^2} + \frac{\left[\frac{d^2Z}{dt^2} - \frac{\dot{f}(t)}{f(t)}\frac{dz}{dt}\right]}{\left(\frac{dZ}{dt}\right)^2}\frac{dB}{dZ} - \frac{4\kappa^2 f^2(t)}{\left(\frac{dZ}{dt}\right)^2} B = 0 \,. \tag{4.31}$$

Next setting

$$\frac{dZ}{dt} = 2\kappa f(t) \ , \tag{4.32}$$

we see that

$$\frac{d^2 Z}{dt^2} - \frac{f(t)}{f(t)} \frac{dZ}{dt} = 0 \ . \tag{4.33}$$

Hence (4.31) reduces to the form

$$\frac{d^2 B}{dZ^2} - B = 0 \ . \tag{4.34a}$$

Similarly, one can show that

$$\frac{d^2 B^*}{dZ^2} - B^* = 0 \ . \tag{4.34b}$$

These are homogeneous differential equations with constant coefficients. Their solutions can be written as

$$B[Z(t)] = ae^{-Z(t)} + be^{Z(t)} \tag{4.35a}$$

and

$$B^*[Z(t)] = ce^{-Z(t)} + de^{Z(t)} \tag{4.35b}$$

where a, b, c and d are constants to be determined. In accordance with (4.32)

$$Z(t) = 2\kappa \int_0^t f(t')\,dt'$$

$$= 2\kappa\beta' \int_0^t \left[1 - \frac{1}{4\beta'^2} \sinh^2(2\kappa\beta' t')\right] dt'$$

or

$$Z(t) = 2\kappa\beta' t + \frac{\kappa t}{4\beta'} - \frac{1}{8\beta'^2} \sinh(2\kappa\beta' t) \cosh(2\kappa\beta' t) \ . \tag{4.36}$$

In view of (4.26), expression (4.35) can be put in the form

$$\alpha(t) = e^{-i\omega t}[ae^{-Z(t)} + be^{Z(t)}] \qquad (4.37a)$$

and

$$\alpha^*(t) = e^{i\omega t}[ce^{-Z(t)} + de^{Z(t)}] \; . \qquad (4.37b)$$

We now determine the integration constant using the boundary conditions $\alpha(0) = \alpha'$ and $\alpha^*(T) = \alpha''^*$ along with (4.25). Thus it follows from (4.37) that

$$\alpha' = a + b \qquad (4.38a)$$

and

$$\alpha''^* e^{-i\omega T} = ce^{-Z(T)} + de^{Z(T)} \; . \qquad (4.38b)$$

In addition, employing (4.25) and (4.38) we have

$$\dot{\alpha}(0) = -i\omega(a+b) - 2\kappa f(0)(c+d) \qquad (4.39a)$$

and

$$\dot{\alpha}^*(0) = i\omega(c+d) - 2\kappa f(0)(a+b) \; . \qquad (4.39b)$$

It also follows from (4.37) that

$$\dot{\alpha}(0) = -i\omega(a+b) + 2\kappa f(0)(b-a) \qquad (4.40a)$$

and

$$\dot{\alpha}^*(0) = i\omega(c+d) + 2\kappa f(0)(d-c) \; . \qquad (4.40b)$$

Comparison of (4.39) and (4.40) shows that

$$a-b = c+d \qquad (4.41a)$$

and

$$a+b = c-d \; , \qquad (4.41b)$$

or

$$c = a \quad \text{and} \quad d = -b. \qquad (4.42)$$

Applying these relations, Eq. (4.38) can be rewritten in the form

$$\alpha' = a + b$$

and

$$\alpha''^{*} e^{-i\omega T} = a e^{-Z(T)} + d e^{Z(T)}$$

from which follows

$$a = \frac{\alpha' e^{Z(T)} + \alpha''^{*} e^{-i\omega T}}{2\cosh[Z(T)]} \tag{4.43a}$$

and

$$b = \frac{\alpha' e^{-Z(T)} - \alpha''^{*} e^{-i\omega T}}{2\cosh[Z(T)]} \; . \tag{4.43b}$$

We finally find the solution of the Euler-Lagrange equations (4.25a) and (4.25b) to be

$$\alpha_c(t) = e^{-i\omega t}[a e^{-Z(t)} + b e^{Z(t)}] \tag{4.44a}$$

and

$$\alpha_c^{*}(t) = e^{i\omega t}[a e^{-Z(t)} - b e^{Z(t)}] \tag{4.44b}$$

Where a and b are given by (4.43). On the other hand, We note that the free classical solutions are

$$\alpha_{c_0}(t) = \alpha' e^{-i\omega t} \tag{4.45a}$$

and

$$\alpha^{*}_{c_0}(t) = \alpha''^{*} e^{i\omega(t-T)} \; . \tag{4.45b}$$

Now we are in position to determine the coherent state propagator action. According to (2.49) the coherent state action can be written as

$$A = -\frac{1}{2}|\alpha''|^2 - \frac{1}{2}|\alpha'|^2 + \alpha'\alpha''^{*}e^{-i\omega T}$$

$$+ \kappa\int_0^T f(t)\left[e^{2i\omega t}\alpha_{c_0}\alpha_c - e^{-2i\omega t}\alpha_{c_0}^{*}\alpha_c^{*}\right] \qquad (4.46)$$

Hence applying (4.44) and (4.45) we have

$$\kappa\int_0^T f(t)\left[e^{2i\omega t}\alpha_{c_0}\alpha_c - e^{-2i\omega t}\alpha^{*}_{c_0}\alpha^{*}_c\right]$$

$$= \kappa\int_0^T f(t)\ [e^{2i\omega t}(\alpha' e^{-i\omega t})\,e^{-i\omega t}(ae^{-Z(t)} + be^{Z(t)})$$

$$- e^{-2i\omega t}(\alpha''^{*}e^{i\omega(t-T)})\,e^{i\omega t}(ae^{-Z(t)} - be^{Z(t)})\,]\,dt$$

$$= \kappa\int_0^T f(t)\left[a\alpha' e^{-Z(t)} + b\alpha' e^{Z(t)} - a\alpha''^{*}e^{-i\omega T - Z(t)} + b\alpha''^{*}e^{-i\omega T + Z(t)}\right]dt\,. \qquad (4.47)$$

Since $dZ = 2\kappa f(t)dt$ and $Z(0) = 0$, we see that

$$\kappa\int_0^T f(t)\left[e^{2i\omega t}\alpha_{c_0}\alpha_c - e^{-2i\omega t}\alpha_{c_0}^{*}\alpha_c^{*}\right] dt$$

$$= \frac{1}{2}\int_0^{Z(T)}\left[a\alpha' e^{-Z(t)} + b\alpha' e^{Z(t)} - a\alpha''^{*}e^{-i\omega T - Z(t)} + b\alpha''^{*}e^{-i\omega T + Z(t)}\right] dZ$$

$$= \frac{1}{2}\,[-a\alpha'(e^{-Z(T)} - 1) + b\alpha'(e^{Z(T)} - 1) + a\alpha''^{*}e^{-i\omega T}(e^{-Z(T)} - 1)$$

$$+ b\alpha''^{*}e^{-i\omega T}(e^{Z(T)} - 1)\,]\,. \qquad (4.48)$$

Substitution of (4.43) into (4.48) leads to

$$\kappa\int_0^T f(t)\left[e^{2i\omega t}\alpha_{c_0}\alpha_c - e^{-2i\omega t}\alpha_{c_0}^{*}\alpha_c^{*}\right] dt$$

$$= -\alpha'\alpha''^{*}e^{-i\omega T} + \alpha'\alpha''^{*}e^{-i\omega T}\mathrm{sech}[Z(T)] + \frac{1}{2}\alpha'^2\tanh[Z(T)]$$

$$-\frac{1}{2}\alpha''^{*2}e^{-2i\omega T}\tanh[Z(T)]\,. \qquad (4.49)$$

Therefore, the coherent state propagator action for the degenerate parametric amplifier under the present approximation takes the form

$$A = -\frac{1}{2}|\alpha''|^2 - \frac{1}{2}|\alpha'|^2 + \frac{1}{2}\alpha'^2\tanh[Z(T)]$$
$$+\alpha'\alpha''^* e^{-i\omega T}sech[Z(T)] - \frac{1}{2}\alpha''^{*2}e^{-2i\omega T}\tanh[Z(T)]\,. \qquad (4.50)$$

Finally, making use of (2.48) along with (4.50), the expression for the coherent state propagator of the system is found to be

$$K(\alpha'',T|\alpha',0) = sech^{\frac{1}{2}}[Z(T)]\exp(-\frac{1}{2}|\alpha''|^2 - \frac{1}{2}|\alpha'|^2$$
$$+\frac{1}{2}\alpha'^2\tanh[Z(T)] + \alpha'\alpha''^* e^{-i\omega T}sech[Z(T)]$$
$$-\frac{1}{2}\alpha''^{*2}e^{-2i\omega T}\tanh[Z(T)])\,. \qquad (4.51)$$

Since we are interested in the case for which the signal mode is initially in the vacuum state, setting $\alpha' = 0$ we have

$$K(\alpha'',T|0,0) = sech^{\frac{1}{2}}[Z(T)]$$
$$\times \exp\left[-\frac{1}{2}|\alpha''|^2 - \frac{1}{2}\alpha''^{*2}e^{-2i\omega T}\tanh[Z(T)]\right]. \qquad (4.52)$$

Furthermore, replacing T and α''^* by t and α^*, (4.52) can be rewritten as

$$K(\alpha,t|0,0) = sech^{\frac{1}{2}}[Z(t)]$$
$$\times \exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}\alpha^{*2}e^{-2i\omega t}\tanh[Z(t)]\right]. \qquad (4.53)$$

We shall use this propagator to investigate the photon statistics and squeezing properties of the signal light.

