COLLEGE OF NATURAL SCIENCE
DEPARTMENT OF MATHEMATICS

# GRADUATE PROJECT REPORT ON FUNDAMENTAL SOLUTIONS and CAUCHY PROBLEMS

Submitted in partial fulfilment of requirements for the Degree of Master of Science in Mathematics

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# Contents

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The undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled **FUNDAMENTAL SOLUTIONS and CAUCHY PROBLEMS** by Berihu Girmay in partial fulfillment of the requirements for the degree of master of Science.

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# Notation

$C$ =Differentiable functions
$C^2$=Twice differentiable functions
$C^\infty$=Infinitely differentiable functions
$N^n$= n-tuple of Natural number
$R^n$= n-tuple of Real number
$\mathbb{C}$=complex number
i.e.=that is
N.B= nota bene
$\langle , \rangle$=Inner product
reg=Region
supp=Support
$F_x$=Fourier transform of a derivative with respect to $x$
$F_y$=Fourier transform of a derivative with respect to $y$
$f * g$=Convolution the functions $f$and$g$
$D(\Omega)$=The set of test functions
$D^{'}(\Omega)$= Distribution
$S(\Omega)$=Schwartz test functions
$S^{'}(\Omega)$=tempered distributions
$L(D)$=linear differential operator
$\partial$=Partial derivatives
$\triangle = \frac{\partial}{\partial x_1^2} + \frac{\partial}{\partial x_2^2} + ... + \frac{\partial}{\partial x_n^2}$

# Abstract

In this report, we discuss about distributions and show how to find fundamental solutions of linear differential operators with constant coefficients with the help of Fourier transform.
Then find the fundamental solution of heat conducting operators and state the properties of this operator.
Using these we state and solve the Cauchy problem for heat conducting operator.

# Introduction

The Theory of distribution functions has exerted a strong influence on the development of the theory of linear differential equations.
**In Mathematics,a fundamental solution** for the Linear Partial Differential Operator $L(D)$ is a formulation of solution in the language of Distribution theory.In terms of the Dirac delta "function" $\delta(x)$,a fundamental solution $\varepsilon$ is the solution of the inhomogeneous equation:

$$L(D)\varepsilon = \delta.$$

Here $\varepsilon$ is assumed to be a distribution. The existence of the fundamental solution for any operator with constant coefficients -the most important case, directly linked to the possibility of using the convolutions to solve an arbitrary right hand side was shown by Bernard Malgrange[2](1953) and Leon Ehrenperies[2](1954).
For equations involving partial derivatives the solution, generally speaking, depends on arbitrary functions.Therefore, to isolate a particular solution describing a real physical process, it is necessary to give additional conditions.These additional conditions are boundary value conditions: or initial and boundary conditions.Cauchy's problem for equations of parabolic type :Initial conditions are given, the region $\Omega$coincides with the whole space$R^n$ and boundary conditions are absent.It is well known that the Fourier Transform plays an important role in finding fundamental solutions of a linear partial differential equation.
In this paper we first introduce distributions and its Fourier Transform.
Next, finding fundamental solutions using the Fourier Transform.In the last section,formulating generalized Cauchy problem for heat conducting equation then finding the generalized and classical solutions of the Cauchy problem for heat conducting equation.

# Chapter 1

# Preliminaries

In this chapter we consider basic properties of function spaces and distributions. Several results and techniques of this chapter are frequently used in later chapters.

## 1.1 Function space and distribution

**Definition 1.1.1.** *A function space is linear space whose points are functions.*

**i.e.**It is a space made of functions.

### 1.1.1 $L^p$ spaces

Let $p$ be a real number and $\Omega$ be a measurable set in $R^n$.
Then$L^p$ space is defined as follows:

**Definition 1.1.2.** *The $L^p$ space is a set of real or complex-valued Lebesgue measurable functions $f(x)$on $\Omega$ that satisfy*

$$\int_{\Omega} |f|^p d\mu < \infty,$$

*for* $1 \leq P < \infty$.

**Example 1.1.1.** $L^3[0,8]$ *consists of all functions $f(x)$ for which the integral*

$$\int_0^8 |f(x)|^3 dx < \infty$$

**Definition 1.1.3.** *A function $f : R^n \to R$ is called (Lebesgue-) integrable,if*

$$\int f dx < \infty$$

Let we define the norm of the $L^p$ space by,
$||f||_{L^p}(\Omega) = (\int_\Omega f^p dx)^{\frac{1}{p}}$, for $1 \leq p < \infty$.
We define the Lebesgue space ,$L^P(\Omega) = \{f : ||f||_{L^p}(\Omega) < \infty\}$

**Example 1.1.2.** $L^1(\Omega), L^2(\Omega)$, *when* $p = 1$ *and* $p = 2$

**Definition 1.1.4.** *Functions in* $L^2(\Omega)$ *are called square-integrable functions*

**properties of $L^p$ spaces**

**Lemma 1.1.1.** *The space $L^p(\Omega)$is a linear space.*

*Proof.* For $f, g \in L^p(\Omega)$ and $\alpha \in R$ or$\mathbb{C}$
we have $\alpha f \in L^p(\Omega)$.
Also, there holds,

$$\begin{aligned}
\int_\Omega |f + g|^p dx &\leq \int_\Omega (|f| + |g|)^p dx \\
&\leq \int_\Omega (2max\{|f|, |g|\})^p dx \\
&= 2^p \int_\Omega max\{|f|^p, |g|^p\} dx \\
&\leq \int_\Omega |f|^p + |g|^p dx < \infty
\end{aligned}$$

□

The space of continuous and compactly supported functions is dense in $L^p(R^n)$.

**Definition 1.1.5.** *(Convergence in the mean).*
*Let $f_n(x)$ be a sequence of functions which belong to $L^p[a, b]$. If there exists a function $f(x) \in L^p$ such that*

$$\lim_{n\to\infty} \int_a^b |f_n - f|^p dx = 0$$

**Definition 1.1.6.** *(Cauchy sequences in $L^p$ spaces).*
*A sequence of functions $f_n(x)$ is said to be a Cauchy sequence if*

$$\lim_{m,n\to\infty} \int_a^b |f_n(x) - g_m(x)|^p dx = 0$$

**Definition 1.1.7.** *(Completeness of an $L^p$ space).*
*An $L^p$ space is said to be complete if every Cauchy sequence in the space converges in the mean to a function in the space.*

**Definition 1.1.8.** *(Continuous linear functional)*
*If $X$ is a linear space then a function $f : X \to \mathbb{C}$ is said to be*

1. linear if
   $f(c_1\varphi_1 + c_2\varphi_2) = c_1 f(\varphi_1) + c_2 f(\varphi_2), \forall \varphi_1, \varphi_2 \in X$ and $c_1, c_2$ constants.

2. continuous if
   $\{\varphi_n\} \subseteq X$such that $\varphi_n \to \varphi$ implies that $f(\varphi_n) \to f(\varphi)$.

### 1.1.2 Test Functions in $D(\Omega)$

Let $\Omega \subset R^n$ be an open set.We recall that if $f$ is a continuous function on $\Omega$,the support of $f$ is the set

$$supp(f) = \overline{\{x : f(x) \neq 0\}}.$$

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, ...., \alpha_n) \in N_0^n$ where $N_0 = \{0, 1, 2, 3, ...\}$and
$N_0^n = \underbrace{N_0 \, x N_0 \, x N_0 \, x ... x \, N_0}_{n.times}$
is called multi index and its length is

$$|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_n.$$

The partial derivative of the function $f(x)$ with the order of
$|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_n$ is defined by

$$D^\alpha f(x) = \frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} . \frac{\partial^{\alpha_2}}{\partial x^{\alpha_2}} \quad . \quad . \quad . \frac{\partial^{\alpha_n}}{\partial x^{\alpha_n}} f(x)$$

where,$D^0 f(x) = f(x)$

**Definition 1.1.9.** *(Class of functions $C^k(\Omega)$)*
*A set of functions $f$,continuous together with the partial derivative $D^\alpha f(x)$, $|\alpha| \leq k (0 \leq k < \infty)$, forms in the region $\Omega$ a class of functions $C^k(\Omega)$.*

A function $f$ in the region $\Omega$ is called $C^\infty(\Omega)$, if $D^\alpha f(x)$ is continuous function in the region $\Omega$ for $\alpha \in N_0^n$.
**N.B** :If $\Omega$ is a domain with smooth boundary then $supp(f)$ is compact in $\Omega$

1. if and only if $f$ vanishes near $\partial\Omega$. **i.e**:$f(x) = 0, \forall x \in \Omega/K, supp(f) \subseteq K$.
   or

2. if its support is closed and bounded set in $\Omega$.

**Definition 1.1.10.** *The spaces of infinitely differentiable functions with compact support in $\Omega$ is defined as*

$$D(\Omega) := \{f : \Omega \to \mathbb{C}; f \in C^\infty(\Omega), and\ \ supp(f)\ \ is\ \ compact\ \ in\ \ \Omega\} = C_c^\infty(\Omega).$$

*This space is called a test function.*
*The set of test functions, the supports of which are contained in the given region $\Omega$, is denoted by $D(\Omega)$; in this way*

$$D(\Omega) \subseteq D(R^n)$$

.

**properties of Test Functions in $D(\Omega)$**

1. Differentiation:
   The operation of differentiation $D^\beta\varphi(x)$ is continuous from $D(\Omega)$ to $D(\Omega)$ if $\varphi_n(x) \to \varphi(x)$ as $n \to \infty$ in $D(\Omega)$ then $D^\beta\varphi_n(x) \to D^\beta\varphi(x)$.

2. Multiplication by a function $a \in C^\infty(R^n)$:
   Let $\varphi(x) \in D(R^n)$ then $a(x)\varphi(x)$, are continuous from $D(R^n)$ into $D(R^n)$.

**Example 1.1.3.**

$$f(x) = \begin{cases} exp\frac{1}{(1-|x|)^2}, |x| < 1 \\ 0, \quad |x| > 1 \end{cases}$$

Show that $f$ is a test function over $R$.
**verification :** If $K \subset\subset V \subset\subset \overline{V} \subseteq R$
where $K$ is compact and $V$ is an open,then$\exists \varphi \in D(\Omega)$
such that $\varphi = 1$ on $K$ and $supp(\varphi) \subseteq V$.

$$supp(f) = \overline{\{x : f(x) \neq 0\}} = [-1, 1]$$

since $[-1, 1]$ is closed and bounded.
Thus $f(x)$has a compact support in $[-1, 1]$and
$f(x)$is infinitely differentiable in this region.
Hence$f(x)$is a test function in$[-1, 1]$

### 1.1.3 Test Functions in $S(\Omega)$

**Definition 1.1.11.** *The set of test functions $S(\Omega)$ all functions the class $C^{\infty}(\Omega)$which decreases as $|x| \to \infty$, together with all their derivatives, faster than any power of $|x|^{-1}$.*

**Definition 1.1.12.** *The Schwartz space $S(\Omega)$ of rapidly decreasing function is the set of infinitely differentiable functions $f : \Omega \to \mathbb{C}$ such that*
$\forall \beta, \beta \in N_0^n$ ,

$$sup \mid x^{\alpha} \frac{\partial^{\beta}}{\partial x^{\beta}} \varphi(x) \mid < +\infty, \forall x \in \Omega. \tag{1.1}$$

**Example 1.1.4.** *Rapidly decreasing function.*

**properties of Test functions in $S(\Omega)$**

1. $S(\Omega)$ is a linear space.
   if $\varphi_n \to \varphi$ as $n \to \infty$ in $D(\Omega)$,then, since the supports of $\varphi_n$ are bounded independently of $n$, the limiting result (1.1 ) is valid for all $\alpha$ and $\beta$, which means that $\varphi_n \to \varphi$ as $n \to \infty$ in $S(\Omega)$ .
   Hence,$D(\Omega) \subseteq S(\Omega)$
   However,$S(\Omega)$ does not coincide with $D(\Omega)$;

   **Example 1.1.5.** *The function $exp(-|x|^2)$ belongs to $S(\Omega)$, but does not belong to $D(\Omega)$.*

2. The operations of differentiation $D^{\beta}(x)$ continuous from $S(\Omega)$ to $S(\Omega)$. This follows from the definition of the convergence of the space $S(\Omega)$.

3. Multiplication:
Let the function $a \in C^{\infty}(R^n)$grow at infinity, together with all its derivatives, no faster than the polynomial

$$|D^{\alpha}a(x)| \leq C_{\alpha}(1+|x|)^{m_{\alpha}} \tag{1.2}$$

We shall denote the set of such functions by $\theta_M$. The operation of multiplication by the function $a \in \theta_M$ is continuous from $S(\Omega)$ to $S(\Omega)$.
**Verification:**If $\varphi \in S(\Omega)$, then $a\varphi \in S(\Omega)$ , and
if $\varphi_n \to \varphi$ as $n \to \infty$ in $S(\Omega)$.
Then $\forall \alpha, \beta$

$$x^{\beta}D^{\alpha}(a(x)\varphi_n(x)) \longrightarrow x^{\beta}D^{\alpha}(a(x)\varphi(x)), x \in R^n$$

that is $a(x)\varphi_n(x) \longrightarrow a(x)\varphi(x)$ as $n \to \infty$ in $S(\Omega)$ .

### 1.1.4 Convergence of Functions in $D(\Omega)$ and in $S(\Omega)$

**Definition 1.1.13.** *The sequence of function $\varphi_1, \varphi_2, \varphi_3, ..., \varphi_n$ from $D(\Omega)$ converges to the $\varphi \in D(\Omega)$ if*

1. $\exists K \subset\subset \Omega : supp(\varphi_n) \subset K, \forall n \in N$
2. $\forall \alpha \in N_0^n, D^{\alpha}\varphi_n(x) \to D^{\alpha}\varphi(x)$ in $D(\Omega), \forall x \in \Omega$.

In this case we shall write : $\varphi_n \to \varphi$ as $n \to \infty$ in $D(\Omega)$.

