THE DYNAMICS OF TWO-LEVEL LASER

By

Fekadu Wolde

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
AT
ADDIS ABABA UNIVERSITY
ADDIS ABABA, ETHIOPIA
FEBRUARY 2016

ADDIS ABABA UNIVERSITY
DEPARTMENT OF
PHYSICS

The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled "**THE DYNAMICS OF TWO-LEVEL LASER**" by **Fekadu Wolde** in partial fulfillment of the requirements for the degree of **Master of Science**.

Dated: February 2016

Supervisor: ____________________
Dr.Deribe Hirpo

Examiners: ____________________
Dr.Fesseha Kassahun

____
Dr.Teshome Senbeta

ADDIS ABABA UNIVERSITY

Date: **February 2016**

Author: **Fekadu Wolde**

Title: **THE DYNAMICS OF TWO-LEVEL LASER**

Department: **Physics**

Degree: **M.Sc.** Convocation: Year: **2016**

Permission is herewith granted to Addis Ababa University to circulate and to have copied for non-commercial purposes, at its discretion, the above title upon the request of individuals or institutions.

Signature of Author

Table of Contents

List of Figures

Abstract

In this thesis, we study the quantum properties of the light generated by a two-level laser in which two-level atoms in a cavity coupled to a vacuum reservoir are pumped to the top level at a rate of r_{ca}. We consider the case in which two-level atoms and the cavity mode are interacting with vacuum reservoir. Employing the Master equation for the system under consideration, we obtained the quantum Langevin equations for the cavity mode and atomic operators. Employing the solution of these equations along with the correlation properties of the noise operators, we calculate the mean and the variance of the photon number of the cavity light. We have found that the variance of the photon number is greater than the mean of the photon number and hence the photon statistics of the cavity light mode is super-Poissonian. Moreover, we determine the quadrature variance and the power spectrum of the cavity light.

Acknowledgements

First of all I would like to express my deepest gratitude and respect to my advisor Dr.Deribe Hirpo for his invaluable support and advise in any part of this thesis work in a friendly manner. I would like to express my gratitude to the Department of Physics for providing the necessary facilities and financial support during my study. Next my thanks tends to the whole of physics instructors who taught me directly and indirectly throughout this programme. I would also like to express my thanks to Dr.Teshome Senbeta for his support in providing and adjusting softwares on my computer for writing this thesis. Finally I would like to thank my elder sister Kibinesh Wolde for her financial and moral support.

Chapter 1

Introduction

It is safe to say that the foundation of quantum optics are built on the concept of a few level atoms available in the cavity. Indeed, the most important concept that will introduce in this thesis is a two-level laser system. A two-level laser is a source of coherent or chaotic light emitted by two level atoms inside a cavity coupled to a vacuum reservoir via a single port mirror. In one model of such a laser, two-level atoms initially in the upper level are injected at a constant rate into a cavity and removed after they have decayed to the lower level due to spontaneous emission [1-3]. In another model the two level atoms available in a cavity are pumped to the upper level by some convenient means of pumping, such as electron bombardment [3,4].

The quantum properties of light generated by two-level laser has been considered by different authors [1-11]. It is found that the light generated by this laser operating well above threshold is coherent and the light generated by the same laser operating bellow threshold is chaotic. In the quantum theory of laser, it is usually takes into consideration the interaction of the atoms inside the cavity with the vacuum reservoir outside the cavity.

In this thesis we study the quantum properties of the light produced by two level

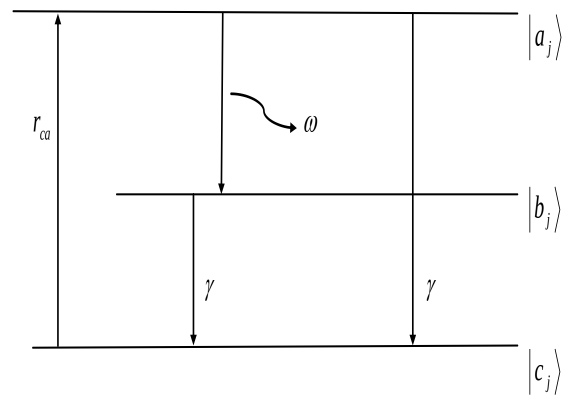

Figure 1.1: A two-level atom.

laser, in which two-level atoms available in a cavity coupled to vacuum reservoir via a single port mirror, are pumped to the upper level at a constant rate r_{ca}. We consider the case in which the two-level atoms in a cavity interact with vacuum reservoir. We denote the upper, bottom and ground level of an atom by $|a_j\rangle$, $|b_j\rangle$ and $|c_j\rangle$ respectively. The two-level atom is pumped from the ground level $|c_j\rangle$ to the upper level $|a_j\rangle$ at a rate r_{ca}. The atom then may make a transition from level $|a_j\rangle$ to $|b_j\rangle$ by emitting a photon of frequency ω, otherwise the atom may decay spontaneously from level $|a_j\rangle$ or $|b_j\rangle$ to the ground level $|c_j\rangle$ at the rate of γ.

Applying the Master equation for the system under consideration we obtain the quantum Langevin equation for the cavity mode and atomic operators. Employing the solutions of these equation along with the correlation properties of noise operator we calculate the mean and variance of the photon number. Finally we evaluate the quadrature variance and power spectrum for the cavity light.

Chapter 2

Operator Dynamics

At normal condition most of the atoms occupies their own lower energy level. When these atoms are pumped, they start to escape to the higher level by some rate. Here these atoms can be de-excited via spontaneous emission or stimulated emission by emitting photons. Photons emitted by spontaneous emission are in all direction, which is not what we want. We are looking for a giant coherent pulse of light known as laser that can only be achieved by stimulated emission. To produce intense laser beam or amplification of light through stimulated emission it requires higher rate of stimulated emission than spontaneous emission.

By considering N two-level atoms that are available in a cavity coupled to a vacuum reservoir via a single port mirror, we study the process in which the atoms as well as the cavity mode interact with the vacuum reservoir. We denote the top, bottom and ground level by $|a_j\rangle$, $|b_j\rangle$ and $|c_j\rangle$, respectively, and the atom is pumped from the ground level $|c_j\rangle$ to the top level $|a_j\rangle$ at the rate of r_{ca}. A two-level atom may make a transition from level $|a_j\rangle$ to $|b_j\rangle$ by emitting a photon of frequency ω , alternatively the atom may decay from level $|a_j\rangle$ or $|b_j\rangle$ spontaneously to the ground level $|c_j\rangle$ at the rate of γ.

The interaction of the cavity mode with a two-level atom can be described by the Hamiltonian:

$$\hat{H} = ig\left(\hat{a}^\dagger|b_j\rangle\langle a_j| - |a_j\rangle\langle b_j|\hat{a}\right). \tag{2.0.1}$$

where g is a coupling constant between the atom and the cavity mode, $\hat{a}$ is the annihilation operator for the cavity mode and $|b_j\rangle\langle a_j|$ is a lowering atomic operator. The Master equation that describe the interaction between the cavity mode and the atom is expressible as

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= -i[\hat{H},\hat{\rho}] + \frac{\kappa}{2}\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right)\\ &+ \frac{\gamma}{2}\left(2|c_j\rangle\langle a_j|\hat{\rho}|a_j\rangle\langle c_j| - |a_j\rangle\langle a_j|\hat{\rho} - \hat{\rho}|a_j\rangle\langle a_j|\right)\\ &+ \frac{\gamma}{2}\left(2|c_j\rangle\langle b_j|\hat{\rho}|b_j\rangle\langle c_j| - |b_j\rangle\langle b_j|\hat{\rho} - \hat{\rho}|b_j\rangle\langle b_j|\right).\end{aligned} \tag{2.0.2}$$

By substituting Eq.(2.0.1) into Eq.(2.0.2), the Master equation have the form

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= g\left(\hat{a}^\dagger\hat{\sigma}_a^j\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^j - \hat{\sigma}_a^{\dagger j}\hat{a}\hat{\rho} + \hat{\rho}\hat{\sigma}_a^{\dagger j}\hat{a}\right)\\ &+ \frac{\kappa}{2}\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right)\\ &+ \frac{\gamma}{2}\left(2\hat{\sigma}_c^j\hat{\rho}\hat{\sigma}_c^{\dagger j} - \hat{\eta}_a^j\hat{\rho} - \hat{\rho}\hat{\eta}_a^j\right)\\ &+ \frac{\gamma}{2}\left(2\hat{\sigma}_b^j\hat{\rho}\hat{\sigma}_b^{\dagger j} - \hat{\eta}_b^j\hat{\rho} - \hat{\rho}\hat{\eta}_b^j\right).\end{aligned} \tag{2.0.3}$$

where

$$\hat{\sigma}_a^j = |b_j\rangle\langle a_j|, \tag{2.0.4}$$

$$\hat{\sigma}_b^j = |c_j\rangle\langle b_j|, \tag{2.0.5}$$

$$\hat{\sigma}_c^j = |c_j\rangle\langle a_j|, \tag{2.0.6}$$

$$\hat{\eta}_a^j = |a_j\rangle\langle a_j|, \tag{2.0.7}$$

$$\hat{\eta}_b^j = |b_j\rangle\langle b_j|, \tag{2.0.8}$$

$$\hat{\eta}_c^j = |c_j\rangle\langle c_j|, \tag{2.0.9}$$

are atomic operators.

For the given two-level atoms in a cavity coupled to a vacuum reservoir via a single port mirror, the time evolution of the expectation value of the cavity mode and atomic operators are evaluated. Employing the relation

$$\frac{d}{dt}\langle \hat{A}\rangle = Tr\left(\frac{d\hat{\rho}}{dt}\hat{A}\right), \tag{2.0.10}$$

along with the Master equation, we thus see that

$$\begin{aligned}\frac{d}{dt}\langle \hat{a}(t)\rangle &= Tr\left(\frac{d\hat{\rho}}{dt}\hat{a}(t)\right)\\ &= gTr\left(\hat{a}^\dagger\hat{\sigma}_a^j\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^j\hat{a} - \hat{\sigma}_a^{\dagger j}\hat{a}\hat{\rho}\hat{a} + \hat{\rho}\hat{\sigma}_a^{\dagger j}\hat{a}^2\right)\\ &+ \frac{k}{2}Tr\left(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2\right)\\ &+ \frac{\gamma}{2}Tr\left(2\hat{\sigma}_c^j\hat{\rho}\hat{\sigma}_c^{\dagger j}\hat{a} - \hat{\eta}_a^j\hat{\rho}\hat{a} - \hat{\rho}\hat{\eta}_a^j\hat{a}\right)\\ &+ \frac{\gamma}{2}Tr\left(2\hat{\sigma}_b^j\hat{\rho}\hat{\sigma}_b^{\dagger j}\hat{a} - \hat{\eta}_b^j\hat{\rho}\hat{a} - \hat{\rho}\hat{\eta}_b^j\hat{a}\right).\end{aligned} \tag{2.0.11}$$

Applying the cyclic properties of trace operators and the relation

$$\hat{\sigma}_b^{\dagger j}\hat{\sigma}_b^j = \hat{\sigma}_a^j\hat{\sigma}_a^{\dagger j} = \hat{\eta}_b^j, \tag{2.0.12}$$

$$\hat{\sigma}_c^{\dagger j}\hat{\sigma}_c^j = \hat{\sigma}_a^{\dagger j}\hat{\sigma}_a^j = \hat{\eta}_a^j, \tag{2.0.13}$$

we can put Eq. (2.0.11) in the form

$$\frac{d}{dt}\langle \hat{a}(t)\rangle = -\frac{\kappa}{2}\langle \hat{a}(t)\rangle + g\langle \hat{\sigma}_a^j(t)\rangle. \tag{2.0.14}$$

Similarly, applying the relation

$$\frac{d}{dt}\langle \hat{\sigma}_a^j\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{\sigma}_a^j), \tag{2.0.15}$$

we see that

$$\begin{aligned}\frac{d}{dt}\langle\hat{\sigma}_a^j(t)\rangle &= gTr\left(\hat{a}^\dagger\hat{\sigma}_a^j\hat{\rho}\hat{\sigma}_a^j - \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^j\hat{\sigma}_a^j - \hat{\sigma}_a^{\dagger j}\hat{a}\hat{\rho}\hat{\sigma}_a^j + \hat{\rho}\hat{\sigma}_a^{\dagger j}\hat{a}\hat{\sigma}_a^j\right)\\ &+ \frac{\kappa}{2}Tr\left(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^j - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{\sigma}_a^j - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{\sigma}_a^j\right)\\ &- \frac{\gamma}{2}Tr\left(2\hat{\sigma}_c^j\hat{\rho}\hat{\sigma}_c^{\dagger j}\hat{\sigma}_a^j - \hat{\eta}_a^j\hat{\rho}\hat{\sigma}_a^j - \hat{\rho}\hat{\eta}_a^j\hat{\sigma}_a^j\right)\\ &- \frac{\gamma}{2}Tr\left(2\hat{\sigma}_b^j\hat{\rho}\hat{\sigma}_b^{\dagger j}\hat{\sigma}_a^j - \hat{\eta}_b^j\hat{\rho}\hat{\sigma}_a^j - \hat{\rho}\hat{\eta}_b^j\hat{\sigma}_a^j\right).\end{aligned} \tag{2.0.16}$$

Employing the cyclic property of trace operator, it can be verified that

$$\begin{aligned}\frac{d}{dt}\langle\hat{\sigma}_a^j(t)\rangle &= gTr(\hat{\rho}\hat{\eta}_b^j\hat{a} - \hat{\rho}\hat{\eta}_a^j) + \frac{\kappa}{2}Tr(2\hat{\rho}\hat{a}^\dagger\hat{a}\hat{\sigma}_a^j - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{\sigma}_a^j - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{\sigma}_a^j)\\ &- \frac{\gamma}{2}Tr(\hat{\rho}\hat{\sigma}_a^j) + \hat{\rho}\hat{\sigma}_a^j),\end{aligned} \tag{2.0.17}$$

or

$$\frac{d}{dt}\langle\hat{\sigma}_a^j(t)\rangle = -\gamma\langle\hat{\sigma}_a^j(t)\rangle + g[\langle\hat{\eta}_a^j(t)\hat{a}(t)\rangle - \langle\hat{\eta}_b^j(t)\hat{a}(t)\rangle], \tag{2.0.18}$$

where

$$\hat{\sigma}_c^{\dagger j}\hat{\sigma}_a^j\hat{\sigma}_c^j = \hat{\sigma}_b^{\dagger j}\hat{\sigma}_a^j\hat{\sigma}_b^j = 0, \tag{2.0.19}$$

$$\hat{\eta}_a^j\hat{\sigma}_a^j = \hat{\sigma}_a^j\hat{\eta}_b^j = 0, \tag{2.0.20}$$

$$\hat{\eta}_b^j\hat{\sigma}_a^j = \hat{\sigma}_a^j\hat{\eta}_a^j = \hat{\sigma}_a^j \tag{2.0.21}$$

has been used.