4.2 Photon Statistics of the Signal Mode

A. The Mean and the variance of the Photon Number

We now seek to calculate the mean and the variance of the photon number for the signal mode. To this end we note that

$$Q(\alpha^*,\alpha,t) = K^*K$$
$$= sech[Z(t)]\exp[-\alpha^*\alpha + A\alpha^{*2} + B\alpha^2] \qquad (4.54)$$

where

$$A = -\frac{1}{2}e^{-2i\omega t}\tanh[Z(t)] \qquad (4.55a)$$

and

$$B = -\frac{1}{2}e^{2i\omega t}\tanh[Z(t)]\,. \qquad (4.55b)$$

We recall that the mean photon number is expressible in the form

$$\langle a^+a\rangle = \int\frac{d^2\alpha}{\pi}\,Q(\alpha^*,\alpha,t)\ \ (\alpha^*\alpha-1) \qquad (4.56)$$

where $\alpha^*\alpha - 1$ is the C- number equivalent of a^+a for the antinormal ordering. Since the Q-function is normalized, we can write (4.56) as

$$\langle a^+a\rangle = \int\frac{d^2\alpha}{\pi}\,Q(\alpha^*,\alpha,t)\,\alpha^*\alpha - 1\,. \qquad (4.57)$$

Using the relation

$$\int\frac{d^2\beta}{\pi}\,e^{-\beta^*\beta + A\beta^{*2} + B\beta^2 + a\beta^* + b\beta} = \frac{1}{(1-4AB)^{\frac{1}{2}}}\,e^{\frac{ab+a^2B+b^2A}{1-4AB}} \qquad (4.58)$$

and (4.54), we see that

$$\int \frac{d^2\alpha}{\pi} Q(\alpha^*, \alpha, t)\, \alpha^*\alpha$$

$$= \lim_{\substack{a\to 0\\ b\to 0}} Sech[Z(t)]\ \frac{\partial^2}{\partial a \partial b} \int \frac{d^2\alpha}{\pi} e^{-\alpha^*\alpha + A\alpha^{*2} + B\alpha^2 + a\alpha^* + b\alpha}$$

$$= \lim_{\substack{a\to 0\\ b\to 0}} sech[Z(t)]\ \frac{\partial^2}{\partial a \partial b} \left[\frac{1}{(1-4AB)^{\frac{1}{2}}}\ e^{\frac{ab + a^2B + b^2A}{1-4AB}} \right]$$

$$= \lim_{\substack{a\to 0\\ b\to 0}} Sech[Z(t)]\ \frac{\partial}{\partial a} \left[\frac{a + 2bA}{(1-4AB)^{\frac{3}{2}}}\ e^{\frac{ab + a^2B + b^2A}{1-4AB}} \right]$$

$$= \lim_{\substack{a\to 0\\ b\to 0}} sech[Z(t)]$$

$$\times \left[\frac{1}{(1-4AB)^{\frac{3}{2}}}\ e^{\frac{ab + a^2B + b^2A}{1-4AB}} + \frac{(a+2bA)\,(b+2aB)}{(1-4AB)^{\frac{5}{2}}}\ e^{\frac{ab + a^2B + b^2A}{1-4AB}} \right]$$

$$= \frac{sech[Z(t)]}{(1-4AB)^{\frac{3}{2}}}\ . \qquad (4.59)$$

Now employing (4.55), we find that

$$AB = \frac{1}{4} \tanh^2[Z(t)] \qquad (4.60)$$

and

$$1 - 4AB = sech^2[Z(t)]\ . \qquad (4.61)$$

Substitution of (4.61) into (4.59) gives

$$\int \frac{d^2\alpha}{\pi} Q(\alpha^*, \alpha, t)\, \alpha^*\alpha = \cosh^2[Z(t)]\ . \qquad (4.62)$$

We then have

$$\langle a^+ a \rangle = \cosh^2[Z(t)] - 1$$

or

$$\langle a^{+}a\rangle = \sinh^2[Z(t)] \,. \qquad (4.63)$$

This represents the mean photon number for the signal light from the degenerate parametric amplifier.

Table 1. Calculated Values of $\bar{n}$ for the degenerate parametric amplifier as a function of $2\kappa\beta' t$ for different values of β'.

$2\kappa\beta' t$	$\bar{n}$ $\beta' = 10$	$\beta' = 100$	$\beta' = 1000$
0.0	0.000000	0.000000	0.00000
0.5	0.271412	0.271539	0.271540
1.0	1.377414	1.381061	1.381097
1.5	4.490084	4.533392	4.533827
2.0	12.76261	13.15014	13.15408
2.5	33.53045	36.57289	36.60465
3.0	78.47049	100.1114	100.3554
3.5	138.8604	271.8097	273.6400
4.0		731.0567	744.6015
4.5		1925.449	2024.248
5.0		4798.511	5498.548
5.5		10296.73	14912.15
6.0		14714.73	40276.46
6.5			107586.6
7.0			278881.7
7.5			666232.0
8.0			1274832
8.5			1334441

We next calculate the variance of the photon number. To this end, we recall that

$$\Delta n^2 = \langle n^2\rangle - \langle n\rangle^2 \qquad (4.64)$$

where

$$n = a^{+}a \qquad (4.65)$$

and for the antinormal ordering

$$n^2 = a^2 a^{+2} - 3aa^{+} + 1 \,. \qquad (4.66)$$

Consequently, the c- number equivalent of n^2 for such ordering is $\alpha^{*2}\alpha^2 - 3\alpha^*\alpha + 1$. Using this one can write

$$\langle n^2\rangle = \int\frac{d^2\alpha}{\pi}\, Q(\alpha^*,\alpha,t)\, (\alpha^{*2}\alpha^2 - 3\alpha^*\alpha + 1)\,. \tag{4.67}$$

We note that

$$\int\frac{d^2\alpha}{\pi}\, Q(\alpha^*,\alpha,t)\, \alpha^{*2}\alpha^2 = Sech[Z(t)]\, \frac{\partial^2}{\partial A\partial B}\int\frac{d^2\alpha}{\pi}\, e^{-\alpha^*\alpha + A\alpha^{*2} + B\alpha^2}\,.$$

On carrying out the integration applying (4.58) we have

$$\begin{aligned}\int\frac{d^2\alpha}{\pi}\, Q(\alpha^*,\alpha,t)\,\alpha^{*2}\alpha^2 &\\ &= Sech[Z(t)]\,\frac{\partial^2}{\partial A\partial B}\left[\frac{1}{(1-4AB)^{\frac{1}{2}}}\right]\\ &= sech[Z(t)]\,\frac{\partial}{\partial A}\left[\frac{2A}{(1-4AB)^{\frac{3}{2}}}\right]\\ &= Sech[Z(t)]\left[\frac{2}{(1-4AB)^{\frac{3}{2}}} + \frac{12AB}{(1-4AB)^{\frac{5}{2}}}\right].\end{aligned} \tag{4.68}$$

Making use of (4.60) and (4.61) one readily obtains

$$\begin{aligned}\int\frac{d^2\alpha}{\pi}\, Q(\alpha^*,\alpha,t)\,\alpha^{*2}\alpha^2 &\\ &= 2\cosh^2[Z(t)] + 3\sinh^2[Z(t)]\cosh^2[Z(t)]\end{aligned} \tag{4.69}$$

In view of (4.62) and (4.69), expression (4.67) can be rewritten as

$$\begin{aligned}\langle n^2\rangle &= 2\cosh^2[Z(t)] + 3\sinh^2[Z(t)]\cosh^2[Z(t)]\\ &\quad - 3\cosh^2[Z(t)] + 1,\end{aligned} \tag{4.70}$$

so that combining this with (4.63) and employing the relation

$$\cosh^2[Z(t)] = \sinh^2[Z(t)] + 1 \tag{4.71}$$

we get

$$\Delta n^2 = 2\bar{n}(1+\bar{n})\,. \tag{4.72}$$

In which $\bar{n} = \sinh^2[Z(t)]$. This represents the variance of the photon number for the signal light. Since the variance of the photon number is greater than the mean photon number, the statistics of the signal photon is super - Poissonian [41].

