Superposed Degenerate Three-Level Lasers

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Degree of Master of Science in Physics (Quantum Optics)

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The undersigned hereby certify that they have read and recommend to the School of Graduate Studies for acceptance a thesis entitled "**Superposed Degenerate Three-Level Lasers**" by **Beyene Abiti** in partial fulfillment of the requirements for the degree of Master of Science in physics (Quantum Optics).

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For those family and friends of mine who have unlimited potential but can not attend the school.

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Abstract

In this thesis we study the statistical and squeezing properties of the light produced by a degenerate three-level laser. Using the solutions of c-number Langevin equations, we have calculated antinormally ordered characteristic function. Then employing the resulting characteristic function, we obtain the Q-function. Applying the Q-function, we have calculated the mean photon number, the variance of phototn number, the photon number distribution and the quadrature variance. We have found that the light mode is 47.9% squeezed below the coherent state level at steady state for $A = 3$ and $\kappa = 0.8$. It is observed that the degree of squeezing increases with the linear gain coefficient(A).

We have determined the Q-function for the superposition of the light beams produced by two degenerate three-level lasers. Using this Q-function, we have calculated the mean number of photons, the variance of photon number, the photon number distribution and quadrature variance. We have found that the steady state mean photon number is a simple sum of the mean photon numbers of the two light beams. The squeezing of the superposed light mode increases with the linear gain coefficient. It is found that for $A = 3$ and $\kappa = 0.8$, the squeezing is 95.8% below the coherent state level.

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Chapter 1

Introduction

Nonclassical properties like squeezing of cavity radiation produced by cascade three-level laser has received a great deal of attention in recent years [1-19]. In quantum optics the annihilation and creation operators used in describing single-mode radiation can be decomposed into two quadrature operators. For single-mode radiation in any state, the product of the fluctuations in the two quadratures satisfies the uncertainity relation. In squeezed state, the quantum noise in one quadrature is below the coherent state level at the expense of enhanced fluctuations in the conjugate quadrature, with the product of the uncertainties in the two quadratures still satisfying the uncertainty relation. Having less noise, squeezed states constitute an important nonclassical resource for information-processing systems like quantum computations, photon detection and in the field of high precision measurments. A squeezed state is now belonging to the selected technolgies for detection of weak signals and in low-noise communication [1, 3, 5]. The squeezed state of light can be generated by some quantum optical systems under certain conditions.

A three-level laser is one source of squeezed light [6, 7]. A degenerate three-level laser is a quantum optical process in which degenerate three-level atoms in cascade configuration and initially prepared in coherent superposition of the top and the bottom levels are injected into a cavity coupled to a vacuum reservoir via single port-mirror. The set of energy levels of an atom consists of an infinite number of discrete levels corrosponding to the bound states of the electron [8]. For a three-level atom, out of these set of energy levels only three-levels interact with electromagnetic radiation. When the three-level atom interacts with radiation, it under goes a transition from top to bottom level via the intermidate level by emitting two photons. If the two photons generated have different frequencies, a two-mode light is generated. In this case the atom is called non-degenerate three-level atom. But, when the frequencies of these photons are equal, the atom generates a single-mode light. For this condition the atom is called degenerate three-level atom.

Three-level lasers in which a cosiderable role is played by the coherent superposition of the top and bottom levels of the injected atoms have been studied by different authors [1, 2, 3 ,5, 6, 7, 9, 10, 11, 12, 17, 18, 19]. These studies show that three-level lasers can generate light in a squeezed state under certain conditions. Ansari [2] has found the quadrature variance of degenerate three-level laser using the steady state solution of the expectation value of cavity mode variables. He found that the cavity mode is in squeezed state if the probability for the injected atoms to be in the bottom levels is larger than the probability to be in the top levels. And almost perfect squeezing can be achieved for slightly high probability for the atoms to be in the bottom levels and for large value of linear gain coefficient. Dawit [9] studied a degenerate three-level laser in which top and bottom levels are coupled by strong coherent light and with half probablity for the atoms to be in the top or bottom levels coupled to a squeeze vacuum reservoir applying the solution of stochastic differential equations. He found that the squeezing increases with the linear gain coefficient.

Alebachew and Fesseha [10] have studied the squeezing properties of the cavity mode produced by a degenerate three-level laser whose cavity contains a parametric amplifier by applying the solution of the stochastic differential equations, with the top and bottom levels of injected atoms coupled by the pump mode emerging from the parametric amplifier. In this study they showed that the optical system generates light in a squeezed state with a maximum interacavity squeezing of 93% below the coherent state level. Tewodros and Fesseha [11] have studied the squeezing properties of the cavity mode produced by a degenerate three-level laser whose cavity contains a parametric amplifier and with the cavity mode driven by strong coherent light and the three-level atoms injected into the cavity are initially prepared in a coherent superposition of the top and bottom levels, with these levels coupled by the pump mode emerging from the parametric amplifier by applying the solution of c-number Langevin equations. Their study showed that the system generates squeezed light with maximum squeezing of 94% below the coherent state level for certain conditions. Recently, Misrak [12] has studied the squeezing properties of cavity mode produced by degenerate three-level laser with parametric amplifier by applying the solution of stochastic differential equations. This study showed that the quantum optical system generates squeezed light and the degree of squeezing increases with the linear gain coefficient with maximum interacavity squeezing of 96.5% below the coherent state level.

In this thesis, we seek to investigate the squeezing and statistical properties for degenerate three-level laser and the superposition of light beams produced by pair of degenerate three-level lasers. we carry out the analysis applying the solutions of c-number Langevin equations associated with the normal ordering. These equations are obtained using the master equation drived in the linear approximation scheme. From the solutions of c-number Langevin equations and the correlation properties of a noise force, we obtain the antinormally ordered characteristic function defined in Hiesenberg picture which is used to find the Q-function of light produced by three-level laser. Employing the Q-function we calculate the mean photon number, the variance of photon number, the photon number distribution and the quadrature variance of light produced by three-level lasers. Furthermore, using the Q-function, we obtain the Q-function for the superposition of two light beams produced by three-level lasers. Upon employing this Q-function, we determine the squeezing and statistical properties of the single-mode light obtained from superposition of two light beams produced by three-level lasers. We then calculate the mean photon number, the variance of photon number, the photon number distribution and the quadrature variance of the superposed light beam.

Chapter 2

Degenerate Three-Level Laser

In this chapter we first seek to get the master equation and c-number Langevin equations for the cavity mode produced by degenerate three-level laser. Using the solutions of the c-number Langevin equations, we find the antinormally ordered characteristic function defined in Hiesenberg picture. Employing the antinormally ordered characteristic function, we obtain the Q-function of light produced by a degenerate three-level laser. Finally, applying the resulting Q-function we calculate the mean photon number, the variance of photon number, the photon number distribution and the quadrature variance of the light beam produced by degenerate three-level laser.

2.1 Master equation

Here we seek to obtain the master equation for a cavity mode of degenerate three-level laser in which a degenerate three-level atoms in a cascade configuration are injected at a constant rate r_a into a cavity coupled to vacuum reservior via a single port-mirror and removed after a large enough decay time τ.

We denote the top, middle and bottom levels by eigen states $|a\rangle, |b\rangle$ and $|c\rangle$ whose enegy is give by $E_a = \hbar\omega_a, E_b = \hbar\omega_b$ and $E_c = \hbar\omega_c$, respectively. We assume the cavity mode to be at resonance with the two transitions $|a\rangle \rightarrow |b\rangle$ and $|b\rangle \rightarrow |c\rangle$, with direct transition between $|a\rangle$ and $|c\rangle$ is to be electric-dipole forbidden.

The interaction of a three-level atom with the cavity mode in the rotating wave approx-

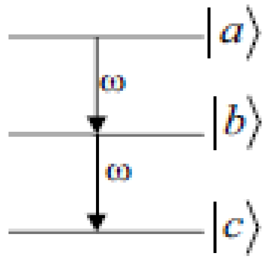

Figure 2.1: Three-level atom in a cascade configuration.

imation and in the interaction picture can be described by the Hamiltonian [1]

$$\hat{H} = ig[(|a\rangle\langle b| + |b\rangle\langle c|)\hat{a} - \hat{a}^{\dagger}(|b\rangle\langle a| + |c\rangle\langle b|)], \tag{2.1.1}$$

where g is the coupling constant and $\hat{a}$ is the annihilation operator for the cavity mode. We take the initial state of a three-level atom to be

$$|\Psi_A(0)\rangle = C_a(0)|a\rangle + C_c(0)|c\rangle,$$

where $C_a(0)$ and $C_c(0)$ are the probablity amplitudes of the three-level atom to be in the upper and bottom levels, respectively. Then the density operator for a single atom has the form

$$\rho_A(0) = \rho_{aa}^{(0)}|a\rangle\langle a| + \rho_{ac}^{(0)}|a\rangle\langle c| + \rho_{ca}^{(0)}|c\rangle\langle a| + \rho_{cc}^{(0)}|c\rangle\langle c|, \tag{2.1.2}$$

with $\rho_{aa}^{(0)} = |C_a|^2$, $\rho_{ac}^{(0)} = C_aC_c^*$, $\rho_{ca}^{(0)} = C_cC_a^*$ and $\rho_{cc}^{(0)} = |C_c|^2$.

Let $\rho_{AR}(t)$ be the density operator for a single atom plus the cavity mode at a time t with the atom injected at a time t_j, such that $(t-\tau) \leq t_j \leq t$. The density operator for all the atoms in the cavity plus the cavity mode at time t can be written as

$$\rho_{AR}(t) = r_a \sum_j \rho_{AR}(t, t')\Delta t_j \tag{2.1.3}$$

where $r_a \Delta t_j$ denotes the number of atoms injected into the cavity in a time Δt_j. Now converting the summation into integration in the limit $\Delta t_j \to 0$, we have

$$\rho_{AR}(t) = r_a \int_{t-\tau}^{t} \rho_{AR}(t,t')dt' \tag{2.1.4}$$

For any function h, if $h = \int_{x_0}^{T} f(x,x')dx'$, then $\frac{d}{dx}h = f(x,T) - f(x,x_0) + \int_{x_0}^{T} \frac{\partial}{\partial x} f(x,x')dx'$. Therefore, Eq. (2.1.4) reduces to

$$\frac{d}{dt}\rho_{AR}(t) = r_a \rho_{AR}(t,t) - r_a \rho_{AR}(t,t-\tau) + r_a \int_{t-\tau}^{t} \frac{\partial}{\partial t}\rho_{AR}(t,t')dt'. \tag{2.1.5}$$

We observe that $\rho_{AR}(t,t)$ the density operator for the cavity mode plus an atom injected at time t. This operator can thus be expressed as

$$\rho_{AR}(t,t) = \rho_A(t)\rho(t) \tag{2.1.6}$$

with $\rho(t)$ is the density operator for the cavity mode alone. We also note that $\rho_{AR}(t-\tau)$ is the density operator for an atom plus the cavity mode at time t, with the atom being removed from the cavity at this time. This operator can also be put in the form

$$\rho_{AR}(t,t-\tau) = \rho_A(t-\tau)\rho(t) \tag{2.1.7}$$

In view of Eqs. (2.1.6) and (2.1.7), Eq.(2.1.5) becomes

$$\frac{d}{dt}\rho_{AR}(t) = r_a(\rho_A(t) - \rho_A(t,t-\tau))\rho(t) + r_a \int_{t-\tau}^{t} \frac{\partial}{\partial t}\rho_{AR}(t,t')dt' \tag{2.1.8}$$

In the absence of damping the cavity mode by a vacuum reservoir, the density operator $\rho_{AR}(t)$ evolves in time according to

$$\frac{\partial}{\partial t}\rho_{AR}(t,t) = -i[\hat{H}, \rho_{AR}(t)],$$

so that using this relation in Eq.(2.1.8) together with Eq.(2.1.4), we find

$$\frac{d}{dt}\rho_{AR}(t) = r_a(\rho_A(t) - \rho_A(t,t-\tau))\rho(t) - i[\hat{H}, \rho_{AR}(t)] \tag{2.1.9}$$

Furthermore, tracing over the atomic variables and taking into account the damping of the cavity mode by a vacuum reservoir together with the fact that

$$Tr\rho_A(t) = Tr\rho_{AR}(t-\tau) = 1,$$

we can show that

$$\frac{d}{dt}\rho = Tr_A(-i[\hat{H}, \rho_{AR}(t)]) + \frac{\kappa}{2}(2\hat{a}\rho\hat{a}^\dagger - \rho\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\rho) \qquad (2.1.10)$$

From Eqs.(2.1.1) and (2.1.10), we get

$$\frac{d}{dt}\rho = gTr_A([(|a\rangle\langle b| + |b\rangle\langle c|)\hat{a} - \hat{a}^\dagger(|b\rangle\langle a| + |c\rangle\langle b|), \rho_{AR}(t)]) + \frac{\kappa}{2}(2\hat{a}\rho\hat{a}^\dagger - \rho\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\rho)$$

