GRADUATE SEMINAR REPORT

ON

STABILITY PROBLEMS IN MATHEMATICAL OPTIMIZATION

By

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PREFACE

Even though the question of Stability is highly considered in different branches of study we are not attempted to present an exhaustive study of stability in this seminar.

We are concerned only to consider Stability of Mathematical optimization problems with primary attention given to Regularization method to solve unstable extermal problems.

This seminar has two parts:-
The first part deals with some basic facts on convex sets, convex functions and related facts.

The second part deals with correctly and incorrectly posed problems and methods of determining (approximating) in correctly posed problems with correct ones.

Before all I like to thank the heavenly father Almighty God, with the help of WHOM this seminar has come to reality.

I am deeply most grateful to my beloved advisor and instructor Prof. Dr. rer.nat.habil.R.Deumlich for his constructive comments, suggestions, hints, valuable advices as well as lots of interesting things for which I don't have space to mention.

I am realy lucky to have him as my advisor.

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Contents

1 Some Facts on convex sets and convex Functions

1.1 Notations

The following notations are going to be used through out this seminar report:

a) U: $= \Re^n$, U: $= \{x \in \Re^n | x \geq 0\}$ or U: $=\{x \in \Re^n |\ a \leq x \leq b\}$.

$||x||: = \sqrt{<x,x>}$ where $x \in \Re^n$

(The Euclidean norm of a vector x)

a) Let $X \subseteq U$, $g: U \rightarrow \Re$ be a function, then we consider the following optimization problem:

Find an $x_o \in X$ such that

$g(x_o) \leq g(x)$ or $g(x_o) = \min_{x \in X} g(x)$.

(P) $g(x) \rightarrow \min\ x \in X$. and we use the following notations

g …. objective function of (p),

X .. feasible set of (p),

$x \in X$ … constraint of (p),

$x_o \in X$ … solution of (p) if $g(x_o) = \min_{x \in X} g(x)$,

M (g, X) the set of all solutions of (p).

c) $\rho(v, X)$ … the distance of the point v from the set X.

$(\rho = \rho(v, X): = \inf_{x \in X} || x - v||)$. (1)

1.2 Some Facts about convex sets

Let S be a vector space, for $x, y \in S$,

$[x, y] : = \{z \in s |\ z = \lambda x + (1-\lambda)y, \lambda \in [0, 1]\}$ is said to be a closed segment.

$(x, y): = \{z \in s | z = \lambda x + (1 - \lambda)y, \lambda \in (0, 1)\}$ is said to be an open segment.

Definition 1.2.1 : Let $K \subseteq S$, then K is said to be a convex set if and only if $x,y \in K$ implies $\lambda x + (1-\lambda)y \in k$, for all $x, y \in K$, for all $\lambda \in [0, 1]$.

i.e. K is convex if and only if $x, y \in K$ implies $[x, y] \subseteq k$.

In $\Re^2$, a line segment, half plane, half line, a whole straight line, a circle a triangle and the entire plane are examples of convex sets.

1.3 Some Facts about convex Functions

Definition 1.3.1: Let $X \subseteq \Re^n$ be a convex set and $g: X \rightarrow \Re$ be a function. Then

a) g is said to be a convex function if and only if

$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda) g(y)$, for all, $y \in X$ and for all $\lambda \in [0, 1]$.

b) g is said to be a strictly convex function if and only if

$g(\lambda x + (1-\lambda)y) < \lambda g(x) + (1-\lambda) g(y)$, for all $x,y \in X$, $x \neq y$, for all $\lambda \in (0, 1)$.

c) g is said to be a strongly convex function if and only if there is a constant $\rho > 0$ such that

$$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda) g(y) - \lambda(1-\lambda)\rho \|x-y\|^2$$

for all $x, y \in X$, $\lambda \in [0, 1]$.

d) g is said to be a concave function if and only if $-g$ is a convex function.

Definition 1.3.2: A matrix B is said to be positive definite if and only if there exists an $\alpha > 0$ such that $\langle x, Bx \rangle \geq \alpha \|x\|^2$ for all $x \in \Re^n$.