Table 2. Calculated Values of Δn^2 for the degenerate parametric amplifier as a function of $2\kappa\beta' t$ for different values of β'.

$2\kappa\beta' t$	Δn^2 $\beta' = 10$	$\beta' = 100$	$\beta' = 1000$
0.0	0.000000	0.000000	0.000000
0.5	0.690152	0.690545	0.690549
1.0	6.549367	6.576781	6.577055
1.5	49.30188	50.17006	50.17882
2.0	351.2936	372.1529	372.3676
2.5	2315.643	2748.299	2753.011
3.0	12472.18	20244.80	20343.10
3.5	38842.17	148304.6	150304.9
4.0		1070350	1110352
4.5		7418555	8199205
5.0		46061022	60479054
5.5		2.12e+08	4.45e+08
6.0		4.33e+08	3.24e+09
6.5			2.31e+10
7.0			1.56e+11
7.5			8.88e+11
8.0			3.25e+12
8.5			3.56e+12

B. The Photon Number Distribution

According to (3.15) the photon number distribution of the signal light is expressible as

$$P_n = \frac{1}{n!}\,\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}\left[Q(\alpha^*,\alpha,t)\,e^{\alpha^*\alpha}\right]_{\alpha^*=\alpha=0} .$$

Thus using (4.54), we have

$$P_n = \frac{sech[Z(t)]}{n!}\,\frac{\partial^n}{\partial\alpha^{*n}}\left[e^{A\alpha^{*2}}\right]\frac{\partial^n}{\partial\alpha^n}\left[e^{B\alpha^2}\right]_{\alpha^*=\alpha=0} . \qquad (4.73)$$

One can easily check that

$$\frac{\partial}{\partial\alpha^*}\left[e^{A\alpha^{*2}}\right] = 2A\alpha^* e^{A\alpha^{*2}}$$

$$\frac{\partial^2}{\partial\alpha^{*2}}\left[e^{A\alpha^{*2}}\right] = 2Ae^{A\alpha^{*2}} + 4A^2\alpha^{*2}e^{A\alpha^{*2}}$$

$$\frac{\partial^3}{\partial\alpha^{*3}}\left[e^{A\alpha^{*2}}\right] = 12A^2\alpha^* e^{A\alpha^{*2}} + 8A^3\alpha^{*3}e^{A\alpha^{*2}}$$

$$\frac{\partial^4}{\partial\alpha^{*4}}\left[e^{A\alpha^{*2}}\right] = 12A^2 e^{A\alpha^{*2}} + 48A^3\alpha^{*2}e^{A\alpha^{*2}} + 16A^4\alpha^{*4}e^{A\alpha^{*2}}$$

$$\frac{\partial^5}{\partial\alpha^{*5}}\left[e^{A\alpha^{*2}}\right] = 120A^3\alpha^* e^{A\alpha^{*2}} + 160A^4\alpha^{*3}e^{A\alpha^{*2}} + 32A^5\alpha^{*5}e^{A\alpha^{*2}}$$

$$\frac{\partial^6}{\partial\alpha^{*6}}\left[e^{A\alpha^{*2}}\right] = 120A^3 e^{A\alpha^{*2}} + 720A^4\alpha^{*2}e^{A\alpha^{*2}} + 480A^5\alpha^{*5}e^{A\alpha^{*2}} + 64A^6\alpha^{*6}e^{A\alpha^{*2}} .$$

Consequently,

$$\left[e^{A\alpha^{*2}}\right]_{\alpha^*=0} = 1$$

$$\frac{\partial}{\partial\alpha^*}\left[e^{A\alpha^{*2}}\right]_{\alpha^*=0} = 0$$

$$\frac{\partial^2}{\partial\alpha^{*2}}\left[e^{A\alpha^{*2}}\right]_{\alpha^*=0} = 2A$$

$$\frac{\partial^3}{\partial\alpha^{*3}}\left[e^{A\alpha^{*2}}\right]_{\alpha^*=0} = 0$$

$$\frac{\partial^4}{\partial\alpha^{*4}}\left[e^{A\alpha^{*2}}\right]_{\alpha^*=0} = 12A^2$$

$$\frac{\partial^5}{\partial\alpha^{*5}}\left[e^{A\alpha^{*2}}\right]_{\alpha^*=0} = 0$$

$$\frac{\partial^6}{\partial\alpha^{*6}}\left[e^{A\alpha^{*2}}\right]_{\alpha^*=0} = 120A^3$$

Similarly, one can show that

$$\left[e^{B\alpha^2}\right]_{\alpha=0} = 1$$

$$\frac{\partial}{\partial\alpha}\left[e^{B\alpha^2}\right]_{\alpha=0} = 0$$

$$\frac{\partial^2}{\partial\alpha^2}\left[e^{B\alpha^2}\right]_{\alpha=0} = 2B$$

$$\frac{\partial^3}{\partial\alpha^3}\left[e^{B\alpha^2}\right]_{\alpha=0} = 0$$

$$\frac{\partial^4}{\partial\alpha^4}\left[e^{B\alpha^2}\right]_{\alpha=0} = 12B$$

$$\frac{\partial^5}{\partial\alpha^5}\left[e^{B\alpha^2}\right]_{\alpha=0} = 0$$

$$\frac{\partial^6}{\partial\alpha^6}\left[e^{B\alpha^2}\right]_{\alpha=0} = 120B\ .$$

Now employing the above results in (4.73) we find that

$$P_0 = \frac{sech[Z(t)]}{0\,!} \tag{4.74a}$$

$$P_2 = \frac{sech[Z(t)]}{2\,!}\ 4AB \tag{4.74b}$$

$$P_4 = \frac{sech[Z(t)]}{4\,!}\ 144A^2B^2 \tag{4.74c}$$

$$P_6 = \frac{sech[Z(t)]}{6\,!}\ 14400A^3B^3 \tag{4.74d}$$

and

$$P_1 = P_3 = P_5 = 0. \tag{4.75}$$

Finally, using (4.60) in expression (4.74) we obtain the following relations for the first four sequences of the photon number distribution :

$$P_0 = \frac{1}{\cosh[Z(t)]} , \qquad (4.76a)$$

$$P_2 = \frac{1}{2} \frac{\sinh^2[Z(t)]}{\cosh^3[Z(t)]} , \qquad (4.76b)$$

$$P_4 = \frac{1 x 3}{2 x 4} \frac{\sinh^4[Z(t)]}{\cosh^5[Z(t)]} , \qquad (4.76c)$$

and

$$P_6 = \frac{1 x 3 x 5}{2 x 4 x 6} \frac{\sinh^6[Z(t)]}{\cosh^7[Z(t)]} . \qquad (4.76d)$$

In general this photon number distribution can be put in the form

$$P_{2n} = \frac{(2n-1)!!}{(2n)!!} \frac{\sinh^{2n}[Z(t)]}{\cosh^{2n+1}[Z(t)]}$$

or

$$P_{2n} = \frac{(2n-1)!!}{(2n)!!} \frac{\bar{n}^n}{(1+\bar{n})^{n+\frac{1}{2}}} . \qquad (4.77)$$

where 1!! = 1(1-2)(1-4) . . ., with -1!! = 1. Expression (4.77) represents the photon number distribution for the signal light. This result confirms the fact that the signal photons are always generated in pairs by the degenerate parametric amplifier.

C. The Photon Count Distribution

According to (3.18) the photon count distribution $P_m(T)$ is expressible as

$$P_m(T) = \frac{\gamma^m}{m!}\int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}\, e^{-\gamma\alpha\beta^*} Q(\alpha^*,\beta)\, \beta^{*m}\alpha^m |\langle\alpha|\beta\rangle|^2 .$$

Applying (4.54) and (2.7) one can write that

$$P_m(T) = Sech[Z(t)]\, \frac{\gamma^m}{m!}\int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi} \exp[-\gamma\alpha\beta^* + A\alpha^{*2} + B\beta^2 + \beta^*\alpha - \alpha^*\alpha - \beta^*\beta]\, \beta^{*m}\alpha^m$$

or

$$P_m(T) = Sech[Z(t)]\, \frac{\gamma^m}{m!}\frac{\partial^m}{\partial C^m}\int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi} \exp[-\alpha^*\alpha + A\alpha^{*2} + C\alpha\beta^* - \beta^*\beta + B\beta^2] \tag{4.78}$$

where

$$C = 1 - \gamma. \tag{4.79}$$

Now carrying out the integration using (4.58) we have

$$\int \frac{d^2\alpha}{\pi} \exp[-\alpha^*\alpha + A\alpha^{*2} + C\beta^*\alpha] = \exp[AC^2\beta^{*2}] , \tag{4.80}$$

so that in view of this result expression (4.78) takes the form

$$P_m(T) = Sech[Z(t)]\, \frac{\gamma^m}{m!}\frac{\partial^m}{\partial C^m}\int \frac{d^2\beta}{\pi} \exp[-\beta^*\beta + B\beta^2 + AC^2\beta^{*2}] . \tag{4.81}$$