In view of the property

$$[A+B, C] = [A, B] + [A, C]$$

and the defination for matrix element

$$\langle\alpha|\rho_{AR}|\beta\rangle = \rho_{\alpha\beta}$$

with $\alpha, \beta = a, b, c$ and the cyclic property of trace operator, the master equation for the cavity mode can be put in the form

$$\frac{d}{dt}\rho = g(\rho_{ab}\hat{a}^\dagger - \hat{a}^\dagger\rho_{ab} + \rho_{bc}\hat{a}^\dagger - \hat{a}^\dagger\rho_{bc} + \hat{a}\rho_{ba} - \rho_{ba}\hat{a} + \hat{a}\rho_{cb} - \rho_{cb}\hat{a}) + \frac{\kappa}{2}(2\hat{a}\rho\hat{a}^\dagger - \rho\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\rho). \qquad (2.1.11)$$

On the other hand, we see from Eq.(2.1.10) that

$$\frac{d}{dt}\rho_{\alpha\beta} = r_a(\langle\alpha|\rho_A(0)|\beta\rangle - \langle\alpha|\rho_A(t, t-\tau)|\beta\rangle\rho(t)) - iTr_A(\langle\alpha|\hat{H}\rho_{AR}|\beta(t)\rangle - \langle\alpha|\rho_{AR}\hat{H}|\beta(t)\rangle) - \gamma\rho_{\alpha\beta},$$

where the term $\rho_{\alpha\beta}$ is added to account for the decay of the atoms due to spontaneous photon emission, here γ, considered to be the same for all the three levels, is the atomic decay constant. We assume that atoms are removed from the cavity after they have decayed to a level other than the middle or the bottom level, we then see that

$$\langle\alpha|\rho_A(t, t-\tau))|\beta\rangle = 0$$

For $\alpha, \beta = a, b$ and c. Then Eq.(2.1.11) reduces to

$$\frac{d}{dt}\rho_{\alpha\beta} = r_a(\langle\alpha|\rho_A(0)|\beta\rangle\rho(t)) - iTr_A(\langle\alpha|\hat{H}\rho_{AR}|\beta(t)\rangle - \langle\alpha|\rho_{AR}\hat{H}|\beta(t)\rangle) - \gamma\rho_{\alpha\beta}. \quad (2.1.12)$$

Upon using the values of $\hat{H}$ and $\rho_A(0)$, we find

$$\frac{d}{dt}\rho_{ab} = g(\rho_{ac}\hat{a}^\dagger + \hat{a}\rho_{bb} - \rho_{aa}\hat{a}) - \gamma\rho_{ab}, \quad (2.1.13)$$

$$\frac{d}{dt}\rho_{bc} = g(\rho_{cc}\hat{a}^\dagger - \rho_{bb}\hat{a} - \hat{a}^\dagger\rho_{ac}) - \gamma\rho_{bc}, \quad (2.1.14)$$

$$\frac{d}{dt}\rho_{aa} = r_a\rho_{aa}^{(0)}\rho + g(\rho_{ab}\hat{a}^\dagger + \hat{a}\rho_{ba}) - \gamma\rho_{aa}, \quad (2.1.15)$$

$$\frac{d}{dt}\rho_{ac} = r_a\rho_{ac}^{(0)}\rho + g(\hat{a}\rho_{bc} - \rho_{ab}\hat{a}) - \gamma\rho_{ac}, \quad (2.1.16)$$

$$\frac{d}{dt}\rho_{bb} = g(\rho_{bc}\hat{a}^\dagger - \hat{a}\rho_{cb} - \hat{a}^\dagger\rho_{aa} - \rho_{ba}\hat{a}) - \gamma\rho_{bb}, \quad (2.1.17)$$

$$\frac{d}{dt}\rho_{cc} = r_a\rho_{cc}^{(0)}\rho + g(\hat{a}^\dagger\rho_{bc} + \rho_{cb}\hat{a}) - \gamma\rho_{cc}. \quad (2.1.18)$$

We confine to linear analysis and this can be achieved by dropping the g-terms in Eqs.(2.1.15), (2.1.16), (2.1.17) and (2.1.18) and applying the first-order approximation. Thus upon dropping the g-terms and applying the first-order approximation scheme, we get

$$\rho_{aa} = \frac{r_a\rho_{aa}^{(0)}\rho}{\gamma}, \quad (2.1.19)$$

$$\rho_{bb} = 0, \quad (2.1.20)$$

$$\rho_{ac} = \frac{r_a\rho_{ac}^{(0)}\rho}{\gamma}, \quad (2.1.21)$$

$$\rho_{cc} = \frac{r_a\rho_{ac}^{(0)}\rho}{\gamma}. \quad (2.1.22)$$

Combination of Eqs.(2.1.13), (2.1.19), (2.1.20) and (2.1.21) as well as Eqs.(2.1.14), (2.1.20), (2.1.21) and (2.1.22) leads to

$$\frac{d}{dt}\rho_{ab} = \frac{r_a g}{\gamma}(\rho_{ac}^{(0)}\rho\hat{a}^\dagger - \rho_{aa}^{(0)}\rho\hat{a}) - \gamma\rho_{ab}, \quad (2.1.23)$$

$$\frac{d}{dt}\rho_{bc} = \frac{r_a g}{\gamma}(\rho_{cc}^{(0)}\hat{a}\rho - \rho_{ac}^{(0)}\hat{a}^\dagger\rho) - \gamma\rho_{bc}. \quad (2.1.24)$$

Using once more the first-order approximation scheme, we can easily find that

$$\rho_{ab} = \frac{r_a g}{\gamma^2}(\rho_{ac}^{(0)}\rho\hat{a}^\dagger - \rho_{aa}^{(0)}\rho\hat{a}), \tag{2.1.25}$$

$$\rho_{bc} = \frac{r_a g}{\gamma^2}(\rho_{cc}^{(0)}\hat{a}\rho - \rho_{ac}^{(0)}\hat{a}^\dagger\rho). \tag{2.1.26}$$

But the complex conjugate of ρ_{ab} and ρ_{bc} are ρ_{ba} and ρ_{cb}, respectively. Hence

$$\rho_{ba} = \frac{r_a g}{\gamma^2}(\hat{a}\rho\rho_{ac}^{(0)} - \hat{a}^\dagger\rho\rho_{aa}^{(0)}) \tag{2.1.27}$$

$$\rho_{cb} = \frac{r_a g}{\gamma^2}(\rho\hat{a}^\dagger\rho_{cc}^{(0)} - \rho\hat{a}\rho_{ca}^{(0)}) \tag{2.1.28}$$

Using Eqs.(2.1.25), (2.1.26), (2.1.27) and (2.1.28) in Eq.(2.1.11), we get

$$\begin{aligned}\frac{d}{dt}\rho \quad &= \frac{A\rho_{aa}^{(0)}}{2}(2\hat{a}^\dagger\rho\hat{a} - \rho\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\rho)\\ &+\frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)(2\hat{a}\rho\hat{a}^\dagger - \rho\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\rho)\\ &+\frac{\rho_{ac}^{(0)}A}{2}(\rho\hat{a}^{\dagger 2} + \hat{a}^{\dagger 2}\rho - 2\hat{a}^\dagger\rho\hat{a}^\dagger)\\ &+\frac{A\rho_{ca}^{(0)}}{2}(\rho\hat{a}^2 + \hat{a}^2\rho - 2\hat{a}\rho\hat{a}),\end{aligned} \tag{2.1.29}$$

where

$$A = \frac{r_a g^2}{\gamma^2} \tag{2.1.30}$$

is the linear gain coefficient. It is worth mentioning that the quantum properties of the light genereted by three-level laser are determined by the master equation (2.1.29). It is easy to observe that with $\rho_{aa}^{(0)} = 1$ and $\rho_{ac}^{(0)} = \rho_{ca}^{(0)} = 0$. This equation reduces to the master equation for a two-level laser operating below threshold.

2.2 c-number Langevin equations

The dynamics of a cavity mode of a three-level laser coupled to vacuum reservior can be described by the quantum Langevin equation in which the time evolution of the cavity mode is carried by the operators. To this end, we seek to find the quantum Langevin equations applying the density operator. The expectation value of any arbitirary operator A is

$$\begin{aligned}\langle A\rangle &= Tr(\rho A)\\ \frac{d\langle A\rangle}{dt} &= Tr(\frac{d\rho}{dt}A)\end{aligned}$$

Hence

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = Tr(\frac{d\rho}{dt}\hat{a}(t)) \tag{2.2.1}$$

In view of Eqs.(2.1.29) and (2.2.1), we find

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\rangle &= \frac{A\rho_{aa}^{(0)}}{2}Tr(2\hat{a}^{\dagger}(t)\rho\hat{a}(t)\hat{a}(t) - \rho\hat{a}(t)\hat{a}^{\dagger}(t)\hat{a}(t) - \hat{a}(t)\hat{a}^{\dagger}(t)\rho\hat{a}(t))\\ &+\frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)Tr(2\hat{a}(t)\rho\hat{a}^{\dagger}(t)\hat{a}(t) - \rho\hat{a}^{\dagger}(t)\hat{a}(t)\hat{a}(t) - \hat{a}^{\dagger}(t)\hat{a}(t)\rho\hat{a}(t))\\ &+\frac{\rho_{ac}^{(0)}A}{2}Tr(\rho\hat{a}^{\dagger 2}(t)\hat{a}(t) + \hat{a}^{\dagger 2}(t)\hat{a}(t)\rho - 2\hat{a}^{\dagger}(t)\rho\hat{a}^{\dagger}(t)\hat{a}(t))\\ &+\frac{A\rho_{ca}^{(0)}}{2}Tr(\rho\hat{a}^{3}(t) + \hat{a}^{3}(t)\rho - 2\hat{a}(t)\rho\hat{a}(t)\hat{a}(t)).\end{aligned}$$

Using the cyclic property of trace operator, we get

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\rangle &= \frac{A\rho_{aa}^{(0)}}{2}\langle\hat{a}(t)\rangle - \frac{1}{2}(A\rho_{cc}^{(0)} + \kappa)\langle\hat{a}(t)\rangle\\ &= -\frac{\mu}{2}\langle\hat{a}(t)\rangle,\end{aligned} \tag{2.2.2}$$

where

$$\mu = A(\rho_{cc}^{(0)} - \rho_{aa}^{(0)}) + \kappa. \tag{2.2.3}$$

We also note that

$$\frac{d}{dt}\langle\hat{a}^{\dagger}(t)\rangle = -\frac{\mu}{2}\langle\hat{a}^{\dagger}(t)\rangle \tag{2.2.4}$$

In the same fashion, one can show that

$$\frac{d}{dt}\langle\hat{a}(t)\hat{a}(t)\rangle = -\mu\langle\hat{a}^{2}(t)\rangle + A\rho_{ac}^{(0)} \tag{2.2.5}$$

and

$$\frac{d}{dt}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle = -\mu\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle + A\rho_{aa}^{(0)}. \tag{2.2.6}$$

We note that the c-number Langevin equations corrosponding to Eqs.(2.2.2), (2.2.4), (2.2.5) and (2.2.6) are

$$\frac{d}{dt}\langle\alpha\rangle = -\frac{\mu}{2}\langle\alpha(t)\rangle, \tag{2.2.7}$$

$$\frac{d}{dt}\langle\alpha^*(t)\rangle = -\frac{\mu}{2}\langle\alpha^*(t)\rangle, \tag{2.2.8}$$

$$\frac{d}{dt}\langle\alpha(t)\alpha(t)\rangle = -\mu\langle\alpha(t)\alpha(t)\rangle + A\rho_{ac}^{(0)}, \tag{2.2.9}$$

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = -\mu\langle\alpha^*(t)\alpha(t)\rangle + A\rho_{aa}^{(0)}. \tag{2.2.10}$$