**Definition 1.1.14.** *The sequence of functions $\varphi_1, \varphi_2, \varphi_3, ..., \varphi_n$ belonging to $S(\Omega)$ converges to the function $\varphi \in S(\Omega)$ as $n \to \infty$ in $S(\Omega)$, if for all $\alpha$ and $\beta$*

$$x^{\beta}D^{\alpha}\varphi_n \Longrightarrow x^{\beta}D^{\alpha}\varphi \ as \ n \to \infty \tag{1.3}$$

### 1.1.5 Distributions in $S^{'}(\Omega)$ and $D^{'}(\Omega)$

**Distribution in $D^{'}(\Omega)$**

**Definition 1.1.15.** *Distribution in $D^{'}(\Omega)$is a class of continuous linear functional that maps a set of test function in $D(\Omega)$ in to the real (complex) numbers.*
***i.e***:*A function $f : D(\Omega) \to \mathbb{C}$ such that*

1. $\langle f, \varphi \rangle \in \mathbb{C}$
2. $\langle f, c_1\varphi_1 + c_2\varphi_2 \rangle = \langle f, c_1\varphi_1 \rangle + \langle f, c_2\varphi_2 \rangle$
3. $\lim_{n\to\infty} \langle f, \varphi_n \rangle = \langle f, \lim_{n\to\infty} \varphi_n \rangle$.
   *where $\varphi_1, \varphi_2$ and $\varphi_n \in D(\Omega), c_1, c_2$ constants.*

The linear space of distributions denoted $D'(\Omega)$ .
**i.e**.$D'(\Omega) = \{f : D(\Omega) \to \mathbb{C}\}$ ,where $f$ is linear and continuous in the region $\Omega$.
The distribution$f$ is a linear function over $D(\Omega)$ ;
if $\varphi, \psi \in D(\Omega)$ and $\lambda$ and $\mu$ are complex numbers, then

$$\langle f, \lambda\varphi + \mu\psi \rangle = \lambda\langle f, \varphi \rangle + \mu\langle f, \psi \rangle$$

The distribution$f \in D^{'}(\Omega)$ is a continuous functional over $D^{'}(\Omega)$ ; if $\varphi_n \longrightarrow \varphi$ as $n \to$ in $D^{'}(\Omega)$, then

$$\langle f, \varphi_n \rangle \to \langle f, \varphi \rangle$$

The set $D^{'}(\Omega)$ is linear if the linear combination $\lambda f + \mu g$ of distribution $f$ and $g$ is defined as a functional acting in accordance with the formula

$$\langle \lambda f + \mu g, \varphi \rangle = \lambda\langle f, \varphi \rangle + \mu\langle g, \varphi \rangle, \forall\varphi \in D(\Omega), \forall\mu, \lambda \in \mathbb{C}$$

**Verification:** The functional $\lambda f + \mu g$ is linear and continuous over $D(\Omega)$,that is, it belongs to $D^{'}(\Omega)$.
if $\varphi, \psi \in D(\Omega)$, and $\alpha, \beta \in \mathbb{C}$, then, according to the definition,

$$\begin{aligned} \langle \lambda f + \mu g, \alpha\varphi + \beta\psi \rangle &= \lambda\langle f, \alpha\varphi + \beta\psi \rangle + \mu\langle g, \alpha\varphi + \beta\psi \rangle \\ &= \alpha[\lambda\langle f, \varphi \rangle + \mu\rangle g, \varphi] + \beta[\lambda\langle f, \psi \rangle + \mu\langle g, \psi \rangle] \\ &= \alpha\langle \lambda f + \mu g, \varphi \rangle + \beta\langle \lambda f + \mu g, \psi \rangle \end{aligned}$$

and so this functional is linear.
Its continuity follows from the continuity of the functionals $f$ and $g$:
If $\varphi_n \to \varphi$ as $n \to \infty$, then

$$\begin{aligned}\langle \lambda f + \mu g, \varphi_n \rangle &= \lambda \langle f, \varphi_n \rangle + \mu \langle g, \varphi_n \rangle \\ &\Rightarrow \lambda \langle f, \varphi \rangle + \mu \langle g, \varphi \rangle \\ &= \langle \lambda f + \mu g, \varphi \rangle\end{aligned}$$

**Definition 1.1.16.** *The sequence of distribution* $f_1, f_2 ... f_n$ *belonging to* $D^{'}(\Omega)$ *converges to the distribution* $f \in D^{'}(\Omega)$ *,if for any* $\varphi \in D(\Omega)$

$$\langle f_n, \varphi \rangle \to \langle f, \varphi \rangle, as \;\; n \to \infty.$$

*We can write as* $f_n \to f$.

**Definition 1.1.17** (Functions as distribution)**.** *Suppose that* $f \in L^1_{loc}(\Omega)$, *then the corresponding distribution*
$f : D(\Omega) \to \mathbb{C}$*is defined by*

$$\langle f, \varphi \rangle = \int_\Omega f(x)\varphi(x)dx, \forall \varphi \in D(\Omega).$$

The distribution $f$ becomes zero in the region $\Omega$ if

$$\langle f, \varphi \rangle = 0, \forall \varphi \in D(\Omega).$$

**Remark 1.1.1.** *supp*$(f)$ *is closed set.*

If supp$(f)$ is a bounded set,then distribution $f$ is said to have a compact support.

**Definition 1.1.18.** *Let* $\Omega$ *be an open set in* $R^n$.
*Then a function* $f : \Omega \to C$ *such that*

$$\int_\Omega |f\varphi| dx < \infty,$$

*for each test function* $\varphi \in D(\Omega)$ *is called locally integrable.*

The set of such functions is denoted by $L^1_{loc(\Omega)}$ The function

**Definition 1.1.19.** *The linear and continuous map $f : D(\Omega) \to R$ defined by*

$$f(\varphi) = \langle f, \varphi\rangle = \int_\Omega f\varphi dx$$

*is called a regular distribution.*

where$f \in L^1_{loc}(\Omega)$ and $\varphi \in D(\Omega)$. And a distributions that is not this form is called singular distribution.

**Example 1.1.6.** *Show that Heaviside function at a*

$$\langle\theta_a, \varphi\rangle = \int_\Omega \theta(x-a)\varphi(x)dx$$

*is a regular distribution,with $\theta(x-a) = H_[a,\infty)(x)$*
***Verification:***
$\theta(x-a) = H_{[a,\infty)}(x)$
$\langle\theta_a, \varphi\rangle = \int_\Omega H_[a,\infty)(x)\varphi(x)dx$
$= \int_a^\infty \varphi(x)dx,$
*where,*

$$H(x) = \begin{cases} 1, x > a \\ 0, x < a, a \in R \end{cases}$$

*Hence $\langle\theta_a, \varphi\rangle = \int_a^\infty 1.\varphi(x)dx$ .*
*By definition,$\theta_a$ is a regular distribution.*

**Example 1.1.7.** *Shows that Dirac delta function is singular distribution.*
*Suppose the contrary ,*
***i.e**: $\exists f \in L^1_{loc}(R)$ with $\delta_{x_0} = u_f$,*
*choose $\varphi \in D(R)$ with $supp(\varphi) \subset \overline{(B(0))}, \varphi(0) = 1.$*

*Define :* $\varphi_l(x) = \varphi(l(x - x_0)), l \in N$ *then*

$$supp(\varphi_l(x)) \subset (B(0))_{\frac{1}{l}}, \varphi(0) = 1$$

$$\begin{aligned} We have 1 = |\langle \delta_{x_0}, \varphi \rangle| &= \int_{(B(0))_{\frac{1}{l}}} f(x)\varphi(l(x - x_0))dx. \\ &\le \int_{(B(0))_{\frac{1}{l}}} |f(x)||\varphi(l(x - x_0))|dx. \\ &\|\varphi\|_{L^\infty} \int_{(B(0))_{\frac{1}{l}}} |f(x)||\varphi(l(x - x_0))|dx \\ &\to 0 as l \to \infty \\ &\implies 1 = 0. contradicts. \end{aligned}$$

**Lemma 1.1.2.** *(Du Bois Reymond). In order that the function $f(x)$ is locally integrable in $\Omega$, should become zero in the region $\Omega$ in the sense of distribution, it is necessary and sufficient that $f(x) = 0$ almost everywhere in $\Omega$.*

**Proof 1.1.1.** *The sufficiency of the condition is evident.*
*Let us prove its necessity.*
*Let $a$be an arbitrary point of the region $\Omega$.*
*There will be a closed sphere $\overline{U}(a, \epsilon)$which is wholly contained in the region $\Omega$ and in which consequently,*
*$f = 0$ in the sense of distribution.*
*Since for each $k = (k_l, k_2 ..., k_n)$the function*
*$\psi_k(x) = exp[\frac{i}{\epsilon}(k, x)]\omega_\epsilon(x - a)$*
*where $\omega_\epsilon$ is a cap-shaped function belongs to $D(\Omega)$, then In this way, all the Fourier coefficients corresponding to the trigonometric system $\{exp(\frac{i}{\epsilon}(k, x))\omega_\epsilon\}$ of the function $f(x)\omega_\epsilon(x - a)$, which is integrable over the sphere $U(a, \epsilon)$, are equal to zero.*
*It follows that this function is equal to zero almost everywhere and, consequently,*

$$f(x) = 0$$

*almost everywhere in this sphere.*
*Since $a$ is an arbitrary point of the region $\Omega$, then $f(x) = 0$ almost everywhere in $\Omega$.*

**Definition 1.1.20.** *The distribution $f$ and $g$ are said to be equal in the region $\Omega$ if $f - g = 0$ ,for $x \in \Omega$.*

Specifically, the distribution $f$ and $g$ are said to be equal if for all $\varphi \in D(\Omega)$,

$$\langle f, \varphi \rangle = \langle g, \varphi \rangle.$$

**Properties of Functions in $D^{'}(\Omega)$**

1. Multiplication of distribution
   Let $f(x)$ be a function locally integrable in $R^n$ and $a(x) \in C^{\infty}(R^n)$. Then for any $\varphi \in D(\Omega)$ the equation

   $$\langle af, \varphi \rangle = \langle f, a\varphi \rangle, \varphi \in D(\Omega). \tag{1.4}$$

   We take eq. (1.4) as the definition of the product $af$ for any $f \in D^{'}(\Omega)$. Since the operation of multiplication by the function $a \in C^{\infty}(R^n)$is linear and continuous from $D(R^n)$ into $D(R^n)$,then the functional $af$,is defined by the right-hand side of eq. (1.4), belongs to $D^{'}(R^n)$.
   If $f \in D^{'}(R^n)$, then the equation

   $$f = \eta f \tag{1.5}$$

   is true, where $\eta$ is any function of the class $C^{\infty}(R^n)$ equal to 1 in the neighborhood of the support of $f$.
   For any $\varphi \in D(\Omega)$ the supports of $f$ and $(1-\eta)\varphi$ do not have common points, and therefore,

   $$\langle f - \eta f, \varphi \rangle = \langle f, (l - \eta)\varphi \rangle = 0$$

2. Differentiation :
   Any distribution is infinitely differentiable, and converging series of distribution can be differentiated term by term an infinite number of times.
   Let $f \in C^k(\Omega)$. Then whenever $\alpha, |\alpha| \leq k$, and $\varphi \in D(\Omega)$ the formula for integration by parts,

   $$\begin{aligned}\langle D^{\alpha} f, \varphi \rangle = \int D^{\alpha} f(x)\varphi(x)dx &= (-1)^{|\alpha|} \int f(x)D^{\alpha}\varphi(x)dx \\ &= (-1)^{|\alpha|}\langle f(x), D^{\alpha}\varphi(x)\rangle is valid.\end{aligned}$$

**Definition 1.1.21.** *If $f \in D^{'}(\Omega)$ is a distribution then the partial derivative of $f$ with respect to the coordinate $\alpha_k$ is define by:*

$$\langle D^{\alpha} f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^{\alpha} \varphi \rangle, \forall \varphi \in D(\Omega).$$

We shall check that $D^{\alpha} f \in D^{'}(\Omega)$.
since $f \in D^{'}(\Omega)$,the functional,$D^{\alpha} f$,definable by the right-hand side of eq. (1.6), is linear:

$$\begin{aligned} \langle D^{\alpha} f, \lambda\varphi + \mu\psi \rangle &= (-1)^{|\alpha|} \langle f, D^{\alpha} \langle \lambda\varphi + \mu\psi \rangle \rangle \\ &= (-1)^{|\alpha|} \langle f, \lambda D^{\alpha} \varphi + \mu D^{\alpha} \psi \rangle \\ &= \lambda (-1)^{|\alpha|} \langle f, D^{\alpha} \varphi \rangle + \mu (-1)^{|\alpha|} \langle f, D^{\alpha} \psi \rangle \\ &= \lambda \langle D^{\alpha} f, \varphi \rangle + \mu \langle D^{\alpha} f, \psi \rangle \end{aligned}$$

and continuous :

$$\begin{aligned} \langle D^{\alpha} f, \varphi_n \rangle &= (-1)^{|\alpha|} \langle f, D^{\alpha} \varphi_n \rangle \\ & (-1)^{|\alpha|} \langle f, D^{\alpha} \varphi \rangle = \langle D^{\alpha} f, \varphi \rangle. \end{aligned}$$

for if $\varphi_n \to \varphi$ as $n \to \infty$ in $D(\Omega)$.
then also $D^{\alpha} \varphi_n \Rightarrow D^{\alpha} \varphi$ as $n \to \infty$ in $D(\Omega)$
We shall denote by $\{D^{\alpha} f(x)\}$ the classical derivative (where it exists). It follows from the definition of the distributional derivative that if the distribution $f \in C^k(\Omega)$, then

$$D^{\alpha} f(x) = \{D^{\alpha} f(x)\}, x \in \Omega, |\alpha| \leq k$$

by the definition.

**properties of Derivative of Functions in $D^{'}(\Omega)$**

1. Any distribution is infinitely differentiable.
   if $f \in D^{'}(\Omega)$, then $D^{\alpha}_{x_i} f \in D^{'}(\Omega)$and $D^{\alpha_i}_{x_i} . D^{\alpha_j}_{x_j} f \in D^{'}(\Omega)$and so on.