Again applying (4.58) one readily gets

$$P_m(T) = Sech[Z(t)]\, \frac{\gamma^m}{m!}\frac{\partial^m}{\partial C^m}\left[\frac{1}{(1-4ABC^2)^{1/2}}\right]. \tag{4.82}$$

In order to put the photon count distribution in some other form, we expand the quantity in brackets. That is

$$[1-4ABC^2]^{-\frac{1}{2}} = \sum_{l=0}^{\infty} \frac{(-\frac{1}{2})!}{l!\,(-\frac{1}{2}-l)!} (4AB)^l C^{2l}. \tag{4.83}$$

Differentiating both sides with respect to "C" m times, we find

$$\frac{\partial^m}{\partial C^m}[1-4ABC^2]^{-\frac{1}{2}}$$

$$= \sum_l \frac{(-\frac{1}{2})!}{l!\,(-\frac{1}{2}-l)!}(2l)(2l-1)\ldots(2l-m+1)\,(4AB)^l\,C^{2l-m}$$

$$= \sum_{l=[\frac{m}{2}]}^{\infty} \frac{(-\frac{1}{2})!}{l!\,(-\frac{1}{2}-l)!}\,\frac{(2l)!}{(2l-m)!}\,(4AB)^l\,C^{2l-m} \qquad (4.84)$$

where [m/2] is equal to m/2 for even m and (m+1)/2 for odd m. Substitution of (4.84) into (4.82) yields

$$P_m(T) = sech[Z(t)]\,\frac{\gamma^m}{m!}$$

$$\times \sum_{l=[\frac{m}{2}]}^{\infty} \frac{(-\frac{1}{2})!}{l!\,(-\frac{1}{2}-l)!}\,\frac{(2l)!}{(2l-m)!}\,(4AB)^l\,C^{2l-m} \qquad (4.85)$$

in which AB and C are given by (4.60) and (4.79), respectively. Thus (4.85) represents an alternative expression for the photon count distribution for the signal light.

4.3 Squeezing of the Signal Mode

We next investigate the squeezing properties of the signal light. To this end, we note that

$$\langle a^2\rangle = sech[Z(t)]\int\frac{d^2\alpha}{\pi}\exp[-\alpha^*\alpha + A\alpha^{*2}+B\alpha^2]\,\alpha^2\,, \qquad (4.86a)$$

so that

$$\langle a^2\rangle = sech[Z(t)]\,\frac{\partial}{\partial B}\int\frac{d^2\alpha}{\pi}\exp[-\alpha^*\alpha + A\alpha^{*2}+B\alpha^2]\,.$$

Now, performing this integration using (4.58) results in

$$\langle a^2\rangle = sech[Z(t)]\,\frac{\partial}{\partial B}\left[\frac{1}{(1-4AB)^{\frac{1}{2}}}\right]$$

or

$$\langle a^2\rangle = sech[Z(t)]\,\frac{2A}{(1-4AB)^{\frac{3}{2}}} \qquad (4.87)$$

On account of (4.55a) and (4.61), this can be rewritten as

$$\langle a^2\rangle = -e^{-2i\omega t}\sinh[Z(t)]\cosh[Z(t)]\,. \qquad (4.88a)$$

Similarly, one can show that

$$\langle a^{+2}\rangle = -e^{2i\omega t}\sinh[Z(t)]\cosh[Z(t)]\,. \qquad (4.88b)$$

Next, we consider

$$\langle a\rangle = sech[Z(t)]\int\frac{d^2\alpha}{\pi}\exp[-\alpha^*\alpha + A\alpha^{*2}+B\alpha^2]\,\alpha\ .$$

This can be expressed in the form

$$\langle a\rangle = \lim_{v\to 0} sech[Z(t)]\,\frac{\partial}{\partial v}\int\frac{d^2\alpha}{\pi}\exp[-\alpha^*\alpha + A\alpha^{*2}+B\alpha^2+v\alpha]\ ,$$

and carrying out the integration we get

$$\langle a\rangle = \lim_{v\to 0} sech[Z(t)\,\frac{\partial}{\partial v}\left[\frac{1}{(1-4AB)^{\frac{1}{2}}}\,e^{\frac{v^2B}{1-4AB}}\right]$$

or

$$\langle a\rangle = 0\ . \qquad (4.89a)$$

Similarly, one can easily verify that

$$\langle a^+\rangle = 0\ . \qquad (4.89b)$$

Finally, with (4.63), (4.88) and (4.89) substituted into (3.26a) there emerges

$$\Delta a_1^2 = \frac{1}{4} + \frac{1}{2}\sinh^2[Z(t)] - \frac{1}{2}\sinh[Z(t)]\cosh[Z(t)]$$

$$= \frac{1}{4} + \frac{1}{8}(e^{Z(t)} - e^{-Z(t)})^2 - \frac{1}{8}(e^{Z(t)} - e^{-Z(t)})(e^{Z(t)} + e^{-Z(t)})$$

or

$$\Delta a_1^2 = \frac{1}{4}e^{-2Z(t)} . \tag{4.90a}$$

And in a similar manner one can easily get

$$\Delta a_2^2 = \frac{1}{4}e^{2Z(t)} . \tag{4.90b}$$

In accordance with (4.21) we observe that for $t < T$, $Z(t) > 0$. Consequently according to (4.90) the quadrature fluctuations satisfy the condition $\Delta a_1 < \frac{1}{2}$ and $\Delta a_2 > \frac{1}{2}$ for $t < T$. This shows that the signal light from the degenerate parametric amplifier is in a squeezed state. Moreover, we realize that the product of the quadrature fluctuations satisfies the minimum uncertainty relation. Therefore the signal light is in the so called squeezed coherent state.

Table 3. Calculated Values of Δa_1 for the degenerate parametric amplifier as a function of $2\kappa\beta' t$ for different values of β'.

$2\kappa\beta' t$	Δa_1 $\beta' = 10$	$\beta' = 100$	$\beta' = 1000$
0.0	0.500000	0.500000	0.500000
0.5	0.303299	0.303266	0.303265
1.0	0.184127	0.183942	0.183940
1.5	0.112055	0.111570	0.111565
2.0	0.068660	0.067677	0.067668
2.5	0.042857	0.041060	0.041043
3.0	0.028133	0.024924	0.024894
3.5	0.021177	0.015150	0.015099
4.0		0.009243	0.009159
4.5		0.005697	0.005556
5.0		0.003609	0.003371
5.5		0.002464	0.002047
6.0		0.002061	0.001246
6.5			0.000762
7.0			0.000473
7.5			0.000306
8.0			0.000221
8.5			0.000216

Table 4. Calculated Values of Δa_1 for the degenerate parametric amplifier as a function of $2\kappa\beta' t$ under the parametric approximation and under the present approximation schemes for β' =100.

$2\kappa\beta' t$	Δa_1 parametric approximation	present approximation
0.0	0.500000	0.500000
0.5	0.303265	0.303266
1.0	0.183940	0.183942
1.5	0.111565	0.111570
2.0	0.067668	0.067677
2.5	0.041042	0.041060
3.0	0.024894	0.024924
3.5	0.015099	0.015150
4.0	0.009158	0.009243
4.5	0.005554	0.005697
5.0	0.003369	0.003609
5.5	0.002043	0.002464
6.0	0.001239	0.002061

5. THE NONDEGENERATE PARAMETRIC AMPLIFIER

There has been considerable interest in the statistical properties of the light produced by the nondegenerate parametric amplifier [35,42]. In the nondegenerate parametric amplifier a pump photon of frequency ω_c interacts with a nonlinear crystal and is down converted into a signal and an idler photon at frequencies ω_a and ω_b, respectively, where $\omega_a + \omega_b = \omega_c$ and $\omega_a \neq \omega_b$. The signal and the idler photons have the same statistical properties and quadrature fluctuations.

This chapter is organized as follows. In section one we derive the coherent state propagator for the system following a similar procedure used for the case of the degenerate parametric amplifier. In section two we study the photon statistics of the signal light. In section three we investigate the quadrature fluctuations of the signal light and also discuss the Heisenberg uncertainty relation for the quadrature fluctuations.