The steady state solutions of the c-number Langevin equations are

$$\langle\alpha(t)\rangle_{ss} = 0,$$

$$\langle\alpha^2(t)\rangle_{ss} = \frac{A\rho_{ac}^{(0)}}{\mu},$$

and

$$\langle\alpha^*(t)\alpha(t)\rangle_{ss} = \frac{A\rho_{aa}^{(0)}}{\mu}.$$

On the basis of Eq.(2.2.7), one can write

$$\frac{d}{dt}\alpha(t) = -\frac{\mu}{2}\alpha(t) + f(t), \tag{2.2.11}$$

where $f(t)$ is a noise force whose correlation properties remain to be detrmined. Taking the expectation value of Eq.(2.2.11) and comparing it with Eq.(2.2.7), we see that the expectation value of the noise force is zero

$$\langle f(t)\rangle = 0 \tag{2.2.12}$$

As $\alpha(t)$ is simply a c-number function, ordering does not affect the c-number Langevin equations. Therefore, we see that

$$\frac{1}{2}\frac{d}{dt}(\alpha(t)\alpha(t)) = \alpha(t)\frac{d\alpha(t)}{dt}$$

But upon multiplying Eq.(2.2.11) by $\alpha(t)$, we find

$$\alpha(t)\frac{d\alpha(t)}{dt} = -\frac{\mu}{2}\alpha^2(t) + \alpha(t)f(t)$$

Hence

$$\frac{1}{2}\frac{d}{dt}(\alpha(t)\alpha(t)) = -\frac{\mu}{2}\alpha^2(t) + \alpha(t)f(t),$$

from which follows

$$\frac{d}{dt}\langle\alpha(t)\alpha(t)\rangle = -\mu\langle\alpha^2(t)\rangle + 2\langle\alpha(t)f(t)\rangle. \tag{2.2.13}$$

In view of Eqs.(2.2.9) and (2.2.13), we see

$$\langle\alpha(t)f(t)\rangle = \frac{A\rho_{ac}^{(0)}}{2}. \tag{2.2.14}$$

The formal solution of Eq.(2.2.11) can be written as

$$\alpha(t) = \alpha(0)e^{-\mu t/2} + \int_0^t e^{-\mu(t-t')/2} f(t')dt' \tag{2.2.15}$$

Upon multiplying by $f(t)$ and taking expectation value, there follows

$$\langle\alpha(t)f(t)\rangle = \langle\alpha(0)f(t)\rangle e^{-\mu t/2} + \int_0^t e^{-\mu(t-t')/2}\langle f(t)f(t')\rangle dt'$$

Noting the fact that a noise force at a time t should not affect system variables at earlier time, we get

$$\langle\alpha(t)f(t)\rangle = \int_0^t e^{-\mu(t-t')/2}\langle f(t)f(t')\rangle dt' \tag{2.2.16}$$

From Eqs.(2.2.14) and (2.2.16), we see that

$$\int_0^t e^{-\mu(t-t')/2}\langle f(t)f(t')\rangle dt' = \frac{A\rho_{ac}^{(0)}}{2}$$

Now on the basis of the relation

$$\int_0^t e^{-a(t-t')}\langle f(t)g(t')\rangle dt' = E$$

We assert that

$$\langle f(t)g(t')\rangle = 2E\delta(t-t'),$$

where E is a constant or some function of time t. We then see that

$$\langle f(t)f(t')\rangle = A\rho_{ac}^{(0)}\delta(t-t'). \qquad (2.2.17)$$

Furthermore, taking the sum of equations obtained on multiplying Eq.(2.2.11) by $\alpha^*(t)$ from the left and its complex conjugate by $\alpha(t)$ from the right, we find

$$\alpha(t)\frac{d\alpha^*(t)}{dt} + \frac{d\alpha(t)}{dt}\alpha^*(t) = -\mu\alpha^*(t)\alpha(t) + \alpha(t)f^*(t) + \alpha^*(t)f(t)$$

$$\frac{d}{dt}\langle\alpha(t)\alpha^*(t)\rangle = -\mu\langle\alpha^*(t)\alpha(t)\rangle + \langle f^*(t)\alpha(t)\rangle + \langle\alpha^*(t)f(t)\rangle \qquad (2.2.18)$$

By comparing Eqs.(2.2.10) with (2.2.18), we find

$$\langle f^*(t)\alpha(t) + \alpha^*(t)f(t)\rangle = A\rho_{aa}^{(0)}. \qquad (2.2.19)$$

In the same way from Eq.(2.2.15) and its complex conjugate, one can easily get

$$\langle f^*(t)\alpha(t)\rangle = \int_0^t e^{-\mu(t-t')/2}\langle f^*(t)f(t')\rangle dt' \qquad (2.2.20)$$

$$\langle\alpha^*(t)f(t)\rangle = \int_0^t e^{-\mu(t-t')/2}\langle f^*(t')f(t)\rangle dt' \qquad (2.2.21)$$

Using Eqs.(2.2.20) and (2.2.21) together with the assumption that

$$\langle f^*(t)f(t')\rangle = \langle f(t')f^*(t)\rangle$$

in with Eq.(2.2.19), we arrive at

$$2\int_0^t e^{-\mu(t-t')/2}\langle f(t')f^*(t)\rangle dt' = A\rho_{aa}^{(0)},$$

from which we assert

$$\langle f^*(t)f(t')\rangle = A\rho_{aa}^{(0)}\delta(t-t') \qquad (2.2.22)$$

It is worth mentioning that Eqs.(2.2.12), (2.2.17) and (2.2.22) represent the correlation properties of the noise force $f(t)$ associated with the normal ordering.

In order to find the solution of Eq.(2.2.11), we introduce a new variable defined by

$$\alpha_\pm(t) = \alpha^*(t) \pm \alpha(t). \qquad (2.2.23)$$

$$\frac{d}{dt}\alpha_{\pm}(t) = \frac{d}{dt}\alpha^*(t) \pm \frac{d}{dt}\alpha(t).$$

Using Eq.(2.2.11) and its complex conjugate, we can get

$$\frac{d}{dt}\alpha_{\pm}(t) = -\frac{\mu}{2}\alpha_{\pm}(t) + f^*(t) \pm f(t). \tag{2.2.24}$$

We see that this equation does not have a well behaved solution for $\kappa < A(\rho_{aa}^{(0)} - \rho_{cc}^{(0)})$. we then define $\kappa = A(\rho_{aa}^{(0)} - \rho_{cc}^{(0)})$ as a threshold condition. For $\kappa > A(\rho_{aa}^{(0)} - \rho_{cc}^{(0)})$ the solution of Eq.(2.2.24) can be written as

$$\alpha_{\pm}(t) = \alpha_{\pm}(0)e^{-\mu t/2} + \int_0^t e^{-\mu(t-t')/2}(f^*(t) \pm f(t))dt' \tag{2.2.25}$$

In view of Eq.(2.2.23), we can find

$$\alpha(t) = A_+(t)\alpha(0) + B_+(t) - B_-(t), \tag{2.2.26}$$

Where

$$A_+(t) = e^{-\mu t/2}$$

and

$$B_{\pm}(t) = \frac{1}{2}\int_0^t e^{-\mu(t-t')/2}(f^*(t) \pm f(t))dt'.$$

The expectation value of α can be developed from Eq.(2.2.26) as follows

$$\langle\alpha(t)\rangle = A_+(t)\langle\alpha(0)\rangle + \langle B_+(t)\rangle - \langle B_-(t)\rangle$$

Assuming the cavity mode initially to be in vacuum state, we have

$$\begin{aligned}
\langle\alpha(t)\rangle &= \langle B_+(t)\rangle - \langle B_-(t)\rangle \\
&= \frac{1}{2}\int_0^t e^{-\mu(t-t')/2}(\langle f^*(t)\rangle + \langle f(t)\rangle)dt' - \frac{1}{2}\int_0^t e^{-\mu(t-t')/2}(\langle f^*(t)\rangle - \langle f(t)\rangle)dt' \\
&= \int_0^t e^{-\mu(t-t')/2}\langle f(t)\rangle dt'
\end{aligned}$$

From Eq.(2.2.12), we have

$$\langle f(t)\rangle = \langle f^*(t)\rangle = 0.$$

Hence,

$$\langle \alpha(t) \rangle = 0. \tag{2.2.27}$$

In view of Eqs.(2.2.7) and (2.2.27), one can see that α is a Gaussian variable with zero mean.

2.3 Q-function

We are now in a position to obtain the Q-function for light produced by a degenerate three-level laser. The Q-function can be expressed in terms of the antinormally ordered characteristic function as [1]

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi^2} \int d^2 z \phi_a(z^*, z, t) e^{z^*\alpha - z\alpha^*}, \tag{2.3.1}$$

where $\phi_a(z^*, z, t)$ is the antinormally ordered characteristic equation defind by

$$\phi_a(z^*, z, t) = Tr(\rho e^{-z^*\hat{a}} e^{z\hat{a}^\dagger}).$$

Using the identity

$$e^A e^B = e^B e^A e^{[A,B]},$$

we have

$$\phi_a(z^*, z, t) = e^{-zz^*} Tr(\rho e^{z\hat{a}^\dagger} e^{-z^*\hat{a}}).$$

This can be written in terms of the c-number variables associated with normal ordering as

$$\phi_a(z^*, z, t) = e^{-zz^*} \langle e^{z\alpha^* - z^*\alpha} \rangle. \tag{2.3.2}$$

Since α is a Gaussian variable with zero mean, one can put Eq.(2.3.2) in the form

$$\begin{aligned} \phi_a(z^*, z, t) &= e^{-zz^*} exp(\frac{1}{2}\langle (z\alpha^* - z^*\alpha)^2 \rangle) \\ &= e^{-zz^*} exp(\langle \alpha^{*2} \rangle z^2/2 + \langle \alpha^2 \rangle z^{*2}/2 - zz^* \langle \alpha^* \alpha \rangle) \end{aligned} \tag{2.3.3}$$

Upon setting

$$\rho_{ac}^{(0)} = \rho_{ca}^{(0)},$$

we see that

$$\langle f(t)f(t')\rangle = \langle f^*(t)f^*(t')\rangle.$$

Thus using Eq.(2.2.26) and its complex conjugate, we get

$$B_+^* = B_+,$$

$$B_-^* = -B_-.$$

With the help of these relations, we find that

$$\langle B_+B_-\rangle = 0,$$

$$\begin{aligned}\langle \alpha^2\rangle \quad &= \langle (B_+ - B_-)^2\rangle \\ &= \langle B_+^2\rangle + \langle B_-^2\rangle - 2\langle B_+B_-\rangle \\ &= \langle B_+^2\rangle + \langle B_-^2\rangle,\end{aligned}$$

$$\langle \alpha^{*2}\rangle = \langle B_+^2\rangle + \langle B_-^2\rangle,$$

and

$$\langle \alpha\alpha^*\rangle = \langle B_+^2\rangle - \langle B_-^2\rangle.$$

But

$$\langle B_+^2\rangle = \frac{1}{2}\langle \int e^{(-\mu(2t-t'-t'')/2)}[f(t')f(t'') + f^*(t')f(t'')]dt'dt''\rangle.$$

In view of Eqs.(2.2.17), (2.2.22) and the property of kronecker delta function

$$\int_p^q f(x)\delta(x-c)dx = \frac{1}{2}f(c), \tag{2.3.4}$$

for $c = q$ or $c = p$, then we find

$$\begin{aligned}\langle B_+^2\rangle \quad &= \frac{A(\rho_{ac}^{(0)} + \rho_{aa}^{(0)})}{4}\int_0^t e^{-\mu(t-t'')/2}dt'', \\ &= \frac{A(\rho_{ac}^{(0)} + \rho_{aa}^{(0)})}{2\mu}(1 - e^{\frac{-\mu t}{2}}),\end{aligned}$$

and

$$\langle B_-^2 \rangle = \frac{A(\rho_{ac}^{(0)} - \rho_{aa}^{(0)})}{2\mu}(1 - e^{\frac{-\mu t}{2}}).$$

Hence,

$$\langle \alpha\alpha^* \rangle = \frac{A\rho_{aa}^{(0)}}{\mu}(1 - e^{-\mu t/2})$$

and

$$\langle \alpha^2 \rangle = \langle \alpha^{*2} \rangle = \frac{A\rho_{ac}^{(0)}}{\mu}(1 - e^{-\mu t/2}).$$

Then

$$\phi_a(z^*, z, t) = e^{-zz^*} exp\bigg(\frac{A\rho_{ac}^{(0)}}{\mu}(1 - e^{-\mu t/2})z^2/2 + \frac{A\rho_{ac}^{(0)}}{\mu}(1 - e^{-\mu t/2})z^{*2}/2 - zz^*\frac{A\rho_{aa}^{(0)}}{\mu}(1 - e^{-\mu t/2})\bigg).$$

It is convenient to introduce a new parameter defind by

$$\rho_{aa}^{(0)} = \frac{1-\eta}{2},$$

where $-1 \leq \eta \leq 1$. Using the fact

$$\rho_{aa}^{(0)} + \rho_{cc}^{(0)} = 1,$$

together with

$$|\rho_{ac}^{(0)}|^2 = \rho_{aa}^{(0)}\rho_{cc}^{(0)}.$$

there follows

$$\rho_{cc}^{(0)} = \frac{1+\eta}{2}$$

and

$$|\rho_{ac}^{(0)}| = \frac{\sqrt{1-\eta^2}}{2}.$$

In view of these relations, the characteristic function reduces to

$$\phi_a(z^*, z, t) = exp[-azz^* + b(z^2 + z^{*2})/2], \tag{2.3.5}$$

where

$$a = 1 + \frac{A(1-\eta)}{2(A\eta + \kappa)}(1 - e^{-\mu t/2}),$$

$$b = \frac{A\sqrt{1-\eta^2}}{2(A\eta+\kappa)}(1-e^{-\mu t/2}).$$

Using Eq.(2.3.5) in Eq.(2.3.1), we get

$$Q(\alpha^*,\alpha,t) = \frac{1}{\pi^2}\int d^2z exp\Big(-azz^* + -\alpha^* z + \alpha z^* + b(z^2+z^{*2})/2\Big) \tag{2.3.6}$$

Carrying out the integration using the identity

$$\int \frac{d^2z}{\pi} exp\Big(-azz^*+bz+cz^*+A'z^2+B'z^{*2}\Big) = \Big[\frac{1}{a^2-4A'B'}\Big]^{1/2} exp\Big[\frac{abc+A'c^2+B'b^2}{a^2-4A'B'}\Big], a>0 \tag{2.3.7}$$

we find

$$Q(\alpha^*,\alpha,t) = \frac{(u^2-v^2)^{1/2}}{\pi} exp\Big(-u\alpha\alpha^* + v(\alpha^2+\alpha^{*2})/2\Big), \tag{2.3.8}$$

in which

$$u = \frac{a}{a^2-b^2}$$

and

$$v = \frac{b}{a^2-b^2}.$$

Now integrating the Q-function over α

$$\int d^2\alpha Q(\alpha,t) = \int d^2\alpha \frac{(u^2-v^2)^{1/2}}{\pi} exp(-u\alpha\alpha^* + v(\alpha^2+\alpha^{*2})/2),$$

so using Eq.(2.3.7) once more, we get

$$\int d^2\alpha Q(\alpha,t) = 1.$$

This shows that the Q-function is normalized.