Example: 1. A quadratic function g given by

$g(x) = \langle x, Bx \rangle + \langle p, x \rangle$ where B is a symmetric positive definite matrix is convex.

Proof : We have $g(\lambda x + (1-\lambda)y) = \lambda^2 \langle x, Bx \rangle + 2\lambda(1-\lambda) \langle x, By \rangle + (1-\lambda)^2 \langle y, By \rangle + \lambda\langle p, x\rangle + (1-\lambda) \langle p,y\rangle$

$$= \lambda g(x) + 1-\lambda)\, g(y) - \lambda(1-\lambda) \langle x-y, B(x-y)\rangle.$$

For $\lambda\in(0, 1)$, $\lambda(1-\lambda)$, $\langle x-y, B(x-y)\rangle \geq 0$ if and only if B is a positive definite matrix.

Hence $g(\lambda x +(1-\lambda)y) = \lambda g(x)+ (1-\lambda)\, g(y) - \lambda(1-\lambda) \langle x - y, B(x-y) \rangle \leq \lambda g(x) + (1-\lambda)g(y)$.

Example 2: A quadratic function g given by $g(x) = \langle x, BX \rangle + \langle p, x\rangle$ where B is a strictly positive definite matrix is a strongly convex.

The strong convexity follows from the relation

$$g(\lambda x +(1-\lambda)y) = \langle \lambda x + (1-\lambda)y, B(\lambda x + (1-\lambda)y)\rangle + \langle p, \lambda x +(1-\lambda)y \rangle$$

$$= \langle \lambda x + (1-\lambda)y, \lambda Bx +(1-\lambda)By \rangle +\langle p, \lambda x + (1-\lambda)y \rangle$$

$$= \lambda^2 \langle x,Bx \rangle + \lambda(1-\lambda) \langle x, By \rangle + \lambda(1-\lambda) \langle y, Bx\rangle +(1-\lambda)^2 \langle y, By \rangle +\lambda\langle p, x\rangle + (1-\lambda)\langle p,y\rangle$$

$$= \lambda^2 \langle x, Bx\rangle + \lambda \langle p, x\rangle + (1-\lambda)^2 \langle y,By\rangle + (1+\lambda) \langle p,y\rangle + \lambda(1-\lambda) \langle x,By\rangle + \lambda(1-\lambda) \langle y,Bx\rangle$$

$$= \lambda.g(x) + (1-\lambda)g(y) - \lambda(1-\lambda) \langle x-y, B(x-y)\rangle$$

By positive definiteness of B we have

$$\langle x - y, B(x - y) \rangle \geq \alpha\|x - y\|^2.$$

were α is the least eigen value of the matrix B. There fore

$$g(\lambda x +(1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y) - \alpha\lambda(1-\lambda) \|x-y\|^2 \text{ where } \lambda \in(0, 1).$$

Consequently the strong convexity follows.

From example 2 it follows that in $\Re$ a quadratic function g, given by $g(x) = x^2 + x$ is a strongly convex. In general any quadratic function g defined on $\Re$ which is given by $g(x) := ax^2 + px$ (where $a \in\Re +$ and $a > 0$, $p \in\Re$) is a strongly convex function.

Proposition 1.3.1 : Let X be convex set and $g: x \rightarrow \Re$ be a convex function. Then for any constant c the set

$Z: = \{x \in X \mid g(x) \leq c\}$ is convex .

Proof: It suffices to show that x, y $\in$ z, implies that $\alpha x + (1-\alpha)\, y \in Z$, $\alpha \in (0, 1)$.

Since g is a convex function we have $g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)\, g(y)$ (1)

By x, y $\in Z$ we have $g(x) \leq c$ and $g(y) \leq c$. (2)

From (1) and (2) we have

$g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)\, g(y) \leq \alpha.\, c + (1-\alpha).\, c = c.$ (3)

From (3) it follows that $\alpha x + (1-\alpha)y \in Z$. //

Proposition 1.3.2: **Let X be a convex set, g: X $\rightarrow \Re$ be a convex function** and $x^* \in X$ is any point of local minimum for problem of minimizing the function g on the set X then x^* is an optimal point.