5.1 The Propagator of the System

The Hamiltonian that describes the nondegenerate parametric amplifier, in the absence of cavity damping, is given by [42]

$$H = \omega_a a^+ a + \omega_b b^+ b + \omega_c c^+ c + i\kappa(abc^+ - a^+ b^+ c) \qquad (5.1)$$

where $a(a^+)$, $b(b^+)$ and $c(c^+)$ are the annihilation (creation) operators for the signal, idler, and pump modes, respectively and κ is a coupling constant between the three modes. The propagator associated with the Hamiltonian (5.1) can not be determined exactly. We then reduce this Hamiltonian to a quadratic form by replacing the pump mode operator c with an approximate c-number function. To this end, we note that the Heisenberg equation of motion for an operator A which does not depend explicitly on time is given by

$$i\frac{dA}{dt} = [A, H] . \tag{5.2}$$

Thus the Heisenberg equations of motion for the operators a, b, and c are

$$\dot{a} = -i\omega_a a - \kappa b^+ c \tag{5.3a}$$

$$\dot{b} = -i\omega_b b - \kappa a^+ c \tag{5.3b}$$

and

$$\dot{c} = -i\omega_c c + \kappa ab . \tag{5.3c}$$

Now as a first approximation, we replace the operator c in (5.3a) and (5.3b) by the free classical solution

$$\gamma_{c_0}(t) = \gamma' e^{-i\omega_c t} \tag{5.4}$$

in which $\gamma = \gamma(0)$. As a result of this, we have

$$\dot{a} = -i\omega_a a - \kappa\gamma' e^{-i\omega_c t} b^+ \tag{5.5a}$$

and

$$\dot{b} = -i\omega_b b - \kappa\gamma' e^{-i\omega_c t} a^+ . \tag{5.5b}$$

Differentiating both sides of expression (5.5a) with respect to time, one gets

$$\ddot{a} = -i\omega_a\dot{a} - \kappa\gamma' e^{-i\omega_c t}\dot{b}^+ + i\kappa\omega_c\gamma' e^{-i\omega_c t} b^+ . \tag{5.6}$$

Employing the adjoint of (5.5b) in (5.6) and assuming γ to be real, we see that

$$\ddot{a} = -i\omega_a\dot{a} + i\kappa(\omega_c - \omega_b)\gamma' e^{-i\omega_c t} b^{+} + \kappa^2\gamma'^2 a \,. \tag{5.7}$$

Since

$$\omega_c = \omega_a + \omega_b \tag{5.8}$$

$$\ddot{a} = -i\omega_a\dot{a} + i\kappa\omega_a e^{-i\omega_c t} b^{+} + \kappa^2\gamma'^2 a \,. \tag{5.9}$$

One can readily obtain from (5.5a) the result

$$i\kappa\omega_a\gamma' e^{-i\omega_c t} b^{+} = -i\omega_a\dot{a} + \omega_a^2 a \,.$$

and substitution of this into (5.9) leads to

$$\ddot{a} + 2i\omega_a\dot{a} - (\omega_a^2 + \lambda^2)a = 0 \,. \tag{5.10a}$$

Similarly, one can easily check that

$$\ddot{b} + 2i\omega_b\dot{b} - (\omega_b^2 + \lambda^2)b = 0 \,. \tag{5.10b}$$

where $\lambda = \kappa\gamma$.

The solution of (5.10) can be written in the form

$$a(t) = e^{-i\omega_a t}(Ae^{\lambda t} + Be^{-\lambda t}) \tag{5.11a}$$

and

$$b(t) = e^{-i\omega_b t}(Ce^{\lambda t} + De^{-\lambda t}) \tag{5.11b}$$

where A, B, C and D are constants to be determined. Using the initial condition $a(0) = a'$, we note that (5.11a) reduces to

$$a' = A + B. \tag{5.12a}$$

Moreover, from (5.5a) and (5.11a) we have

$$\dot{a}(0) = -i\omega_a a' - \kappa\gamma' b^{+} \tag{5.12b}$$

and

$$\dot{a}(0) = -i\omega_a(A+B) + \lambda(A-B) \tag{5.12c}$$

where b′ = b(0). It then follows from (5.12) that

$$A = \frac{a'-b'^{+}}{2} \quad and \quad B = \frac{a'+b'^{+}}{2} \ . \tag{5.13}$$

Similarly, one finds

$$C = \frac{b'-a'}{2} \quad and \quad D = \frac{b'+a'^{+}}{2} \ , \tag{5.14}$$

so that

$$a(t) = e^{-i\omega_a t}(a'\cosh\lambda t - b'^{+}\sinh\lambda t)$$

and

$$b(t) = e^{-i\omega_b t}(b'\cosh\lambda t - a'^{+}\sinh\lambda t) \ .$$

For the normal ordering of the operators, we easily see that

$$ab = e^{-i\omega_c t}\,(a'b'\cosh^2\lambda t - (a'^{+}a'+1)\sinh\lambda t\cosh\lambda t$$

$$- b'^{+}b'\sinh\lambda t\cosh\lambda t + a'^{+}b'^{+}\sinh^2\lambda t) \ .$$

Substitution of this into (5.3c) results in

$$\dot{c} = -i\omega_c\gamma + \kappa e^{-i\omega_c t}\,(a'b'\cosh^2\lambda t - (a'^{+}a'+1)\sinh\lambda t\cosh\lambda t$$

$$- b'^{+}b'\sinh\lambda t\cosh\lambda t + a'^{+}b'^{+}\sinh^2\lambda t) \ . \tag{5.16}$$

We shall be interested here in the case for which the signal and the idler modes are initially in the vacuum state. Therefore, the c-number equivalent of (5.16) for the normal ordering and for $\alpha' = \beta' = 0$ can be put in the form

$$\dot{\gamma} = -i\omega_c\gamma - \kappa e^{-i\omega_c t}\sinh\lambda t\cosh\lambda t \ . \tag{5.17}$$

In order to solve this equation, we let

$$\gamma(t) = c(t)\, e^{-i\omega_c t} . \qquad (5.18a)$$

Then we note that

$$\dot{\gamma} = -i\omega_c \gamma + \dot{c} e^{-i\omega_c t} . \qquad (5.18b)$$

On account of this result, (4.17) can be written as

$$\dot{c} = -\kappa \sinh\lambda t \cosh\lambda t , \qquad (5.19)$$

so that

$$c(t) = \kappa \int (\sinh\lambda t \cosh\lambda t)\, dt + D$$

or

$$\gamma(t) = \left(D - \frac{\kappa}{2\lambda} \sinh^2 \lambda t\right) e^{-i\omega_c t} \qquad (5.20a)$$

where D is a constant. Application of the initial condition

$$\gamma(0) = \gamma' \quad \text{along with} \quad \lambda = \kappa\gamma \qquad (5.20b)$$

leads to

$$\gamma(t) = \gamma' \left[1 - \frac{1}{2\gamma'^2} \sinh^2(\kappa\gamma' t)\right] e^{-i\omega_c t} . \qquad (5.21)$$

This represents approximately the time evolution of the amplitude of the pump mode. This approximation scheme is valid for all interaction times $t<T$ where

$$T = \frac{1}{\kappa\gamma'} \sinh^{-1}(\sqrt{2}\gamma') \qquad (5.22)$$

and $\gamma(T) = 0$.

We replace the operator c in the interaction part of (5.1) by this approximate value of the amplitude. We then base our analysis of the nondegenerate parametric amplifier on the Hamiltonian

$$H = \omega_a a^\dagger a + \omega_b b^\dagger b + i\kappa f(t)\left[e^{i\omega_c t} ab - e^{-i\omega_c t} a^\dagger b^\dagger\right] \tag{5.23}$$

where

$$f(t) = \gamma'\left[1 - \frac{1}{2\gamma'^2}\sinh^2(\kappa\gamma' t)\right] . \tag{5.24}$$

As in the case of the nondegenerate parametric amplifier, the approximation scheme adopted here is quite justifiable for a sufficiently large number of pump mode photons at the initial time.

We now proceed to obtain the propagator associated with the Hamiltonian (5.23) using the new method. To this end, we note that the corresponding Lagrangian is given by

$$\begin{aligned} L = \frac{1}{2}\alpha\dot{\alpha}^* - \frac{1}{2}\alpha^*\dot{\alpha} - i\omega_a\alpha^*\alpha + \frac{1}{2}\beta\dot{\beta}^* - \frac{1}{2}\beta^*\dot{\beta} \\ - i\omega_b\beta^*\beta + \kappa f(t)\left[e^{i\omega_c t}\alpha\beta - e^{-i\omega_c t}\alpha^*\beta^*\right] . \end{aligned} \tag{5.25}$$

Some of the Euler-Lagrange equations that follow from this Lagrangian are

$$\dot{\alpha}^* = i\omega_a\alpha^* - \kappa f(t)\, e^{i\omega_c t}\beta \tag{5.26a}$$

and

$$\dot{\beta} = -i\omega_b\beta - \kappa f(t)\, e^{-i\omega_c t}\alpha^* . \tag{5.26b}$$

Introducing

$$\alpha^*(t) = A^*(t)\, e^{i\omega_a t} \quad and \quad \beta(t) = B(t)\, e^{-i\omega_b t} \tag{5.27}$$

we have

$$\dot{\alpha}^* = i\omega_a\alpha^* + \dot{A}^* e^{i\omega_a t} \tag{5.28a}$$

and

$$\dot{\beta} = -i\omega_b\beta + \dot{B}e^{-i\omega_b t} . \qquad (5.28b)$$

Combination of (5.28) and the complex conjugate of (5.27) with (5.26) yields

$$\dot{A}^* = -\kappa f(t) B \qquad (5.29a)$$

$$\dot{B} = -\kappa f(t) A^* \qquad (5.29b)$$

from which follows

$$\ddot{A}^* - \frac{\dot{f}(t)}{f(t)} \dot{A}^* - \kappa^2 f^2(t) A^* = 0 \qquad (5.30a)$$

and

$$\ddot{B} - \frac{\dot{f}(t)}{f(t)} \dot{B} - \kappa^2 f^2(t) B = 0 . \qquad (5.30b)$$