2. The result of differentiation does not depend on the order of differentiation;
   For example:
   $$D^{\alpha_1}(D^{\alpha_2} f) = D^{\alpha_2}(D^{\alpha_1} f) = D^{(1,1)} f \tag{1.6}$$

In general,

$$D^{\alpha+\beta}f = D^{\alpha}(D^{\beta}f) = D^{\beta}(D^{\alpha}f) \tag{1.7}$$

3. If $f \in D^{'}(\Omega)$ and $a \in C^{\infty}(R^{n})$, then Leibnitz' formula for differentiation of the product $af$ is valid.
   For example: $D^{\alpha_1}(af) = fD^{\alpha_1}(a) + aD^{\alpha_1}(f)$
   if $\varphi$ is any basic function, then

$$\begin{aligned}
\langle D^{\alpha_1}(af), \varphi\rangle &= -\langle af, D^{\alpha_1}\varphi\rangle \\
&= -\langle f, aD^{\alpha_1}\varphi\rangle \\
&= -\langle f, D^{\alpha_1}(a\varphi) - \varphi D^{\alpha_1}a\rangle \\
&= -\langle f, D^{\alpha_1}(a\varphi)\rangle + \langle f, \varphi D^{\alpha_1}a\rangle \\
&= \langle D^{\alpha_1}f, a\varphi\rangle + \langle fD^{\alpha_1}a, \varphi\rangle \\
&= \langle aD^{\alpha_1}f, \varphi\rangle + \langle fD^{\alpha_1}a, \varphi\rangle \\
&= \langle aD^{\alpha_1}f + fD^{\alpha_1}a, \varphi\rangle \\
D^{\alpha_1}(af) &= fD^{\alpha_1}(a) + aD^{\alpha_1}(f)
\end{aligned}$$

4. If the distribution $f = 0$ for $x \in \Omega$, then also $D^{\alpha}f = 0$ for $x \in \Omega$.
   So that $supp(D^{\alpha}f) \subset supp(f)$,
   if $\varphi \in D(\Omega)$ then $D^{\alpha}f \in D(\Omega)$and

$$\begin{aligned}
\langle D^{\alpha}f, \varphi\rangle &= (-1)^{|\alpha|}\langle f, D^{\alpha}\varphi\rangle = 0 \\
&\Longrightarrow D^{\alpha}f = 0.
\end{aligned}$$

5. The operation of differentiation is continuous from $D^{'}(\Omega)$into$D^{'}(\Omega)$.
   That is,if $f_n \to f$ as $n \to \infty$ in $D^{'}(\Omega)$, then $D^{\alpha}f_n \to D^{\alpha}f$ as $n \to \infty$ in $D^{'}(\Omega)$. Indeed, according to the definition of convergence in the space $D^{'}(\Omega)$,
   $\forall\varphi \in D(\Omega)$.
   we have,

$$\begin{aligned}
\langle D^{\alpha}f_n, \varphi\rangle &= (-1)^{|\alpha|}\langle f_n, D^{\alpha}\varphi\rangle \\
&\Rightarrow (-1)^{|\alpha|}\langle f, D^{\alpha}\varphi\rangle asn \to \infty \\
&= \langle D^{\alpha}f, \varphi\rangle
\end{aligned}$$

   which shows that $D^{\alpha}f_n \longrightarrow D^{\alpha}f$, $n \longrightarrow \infty$ in $D^{'}(\Omega)$

**Example 1.1.8.** *The derivative of Dirac delta function has a distributional derivative.*
*Defined by :* $\langle \delta', \varphi \rangle = -\langle \delta, \varphi' \rangle,$
*where* $\langle \delta, \varphi \rangle = \varphi(0).$
***Verification:****Let* $\varphi \in D(R)$

$$H(x) = \begin{cases} 1, x > 0 \\ 0, x \leq 0. \end{cases}$$

*then*

$$\begin{aligned} \langle H', \varphi \rangle &= -\langle H, \varphi' \rangle \\ &= \int H(x)\varphi'(x)dx. \\ &= \langle H, \varphi' \rangle \\ &= -\int \varphi' dx \\ &= \varphi(0). \\ &= \langle \delta, \varphi \rangle \end{aligned}$$

*So* $H' = \delta.$
*Hence* $\langle \delta', \varphi \rangle = -\langle \delta, \varphi' \rangle = \varphi'(0)$ *and Since* $\varphi \in D(R)$

$$\Rightarrow \varphi' \in D(R) \quad \textit{and } \varphi'(0) \textit{ satisfies linearity and continuity.}$$

In this case we say that the function $g$ is the weakly or distributional derivative of the function $f$ and we write $g(x) = D^\alpha f.$
From this the step function is not weakly differentiable but it has distributional derivative.

**Distribution in** $S^{'}(\Omega)$

**Definition 1.1.22.** *Each linear functional over the space of test functions* $S(\Omega)$*is known as a distribution of rapidly decreasing functions (tempered distribution).*
*That is*

$$S^{'}(\Omega) = \{f : S(\Omega) \to \mathbb{C}\}$$

*It is the set of all distribution of rapidly decreasing functions.*
***i.e****: denoted by* $S'(\Omega).$

$S'(\Omega) \subseteq S'(R^n)$

**Definition 1.1.23.** *The sequence of distribution* $f_1, f_2, f_3, ..., f_n$ *belonging to* $S'(\Omega)$ *converges to the distribution* $f \in S'(\Omega)$, $f_n \to f$ *as* $n \to \infty$ $S'(\Omega)$ *, if* $\forall \varphi \in S'(\Omega)$,

$$\langle f_n, \varphi \rangle \longrightarrow \langle f, \varphi \rangle, as \ \ n \to \infty.$$

if $f \in S'$, then $f \in D^{'}(\Omega)$ .
since $D(\Omega) \subseteq S(\Omega)$and the convergence in $S(\Omega)$ follows from the convergence in $D(\Omega)$.
Further, if $f_n \longrightarrow f$ as $n \longrightarrow \infty$ in $S^{'}(\Omega)$
then $\langle f_n, \varphi \rangle \longrightarrow \langle f, \varphi \rangle$ as $n \to \infty, \forall \varphi \in D(\Omega) \subseteq S(\Omega)$
and consequently,$f_n \to f$ as $n \to \infty$ in $D^{'}(\Omega)$

**Properties of Functions in** $S^{'}(\Omega)$

1. Differentiation:
   If $f \in S^{'}(\Omega)$, then each derivative $D^{\alpha} f \in S^{'}(\Omega)$.
   since the operation of differentiation $D^{\alpha} f$ is continuous from $S^{'}(\Omega)$ into $S^{'}(\Omega)$ using integration by parts, the right-hand side of the equation

   $$\langle D^{\alpha} f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^{\alpha} \varphi \rangle$$

   is a linear continuous functionals over $S^{'}(\Omega)$.

2. Multiplication:
   If$f \in S^{'}(\Omega)$and $a \in \theta_M$, then $af \in S^{'}(\Omega)$.
   since the operation of multiplication by the function $a$ belonging to $\theta_M$ is continuous from $S^{'}(\Omega)$into $S^{'}(\Omega)$,the right-hand side of the equation

   $$\langle af, \varphi \rangle = \langle f, a\varphi \rangle$$

   is a linear continuous over $S^{'}(\Omega)$

### 1.1.6 Convolution of Distribution

Let $f(x)$and $g(x)$ be a functions locally integrable in $R^n$ where the function

$$h(x) = \int |g(y)f(x-y)|dy$$

is called locally integrable in $R^n$.

**Definition 1.1.24.**

$$\textit{The function} \ \ (f*g) = \int f(y)g(x-y)dy = \int g(y)f(x-y)dy = (g*f) \tag{1.8}$$

*is known as the convolution of the functions $f$ and $g$.*

The function (1.11)is locally integrable in $R^n$ and therefore defines a regular distribution acting on the test function $\varphi \in D(\Omega)$.
**i.e:**

$$\begin{aligned}\langle h, \varphi\rangle = \int (f*g)(z)\varphi(z)dz &= \int g(y)f(z-x)\varphi(z)dz \\ &= \int g(y)[\int f(z-x)\varphi(z)dz]dy \\ &= \int g(y)[\int f(x)\varphi(x+y)dx]dy\end{aligned}$$

By Fubinis theorem .

#### Conditions for the exists of a Convolution

If condition of locally integrability of the function $h(x)$is satisfied and so the convolution $f*g$ exists and defined by:

1. If One of the functions $f$ or $g$ has compact support.
   **i.e:** $supp(g) \subset K \subset\subset \Omega$;

$$\int_\Omega h(x)dx = \int_k |g(y)| \int_\Omega |f(x-y)dxdy \leq \int_k |g(y)| \int_{k+\Omega} |f(z)dxdy < \infty.$$

2. The functions $f$ and $g$ became zero when $x<0,(n=1)$:

$$\begin{aligned}\int_R h(x)dx &= \int_0^\infty \int_0^x |g(y)|f(x-y)dydx\\ &= \int_0^R |g(y)| \int_y^R |f(x-y)|dxdy\\ &\leq \int_0^R |g(y)| \int_y^R dy|f(z)|dz < \infty.\end{aligned}$$

3. The functions $f$ and $g$ are integrable over $R^n$:

$$\begin{aligned}\int_\Omega h(x)dx &= \int |g(y)| \int_\Omega |f(x-y)|dxdy\\ &= \int |g(y)|dy \int |f(x)|dz < \infty.\end{aligned}$$

**Remark 1.1.2.** *convolution of all pairs of distribution of $f$ and $g$ does not exist always.*

If the convolution $f*g$ exists then there is also a convolution $g*f$
**i.e:**$f*g=g*f$ commutative from the definition.

$$\begin{aligned}\langle f(x)*g(x),\varphi\rangle &= \langle f(x)*g(y),\varphi(x+y)\rangle\\ &= \lim_{n\to\infty}\langle f(x)*g(x),\varphi(x+y)\rangle.\langle g(y)*f(x),\varphi(x+y)\rangle\\ &= \langle f(x)*g(y),\varphi(x+y)\rangle,\varphi\in D(R^n).\end{aligned}$$

**Theorem 1.1.1.** *Let $f$ be an arbitrary function and $g$ be a distribution with compact support.Then the convolution $f*g$ exists in $D^{'}(R^n)$ and appears in the form*

$$\langle g*f,\varphi\rangle = \langle f(x)*g(y),\eta(y)\varphi(x+y)\rangle, \forall\varphi\in D(R^n). \tag{1.9}$$

*Where $\eta$ is any test function equal to 1 in the neighbourhood of the support of $g$. For this the convolution is continuous with respect to $f$ and $g$ separately.*

1. If $f_n\to f$ as $n\to\infty$ in $D^{'}(R^n)$ then $f_n*g\to f*g$ as $n\to\infty$ in $D^{'}(R^n)$.

2. If $g_n \to g$ as $n \to \infty$ in $D^{'}(R^n)$ and for a certain $u, supp(g_n) \subset \Omega_u$ then $f * g_n \to f * g$ as $n \to \infty$ in $D^{'}(R^n)$ .

*Proof.* Let $supp(g_n) \subset \Omega_u$ and let $y$ be a function in $D(R^n) = 1$ in the neighbourhood of $supp(g) \subset \Omega_u$ and $supp(y) \subset \Omega_u$.
Let $\varphi \in D(R^n), supp(\varphi) \subset \Omega_u$and $y_n(x), n = 1, 2, 3, .....$be sequence of functions belonging to $D(R^n)$and converging to 1 in $R^n$.

$$Then\ for\ all\ sufficient\ large\ n, \eta(y), y_n(x) = \eta(y)\varphi(x + y). \tag{1.10}$$

In (2) the condition $supp(g_n) \subset \Omega_u$ makes it possible to choose an auxiliary function $\eta$ which does not depends on $\Omega$.

□

**Corollary 1.1.1.** *The convolution of any distribution $f$with the $\delta$ function exists and is equal to $f$*

$$f * \delta = \delta * f = f \tag{1.11}$$

using the equation in the thm(1.1.1).
Let $\forall \varphi \in D(\Omega)$

$$\langle f * \delta, \varphi\rangle = \langle f(x).\delta(y), \eta(y)\varphi(x + y)\rangle$$
$$= \langle f(x), \langle \delta(y), \eta(y)\varphi(x + y)\rangle\rangle$$
$$= \langle f, \varphi\rangle$$
$$\Longrightarrow f * \delta = \delta * f = f$$

**Derivative of convolution of Distributions**

If the convolution $f * g$ exists then the convolution $D^\alpha f * g$ and $f * D^\alpha g$ exists.More over

$$D^\alpha f * g = D^\alpha(f * g) = f * D^\alpha g. \tag{1.12}$$

*Proof.* Let $\varphi \in D(R)$and $y_n(x, y), n = 1, 2, 3, ....$ be sequence of functions belonging to $D(R^{2n})$ and converging to 1 in $R^{2n}$.
Then the sequence $\eta_n + \frac{\partial \eta_n}{\partial x_i}$ ,$i = 1, 2, 3....$and $n = 1, 2, 3, ....$ of function belonging to $D(R^{2n})$ also converging to 1 in $R^{2n}$.