2.4 Photon statistics

The statistical properties of a light beam is described in terms of the mean photon number, the variance of the photon number and the photon number distribution. Here we wish to calculate the mean photon number, the variance of photon number and the photon number distribution of the light generated by degenerate three-level laser employing the Q-function.

2.4.1 The mean photon number

The number of photons for degenerate three-level laser light beam is represented by the operator $\hat{n} = \hat{a}^\dagger\hat{a}$ and the mean photon number for this light beam is expressible as $\langle\hat{n}\rangle$=$\langle\hat{a}^\dagger\hat{a}\rangle$. But $\langle\hat{a}^n\hat{a}^{\dagger m}\rangle$ is product of operators in the antinormal order (antinormal moments) which can be evaluated using Q-function as follows .

$$\langle\hat{a}^n\hat{a}^{\dagger m}\rangle = Tr(\rho\hat{a}^n\hat{a}^{\dagger m})$$

From the cyclic property of trace operator

$$\langle\hat{a}^n\hat{a}^{\dagger m}\rangle = Tr(\hat{a}^{\dagger m}\rho\hat{a}^n),$$

the completness relation for coherent states of light leads to

$$\begin{aligned}\langle\hat{a}^n\hat{a}^{\dagger m}\rangle \quad &= Tr\bigg(\int \frac{d^2\alpha}{\pi}\hat{a}^{\dagger m}\rho\hat{a}^n|\alpha\rangle\langle\alpha|\bigg) \\ &= \int \frac{d^2\alpha}{\pi}Tr\bigg(\hat{a}^{\dagger m}\rho\hat{a}^n|\alpha\rangle\langle\alpha|\bigg) \\ &= \int \frac{d^2\alpha}{\pi}\langle\alpha|\hat{a}^{\dagger m}\rho\hat{a}^n|\alpha\rangle \\ &= \int d^2\alpha\frac{\langle\alpha|\rho|\alpha\rangle}{\pi}\alpha^{*m}\alpha^n \\ &= \int d^2\alpha Q(\alpha,\alpha^*)\alpha^{*m}\alpha^n.\end{aligned}$$

in which

$$Q = \frac{\langle\alpha|\rho|\alpha\rangle}{\pi}$$

is the Q-function and then follows

$$\bar{n} = \int d^2\alpha Q(\alpha,\alpha^*)(\alpha^*\alpha - 1)$$

where $\alpha^*\alpha - 1$ is the c-number function corrosponding to the number operator $\hat{n}$ in the antinormal order.

Using Eq.(2.3.8), we get

$$\begin{aligned}
\bar{n} &= \int d^2\alpha Q(\alpha,\alpha^*)\alpha^*\alpha - \int d^2\alpha Q(\alpha,\alpha^*) \\
&= \left((u^2-v^2)^{1/2}\int \frac{d^2\alpha}{\pi} exp(-u\alpha\alpha^* + v(\alpha^2+\alpha^{*2})/2)\alpha^*\alpha\right) - 1 \\
&= (u^2-v^2)^{1/2}\frac{-\partial}{\partial u}\left(\int \frac{d^2\alpha}{\pi} exp(-u\alpha\alpha^* + v(\alpha^2+\alpha^{*2})/2)\right) - 1,
\end{aligned}$$

In view of Eq.(2.3.7), this reduces to

$$\bar{n} = \frac{u}{u^2-v^2} - 1, \tag{2.4.1}$$

The mean number of photons for the light beam at steady state is

$$\bar{n} = \frac{A(1-\eta)}{2(A\eta+\kappa)}. \tag{2.4.2}$$

which is identical with the expresion obtaind by Fesseha [1] upon direct use of steady state solutions of c-number Langevin equations.

2.4.2 The variance of photon number

In this section we find the variance of the photon number, whose square root give us the uncertainity in photon number for the light beam, employing the Q-function. The variance of photon number for light beam is expressed as [1]

$$(\Delta n)^2 = \langle \hat{n}^2 \rangle - \bar{n}^2 \tag{2.4.3}$$

Since the mean photon number is found earlier, the only unknown is $\langle \hat{n}^2 \rangle$ which can be also obtained using the Q-function of the laser light beam. This needs the c-number function corrosponding to the operator in the antinormal order. To do so, the number operator is expressed in terms of the annihilation and creation operators in the antinormal order using the commutation relation $a^{\dagger}\hat{a} = \hat{a}\hat{a}^{\dagger} - 1$ as follows

$$\begin{aligned}\langle \hat{n}^2 \rangle &= \langle \hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}\hat{a} \rangle \\ &= \langle (\hat{a}\hat{a}^{\dagger} - 1)^2 \rangle \\ &= \langle \hat{a}\hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger} - 2\hat{a}\hat{a}^{\dagger} + 1 \rangle \\ &= \langle \hat{a}^2\hat{a}^{\dagger 2} - 3\hat{a}\hat{a}^{\dagger} + 1 \rangle \\ &= \langle \hat{a}^2\hat{a}^{\dagger 2} \rangle - 3\bar{n} - 2.\end{aligned}$$

But

$$\begin{aligned}\langle \hat{a}^2\hat{a}^{\dagger 2} \rangle &= \int d^2\alpha Q(\alpha, \alpha^*, t)\alpha^2\alpha^{*2} \\ &= (u^2 - v^2)^{1/2} \int \frac{d^2\alpha}{\pi} exp\Big(- u\alpha\alpha^* + v(\alpha^2 + \alpha^{*2})/2 \Big)\alpha^2\alpha^{*2} \\ &= (u^2 - v^2)^{1/2} \frac{\partial^2}{\partial u^2}\Big(\int \frac{d^2\alpha}{\pi} exp(-u\alpha\alpha^* + v(\frac{\alpha^2}{2} + \frac{\alpha^{*2}}{2})) \Big) \\ &= (u^2 - v^2)^{1/2} \frac{\partial^2}{\partial u^2}\Big(\frac{1}{\sqrt{u^2 - v^2}} \Big),\end{aligned}$$

Carrying out the differentiation gives

$$\langle a^2\hat{a}^{\dagger 2} \rangle = \frac{2u^2 + v^2}{(u^2 - v^2)^2}. \tag{2.4.4}$$

From Eqs.(2.4.3), (2.4.4) and (2.4.5) the variance of photon number is

$$(\Delta n)^2 = \frac{2u^2 + v^2}{(u^2 - v^2)^2} - \bar{n}^2 - 3\bar{n} - 2,$$

with steady state value

$$(\Delta n)^2_{ss} = \frac{A(1 - \eta)(A + A\eta + \kappa)}{2(A\eta + \kappa)^2}. \tag{2.4.5}$$

which can be expressed in terms of $\bar{n}$ as

$$(\Delta n)^2_{ss} = \bar{n}(1 + \frac{A}{A\eta + \kappa}).$$

This shows that the photon statistics is super-Poissonian for all values of η in the interval $0 \leq \eta \leq 1$ as can be seen in Fig.2.2

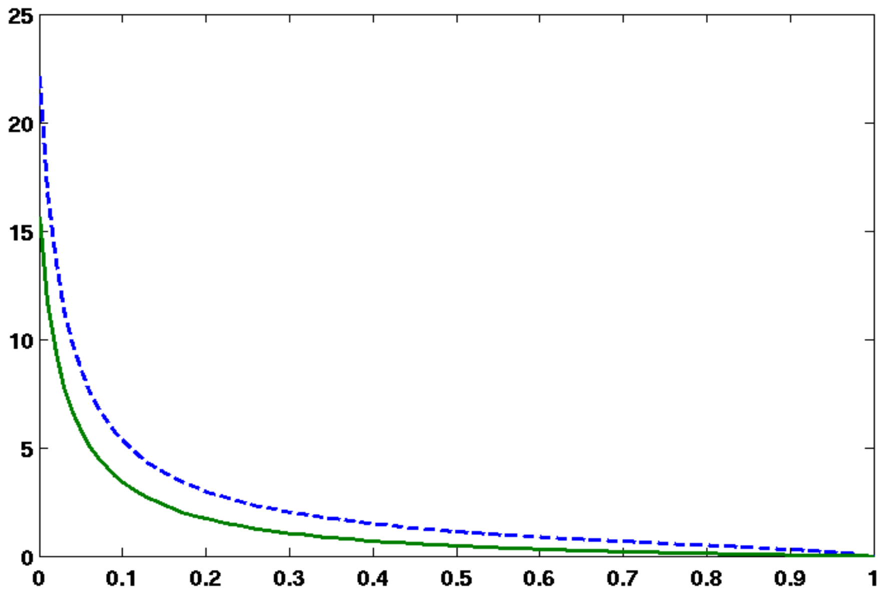

Figure 2.2: Plots of the mean number of photons (solid line) and the uncertainity in number of photons (broken line) at steady state versus η for $\kappa = 0.8$ and $A = 25$.

2.4.3 The photon number distribution

We now seek to study the photon number distribution for the light beam employing the Q-function (2.3.8). The photon number distribution of any light is expressible in terms of the Q-function as [1]

$$\begin{aligned} P(n,t) &= \frac{\pi}{n!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^{n}}\Big[Q(\alpha^*,\alpha,t)e^{\alpha^*\alpha}\Big]_{\alpha=\alpha^*=0} \\ &= \frac{(u^2-v^2)^{1/2}}{n!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^{n}}\Big[exp((1-u)\alpha\alpha^* + v(\alpha^2+\alpha^{*2})/2)\Big]_{\alpha=\alpha^*=0.} \end{aligned} \quad (2.4.6)$$

Upon using the power series expansion, one finds

$$e^{(1-u)\alpha^*\alpha} = \sum_l \frac{(1-u)^l\alpha^l\alpha^{*l}}{l!}$$

$$e^{v\alpha^2} = \sum_j \frac{v^j\alpha^{2j}}{j!}$$

$$e^{v\alpha^{*2}} = \sum_r \frac{(v)^r\alpha^{*2r}}{r!}$$

so that

$$P(n,t) = \frac{(u^2-v^2)^{1/2}}{n!}\sum_{ljr}\frac{(v)^{r+j}(1-u)^l}{l!r!j!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^{n}}\Big[\alpha^{2j+l}\alpha^{*2r+l}\Big]_{\alpha=\alpha^*=0}.$$

By the help of the relation

$$\frac{\partial^n x^m}{\partial \alpha^n} = \frac{m!}{(m-n)!} x^{m-n},$$

we get

$$P(n,t) = \frac{(u^2 - v^2)^{1/2}}{n!} \sum_{ljr} \frac{(v)^{r+j}(1-u)^l (2j+l)!(2r+l)!}{l!r!j!(2j+l-n)!(2r+l-n)!} \left[\alpha^{2j+l-n} \alpha^{*2r+l-n} \right]_{\alpha=\alpha^*=0}. \tag{2.4.7}$$

If we apply the condition $\alpha = \alpha^* = 0$, the photon number distribution function under Eq.(2.4.8) vanishes. This function will have a non-zero value only for the condition $2r+l = n$ and $2j+l = n$, from which follows $l = n-2r$, $l = n-2j$ and $j = r$

$$P(n,t) = (u^2 - v^2)^{1/2} \sum_{ljr} \frac{(v)^{r+j}(1-u)^l (2j+l)!(2r+l)!}{l!r!j!(2j+l-n)!(2r+l-n)!} \delta_{2j+l,n} \delta_{2r+l,n}$$

From the property of kronecker delta function, we can rewrite it to get

$$P(n,t) = (u^2 - v^2)^{1/2} \sum_{j}^{[n]} n! \frac{(v)^j (1-u)^{n-2j}}{j!(n-2j)!} \sum_{r}^{[n]} \frac{(v)^r}{r!}$$

To avoid the factorial of a negative number we set $n - 2j \geq 0$, then

$$r = j \leq n/2$$

hence

$$P(n,t) = (u^2 - v^2)^{1/2} \sum_{r=0}^{[n]} \frac{n!(1-u)^{n-2r} v^{2r}}{2^{2r}(r!)^2 (n-2r)!},$$

with a steady state value

$$\begin{aligned} P(n,t) &= \left[\frac{2(A\eta + \kappa)^2}{A^2\eta^2(1+\eta) + 2A(1+\eta)\kappa + 2\kappa^2} \right]^{1/2} \\ &\times \sum_{r=0}^{[n]} \frac{n! \left(\frac{A(1-\eta)+\kappa}{A^2\eta^2(1+\eta)+2A(1+\eta)\kappa+2\kappa^2} \right)^{n-2r} \left(\frac{A\sqrt{1-\eta^2}(A\eta+\kappa)}{A^2\eta^2(1+\eta)+2A(1+\eta)\kappa+2\kappa^2} \right)^{2r}}{2^{2r}(r!)^2(n-2r)!}, \end{aligned}$$

where $[n] = \frac{n}{2}$ for even n and $[n] = \frac{n-1}{2}$ for odd n.