Proof: Suppose x^* is not optimal, then there is a point $x' \in X$ such that $g(x') < g(x^*)$

Let now $x: = \alpha x' + (1-\alpha)x^*$, $\alpha \in [0, 1]$

Then by the convexity of X we have $x \in X$. Further more by the convexity of g we have

$$g(x) = g(\alpha x' + (1-\alpha)x^*) \leq \alpha g(x') + (1-\alpha)\, g(x^*)$$
$$\leq \alpha g(x^*) + (1-\alpha)\, g(x^*) = g(x^*)$$

which implies that $g(x) < g(x^*)$.

But the last result contradicts that x^* is a local minimum point.

Since for sufficiently small α the point x belongs to an arbitrary small neighborhood of the point x^*. //

Proposition 1.3.3 : Let X be a convex set and $g: X \to \Re$ be a convex function. Then the set of optimal points i.e M(g, X) is a convex set .

proof: suppose $x_1, x_2 \in M(g, X)$

since $M(g, X) \subseteq X$ and X is a convex set we have

$z := \lambda x_1 + (1-\lambda)x_2 \in X$, for all $\lambda \in [0, 1,]$

By the convexity of g we have

$g(z) = g(\lambda x_1 + (1-\lambda)x_2) = \lambda g(x_1) + (1-\lambda)g(x_2)$

But since $x_1, x_2 \in M(g, X)$ we have $g(x_1) = g(x_2) =: c$

Then we get $g(z) = g(\lambda x_1 + (1-\lambda)x_2) \leq \lambda g(x_1) + (1-\lambda)g(x_2) = c$.

Thus $g(z) \leq c$.

On the other hand since c is a minimal value it clearly holds that

$c \leq g(z)$.

There fore $g(z) = c$.

Consequently $z \in M(g, X)$. //

Proposition 1.3.4: Let X be a convex set, $g: X \to \Re$ be a strictly convex function and $x^* \in X$ is an optimal point.

i.e. $g(x^*) = \min_{x \in X} g(x)$, then $g(x) > g(x^*)$ for all $\underset{x \neq x^*}{x \in X}$ and consequently, the point x^* is unique.

Proof :Assume that there is a point $x' \in X$, $x' \neq x^*$ such that $g(x') = g(x^*) =: c$.

Then given $\alpha \in (0, 1)$ the point $x = \alpha x' + (1-\alpha)x^* \in X$

By the strict convexity of g we have

$$g(x) = g(\alpha x' + (1-\alpha)x^*) < \alpha g(x') + (1-\alpha)g(x^*)$$

$$= \alpha . c + (1-\alpha) . c = c.$$

Which contradicts the optimality of the point x^*. //

Proposition 1.3.5: Let X be a convex and closed set g: X $\rightarrow \Re$ be a strongly convex function and continuous, then for any point y $\in$X the set X_o: = {x $\in$X |g(x) $\leq$ g(y)} is bounded and there is unique point $x_o \in X_o$ such that $g(x_o) \leq g(x)$ for all x $\in$X, i.e.

$g(x_o) = \min_{x \in X} g(x)$.

<u>Proof</u>: Let y be an arbitrary point in X but fixed.

By the continuity of the function g on X for $\rho > 0$ there is $\varepsilon > 0$ such that $|g(x) - g(y)| \leq \rho$ for all x, y $\in$V where V: = {x $\in$X| $\|x-y\| \leq \varepsilon$} (1)

From (1) we get $g(x) \geq g(y) - \rho$ for all x $\in$V.

Let x $\in$X\V and λ: ε. $\|x-y\|^{-1}$ then, $\lambda \leq 1$ as $\|x-y\|^{-1} > \varepsilon$, for all x $\in$X\V.