Furthermore, using the relation

$$\frac{d}{dt}\langle\hat{\eta}_a^j(t)\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{\eta}_a^j(t)), \tag{2.0.22}$$

along with Eq. (2.0.3) and the cyclic properties of trace operators we can find that

$$\begin{aligned}\frac{d}{dt}\langle\hat{\eta}_a^j(t)\rangle &= gTr\left(\hat{\rho}\hat{a}^\dagger\hat{\eta}_a^j\hat{\sigma}_a^j - \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^j\hat{\eta}_a^j - \hat{\rho}\hat{\eta}_a^j\hat{\sigma}_a^{\dagger j}\hat{a} + \hat{\rho}\hat{\sigma}_a^{\dagger j}\hat{\eta}_a^j\hat{a}\right)\\ &+ \frac{\kappa}{2}Tr\left(2\hat{\rho}\hat{a}^\dagger\hat{a}\hat{\eta}_a^j - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{\eta}_a^j - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{\eta}_a^j\right)\\ &+ \frac{\gamma}{2}Tr\left(2\hat{\rho}\hat{\sigma}_c^{\dagger j}\hat{\eta}_a^j\hat{\sigma}_c^j - \hat{\rho}\hat{\eta}_a^j\hat{\eta}_a^j - \hat{\rho}\hat{\eta}_a^j\hat{\eta}_a^j\right)\\ &+ \frac{\gamma}{2}Tr\left(2\hat{\rho}\hat{\sigma}_b^{\dagger j}\hat{\eta}_a^j\hat{\sigma}_b^j - \hat{\rho}\hat{\eta}_a^j\hat{\eta}_b^j - \hat{\rho}\hat{\eta}_b^j\hat{\eta}_a^j\right).\end{aligned} \tag{2.0.23}$$

This equation can be written in simple form as

$$\frac{d}{dt}\langle\hat{\eta}_a^j(t)\rangle = -gTr\left[\hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^j + \hat{\rho}\hat{\sigma}_a^{\dagger j}\hat{a}\right] - \gamma Tr\left(\hat{\rho}\hat{\eta}_a^j\right), \tag{2.0.24}$$

or

$$\frac{d}{dt}\langle\hat{\eta}_a^j(t)\rangle = -\gamma\langle\hat{\eta}_a^j(t)\rangle - g\left[\langle\hat{a}^\dagger(t)\hat{\sigma}_a^j(t)\rangle + \langle\hat{\sigma}_a^{\dagger j}(t)\hat{a}(t)\rangle\right], \tag{2.0.25}$$

where

$$\hat{\sigma}_a^{\dagger j}\hat{\eta}_a^j = \hat{\eta}_a^j\hat{\eta}_b^j = \hat{\eta}_b^j\hat{\eta}_a^j = 0, \tag{2.0.26}$$

$$\hat{\sigma}_c^{\dagger j}\hat{\eta}_a^j\hat{\sigma}_c^j = \hat{\sigma}_b^{\dagger j}\hat{\eta}_a^j\hat{\sigma}_b^j = 0, \tag{2.0.27}$$

$$\hat{\eta}_a^j\hat{\eta}_a^j = \hat{\eta}_a^j \tag{2.0.28}$$

has been used.

In a similar procedure, we find that

$$\frac{d}{dt}\langle\hat{\sigma}_b^j(t)\rangle = -\frac{\gamma}{2}\langle\hat{\sigma}_b^j(t)\rangle + g\langle\hat{a}^\dagger(t)\hat{\sigma}_c^j(t)\rangle, \tag{2.0.29}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^j(t)\rangle = -\frac{\gamma}{2}\langle\hat{\sigma}_c^j(t)\rangle + g\langle\hat{\eta}_b^j(t)\hat{a}(t)\rangle, \tag{2.0.30}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^j(t)\rangle = -\gamma\langle\hat{\eta}_b^j\rangle + g\left[\langle\hat{a}^\dagger(t)\hat{\sigma}_a^j(t)\rangle + \langle\hat{\sigma}_a^{\dagger j}(t)\hat{a}(t)\rangle\right], \tag{2.0.31}$$

$$\frac{d}{dt}\langle\hat{\eta}_c^j(t)\rangle = \gamma\left[\langle\hat{\eta}_a^j(t)\rangle + \langle\hat{\eta}_b^j(t)\rangle\right]. \tag{2.0.32}$$

On the basis of Eq. (2.0.14) we can write

$$\frac{d}{dt}\hat{a}(t) = -\frac{\kappa}{2}\hat{a}(t) + g\hat{\sigma}_a^j(t) + \hat{g}_a(t), \tag{2.0.33}$$

where $\hat{g}_a(t)$ is the noise operator associated with the cavity mode when the cavity mode is interacting with a single two-level atom, whose correlation properties remain to be determined.

We note that Eq. (2.0.14) and the expectation value of Eq. (2.0.33) will have the same form if

$$\langle \hat{g}_a(t) \rangle = 0. \tag{2.0.34}$$

Applying the large time approximation scheme to Eq. (2.0.33), we obtain

$$\hat{a}(t) = \frac{2g}{\kappa}\hat{\sigma}_a^j(t) + \frac{2}{\kappa}\hat{g}_a(t). \tag{2.0.35}$$

Then substitution of Eq. (2.0.35) in each of Eqs. (2.0.18), (2.0.25) (2.0.29), (2.0.30) and (2.0.31) gives

$$\frac{d}{dt}\langle \hat{\sigma}_a^j(t) \rangle = -(\gamma + \frac{\gamma_c}{2})\langle \hat{\sigma}_a^j(t) \rangle + \frac{2g}{\kappa}\left[\langle \hat{\eta}_a^j(t)\hat{g}_a(t) \rangle - \langle \hat{\eta}_b^j(t)\hat{g}_a(t) \rangle\right], \tag{2.0.36}$$

$$\frac{d}{dt}\langle \hat{\sigma}_b^j(t) \rangle = -\frac{\gamma}{2}\langle \hat{\sigma}_b^j(t) \rangle + \frac{2g}{\kappa}\langle \hat{g}_a(t)\hat{\sigma}_c^j(t) \rangle, \tag{2.0.37}$$

$$\frac{d}{dt}\langle \hat{\sigma}_c^j(t) \rangle = -\gamma\langle \hat{\sigma}_c^j(t) \rangle - \frac{2g}{\kappa}\langle \hat{\sigma}_b^j(t)\hat{g}_a(t) \rangle, \tag{2.0.38}$$

$$\frac{d}{dt}\langle \hat{\eta}_a^j(t) \rangle = -(\gamma + \gamma_c)\langle \hat{\eta}_a^j(t) \rangle - \frac{2g}{\kappa}[\langle \hat{g}_a^\dagger(t)\hat{\sigma}_a^j(t) \rangle + \langle \hat{\sigma}_a^{\dagger j}(t)\hat{g}_a(t)], \tag{2.0.39}$$

$$\frac{d}{dt}\langle \hat{\eta}_b^j(t) \rangle = -\gamma\langle \hat{\eta}_b^j(t) \rangle + \gamma_c\langle \hat{\eta}_a^j(t) \rangle + \frac{2g}{\kappa}[\langle \hat{g}_a^\dagger(t)\hat{\sigma}_a^j(t) \rangle + \langle \hat{\sigma}_a^{\dagger j}(t)\hat{g}_a(t)], \tag{2.0.40}$$

where

$$\gamma_c = \frac{4g^2}{\kappa} \tag{2.0.41}$$

is the rate of stimulated emission [1].

We now seek to evaluate the expectation values of the product of atomic operators and cavity mode noise operators. To this end we write on the basis of Eq. (2.0.18), that

$$\frac{d}{dt}\hat{\sigma}_a^j(t) = -\gamma\hat{\sigma}_a^j(t) + g[\hat{\eta}_a^j(t)\hat{a}(t) - \hat{\eta}_b^j(t)\hat{a}(t)] + \hat{f}_a(t), \tag{2.0.42}$$

where $\hat{f}_a(t)$ is the noise operator with vanishing mean corresponding to atomic operator $\hat{\sigma}_a^j(t)$.

Then using the relation

$$\frac{d}{dt}\langle\hat{a}(t)\hat{\sigma}_a^j(t)\rangle = \left\langle\frac{d\hat{a}(t)}{dt}\hat{\sigma}_a^j(t)\right\rangle + \left\langle\hat{a}(t)\frac{d\hat{\sigma}_a^j(t)}{dt}\right\rangle, \tag{2.0.43}$$

along with Eq. (2.0.33) and (2.0.42), one can verify that

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\hat{\sigma}_a^j(t)\rangle = &-(\frac{\kappa}{2}+\gamma)\langle\hat{a}(t)\hat{\sigma}_a^j(t)\rangle + \langle\hat{g}_a(t)\hat{\sigma}_a^j(t)\rangle + \langle\hat{a}(t)\hat{f}_a(t)\rangle\\ &+ g[\langle\hat{\eta}_a^j(t)\hat{a}^2(t)\rangle - \langle\hat{\eta}_b^j(t)\hat{a}^2(t)\rangle],\end{aligned} \tag{2.0.44}$$

On the other hand, employing the relation

$$\frac{d}{dt}\langle\hat{a}(t)\hat{\sigma}_a^j(t)\rangle = Tr\left(\frac{d\hat{\rho}}{dt}\hat{a}(t)\hat{\sigma}_a^j(t)\right), \tag{2.0.45}$$

along with the Master equation, we arrive at

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}\hat{\sigma}_a^j\rangle = &\, gTr[\hat{a}^\dagger\hat{\sigma}_a^j\hat{\rho}\hat{a}\hat{\sigma}_a^j - \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^j\hat{a}\hat{\sigma}_a^j - \hat{\sigma}_a^{\dagger j}\hat{a}\hat{\rho}\hat{a}\hat{\sigma}_a^j + \hat{\rho}\hat{\sigma}_a^{\dagger j}\hat{a}^2\hat{\sigma}_a^j]\\ &+\frac{\kappa}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\hat{\sigma}_a^j - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}\hat{\sigma}_a^j - \hat{\rho}\hat{a}^\dagger\hat{a}^2\hat{\sigma}_a^j)\\ &+\frac{\gamma}{2}Tr(2\hat{\sigma}_c^j\hat{\rho}\hat{\sigma}_c^{\dagger j}\hat{a}\hat{\sigma}_a^j - \hat{\eta}_a^j\hat{\rho}\hat{a}\hat{\sigma}_a^j - \hat{\rho}\hat{\eta}_a^j\hat{a}\hat{\sigma}_a^j)\\ &+\frac{\gamma}{2}Tr(2\hat{\sigma}_b^j\hat{\rho}\hat{\sigma}_b^{\dagger j}\hat{a}\hat{\sigma}_a^j - \hat{\eta}_b^j\hat{\rho}\hat{a}\hat{\sigma}_a^j - \hat{\rho}\hat{\eta}_b^j\hat{a}\hat{\sigma}_a^j).\end{aligned} \tag{2.0.46}$$

Applying the cyclic properties of trace operators, one can readily obtain

$$\frac{d}{dt}\langle\hat{a}(t)\hat{\sigma}_a^j(t)\rangle = -(\frac{\kappa}{2}+\gamma)\langle\hat{a}(t)\hat{\sigma}_a^j(t)\rangle + g[\langle\hat{\eta}_a^j(t)\hat{a}^2(t)\rangle - \langle\hat{\eta}_b^j(t)\hat{a}^2(t)\rangle]. \tag{2.0.47}$$

Now comparison of Eqs. (2.0.44) and (2.0.47) shows that

$$\langle\hat{g}_a(t)\hat{\sigma}_a^j(t)\rangle + \langle\hat{a}(t)\hat{f}_a(t)\rangle = 0. \tag{2.0.48}$$

Furthermore, up on multiplying the solution of Eq. (2.0.33) by $\hat{f}_a(t)$ from the right and taking the expectation value of the resulting equation, we obtain

$$\begin{aligned}\langle\hat{a}(t)\hat{f}_a(t)\rangle = &\langle\hat{a}(0)\hat{f}_a(t)\rangle e^{-\frac{\kappa}{2}t} + \int_0^t e^{\frac{-\kappa}{2}(t-t')}\langle\hat{\sigma}_a^j(t')\hat{f}_a(t)\rangle dt'\\ &+\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a(t')\hat{f}_a(t)\rangle dt'.\end{aligned} \tag{2.0.49}$$

On account of the fact that, the noise operator at a given time does not affect the atomic or cavity mode operator at earlier time, Eq. (2.0.49) turns out to be

$$\langle \hat{a}(t)\hat{f}_a(t)\rangle = \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle \hat{g}_a(t')\hat{f}_a(t)\rangle dt'. \quad (2.0.50)$$

Similarly, by multiplying the solution of Eq. (2.0.42) by $\hat{g}_{aj}(t)$ from the left and taking the expectation value of the resulting equation, we obtain

$$\langle \hat{g}_a(t)\hat{\sigma}_a^j(t)\rangle = \int_0^t e^{-(\gamma+\frac{\gamma_c}{2})(t-t')}\langle \hat{g}_a(t)\hat{f}_a(t')\rangle dt'. \quad (2.0.51)$$

Assuming the cavity mode noise operators and atomic noise operators are independent and do not correlate to each other, we can write

$$\langle \hat{g}_a(t')\hat{f}_a(t)\rangle = \langle \hat{g}_a(t')\rangle\langle \hat{f}_a(t)\rangle = 0, \quad (2.0.52)$$

and

$$\langle \hat{g}_a(t)\hat{f}_a(t')\rangle = \langle \hat{g}_a(t)\rangle\langle \hat{f}_a(t')\rangle = 0. \quad (2.0.53)$$

In view of these results, Eqs. (2.0.50) and (2.0.51) reduces to

$$\langle \hat{a}(t)\hat{f}_a(t)\rangle = 0, \quad (2.0.54)$$

and

$$\langle \hat{g}_a(t)\hat{\sigma}_a^j(t)\rangle = 0. \quad (2.0.55)$$

Following a similar procedure one can readily obtain that

$$\langle \hat{\sigma}_a^j(t)\hat{g}_a(t)\rangle = 0, \quad (2.0.56)$$

$$\langle \hat{g}_a^\dagger(t)\hat{\sigma}_a^j(t)\rangle = 0, \quad (2.0.57)$$

$$\langle \hat{\sigma}_a^{\dagger j}(t)\hat{g}_a(t)\rangle = 0, \quad (2.0.58)$$

$$\langle \hat{\sigma}_b^j(t)\hat{g}_a(t)\rangle = 0, \tag{2.0.59}$$

$$\langle \hat{\sigma}_b^{\dagger j}(t)\hat{g}_a(t)\rangle = 0, \tag{2.0.60}$$

$$\langle \hat{g}_a(t)\hat{\sigma}_c^j(t)\rangle = 0, \tag{2.0.61}$$

$$\langle \hat{\eta}_a^j(t)\hat{g}_a(t)\rangle = 0, \tag{2.0.62}$$

$$\langle \hat{\eta}_b^j(t)\hat{g}_a(t)\rangle = 0. \tag{2.0.63}$$

From these results we may notice that the cavity mode noise operator $\hat{g}_a(t)$ and atomic operators do not correlate to each other. With these outputs of the correlation properties of the product of atomic operators and the cavity mode noise operators, we reset Eqs. (2.0.32), (2.0.36), (2.0.37), (2.0.38), (2.0.39) and (2.0.40) in the form:

$$\frac{d}{dt}\langle \hat{\sigma}_a^j(t)\rangle = -(\gamma + \frac{\gamma_c}{2})\langle \hat{\sigma}_a^j(t)\rangle, \tag{2.0.64}$$

$$\frac{d}{dt}\langle \hat{\sigma}_b^j(t)\rangle = -\frac{\gamma}{2}\langle \hat{\sigma}_b^j(t)\rangle, \tag{2.0.65}$$

$$\frac{d}{dt}\langle \hat{\sigma}_c^j(t)\rangle = -\gamma\langle \hat{\sigma}_c^j(t)\rangle, \tag{2.0.66}$$

$$\frac{d}{dt}\langle \hat{\eta}_a^j(t)\rangle = -(\gamma + \gamma_c)\langle \hat{\eta}_a^j(t)\rangle, \tag{2.0.67}$$

$$\frac{d}{dt}\langle \hat{\eta}_b^j(t)\rangle = -\gamma\langle \hat{\eta}_b^j(t)\rangle + \gamma_c\langle \hat{\eta}_a^j(t)\rangle, \tag{2.0.68}$$

$$\frac{d}{dt}\langle \hat{\eta}_c^j(t)\rangle = \gamma\left(\langle \hat{\eta}_b^j(t)\rangle + \langle \hat{\eta}_a^j(t)\rangle\right). \tag{2.0.69}$$