This is the photon number distribution for the light produced by degenerate three-level lasers which shows that the photon number distribution decreases with the number of photons [1].

2.5 Quadrature variance

The squeezing properties of single mode light are described by two quadrature operators defined as

$$\hat{a}_+ = \hat{a} + \hat{a}^\dagger$$

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}),$$

where $\hat{a}_+$ and $\hat{a}_-$ are Hermitian operators representing the physical quantities called plus and minus quadratures, respectively while $\hat{a}^\dagger$, $\hat{a}$ are the creation and annihilation oprators of light obtaind from degenerate three-level laser. The quadrature variance can be expressed in terms of the quadrature operators as

$$(\Delta a_\pm)^2 = \langle \hat{a}_\pm^2 \rangle - \langle \hat{a}_\pm \rangle^2.$$

The explicit form of quadrature variance for the plus quadrature can be expressed in terms of the creation and annihilation operators as

$$(\Delta a_+)^2 = 1 + \langle \hat{a}^2 \rangle + \langle \hat{a}^{\dagger 2} \rangle + 2\langle \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a} \rangle^2 - \langle \hat{a}^\dagger \rangle^2 - 2\langle \hat{a} \rangle \langle \hat{a}^\dagger \rangle \qquad (2.5.1)$$

In the same way quadrature variance of the minus quadrature will be

$$(\Delta a_-)^2 = 1 + 2\langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a} \rangle^2 + \langle \hat{a}^\dagger \rangle^2 - \langle \hat{a}^2 \rangle - \langle \hat{a}^{\dagger 2} \rangle - 2\langle \hat{a} \rangle \langle \hat{a}^\dagger \rangle. \qquad (2.5.2)$$

But $\langle \hat{a}^\dagger \hat{a} \rangle = \bar{n}$. We can evaluate the remaining expectation values using the Q-function of light beam and the c-number variable corrosponding to each operator or product of operators as follows

$$\langle \hat{a} \rangle = \int d^2\alpha Q(\alpha, \alpha^*) \alpha$$

in which α is the c-number variable corrosponding to the annihilation operator $\hat{a}$. Upon using Eq.(2.3.8)

$$\begin{aligned}\langle\hat{a}\rangle &= (u^2-v^2)^{1/2}\int\frac{d^2\alpha}{\pi}exp(-u\alpha\alpha^*+v(\alpha^2+\alpha^{*2})/2)\alpha \\ &= \frac{\partial}{\partial p}\Bigg[(u^2-v^2)^{1/2}\int\frac{d^2\alpha}{\pi}exp(-u\alpha\alpha^*+p\alpha+v(\alpha^2+\alpha^{*2})/2)\alpha^2\alpha^{*2}]\Bigg]_{p=0} \\ &= \frac{(u^2-v^2)^{1/2}}{\sqrt{u^2-v^2}}\frac{\partial}{\partial p}\left[exp\frac{vp^2}{2(u^2-v^2)}\right]_{p=0} \\ &= 0. \end{aligned} \tag{2.5.3}$$

Similarily

$$\langle\hat{a}^\dagger\rangle = 0. \tag{2.5.4}$$

$$\begin{aligned}\langle\hat{a}^2\rangle &= (u^2-v^2)^{1/2}(\int\frac{d^2\alpha}{\pi}exp[-u\alpha\alpha^*+v(\frac{\alpha^2}{2}+\frac{\alpha^{*2}}{2})]\alpha^2 \\ &= (u^2-v^2)^{1/2}\frac{\partial^2}{\partial q^2}\left[\int\frac{d^2\alpha}{\pi}exp[-u\alpha\alpha^*+q\alpha+v(\frac{\alpha^2}{2}+\frac{\alpha^{*2}}{2})]\right]_{q=0} \\ &= \frac{\partial^2}{\partial q^2}\left[exp\frac{vq^2}{2(u^2-v^2)}\right]_{q=0} \\ &= \frac{\partial}{\partial q}\left[\frac{vq}{u^2-v^2}exp\left(\frac{vq^2}{2(u^2-v^2)}\right)\right]_{q=0} \\ &= \frac{v}{(u^2-v^2)^2}. \end{aligned} \tag{2.5.5}$$

In the same way

$$\langle\hat{a}^{\dagger 2}\rangle = \frac{v}{(u^2-v^2)^2}. \tag{2.5.6}$$

Applying Eqs.(2.5.3), (2.5.4), (2.5.5) and (2.5.6) in Eq.(2.5.1), the quadrature variance for the plus quadrature becomes

$$(\Delta a_+)^2 = 1+2\bar{n}+\frac{2v}{(u^2-v^2)^2}, \tag{2.5.7}$$

with steady state value

$$(\Delta a_+)^2 = \frac{A+A\sqrt{1-\eta^2}+\kappa}{A\eta+\kappa}. \tag{2.5.8}$$

Applying Eqs.(2.5.3), (2.5.4), (2.5.5) and (2.5.6) in Eq.(2.5.2), the quadrature variance for the minus quadrature

$$(\Delta a_{-})^2 = 1 + 2\bar{n} - \frac{2v}{(u^2 - v^2)^2}, \tag{2.5.9}$$

with steady state value

$$(\Delta a_{-})^2 = \frac{A - A\sqrt{1-\eta^2} + \kappa}{A\eta + \kappa}. \tag{2.5.10}$$

By looking at Eqs. (2.5.8) and (2.5.10), we are not able to any thing about the squeezing of the laser light. However, we can draw the graph of quadrature variance againest η for some value of A and κ to see the squeezing clearly. Fig. 2.3 indicates that the fluctuations in the minus quadrature are below the vacuum level with enhanced fluctuations in the plus quadrature. This verifies that the light mode is in a squeezed state.

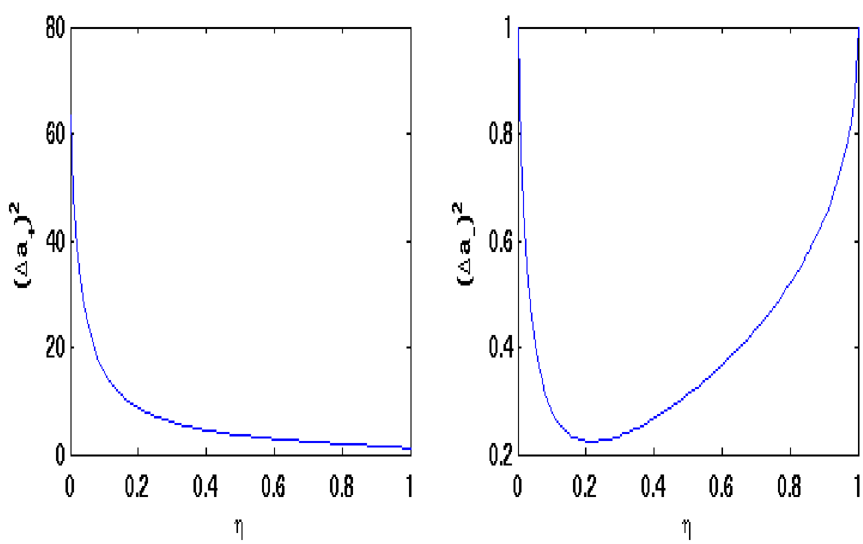

Figure 2.3: Plots of $(\Delta a_{+})^2$ and $(\Delta a_{-})^2$ versus η for $\kappa = 0.8$ and $A = 25$.

Fig. 2.4 shows that the degree of squeezing increases with the linear gain coefficient (A). It appears that almost perfect squeezing could be achieved by taking large values of

A and for small values of η. Furthermore, for $A = 100$, one can see that $(\Delta a_-)^2 = 0.1191$, which occurs at $\eta = 0.1111$. Hence a squeezed light is generated with squeezing of 88.09% below the coherent state level.

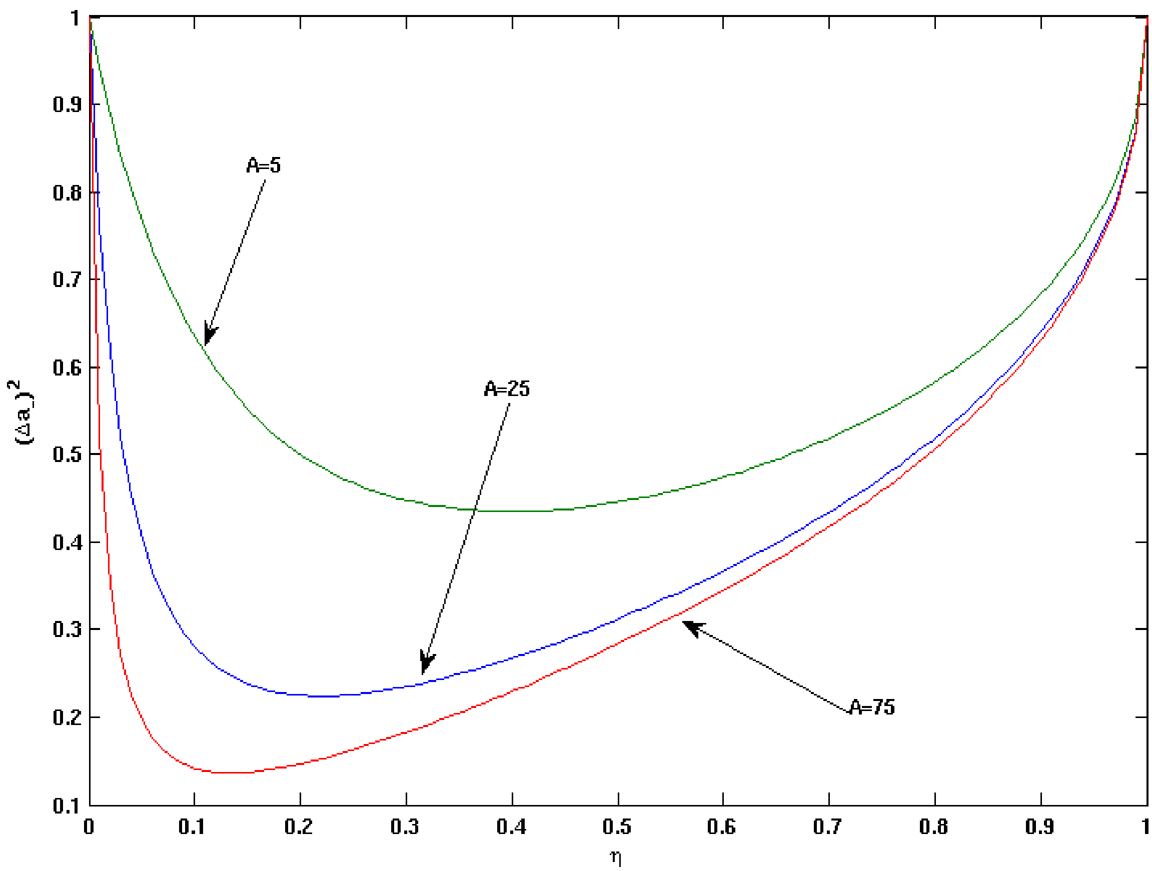

Figure 2.4: Plots of $(\Delta a_-)^2$ at steady state versus η for $\kappa = 0.8$ and $A = 5, 25, 75$.