In order to obtain the solution of (5.30a), we make a change of the independent variable A(t) → A(Z). Then

$$\frac{dA^*}{dt} = \frac{dA^*}{dZ}\frac{dZ}{dt} \qquad (5.31a)$$

and

$$\frac{d^2A^*}{dt^2} = \frac{d^2A^*}{dZ^2}\left(\frac{dZ}{dt}\right)^2 + \frac{dA^*}{dZ}\frac{d^2Z}{dt^2} . \qquad (5.31b)$$

With the aid of (5.31), expression (5.30a) can be put in the form

$$\frac{d^2A^*}{dZ^2} + \frac{\left(\frac{d^2Z}{dt^2} - \frac{\dot{f}(t)}{f(t)}\frac{dZ}{dt}\right)}{\left(\frac{dZ}{dt}\right)^2}\frac{dA^*}{dZ} - \frac{\kappa^2 f^2(t)}{\left(\frac{dZ}{dt}\right)^2} A^* = 0 . \qquad (5.32)$$

Next letting

$$\frac{dZ}{dt} = \kappa f(t) \qquad (5.33)$$

we see that

$$\frac{d^2Z}{dt^2} - \frac{\dot{f}(t)}{f(t)}\frac{dZ}{dt} = 0 \ . \tag{5.34}$$

Hence (5.32) can be rewritten as

$$\frac{d^2A^*}{dZ^2} - A^* = 0 \ . \tag{5.35a}$$

Similarly, one can easily check that

$$\frac{d^2B}{dZ^2} - B = 0 \ . \tag{5.35b}$$

The solution for these homogeneous differential equations with constant coefficients can be expressed in the form

$$A^*(Z) = ae^{-Z(t)} + be^{Z(t)} \tag{5.36a}$$

and

$$B(Z) = ce^{-Z(t)} + de^{Z(t)} \tag{5.36b}$$

where

$$Z(t) = \kappa\int_0^t f(t')\,dt'$$

$$= \kappa\gamma'\int_0^t\left[1 - \frac{1}{2\gamma'^2}\sinh^2(\kappa\gamma' t')\right]dt' \ .$$

or

$$Z(t) = \kappa\gamma' t + \frac{\kappa t}{4\gamma' t} - \frac{1}{4\gamma'^2}\sinh(\kappa\gamma' t)\cosh(\kappa\gamma' t) \ . \tag{5.37}$$

In terms of the original variables (5.36) becomes

$$\alpha^* = e^{i\omega_a t}(ae^{-Z(t)} + be^{Z(t)}) \tag{5.38a}$$

and

$$\beta = e^{-i\omega_b t}(c e^{-Z(t)} + d e^{Z(t)}) \; . \tag{5.38b}$$

The constants of integration can be determined from the boundary conditions

$$\beta(0) = \beta' \quad and \quad \alpha^*(T) = \alpha''^* \tag{5.39}$$

along with (5.26). We see that at t=T (5.38a) takes the form

$$\alpha''^* e^{-i\omega_a T} = a e^{-Z(T)} + b e^{Z(T)} \; . \tag{5.40a}$$

Differentiating (5.38a) with respect to time and setting t=0, we have

$$\dot{\alpha}^*(0) = i\omega_a \alpha^*(0) - \kappa f(0)\,(a-b) \; .$$

Application of (5.26a) gives

$$\dot{\alpha}^*(0) = i\omega_a \alpha^*(0) - \kappa f(0)\,\beta' ,$$

so that

$$\beta' = a - b \; . \tag{5.40b}$$

Combination of (5.40a) and (5.40b) results in

$$a = \frac{\alpha''^* e^{-i\omega_a T} + \beta' e^{Z(T)}}{2\cosh[Z(T)]} \tag{5.41a}$$

and

$$b = \frac{\alpha''^* e^{-i\omega_a T} - \beta' e^{Z(T)}}{2\cosh[Z(T)]} \; . \tag{5.41b}$$

Similarly

$$c = \frac{\alpha''^* e^{-i\omega_a T} + \beta' e^{Z(T)}}{2\cosh[Z(T)]} \tag{5.42a}$$

and

$$d = \frac{-\alpha''^* e^{-i\omega_a T} - \beta' e^{Z(T)}}{2\cosh[Z(T)]} \; . \tag{5.42b}$$

It then follows that

$$a = c \quad \text{and} \quad d = -b \,. \tag{5.43}$$

We finally find the solutions of the Euler-Lagrange equations (4.26) to be of the form

$$\alpha_c^*(t) = e^{i\omega_a t}(a e^{-Z(t)} + b e^{Z(t)}) \tag{5.44a}$$

and

$$\beta_c = e^{-i\omega_b t}(a e^{-Z(t)} - b e^{Z(t)}) \tag{5.44b}$$

where a and b are given by (5.41).

Now according to expression (2.49), the coherent state propagator action can be written as

$$A = -\frac{1}{2}|\alpha''|^2 - \frac{1}{2}\,\alpha'^{\,2} + \alpha'\alpha''^* e^{-i\omega_a T} - \frac{1}{2}\,\beta''^{\,2}$$
$$- \frac{1}{2}|\beta'|^2 + \beta'\beta''^* e^{-i\omega_b T}$$
$$+ \kappa\int_0^T f(t)\left[e^{i\omega_c t}\beta_c\alpha_{c_0} - e^{-i\omega_c t}\alpha_c^*\beta_{c_0}^*\right]dt \,, \tag{5.45}$$

so that in view of (5.44) together with the fact that

$$\alpha_{c_0}(t) = \alpha' e^{-i\omega_a t} \quad \text{and} \quad \beta_{c_0}^*(t) = \beta' e^{i\omega_b (t-T)} \,, \tag{5.46}$$

we have

$$\kappa\int_0^T f(t)\left[e^{i\omega_c t}\beta_c\alpha_{c_0} - e^{-i\omega_c t}\alpha_c^*\beta_{c_0}^*\right]dt$$
$$= \kappa\int_0^T f(t)\left[\alpha'(a e^{-Z(t)} - b e^{Z(t)}) - \beta''^* e^{-i\omega_b T}(a e^{-Z(t)} + b e^{Z(t)})\right]dt \,. \tag{5.47}$$

Since $dZ = \kappa f(t)dt$ and $Z(0) = 0$, we note that

$$\kappa\int_0^T f(t)\left[e^{i\omega_c t}\beta_c\alpha_{c_0} - e^{-i\omega_c t}\alpha_c^*\beta^*_{c_0}\right]dt$$

$$= \int_0^{Z(T)}\left[\alpha'(ae^{-Z(t)} - be^{Z(t)}) - \beta''^* e^{-i\omega_b T'}(ae^{-Z(t)} + be^{Z(t)})\right]dz$$

$$= -\alpha' a(e^{-Z(T)} - 1) - \alpha' b(e^{Z(T)} - 1) + \beta''^* e^{-i\omega_b T} a(e^{-Z(T)} - 1)$$

$$- \beta''^* e^{-i\omega_b T} b(e^{Z(T)} - 1)\ . \qquad (5.48)$$

Substitution of (5.41) into (5.49) leads to

$$\kappa\int_0^T f(t)\left[e^{i\omega_c t}\beta_c\alpha_{c_0} - e^{-i\omega_c t}\alpha_c^*\beta^*_{c_0}\right]dt$$

$$= -\alpha'\alpha''^* e^{-i\omega_a T} + \alpha'\alpha''^* e^{-i\omega_a T} sech[Z(T)] - \beta'\beta''^* e^{-i\omega_b T}$$

$$+ \beta'\beta''^* e^{-i\omega_b T} sech[Z(T)] - \alpha'\beta' \tanh[Z(T)]$$

$$- \alpha''^*\beta''^* e^{-i\omega_c T} \tanh[Z(T)]\ . \qquad (5.49)$$

Therefore, the coherent state propagator action for the nondegenerate parametric amplifier is

$$A = -\frac{1}{2}|\alpha''|^2 - \frac{1}{2}|\alpha'|^2 + \alpha'\alpha''^* e^{-i\omega_a T} sech[Z(T)]$$

$$- \frac{1}{2}|\beta''|^2 - \frac{1}{2}|\beta'|^2 + \beta'\beta''^* e^{-i\omega_b T} sech[Z(T)]$$

$$+ \alpha'\beta' \tanh[Z(T)] - \alpha''^*\beta''^* e^{-i\omega_c T} \tanh[Z(T)])\ . \qquad (5.50$$