By the strong convexity of g we have

$g(\lambda x+(1-\lambda)y) \leq \lambda g(x) + (1-\lambda) g(y) - \lambda(1-\lambda) \rho\|x-y\|^2$ (2)

From (2) we get $\lambda g(x) \geq g(\lambda x + (1-\lambda)y) - (1-\lambda)g(y) + \lambda(1-\lambda)\rho \|x-y\|^2$ (3)

Now let z: = $\lambda x + (1-\lambda)y = y+\lambda(x-y)$. (4)

From (4) we get $\|z-y\| \leq \lambda \|x-y\| = \varepsilon.\|x-y\|^{-1}. \|x-y\| = \varepsilon$, which gives that z $\in$V. (5)

From (1), (3) and (5) we get

$$\lambda g(x) \geq -\rho+\lambda g(y) + \lambda(1-\lambda) \rho\|x-y\|^2 \quad (6)$$

Dividing both sides of (6) by λ gives

$$g(x) \geq \frac{-\rho}{\lambda} + g(y) + (1-\lambda) \rho\|x - y\|^2 \quad (7)$$

Substituting the value of $\lambda = \varepsilon \|x-y\|^{-1}$ in (7) we get

$$g(x) \geq g(y) + (1-\varepsilon\|x-y\|^{-1})\rho \|x-y\|^2 - \frac{\rho}{\varepsilon.\|x-y\|^{-1}}$$

$$\geq g(y) + \rho.\|x-y\|^2 - \varepsilon\rho\|x-y\| \frac{-\rho}{\lambda} \|x-y\|$$

$$\geq g(y) - \rho\left(\varepsilon + \frac{1}{\varepsilon}\right) \|x-y\| + \rho \|x-y\|^2$$

There fore

$$g(x) \geq g(y) - \rho \left(\varepsilon + \frac{1}{\varepsilon}\right) \|x-y\| + \rho \|x-y\|^2 \quad (8)$$

From (8) we get that if the set X is not bounded then $g(x) \to \infty$ as $||x|| \to \infty$.

This we show now

By $g(x) \geq g(y) - \rho\left(\varepsilon + \frac{1}{\varepsilon}\right) ||x-y|| + \rho\, ||x-y||^2$.

$= g(y) + ||x-y||\, [\rho(||x - y||) - \rho \left(\varepsilon + \frac{1}{\varepsilon}\right)\ ]$

$\geq g(y) + (||x|| - ||y||)\, (\rho(||x|| - ||y|| - \rho\left(\varepsilon + \frac{1}{\varepsilon}\right)\ )$

But if $||x|| \to \infty$ then $||x|| - ||y|| \to \infty$ as $g(y)$, $||y||$, $\rho\left(\varepsilon + \frac{1}{\varepsilon}\right)$ are fixed constants and $\rho > 0$,

Therefore $g(x) \to \infty$ as $||x|| \to \infty$.

Now we show that the set X_o is bounded

To prove this we assume to the contrary (indirect proof)

Suppose the set X_o is not bounded, then there is a sequence $(x_k) \in X_o$ such that $||x_k|| \to \infty$ as $k \to \infty$.

Therefore there exists an index $k_o = k_o(y)$ such that $g(x_k) > g(y)$ for all $k \geq k_o$ which follows from the continuity of the function g.

Consequently $x_k \in X_o$, for all $k \geq k_o$ contradicting the assumption that $x_k \in X_o$.

Therefore the set X_o is bounded.

Clearly the point x_o exists because a continuous function on a closed and bounded set has a minimal point i.e $x_o \in M(g, X_o)$.

But since g is a convex function on a convex set X by proposition 1.3.2 $x_o \in M(g, X)$.

More over the point x_o is unique because a strongly convex function is at the same time strictly convex.

<u>Proposition 1.3.6</u>: Let X be a convex and closed set, $g: X \to \Re$ is a strongly convex function, then for all $x \in X$ the following inequality is valid.

$||x - x_o||^2 \leq \frac{2}{\rho}\ ||x - x_o||^2$ where $x_o \in M(g, X)$.