Chapter 3

Superposition of Two Laser Light Beams

In this chapter we first seek to get the Q-function for the superposition of the light beams produced by a pair of degenerate three-level lasers. Then applying the Q-function, we calculate the mean photon number, the variance of photon number, the photon number distribution and the quadrature variance.

3.1 Q-function

A degenerate three-level laser is the source of light emitted by three-level atoms in a cavity coupled to a vacuum reservoir via a single port-mirror. Here we wish to obtain the Q-function for the superposition of two light beams produced by three-level lasers. The Q-function is used to describe the superposition of two light beams with the same frequency but may be in the same or different states. Let $\rho(\hat{a}^\dagger, \hat{a})$ be the density operator for a certain light beam. Then upon expanding this operator in normal order and using the completness relation for coherent states, we get

$$\hat{\rho}' = \int d^2\beta Q(\beta^*, \beta + \frac{\partial}{\partial\beta})|\beta\rangle\langle\beta|,$$

where $\hat{\rho}'$ is the density operator for the first light beam.

If we inject a light beam into the cavity which initially contains a light beam of the same

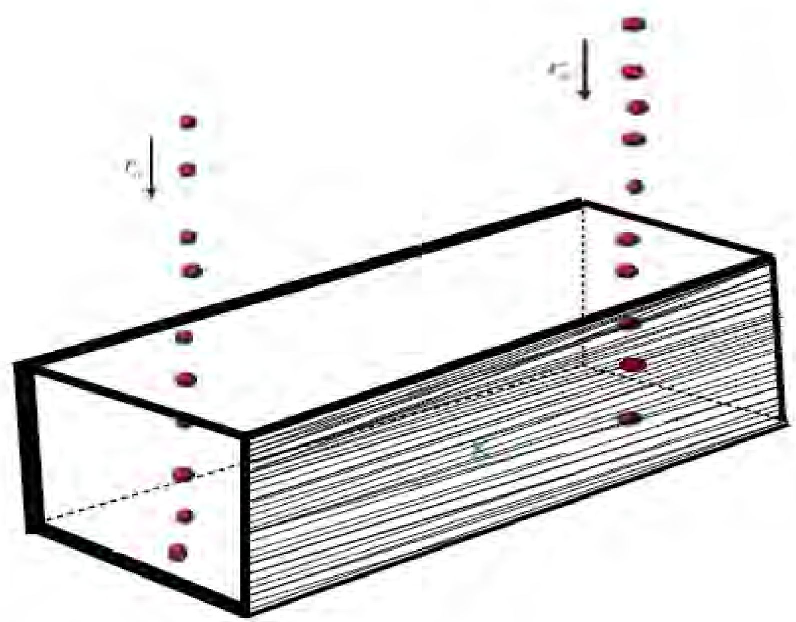

Figure 3.1: Superposition of light beams emitted from two degenerate three-level atoms injected into a cavity at a rate r_a.

frequency (Fig 3.1), the density operator for the superposition of the two light beams in the cavity is given as [1]

$$\hat{\rho} = \int d^2\gamma Q(\gamma^*, \gamma + \frac{\partial}{\partial\gamma^*})\hat{D}(\gamma)\hat{\rho}'\hat{D}^\dagger(\gamma)$$

using the value of $\hat{\rho}'$, we find

$$\hat{\rho} = \int d^2\gamma d^2\beta Q(\gamma^*, \gamma + \frac{\partial}{\partial\gamma})Q(\beta^*, \beta + \frac{\partial}{\partial\beta})|\beta + \gamma\rangle\langle\beta + \gamma|.$$

Then, the Q-function for the superposition of the two light beams from three-level lasers turns out to be

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi}\int d^2\beta d^2\gamma Q(\beta^*, \beta + \frac{\partial}{\partial\beta^*})Q(\gamma^*, \gamma + \frac{\partial}{\partial\gamma^*})$$
$$\times \quad exp[-\alpha\alpha^* - \beta\beta^* - \gamma\gamma^* + \alpha^*\beta + \alpha\beta^* + \alpha^*\gamma + \alpha\gamma^* - \beta^*\gamma - \beta\gamma^*]. \quad (3.1.1)$$

which can be rewritten as

$$Q(\alpha^*,\alpha,t) \quad = \frac{1}{\pi}\int d^2\beta d^2\gamma exp[-\alpha^*(\alpha-\beta-\gamma)]$$
$$\times Q(\beta^*,\beta+\frac{\partial}{\partial\beta})exp[\beta^*(\alpha-\beta-\gamma)]$$
$$\times Q(\gamma^*,\gamma+\frac{\partial}{\partial\gamma})exp[\gamma^*(\alpha-\beta-\gamma)] \tag{3.1.2}$$

Using the binomial theorem

$$\left(x+\frac{d}{dy}\right)^l = \sum_j^l (x)^{l-j}\left(\frac{d}{dy}\right)^j.$$

we readily find

$$Q(\beta^*,\beta+\frac{\partial}{\partial\beta})exp[\beta^*(\alpha-\beta-\gamma)] = Q(\beta^*,\alpha-\gamma)exp[\beta^*(\alpha-\beta-\gamma)],$$

and

$$Q(\gamma^*,\gamma+\frac{\partial}{\partial\gamma})exp[\gamma^*(\alpha-\beta-\gamma)] = Q(\gamma^*,\alpha-\beta)exp[\gamma^*(\alpha-\beta-\gamma)].$$

In view of the preceeding two equations, Eq.(3.1.2) takes the form

$$Q(\alpha^*,\alpha,t) \quad = \frac{1}{\pi}\int d^2\beta d^2\gamma Q(\beta^*,\alpha-\gamma)Q(\gamma^*,\alpha-\beta) \tag{3.1.3}$$
$$\times exp[-\alpha^*\alpha-\beta^*\beta-\gamma^*\gamma+\alpha^*\beta+\alpha\beta^*+\alpha^*\gamma+\alpha\gamma^*-\beta^*\gamma-\beta\gamma^*]$$

Let $Q(\gamma^*,\gamma)$ and $Q(\beta^*,\beta)$ be the Q-function of the first and second light beams, respectively. Using

$$u_1 = \frac{a_1}{a_1^2-b_1^2},$$
$$v_1 = \frac{b_1}{a_1^2-b_1^2}$$

for the first light beam and

$$u_2 = \frac{a_2}{a_2^2-b_2^2},$$
$$v_2 = \frac{b_2}{a_2^2-b_2^2}$$

for the second light beam. Together with the notion of the Q-function for the light beam given by Eq.(2.3.8). The Q-function of the first light beam becomes

$$Q(\gamma^*,\gamma,t) = \frac{(u_1^2 - v_1^2)^{1/2}}{\pi} exp(-u_1\gamma\gamma^* + v_1(\gamma^2+\gamma^{*2})/2), \quad (3.1.4)$$

then

$$Q(\gamma^*,\alpha-\beta) = \frac{(u_1^2 - v_1^2)^{1/2}}{\pi} exp(-u_1(\alpha\gamma^* - \gamma^*\beta) + v_1(\alpha^2+\gamma^{*2}+\beta^2-2\alpha\beta)/2) \quad (3.1.5)$$

In the same way, the Q-function of the second light beam is given as

$$Q(\beta^*,\alpha-\gamma) = \frac{(u_2^2 - v_2^2)^{1/2}}{\pi} exp(-u_2(\alpha\beta^* - \beta^*\gamma) + v_2(\alpha^2+\gamma^2+\beta^{*2}-2\alpha\gamma)/2) \quad (3.1.6)$$

In view of Eqs.(3.1.5) and (3.1.6) together with taking two identical light beams in which $u_1 = u_2 = u$ and $v_1 = v_2 = v$, Eq.(3.1.3) can be rewritten as

$$\begin{aligned} Q(\alpha^*,\alpha,t) &= \frac{[(u^2-v^2)^2]^{1/2}}{\pi} exp(-\alpha^*\alpha + (2v)\alpha^2/2) \\ &\times \int \frac{d^2\beta}{\pi} exp\Big[-\beta^*\beta + (\alpha^* - v\alpha)\beta + (\alpha - u\alpha)\beta^* + v\beta^{*2}/2 + v\beta^2/2\Big] \\ &\times \int \frac{d^2\gamma}{\pi} exp\Big[-\gamma^*\gamma + (\alpha^* - \beta^* + u\beta^* - v\alpha)\gamma + (\alpha - \beta - u(\alpha-\beta))\gamma^* + v\gamma^{*2}/2 + v\gamma^2/2\Big], \end{aligned}$$

Upon integrating over γ using Eq.(2.3.7), we find

$$\begin{aligned} Q(\alpha^*,\alpha,t) = & \frac{1}{\pi}\left[\frac{(u^2-v^2)^2}{1-v^2}\right]^{1/2} \\ & exp\left[\frac{-u\alpha^*\alpha}{1-v^2} + (\frac{-v^3+vu^2+v}{1-v^2})\frac{\alpha^2}{2} + \frac{v}{1-v^2}\frac{\alpha^{*2}}{2}\right] \\ & \times \int \frac{d^2\beta}{\pi} exp\Big[-\beta^*\beta(\frac{2u-u^2-v^2}{1-v^2}) \\ & +\beta\frac{\alpha^*(u-v^2)+\alpha(v^3+vu-vu^2-v)}{1-v^2} \\ & +\beta^*\frac{\alpha^*(vu-v)+\alpha(u-u^2)}{1-v^2} \\ & +\frac{2v-v^3+vu^2-2vu}{1-v^2}(\frac{\beta^{*2}}{2}+\frac{\beta^2}{2})\Big]. \end{aligned}$$

(3.1.8)

Similarily, integrating Eq.(3.1.8) over β gives the Q-function for the superposition of two light beams which is

$$Q(\alpha^*,\alpha,t)=\frac{R}{\pi}exp[-M\alpha\alpha^*+N(\frac{\alpha^2}{2}+\frac{\alpha^{*2}}{2})] \tag{3.1.9}$$

in which

$$R=\sqrt{\frac{u^2-v^2}{4-4u+u^2-v^2}},$$
$$M=\frac{2u-u^2+v^2}{4-4u+u^2-v^2},$$

and

$$N=\frac{2v}{4-4u+u^2-v^2}.$$

Integrating the Q-function over α

$$\int d^2\alpha Q(\alpha,t)=R\int\frac{d^2\alpha}{\pi}exp[-M\alpha\alpha^*+N(\frac{\alpha^2}{2}+\frac{\alpha^{*2}}{2})],$$

In view of Eq.(2.3.7), the integration gives

$$\int d^2\alpha Q(\alpha,t)=1.$$

Hence, the Q-function for the superposition of the light beams produced by a pair of degenerate three-level laser light beams is normalized.

3.2 Photon statistics

Here we wish to calculate the photon number distribution, the mean photon number and variance of the photon number for the superposition of two light beams employing the Q-function.

3.2.1 The mean photon number

The mean number of photons for the superposition of two light beams obtained from superposition of two identical degenerate three-level lasers is expressible as

$$\bar{n}=\int d^2\alpha Q(\alpha,\alpha^*)(\alpha^*\alpha-1)$$

where $\alpha^*\alpha - 1$ is the c-number function corrosponding to the operator $\hat{n}$ in the antinormal order.