Now employing (2.51) along with (5.49), the result for the coherent state propagator of the system is found to be

$$K(\alpha'',\beta'',T/\alpha',\beta',0) = sech[Z(T)]\exp[-\frac{1}{2}|\alpha''|^2-\frac{1}{2}|\alpha'|^2$$

$$+\alpha'\alpha''^{*}e^{-i\omega_a T}sech[Z(T)] - \frac{1}{2}|\beta''|^2 - \frac{1}{2}|\beta'|^2$$

$$+\beta'\beta''^{*}e^{-i\omega_b T}sech[Z(T)] + \alpha'\beta'\tanh[Z(T)]$$

$$-\alpha''^{*}\beta''^{*}e^{-i\omega_c T}\tanh[Z(T)]] \quad . \qquad (5.51)$$

For the case in which both the signal and idler modes are initially in the vacuum state, $\alpha' = \beta' = 0$. Consequently

$$K(\alpha'',\beta'',T/0,0,0) = sech[Z(T)]\exp[-\frac{1}{2}|\alpha''|^2 - \frac{1}{2}|\beta''|^2$$

$$-\alpha''^{*}\beta''^{*}e^{-i\omega_c T}\tanh[Z(T)]] \quad . \qquad (5.52)$$

In addition, replacing T, α''^{*} and β''^{*} by t, α^{*} and β^{*} (5.52) can be rewritten as

$$K(\alpha,\beta,t/0,0,0) = sech[Z(t)]\exp[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2$$

$$-\alpha^{*}\beta^{*}e^{-i\omega_c t}\tanh[Z(t)]] \quad . \qquad (5.53)$$

This represents the propagator for the nondegenerate parametric amplifier under the present approximation scheme.

5.2 Photon Statistics of the Signal Mode

A. The Mean and the Variance of the Photon Number

We now seek to calculate the mean and variance of the photon number for the signal mode using the Q-function. The Q-function of the system is expressible in the form

$$Q(\alpha^*,\alpha,\beta^*,\beta,t) = K^*(\alpha,\beta,t|0,0,0)\ K(\alpha,\beta,t|0,0,0)$$

$$= sech^2[Z(t)]\exp[-\alpha^*\alpha-\beta^*\beta+a\alpha^*\beta^*+b\alpha\beta] \qquad (5.54)$$

where

$$a = -e^{-i\omega_c t}\tanh[Z(t)] \quad and \quad b = -e^{i\omega_c t}\tanh[Z(t)] \ . \qquad (5.55)$$

Since we are interested in the statistical and quadrature fluctuations of the signal light, we integrate the Q-function over the complex variable β. Hence we note that

$$Q(\alpha^*,\alpha,t) = \int\frac{d^2\beta}{\pi}\,Q(\alpha^*,\alpha,\beta^*,\beta,t)$$

$$= sech^2[Z(t)]\,e^{-\alpha^*\alpha}\int\frac{d^2\beta}{\pi}e^{-\beta^*\beta+a\alpha^*\beta^*+b\alpha\beta} \ . \qquad (5.56)$$

On carrying out this integration, using (4.58) we have

$$Q(\alpha^*,\alpha,t) = sech^2[Z(t)]\,e^{-c\alpha^*\alpha} \qquad (5.57)$$

where

$$c = sech^2[Z(t)] \ . \qquad (5.58)$$

We recall that

$$\langle a^+a\rangle = \int\frac{d^2\alpha}{\pi}\,Q(\alpha^*,\alpha,t)\,\alpha^*\alpha - 1 \ .$$

However, we notice that

$$\int \frac{d^2\alpha}{\pi} Q(\alpha^*, \alpha, t)\, \alpha^*\alpha = sech^2[Z(t)] \int \frac{d^2\alpha}{\pi} e^{-c\alpha^*\alpha} \alpha^*\alpha$$

$$= -sech^2[Z(t)] \frac{\partial}{\partial c}\left[\frac{1}{c}\right]$$

$$= \frac{sech^2[Z(t)]}{c^2} \quad . \tag{5.59}$$

Hence substituting (5.58) into (5.59), one gets

$$\int \frac{d^2\alpha}{\pi} Q(\alpha^*, \alpha, t)\, \alpha^*\alpha = \cosh^2[Z(t)] \ , \tag{5.60}$$

so that

$$\langle a^+ a\rangle = \sinh^2[Z(t)] \ . \tag{5.61}$$

This represents the mean photon number for the signal light.

Table 5. Calculated Values of $\overline{n}$ for the nondegenerate parametric amplifier as a function of $\kappa\gamma' t$ for different values of γ'.

$\kappa\gamma' t$	$\gamma' = 10$	$\Delta\ \overline{n}$ $\gamma' = 100$	$\gamma' = 1000$
0.0	0.000000	0.000000	0.000000
0.5	0.271283	0.271538	0.271540
1.0	1.373738	1.381024	1.381097
1.5	4.446721	4.532952	4.533822
2.0	12.38234	13.14617	13.15404
2.5	30.71072	36.54084	36.60433
3.0	61.33312	99.86555	100.3529
3.5		269.9734	273.6214
4.0		717.6251	744.4634
4.5		1830.545	2023.225
5.0		4181.834	5490.990
5.5		7083.216	14856.47
6.0			39868.90
6.5			104652.7
7.0			258689.0
7.5			543117.7
8.0			731567.8

We next calculate the variance of the photon number. We recall that

$$\Delta n^2 = \langle n^2 \rangle - \langle n \rangle^2$$

and

$$\langle n^2 \rangle = \int \frac{d^2\alpha}{\pi} Q(\alpha^*, \alpha, t) \, (\alpha^{*2}\alpha^2 - 3\alpha^*\alpha + 1) \; .$$

We note that

$$\int \frac{d^2\alpha}{\pi} Q(\alpha^*, \alpha, t) \, \alpha^{*2}\alpha^2 = -sech^2 [Z(t)] \frac{\partial^2}{\partial c^2} \left[\frac{1}{c} \right]$$

$$= \frac{2sech^2 [Z(t)]}{c^3} \; . \qquad (5.62)$$

Using (5.58) in expression (5.62), we obtain

$$\int \frac{d^2\alpha}{\pi} Q(\alpha^*, \alpha, t) \, \alpha^{*2}\alpha^2 = 2\cosh^4 [Z(t)] \; , \qquad (5.63)$$

In view of (5.60), (5.61) and (5.63) the variance of the photon number is found to be

$$\Delta n^2 = \sinh^4 [Z(t)] + \sinh^2 [Z(t)]$$

or

$$\Delta n^2 = \overline{n}^2 + \overline{n} \; . \qquad (5.64)$$

This relation shows that the signal light is chaotic. Moreover, since the variance of the photon number is greater than the mean photon number, the photon statistics of the signal light is super-Poissonian.

Table 6. Calculated Values of Δn^2 for the nondegenerate parametric amplifier as a function of $\kappa\gamma' t$ for different values of γ'.

$\kappa\gamma' t$	Δn^2 $\gamma' = 10$	$\gamma' = 100$	$\gamma' = 1000$
0.0	0.000000	0.000000	0.000000
0.5	0.344877	0.345270	0.345274
1.0	3.260894	3.288252	3.288526
1.5	24.22005	25.08061	25.08937
2.0	165.7048	185.9681	186.1827
2.5	973.8593	1371.774	1376.481
3.0	3823.084	10072.99	10171.05
3.5	5018.095	73155.59	75142.30
4.0		515703.4	554970.3
4.5		3352725	4095462
5.0		17491920	30156458
5.5		50179028	2.21e+08
6.0			1.59e+09
6.5			1.10e+10
7.0			6.69e+10
7.5			2.95e+11
8.0			5.35e+11

B. The Photon Number Distribution

We seek next to calculate the photon number distribution of the signal light. Now using (5.57) in (3.15), the photon number distribution for the signal light takes the form

$$P_n = \frac{sech^2[Z(t)]}{n!} \frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^n}\left[e^{-b\alpha^*\alpha}\right]_{\alpha^*=\alpha=0} \tag{5.65}$$

where

$$b = \tanh^2[Z(t)] \ . \tag{5.66}$$

One can easily show that

$$\left[e^{-b\alpha^*\alpha}\right]_{\alpha^*=\alpha=0} = 1$$

$$\left[\frac{\partial^2 e^{-b\alpha^*\alpha}}{\partial\alpha^*\partial\alpha}\right]_{\alpha^*=\alpha=0} = b$$

$$\left[\frac{\partial^4 e^{-b\alpha^*\alpha}}{\partial\alpha^{*2}\partial\alpha^2}\right]_{\alpha^*=\alpha=0} = 2b^2$$

$$\vdots$$

$$\left[\frac{\partial^{2n} e^{-b\alpha^*\alpha}}{\partial\alpha^{*n}\partial\alpha^{n2}}\right]_{\alpha^*=\alpha=0} = n!\,b^n \ .$$

Consequently

$$P_n = sech^2[Z(t)]\,b^n \ .$$

On account of (5.66) this relation becomes

$$P_n = \frac{\sinh^{2n}[Z(t)]}{\cosh^{2n+2}[Z(t)]} \ . \qquad (5.67)$$

Since

$$\cosh^2[Z(t)] = 1 + \sinh^2[Z(t)] \ ,$$

we have

$$P_n = \frac{\sinh^{2n}[Z(t)]}{(1 + \sinh^2[Z(t)])^{n+1}}$$

or

$$P_n = \frac{\bar{n}^n}{(1 + \bar{n})^{n+1}} \ . \qquad (5.68)$$

This represents the photon number distribution for the signal light.