We now proceed to evaluate the correlation properties of the cavity mode noise operator as well as the atomic noise operators. Employing the relation

$$\frac{d}{dt}\langle \hat{a}(t)\hat{a}(t)\rangle = Tr\left(\frac{d\hat{\rho}}{dt}\hat{a}(t)\hat{a}(t)\right), \tag{2.0.70}$$

along with the Master equation described by Eq. (2.0.3), we obtain

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\hat{a}(t)\rangle &= Tr\left(\frac{d\hat{\rho}}{dt}\hat{a}(t)\hat{a}(t)\right)\\ &= gTr\left(\hat{a}^{\dagger}\hat{\sigma}_a^j\hat{\rho}\hat{a}^2-\hat{\rho}\hat{a}^{\dagger}\hat{\sigma}_a^j\hat{a}^2-\hat{\sigma}_a^{\dagger j}\hat{a}\hat{\rho}\hat{a}^2+\hat{\rho}\hat{\sigma}_a^{\dagger j}\hat{a}^3\right)\\ &+\frac{\kappa}{2}Tr\left(2\hat{a}\hat{\rho}\hat{a}^{\dagger}\hat{a}^2-\hat{a}^{\dagger}\hat{a}\hat{\rho}\hat{a}^2-\hat{\rho}\hat{a}^{\dagger}\hat{a}^3\right)\\ &+\frac{\gamma}{2}Tr\left(2\hat{\sigma}_c^j\hat{\rho}\hat{\sigma}_c^{\dagger j}\hat{a}^2-\hat{\eta}_a^j\hat{\rho}\hat{a}^2-\hat{\rho}\hat{\eta}_a^j\hat{a}^2\right)\\ &+\frac{\gamma}{2}Tr\left(2\hat{\sigma}_b^j\hat{\rho}\hat{\sigma}_b^{\dagger j}\hat{a}^2-\hat{\eta}_b^j\hat{\rho}\hat{a}^2-\hat{\rho}\hat{\eta}_b^j\hat{a}^2\right).\end{aligned} \tag{2.0.71}$$

Applying the cyclic properties of trace operators in Eq. (2.0.71) we get

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\hat{a}(t)\rangle &= gTr\left(\hat{\rho}\hat{a}^2\hat{a}^{\dagger}\hat{\sigma}_a^j-\hat{\rho}\hat{a}^{\dagger}\hat{a}^2\hat{\sigma}_a^j\right)\\ &+\frac{\kappa}{2}Tr\left(2\hat{\rho}\hat{a}^{\dagger}\hat{a}^3-\hat{\rho}\hat{a}^2\hat{a}^{\dagger}\hat{a}-\hat{\rho}\hat{a}^{\dagger}\hat{a}^3\right)\\ &+\frac{\gamma}{2}Tr\left(2\hat{\rho}\hat{a}^2\hat{\sigma}_c^{\dagger j}\hat{\sigma}_c^j-\hat{\rho}\hat{a}^2\hat{\eta}_a^j-\hat{\rho}\hat{a}^2\hat{\eta}_a^j\right)\\ &+\frac{\gamma}{2}Tr\left(2\hat{\rho}\hat{a}^2\hat{\sigma}_b^{\dagger j}\hat{\sigma}_b^j-\hat{\rho}\hat{a}^2\hat{\eta}_b^j-\hat{\rho}\hat{a}^2\hat{\eta}_b^j\right)\end{aligned} \tag{2.0.72}$$

In simple form

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\hat{a}(t)\rangle &= gTr\left(\hat{\rho}\hat{a}(1+\hat{a}^{\dagger}\hat{a})\hat{\sigma}_a^j-\hat{\rho}(\hat{a}\hat{a}^{\dagger}-1)\hat{a}\hat{\sigma}_a^j\right)\\ &+\frac{\kappa}{2}Tr\left(\hat{\rho}(\hat{a}\hat{a}^{\dagger}-1)\hat{a}\hat{a}-\hat{\rho}\hat{a}(1+\hat{a}^{\dagger}\hat{a})\hat{a}\right)\\ &+\frac{\gamma}{2}Tr\left(2\hat{\rho}\hat{a}\hat{a}\hat{\eta}_a^j-\hat{\rho}\hat{a}\hat{a}\hat{\eta}_a^j-\hat{\rho}\hat{a}\hat{a}\hat{\eta}_a^j\right)\\ &+\frac{\gamma}{2}Tr\left(2\hat{\rho}\hat{a}\hat{a}\hat{\eta}_b^j-\hat{\rho}\hat{a}\hat{a}\hat{\eta}_b^j-\hat{\rho}\hat{a}\hat{a}\hat{\eta}_b^j\right).\\ &= gTr\left(\hat{\rho}\hat{a}\hat{\sigma}_a^j+\hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a}\hat{\sigma}_a^j-\hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a}\hat{\sigma}_a^j+\hat{\rho}\hat{a}\hat{\sigma}_a^j\right)\\ &+\frac{\kappa}{2}Tr\left(\hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a}\hat{a}-\hat{\rho}\hat{a}\hat{a}-\hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a}\hat{a}-\hat{\rho}\hat{a}\hat{a}\right).\end{aligned} \tag{2.0.73}$$

Simplifying this equation leads to

$$\frac{d}{dt}\langle\hat{a}(t)\hat{a}(t)\rangle = -\kappa Tr\left(\hat{\rho}\hat{a}(t)\hat{a}(t)\right)+2gTr\left(\hat{\rho}\hat{a}(t)\hat{\sigma}_a^j(t)\right), \tag{2.0.74}$$

or

$$\frac{d}{dt}\langle\hat{a}(t)\hat{a}(t)\rangle = -\kappa\langle\hat{a}(t)\hat{a}(t)\rangle+2g\langle\hat{a}(t)\hat{\sigma}_a^j(t)\rangle. \tag{2.0.75}$$

On the other hand, using the relation

$$\frac{d}{dt}\langle\hat{a}(t)\hat{a}(t)\rangle = \left\langle\frac{d\hat{a}(t)}{dt}\hat{a}(t)\right\rangle + \left\langle\hat{a}(t)\frac{d\hat{a}(t)}{dt}\right\rangle, \tag{2.0.76}$$

along with Eq. (2.0.33) we arrive at

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\hat{a}(t)\rangle &= -\frac{\kappa}{2}\langle\hat{a}(t)\hat{a}(t)\rangle + g\langle\hat{a}(t)\hat{\sigma}_a^j(t)\rangle + \langle\hat{g}_a(t)\hat{a}(t)\rangle \\ &\quad - \frac{\kappa}{2}\langle\hat{a}(t)\hat{a}(t)\rangle + g\langle\hat{a}(t)\hat{\sigma}_a^j(t)\rangle + \langle\hat{a}(t)\hat{g}_a(t)\rangle \\ &= -\kappa\langle\hat{a}(t)\hat{a}(t)\rangle + 2g\langle\hat{a}(t)\hat{\sigma}_a^j(t)\rangle + (\langle\hat{g}_a(t)\hat{a}(t)\rangle + \langle\hat{a}(t)\hat{g}_a(t)\rangle).\end{aligned} \tag{2.0.77}$$

Upon comparing Eqs. (2.0.75) and (2.0.77), we realize that

$$\langle\hat{g}_a(t)\hat{a}(t)\rangle + \langle\hat{a}(t)\hat{g}_a(t)\rangle = 0. \tag{2.0.78}$$

A formal solution of Eq. (2.0.33) is expressible as

$$\hat{a}(t) = \hat{a}(0)e^{-\frac{\kappa}{2}t} + \int_0^t e^{\frac{-\kappa}{2}(t-t')}(g\hat{\sigma}_a^j(t') + \hat{g}_a(t'))dt'. \tag{2.0.79}$$

Upon multiplying Eq.(2.0.79) by $\hat{g}_a(t)$ from the left and taking the expectation value of the resulting equation gives

$$\begin{aligned}\langle\hat{g}_a(t)\hat{a}(t)\rangle &= \langle\hat{g}_a(t)\hat{a}(0)\rangle e^{-\frac{\kappa}{2}t} \\ &\quad + \int_0^t e^{-\frac{\kappa}{2}(t-t')}(g\langle\hat{g}_a(t)\hat{\sigma}_a^j(t')\rangle + \langle\hat{g}_a(t)\hat{g}_a(t')\rangle)dt'.\end{aligned} \tag{2.0.80}$$

Here the noise operator at a certain time should not affect the cavity mode operator at earlier time. It then follows that

$$\langle\hat{g}_a(t)\hat{a}(0)\rangle = 0. \tag{2.0.81}$$

In view of Eqs. (2.0.55) and (2.0.81), we can put Eq. (2.0.80) in the form

$$\langle\hat{g}_a(t)\hat{a}(t)\rangle = \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a(t)\hat{g}_a(t')\rangle dt'. \tag{2.0.82}$$

Moreover, multiplying Eq.(2.0.79) by $\hat{g}_a(t)$ from the right and taking into account the assertion that a noise operator at some time should not affect the atomic operator at earlier time, we obtain

$$\langle \hat{a}(t)\hat{g}_a(t)\rangle = \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle \hat{g}_a(t')\hat{g}_a(t)\rangle dt'. \tag{2.0.83}$$

Combining Eq. (2.0.82) and (2.0.83) in Eq. (2.0.78), leads to

$$\int_0^t e^{\frac{-\kappa}{2}(t-t')}\langle \hat{g}_a(t)\hat{g}_a(t')\rangle dt' + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle \hat{g}_a(t')\hat{g}_a(t)\rangle dt' = 0, \tag{2.0.84}$$

and assuming that

$$\langle \hat{g}_a(t)\hat{g}_a(t')\rangle = \langle \hat{g}_a(t')\hat{g}_a(t)\rangle, \tag{2.0.85}$$

we have

$$2\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle \hat{g}_a(t)\hat{g}_a(t')\rangle dt' = 0. \tag{2.0.86}$$

Here on the basis of this result, we assert that

$$\langle \hat{g}_a(t)\hat{g}_a(t')\rangle = 0. \tag{2.0.87}$$

Furthermore, employing the relation

$$\frac{d}{dt}\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle = Tr\left(\frac{d\hat{\rho}}{dt}\hat{a}^\dagger(t)\hat{a}(t)\right), \tag{2.0.88}$$

along with Eq. (2.0.3), we find that

$$\begin{aligned}\frac{d}{dt}\langle \hat{a}^\dagger\hat{a})\rangle = {} & gTr\left(\hat{a}^\dagger\hat{\sigma}_a^j\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^j\hat{a}^\dagger\hat{a} - \hat{\sigma}_a^{\dagger j}\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} + \hat{\rho}\hat{\sigma}_a^{\dagger j}\hat{a}\hat{a}^\dagger\hat{a}\right)\\ & + \frac{\kappa}{2}Tr\left(2\hat{a}\hat{\rho}\hat{a}^{\dagger 2}\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\right)\\ & + \frac{\gamma}{2}Tr\left(2\hat{\sigma}_c^j\hat{\rho}\hat{\sigma}_c^{\dagger j}\hat{a}^\dagger\hat{a} - \hat{\eta}_a^j\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{\eta}_a^j\hat{a}^\dagger\hat{a}\right)\\ & + \frac{\gamma}{2}Tr\left(2\hat{\sigma}_b^j\hat{\rho}\hat{\sigma}_b^{\dagger j}\hat{a}^\dagger\hat{a} - \hat{\eta}_b^j\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{\eta}_b^j\hat{a}^\dagger\hat{a}\right).\end{aligned} \tag{2.0.89}$$

Employing the cyclic properties of trace operators in Eq. (2.0.89), follows that

$$\begin{aligned}\frac{d}{dt}\langle \hat{a}^\dagger \hat{a})\rangle &= gTr\left(\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\sigma}_a^j - \hat{\rho}\hat{a}^{\dagger 2}\hat{a}\hat{\sigma}_a^j - \hat{\rho}\hat{a}^\dagger\hat{a}^2\hat{\sigma}_a^{\dagger j} + \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}\hat{\sigma}_a^{\dagger j}\right)\\ &+ \frac{\kappa}{2}Tr\left(2\hat{\rho}\hat{a}^{\dagger 2}\hat{a}^2 - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\right)\\ &+ \frac{\gamma}{2}Tr\left(2\hat{\rho}\hat{a}^\dagger\hat{a}\hat{\sigma}_c^{\dagger j}\hat{\sigma}_c^j - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{\eta}_a^j - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{\eta}_a^j\right)\\ &+ \frac{\gamma}{2}Tr\left(2\hat{\rho}\hat{a}^\dagger\hat{a}\hat{\sigma}_b^{\dagger j}\hat{\sigma}_b^j - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{\eta}_b^j - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{\eta}_b^j\right)\\ &= gTr\left(\hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^j + \hat{\rho}\hat{\sigma}_a^{\dagger j}\hat{a}\right) - \kappa Tr\left(\hat{\rho}\hat{a}^\dagger\hat{a}\right),\end{aligned} \tag{2.0.90}$$

or

$$\frac{d}{dt}\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle = -\kappa\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle + g\left(\langle \hat{a}^\dagger(t)\hat{\sigma}_a^j(t)\rangle + \langle \hat{\sigma}_a^{\dagger j}(t)\hat{a}(t)\rangle\right). \tag{2.0.91}$$

On the other hand, using the relation

$$\frac{d}{dt}\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle = \left\langle \frac{d\hat{a}^\dagger(t)}{dt}\hat{a}(t)\right\rangle + \left\langle \hat{a}^\dagger(t)\frac{d\hat{a}(t)}{dt}\right\rangle, \tag{2.0.92}$$

along with Eq. (2.0.33) and its conjugate, we obtain

$$\begin{aligned}\frac{d}{dt}\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle &= -\kappa\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle + g\left(\langle \hat{a}^\dagger(t)\hat{\sigma}_a^j(t)\rangle + \langle \hat{\sigma}^{\dagger j}(t)\hat{a}(t)\rangle\right)\\ &+ \left(\langle \hat{g}_a^\dagger(t)\hat{a}(t)\rangle + \langle \hat{a}^\dagger(t)\hat{g}_a(t)\rangle\right).\end{aligned} \tag{2.0.93}$$

Comparison of Eq. (2.0.91) and (2.0.93), leads to

$$\langle \hat{g}_a^\dagger(t)\hat{a}(t)\rangle + \langle \hat{a}^\dagger(t)\hat{g}_a(t)\rangle = 0. \tag{2.0.94}$$