Using Eq.(3.1.9), we get

$$\begin{aligned}\bar{n} &= \int d^2\alpha Q(\alpha,\alpha^*)\alpha^*\alpha - \int d^2\alpha Q(\alpha,\alpha^*)\\ &= R\bigg(\int \frac{d^2\alpha}{\pi} exp[-M\alpha\alpha^* + N(\frac{\alpha^2}{2} + \frac{\alpha^{*2}}{2})]\alpha^*\alpha\bigg) - 1\\ &= R\frac{-\partial}{\partial M}\bigg(\int \frac{d^2\alpha}{\pi} exp[-M\alpha\alpha^* + N(\frac{\alpha^2}{2} + \frac{\alpha^{*2}}{2})]\bigg) - 1.\end{aligned}$$

In view of Eq.(2.3.7), this reduces to

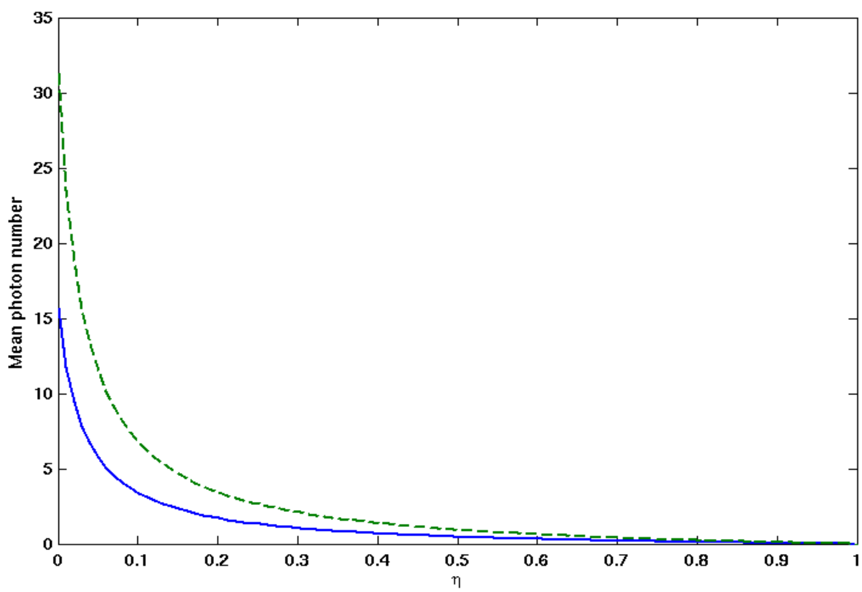

Figure 3.2: Plots of the steady state mean number of photons for superposed light (broken line) and for single light (solid line) versus η for $\kappa = 0.8$ and $A = 25$.

$$\bar{n} = \frac{RM}{\sqrt{(M^2 - N^2)^3}} - 1,$$

But

$$R = \sqrt{M^2 - N^2},$$

then

$$\bar{n} = \frac{M}{M^2 - N^2} - 1, \tag{3.2.1}$$

which is the mean number of photons for the superposition of two light beams obtained from identical degenerate three-level lasers with steady state value

$$\bar{n} = \frac{A(1-\eta)}{A\eta + \kappa}. \tag{3.2.2}$$

which is a simple sum of the mean photon numbers of the two light beams as it can also be seen in Fig.3.2.

3.2.2 The variance of photon number

The variance of photon number for a light beam is expressed as

$$(\Delta n)^2 = \langle \hat{n}^2 \rangle - \bar{n}^2 \tag{3.2.3}$$

But from previous chapter

$$\langle \hat{n}^2 \rangle = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - 3\bar{n} - 2 \tag{3.2.4}$$

and

$$\begin{aligned} \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle &= \int d^2\alpha Q(\alpha, \alpha^*, t) \alpha^2 \alpha^{*2} \\ &= R \int \frac{d^2\alpha}{\pi} exp(-M\alpha\alpha^* + N(\frac{\alpha^2}{2} + \frac{\alpha^{*2}}{2}))\alpha^2\alpha^{*2} \\ &= R\frac{\partial^2}{\partial M^2}\Big(\int \frac{d^2\alpha}{\pi} exp(-M\alpha\alpha^* + N(\frac{\alpha^2}{2} + \frac{\alpha^{*2}}{2}))\Big) \\ &= R\frac{\partial^2}{\partial M^2}\Big(\frac{1}{\sqrt{M^2 - N^2}}\Big), \end{aligned}$$

there follows

$$\langle a^2 \hat{a}^{\dagger 2} \rangle = \frac{R(2M^2 + N^2)}{(M^2 - N^2)^{5/2}} \tag{3.2.5}$$

From Eqs.(3.2.3), (3.2.4) and (3.2.5) the variance of photon number is

$$(\Delta n)^2 = \frac{R(2M^2 + N^2)}{(M^2 - N^2)^{5/2}} - \bar{n}^2 - 3\bar{n} - 2,$$

with steady state value

$$(\Delta n)^2_{ss} = \frac{2A\eta\kappa + \kappa^2 - A^2(-2 + \eta^2)}{(A\eta + \kappa)^2} - \bar{n}^2 - 3\bar{n} - 2.$$

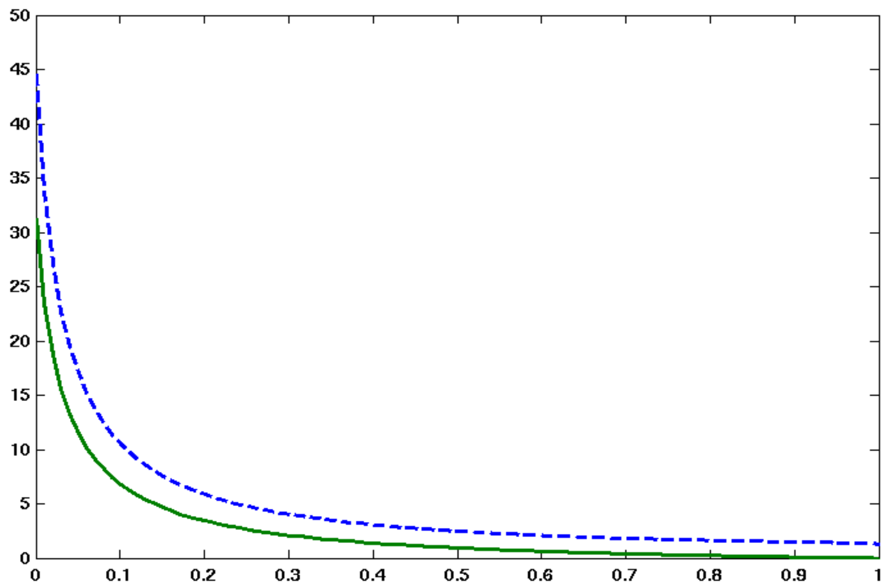

Figure 3.3: Plots of the mean number of photons (solid line) and the uncertainity in photon number (broken line) at steady state versus η for $\kappa = 0.8$ and $A = 25$.

Using Eq.(3.2.2), we find

$$(\Delta n)^2_{ss} = \frac{A(1-\eta)(A(2+\eta)+\kappa)}{(A\eta+\kappa)^2} = \bar{n}(1+\frac{2A}{A\eta+\kappa}). \tag{3.2.6}$$

from which one could easily see that for $0 < \eta < 1$, the photon statistics is super-Poissonian which is also observed clearly in Fig.3.3.

3.2.3 The photon number distribution

We now seek to study the photon number distribution for the light obtained from the superposition of two light beams generated by identical degenerate three-level lasers employing the Q-function (3.1.9). The photon number distribution of any light is expressible in terms of the Q-function as

$$P(n,t) = \frac{\pi}{n!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^{n}}\left[Q(\alpha^*,\alpha,t)e^{\alpha^*\alpha}\right]_{\alpha=\alpha^*=0}$$
$$= \frac{R}{n!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^{n}}\left[exp[(1-M)\alpha\alpha^* + N(\frac{\alpha^2}{2}+\frac{\alpha^{*2}}{2})]\right]_{\alpha=\alpha^*=0}. \qquad (3.2.7)$$

Upon using the power series expansion, one finds

$$e^{(1-M)\alpha^*\alpha} = \sum_l \frac{(1-M)^l\alpha^l\alpha^{*l}}{l!}$$

$$e^{\frac{N}{2}\alpha^2} = \sum_j \frac{(\frac{N}{2})^j\alpha^{2j}}{j!}$$

$$e^{\frac{N}{2}\alpha^{*2}} = \sum_r \frac{(\frac{N}{2})^r\alpha^{*2r}}{r!}$$

so that

$$P(n,t) = \frac{R}{n!}\sum_{ljr}\frac{(\frac{N}{2})^{r+j}(1-M)^l}{l!r!j!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^{n}}\left[\alpha^{2j+l}\alpha^{*2r+l}\right]_{\alpha=\alpha^*=0}.$$

By the help of the relation

$$\frac{\partial^n x^m}{\partial\alpha^n} = \frac{m!}{(m-n)!}x^{m-n},$$

we get

$$P(n,t) = \frac{R}{n!}\sum_{ljr}\frac{(\frac{N}{2})^{r+j}(1-M)^l(2j+l)!(2r+l)!}{l!r!j!(2j+l-n)!(2r+l-n)!}\left[\alpha^{2j+l-n}\alpha^{*2r+l-n}\right]_{\alpha=\alpha^*=0}. \qquad (3.2.8)$$

If we apply the condition $\alpha = \alpha^* = 0$, the photon number distribution function under Eq.(3.2.8) vanishes. This function will have a non-zero value only for the condition $2r+l = n$ and $2j+l = n$, from which follows $l = n-2r$, $l = n-2j$ and $j = r$

$$P(n,t) = R\sum_{ljr} n!\frac{(\frac{N}{2})^{r+j}(1-M)^l}{r!j!(n-2j)!}\delta_{2j+l,n}\delta_{2r+l,n}$$

From the property of kronecker delta function, we get

$$P(n,t) = R\sum_j^{[n]} n!\frac{(\frac{N}{2})^j(1-M)^{n-2j}}{j!(n-2j)!}\sum_r^{[n]}\frac{(\frac{N}{2})^r}{r!}$$

To avoid the factorial of a negative number we set $n-2j \geq 0$, then

$$r = j \leq n/2$$

hence

$$P(n,t) = R\sum_{r=0}^{[n]} \frac{n!(1-M)^{n-2r}N^{2r}}{2^{2r}(r!)^2(n-2r)!}. \tag{3.2.9}$$

Using the respective values of R, M and N, we arrive at

$$P(n,t) = \left[\frac{u^2-v^2}{4-4u+u^2-v^2}\right]^{1/2} \sum_{r=0}^{[n]} \frac{n!\left(\frac{4-6u+2u^2-2v^2}{4-4u+u^2-v^2}\right)^{n-2r}\left(\frac{2v}{4-4u+u^2-v^2}\right)^{2r}}{2^{2r}(r!)^2(n-2r)!}, \tag{3.2.10}$$

with steady state value

$$P(n,t) = \left[\frac{(A\eta+\kappa)^2}{A^2\eta^2+2A\kappa+2A\kappa+\kappa^2}\right]^{1/2} \sum_{r=0}^{[n]} \frac{n!\left(\frac{A(1-\eta)(-A\eta+\kappa)}{A^2\eta^2+2A\kappa+2A\kappa+\kappa^2}\right)^{n-2r}\left(\frac{A(A\eta+\kappa)\sqrt{1-\eta^2}}{A^2\eta^2+2A\kappa+2A\kappa+\kappa^2}\right)^{2r}}{2^{2r}(r!)^2(n-2r)!}, \tag{3.2.11}$$

where $[n] = \frac{n}{2}$ for even n and $[n] = \frac{n-1}{2}$ for odd n.

This is the photon number distribution for the superposition of two light beams produced by degenerate three-level lasers which has the same form as in the case of light generated by degenerate three-level laser coupled to a vacuum reservior in chapter two. From which we can check that the photon number distribution decreases with the number of photons.

3.3 Quadrature variance

We now seek to study the squeezing property of a single-mode light obtaind from the superposition of two light beams produced by three-level lasers. We then calculate the quadrature variance of this light. The squeezing properties of single-mode light are described by two quadrature operators defined as

$$\hat{a}_+ = \hat{a} + \hat{a}^\dagger$$

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}),$$

where $\hat{a}_+$ and $\hat{a}_-$ are Hermitian operators representing the physical quantities called plus and minus quadratures, respectively while $\hat{a}^\dagger$, $\hat{a}$ are the creation and annihilation oprators of light obtained from the superposition of two light beams. The quadrature variance can be expressed in terms of the quadrature operators as

$$(\Delta a_\pm)^2 = \langle \hat{a}_\pm^2 \rangle - \langle \hat{a}_\pm \rangle^2.$$

The explicit form of quadrature variance for the plus quadrature can be expressed in terms of the creation and annihilation operators as

$$(\Delta a_+)^2 = 1 + \langle \hat{a}^2 \rangle + \langle \hat{a}^{\dagger 2} \rangle + 2\langle \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a} \rangle^2 - \langle \hat{a}^\dagger \rangle^2 - 2\langle \hat{a} \rangle \langle \hat{a}^\dagger \rangle \tag{3.3.1}$$

In the same way quadrature variance of the minus quadrature will be

$$(\Delta a_-)^2 = 1 + 2\langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a} \rangle^2 + \langle \hat{a}^\dagger \rangle^2 - \langle \hat{a}^2 \rangle - \langle \hat{a}^{\dagger 2} \rangle - 2\langle \hat{a} \rangle \langle \hat{a}^\dagger \rangle. \tag{3.3.2}$$

But

$$\langle \hat{a} \rangle = \int d^2\alpha Q(\alpha, \alpha^*)\alpha,$$

in which α is the c-number variable corrosponding to the annihilation operator $\hat{a}$. Upon using Eq.(3.1.9)

$$\begin{aligned} \langle \hat{a} \rangle \quad &= R\int \frac{d^2\alpha}{\pi} exp[-M\alpha\alpha^* + N(\frac{\alpha^2}{2} + \frac{\alpha^{*2}}{2})]\alpha \\ &= R\frac{\partial}{\partial p}\bigg[\int \frac{d^2\alpha}{\pi} exp[-M\alpha\alpha^* + p\alpha + N(\frac{\alpha^2}{2} + \frac{\alpha^{*2}}{2})]\bigg]_{p=0} \\ &= \frac{R}{\sqrt{M^2 - N^2}}\frac{\partial}{\partial p}\bigg[exp\bigg(\frac{Np^2}{2(M^2 - N^2)}\bigg)\bigg]_{p=0} \\ &= 0. \end{aligned} \tag{3.3.3}$$