From this and the existence of convolution $f * g$
we obtained

$$
\begin{aligned}
\langle \frac{\partial(g * f)}{\partial x_i}, \varphi \rangle &= -\langle f * g, \frac{\partial \varphi}{\partial x_i} \rangle \\
&= \langle f(x) * g(y), \frac{\partial \varphi(x+y)}{\partial x_i} \rangle \\
&= \lim_{n\to\infty} \langle f(x) * g(y), \eta_n(x,y) \frac{\partial \varphi(x+y)}{\partial x_i} \rangle \\
&= \lim_{n\to\infty} \langle f(x) * g(y), \eta_n(x,y) \frac{\partial \varphi(x+y)}{\partial x_i} - \frac{\partial \eta_n(x,y)}{\partial x_i} \varphi(x+y) \rangle \\
&= \lim_{n\to\infty} \langle \frac{\partial f(x)}{\partial x_i} * g(y), (\eta_n \varphi(x+y)) \rangle + \lim_{n\to\infty} \langle f(x) * g(y), (\eta_n + \frac{\partial \eta_n}{\partial x_i}) \varphi(x+y) \rangle \\
&- \lim_{n\to\infty} \langle f(x) * g(y), \eta_n \varphi(x+y) \rangle \\
&= \langle \frac{\partial f(x)}{\partial x_i} * g(y), \varphi \rangle + \langle f(x) * g(y), \varphi \rangle - \langle f(x) * g(y), \varphi \rangle
\end{aligned}
$$

Therefore, $\langle \frac{\partial f(x) * g(y)}{\partial x_i}, \varphi \rangle = \langle \frac{\partial f(x)}{\partial x_i} * g(y), \varphi \rangle$. □

We note that the existence of the convolutions $D^\alpha f * g$ and $f * D^\alpha g$
$|\alpha| \geq 1$ is not sufficient for the existence of the convolution $f * g$ ;
specifically, these convolutions need not be equal :
For instance,
$\theta' * 1 = \delta$ but, $\theta * 1' = 0$

## 1.2 Fourier Transform of Tempered Distributions

### 1.2.1 Fourier transform of a Test Function belonging to $S(R^n)$

**Definition 1.2.1.** *Since test function belonging to $S(R^n)$ are absolutely integrable over $R^n$, then the operation of the Fourier transform $f$ is defined over them by*

$$F[\varphi](z) = \int_\Omega \varphi(x) exp(izx) dx, \varphi \in S(R^n).$$

For this function $F[\varphi](z)$ the Fourier transform of the function $\varphi(x)$,is bounded and continuous in $R^n$ ,the test function $\varphi(x)$decrease at infinity faster than any power of $|x|^{-1}$,therefore its Fourier transform may be differentiable under the integral sign,

$$D^\alpha F[\varphi](z) = \int_\Omega \varphi(x) D^\alpha (exp(izx)dx = \int_\Omega (-iz)^\alpha \varphi(x) exp(izx)dx = F[(-ix)^\alpha \varphi](z) \tag{1.13}$$

From this $F[\varphi] \in C^\infty(R^n)$.
The Fourier transform of each derivative $D^\alpha \varphi$ has the same properties.
Hence

$$F[D^\alpha \varphi](z) = \int_\Omega D^\alpha \varphi(z) exp(izx)dx$$
$$= (-iz)^\alpha F[\varphi](z).$$

Finally,$z^\beta D^\alpha F[\varphi](z) = z^\beta[(ix)^\alpha \varphi(x)](z) = (i)^{|\alpha|+|\beta|} F[D^\alpha (x)^\beta \varphi](z)$.

**Remark 1.2.1.** *For all $\beta, \alpha \in N_0^n$ the magnitude $z^\beta D^\alpha F[\varphi](z)$are uniformly bounded with respect to $z \in R^n$,*

$|Z^\beta D^\alpha F[\varphi](z)| \leq \int |D^\beta (x)^\beta \varphi| dx$.
This means that $[\varphi](z) \in S(R^n)$.

So the Fourier transform maps the space $S(R^n)$ in to it self.
**N.B:**The space of test function $D(R^n)$ is not mapped in to itself by the Fourier transform because if we take a function $f$ from $D(R^n)$then if we transform this function using Fourier transformation ,we may not get $F[f] \in D(R^n)$.The function $\varphi(x) \in S(R^n)$ may be expressed in terms of its Fourier transform $F[\varphi](z)$ by means of the operation of the inverse Fourier transform of $F[\varphi](z)$

$$F^{-1} = \varphi(x) = F^{-1}[F[\varphi](z)] = F[F^{-1}[\varphi](z)] \tag{1.14}$$

Where

$$\begin{aligned}
F^{-1}[\varphi(x)] &= \frac{1}{(2\pi)^n}\int_\Omega \psi(z)exp(izx)dz \\
&= \frac{1}{(2\pi)^n}\int_\Omega F[\psi](-x)dx \\
&= \frac{1}{(2\pi)^n}\int_\Omega (-z)exp(izx)dz \\
&= \frac{1}{(2\pi)^n}F[\psi](-z).
\end{aligned} \tag{1.15}$$

From this each function $\varphi(x) \in S(R^n)$ is a Fourier transform of the function

$$\psi = F^{-1}[\varphi(x)] \in S(R^n),$$

This means that the Fourier transforms maps $S(R^n)$ to $S(R^n)$.

**Lemma 1.2.1.** *The Fourier transform operationF is continuous from $S(R^n)$ on to $S(R^n)$.*

*Proof.* Let $\varphi_n \to \varphi$ as $n \to \infty$ in $S(R^n)$ then applying the inequality above the functions $\varphi_n \to \varphi$ for all $\beta, \alpha$
we obtain

$$\begin{aligned}
|Z^\beta D^\alpha F[\varphi_n - \varphi](z)| &\leq \int |D^\beta(x)^\alpha(\varphi_n - \varphi)|dx \\
&\leq supp|D^\beta(x)^\alpha(\varphi_n - \varphi)|(1+|x|)^n \int \frac{dx}{(1+|x|)^{n+1}}, x \in R^n \\
&\Rightarrow |z^\beta D^\alpha F[\varphi_n](z)| - |z^\beta D^\alpha F[\varphi(z)| \to 0as\ n \to \infty, z \in R^n \\
&\therefore F[\varphi_n] \to F[\varphi], as\ \ n \to \infty
\end{aligned}$$

□

The operation of the inverse Fourier transformation $F^{-1}$ has the properties.

### 1.2.2 The Fourier transform of Tempered Distribution.

Let $f$ be an integrable function over $R^n$ then its Fourier transform

$$F[f](z) = \int_\Omega f(x)exp(izx)dx$$

$$\leq \int |f(x)|dx < \infty$$
$$\Rightarrow F[f](z) \leq \int |f(x)|dx < \infty$$

is a continuous and bounded function in $R^n$ and
consequently define a distribution belonging to $S(R^n)$.

$$\langle F[f], \varphi\rangle = \int F[f](z)\varphi(z)dz, \varphi(z) \in S(R^n).$$

Concerning the change of integration,we transform the last integral

$$\begin{aligned}\int F[f](z)\varphi(z)dz &= \int \int_\Omega f(x)exp(izx)dx]\varphi(z)dz \\ &= \int f(x) \int_\Omega exp(izx)\varphi(z)dz \\ &= \int f(x)F[\varphi](z)\varphi(x)dx\end{aligned}$$

From this , $\langle F[f], \varphi\rangle = \langle f, \langle F[\varphi], \varphi\rangle \in S(R^n)$.
We can take this equation as the definition of the Fourier transform $F[f]$ of any tempered distribution
$\langle F[f], \varphi\rangle = \langle f, F[\varphi]\rangle$and $\varphi(x) \in S(R^n)$.
This equation defines a linear continuous function over $S(R^n)$,then this implies that $F[f] \in S(R^n)$.
since $F[\varphi] \in S(R^n)$ $\langle f, F[\varphi]\rangle$ is a linear continuous functional.
Let $\varphi_n \rightarrow \varphi \; as \; n \rightarrow \infty \; in \; S(R^n)$
According the Lemma (1.2.1), $F[\varphi_n] \rightarrow F[\varphi]$ as $n \rightarrow \infty$ in $S(R^n)$.
And since $f \in S(R^n)$,

$$\langle f, F[\varphi_n] \Rightarrow \langle f, F[\varphi]\rangle,$$

as $n \rightarrow \infty$.
So $\langle f, F[\varphi]\rangle$ is continuous over$S(R^n)$.
In this way the operation of the Fourier transformation $F$ maps the space

$S(R^n)$ in to $S(R^n)$.
We need to show that $f$ is continuous from$S(R^n)$in to $S(R^n)$.
Let $f_n \to f$ as $n \to \infty$ in $S(R^n)$ and since $F[\varphi_n] \to F[\varphi]$ ,as $n \to \infty$ then

$$\begin{aligned}\langle F[f_n], \varphi\rangle &= \langle f_n, F[\varphi]\rangle \\ &= \langle f, F[\varphi]\rangle \\ &= \langle F[f], \varphi\rangle, as\, n \to \infty, \forall \varphi \in S(R^n)\end{aligned}$$

we define the inverse Fourier transformation by

$$F^{-1}[\varphi](x) = \frac{1}{(2\pi)^n} F[f(-z)] = \frac{1}{(2\pi)^n} \int_\Omega f(-z) exp(izx) dz$$

Let $\forall \varphi \in S(R^n)$ , we obtain the equation

$$\begin{aligned}\langle F^{-1}[F[f]], \varphi\rangle &= \frac{1}{(2\pi)^n} \langle F^{-1} F[f(-z)], \varphi\rangle \\ &= \frac{1}{(2\pi)^n} \langle F[f(-z)], F[\varphi]\rangle \\ &= \frac{1}{(2\pi)^n} \langle F[f], F[\varphi](-z)\rangle. \\ &= \langle F[f], F^{-1}[\varphi] = \langle f, F[F^{-1}[\varphi]]\rangle = \langle f, \varphi\rangle. \\ \langle f, F^{-1}[F[\varphi]]\rangle &= \langle F[F^{-1}[f]], \varphi\rangle.\end{aligned}$$

From this distribution $f \in S(R^n)$ is a Fourier transform of the distribution $g = F^{-1}[f] \in S(R^n)$.This means $F$ and $F^{-1}$ maps $S(R^n)$ on to$S(R^n)$ .

**properties of the Fourier Transform of Tempered Distribution.**

1. Differentiability of the Fourier Transform
   If $f \in S'(R^n)$ then

$$\begin{aligned}D^\alpha F[f](z) &= \int_\Omega f(x) D^\alpha exp(izx) dx \\ &= \int_\Omega (ix)^{|\alpha|} f(x) exp(izx) dx \\ &= F[(ix)^\alpha f]\end{aligned}$$

**i.e:** $D^\alpha F[f] = F[(ix)^\alpha f]$.
**Verification:** Let $\varphi \in S(R^n)$ then we obtain

$$\begin{aligned}\langle D^\alpha F[f], \varphi\rangle &= (-1)^{-1}\langle F[f], D^\alpha \varphi\rangle \\ &= (-1)^\alpha \langle f, F[D^\alpha \phi]\rangle. \\ &\Rightarrow \langle D^\alpha F[f], \varphi\rangle \\ &= (-1)^{|\alpha|}\langle f, (ix)^\alpha F[\phi]\rangle \\ &= \langle (ix)^\alpha f, F[\varphi]\rangle \\ &= \langle F[(ix)^\alpha f], \varphi\rangle. \\ &\therefore D^\alpha F[f] = F[(ix)^\alpha f]\end{aligned}$$

2. Fourier Transform of a Derivative
   If $f \in S^{'}(R^n)$ then,
$$F[D^\alpha f] = (-iz)^\alpha F[f] \tag{1.16}$$
   Let $\varphi \in S^{'}(R^n)$,we obtaine

$$\begin{aligned}\langle F[D^\alpha f], \varphi\rangle &= \langle D^\alpha f, F[\varphi]\rangle \\ &= (-1)^{|\alpha|}\langle f, D^\alpha F[\varphi]\rangle \\ &= (-1)^{|\alpha|}\langle f, F[(-iz)^\alpha \varphi]\rangle \\ &= (-1)^{|\alpha|}\langle F[f], (-iz)^\alpha \varphi\rangle \\ &= \langle (-iz)^{|\alpha|} F[f], \varphi\rangle \\ \Longrightarrow F[D^\alpha f] &= (-iz)^\alpha F[f]\end{aligned}$$

3. Fourier Transform of Inner product
   If $f \in S^{'}(R^n)$and $g \in S^{'}(R^n)$then,

$$\begin{aligned}F[f(x).g(y)] &= F_x[f(x).F[g](\eta)] \\ &= F_y[F[f](z).g(y)] = F[f](z).F[g](\eta)\end{aligned}$$

   for all $\varphi(z, \eta) \in S^{'}(R^{n+m})$ we have

$$\begin{aligned}\langle F[f(x).g(y)], \varphi\rangle &= \langle f, F[\varphi]\rangle \\ &= \langle f(x), g(y), F_\eta F_z[\varphi]\rangle \\ &= \langle f(x), \langle F[g], F_z[\varphi]\rangle\rangle\end{aligned}$$

**Fourier Transform of Convolution of Distribution**

**Proposition 1.2.1.** *Let $f \in S'(R^n)$ and $g$ be a finite distribution with compact support in the $R^n$ .Then*

$$F[f * g] = F[f].F[g].$$

*Proof.* Since convolution $f * g \in S'(R^n)$and appears in the form

$$\langle f * g, \varphi \rangle = \langle f(x), \langle g(y), \eta(y)\varphi(x+y)\rangle\rangle, \varphi \in S(R^n)$$

where,$\eta(y) = 1$ in the neighborhoods of $supp(f)$.
From this we obtain

$$\begin{aligned}
\langle F[f * g], \varphi \rangle &= \langle f * g, F[\varphi]\rangle, \forall \varphi \in S(R^n) \\
&= \langle f(x), \langle g(y), \eta(y) \int_\Omega \varphi(z) exp(iz(x+y))dz\rangle\rangle \\
&= \langle f, \int \langle g(y), \eta(y) exp(izy).\varphi(z) exp(izx)dz\rangle\rangle \\
&\langle f, \int F[g](z)(z) exp(izx)dz\rangle \\
&= \langle f, F[F[g](\varphi)]\rangle. \\
&= \langle F[f], F[g](\varphi)\rangle \\
&= \langle F[g]F[f], \varphi\rangle. \\
\Rightarrow F[f * g] &= F[f].F[g]
\end{aligned}$$

□

**Example 1.2.1.** *Consider the ordinary differential equation*

$$\frac{\partial^2}{\partial x^2} u - u = f(x), x \in R.$$

Assume that $u$ and its derivative,and $f$ are sufficiently smooth and rapidly decreasing at infinity.
We take the transform of both sides ,we obtain

$$(-iz)^2 \widetilde{u} - \widetilde{u} = \widetilde{f}$$

$$\widetilde{u(z)} = -\frac{1}{1+z^2} \widetilde{f(z)}$$

where,$\widetilde{u(z)} = F[u]$ and $\widetilde{f(z)} = F[f]$.
In the transform domain the solution is a product of a transforms and so apply the convolution theorem .Since $F[\frac{1}{2}exp(-|x|)] = \frac{1}{1+z^2}$.

$$\therefore u(x) = -\frac{1}{2}exp(-|x|) * f(x) = -\frac{1}{2}\int_R exp(-|x-y|)f(y)dy.$$

This is a solution if $f$ is continuous and integrable.