Multiplying Eq. (2.0.79) by $\hat{g}_a^\dagger(t)$ from the left and taking the expectation value of the resulting equation, we find that

$$\begin{aligned}\langle \hat{g}_a^\dagger(t)\hat{a}(t)\rangle &= \langle \hat{g}_a^\dagger(t)\hat{a}(0)\rangle e^{-\frac{\kappa}{2}t} + g\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle \hat{g}_a^\dagger(t)\hat{\sigma}_a^j(t')\rangle dt'\\ &+ \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle \hat{g}_a^\dagger(t)\hat{g}_a(t')\rangle dt'.\end{aligned} \tag{2.0.95}$$

Since the noise operator at a certain time should not affect the cavity mode operator at earlier time, one can write

$$\langle \hat{g}_a^\dagger(t)\hat{a}(0)\rangle = \langle \hat{g}_a^\dagger(t)\rangle\langle \hat{a}(0)\rangle = 0. \tag{2.0.96}$$

In view of Eq. (2.0.57) and (2.0.96), Eq. (2.0.95) reduces to

$$\langle\hat{g}_a^\dagger(t)\hat{a}(t)\rangle = \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a^\dagger(t)\hat{g}_a(t')\rangle dt'. \tag{2.0.97}$$

Following the same procedure one can verify that

$$\langle\hat{a}^\dagger(t)\hat{g}_a(t)\rangle = \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a^\dagger(t')\hat{g}_a(t)\rangle dt'. \tag{2.0.98}$$

Taking into account Eqs. (2.0.97) and (2.0.98) along with Eq. (2.0.94), we see that

$$\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a^\dagger(t)\hat{g}_a(t')\rangle dt' + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a^\dagger(t')\hat{g}_a(t)\rangle dt' = 0, \tag{2.0.99}$$

and assuming

$$\langle\hat{g}_a^\dagger(t)\hat{g}_a(t')\rangle = \langle\hat{g}_a^\dagger(t')\hat{g}_a(t)\rangle,$$

it then follows that

$$2\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a^\dagger(t)\hat{g}_a(t')\rangle dt' = 0. \tag{2.0.100}$$

On the basis of Eq. (2.0.100), we assert that

$$\langle\hat{g}_a^\dagger(t)\hat{g}_a(t')\rangle = 0. \tag{2.0.101}$$

Moreover, employing the relation

$$\frac{d}{dt}\langle\hat{a}(t)\hat{a}^\dagger(t)\rangle = Tr\left(\frac{d\hat{\rho}}{dt}\hat{a}(t)\hat{a}^\dagger(t)\right), \tag{2.0.102}$$

along with the Master equation we find that

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}\hat{a}^\dagger\rangle &= gTr\left(\hat{a}^\dagger\sigma_a^j\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^j\hat{a}\hat{a}^\dagger - \hat{\sigma}_a^{\dagger j}\hat{a}\hat{\rho}\hat{a}\hat{a}^\dagger + \hat{\rho}\hat{\sigma}_a^{\dagger j}\hat{a}\hat{a}\hat{a}^\dagger\right)\\ &+ \frac{\kappa}{2}Tr\left(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}\hat{a}^\dagger\right)\\ &+ \frac{\gamma}{2}Tr\left(2\hat{\sigma}_c^j\hat{\rho}\hat{\sigma}_c^{\dagger j}\hat{a}\hat{a}^\dagger - \hat{\eta}_a^j\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{\eta}_a^j\hat{a}\hat{a}^\dagger\right)\\ &+ \frac{\gamma}{2}Tr\left(2\hat{\sigma}_b^j\hat{\rho}\hat{\sigma}_b^{\dagger j}\hat{a}\hat{a}^\dagger - \hat{\eta}_b^j\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{\eta}_b^j\right).\end{aligned} \tag{2.0.103}$$

Similarly, applying the cyclic property of trace operator in this equation, we see that

$$\frac{d}{dt}\langle\hat{a}(t)\hat{a}^{\dagger}(t)\rangle = -\frac{\kappa}{2}Tr\left(\hat{\rho}\hat{a}(t)\hat{a}^{\dagger}(t) - \hat{\rho}\right) + gTr\left(\hat{\rho}\hat{a}^{\dagger}(t)\hat{\sigma}_a^j(t) + \hat{\rho}\hat{\sigma}_a^{\dagger j}(t)\hat{a}(t)\right), \quad (2.0.104)$$

or

$$\frac{d}{dt}\langle\hat{a}(t)\hat{a}^{\dagger}(t)\rangle = -\kappa\langle\hat{\rho}\hat{a}(t)\hat{a}^{\dagger}(t)\rangle + g\left(\langle\hat{\rho}\hat{a}^{\dagger}(t)\hat{\sigma}_a^j(t)\rangle + \langle\hat{\rho}\hat{\sigma}_a^{\dagger j}(t)\hat{a}(t)\rangle\right) + \kappa. \quad (2.0.105)$$

On the other hand, using the relation

$$\frac{d}{dt}\langle\hat{a}(t)\hat{a}^{\dagger}(t)\rangle = \left\langle\frac{d\hat{a}(t)}{dt}\hat{a}^{\dagger}(t)\right\rangle + \left\langle\hat{a}(t)\frac{d\hat{a}^{\dagger}(t)}{dt}\right\rangle, \quad (2.0.106)$$

along with Eq. (2.0.33) and its conjugate, we find that

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\hat{a}^{\dagger}(t)\rangle &= -\kappa\langle\hat{a}(t)\hat{a}^{\dagger}(t)\rangle + g\left(\langle\hat{a}^{\dagger}(t)\hat{\sigma}_a^j(t)\rangle + \langle\hat{\sigma}_a^{\dagger j}(t)\hat{a}(t)\rangle\right)\\ &\quad + \left(\langle\hat{g}_a(t)\hat{a}^{\dagger}(t)\rangle + \langle\hat{a}(t)\hat{g}_a^{\dagger}(t)\rangle\right).\end{aligned} \quad (2.0.107)$$

Comparing Eqs. (2.0.105) and (2.0.107), we find

$$\langle\hat{g}_a(t)\hat{a}^{\dagger}(t)\rangle + \langle\hat{a}(t)\hat{g}_a^{\dagger}(t)\rangle = \kappa. \quad (2.0.108)$$

Multiplying the conjugate of Eq.(2.0.79) by $\hat{g}_a(t)$ from the left and taking the expectation value of the resulting equation, we obtain that

$$\begin{aligned}\langle\hat{g}_a(t)\hat{a}^{\dagger}(t)\rangle &= \langle\hat{g}_a(t)\hat{a}^{\dagger}(0)\rangle e^{-\frac{\kappa}{2}t}\\ &\quad + g\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a(t)\hat{\sigma}_a^{\dagger j}(t')\rangle dt'\\ &\quad + \int_0^t e^{-\frac{k}{2}(t-t')}\langle\hat{g}_a(t)\hat{g}_a^{\dagger}(t')\rangle dt'.\end{aligned} \quad (2.0.109)$$

In view of Eq. (2.0.56) and the fact that the noise operator at a certain time should not affect the cavity mode operator at earlier times, Eq. (2.0.109) reduces to

$$\langle\hat{g}_a(t)\hat{a}^{\dagger}(t)\rangle = \int_0^t e^{-\frac{k}{2}(t-t')}\langle\hat{g}_a(t)\hat{g}_a^{\dagger}(t')\rangle dt'. \quad (2.0.110)$$

Following a similar procedure, we see that

$$\langle \hat{a}(t)\hat{g}_a^\dagger(t)\rangle = \int_0^t e^{-\frac{k}{2}(t-t')}\langle \hat{g}_a(t')\hat{g}_a^\dagger(t)\rangle dt'. \tag{2.0.111}$$

On account of Eqs. (2.0.110) and (2.0.111), we can put Eq. (2.0.108) in the form

$$\int_0^t e^{-\frac{k}{2}(t-t')}\langle \hat{g}_a(t)\hat{g}_a^\dagger(t')\rangle dt' + \int_0^t e^{-\frac{k}{2}(t-t')}\langle \hat{g}_a(t')\hat{g}_a^\dagger(t)\rangle dt' = \kappa. \tag{2.0.112}$$

Now by assuming

$$\langle \hat{g}_a(t)\hat{g}_a^\dagger(t')\rangle = \langle \hat{g}_a(t')\hat{g}_a^\dagger(t)\rangle, \tag{2.0.113}$$

we note that

$$2\int_0^t e^{-\frac{k}{2}(t-t')}\langle \hat{g}_a(t)\hat{g}_a^\dagger(t')\rangle dt' = \kappa, \tag{2.0.114}$$

On the basis of which we may assert that

$$\langle \hat{g}_a(t)\hat{g}_a^\dagger(t')\rangle = \kappa\delta(t-t'). \tag{2.0.115}$$

Now we want to extend our analysis to the dynamics of N two-level atoms available in the cavity. To this end we sum up Eqs. (2.0.64), (2.0.65), (2.0.66), (2.0.67), (2.0.68) and (2.0.69) over the given N two-level atoms. It then follows that we obtain

$$\frac{d}{dt}\langle \hat{m}_a(t)\rangle = -(\gamma + \frac{\gamma_c}{2})\langle \hat{m}_a(t)\rangle, \tag{2.0.116}$$

$$\frac{d}{dt}\langle \hat{m}_b(t)\rangle = -\frac{\gamma}{2}\langle \hat{m}_b(t)\rangle, \tag{2.0.117}$$

$$\frac{d}{dt}\langle \hat{m}_c(t)\rangle = -\gamma\langle \hat{m}_c(t)\rangle, \tag{2.0.118}$$

$$\frac{d}{dt}\langle \hat{N}_a(t)\rangle = -(\gamma + \gamma_c)\langle \hat{N}_a(t)\rangle, \tag{2.0.119}$$

$$\frac{d}{dt}\langle \hat{N}_b(t)\rangle = -\gamma\langle \hat{N}_b(t)\rangle + \gamma_c\langle \hat{N}_a(t)\rangle, \tag{2.0.120}$$

$$\frac{d}{dt}\langle \hat{N}_c(t)\rangle = \gamma(\langle \hat{N}_b(t)\rangle + \langle \hat{N}_a(t)\rangle), \tag{2.0.121}$$

where

$$\hat{m}_a = \sum_{j=1}^{N} \hat{\sigma}_a^j = N\hat{\sigma}_a = N|b\rangle\langle a|, \tag{2.0.122}$$

$$\hat{m}_b = \sum_{j=1}^{N} \hat{\sigma}_b^j = N\hat{\sigma}_b = N|c\rangle\langle b|, \tag{2.0.123}$$

$$\hat{m}_c = \sum_{j=1}^{N} \hat{\sigma}_c^j = N\hat{\sigma}_c = N|c\rangle\langle a|, \tag{2.0.124}$$

$$\hat{N}_a = \sum_{j=1}^{N} \hat{\eta}_a^j = N\hat{\eta}_a = N|a\rangle\langle a|, \tag{2.0.125}$$

$$\hat{N}_b = \sum_{j=1}^{N} \hat{\eta}_b^j = N\hat{\eta}_b = N|b\rangle\langle b|, \tag{2.0.126}$$

$$\hat{N}_c = \sum_{j=1}^{N} \hat{\eta}_c^j = N\hat{\eta}_c = N|c\rangle\langle c|, \tag{2.0.127}$$

in which $\hat{N}_a$, $\hat{N}_b$ and $\hat{N}_c$ are atomic operators representing the number of atoms in the upper, lower and ground levels, respectively.

Moreover, in the presence of N two-level atoms in the cavity, we can write Eq. (2.0.33) in the form [1]

$$\frac{d}{dt}\hat{a}(t) = -\frac{\kappa}{2}\hat{a}(t) + \lambda\hat{m}_a(t) + \beta\hat{g}_a(t), \tag{2.0.128}$$

where λ and β are constants to be determined.

Taking Eqn. (2.0.35) into consideration, we see that

$$\begin{aligned} [\hat{a}, \hat{a}^\dagger]_j &= \frac{4g^2}{\kappa^2}\left[\hat{\sigma}_a^j, \hat{\sigma}_a^{\dagger j}\right] + \frac{4}{\kappa^2}[\hat{g}_a(t), \hat{g}_a^\dagger(t)] \\ &= \frac{4g^2}{\kappa^2}\left(\hat{\eta}_b^j - \hat{\eta}_a^j\right) + \frac{4}{\kappa^2}[\hat{g}_a(t), \hat{g}_a^\dagger(t)]. \end{aligned} \tag{2.0.129}$$

For N two-level atoms in the cavity we sum up Eq. (2.0.129) and obtain

$$[\hat{a}, \hat{a}^\dagger] = \frac{4g^2}{\kappa^2}\left(\hat{N}_b - \hat{N}_a\right) + \frac{4N}{\kappa^2}\left[\hat{g}_a(t), \hat{g}_a^\dagger(t)\right]. \tag{2.0.130}$$

On the other hand, writing the steady-state solution of Eq. (2.0.128) as

$$\hat{a}(t) = \frac{2\lambda}{\kappa}\hat{m}_a(t) + \frac{2\beta}{\kappa}\hat{g}_a(t), \tag{2.0.131}$$

we see that

$$\begin{aligned}[\hat{a}, \hat{a}^\dagger] &= \frac{4\lambda^2}{\kappa^2}\left[\hat{m}_a, \hat{m}_a^\dagger\right] + \frac{4\beta^2}{\kappa^2}\left[\hat{g}_a(t), \hat{g}_a^\dagger(t)\right] \\ &= \frac{4\lambda^2}{\kappa^2}\left(\hat{m}_a\hat{m}_a^\dagger - \hat{m}_a^\dagger\hat{m}_a\right) + \frac{4\beta^2}{\kappa^2}\left[\hat{g}_a(t), \hat{g}_a^\dagger(t)\right],\end{aligned} \tag{2.0.132}$$

and with the aid of Eq. (2.0.122), it follows that

$$[\hat{a}, \hat{a}^\dagger] = \frac{4\lambda^2}{\kappa^2}\left(N^2|b\rangle\langle a|a\rangle\langle b| - N^2|a\rangle\langle b|b\rangle\langle a|\right) + \frac{4\beta^2}{\kappa^2}\left[\hat{g}_a(t), \hat{g}_a^\dagger(t)\right], \tag{2.0.133}$$

or

$$[\hat{a}, \hat{a}^\dagger] = \frac{4N\lambda^2}{\kappa^2}\left(\hat{N}_b - \hat{N}_a\right) + \frac{4\beta^2}{\kappa^2}\left[\hat{g}_a(t), \hat{g}_a^\dagger(t)\right]. \tag{2.0.134}$$

Upon comparing Eq. (2.0.130) and (2.0.134), the value of λ and β appears to be

$$\lambda = \frac{g}{\sqrt{N}} \tag{2.0.135}$$

and

$$\beta = \sqrt{N}. \tag{2.0.136}$$

In view of this result, Eq. (2.0.128) is expressible in the form

$$\frac{d}{dt}\hat{a}(t) = -\frac{\kappa}{2}\hat{a}(t) + \frac{g}{\sqrt{N}}\hat{m}_a(t) + \hat{G}_a(t). \tag{2.0.137}$$

where

$$\hat{G}_a(t) = \sqrt{N}\hat{g}_a(t) \tag{2.0.138}$$

is the noise operator when the cavity mode is interacting with N two-level atoms. When the two-level atoms are pumped to the upper level at a rate of r_{ca} the process of pumping affects the time evolution of the expectation values of operators $\langle \hat{N}_a \rangle$ and $\langle \hat{N}_c \rangle$. Then the time evolution of the expectation value of $\langle \hat{N}_a \rangle$ is increased by the rate of $r_{ca}\langle \hat{N}_c \rangle$ but that of $\langle \hat{N}_c \rangle$ decreased by the same rate [1,11].