Proof: By strong convexity of g we have for $\lambda = \frac{1}{2}$,

$$g\left(\frac{1}{2}x+\frac{1}{2}x_0\right) \le \frac{1}{2}g(x) + \frac{1}{2}g(x_0) - \frac{1}{4}\rho \|x - x_0\|^2 \tag{1}$$

From (1) we get

$$g\left(\frac{1}{2}x+\frac{1}{2}x_0\right) - \frac{1}{2}g(x_0) + \frac{1}{4}\rho \|x - x_0\|^2 < \frac{1}{2}g(x) \tag{2}$$

This implies

$$g(x_0) - \frac{1}{2}\, g(x_0) + \frac{1}{4}\rho \|x - x_0\|^2 < \frac{1}{2}g(x) \tag{3}$$

(3) follows from $g(x_0) \le g\left(\frac{1}{2}x+\frac{1}{2}x_0\right)$

From (3) we get

$$\frac{1}{2}g(x_0) + \frac{1}{4}\rho \|x - x_0\|^2 \le \frac{1}{2}g(x) \tag{4}$$

From (4) we have

$$\frac{1}{4}\rho \|x - x_0\|^2 \le \frac{1}{2}(g(x) - g(x_0)) \tag{5}$$

From (5) we conclude that

$$\|x - x_0\|^2 \le \frac{1}{2}(g(x) - g(x_0)).$$

Suppose the following optimization problem is give

(p) $\quad g(x) \to \min,\ x \in X \subseteq U \subseteq \Re^n.$ (1)

Definition 1.3.2: A continuous function $\frac{1}{\beta}\psi$ defined for all $x \in U$ and $\beta > 0$ will be called a penalty if ψ is such that $\psi(x) = 0$ for all $x \in X$

$\psi(x) > 0$ for all $x \in U\backslash X$ (2)

A function $\psi(x, B) = \beta g(x) + \psi(x)$ defined on the set U for all $\beta > 0$ will be called a penalty function of the problem (1) if ψ satisfies the conditions in (2).

Definition 1.3.3: Let X be a vector space $K \subseteq X$ and f: $K \rightarrow (-\infty, \infty)$ then, for $\alpha \in \Re$ the set $N_\alpha(f) := \{x \in K \mid f(x) \leq \alpha\}$ is said to be level set of f.

Remark: If f is a convex function, then the level sets $N_\alpha(f)$ are convex sets for all $\alpha \in \Re$.

Obviously for x, y $\in N_\alpha(f)$ and $\lambda \in (0, 1,)$ by the convexity of f we have

$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \leq \lambda.\alpha + (1-\lambda)y = \alpha$.

Theorem 1.3.1: Let $X \subseteq \Re^n$ and f: $X \rightarrow \Re$ be convex and continuous further more let the set X be convex. If f possesses a single minimum point $x_0 \in X$,

i.e. $\{x_0\} = M(f, X)$ then each level set $N_c(f)$ is bounded and closed, i.e each level set is compact.

Proof: We assume $N_c(f)$ is not bounded. Then there is a sequence (x_k) in $N_c(f)$ such that $\|x_k\| \xrightarrow{k\to\infty} \infty$. So we have $f(x_0) \leq f(x_k) \leq c$ for all $k \in N$ i.e $x_0 \in N_c(f)$ or

$$0 \leq f(x_k) - f(x_0) \leq c - f(x_0) = : c_1$$

Since $N_c(f)$ is convex we get $\lambda x_k + (1-\lambda)x_0 \in N_c(f)$ for all $\lambda \in (0,1)$ we have by the convexity of f (1)

$$f(\lambda x_k + (1-\lambda)x_0) \leq \lambda f(x_k) + (1-\lambda) f(x_0) = f(x_0) + \lambda \underbrace{(f(x_k) - f(x_0))}_{\leq c_1}$$

$$\leq f(x_0) + \lambda c_1, \text{ for all } \lambda \in (0,1).$$

Now we choose k_0 sufficiently great, Then we have

$$\|x_k\| > 1 \quad \text{for all } k \geq k_0$$

we set $\lambda = \dfrac{1}{\|x_k\|} \in (0, 1)$ for all $k \geq k_0$.