**Example 1.2.2.** *Find the solution of the pure initial value problem for the diffusion equation*

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}, x \in R, t > 0 \, and \, u(x,0) = f(x).$$

We use Fourier Transform and assume $u$ and $f$ are in $S(R)$.Then we obtain

$$\widetilde{\frac{\partial u}{\partial t}} = -z^2 \; a \; \widetilde{\frac{\partial^2 u}{\partial x^2}}$$
$$\widetilde{u}_t = -z^2 \; a \; \widetilde{u},$$

which is an ordinary differential equation in $t$ for $\widetilde{u}(x,t)$, with $z$ as parameter. Its solution is $\widetilde{u}(z,t) = K(z)exp(-z^2at)$ The initial condition gives,$\widetilde{u}(z,0) = \widetilde{f}(z)$ and so $K(z) = \widetilde{f}(z)$.

$$\therefore \widetilde{u}(z,t) = \widetilde{f}(z)exp(-z^2at).$$

Using the inverse Fourier Transform and Convolution theorem,we obtian

$$u(x,t) = \int_R \frac{1}{\sqrt{4\pi at}}exp(\frac{-(x-y)^2}{4at})f(y)dy$$

# Chapter 2

# Fundamental solutions

## 2.1 Fundamental solutions of Linear differential operators

To constructs the fundamental solutions of a linear differential operators with constant coefficients ,we will use the Fourier transform method.
Naturally ,only fundamental solutions of rapidly decreasing functions can be obtained in this way.

**Definition 2.1.1.**

$$Let \sum_{|\alpha|=0}^{k} c_\alpha(x)D^\alpha u = f(x), f(x) \in D'(R^n) \tag{2.1}$$

*be a linear differential equation of order* $k = |\alpha|$*with coefficients* $c_\alpha \in C^\infty(\Omega)$.
*Introducing the linear differential operator,*

$$L(x, D) = \sum_{|\alpha|=0}^{k} c_\alpha(x)D^\alpha, D = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, ..., \frac{\partial}{\partial x_n}).$$

*We can write this equation as:*

$$L(x, D)u = f(x). \tag{2.2}$$

**Definition 2.1.2.** *Each distribution* $u \in D'(\Omega)$*which satisfies eq.(2.2)in the region* $\Omega$ *in the distribution sense,* ***i.e:***

$$\langle L(x, D)u, \varphi\rangle = \langle f, \varphi\rangle, \tag{2.3}$$

$\forall\varphi \in D(\Omega)$*is known as generalized solution of eq.(2.1)in the region* $\Omega$.

Equation (2.3)is equivalent to

$$\langle u, L^*(x, D)\varphi\rangle = \langle f, \varphi\rangle, \forall\varphi \in D(\Omega). \tag{2.4}$$

$$where, L^*(x, D)\varphi = \sum_{|\alpha|=0}^{k} (-1)^{|\alpha|} D^\alpha(c_\alpha\varphi). \tag{2.5}$$

**Verification:**

$$\begin{aligned}
\langle L(x, D)u, \varphi\rangle &= \langle \sum_{|\alpha|=0}^{k} c_\alpha D^\alpha u, \varphi\rangle \\
&= \sum_{|\alpha|=0}^{k} \langle c_\alpha D^\alpha u, \varphi\rangle \\
&= \sum_{|\alpha|=0}^{k} \langle D^\alpha u, c_\alpha \phi\rangle \\
&= \sum_{|\alpha|=0}^{k} (-1)^{|\alpha|} \langle u, D^\alpha c_\alpha \varphi\rangle. \\
&= \langle u, \sum_{|\alpha|=0}^{k} (-1)^{|\alpha|} D^\alpha c_\alpha \varphi\rangle \\
&= \langle u, L^*(x, D)\varphi\rangle
\end{aligned}$$

Hence,$\langle L(x, D)u, \varphi\rangle = \langle u, L^*(x, D)\varphi\rangle = \langle f, \varphi\rangle$.

**Remark 2.1.1.** *Every classical solution is also generalized solution.*

**Lemma 2.1.1.** *If the generalized solution $u(x)$of eq.(2.1)in the region $\Omega$ belongs to the class of $C^k(\Omega)$and $f \in C(\Omega)$,then it is also classical solution of this equation in the region $\Omega$.*

*Proof.* Since $u \in D'(\Omega) \bigcap C^k(\Omega)$,the classical derivative and distributional derivatives of the function $u$ up to order $k$ coincides in the region $\Omega$.
Since $u$ is the generalized solution of eq.(2.1)in the region $\Omega$,then the function $L(x, D)u - f$ which is continuous in $\Omega$ and vanishes in region $\Omega$ in sense of distributions.According to Bois Reymond's lemma,$L(x, D)u - f = 0$ at all points of $\Omega$.Hence $u$ satisfies eq.(2.1) in the region $\Omega$ in classical sense.
Thus $u$ is a classical solution in $\Omega$. □

**Definition 2.1.3.** *Let $L$ be a linear differential operator with constant coefficients $c_\alpha(x) = c_\alpha$,*

$$L(D) = \sum_{|\alpha|=0}^{k} c_\alpha D^\alpha, L^*(D) = L(-D) \tag{2.6}$$

**Definition 2.1.4.** *A distribution $\varepsilon \in D'(R^n)$,that satisfies the equation*

$$L(D)\varepsilon = \delta(x) \tag{2.7}$$

*in the region $R^n$ is said to be the fundamental solution of the differential operator $L(D)$.*

A fundamental solution $\varepsilon(x)$ of the operator $L(D)$is generally not unique, it is determined to with in a term $\varepsilon_0(x)$,which is an arbitrary solution of the homogeneous equation $L(D) = 0$.
In fact,the distribution $\varepsilon(x) + \varepsilon_0(x)$ is also a fundamental solution of the

$$L(D) : L(D)(\varepsilon(x) + \varepsilon_0(x)) = L(D)\varepsilon(x) + L(D)\varepsilon_0(x) = \delta(x).$$

**Lemma 2.1.2.** *In order to that the distribution $\varepsilon \in S'(\Omega)$,should be the fundamental solution of the operator $L(D)$,it is necessary and sufficient that its Fourier transforms $F[\varepsilon]$ satisfies the equation*

$$L(-iz)F[\varepsilon] = 1 \tag{2.8}$$

*Proof.* Let $\varepsilon \in S'(\Omega)$ be a fundamental solution of $L(D)$.
Applying the Fourier transform to both sides of eq.(2.7)
We obtain $F[L(D)\varepsilon] = F[\delta]$

$$F[\sum_{|\alpha|=0}^{k} c_\alpha D^\alpha \varepsilon] = 1 \tag{2.9}$$

$$\begin{aligned} F[\sum_{|\alpha|=0}^{k} c_\alpha D^\alpha \varepsilon] &= \sum_{|\alpha|=0}^{k} c_\alpha F[D^\alpha \varepsilon] \\ &= 1 \end{aligned}$$

By the property of Fourier transform,

$$\sum_{|\alpha|=0}^{k} c_\alpha (-iz)^\alpha F[\varepsilon] = L(-iz)F[\varepsilon] = 1 \tag{2.10}$$

This shows that $F[\varepsilon]$ satisfies eq.(2.8).
**conversely:**if $\varepsilon \in S'(\Omega)$ satisfies eq.(2.8),then by eq.(2.10)it satisfies eq.(2.9) which implies that satisfies the equation $L(D)\varepsilon = \delta(x)$.
$\therefore \varepsilon \in S'$ is a fundamental solution of the operator $L(D)$.
The lemma we have just proved reduces the problem of constructing fundamental solutions of rapidly decreasing functions for linear differential operators with constant coefficients to solving in $S'$ algebraic equations of the type

$$P(z)x = 1 \tag{2.11}$$

with $P$ an arbitrary polynomial.
As we have seen eq.(2.11),each of its solution belonging to $D'(\Omega)$(if such a solution exist)must coincide with the function $\frac{1}{P(Z)}$ out side the set $N_P$ of the zeros of the polynomial $P(z)$.
**i.e:** $N_P = \{z : P(z) = 0\}$.
This implies that if $N_p \neq \emptyset$,eq.(2.11) has no unique solution: the various solutions differ from each other by a distribution with its support in $N_P$. □

**Example 2.1.1.** *The distributions* ,$\frac{1}{z+i0}$, $\frac{1}{z-i0}$ *and* $-\rho\frac{1}{z}$ *which differs from each other by a term of the type constant* $\delta(z)$ *are different solutions the equation* $zx = 1$.

If the $\frac{1}{p(z)}$ is locally integrable in $R^n$,then it is a solution in $S'$ of the eq.(2.11). Equation (2.11)is solvable in $S'$ if $P(z) \neq 0$.
**Note that:**Any solution of eq.(2.11)belongs to $S'$denoted by $reg\frac{1}{P(z)}$. The construction of such a solution depends on a great extent on the structure of the set $N_p$ and can be carried out for each concert polynomial $P$.
Thus,eq.(2.8)is always solvable in $S'(\Omega)$ or $F[\varepsilon] = reg\frac{1}{L(-iz)}$.
Consequently ,each linear differential operator $L(D)$with constant coefficients has a fundamental solution of rapidly decreasing,this solution is given by

$$\varepsilon = F^{-1}[reg\frac{1}{L(-iz)}] = \frac{1}{(2\pi)^2}F[reg\frac{1}{L(iz)}] \tag{2.12}$$

**Theorem 2.1.1.** *Let $f \in D'(\Omega)$ be such that the convolution $\varepsilon * f$ exists in $D'(\Omega)$. Then*

$$L(D)u = f(x) \tag{2.13}$$

*has a solution in $D'(\Omega)$.*
***i.e:*** $u = \varepsilon * f$.
*This solution is unique in the class of distributions belonging to $D'(\Omega)$ for which a convolution with $\varepsilon$ exists.*

*Proof.* Using the formula for differentiating the convolution and equation

$$\begin{aligned} L(x, D)u &= \delta(x) \\ L(D)(\varepsilon * f) &= \sum_{|\alpha|=0}^{k} c_\alpha D^\alpha(\varepsilon * f) \\ &= (\sum_{|\alpha|=0}^{k} c_\alpha D^\alpha \varepsilon) * f \\ L(D)\varepsilon * f &= \delta * f \\ &= f \end{aligned}$$

$\therefore u = \varepsilon * f$ is a solution of eq.(2.13) and has only one zero in $D'(\Omega)$.
uniqueness of such a solution in the distribution belonging to $D'(\Omega)$ whose convolution with $\varepsilon$ exists in $D'(\Omega)$.For this it is sufficient to establish that

the corresponding homogeneous equation,

$$\begin{aligned} L(D) &= 0 \\ u &= u * \delta \\ &= u * L(D)\varepsilon \\ &= L(D)u * \varepsilon = 0. \end{aligned}$$

□

**Corollary 2.1.1.** *Every nonzero linear differential operator with constant coefficients has a fundamental solution of distribution.*

### 2.1.1 Method of Descent

we consider a linear differential equation with constant coefficients in the space $R^{n+1}$ of variables

$$(x,t) = (x_1, x_2, x_3, ..., x_1, t), L(D, \frac{\partial}{\partial t})u = f(x).\delta(t), f \in D'(R) \tag{2.14}$$

where $L(D, \frac{\partial}{\partial t}) = \sum_{q=0}^{p} \frac{\partial^q}{\partial t^q} L_q D + L_0 D$,and the $L_q(D)$are differential operators in the variables $x$.
Let the distribution $u \in D'(R^{n+1})$allow continuation on functions of the type $\varphi(x)1(t)$,where $\varphi \in D'(R^n)$,in the following sense:

1. what ever the sequence of the test function $\eta_k(t), k = 1, 2, 3, .., $ in $D(R^1)$ and converging to 1 in $R^1$ there exists the limits

$$\lim_{k\to\infty} \langle u, \varphi(x)\eta_k(t)\rangle = \langle u, \varphi(x)1(t)\rangle \tag{2.15}$$

2. and this limit does not depend on the sequence$\{\eta_k\}$.

we denote eq.(2.15)by $u_0$

$$\textbf{i.e}: \langle u_0, \varphi\rangle = \langle u, \varphi(x)1(t)\rangle = \lim_{k\to\infty} \langle u, \varphi(x)\eta_k(t)\rangle, \forall \varphi \in D(R^n). \tag{2.16}$$

For any $k$ the function $\langle u, \varphi(x)\eta_k(t)\rangle$is linear and continuous on $D(R^n)$.
That is belongs to $D'(R^n)$.
By the completeness of space $D'(R^n)$ then $u_0 \in D'(R^n)$

**Theorem 2.1.2.** *If the solution $u \in D'(R^n)$ of the eq.(2.15)admits of the continuation eq.(2.16),the distribution $u_0$ belonging to $D'(R^n)$ satisfies the equation*

$$L_0(D)u_0 = f(x). \tag{2.17}$$

*Proof.* Let the $\eta_k(t), k = 1.2, 3, ..$constitute a sequence of functions belonging to $D(R^1)$ that converges to1 in $R^1$ .
Then at $q = 1, 2, 3, ..$the sequences of the functions $\eta_k(t) + \eta_k^{(q)}(t), k = 1, 2..$ also converges to 1 in $R^1$ and consequently ,for all $\varphi \in D(R^n)$ we have

$$\lim_{k\to\infty} \langle u, \varphi(x)\eta_k^{(q)}(t)\rangle = \lim_{k\to\infty} \langle u, \varphi(x)[\eta_k(t) + \eta_k^{(q)}(t)]\rangle - \lim_{k\to\infty} \langle u, \varphi(x)\eta_k(t)\rangle$$

$$= \langle u_0, \varphi(x)\rangle - \langle u_0, \varphi(x)\rangle = 0 \tag{2.18}$$

After this verify that $u_0 \in D(R^n)$ satisfying equation.(2.17).

$$\begin{aligned}
\langle L_0(D)u_0, \varphi\rangle &= \langle u_0, L_0(-D)\varphi\rangle \\
&= \lim_{k\to\infty} \langle u, L_0(-D)\varphi(x)\eta_k(t)\rangle. \\
&= \lim_{k\to\infty} \langle u, L_0(-D)\varphi(x)\eta_k(t) + \sum_{q=1}^{p}(-1)^q L_q(-D)\varphi(x)_k^{(q)}(t)\rangle \\
&= \lim_{k\to\infty} \langle u, L(-D, \frac{-\partial}{\partial t})\varphi(x)\eta_k(t)\rangle \\
&= \lim_{k\to\infty} \langle u, L(D, \frac{\partial}{\partial t})u, \varphi(x)\eta_k(t)\rangle \\
&= \lim_{k\to\infty} \langle f(x).\delta(t), \varphi(x)\eta_k(t)\rangle \\
&= \lim_{k\to\infty} \langle f(x), \varphi(x)\eta_k(0)\rangle = \langle f, \varphi\rangle
\end{aligned}$$

Hence ,$\langle L_0(D)u_0, \varphi\rangle = \langle f, \varphi\rangle$

$$\therefore u_0 \in D(R^n) \ satisfies \ eq.(2.17)$$

□

The method we have just developed for obtaining the solution $u_0(x)$ eq.(2.17) in $n$ variables in terms of the solution $u(x, t)$ of eq.(2.14) in

$n+1$ variables is known as the descent method with respect to the variable $t$.
The method of descent is especially use full in constructing fundamental solutions of a differential operator.
Indeed ,applying the theorem to the case where $f(x) = \delta(x)$,we obtain the following assertion ;
if $\varepsilon(x,t)$ is a fundamental solution of the operator $L(D, \frac{\partial}{\partial t})$ and admits of the continuation $\varepsilon_0$ of the type eq.(2.16),the distribution

$$\langle \varepsilon_0, \varphi \rangle = \langle \varepsilon, \varphi(x)1(t) \rangle, \forall \varphi \in D(R^n) \tag{2.19}$$

is a fundamental solution of the operator $L_0(D)$.
In particular,if $\varepsilon(x,t)$ is such that

$$\int_\Omega |\varepsilon(x,t)|dt < \infty, \forall \Omega \subset R^n, \text{then}$$

$$\varepsilon_0(x,t) = \int_R |\varepsilon(x,t)|dt \tag{2.20}$$

The fundamental solutions of $\varepsilon_0$ and $\varepsilon$ satisfy the condition
$\varepsilon_0(x).1(t) = \varepsilon * [\delta(t).1(t)]$.