Incorporating the effect of pumping, we can rewrite Eqs. (2.0.119), (2.0.120) and (2.0.121) as

$$\frac{d}{dt}\langle \hat{N}_a(t) \rangle = -(\gamma + \gamma_c)\langle \hat{N}_a(t) \rangle + r_{ca}\langle \hat{N}_c(t) \rangle, \tag{2.0.139}$$

$$\frac{d}{dt}\langle \hat{N}_b(t) \rangle = -\gamma\langle \hat{N}_b(t) \rangle + \gamma_c\langle \hat{N}_a(t) \rangle, \tag{2.0.140}$$

$$\frac{d}{dt}\langle \hat{N}_c(t) \rangle = \gamma(\langle \hat{N}_a(t) \rangle + \langle \hat{N}_b(t) \rangle) - r_{ca}\langle \hat{N}_c(t) \rangle. \tag{2.0.141}$$

The steady-state solutions of Eqs. (2.0.142), (2.0.143) and (2.0.144) are expressible as

$$\langle \hat{N}_a \rangle = \frac{r_{ca}}{(\gamma + \gamma_c)}\langle \hat{N}_c \rangle, \tag{2.0.142}$$

$$\langle \hat{N}_b \rangle = \frac{\gamma_c r_{ca}}{\gamma(\gamma + \gamma_c)}\langle \hat{N}_c \rangle, \tag{2.0.143}$$

and

$$\langle \hat{N}_c \rangle = \frac{\gamma}{\gamma + r_{ca}}N. \tag{2.0.144}$$

On the basis of Eqs. (2.0.116) and (2.0.117), we can write

$$\frac{d}{dt}\hat{m}_a(t) = -(\gamma + \frac{\gamma_c}{2})\hat{m}_a(t) + \hat{f}_a(t), \tag{2.0.145}$$

$$\frac{d}{dt}\hat{m}_b(t) = -\frac{\gamma}{2}\hat{m}_b(t) + \hat{f}_b(t), \tag{2.0.146}$$

where $\hat{f}_a(t)$ and $\hat{f}_b(t)$ are the noise operator associated to atomic operators $\hat{m}_a(t)$ and $\hat{m}_b(t)$, respectively.

We note that Eq. (2.0.116) and the expectation value of Eq. (2.0.145) will be equal as well as Eq. (2.0.117) and the expectation value of Eq. (2.0.146), if

$$\langle \hat{f}_a(t) \rangle = 0, \tag{2.0.147}$$

$$\langle \hat{f}_b(t) \rangle = 0. \tag{2.0.148}$$

We now proceed to evaluate the correlation properties of $\hat{f}_a(t)$ and $\hat{f}_b(t)$. Employing the relation

$$\frac{d}{dt}\langle \hat{m}_a(t)\hat{m}_a(t) \rangle = \left\langle \frac{d\hat{m}_a(t)}{dt}\hat{m}_a(t) \right\rangle + \left\langle \hat{m}_a(t)\frac{d\hat{m}_a(t)}{dt} \right\rangle, \tag{2.0.149}$$

along with Eq. (2.0.145) and its conjugate, we see that

$$\begin{aligned} \frac{d}{dt}\langle \hat{m}_a(t)\hat{m}_a(t) \rangle = &-(\gamma + \frac{\gamma_c}{2})\langle \hat{m}_a(t)\hat{m}_a(t) \rangle + \langle \hat{f}_a(t)\hat{m}_a(t) \rangle \\ &- (\gamma + \frac{\gamma_c}{2})\langle \hat{m}_a(t)\hat{m}_a(t) \rangle + \langle \hat{m}_a(t)\hat{f}_a(t) \rangle. \end{aligned} \tag{2.0.150}$$

In view of Eq. (2.0.122), this equation turns to

$$\langle \hat{f}_a(t)\hat{m}_a(t) \rangle + \langle \hat{m}_a(t)\hat{f}_a(t) \rangle = 0. \tag{2.0.151}$$

On the other hand, by writing the formal solution of Eq. (2.0.145) as

$$\hat{m}_a(t) = \hat{m}_a(0)e^{-(\gamma+\frac{\gamma_c}{2})t} + \int_0^t e^{-(\gamma+\frac{\gamma_c}{2})(t-t')}\hat{f}_a(t')dt', \tag{2.0.152}$$

and multiplying this equation by $\hat{f}_a(t)$ from the left and taking the expectation value of the resulting equation, we see that

$$\begin{aligned} \langle \hat{f}_a(t)\hat{m}_a(t) \rangle &= \langle \hat{f}_a(t)\hat{m}_a(0) \rangle e^{-(\gamma+\frac{\gamma_c}{2})t} + \int_0^t e^{-(\gamma+\frac{\gamma_c}{2})(t-t')}\langle \hat{f}_a(t)\hat{f}_a(t') \rangle dt' \\ &= \int_0^t e^{-(\gamma+\frac{\gamma_c}{2})(t-t')}\langle \hat{f}_a(t)\hat{f}_a(t') \rangle dt', \end{aligned} \tag{2.0.153}$$

where, the fact *the noise operator at a certain time do not affect the atomic operator at earlier time* has been used.

Similarly one can readily find that

$$\langle \hat{m}_a(t)\hat{f}_a(t) \rangle = \int_0^t e^{-(\gamma+\frac{\gamma_c}{2})(t-t')}\langle \hat{f}_a(t')\hat{f}_a(t) \rangle dt'. \tag{2.0.154}$$

Combining Eqs. (2.0.153), (2.0.154) and (2.0.151), it follows that

$$\int_0^t e^{-(\gamma+\frac{\gamma_c}{2})(t-t')}\langle\hat{f}_a(t)\hat{f}_a(t')\rangle dt' + \int_0^t e^{-(\gamma+\frac{\gamma_c}{2})(t-t')}\langle\hat{f}_a(t')\hat{f}_a(t)\rangle dt' = 0. \qquad (2.0.155)$$

and assuming

$$\langle\hat{f}_a(t)\hat{f}_a(t')\rangle = \langle\hat{f}_a(t')\hat{f}_a(t)\rangle, \qquad (2.0.156)$$

we see that

$$2\int_0^t e^{-(\gamma+\frac{\gamma_c}{2})(t-t')}\langle\hat{f}_a(t)\hat{f}_a(t')\rangle dt' = 0. \qquad (2.0.157)$$

From this equation we assert that

$$\langle\hat{f}_a(t)\hat{f}_a(t')\rangle = 0. \qquad (2.0.158)$$

Furthermore, employing Eq. (2.0.145) along with its conjugate in the relation

$$\frac{d}{dt}\langle\hat{m}_a^\dagger(t)\hat{m}_a(t)\rangle = \left\langle\frac{d\hat{m}_a^\dagger(t)}{dt}\hat{m}_a(t)\right\rangle + \left\langle\hat{m}_a^\dagger(t)\frac{d\hat{m}_a(t)}{dt}\right\rangle, \qquad (2.0.159)$$

we see that

$$\begin{aligned}\frac{d}{dt}\langle\hat{m}_a^\dagger(t)\hat{m}_a(t)\rangle &= -2(\gamma+\frac{\gamma_c}{2})\langle\hat{m}_a^\dagger(t)\hat{m}_a(t)\rangle \\ &\quad + (\langle\hat{f}_a^\dagger(t)\hat{m}_a(t)\rangle + \langle\hat{m}_a^\dagger(t)\hat{f}_a(t)\rangle.\end{aligned} \qquad (2.0.160)$$

In view of Eq. (2.0.122), this equation goes over into

$$\frac{d}{dt}\langle\hat{N}_a(t)\rangle = -2(\gamma+\frac{\gamma_c}{2})\langle\hat{N}_a(t)\rangle + \frac{1}{N}\left(\langle\hat{f}_a^\dagger(t)\hat{m}_a(t)\rangle + \langle\hat{m}_a^\dagger(t)\hat{f}_a(t)\rangle\right). \qquad (2.0.161)$$

By comparing Eq. (2.0.139) and (2.0.161), we obtain

$$\langle\hat{f}_a^\dagger(t)\hat{m}_a(t)\rangle + \langle\hat{m}_a^\dagger(t)\hat{f}_a(t)\rangle = N\left(r_{ca}\langle\hat{N}_c\rangle + (\gamma+\frac{\gamma_c}{2})\langle\hat{N}_a\rangle\right). \qquad (2.0.162)$$

In view of Eqs. (2.0.143) and (2.0.144), we get

$$\langle\hat{f}_a^\dagger(t)\hat{m}_a(t)\rangle + \langle\hat{m}_a^\dagger(t)\hat{f}_a(t)\rangle = \frac{(2\gamma+\frac{3\gamma_c}{2})N^2}{(1+\frac{\gamma_c}{\gamma})(1+\frac{\gamma}{r_{ca}})}. \qquad (2.0.163)$$

On the other hand, multiplying Eq. (2.0.152) by $\hat{f}^{\dagger}(t)$ from the left and taking the expectation value of the resulting equation, gives

$$\langle\hat{f}_a^{\dagger}(t)\hat{m}_a(t)\rangle = \langle\hat{f}_a^{\dagger}(t)\hat{m}_a(0)\rangle e^{-(\gamma+\frac{\gamma_c}{2})t} + \int_0^t e^{-(\gamma+\frac{\gamma_c}{2})(t-t')}\langle\hat{f}_a^{\dagger}(t)\hat{f}_a(t')\rangle dt'. \quad (2.0.164)$$

Since the noise operator at a given time should not affect the atomic operator at the earlier time. As a result of this, Eq. (2.0.164) reduces to

$$\langle\hat{f}_a^{\dagger}(t)\hat{m}_a(t)\rangle = \int_0^t e^{-(\gamma+\frac{\gamma_c}{2})(t-t')}\langle\hat{f}_a^{\dagger}(t)\hat{f}_a(t')\rangle dt'. \quad (2.0.165)$$

Similarly, multiplying the conjugate of Eq. (2.0.152) by $\hat{f}_a(t)$ from the right and taking the expectation value of the resulting equation, yields

$$\langle\hat{m}_a^{\dagger}(t)\hat{f}_a(t)\rangle = \int_0^t e^{-(\gamma+\frac{\gamma_c}{2})(t-t')}\langle\hat{f}_a^{\dagger}(t')\hat{f}_a(t)\rangle dt'. \quad (2.0.166)$$

Substitution of Eqs. (2.0.165) and (2.0.166) into (2.0.163), gives

$$\int_0^t e^{-(\gamma+\frac{\gamma_c}{2})(t-t')}\left(\langle\hat{f}_a^{\dagger}(t)\hat{f}_a(t')\rangle + \langle\hat{f}_a^{\dagger}(t')\hat{f}_a(t)\rangle\right)dt' = \frac{(2\gamma+\frac{3\gamma_c}{2})N^2}{(1+\frac{\gamma_c}{\gamma})(1+\frac{\gamma}{r_{ca}})}. \quad (2.0.167)$$

Then, by assuming

$$\langle\hat{f}_a^{\dagger}(t)\hat{f}_a(t')\rangle = \langle\hat{f}_a^{\dagger}(t')\hat{f}_a(t)\rangle, \quad (2.0.168)$$

one can put Eq. (2.0.167) in the form

$$2\int_0^t e^{-(\gamma+\frac{\gamma_c}{2})(t-t')}\langle\hat{f}_a^{\dagger}(t)\hat{f}_a(t')\rangle dt' = \frac{(2\gamma+\frac{3\gamma_c}{2})N^2}{(1+\frac{\gamma_c}{\gamma})(1+\frac{\gamma}{r_{ca}})}. \quad (2.0.169)$$

From this equation we may assert that

$$\langle\hat{f}_a^{\dagger}(t)\hat{f}_a(t')\rangle = \frac{(2\gamma+\frac{3\gamma_c}{2})N^2}{(1+\frac{\gamma_c}{\gamma})(1+\frac{\gamma}{r_{ca}})}\delta(t-t'). \quad (2.0.170)$$

Furthermore, introducing Eq. (2.0.146) along with its conjugate into the relation

$$\frac{d}{dt}\langle\hat{m}_b^{\dagger}(t)\hat{m}_b(t)\rangle = \left\langle\frac{d\hat{m}_b^{\dagger}(t)}{dt}\hat{m}_b(t)\right\rangle + \left\langle\hat{m}_b^{\dagger}(t)\frac{d\hat{m}_b(t)}{dt}\right\rangle, \quad (2.0.171)$$

we see that

$$\begin{aligned}\frac{d}{dt}\langle\hat{m}_b^\dagger(t)\hat{m}_b(t)\rangle &= -\gamma\langle\hat{m}_b^\dagger(t)\hat{m}_b(t)\rangle \\ &\quad + \left(\langle\hat{f}_b^\dagger(t)\hat{m}_b(t)\rangle + \langle\hat{m}_b^\dagger(t)\hat{f}_b(t)\rangle\right).\end{aligned} \tag{2.0.172}$$

Employing Eq. (2.0.123) in this equation, we find that

$$\frac{d}{dt}\langle\hat{N}_b(t)\rangle = -\gamma\langle\hat{N}_b(t)\rangle + \frac{1}{N}\left(\langle\hat{f}_b^\dagger(t)\hat{m}_b(t)\rangle + \langle\hat{m}_b^\dagger(t)\hat{f}_b(t)\rangle\right). \tag{2.0.173}$$

By comparing Eqs. (2.0.120) and (2.0.173), we have got

$$\langle\hat{f}_b^\dagger(t)\hat{m}_b(t)\rangle + \langle\hat{m}_b^\dagger(t)\hat{f}_b(t)\rangle = \gamma_c N\langle\hat{N}_a(t)\rangle. \tag{2.0.174}$$

In view of Eqs. (2.0.142) and (2.0.144), Eq. (2.0.174) become

$$\langle\hat{f}_b^\dagger(t)\hat{m}_b(t)\rangle + \langle\hat{m}_b^\dagger(t)\hat{f}_b(t)\rangle = \frac{\gamma_c N^2}{(1+\frac{\gamma_c}{\gamma})(1+\frac{\gamma}{r_{ca}})}. \tag{2.0.175}$$

A formal solution of Eq. (2.0.146) can be written as

$$\hat{m}_b(t) = \hat{m}_b(0)e^{-\frac{\gamma}{2}t} + \int_0^t e^{-\frac{\gamma}{2}(t-t')}\hat{f}_b(t')dt'. \tag{2.0.176}$$

Multiplying this equation by $\hat{f}_b^\dagger(t)$ from the left and taking expectation value of the resulting equation, gives

$$\langle\hat{f}_b^\dagger(t)\hat{m}_b(t)\rangle = \langle\hat{f}_b^\dagger(t)\hat{m}_b(0)\rangle + \int_0^t e^{-\frac{\gamma}{2}(t-t')}\langle\hat{f}_b^\dagger(t)\hat{f}_b(t')\rangle dt'. \tag{2.0.177}$$

Because of the noise operator at a certain time should not affect the atomic operator at the earlier time, Eq. (2.0.177) reduces to

$$\langle\hat{f}_b^\dagger(t)\hat{m}_b(t)\rangle = \int_0^t e^{-\frac{\gamma}{2}(t-t')}\langle\hat{f}_b^\dagger(t)\hat{f}_b(t')\rangle dt', \tag{2.0.178}$$

similarly

$$\langle\hat{m}_b^\dagger(t)\hat{f}_b(t)\rangle = \int_0^t e^{-\frac{\gamma}{2}(t-t')}\langle\hat{f}_b^\dagger(t')\hat{f}_b(t)\rangle dt'. \tag{2.0.179}$$

Combining Eq. (2.0.178) and (2.0.179) through Eq. (2.0.175), gives

$$\int_0^t e^{-\frac{\gamma}{2}(t-t')} \left(\langle \hat{f}_b^\dagger(t)\hat{f}_b(t')\rangle + \langle \hat{f}_b^\dagger(t')\hat{f}_b(t)\rangle\right) dt' = \gamma_c N \langle \hat{N}_a(t)\rangle. \tag{2.0.180}$$

Now by assuming

$$\langle \hat{f}_b^\dagger(t)\hat{f}_b(t')\rangle = \langle \hat{f}_b^\dagger(t')\hat{f}_b(t)\rangle, \tag{2.0.181}$$

we can write Eq. (2.0.180) as

$$2\int_0^t e^{-\frac{\gamma}{2}(t-t')} \langle \hat{f}_b^\dagger(t)\hat{f}_b(t')\rangle dt' = \gamma_c N \langle \hat{N}_a(t)\rangle. \tag{2.0.182}$$

From this result we may assert that

$$\langle \hat{f}_b^\dagger(t)\hat{f}_b(t')\rangle = \gamma_c N \langle \hat{N}_a(t)\rangle \delta(t-t'), \tag{2.0.183}$$

where the value of $\langle \hat{N}_a(t)\rangle$ is given in Eq. (2.0.142).