$$\text{Then } f\left(\frac{x_k}{\|x_k\|} + \left(1 - \frac{1}{\|x_k\|}\right)x_o\right) \leq \frac{1}{\|x_k\|} f(x_k) + \left(1 - \frac{1}{\|x_k\|}\right) f(x_0) \qquad (2)$$

$$\leq f(x_0) + \frac{1}{\|x_k\|} c_1$$

we set $z_k := \frac{x_k}{\|x_k\|} + \left(1 - \frac{1}{\|x_k\|}\right) x_0 = \lambda x_k + (1-\lambda) x_0$

Then we get

$$\|z_k\| = \|\lambda x_k + (1-\lambda) x_0\| \le \underbrace{\|\lambda x_k\|}_{=\left\|\frac{x_k}{\|x_k\|}\right\| = 1} + \|(1-\lambda) x_0\| = 1 + \|x_0\| - \lambda\|x_0\|, \ (\lambda \in (0,1)) \ .$$

So we have $\|\lambda x_k\| - 1\|(1-\lambda) x_0\| = 1 + \|x_0\| - \lambda\|x_0\| \le 1 + \|x_0\| \ \lambda \in (0,1)$

This gives that (z_k) is bounded.

Therefore, there is a bounded sequence (z_{ki}) i.e. $z_{ki} \to \tilde{z} \in N_c(f)$

Since f is continuous (assumption) $N_c(f)$ is closed.

From (2) we get

$$f(\tilde{z}) = f\left(\lim_{i\to\infty} z_{ki}\right) = \lim_{i\to\infty} f(z_{ki}) = \lim_{i\to\infty} f\left(\frac{x_{ki}}{\|x_{ki}\|} + \left(1 - \frac{1}{\|x_{ki}\|}\right) x_o\right)$$

$$\le \underbrace{\lim_{i\to\infty} \frac{1}{\|x_{ki}\|} f\Big(\underbrace{x_{ki}}_{\le c}\Big)}_{\to 0} + \lim_{i\to\infty}\left(1 - \frac{1}{\|x_{ki}\|}\right) f(x_0) = f(x_0).$$

Hence $f(\tilde{z}) \le f(x_0)$.

But since $x_0 \in M(f, X)$ we have $f(x_0) \le f(\tilde{z})$

i.e $f(\tilde{z}) = f(x_0)$.

Since x_0 is the only minimum point of f on X we have $\tilde{z} = x_0$.

On the other hand

$$\|z_k - x_0\|^2 = \|\lambda x_k + (1-\lambda) x_0 - x_0\|^2 = \|\lambda x_k - \lambda x_0\|^2$$

$$= \lambda^2[2(\|x_k\|^2 + \|x_0\|^2) - \|x_k + x_0\|^2] \quad \text{(parallelogram equation)},$$

$$\ge \lambda^2 [2\|x_k\|^2 + 2\|x_0\|^2 - (\|x_k\| + \|x_0\|)^2] \qquad \text{(Triangle inequality)}$$

$$= \lambda^2 [\|x_k\|^2 - 2\|x_k\|.\|x_0\| + \|x_0\|^2]$$

$$= \lambda^2 \|x_k\|^2 + \lambda^2 \|x_0\|^2 - 2\lambda^2\|x_k\|.\|x_0\|$$

$$= 1 + \frac{\|x_0\|^2}{\|x_k\|^2} - 2\frac{\|x_0\|}{\|x_k\|} \tag{3}$$

(3) is also true for the elements of the subsequence

i.e

$$\|z_{ki} - x_0\|^2 \geq 1 + \frac{\|x_0\|^2}{\|x_k\|^2} - 2\frac{\|x_0\|}{\|x_k\|} \xrightarrow{i\to\infty} 1$$

So we get

$$\|\tilde{z} - x_0\|^2 = \lim_{i\to\infty} \|z_{ik} - x_0\|^2 \geq 1.$$

i.e $\tilde{z} \neq x_0$.

But this a contradiction. //

2. Correctly and Incorrectly posed problems

In most of practical problems whose mathematical model are mathematical optimization problems the initial condition is of approximate character.

There fore, we identify problems and numerical minimization method for which small perturbation in the initial conditions does not produce strong effect on the solution.