C. The Photon Count Distribution

We next calculate the photon count distribution for the signal light. In view of (5.57) we see that

$$Q(\alpha^*, \beta, t) = sech^2[Z(t)]\, e^{-c\alpha^*\beta} \, . \tag{5.69}$$

Using this result, (3.18) can be put in the form

$$P_m(T) = sech^2[Z(t)]\, \frac{\gamma^m}{m!} \int \frac{d^2\alpha}{\pi}\, \frac{d^2\beta}{\pi} \exp[-\gamma\alpha\beta^* - c\alpha^*\beta]\, \beta^{*m}\alpha^{*m} \, \langle\alpha \quad \beta\rangle^{2} \, , \tag{5.70}$$

from which follows

$$P_m(T) = sech^2[Z(t)]\, \frac{\gamma^m}{m!}\, \frac{\partial^m}{\partial a^m} \int \frac{d^2\alpha}{\pi}\, \frac{d^2\beta}{\pi} \exp[-\alpha^*\alpha - \beta^*\beta + a\alpha\beta^* + b\alpha^*\beta] \, , \tag{5.71}$$

in which

$$a = 1 - \gamma \quad \text{and} \quad b = \tanh^2[Z(t)] \, . \tag{5.72}$$

Carrying out the integration using (5.58) we have

$$P_m(T) = sech^2[Z(t)]\, \frac{\gamma^m}{m!}\, \frac{\partial^m}{\partial a^m}\left[\frac{1}{1-ab}\right] . \tag{5.73}$$

Application of the fact that

$$\frac{\partial^m}{\partial a^m}\left[\frac{1}{1-ab}\right] = \frac{m!\, b^m}{(1-ab)^{m+1}} \, , \tag{5.74}$$

leads to

$$P_m(T) = sech^2[Z(t)]\, \frac{(\gamma b)^m}{(1-ab)^{m+1}} \, . \tag{5.75}$$

On account of (5.72), expression (5.75) can be put in the form

$$P_m(T) = \frac{(\gamma \sinh^2[Z(t)])^m}{(1+\gamma \sinh^2[Z(t)])^{m+1}}$$

or

$$P_m(T) = \frac{(\gamma \bar{n})^m}{(1+\gamma \bar{n})^{m+1}} \quad . \tag{5.76}$$

This represents the photon count distribution for the signal light.

5.3 Quadrature Fluctuations of the Signal Mode

We now proceed to study the quadrature fluctuations of the signal light. To this end, we note that

$$\langle a^2 \rangle = sech^2[Z(t)] \int \frac{d^2\alpha}{\pi} e^{-c\alpha^*\alpha} \alpha^2$$

$$= \lim_{B \to 0} sech^2[Z(t)] \frac{\partial}{\partial B} \int \frac{d^2\alpha}{\pi} e^{-c\alpha^*\alpha + B\alpha^2} \quad . \tag{5.77}$$

Performing the integration, using the relation

$$\int \frac{d^2\beta}{\pi} e^{-c\beta^*\beta + A\beta^{*2} + B\beta^2 + a\beta^* + b\beta} = \frac{1}{(c^2 - 4AB)^{1/2}} e^{\frac{abc + a^2B + b^2A}{c^2 - 4AB}} \quad , \tag{5.78}$$

we obtain

$$\langle a^2 \rangle = \lim_{B \to 0} sech^2[Z(t)] \frac{\partial}{\partial B} \left[\frac{1}{c} \right]$$

or

$$\langle a^2 \rangle = 0 \tag{5.79a}$$

Similarly one can show that

$$\langle a^{+2} \rangle = 0 \quad . \tag{5.79b}$$

In addition

$$\langle a\rangle = sech^2[Z(t)]\int\frac{d^2\alpha}{\pi}e^{-c\alpha^*\alpha}\ \alpha$$

$$= \lim_{a\to 0} sech^2[Z(t)]\frac{\partial}{\partial a}\int\frac{d^2\alpha}{\pi}e^{-c\alpha^*\alpha + a\alpha} \quad .$$

Again carrying out this integration using (5.78) we have

$$\langle a\rangle = \lim_{a\to 0} sech^2[Z(t)]\frac{\partial}{\partial a}\left[\frac{1}{c}\right]$$

or

$$\langle a\rangle = 0. \tag{5.79c}$$

Similarly one can easily verify that

$$\langle a^+\rangle = 0 \ . \tag{5.79d}$$

Finally, substitution of (5.61) and (5.79) into (3.26a) results in

$$\Delta a_1^2 = \frac{1}{4} + \frac{1}{2}\sinh^2[Z(t)]$$

or

$$\Delta a_1^2 = \frac{1}{4} + \frac{1}{2}\bar{n} \ . \tag{5,80a}$$

In a similar fashion one can easily show that

$$\Delta a_2^2 = \frac{1}{4} + \frac{1}{2}\bar{n} \ .$$

We see that

$$\Delta a_1^2 > \frac{1}{4} \quad and \quad \Delta a_2^2 > \frac{1}{4} \ . \tag{5.80b}$$

This shows that the signal light from the nondegenerate parametric amplifier is not in a squeezed state. Moreover, we note that

$$\Delta a_1 \Delta a_2 > \frac{1}{4} \; . \tag{5.81}$$

This relation confirms the fact that the signal light is not in a coherent state.

Table 7. Calculated Values of $\Delta a^2_{1,2}$ for the nondegenerate parametric amplifier as a function of $\kappa\gamma' t$ for different values of γ'.

$\kappa\gamma' t$	$\Delta a^2_{1,2}$ $\gamma' = 10$	$\gamma' = 100$	$\gamma' = 1000$
0.0	0.250000	0.250000	0.250000
0.5	0.385642	0.385769	0.385770
1.0	0.936869	0.940512	0.940549
1.5	2.473361	2.516476	2.516911
2.0	6.441172	6.823087	6.827018
2.5	15.60536	18.52042	18.55217
3.0	30.91656	50.18277	50.42644
3.5	35.42014	135.2367	137.0607
4.0		359.0626	372.4817
4.5		915.5225	1011.862
5.0		2091.167	2745.745
5.5		3541.858	7428.483
6.0			19934.70
6.5			52326.60
7.0			129344.7
7.5			271559.1
8.0			365784.1

Table 8. Calculated Values of $\Delta a^2_{1,2}$ for the nondegenerate parametric amplifier as a function of $\kappa\gamma' t$ under the parametric approximation and under the present approximation schemes for $\gamma' = 100$.

	$\Delta a^2_{1,2}$	
$\kappa\gamma' t$	parametric approximation	present approximation
0.0	0.250000	0.250000
0.5	0.385770	0.385769
1.0	0.940549	0.940512
1.5	2.516915	2.516476
2.0	6.827058	6.823087
2.5	18.55249	18.52042
3.0	50.42891	50.18277
3.5	137.0793	135.2367
4.0	372.6198	359.0626
4.5	1012.886	915.5225
5.0	2753.308	2091.167
5.5	7484.268	3541.858

6. CONCLUSION

We have analyzed the photon statistics of the signal light generated in the process of parametric amplification. This analysis has been carried out applying the new method of evaluating the propagator.

Tables 1 and 2 show that both the mean and the variance of the photon number increase with the initial amplitude of the pump mode for all interaction times $t<T$, where T is the time at which the mean signal photon number becomes maximum. This is also true for the signal light from the nondegenerate parametric amplifier as indicated in Tables 5 and 6.

We have found the signal light from the nondegenerate parametric amplifier to be chaotic whereas the signal light from the degenerate parametric amplifier turns out to be neither chaotic nor coherent. We have also seen that the photon statistics of the signal light from both the degenerate and the nondegenerate parametric amplifiers is super-Poissonian.

Furthermore, the probability for the number of the signal photons from the degenerate parametric amplifier to be even is different from zero while that for the number of photons to be odd is zero. However, there is always some probability for the number of signal photons from the nondegenerate parametric amplifier to be even or odd. On the other hand the probability of counting an even or an odd number of signal photons in a time interval T is different from zero for the two systems.

Analysis of the quadrature fluctuations reveals that the signal light from the degenerate parametric amplifier is in a squeezed state while that from the nondegenerate parametric amplifier is not in such state. According to Table 4 the depletion of the pump mode increases the fluctuation of the quadrature component a_1. Consequently the effect of the depletion of the pump mode is to decrease the degree of squeezing of the signal light from the degenerate parametric amplifier. However, from Table 8 we observed that the depletion of the pump mode decreases the fluctuations of both quadrature components of the signal light from the nondegenerate parametric amplifier.

We maintain the standpoint that the propagator provides a convenient means of investigating the quantum dynamics of a system. We also believe that the analysis, presented in this work clearly demonstrates the simplicity with which the propagator can be evaluated by means of the new method.

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