Furthermore, using the relation

$$\frac{d}{dt}\langle \hat{m}_b \hat{m}_b^\dagger\rangle = \left\langle \frac{d\hat{m}_b}{dt}\hat{m}_b^\dagger \right\rangle + \left\langle \hat{m}_b \frac{d\hat{m}_b^\dagger}{dt} \right\rangle, \tag{2.0.184}$$

along with Eq. (2.0.146) and its conjugate, we arrive at

$$\frac{d}{dt}\langle \hat{m}_b \hat{m}_b^\dagger\rangle = -\gamma \langle \hat{m}_b \hat{m}_b^\dagger\rangle + \langle \hat{f}_b(t)\hat{m}_b^\dagger(t)\rangle + \langle \hat{m}_b(t)\hat{f}_b^\dagger(t)\rangle. \tag{2.0.185}$$

Employing Eqs. (2.0.123) in Eq. (2.0.185), we obtain

$$\frac{d}{dt}\langle \hat{N}_c(t)\rangle = -\gamma \langle \hat{N}_c(t)\rangle + \frac{1}{N}\left(\langle \hat{f}_b(t)\hat{m}_b^\dagger(t)\rangle + \langle \hat{m}_b(t)\hat{f}_b^\dagger(t)\rangle\right). \tag{2.0.186}$$

In addition, using the completeness relation [12]

$$\hat{\eta}_a^j + \hat{\eta}_b^j + \hat{\eta}_c^j = \hat{I} \tag{2.0.187}$$

we see that

$$\langle \hat{N}_a\rangle + \langle \hat{N}_b\rangle + \langle \hat{N}_c\rangle = N, \tag{2.0.188}$$

By comparing Eq. (2.0.121) with (2.0.186) along with Eq. (2.0.188), we obtain

$$\langle \hat{f}_b(t)\hat{m}_b^\dagger(t)\rangle + \langle \hat{m}_b(t)\hat{f}_b^\dagger(t)\rangle = \gamma N^2 - r_{ca}N\langle \hat{N}_c\rangle. \tag{2.0.189}$$

Substituting Eq. (2.0.144) into (2.0.189), we find

$$\langle \hat{f}_b(t)\hat{m}_b^\dagger(t)\rangle + \langle \hat{m}_b(t)\hat{f}_b^\dagger(t)\rangle = \frac{\gamma^2 N^2}{\gamma + r_{ca}}. \tag{2.0.190}$$

Multiplying the conjugate of Eq. (2.0.176) by $\hat{f}_b(t)$ from the left and taking the expectation value of the resulting equation, gives

$$\langle \hat{f}_b(t)\hat{m}_b^\dagger(t)\rangle = \langle \hat{f}_b(t)\hat{m}_b^\dagger(0)\rangle e^{-\frac{\gamma}{2}t} + \int_0^t e^{-\frac{\gamma}{2}(t-t')}\langle \hat{f}_b(t)\hat{f}_b^\dagger(t')\rangle dt'. \tag{2.0.191}$$

Since the noise operator at a certain time do not affect the atomic operator at a later time, so that Eq.(2.0.191) reduces to

$$\langle \hat{f}_b(t)\hat{m}_b^\dagger(t)\rangle = \int_0^t e^{-\frac{\gamma}{2}(t-t')}\langle \hat{f}_b(t)\hat{f}_b^\dagger(t')\rangle dt'. \tag{2.0.192}$$

Similarly, multiplying Eq. (2.0.176) by $\hat{f}_b^\dagger(t)$ from the right and taking its expectation value yields

$$\langle \hat{m}_b(t)\hat{f}_b^\dagger(t)\rangle = \int_0^t e^{-\frac{\gamma}{2}(t-t')}\langle \hat{f}_b(t')\hat{f}_b^\dagger(t)\rangle dt'. \tag{2.0.193}$$

In view of Eqs. (2.0.192) and (2.0.193), we can write Eq. (2.0.190) in the form

$$\int_0^t e^{-\frac{\gamma}{2}(t-t')}\left(\langle \hat{f}_b(t)\hat{f}_b^\dagger(t')\rangle + \langle \hat{f}_b(t')\hat{f}_b^\dagger(t)\rangle\right) dt' = \frac{\gamma^2 N^2}{\gamma + r_{ca}}. \tag{2.0.194}$$

here by assuming

$$\langle \hat{f}_b(t)\hat{f}_b^\dagger(t')\rangle = \langle \hat{f}_b(t')\hat{f}_b^\dagger(t)\rangle, \tag{2.0.195}$$

we see that

$$2\int_0^t e^{-\frac{\gamma}{2}(t-t')}\langle \hat{f}_b(t)\hat{f}_b^\dagger(t')\rangle dt' = \frac{\gamma^2 N^2}{\gamma + r_{ca}}. \tag{2.0.196}$$

From the this equation we assert that

$$\langle \hat{f}_b(t)\hat{f}_b^\dagger(t')\rangle = \frac{\gamma^2 N^2}{\gamma + r_{ca}}\delta(t-t'). \tag{2.0.197}$$

Following the same procedure, one can readily obtain that

$$\langle \hat{f}_a(t)\hat{f}_a^\dagger(t')\rangle = \frac{(2\gamma+\gamma_c)N^2}{(1+\frac{\gamma}{\gamma_c})(1+\frac{\gamma}{r_{ca}})}\delta(t-t'), \tag{2.0.198}$$

$$\langle \hat{f}_b(t)\hat{f}_b(t')\rangle = 0, \tag{2.0.199}$$

$$\langle \hat{f}_a(t)\hat{f}_b(t')\rangle = 0, \tag{2.0.200}$$

$$\langle \hat{f}_a^\dagger(t)\hat{f}_b(t')\rangle = 0, \tag{2.0.201}$$

$$\langle \hat{f}_a(t)\hat{f}_b^\dagger(t')\rangle = 0, \tag{2.0.202}$$

Finally, employing Eq. (2.0.137) and its conjugate along with the Master equation for the total of N two-level atoms available in the cavity, the correlation properties of cavity mode noise operator for the given N two-level atoms found to be

$$\langle \hat{G}_a(t)\hat{G}_a(t')\rangle = \sum_1^N \langle \hat{g}_a(t)\hat{g}_a(t')\rangle = 0 \tag{2.0.203}$$

$$\langle \hat{G}_a^\dagger(t)\hat{G}_a(t')\rangle = \sum_1^N \langle \hat{g}_a^\dagger(t)\hat{g}_a(t')\rangle = 0 \tag{2.0.204}$$

$$\langle \hat{G}_a(t)\hat{G}_a^\dagger(t')\rangle = \sum_1^N \langle \hat{g}_a(t)\hat{g}_a^\dagger(t')\rangle = \kappa N\delta(t-t') \tag{2.0.205}$$

Chapter 3

Photon Statistics

3.1 The mean of the photon number

The mean of the photon number of the cavity light is obtainable employing the solution of Eq. (2.0.137)

$$\hat{a}(t) = \hat{a}(0)e^{-\frac{\kappa}{2}t} + \frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(t-t')}\hat{m}_a(t')dt' + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\hat{G}_a(t')dt', \tag{3.1.1}$$

In view of Eqs. (3.1.1) along with its conjugate, the mean of the photon number of the cavity light is expressible as

$$\begin{aligned}
\bar{n} &= \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle \\
&= \langle \hat{a}^\dagger(0)\hat{a}(0)\rangle e^{-\kappa t} + \frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle \hat{a}^\dagger(0)\hat{m}_a(t')\rangle dt' + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle \hat{a}^\dagger(0)\hat{G}_a(t')\rangle dt' \\
&+ \frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(t-t'')}\langle \hat{m}_a^\dagger(t'')\hat{a}(0)\rangle dt'' + \frac{g^2}{N}\int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle \hat{m}_a^\dagger(t'')\hat{m}_a(t')\rangle dt'dt'' \\
&+ \frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle \hat{m}_a^\dagger(t'')\hat{G}_a(t')\rangle dt'dt'' + \int_0^t e^{-\frac{\kappa}{2}(2t-t')}\langle \hat{G}_a^\dagger(t'')\hat{a}(0)\rangle dt'' \\
&+ \frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle \hat{G}_a^\dagger(t'')\hat{m}_a(t')\rangle dt'dt'' + \int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle \hat{G}_a^\dagger(t'')\hat{G}_a(t')\rangle dt'dt''.
\end{aligned} \tag{3.1.2}$$

Now assuming that the cavity mode operators and atomic operators do not correlate at the initial time, we can write

$$\langle \hat{a}^\dagger(0)\hat{m}_a(t')\rangle = \langle \hat{a}^\dagger(0)\rangle\langle \hat{m}_a(t')\rangle, \tag{3.1.3}$$

$$\langle \hat{m}^\dagger(t'')\hat{a}(0)\rangle = \langle \hat{m}_a^\dagger(t'')\rangle\langle \hat{a}(0)\rangle. \tag{3.1.4}$$

For a cavity mode initially in a vacuum state, we note that

$$\langle \hat{a}(0)\rangle = 0. \tag{3.1.5}$$

In view of this result Eq. (3.1.3) and (3.1.4) reduces to

$$\langle \hat{a}^\dagger(0)\hat{m}_a(t')\rangle = \langle \hat{a}^\dagger(0)\rangle\langle \hat{m}_a(t')\rangle = 0, \tag{3.1.6}$$

and

$$\langle \hat{m}^\dagger(t'')\hat{a}(0)\rangle = \langle \hat{m}_a^\dagger(t'')\rangle\langle \hat{a}(0)\rangle = 0. \tag{3.1.7}$$

And due to the fact that the noise operator at a certain time do not affect the cavity mode operator at the earlier time, we see that

$$\langle \hat{a}^\dagger(0)\hat{G}_a(t')\rangle = \langle \hat{G}_a^\dagger(t'')\hat{a}(0)\rangle = 0. \tag{3.1.8}$$

For the cavity mode assumed to be initially in a vacuum state, we see that

$$\langle \hat{a}^\dagger(0)\hat{a}(0)\rangle = 0. \tag{3.1.9}$$

Now with the aid of Eqs. (3.1.6), (3.1.7), (3.1.8) and (3.1.9) along with Eqs. (2.0.57) and (2.0.58), we can express the mean photon number as

$$\begin{aligned}\bar{n} = \frac{g^2}{N}\int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle \hat{m}_a^\dagger(t'')\hat{m}_a(t')\rangle dt'dt'' \\ + \int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle \hat{G}_a^\dagger(t'')\hat{G}_a(t')\rangle dt'dt'' \\ .\end{aligned} \tag{3.1.10}$$

On account of Eq. (2.0.204), we can see that

$$\bar{n} = \frac{g^2}{N}\int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle\hat{m}_a^\dagger(t'')\hat{m}_a(t')\rangle dt'dt''. \tag{3.1.11}$$

This relation can be rewritten as

$$\bar{n} = \left\langle \frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(t-t')}\hat{m}_a^\dagger(t'')dt''\frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(t-t')}\hat{m}_a(t')dt'\right\rangle, \tag{3.1.12}$$

or

$$\bar{n} = \langle\hat{a}'^\dagger(t)\hat{a}'(t)\rangle, \tag{3.1.13}$$

where

$$\hat{a}'(t) = \frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(t-t')}\hat{m}_a(t')dt'. \tag{3.1.14}$$

Upon differentiating Eq. (3.1.14) with respect to t, we obtain

$$\frac{d}{dt}\hat{a}'(t) = -\frac{\kappa}{2}\frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}t}\int_0^t e^{\frac{\kappa}{2}t'}\hat{m}_a(t')dt' + \frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}t}\frac{d}{dt}\int_0^t e^{\frac{\kappa}{2}t'}\hat{m}_a(t')dt'. \tag{3.1.15}$$

Now applying the relation

$$\frac{d}{dx}\int_a^x f(x,x')dx' = f(x,x) - f(x,a) + \int_a^x \frac{d}{dx}f(x.x')dx', \tag{3.1.16}$$

we see that

$$\frac{d}{dt}\int_0^t e^{\frac{\kappa}{2}t'}\hat{m}_a(t')dt' = e^{\frac{\kappa}{2}t}\hat{m}_a(t) - \hat{m}_a(0) + \int_0^t \frac{d}{dt}e^{\frac{\kappa}{2}t'}\hat{m}_a(t')dt', \tag{3.1.17}$$

or

$$\frac{d}{dt}\int_0^t e^{\frac{\kappa}{2}t'}\hat{m}_a(t')dt' = e^{\frac{\kappa}{2}t}\hat{m}_a(t) - \hat{m}_a(0). \tag{3.1.18}$$

Using Eq. (3.1.18) in (3.1.15), we obtain

$$\frac{d}{dt}\hat{a}'(t) = -\frac{\kappa}{2}\hat{a}'(t) + \frac{g}{\sqrt{N}}(\hat{m}_a(t) - \hat{m}_a(0)e^{-\frac{\kappa}{2}t}). \tag{3.1.19}$$

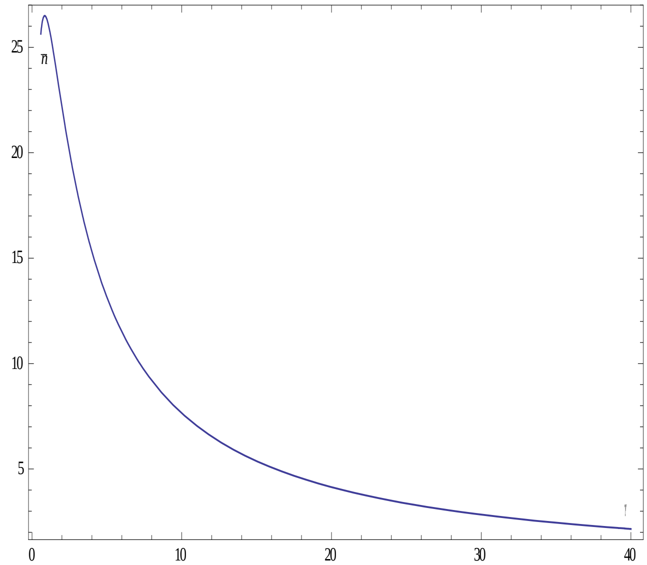

Figure 3.1: A plot of the mean of the photon number Eq. (3.1.22) versus the rate of spontaneous emission (γ), for $\gamma_c = 0.7$, $r_{ca} = 0.9$, $\kappa = 0.4$ and N=50

.