2.1 The exact (Unperturbed) Problem

Let $U \subseteq \Re^n$, $g: U \to \Re$

Then we consider the following optimization problem

(p) $g(x) \to \min,\ x \in X,$

$X: = \{x \in U \mid f(x) \geq 0\},$

f is vector valued function, i.e. (1)

$f = (f_1, f_2, \ldots, f_m),$

$f_i: U \to \Re,\ i = 1, 2, \ldots m.$

Then (p) is called the exact (unperturbed) problem.

2.2 The initial information

In practice the information about the functions g and f is of approximate character, i.e instead of g and f we know some perturbed functions g_ε and f_ε on the set U such that, g_ε and f_ε belong to the set.

$p_\varepsilon(g, f): = \{g_\varepsilon, f_\varepsilon \mid |g(x) - g_\varepsilon(x)| \leq \varepsilon, \quad \|f(x) - f_\varepsilon(x)\| \leq \varepsilon,\ x \in U\}.$

(where ε is parameter of perturbation).

There fore, the solution y to the unperturbed problem is determined from the solution y_ε of perturbed problem.

2.3 The approximate (Perturbed) problem

Let $U \subseteq \Re^n$, $g_\varepsilon: U \to \Re$.

Then we consider the following optimization problem.

(p_ε) $g_\varepsilon(x) \to \min,\ x \in X_\varepsilon,$

$X_\varepsilon: = \{x \in U \mid f_\varepsilon(x) \geq 0\},$

f_ε is a vector function. i.e. (2)

$f_\varepsilon^T(x) = (f_{\varepsilon_1}(x),\ f_{\varepsilon_2}(x),\ \ldots,\ f_{\varepsilon_m}(x))^T,$

$f_{\varepsilon_i}: U \to \Re,\ i = 1, 2, \ldots, m.$

Then (p_ε) is called approximate (perturbed) problem.

2.4. Correctness

Let the un perturbed problem (p) be given.

Let $Y: = M(g, X)$ be non empty,

$Y_\varepsilon: = M(g_\varepsilon, X_\varepsilon)$ be non empty,

$Z_\varepsilon: = Y \cup Y_\varepsilon.$

Definition 2.4.1: The problem (p) is said to be:-

a) weakly correct if $\lim\limits_{\varepsilon \to 0} \sup\limits_{y_\varepsilon \in Y_\varepsilon} \inf\limits_{y \in Y} \|y_\varepsilon - y\| = 0.$

b) correct (correctly posed) if

$\lim\limits_{\varepsilon \to 0} \sup\limits_{z', z'' \in z_\varepsilon} \|z' - z''\| = 0$

c) In correct (Incorrectly posed) if

$\lim\limits_{\varepsilon \to 0} \sup\limits_{z', z'' \in z_\varepsilon} \|z' - z''\| = 0$ is not satisfied.

<u>**Remark**</u>: Obviously from definition 2.4.1 a) we have for $\varepsilon \to 0$ the family of the set Y_ε belongs to the set Y, i.e. given $\delta > 0$ there is $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and $y_\varepsilon \in Y_\varepsilon$ there is $y \in Y$ such that

$\|y_\varepsilon - y\| \leq \delta.$

From this it trivially follows that for any $\delta > 0$ there is $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and $g_\varepsilon \in p_\varepsilon$ we have

$$|g_\varepsilon(y_\varepsilon) - g(y)| \leq \delta. \quad (1)$$

we will show now that (1) is valid.

By $g_\varepsilon \in p_\varepsilon(g)$ we get

$$|g_\varepsilon(x) - g(x)| \leq \varepsilon,\ x \in U;$$

If trivially follows that (2)

$$|g_\varepsilon(y) - g(y)| \leq \varepsilon,\ y \in Y.$$

From $\|y_\varepsilon - y\| \leq \delta$ and (2) we get

$$|g_\varepsilon(y_\varepsilon) - g(y)| \leq \varepsilon.$$

By setting $\delta := \varepsilon$ we get (1) i.e.

$$|g_\varepsilon(y_\varepsilon) - g(y)| \leq \delta.$$

There fore, in connection with this we have that weak correctness implies correctness i.e.

$$\lim_{\varepsilon \to 0} g_\varepsilon(y_\varepsilon) = g(y).$$

From definition 2.4.1. b) we have that the family of the set Y_ε belongs to a set Y consisting a single element y.