## 2.2 Fundamental solutions of a Linear Differential equation with ordinary derivatives

$$Let \quad \sum_{k=0}^{k} a_k(t) \frac{\partial^k}{\partial t^k} \varepsilon = \delta(t) \tag{2.21}$$

Then the fundamental solution of this operator is expressed by

$$\varepsilon(t) = \theta(t)z(t),$$

where,$z(t)$ satisfies the homogeneous equation $Lz = 0$ and the initial conditions and $\theta(t)$ is a heaviside function

$$z(0) = z'(0) = ..... = z^{(k-2)}(0) = 0$$
$$z^{(k-1)}(0) = 1.$$

satisfies the $L\varepsilon = \delta(t)$.Specifically,the function $\varepsilon_1(t) = \theta(t)exp(-at)$ and $\varepsilon_2(t) = \theta(t)\frac{\sin(at)}{a}$ are fundamental solutions of the operator

$$\frac{\partial}{\partial t} + a \qquad and \quad \frac{\partial^2}{\partial t^2} + a^2 \quad \text{respectively .}$$

**Verification:** since $Lz = 0$,then

$$\begin{aligned} \frac{\partial}{\partial t} z + az &= 0 \\ \Rightarrow \frac{\partial}{\partial t} z &= -az \\ \frac{\partial z}{z} &= -a\partial t \\ \ln|z(t)| &= -at + c \\ z(t) &= exp(-at), c = 0 \end{aligned}$$

$$Hence\ \ \varepsilon_1(t) = \theta(t)exp(-at) \tag{2.22}$$

is a fundamental solution $\frac{\partial}{\partial t}\varepsilon(t) + a\varepsilon(t) = \delta(t)$
Again, $\frac{\partial^2}{\partial t^2} z + a^2 z = 0$
Let $z(t) = exp(rt)$ be a solution.

Then,

$$\begin{aligned} r^2 + a^2 &= 0 \\ \Rightarrow r &= ia \\ \Rightarrow z(t) &= exp(iat) \end{aligned}$$

Using Euler's formula, $z(t) = c_1 \cos(at) + c_2 \sin(at)$ but,

$$\begin{aligned} z(0) = 0 &\Rightarrow c_1 = 0 \\ Hence, z(t) &= c_2 \sin(at) \end{aligned}$$

Let $c_2 = \frac{1}{a}$
Thus,$z(t) = \frac{1}{a}\sin(at)$ is a solution of $Lz = 0$.

$$\therefore \varepsilon_2(t) = \theta(t)\frac{1}{a}\sin(at) \tag{2.23}$$

is a fundamental solution for $\frac{\partial^2}{\partial t^2}\varepsilon + a^2\varepsilon = \delta(t)$

## 2.3 Fundamental solutions of wave operator

Let $P\varepsilon_k = \delta(x,t)$ be a wave operator, where $P = \frac{\partial}{\partial t^2} + \triangle$.
Applying the Fourier transform $F_x$to this equation.
For the distribution $\varepsilon_k$,

$$F_x[\varepsilon_k] = \widetilde{\varepsilon_k}(z,t) \text{ as before.}$$

$$\frac{\partial^2}{\partial t^2}\widetilde{\varepsilon_k}(z,t) + a^2|z|^2\widetilde{\varepsilon_k}(z,t) = 1(z).\delta(t) \tag{2.24}$$

Using eq.(2.22)in which substitute $a^2|z|^2$ for $a$,

$$\widetilde{\varepsilon_k}(z,t) = \theta(t)\frac{\sin a|z|t}{a|z|}$$

is the solution of eq.(2.24) in $S'(\Omega)$.

Hence ,applying inverse Fourier transform,then

$$\widetilde{\varepsilon_k}(z,t) = F_z^{-1}[\widetilde{\varepsilon_k}(z,t)] = \theta(t)F_z^{-1}[\frac{\sin a|z|t}{az}] \tag{2.25}$$

Suppose $k = 3$
Let $\delta_{s_a}(x)$ be a simple layer on the sphere $\delta_{s_a}$ in $R^3$.
Then

$$F[\delta_{sa}] = 4\pi a\frac{\sin a|z|t}{az}.$$

Using this $F^{-1}[\frac{\sin a|z|t}{az}] = \frac{1}{4\pi at}\delta_{s_{at}}(x)$

combining with eq.(2.25),

$$\varepsilon_3(x,t) = \frac{\theta(t)}{4\pi at}\delta_{s_{at}}(x).$$

$$\varepsilon_3(x,t) = \frac{\theta(t)}{2\pi a}\delta(a^2t^2 - |z|^2) \tag{2.26}$$

where the distribution function $\varepsilon_3$ acts according to the rule

$$= \langle\varepsilon_3,\varphi\rangle = \frac{1}{4\pi a^2}\int_0^\infty \langle\delta_{s_{at}},\varphi\rangle\frac{dt}{t} = \frac{1}{4\pi a^2}\int_0^\infty \frac{1}{t}\int_{s_{at}}\varphi(x,t)ds_x dt \tag{2.27}$$

To obtain the fundamental solution $\varepsilon_2(x,t)$,
with $x = (x_1, x_2)$,we will use the method of descent with respect to the variable $x_3$.
For this we must show that $\varepsilon_3(x, x_3, t)$ admits of the continuation eq.(2.14) on the functions of the form $\varphi(x,t).1(x_3)$,with $\varphi \in D(R^3)$.
Suppose that $\eta_k, k = 1,2,3,...$belongs to $D(R^1)$ and the sequence of the $\eta_k(x_3)$ tends to 1 in $R^1$.then using eq.(2.27),$\forall\varphi \in D(R^3)$, we have

$$\lim_{k\to\infty}\langle\varepsilon_3, \varphi(x,t)\eta_k(x_3)\rangle = \lim_{k\to\infty}\frac{1}{4\pi a^2}\int_0^\infty \frac{1}{t}\int_{s_{at}}\varphi(x,t)\eta_k(x_3)dsdt$$
$$= \frac{1}{4\pi a^2}\int_0^\infty \frac{1}{t}\int_{s_{at}}\varphi(x,t)dsdt$$
$$=\langle\varepsilon_3, \varphi(x,t)1(x_3)\rangle$$

so that this limit exists and does not depends on the $\{\eta_k\}$.
Hence ,applying eq.(2.18)

$$\langle\varepsilon_2, \varphi\rangle = \langle\varepsilon_3, \varphi(x,t).1(x_3)\rangle.$$
$$= \frac{1}{4\pi a^2}\int_0^\infty \frac{1}{t}\int_{s_{at}}\varphi(x,t)dsdt.$$

Finally,let us transform the integral over $s_{at}$.
Since $\varphi$ does not depend on $x_3$ ,replacing the surface integral over the spherical surface $s_{at} = [|x|^2 + x_3^2 = a^2t^2]$.
By twice integral over the circle $|x| < at$.

we obtain, $\langle\varepsilon_2, \varphi\rangle = \frac{1}{2\pi a}\int_0^\infty \int_{|x|<at}\frac{\varphi(x,t)}{\sqrt{a^2t^2-|x|^2}}dsdt$

$$= \frac{1}{2\pi a}\int \frac{\theta(t)}{\sqrt{a^2t^2-|x|^2}}\varphi(x,t)dxdt$$

$$\therefore \varepsilon_2(x,t) = \frac{1}{2\pi a}\frac{\theta(at-|x|)}{\sqrt{a^2t^2-|x|^2}}$$

be a fundamental solution of the wave equation with $n = 2$.
Similarly,using the eq.(2.19),we obtain by the method of descent with respect to $x_2$ ,for the fundamental solution $\varepsilon_1(x,t)$:

$$\varepsilon_1(x,t) = \int_R \varepsilon_2(x, x_2, t)dx_2$$
$$= \frac{1}{2\pi a}\int_{-\infty}^{\infty}\frac{\theta(at-\sqrt{x^2+x_2^2})}{\sqrt{a^2t^2-x^2-x_2^2}}dx_2$$

$$= \frac{\theta(at-|x|)}{\pi a} \int_0^{\sqrt{a^2t^2-x^2}} \frac{1}{\sqrt{a^2t^2-x^2-x_2^2}} dx_2$$

$$= \frac{\theta(at-|x|)}{\pi a} \int_0^1 \frac{1}{\sqrt{1-u}} du$$

$$\varepsilon_1(x,t) = \frac{1}{2a}\theta(at-|x|)$$

Hence, $\varepsilon_1(x,t) = \frac{1}{2a}\theta(at-|x|)$ is a fundamental solution of the wave equation with $n = 1$.

## 2.4 Fundamental solutions of Laplace's operator

$$Let \ \triangle\varepsilon_k = \delta(x) \ be \ a \ Laplace's\,operator \tag{2.28}$$

We can find the fundamental solution of the Laplace's operator by applying the Fourier transform to eq.(2.28)

$$F[\triangle\varepsilon_k] = F[\delta(x)]$$

$$= -|z|^2F[\varepsilon_k] = F[\delta(x)] = 1 \tag{2.29}$$

Let $k = 2$.
we will verify that the distribution $-\rho\frac{1}{|z|^2}$ satisfies eq.(2.25).

$$\langle|z|^2\rho\frac{1}{|z|^2}, \varphi\rangle = \langle\rho\frac{1}{|z|^2}, |z|^2\varphi\rangle$$

$$= \int_{|z|<1} \frac{|z|^2\varphi(z)-|z|^2\varphi(z)|_{z=0}}{|z|^2} dz + \int_{|z|>1} \frac{|z|^2\varphi(z)}{|z|^2}dz$$

$$= \int \varphi(z)dz = \langle 1, \varphi\rangle, \forall\varphi \in S$$

According to eq.(2.11)
$F[\varepsilon_2] = reg\frac{1}{-|z|^2} = -\rho\frac{1}{|z|^2}$,
Then applying the inverse Fourier transform.

$$\varepsilon_2(x) = F^{-1}[-\rho\frac{1}{|z|^2}]$$

$$= \frac{-1}{2\pi^2}F[\rho\frac{1}{|z|^2}]$$

$$= \frac{-1}{4(\pi)^2}. - 2\pi\ln|x| - 2\pi c_0$$

$$= \frac{\ln|x|}{2\pi} + 2\pi c_0$$

since a constant satisfies the homogeneous Laplace's equation,
$\varepsilon_2(x) = \frac{\ln|x|}{2\pi}$ is a fundamental solution of Laplace's operator,when $k = 2$.
Now let $k \geq 3$ ,the function $\frac{-1}{|z|^2}$ is then locally integrable in $R^n$.
Hence $F[\varepsilon_k] = \frac{-1}{|z|^2}$.
Applying inverse Fourier transform

$$\varepsilon_k = -F^{-1}[\tfrac{1}{|z|^2}],$$

putting $k = 3$, $F[\frac{1}{|z|^2}]$ is given by

$$\langle F[\tfrac{1}{|z|^2}], \varphi\rangle = \langle \tfrac{1}{|z|^2}, F[\varphi]\rangle$$

$$\int \tfrac{1}{|z|^2} F[\varphi]dx$$

$$\Rightarrow \lim_{k\to\infty} \int_{|x|<\infty} \tfrac{1}{|x|^2} \int \varphi(z) exp(izx)dzdx$$

$$\Rightarrow \lim_{k\to\infty} \int \varphi(z) \int_{|x|<\infty} \tfrac{1}{|x|^2} exp(izx)dxdz$$

$$\Rightarrow \lim_{k\to\infty} \int \varphi(z) \int_0^R \int_0^\pi \int_0^{2\pi} \tfrac{exp(i|z|\rho\cos\theta)}{\rho^2} \rho^2 d\omega \sin\theta d\theta d\rho dz$$

$$= 2\pi \lim_{k\to\infty} \int \varphi(z) \int_0^R \int_{-1}^1 exp(iz\rho\mu)d\mu d\rho dz$$

$$4\pi \lim_{k\to\infty} \tfrac{\varphi(z)}{|z|} \int_0^R \tfrac{\sin|Z|\rho}{\rho} d\rho dz$$

$$\text{since } |z||\int_R^\infty \tfrac{\sin|Z|\rho}{\rho}d\rho| = |\tfrac{\cos|z|R}{R} - \int_R^\infty \tfrac{\cos|z|\rho}{\rho^2}d\rho|$$

$$< \tfrac{1}{R}\int_R \tfrac{1}{\rho^2}d\rho = \tfrac{2}{R}$$

$$\int_R^\infty \tfrac{\sin|Z|\rho}{\rho}d\rho = \tfrac{\pi}{2}, |z| \neq 0$$

Hence,$\langle F[\frac{1}{|x|^2}], \varphi\rangle = 4\pi \int \frac{\varphi(z)}{|z|^2}.|z| \int_0^\infty \frac{\sin|z|\rho}{\rho} d\rho dz$

$$= 2(\pi)^2 \int \tfrac{\varphi(z)}{|z|^2}$$

$$\langle F[\frac{1}{|x|^2}], \varphi\rangle = \langle \frac{2\pi^2}{|z|^2}, \varphi\rangle \tag{2.30}$$

Using this ,we obtain

$$\varepsilon_3(x) = \tfrac{-1}{(2\pi)^3} F[\tfrac{1}{|z|^2}] = \tfrac{-2\pi^2}{8\pi^3|x|} = \tfrac{-1}{4\pi|x|}$$

for $k \neq 3$ the $\varepsilon_k(x)$are calculated along the same lines.
By Method of descent with respect to the variable t from the fundamental solution of wave operator.
combining (2.20)with (2.25)at $a = 1$

$$\frac{\partial^2}{\partial x^2}\varepsilon_k(x) = \int_R \varepsilon(x,t)dt$$
$$= -\int_0^\infty \frac{1}{(2\sqrt{\pi t})^n} exp(\frac{-|x|^2}{4t})dt$$
$$= \frac{-|x|^{2-n}}{4\pi^{\frac{k}{2}}} \int_0^\infty exp(u).u^{-2+\frac{k}{2}}$$
$$= \frac{\Gamma(-1+\frac{k}{2})-|x|^{2-k}}{4\pi^{\frac{k}{2}}}$$

$$Thus, \varepsilon_k(x) = \frac{-1}{(k-2)\sigma_k^{2-k}}|x|^{2-k}, k \geq 3 \tag{2.31}$$

where, $\sigma_k^{2-k}$ is the area of a unit surface a sphere.