Applying the large time approximation scheme to Eq. (3.1.19), we see that

$$\hat{a}'(t) = \frac{2g}{\kappa\sqrt{N}}\hat{m}_a(t). \tag{3.1.20}$$

Upon introducing Eq. (3.1.20) and its conjugate into (3.1.13), we get

$$\bar{n} = \frac{4g^2}{\kappa^2}\langle\hat{N}_a\rangle = \frac{\gamma_c}{\kappa}\langle\hat{N}_a\rangle. \tag{3.1.21}$$

On account of Eqs. (2.0.142) and (2.0.144), the steady-state mean of the photon number appears to be

$$\bar{n} = \frac{\gamma_c}{\kappa}\frac{\gamma r_{ca} N}{(\gamma+\gamma_c)(\gamma+r_{ca})}. \tag{3.1.22}$$

We notice from Fig.3.1 that the mean of the photon number decreases as spontaneous emission dominate stimulated emission as can be seen from Eq. (3.1.22).

3.2 The variance of the photon number

The variance of the photon number for the cavity light can be written as

$$\begin{aligned}(\triangle n)^2 &= \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 \\ &= \langle \hat{a}^\dagger(t)\hat{a}(t)\hat{a}^\dagger(t)\hat{a}(t) \rangle - \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle^2.\end{aligned} \tag{3.2.1}$$

Assuming the atoms are initially in the ground level and taking Eq. (2.0.147) into account, the expectation value of Eq. (2.0.152) becomes

$$\langle \hat{m}_a(t) \rangle = 0. \tag{3.2.2}$$

Moreover, the expectation value of Eq. (3.1.1) is expressible as

$$\begin{aligned}\langle \hat{a}(t) \rangle = \langle \hat{a}(0) \rangle e^{-\frac{\kappa}{2}t} + \frac{g}{\sqrt{N}} \int_0^t e^{-\frac{\kappa}{2}(t-t')} \langle \hat{m}_a(t') \rangle dt' \\ + \int_0^t e^{-\frac{\kappa}{2}(t-t')} \langle \hat{G}_a(t') \rangle dt'.\end{aligned} \tag{3.2.3}$$

On account of Eqs. (2.0.34), (3.1.5) and (3.2.2), Eq (3.2.3) reduces to

$$\langle \hat{a}(t) \rangle = 0. \tag{3.2.4}$$

Similarly, we can see that

$$\langle \hat{a}^\dagger(t) \rangle = 0. \tag{3.2.5}$$

From these results we note that $\hat{a}(t)$ is a Gaussian variable with vanishing mean. We can thus write

$$\begin{aligned}\langle \hat{a}^\dagger(t)\hat{a}(t)\hat{a}^\dagger(t)\hat{a}(t) \rangle = \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle + \langle \hat{a}^{\dagger 2}(t) \rangle \langle \hat{a}^2(t) \rangle \\ + \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle \langle \hat{a}(t)\hat{a}^\dagger(t) \rangle\end{aligned} \tag{3.2.6}$$

The variance of the photon number turn out to be

$$(\triangle n)^2 = \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle \langle \hat{a}(t)\hat{a}^\dagger(t) \rangle + \langle \hat{a}^{\dagger 2}(t) \rangle \langle \hat{a}^2(t) \rangle. \tag{3.2.7}$$

In view of Eqs. (3.1.1) along with its conjugate, we can find

$$\begin{aligned}\langle\hat{a}(t)\hat{a}^\dagger(t)\rangle &= \langle\hat{a}(0)\hat{a}^\dagger(0)\rangle e^{-\kappa t} + \frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(2t-t'')}\langle\hat{a}(0)\hat{m}_a^\dagger(t'')\rangle dt'' \\ &+ \int_0^t e^{-\frac{\kappa}{2}(2t-t'')}\langle\hat{a}(0)\hat{G}^\dagger(t'')\rangle dt'' + \frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(2t-t')}\langle\hat{m}_a(t')\hat{a}^\dagger(0)\rangle dt' \\ &+ \frac{g^2}{N}\int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle\hat{m}_a(t')\hat{m}_a^\dagger(t'')\rangle dt'dt'' + \frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle\hat{m}_a(t')\hat{G}_a^\dagger(t'')\rangle dt'dt'' \\ &+ \int_0^t e^{-\frac{\kappa}{2}(2t-t')}\langle\hat{G}_a(t')\hat{a}^\dagger(0)\rangle dt' + \frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle\hat{G}_a(t')\hat{m}_a^\dagger(t'')\rangle dt'dt'' \\ &+ \int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle\hat{G}_a(t')\hat{G}_a^\dagger(t'')\rangle dt'dt''.\end{aligned} \tag{3.2.8}$$

We know that the noise operator at a certain time do not affect the cavity mode operator and atomic operator at the earlier time. We can thus write

$$\langle\hat{a}(0)\hat{G}^\dagger(t'')\rangle = \langle\hat{G}_a(t')\hat{a}^\dagger(0)\rangle = 0. \tag{3.2.9}$$

Moreover, in view of the relation

$$\langle\hat{m}_a(t')\hat{G}_a^\dagger(t'')\rangle = \langle\hat{G}_a(t')\hat{m}_a^\dagger(t'')\rangle = 0, \tag{3.2.10}$$

and

$$\langle\hat{a}(0)\hat{a}^\dagger(0)\rangle = 1, \tag{3.2.11}$$

along with Eqs. (3.1.7) and (3.1.8), we can find Eq. (3.2.8) in simple form

$$\begin{aligned}\langle\hat{a}(t)\hat{a}^\dagger(t)\rangle = \frac{g^2}{N}\int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle\hat{m}_a(t')\hat{m}_a^\dagger(t'')\rangle dt'dt'' \\ \int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle\hat{G}_a(t')\hat{G}_a^\dagger(t'')\rangle dt'dt' + e^{-\kappa t}.\end{aligned} \tag{3.2.12}$$

On account of Eq. (2.0.205), we can find Eq. (3.2.12), we see that

$$\begin{aligned}\langle\hat{a}(t)\hat{a}^\dagger(t)\rangle = \langle\frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(t-t')}\hat{m}_a(t')dt'\frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(t-t'')}\hat{m}_a^\dagger(t'')dt''\rangle \\ + \kappa N\int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\delta(t'-t'')dt'dt'' + e^{-\kappa t},\end{aligned} \tag{3.2.13}$$

or

$$\langle \hat{a}(t)\hat{a}^{\dagger}(t)\rangle = \langle \hat{a}'(t)\hat{a}'^{\dagger}(t)\rangle + \kappa N \int_0^t e^{-\kappa(t-t')}dt' + e^{-\kappa t}, \tag{3.2.14}$$

where $\hat{a}'(t)$ is given by Eq. (3.1.14).

Substitution of Eqs. (3.1.20) and its conjugate into Eq. (3.2.14) and carrying out the integration results in

$$\langle \hat{a}(t)\hat{a}^{\dagger}(t)\rangle = \frac{4g^2}{\kappa^2}\langle \hat{N}_b\rangle + (1-N)e^{-\kappa t} + N \tag{3.2.15}$$

Employing Eq. (2.0.143) along with (2.0.144) in Eq. (3.2.15) at steady state, we see that

$$\langle \hat{a}(t)\hat{a}^{\dagger}(t)\rangle = \frac{\gamma_c}{\kappa}\frac{\gamma_c r_{ca} N}{(\gamma+\gamma_c)(\gamma+r_{ca})} + N. \tag{3.2.16}$$

In a similar manner, it can be verified that

$$\langle \hat{a}^2(t)\rangle = 0, \tag{3.2.17}$$

and

$$\langle \hat{a}^{\dagger 2}(t)\rangle = 0. \tag{3.2.18}$$

Taking Eqs. (3.2.7), (3.2.16), (3.2.17) and (3.2.18) into account the variance of the photon number has the form

$$(\triangle n)^2 = \bar{n}(\frac{\gamma_c}{\kappa}\frac{\gamma_c r_{ca} N}{(\gamma+\gamma_c)(\gamma+r_{ca})} + 1)N. \tag{3.2.19}$$

In view of Eq. (3.1.22), the variance can be written as

$$(\triangle n)^2 = (\frac{\gamma_c}{\gamma}\bar{n} + 1)\bar{n}N. \tag{3.2.20}$$

We notice from Eqs. (3.2.20) that the variance of the photon number is greater than the mean of the photon number as can be seen in Fig. (3.2).

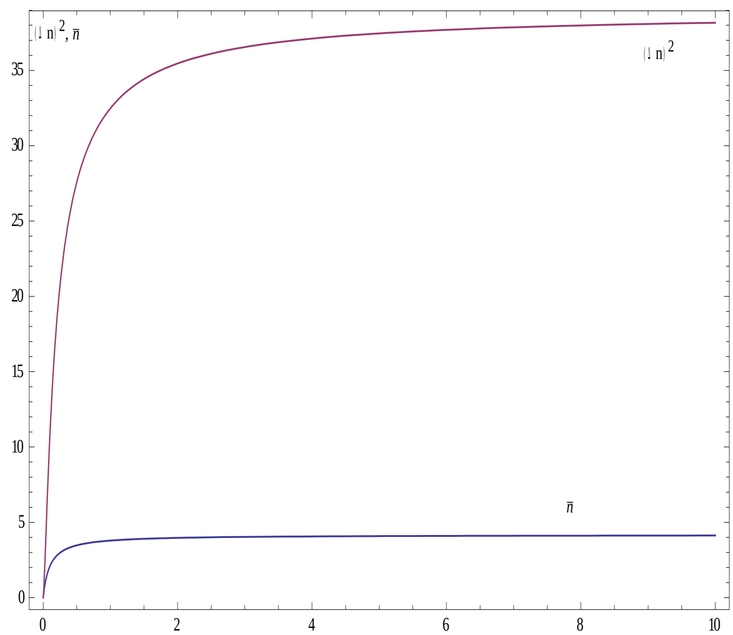

Figure 3.2: A plot of the *mean* and the *variance of the photon number* [Eqs. (3.1.22) and (3.2.19)] versus the *pumping rate* r_{ca} for N=50, $\gamma_c = 0.8$, $\gamma = 0.2$, and $\kappa = 0.4$.

Chapter 4

Quadrature Variance and Power Spectrum

4.1 Quadrature variance

Next we are going to evaluate the quadrature variance for the plus and minus quadrature operators defined by

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a} \tag{4.1.1}$$

and

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}) \tag{4.1.2}$$

These operators are Hermitian and satisfy the commutation relation

$$[\hat{a}_+, \hat{a}_-] = [\hat{a}^\dagger + \hat{a}, i(\hat{a}^\dagger - \hat{a})] \tag{4.1.3}$$

Using Eqs. (4.1.1) and (4.1.2) in Eq. (4.1.3), we can find that

$$[\hat{a}_+, \hat{a}_-] = 2i[\hat{a}, \hat{a}^\dagger] \tag{4.1.4}$$

For the cavity light the variance of the two quadrature is defined by

$$(\triangle a_\pm)^2 = \langle \hat{a}_\pm^2 \rangle - \langle \hat{a}_\pm \rangle^2 \tag{4.1.5}$$

In view of Eq. (4.1.1) and (4.1.5) the variance of the plus quadrature becomes

$$(\triangle a_+)^2 = \langle \hat{a}^\dagger \hat{a}\rangle + \langle \hat{a}\hat{a}^\dagger\rangle + \langle \hat{a}^{\dagger 2}\rangle + \langle \hat{a}^2\rangle - \langle \hat{a}^\dagger\rangle^2 - \langle \hat{a}\rangle^2 - \langle \hat{a}^\dagger\rangle\langle \hat{a}\rangle - \langle \hat{a}\rangle\langle \hat{a}^\dagger\rangle. \quad (4.1.6)$$

Substitution of Eqs. (3.2.15) and (3.2.16) in Eq. (4.1.6) gives

$$(\triangle a_+)^2 = \langle \hat{a}^\dagger \hat{a}\rangle + \langle \hat{a}\hat{a}^\dagger\rangle + \langle \hat{a}^\dagger\rangle^2 - \langle \hat{a}\rangle^2 - \langle \hat{a}^\dagger\rangle\langle \hat{a}\rangle - \langle \hat{a}\rangle\langle \hat{a}^\dagger\rangle \quad (4.1.7)$$

In view of Eqs. (3.2.4) and (3.2.5), Eq. (4.1.7) reduces to

$$(\triangle a_+)^2 = \langle \hat{a}^\dagger \hat{a}\rangle + \langle \hat{a}\hat{a}^\dagger\rangle \quad (4.1.8)$$

Now using Eqs. (3.1.22) and (3.2.16) in Eq. (4.1.8). the variance of the plus quadrature is expressible as

$$\begin{aligned}(\triangle a_+)^2 &= \frac{\gamma_c}{\kappa}\left(\frac{\gamma r_{ca} N}{(\gamma+\gamma_c)(\gamma+r_{ca})} + \frac{\gamma_c r_{ca} N}{(\gamma+\gamma_c)(\gamma+r_{ca})}\right) + N \\ &= \frac{\gamma_c}{\kappa}\left(\frac{r_{ca} N}{\gamma + r_{ca}}\right) + N\end{aligned} \quad (4.1.9)$$

Similarly, the variance of the minus quadrature found to be

$$(\triangle a_-)^2 = \frac{\gamma_c}{\kappa}\left(\frac{r_{ca} N}{\gamma + r_{ca}}\right) + N \quad (4.1.10)$$

We notice from Eqs. (4.1.9) and (4.1.10) that the plus and minus quadrature variances are equal and we can write that

$$(\triangle a_\pm)^2 = \frac{\gamma_c}{\kappa}\left(\frac{r_{ca} N}{\gamma + r_{ca}}\right) + N \quad (4.1.11)$$

The uncertainty relation for the two quadrature operators can be written as

$$\triangle a_+ \triangle a_- \geq \frac{1}{2}|\langle[\hat{a}_+, \hat{a}_-]\rangle| \quad (4.1.12)$$

and for the minimum uncertainty relation, we have

$$\triangle a_+ \triangle a_- = \frac{1}{2}|\langle[\hat{a}_+, \hat{a}_-]\rangle| \quad (4.1.13)$$

So that using Eq. (4.1.3), we obtain

$$\triangle a_+ \triangle a_- = |\langle \hat{a}\hat{a}^\dagger \rangle - \langle \hat{a}^\dagger \hat{a} \rangle| \tag{4.1.14}$$

Upon introducing Eqs. (3.1.22) and (3.2.16) in Eq. (4.1.14), there follows

$$\triangle a_+ \triangle a_- = \frac{\gamma_c}{\kappa}\left(\frac{\gamma_c r_{ca} N}{(\gamma + \gamma_c)(\gamma + r_{ca})} - \frac{\gamma r_{ca} N}{(\gamma + \gamma_c)(\gamma + r_{ca})}\right) + N \tag{4.1.15}$$

or

$$\triangle a_+ \triangle a_- = \frac{\gamma_c}{\kappa}\left(\frac{\gamma_c - \gamma}{\gamma + \gamma_c}\right)\left(\frac{r_{ca} N}{\gamma + r_{ca}}\right) + N \tag{4.1.16}$$

Here, for $\gamma \ll \gamma_c$, we have $\gamma_c - \gamma \approx \gamma_c$ as well as $\gamma_c + \gamma \approx \gamma_c$. As a result of these conditions, Eq. (4.1.16) reduces to

$$\triangle a_+ \triangle a_- = \frac{\gamma_c}{\kappa}\left(\frac{r_{ca} N}{\gamma + r_{ca}}\right) + N \tag{4.1.17}$$

In view of Eqs. (4.1.11) and (4.1.17) we see that the uncertainties in the two quadrature are equal and their product satisfies the minimum uncertainty relation. Hence for $\gamma \ll \gamma_c$, the light produced by the two level laser is coherent[1].