Similarily

$$\langle \hat{a}^\dagger \rangle = 0 \tag{3.3.4}$$

$$\begin{aligned}\langle \hat{a}^2 \rangle &= R(\int \frac{d^2\alpha}{\pi} exp[-M\alpha\alpha^* + N(\frac{\alpha^2}{2} + \frac{\alpha^{*2}}{2})]\alpha^2 \\ &= R\frac{\partial^2}{\partial q^2}\Big[\int \frac{d^2\alpha}{\pi} exp[-M\alpha\alpha^* + q\alpha + N(\frac{\alpha^2}{2} + \frac{\alpha^{*2}}{2})]\Big]_{q=0} \\ &= \frac{R}{\sqrt{M^2 - N^2}}\frac{\partial^2}{\partial q^2}\Big[exp\frac{Nq^2}{2(M^2 - N^2)}\Big]_{q=0} \\ &= \frac{R}{\sqrt{M^2 - N^2}}\frac{\partial}{\partial q}\Big[\frac{Nq}{M^2 - N^2} exp\Big(\frac{Nq^2}{2(M^2 - N^2)}\Big)\Big]_{q=0} \\ &= \frac{NR}{(M^2 - N^2)^{3/2}}. \end{aligned} \tag{3.3.5}$$

In the same way

$$\langle \hat{a}^{\dagger 2} \rangle = \frac{NR}{(M^2 - N^2)^{3/2}}. \tag{3.3.6}$$

Applying Eqs.(3.3.3), (3.3.4), (3.3.5) and (3.3.6) in Eq.(3.3.1), the quadrature variance for the plus quadrature becomes

$$(\Delta a_+)^2 = 1 + 2\bar{n} + \frac{2NR}{(M^2 - N^2)^{3/2}}, \tag{3.3.7}$$

with steady state value

$$(\Delta a_+)^2 = \frac{A\eta + 2A(1 - \eta + \sqrt{1 - \eta^2}) + \kappa}{A\eta + \kappa}. \tag{3.3.8}$$

Applying Eqs.(3.3.3), (3.3.4), (3.3.5) and (3.3.6) in Eq.(3.3.2), the quadrature variance for the minus quadrature

$$(\Delta a_-)^2 = 1 + 2\bar{n} - \frac{2NR}{(M^2 - N^2)^{3/2}}, \tag{3.3.9}$$

with a steady state value

$$(\Delta a_-)^2 = \frac{A\eta + 2A(1 - \eta - \sqrt{1 - \eta^2}) + \kappa}{A\eta + \kappa}. \tag{3.3.10}$$

From a direct look at the Eqs.(3.3.8) and (3.3.10), we could not judge the squeezing of properties of the light. However, we can draw the graph of quadrature variance againest η for some value of A and κ. It is observed that the light mode is in a squeezed state (Fig. 3.4). Of course, the squeezing occurs in the minus quadrature.

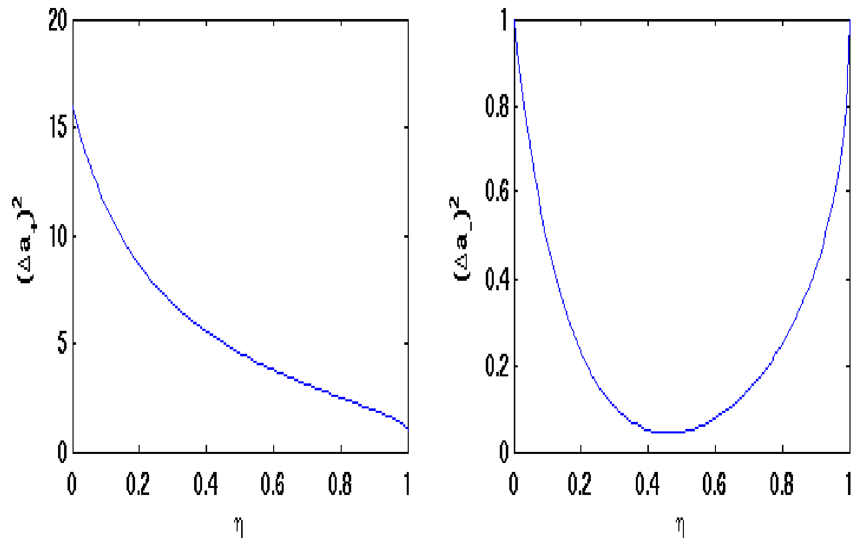

Figure 3.4: Plots of $(\Delta a_+)^2$ and $(\Delta a_-)^2$ versus η for $\kappa = 0.8$ and $A = 3$.

Fig. 3.5 is the quadrature variance for superposition of the light beams produced by a pair of degenerate three-level lasers for different values of A. This figure shows that the degree of squeezing increases with the linear gain coefficient. It appears that almost perfect squeezing could be achieved by taking large values of A with maximum value $A = 3$ and for small values of η. Moreover, the minimum value of quadrature variance for $A = 3$ and $\kappa = 0.8$ is found to be 0.0425 which occur at $\eta = 0.4545$. This implies that the maximum squeezing is 95.8% below the coherent state level.

Fig. 3.6 shows that for $A = 3$ and $\kappa = 0.8$ the quadrature variance of the minus quadrature is 0.5213 which occurs at $\eta = 0.4545$. In other words, the degenerate three-level laser generate squeezed light with a squeezing of 47.9%. Besides, the superposition of two light beams generate a squeezed light with quadrature squeezing of 95.8% for the same values of A, κ and η. From this we can see that the superposition of two light beams

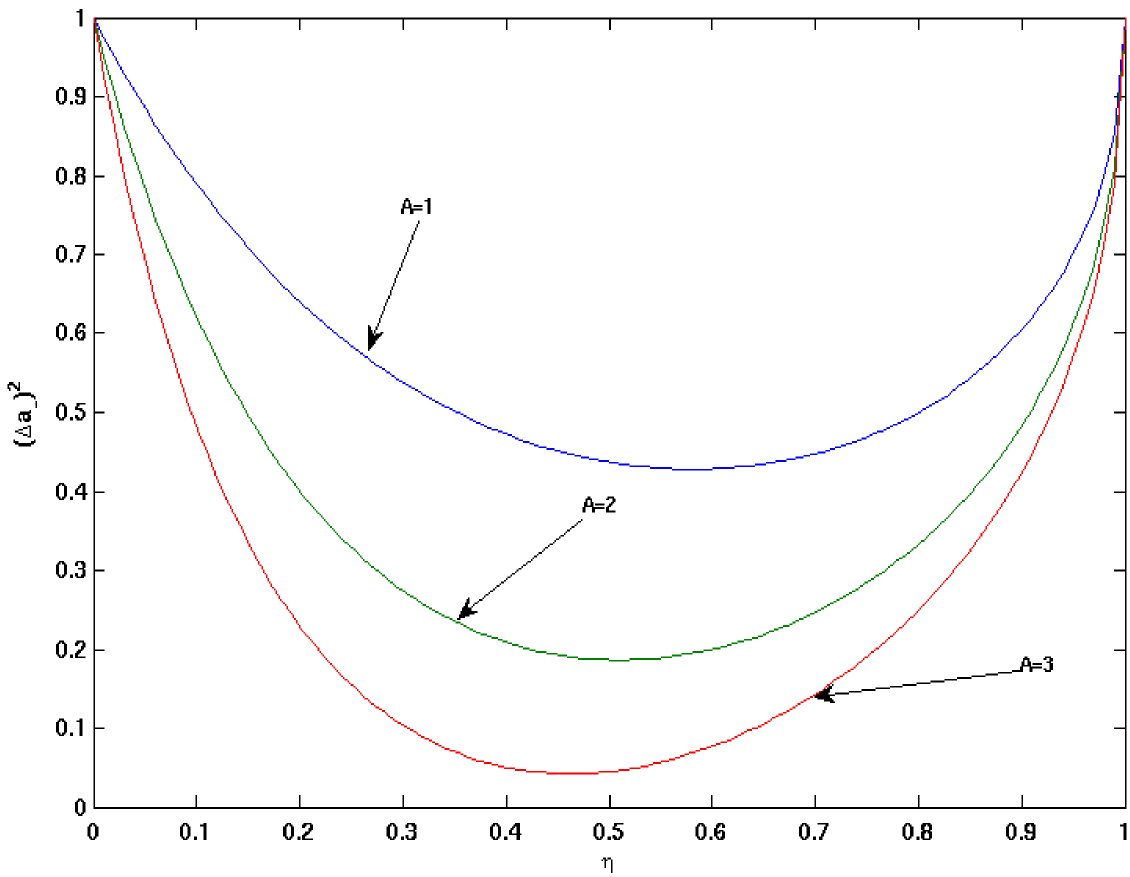

Figure 3.5: Plots of $(\Delta a_-)^2$ versus η for $\kappa = 0.8$ and $A = 1, 2, 3$.

changes the quadrature squeezing. For our specific case it is found that when we produce a single-mode light from the superposition of the two light beams, the quadrature squeezing doubles which can be seen in Fig. 3.6.

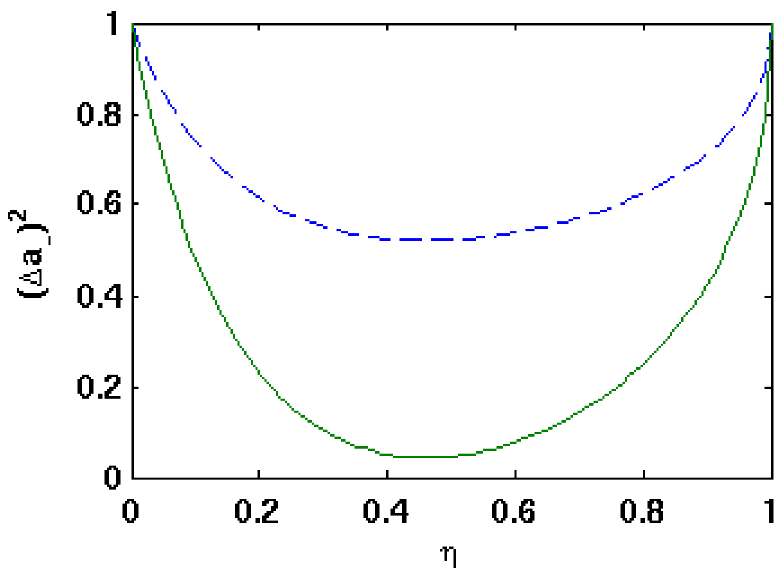

Figure 3.6: Plots of $(\Delta a_-)^2$ (for $\kappa = 0.8$ and $A = 3$) versus η for the light produced by single degenerate three-levle laser (broken line) and the light beam produced by a pair of three-level lasers (solid line).

Chapter 4

Conclusion

In this thesis we have seen the squeezing and statistical properties of the light generated by degenerate three-level laser in which degenerate three-level atoms in a cascade configuration and initially prepared in a coherent superposition of the top and bottom levels are injected into a cavity coupled to vacuum reservoir via a single port-mirror. Applying the linear approximation scheme we found the master equation for a light produced by degenerate three-level laser from which we obtained the solutions of c-number Langevin equations. Employing these solutions we found the antinormally ordered characteristic function which was used to find the Q-function of a light beam generated by degenerate three-level laser.

Upon applying the Q-function we calculated the photon statistics of the light and it appears that the photon statistics is super-Poissonian while the photon number distribution decreases with the photon number. We have calculated quadrature variance for $A = 3$ and $\kappa = 0.8$ at steady state to be 0.5213 with a squeezing of 47.9% which occurs at $\eta = 0.4545$. It is observed that the quadrature squeezing increases with the linear gain coefficient.

Moreover, we have calculated the Q-function for the superposition of two light beams. Applying this function we calculated the mean photon number which appears as a simple sum of the steady state mean photon numbers of the two light beams. We have also calculated the variance of photon number at steady state and found that the photon statistics for the superposition of two light beams is super-Poissonian. The most interesting observation here is that the quadrature variance for the superposition of two identical light beams decreases with a linear gain cofficient having minimum positive value of 0.0425 for $A = 3$, $\kappa = 0.8$ and $\eta = 0.4545$. After this point the quadrature variance has negative values. It needs further investigation why the quadrature variacne is negative for the value of $A > 3$. The light produced from the superposition of two light beams is in a squeezed state with a maximum squeezing of 95.8% below the coherent state level for the same values of A, κ and η.

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DECLARATION

This thesis is my original work, has not been presented for a degree in any other University and that all the sources of material used for the thesis have been dully acknowledged.

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