## 2.5 Fundamental solutions of Heat Conducting operator

Let
$$\frac{\partial}{\partial t}\varepsilon - a^2\triangle\varepsilon = \delta(x,t) \tag{2.32}$$
To find the fundamental solution of this operator we apply the Fourier transform $F_x$ to eq.(2.32). This yields ,

$$F_x[\frac{\partial}{\partial t}\varepsilon] - a^2F_x[\triangle\varepsilon] = F_x[\delta(x,t)],$$

using the properties of Fourier transform,

$$F_x[\delta(x,t)] = F[\delta](z).\delta(t) = 1(z)\delta(t),$$
$$F_x[\frac{\partial}{\partial t}\varepsilon] = \frac{\partial}{\partial t}F[\varepsilon]$$
$$F_x[\triangle\varepsilon] = -|z|^2F_x[\varepsilon].$$

Let the distribution $\widetilde{\varepsilon}(z,t) = F_x[\varepsilon](z,t)$.
We have the following equation:

$$\frac{\partial}{\partial t}\widetilde{\varepsilon}(z,t) + a^2|z|^2\widetilde{\varepsilon}(z,t) = 1(z)\delta(t) \tag{2.33}$$

Using method of descent and eq.(2.22)in which we substitute $a^2|z|^2$ for $a$,we conclude that
the function

$$\widetilde{\varepsilon}(z,t) = \theta(t)exp(-a^2|z|^2t)$$

is the solution to eq.(2.33) in $S'(R^n)$.
Hence applying the inverse Fourier transform $F_z^{-1}$,

$$\varepsilon(x,t) = F_z^{-1}[\widetilde{\varepsilon}(z,t)]$$

$$= F_z^{-1}[\theta(t)exp(-a^2|z|^2t)]$$

$$= \tfrac{\theta(t)}{(2\pi)^n}\int exp(-a^2|z|^2t - iz)dz$$

$$\varepsilon(x,t) = \frac{\theta(t)}{(2a\sqrt{\pi t})^n}exp(\frac{-|x|^2}{4a^2t}) \qquad (2.34)$$

is the fundamental solution of heat conducting operators.

## 2.6 Properties of the Fundamental solutions of the Heat Conducting operators

The solution of the Cauchy problem for the Heat Conducting operators equation is constructed by the theory of distribution.

### 2.6.1 Heat potential

From section (2.5),
$\varepsilon(x,t) = \frac{\theta(t)}{(2a\sqrt{\pi t})^n}exp(\frac{-|x|^2}{4a^2t})$ is the fundamental solution of the heat conducting operators.
This function is non-negative becomes zero when $t < 0$,is infinitely differentiable for $(x,t) \neq (0,0)$ ,and locally integrable in $R^{n+1}$. Moreover,

$$\int \varepsilon(x,t) = 1, t > 0 \qquad (2.35)$$

**Verification:** The function $\varepsilon(x,t)$ is locally integrable in $R^{n+1}$,
since $\varepsilon(x,t) = 0$,for $t < 0$ $\varepsilon(x,t) \geq 0, t \geq 0$

$$\int \varepsilon(x,t)dx = \frac{1}{(2a\sqrt{\pi t})^n}\int exp(\frac{-|x|^2}{4a^2t})dx$$
$$= \prod_{i=1}^{m}\frac{1}{\sqrt{\pi}}\int_R exp(-z_i^2)dz_i^2$$
$$= 1$$

Hence,$\int \varepsilon(x,t) = 1, t > 0$. And

$$\varepsilon(x,t) \to \delta(t), t \to 0^+ in\ \ D(R^{n+1}). \tag{2.36}$$

**verification:**If $t > 0$, then $\varepsilon \in C^\infty$, and therefore

$$\frac{\partial\varepsilon}{\partial t} = (\frac{-|x|^2}{4a^2t} - \frac{m}{2t})\varepsilon$$
$$= \frac{\partial\varepsilon}{\partial x_i} = -\frac{x_i}{2at^2}\varepsilon, \frac{\partial^2\varepsilon}{\partial x_i^2} = (\frac{x^2}{4a^4t} - \frac{1}{2a^2t})\varepsilon$$
$$\frac{\partial}{\partial t}\varepsilon - a^2\triangle\varepsilon = (\frac{|x|^2}{4a^4t} - \frac{k}{2t})\varepsilon - (\frac{x^2}{4a^4t} - \frac{k}{2t})\varepsilon = 0 \tag{2.37}$$

Let $\varphi \in D(R^{n+1})$. Taking eq.(2.29) into account, we obtain

$$\langle\frac{\partial}{\partial t}\varepsilon - a^2\triangle\varepsilon, \varphi\rangle = -\langle\varepsilon, \frac{\partial}{\partial t}\varphi + a^2\triangle\varphi\rangle$$
$$= \int_0^\infty \int \varepsilon(x,t)(\frac{\partial}{\partial t}\varphi + a^2\triangle\varphi)dxdt$$
$$-\lim_{\epsilon\to 0^+}\int_\epsilon^\infty \int \varepsilon(x,t)(\frac{\partial}{\partial t}\varphi + a^2\triangle\varphi)dxdt$$
$$= -\lim_{\epsilon\to 0^+}[\int \varepsilon(x,\epsilon)\varphi(x,\epsilon)dx + \int_\epsilon^\infty \int(\frac{\partial\varepsilon}{\partial t} - a^2\triangle\varepsilon)\varphi dxdt]$$
$$= \lim_{\epsilon\to 0^+}\int \varepsilon(x,\epsilon)\varphi(x,0)dx + \lim_{\epsilon\to 0^+}\int \varepsilon(x,\epsilon)[\varphi(x,\epsilon) - \varphi(x,0)]dx$$
$$\langle\frac{\partial}{\partial t}\varepsilon - a^2\triangle\varepsilon, \varphi\rangle = \lim_{\epsilon\to 0^+}\int \varepsilon(x,\epsilon)\varphi(x,0)dx \tag{2.38}$$

since, by virtue of eq. (2.35),
$|\int \varepsilon(x,\epsilon)[\varphi(x,\epsilon) - \varphi(x,0)]dx| \le K\epsilon\int \varepsilon(x,\epsilon)dx = K\epsilon$

We shall now prove the result

$$\varepsilon(x,t) = \frac{1}{(4\pi a^2t)^{\frac{n}{2}}}exp(\frac{-|x|^2}{4a^2t}) \to \delta(t), t \to 0^+\ \ in\ \ D^{'}(R^n) \tag{2.39}$$

let $\varphi(x) \in D(R^n)$. Then, taking into consideration that

$$|\int \varepsilon(x,\epsilon)[\varphi(x,\epsilon)-\varphi(x,0)]dx| \leq \frac{K}{(4\pi a^2 t)^{\frac{n}{2}}}\int exp(\frac{-|x|^2}{4a^2t})|x|dx$$
$$= \frac{K\sigma_k}{(4\pi a^2 t)^{\frac{k}{2}}}\int_0^\infty exp(\frac{-r^2}{4a^2t})r^k dr$$
$$= K\cdot\sqrt{t}\int_0^\infty exp(-u^2).u^k du = C\sqrt{t}$$

due to eq.(2.35) we obtain as $t \to 0^+$ the result eq.(2.36) :

$$\langle \varepsilon(x,t), \varphi(x)\rangle = \int \varepsilon(x,t)\varphi(x)dx$$
$$= \varphi(0)\int \varepsilon(x,t)dx + \int \varepsilon(x,t)[\varphi(x)-\varphi(0)]dx \to \varphi(0) = \langle \delta, \varphi\rangle.$$
$$\Rightarrow \varepsilon(x,t) \to \delta(t), t \to 0^+ in \ \ D(R^{n+1}).$$

The fundamental solution $\varepsilon(x,t)$ gives the temperature distribution from the momentary point source $\delta(x).\delta(t)$.
Since $\varepsilon(x,t) > 0, \forall t > 0$,and $x \in R^n$,it follows that heat diffused with infinite velocity.
Let the distribution function $f \in D'(R^{n+1})$ becomes zero when $t < 0$.

**Definition 2.6.1.** *The distribution function* $V = \varepsilon * f$,*where* $\varepsilon$ *is a fundamental solution of heat conduction equation ,is said to be the heat potential with density* $f$.

**Proposition 2.6.1.** *If the heat potential* $V$ *exists in* $D(R^{n+1})$,*then it satisfies the heat conduction equation*

$$\frac{\partial V}{\partial t} = a^2 \triangle V + f(x,t) \tag{2.40}$$

**Verification:**By the thm1.1.1.

$$\frac{\partial}{\partial t}\varepsilon * f = a^2 \frac{\partial^2}{\partial t^2}\varepsilon * f + f(x,t)$$

Since $f$ is a distribution function with compact and becomes zero when $t < 0$,then ,the heat potential is known to exists in $D'(R^{n+1})$.
We shall distinguish another class of densities $f$ for which the heat potential exists.

Let $M$ be a class of functions which become zero for $t < 0$ and which are bounded in each strip $0 \leq t \leq T$.

**Theorem 2.6.1.** *If $f \in M$, then the heat potential $V$ with a density $f$ exists in $M$ and is expressed by the formula*

$$V(x,t) = \int_0^t \int_{R^n} \frac{f(z,\tau)}{2a\sqrt{\pi(t-\tau)}^n} exp(-\frac{|x-z|^2}{4a^2(t-\tau)})dzd\tau \quad (2.41)$$

The potential $V$ satisfies the estimate

$$|V(x,t)| \leq t \sup |f(z,t)|, t > 0^+, \qquad 0 \leq \tau \leq t, \quad (2.42)$$

$z \in R^n$ and the initial condition : for any fixed $x \in R^n$

$$V(x,t) \to 0 \quad (2.43)$$

$$as \;\; t \to 0^+$$

*Proof.* Since the function $\varepsilon$ and $f$ are locally integrable in $R^{n+1}$, then

$$\varepsilon * f = \int_0^t \int_{R^n} f(z,\tau)\varepsilon(x-z,t-\tau)dzd\tau$$

exists and is a locally integrable function in $R^{n+1}$,provided that the function

$$h(x,t) = \int_0^t \int_{R^n} |f(z,\tau)|\varepsilon(x-z,t-\tau)dzd\tau$$

is locally integrable in $R^{n+1}$.
We shall check that this condition is satisfied.
Since $h = 0$ when $t < 0$, then it is sufficient to establish that the function h satisfies eq. (2.42) for $t > 0$. This follows from eq. ( 2.27) by Fubini's theorem,

$$h(x,t) \leq \sup_{0 \leq \tau \leq t} |f(z,\tau)| \int_0^t \int_{R^n} \varepsilon(x-z,t-\tau)dzd\tau,$$

$$= t \sup_{0 \leq \tau \leq t} |f(z,\tau)|, t > 0, \quad (2.44)$$

In this way, the heat potential $V = \varepsilon * f$ is represented by eq. (2.41).
Since $|V| \leq h$ when $t < 0$, then this potential becomes zero for $t < 0$ and, by virtue of eq.(2.44), satisfies eq. (2.42).
This means that $V \in$M .
It follows from eq. (2.42) that $V$ satisfies the initial condition in the sense of eq.(2.43). □

### 2.6.2 Surface Heat Potential

**Definition 2.6.2.** *The heat potential* $V^{(O)}$ *with a density*

$$f = u_o(x).\delta(t)$$

*is known as the surface heat potential (of a simple layer with a density* $u_o$*),*

$$V^{(O)} = \varepsilon * u_o(x).\delta(t) = \varepsilon(x,t) * u_o(x)$$

If $u_o$ is of compact support in $R^n$, then the surface heat potential $V^{(O)}$is known to exist in $D'(R^{n+1})$.