4.2 Power spectrum

The power spectrum of a single mode light with central frequency of ω_o is expressible as [1]

$$P(\omega) = \frac{1}{\pi} Re \int_0^\infty \langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss} e^{-i(\omega-\omega_o)\tau} d\tau \tag{4.2.1}$$

where ss stands for steady state.

Upon integrating both sides of Eq. (4.2.1) over ω, we readily get

$$\int_{-\infty}^{\infty} P(\omega) d\omega = \bar{n} \tag{4.2.2}$$

in which $\bar{n}$ is the steady state mean photon number of the cavity light.

From this result, we observe that $P(\omega)d\omega$ is the steady state mean photon number in the interval between ω and $\omega + d\omega$ [2].

We now proceed to calculate the two-time correlation function that appears in Eq. (4.2.1) for the cavity light. To this end, we realize that the solution of Eq. (2.0.137) can be written as

$$\hat{a}(t+\tau) = \hat{a}(t)e^{-\frac{\kappa}{2}\tau} + \frac{g}{\sqrt{N}}\int_0^\tau e^{-\frac{\kappa}{2}(\tau+\tau')}\hat{m}_a(t+\tau')d\tau' + \int_0^\tau e^{\frac{\kappa}{2}(\tau+\tau')}\hat{G}_a(t+\tau')d\tau' \quad (4.2.3)$$

On the other hand, by rewriting Eq. (2.0.145) as

$$\frac{d}{dt}\hat{m}_a(t) = -\mu\hat{m}_a(t) + \hat{f}_a(t) \quad (4.2.4)$$

we can set its solution in the form

$$\hat{m}_a(t+\tau) = \hat{m}_a(t)e^{-\mu\tau} + e^{-\mu\tau}\int_0^\tau e^{\mu\tau'}\hat{f}_a(t+\tau')d\tau' \quad (4.2.5)$$

where

$$\mu = \gamma + \frac{\gamma_c}{2} \quad (4.2.6)$$

By rewriting Eq. (4.2.5) as

$$\hat{m}_a(t+\tau') = \hat{m}_a(t)e^{-\mu\tau'} + e^{-\mu\tau'}\int_0^{\tau'} e^{\mu\tau''}\hat{f}_a(t+\tau'')d\tau'' \quad (4.2.7)$$

and on introducing this equation into Eq. (4.2.3), we have

$$\begin{aligned}\hat{a}(t+\tau) = {} & \hat{a}(t)e^{-\frac{\kappa}{2}\tau} + \frac{g}{\sqrt{N}}\hat{m}_a(t)e^{-\frac{\kappa}{2}\tau}\int_0^\tau e^{-(\mu-\frac{\kappa}{2})\tau'}d\tau' \\ & + \frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}\tau}\int_0^\tau e^{-(\mu-\frac{\kappa}{2})\tau'}\int_0^{\tau'} e^{\mu\tau''}\hat{f}_a(t+\tau'')d\tau' d\tau'' \\ & + \int_0^\tau e^{\frac{\kappa}{2}(\tau+\tau')}\hat{G}_a(t+\tau')d\tau' \end{aligned} \quad (4.2.8)$$

Then, multiplying both sides of Eq. (4.2.8) by $\hat{a}^\dagger(t)$ from the left and taking the expectation value of the resulting equation, we find

$$\begin{aligned}\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle &= \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle e^{-\frac{\kappa}{2}\tau} + \frac{g}{\sqrt{N}}\langle\hat{a}^\dagger(t)\hat{m}_a(t)\rangle e^{-\frac{\kappa}{2}\tau}\int_0^\tau e^{-(\mu-\frac{\kappa}{2})\tau'}d\tau' \\ &+ \frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}\tau}\int_0^\tau e^{-(\mu-\frac{\kappa}{2})\tau'}\int_0^{\tau'} e^{\mu\tau''}\langle\hat{a}^\dagger(t)\hat{f}_a(t+\tau'')\rangle d\tau' d\tau'' \\ &+ \int_0^\tau e^{\frac{\kappa}{2}(\tau+\tau')}\langle\hat{a}^\dagger(t)\hat{G}_a(t+\tau')\rangle d\tau'\end{aligned} \tag{4.2.9}$$

In view of the fact that,

$$\langle\hat{a}^\dagger(t)\hat{f}_a(t+\tau'')\rangle = 0 \tag{4.2.10}$$

and

$$\langle\hat{a}^\dagger(t)\hat{G}_a(t+\tau')\rangle = 0 \tag{4.2.11}$$

Eq. (4.2.9) reduces to

$$\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle e^{-\frac{\kappa}{2}\tau} + \frac{g}{\sqrt{N}}\langle\hat{a}^\dagger(t)\hat{m}_a(t)\rangle e^{-\frac{\kappa}{2}\tau}\int_0^\tau e^{-(\mu-\frac{\kappa}{2})\tau'}d\tau' \tag{4.2.12}$$

Thus on carrying out the integration, we arrive at

$$\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle e^{-\frac{\kappa}{2}\tau} + \frac{g}{\sqrt{N}}\langle\hat{a}^\dagger(t)\hat{m}_a(t)\rangle(\frac{1}{\frac{\kappa}{2}-\mu})(e^{-\mu\tau} - e^{-\frac{\kappa}{2}\tau}) \tag{4.2.13}$$

Applying the large time approximation scheme to Eq. (2.0.137), we obtain

$$\hat{a}(t) = \frac{2g}{\kappa\sqrt{N}}\hat{m}_a(t) + \frac{2}{\kappa}\hat{G}_a(t) \tag{4.2.14}$$

Using this equation we can see that

$$\langle\hat{a}^\dagger(t)\hat{m}_a(t)\rangle = \frac{2g}{\kappa\sqrt{N}}\langle\hat{m}_a^\dagger(t)\hat{m}_a(t)\rangle + \frac{2}{\kappa}\langle\hat{G}_a^\dagger(t)\hat{m}_a(t)\rangle \tag{4.2.15}$$

In view of Eqs. (2.0.57) and (2.0.122), along with (3.1.21), one can find Eq. (4.2.15) in reduced form

$$\begin{aligned}\langle\hat{a}^\dagger(t)\hat{m}_a(t)\rangle &= \frac{2gN}{\kappa\sqrt{N}}\langle\hat{N}_a(t)\rangle \\ &= \frac{\kappa N}{2g\sqrt{N}}\bar{n}\end{aligned} \tag{4.2.16}$$

Substitution of Eq. (4.2.16) into (4.2.13), gives

$$\langle \hat{a}^{\dagger}(t)\hat{a}(t+\tau)\rangle = \bar{n}e^{-\frac{\kappa}{2}\tau} + \bar{n}\left(\frac{\kappa}{\kappa-2\mu}\right)(e^{-\mu\tau} - e^{-\frac{\kappa}{2}\tau}) \tag{4.2.17}$$

or

$$\langle \hat{a}^{\dagger}(t)\hat{a}(t+\tau)\rangle = \bar{n}\left(\frac{\kappa}{\kappa-2\mu}e^{-\mu\tau} - \frac{2\mu}{\kappa-2\mu}e^{-\frac{\kappa}{2}\tau}\right) \tag{4.2.18}$$

Introduction of this equation into Eq. (4.2.1) yields

$$P(\omega) = \frac{1}{\pi}\bar{n}Re\left[\frac{\kappa}{\kappa-2\mu}\int_0^{\infty} e^{-(\mu+i(\omega-\omega_o))\tau}d\tau - \frac{2\mu}{\kappa-2\mu}\int_0^{\infty} e^{-(\frac{\kappa}{2}+i(\omega-\omega_o))\tau}d\tau\right] \tag{4.2.19}$$

Upon carrying out the integration of we find that

$$P(\omega) = \frac{1}{\pi}\bar{n}Re\left[\frac{\kappa}{\kappa-2\mu}\left(\frac{1}{\mu+i(\omega-\omega_o)}\right) - \frac{2\mu}{\kappa-2\mu}\left(\frac{1}{\frac{\kappa}{2}+i(\omega-\omega_o)}\right)\right] \tag{4.2.20}$$

It then follows that

$$\begin{aligned} P(\omega) = {}& \frac{1}{\pi(\kappa-2\mu)}\bar{n}Re[\frac{\kappa\mu}{\kappa^2+(\omega-\omega_o)^2} - \frac{\kappa\mu}{(\kappa/2)^2+(\omega-\omega_o)^2} \\ & - i(\omega-\omega_o)(\frac{\kappa}{\mu^2+(\omega-\omega_o)^2} - \frac{2\mu}{(\kappa/2)^2+(\omega-\omega_o)^2})] \end{aligned} \tag{4.2.21}$$

The real part of Eq.(4.2.21) has the form

$$P(\omega) = \frac{1}{\pi(\kappa-2\mu)}\bar{n}\left[\frac{\kappa\mu}{\mu^2+(\omega-\omega_o)^2} - \frac{\kappa\mu}{\frac{\kappa^2}{4}+(\omega-\omega_o)^2}\right] \tag{4.2.22}$$

or

$$P(\omega) = \frac{\kappa\bar{n}}{\kappa-2\mu}\left[\frac{\mu/\pi}{\mu^2+(\omega-\omega_o)^2}\right] - \frac{\mu\bar{n}}{\kappa-2\mu}\left[\frac{\kappa/\pi}{(\kappa/2)^2+(\omega-\omega_o)^2}\right] \tag{4.2.23}$$

On the basis of Eq. (4.2.2) we realize that the mean photon number in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$ is expressible as

$$\bar{n}_{\pm\lambda} = \int_{-\lambda}^{\lambda} P(\omega')d\omega' \tag{4.2.24}$$

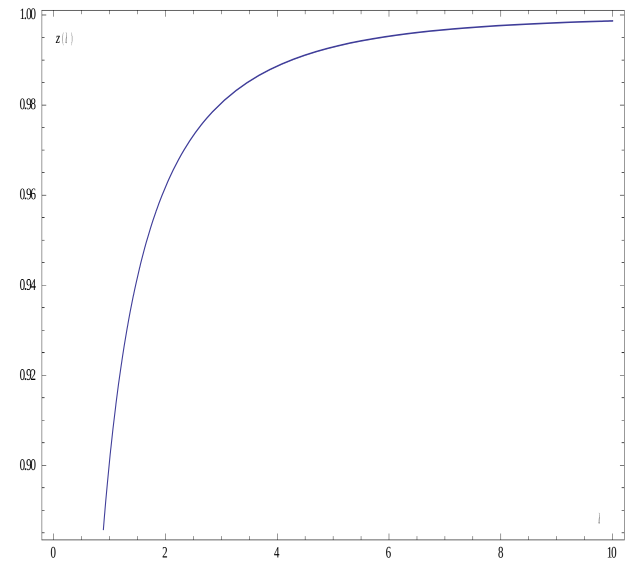

Figure 4.1: A plot of Eq. (4.2.28) versus λ for $\mu = 0.6$ and $\kappa = 0.4$

in which

$$\omega' = \omega - \omega_o \tag{4.2.25}$$

Therefore upon substituting Eq. (4.2.23) into (4.2.24) and carrying out the integration applying the relation

$$\int_{-\lambda}^{\lambda} \frac{d\omega}{\omega^2 + b^2} = \frac{2}{b} \tan^{-1}\left(\frac{\lambda}{b}\right) \tag{4.2.26}$$

we find that

$$\bar{n}_{\pm\lambda} = \bar{n} z(\lambda) \tag{4.2.27}$$

where

$$z(\lambda) = \frac{2\kappa/\pi}{\kappa - 2\mu} \tan^{-1}\left(\frac{\lambda}{\mu}\right) - \frac{4\mu/\pi}{\kappa - 2\mu} \tan^{-1}\left(\frac{2\lambda}{\kappa}\right) \tag{4.2.28}$$

From Fig. 4.1, we can read that z(1)=0.893, z(2)=0.947, z(3)=0.959, z(5)=0.960. Then the combination of these results with that of Eq. (4.2.27) gives

$\bar{n}_{\pm 1} = 0.893\bar{n}$, $\bar{n}_{\pm 2} = 0.947\bar{n}$, $\bar{n}_{\pm 3} = 0.959\bar{n}$, $\bar{n}_{\pm 5} = 0.960\bar{n}$. From this we notice that the mean of the photon number approaches to the total mean photon number for a relatively small frequency interval.

Chapter 5

Conclusion

In this thesis we have addressed the quantum properties of the light emitted by a two-level laser. Applying the pertinent Master equation and following the approach described in Ref.[1,12] we obtained the quantum Langevin equations for the cavity mode and atomic operators. Using the solution of the quantum Langevin equation along with the correlation properties of noise operators associated with cavity mode operators and atomic operators, we calculated the mean of the photon number and the variance of the photon number for the cavity light. Our result shows that, the variance of the photon number is greater than the mean of the photon number. Thus the photon statistics is Super-Poissonian. Moreover we have calculated the quadrature variance and the power spectrum of the cavity light. From this we have seen that the uncertainty in the two quadratures are equal and their product satisfies the minimum uncertainty relation for $\gamma \ll \gamma_c$. Thus the light produced by the two-level laser is coherent for $\gamma \ll \gamma_c$.

Bibliography

[1] Fesseha Kassahun, arXiv:1209.4723v1 (2012).

[2] D.F Walls, and J.G. Milburn, *Quantum Optics* (Springer-Verlag, Berlin, 1994).

[3] Marlan O. Scully and M. Subail Zubairy, *Quantum optics*, (Cambridge University Press 1997).

[4] R. Loudon, *The Quantum Theory of Light*, (Oxford University Press, New York, 2000).

[5] Wolfgang P. Schleich, *Quantum Optics in Phase Space*, (WILEY-VCH Verlag Berlin GmbH, Berlin, 2001).

[6] Miguel Orsyzag, *Quantum Optics*, (Santiago, Chile, 2007)

[7] F. W. Cummings, Phys. Rev. 140:**A1051-A1056**, (1965).

[8] L. Allen, and J. H. Eberly. *Optical Resonance and Two-level Atoms*, Wiley, New York, (1975).

[9] Svelto Orazio, *Principles of Lasers*, 4th ed. Trans. David Hanna. Springer. ISBN 0-306-45748-2, (1998).

[10] Warner Vogel and Dirk-Gunnar Welsch, *Quantum Optics*, (WILEY-VCH Verlag GmbH and Co.KGaA, Weinheim, 2006).

[11] Stephen M. Barnet and Paul M. Radmore, *Methods in Theoretical Quantum Optics*, (Clarendon Press, Oxford, 1997).

[12] Fesseha Kassahun, arXiv:1105.1438v3 (2012).

[13] Misrak Getahun, (PhD Dissertation, Addis Ababa University, 2009).

[14] Fesseha Kassahun, *Refined Quantum Analysis of Light(CreateSpace Independent Publishing Plateform, 2014).*