**Theorem 2.6.2.** *If* $u_o(x)$ *is a function bounded in* $R^n$ *then the surface heat potential* $V^{(O)}$ *exists in* $M$*, belongs to the class* $C^{\infty}(t > 0)$*, is represented by Poisson's integral*

$$V^{(O)}(x,t) = \frac{\theta(t)}{(2a\sqrt{\pi t})^n} \int_{R^n} u_o(z) exp(\frac{-|x-z|^2}{4a^2t}) dz \tag{2.45}$$

*and satisfies the inequality*

$$|V^{(O)}(x,t) \leq \sup |u_o(z)|, z \in R^n, t > 0 \tag{2.46}$$

*If, moreover, the function* $u_o(x)$ *is continuous in* $R^n$*, then the potential* $V^{(O)}$ *satisfies the initial condition : for each* $x \in R^n$

$$V^{(O)}(x,t) \to u_o(x), t \to 0^+ \tag{2.47}$$

*Proof.* Since the function

$$h(x,t) = \int |u_o(z)| \varepsilon(x-z,t) dz$$

becomes zero for $t < 0$ and, for $t > 0$, by eq.(2.35 ), satisfies eq. (2.46) :

$$\begin{aligned} h(x,t) &\leq \sup |u_o(z)| \varepsilon(x-z,t)dz \\ &= \sup |u_o(z)| \end{aligned}$$

then this function is locally integrable in $R^{n+1}$.
Consequently, the surface heat potential
$V^{(O)} = \varepsilon(x,t) * u_o(x)$ is represented by eq. (2.45):

$$V^{(O)}(x,t) = \int u_o(z)\varepsilon(x-z,t)dz \tag{2.48}$$

becomes zero for $t < 0$, and, by virtue of the inequality

$$|V^{(O)}| \leq h,$$

satisfies eq. (2.46).This means that $V^{(O)} \in M$.
Moreover, it follows from formula eq.(2.45)that $V^{(O)} \in C^\infty(t > 0)$.
Now, let $u_o$ be a continuous function bounded in $R^n$.
we deduce initial condition eq.(2.47)for the potential $V^{(O)}$ :

$$V^{(O)}(x,t) = \langle \varepsilon(x-z,t), u_o(z) \rangle$$

$$\begin{aligned} &\to \langle \delta(x-z), u_o(z) \rangle \\ &= u_o(x), t \to 0^+ \end{aligned}$$

□

**N.B:** Formula eq.(2.45) follows formally from formula eq.(2.41) if we set $f(z,\tau) = u_0(z).\delta(\tau)$and $\delta(\tau)$is integrated.

# Chapter 3

# The Cauchy problem for the Heat Conducting Equation

## 3.1 The Cauchy problem for the Ordinary Differential equations with constant coefficients

Consider the Cauchy problem for an ordinary differential equations with constant coefficients

$$Lu \equiv u^k + a_1 u^{k-1} + .... + a_k u = f(t), t > 0 \tag{3.1}$$

$$u^n(0) = u_n, n = 0, 1, 2..., k-1 \tag{3.2}$$

Where $f \in C(t \geq 0)$.
Suppose that $u(t)$ is the solution of the Cauchy problem (3.1),(3.2).
We will continue the functions $u(t)$ and $f(t)$to zero when $t < 0$.
Using the Fourier transform,we denote,
$F[u] = \widetilde{u}$ and $F[f] = \widetilde{f}$ respectively and the initial conditions.
We obtain

$$\widetilde{u}^{(n)} = \{u^n(t)\} + \sum_{j=0}^{n-1} u_j \delta^{n-j-1}(t), n = 1, 2, ...k$$

combining this with eq.(3.2),we conclude that
$\widetilde{Lu} = \{\widetilde{Lu}(t)\} u_0 \delta^{k-1} + (a_1 u_0 + u_1)\delta^{k-2} + .. + (a_{k-1}u_0 + ...a_1 u_{k-2} + u_{k-1})\delta(t)$

$= f(t) + \sum_{n=0}^{k-1} c_n \delta^n,$

where

$$c_0 = a_{k-1}u_0 + ...a_1 u_{k-2} + u_{k-1}$$
$$c_{k-2} = a_1 u_0 + u_1$$
$$c_{k-1} = u_0$$

Thus,in the distribution sense the function $\widetilde{u}$ satisfies in $R$ the differential equation.

$$\widetilde{Lu} = f(t) + \sum_{n=0}^{k-1} c_n \ \delta^n(t) \tag{3.3}$$

Let us construct the solution eq.(3.1).
The function $\varepsilon(t) = \theta(t)z(t)$ where $Lz = 0$ and

$$z(0) = z'(0) = .... = z^{k-2}(0) = 0, z^{k-1}(0) = 1 \tag{3.4}$$

is the fundamental solution of the operator $L$.
Since $\varepsilon$ and eq.(3.1) belongs to the convolution of distribution function.
By the thm1.1.1. ,the solution eq.(3.1)exists and is unique in $D'(R)$ and expressed by the convolution of

$$\widetilde{u} = \varepsilon * (f(t) + \textstyle\sum_{n=0}^{k-1} c_n \ \delta^n(t))$$

$$= \varepsilon * f + \textstyle\sum_{n=0}^{k-1} c_n \ \delta^n(t)$$

$$= \theta(t) \int_0^t z(t-\tau)f(\tau)d\tau + \theta(t) \sum_{n=0}^{k-1} c_n \ z^n(t) \tag{3.5}$$

Here, we have $\varepsilon^n(t) = [\theta(t)z(t)]^n = \theta(t)z^n(t), n = 0, 1, 2, ..., k-1$ by the eq.(3.4).

$$Hence, u(t) = \int_0^t z(t-\tau)f(\tau)d\tau + \sum_{n=0}^{k-1} c_n \ z^n(t) \tag{3.6}$$

is the solution of the Cauchy problem.

In particular,

$$u^{''} + a^2 u = f(t)$$
$$u_{|t=0} = u_0, u^{'}_{|t=0} = u_1$$

where $f \in C(t \geq 0)$.
Let $u(t)$ be the solution of this problem and the function $f(t) = 0, t > 0$; using the Fourier transform $\widetilde{u}$ and $\widetilde{f}$ denotes the Fourier transform of $u(t)$ and $f(t)$ respectively.
Then, using formula eq.(3.5) and since

$$\varepsilon(t) = \theta(t)\frac{\sin at}{a}$$

is the fundamental solution of the operator

$$u^{''} + a^2 u = \delta,$$

becomes zero when $t < 0$,then its convolution with right hand side of eq.(3.5)exists in $D'(R)$ and becomes zero for $t < 0$, by theorem1.1.1.
So,by thm2.1.1,the solution of eq.(3.5)exists and is unique in the class of distribution functions belonging $D^{'}(R)$ which becomes zero when $t < 0$,and this solution is expressed by the convolution,

$$u(t) = \frac{1}{a}\int_0^t f(\tau)\sin a(t-\tau)d\tau + u_0\cos(at) + u_1\frac{\sin at}{a} \qquad (3.7)$$

is the solution of the given Cauchy problem.

## 3.2 Formulation of the generalized Cauchy Problem for the Heat Conducting Equation

The method for solving the Cauchy problem which was set out for the ordinary linear differential equation is also used in the solution of the Cauchy problem for the heat conducting equation.

$$\frac{\partial u}{\partial t} = a^2 \triangle u + f(x,t) \tag{3.8}$$

$$u_{|t=0} = u_0 \tag{3.9}$$

Consider $f \in C(t \geq 0)$ and $u_0 \in C(R^n)$. Suppose there is a classical solution $u(x,t)$ of this problem.
This means that $u \in C^2(t >) \cap C(t > 0)$ and eq.(3.8)and the initial condition eq.(3.9) for $t \to 0$ are satisfied this Cauchy problem.
Using the Fourier transform, and suppose

$$\widetilde{u} = \begin{cases} u, t > 0 \\ 0, t < 0 \end{cases}$$

and

$$\widetilde{f} = \begin{cases} f, t > 0 \\ 0, t < 0 \end{cases}$$

$$F[\tfrac{\partial u}{\partial t}] = F[a^2 \triangle u + f(x,t)]$$

$$\tfrac{\partial}{\partial t} F[u] = a^2 z^2 F[u] + F[f(x,t)]$$

where, $z$is a parameter.

Let $F[u] = \widetilde{u}$ and $F[f(x,t)] = \widetilde{f} + u_0(x).\delta(t)$

$$\frac{\partial \widetilde{u}}{\partial t} = a^2 z^2 \widetilde{u} + \widetilde{f} + u_0(x).\delta(t) \tag{3.10}$$

Here the initial distribution $u_0(x)$ for the function $\widetilde{u}(x,t)$ serves as the external source of the type of a simple layer $u_0.\delta(t)$ acting instantaneously and the classical solution of the Cauchy problem (3.8)-(3.9) are contained among the those solution of eq.(3.10)which becomes zero for $t < 0$.

$$\frac{\partial u}{\partial t} = a^2 \triangle u + f(x,t) + u_0.\delta(t) \tag{3.11}$$

Eq.(3.11)is the generalized Cauchy problem for the equation of heat conduction with the source $f \in D'(R^{n+1})$ and the initial distribution function $u_0 \in D'(R^{n+1})$

## 3.3 Generalized Solution of the Cauchy Problem for the Heat Conducting Equation

**Definition 3.3.1.** *The generalized Cauchy problem for the heat conducting equation with source $f \in D^{'}(R^{n+1})$ ,$f = 0$, for $t < 0$,is the problem finding the generalized solution$u(x,t)$in $R^{n+1}$ of the heat conducting equation*

$$\frac{\partial u}{\partial t} = a^2 \triangle u + f(x,t) + u_0.\delta(t). \tag{3.12}$$

We can find generalized solution of the generalized Cauchy problem for the heat conducting equation of eq.(3.12)using Fourier transform.

$$f[\frac{\partial u}{\partial t}] = F[a^2 \triangle u + f(x,t) + u_0.\delta(t)]$$
$$\frac{\partial \widetilde{u}}{\partial t} = a^2|z|^2\widetilde{u} + \widetilde{f}(x,t) + \widetilde{u}_0$$

The solution is given by

$$\widetilde{u}(z,t) = exp(-a^2|z|^2 t)\widetilde{u}_0(z) + \int_0^t exp(-a^2|z|^2(t-\tau))\widetilde{f}(z,\tau)d\tau.$$

Applying the inverse Fourier transform and change of variables,

$$u(x,t) = \frac{1}{(2\pi)^n}\int_{R^n} exp(ix.z)(exp(-a^2|z|^2 t))\widetilde{u}_0(z) + \int_0^t exp(-a^2|z|^2(t-\tau))\widetilde{f}(z,\tau)d\tau)dz$$

Using the properties of Fourier transform and inverse Fourier transform,we obtained,

$$u(x,t) = \frac{1}{(2a\sqrt{\pi t})^n}\int_{R^n} u_0(y)exp(-\frac{|y-x|^2}{4a^2 t})dy + \int_0^t \int_{R^n} \frac{f(z,\tau)}{(2a\sqrt{\pi(t-\tau)})^n} exp(-\frac{|x-z|^2}{4a^2(t-\tau)})dyd\tau$$

is solution of the generalized Cauchy problem for heat equation.

**Theorem 3.3.1.** *Let $f \in M$ and $u_o$ be a function bounded in $R^n$. Then the solution of the corresponding generalized Cauchy problem exists and is unique in the class $M$ and appears in the form of a sum of the two heat potentials*

$$u(x,t) = V(x,t) + V^{(O)}(x,t) \tag{3.13}$$

where the potentials $V$ and $V^{(0)}$ are expressed by the equations (3.4) and (3.8). The solution depends continuously on $f$ and $u_o$ in the following sense :
If

$$|f - \widetilde{f}| \leq \epsilon, |u_0 - \widetilde{u_0}| \leq \epsilon_0$$

then the corresponding solutions $u$ and $\widetilde{u}$ in any strip $0 \leq t \leq T$ satisfy the inequality

$$|u(x,t) - \widetilde{u}(x,t)| \leq T_\epsilon + \epsilon_0 \tag{3.14}$$

Moreover, if $u \in C(R^n)$, then the solution $u(x,t)$ which has been constructed satisfies the initial condition : for each $x \in R^n$

$$u(x,t) \rightarrow u_o(x), t \rightarrow O^+ \tag{3.15}$$

*Proof.* By the conditions of the theorem, the convolution of $\varepsilon$ with the right-hand side of eq. (3.16) exists in $M$ and eq.(3.8),(3.9)and appears in the form of a sum eq.(3.15) of the two heat potentials $V$ and $V^{(O)}$ and these potentials are expressed by the formulas eq.(2.47) and eq.(2.49), respectively.
In this way, according to the thm1.1.1., formula eq.(2.23) gives a solution of the generalized Cauchy problem for the heat conduction equation, and this solution is unique in the class $M$.
The continuous dependence of the solution on the data of the problem $f$ and $u_0$ follows from estimates eq.(3.5) and eq.(3.9). The initial condition follows from eq.(3.6) and eq.(3.10). □

**N.B:** A necessary condition to find the solution of the generalized Cauchy problem is that $f$ must become zero for $t < O$.

## 3.4 Classical Solution of the Cauchy Problem for the Heat Conducting Equation

Consider eq.(3.8) and eq.(3.9) with $f = 0$.
Since $u(x,t)$ is a solution the equation then using the Fourier transform ,we obtained

$$u(x,t) = \frac{1}{(2a\sqrt{\pi t})^n} \int_{R^n} u_o(z) e^{\frac{-|x-z|^2}{4a^2 t}} dz \tag{3.16}$$

This solution belongs to the class $C^\infty(t > 0)$ and therefore satisfies the homogeneous heat conduction equation for $t > 0$ in the classical sense. If the function $u_o$ is continuous and bounded in $R^n$, then, using formula eq.(3.16), it is easy to see that $u \in C(t > 0)$. Moreover, by the theorem just proved, this solution is unique, belongs to the class $M$, satisfies initial condition , and depends continuously on $u_o$ . So in this case Poisson's integral eq.(3.16) gives the classical solution of the Cauchy problem for the heat conducting equation, and the classical formulation of this problem is correct, while the intersection $C^\infty(t > 0) \cap C(t > 0) \cap M$is the correctness class.The correctness of the solution of the Cauchy problem for the heat conduction equation may be established in a wider class, namely the class of functions which satisfy in any strip $0 \leq t \leq T$ the equation $|u(x,t) \leq C_T e^{(a_T |x|^2)}$. To say the solution of the Cauchy problem for the heat conduction equation is correctness,if these problems must satisfy the following

1. The solution must exist within a certain class of functions $M_1$.
2. The solution must be unique within a certain class of functions $M_2$.
3. The solution must depend continuously on the data of the problem (initial and boundary data, inhomogeneous term, coefficients of the equation).

# Summary

In chapter1,we discussed about $L^p$spaces and properties of $L^p$spaces ,test functions,properties of test functions ,distributions functions and properties of distributions functions ,Fourier transform method,properties of Fourier transform method and inverse Fourier transform method .

In chapter2,we discussed about fundamental solutions of ordinary differential equations and fundamental solutions of wave operator,Laplace operator and heat conducting operators using Fourier transform . And properties of fundamental solutions of heat conducting operators. **i.e** About heat potential and surface heat potential of heat conducting operators.

And finally in chapter 3,we state and solve Cauchy problem for heat conducting operators.

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