Obviously a correct problem is always weakly correct but not conversely.

Now we consider the following examples

Example 1

$g(x) = -x_1 - x_2 \to \min\ x \in S$

$S := \{x \in \Re^2\ |x_1 + x_2 \leq 1, x_1, x_2 \geq 0\}$

and $g(x) = -x_1 - x_2 \to \min,\ x \in S'$

$S' = \{x \in \Re^2\ |\ (1 + |\varepsilon| + \varepsilon)\, x_1 + (1 + |\varepsilon| - \varepsilon)\, x_2 \leq 1)\ x_1 \geq 0, x_2 \geq 0\ \}.$

Solution

case 1: suppose $\varepsilon = 0$

Then $Y = \{[a, b]\}$

$Y_\varepsilon = \{[a, b]\}$

$z_\varepsilon = \{[a, b]\}$

Figure 1 illustrates this fact

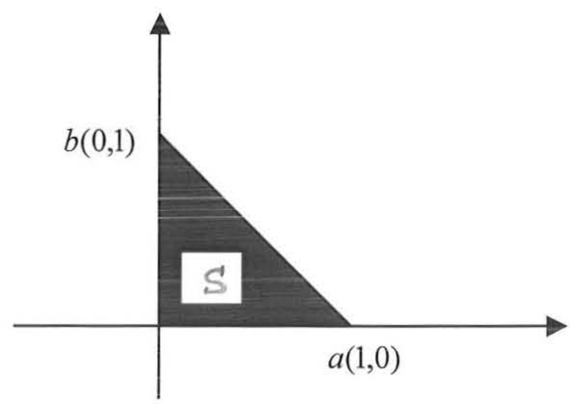

Figure 1

Case 2: Suppose $\varepsilon > 0$

Then

$(1+|\varepsilon| + |\varepsilon)\ x_1 + (1 + |\varepsilon| - \varepsilon)x_2 = 1$ is equivalent to

$(1+2\varepsilon)\ x_1 + x_2 = 1$

or $x_2 = -(1+2\varepsilon)\ x_1 + 1$

In this case

$Y = \{[a, b]\}$

$Y_\varepsilon = \{(0, 1)\}$.

$z_\varepsilon = YuY_\varepsilon = \{[a, b]\}$.

Figure 2 illustrates this

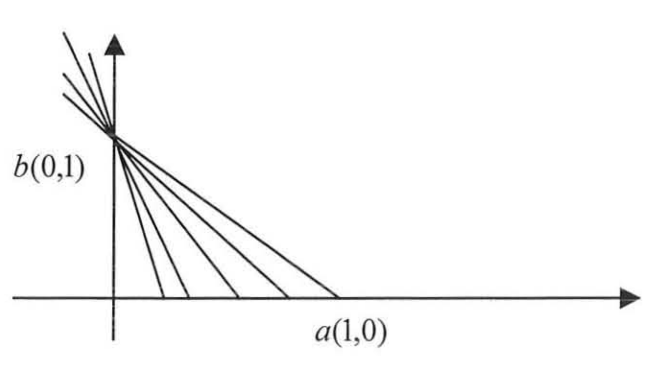

Figure 2

Case 3 : suppose $\varepsilon < 0$

Then $(1+|\varepsilon| + \varepsilon)\, x_1 + (1+ |\varepsilon| - \varepsilon)\, x_2 = 1$ is equivalent to $x_1 + (1 - 2\varepsilon)\, x_2 = 1$

$$\text{or} \quad x_2 = \frac{-x_1}{1-2\varepsilon} + \frac{1}{1-2\varepsilon}$$

We have the following generalization from above considerations

$Y_\varepsilon = \{[a, b]\}$ if $\varepsilon = 0$

$Y_\varepsilon = \{0, 1)\, \}$ if $\varepsilon > 0$

$Y_\varepsilon = \{(1, 0)\}$ if $\varepsilon < 0$

$Y = \{[a, b]\}$

$z_\varepsilon = Y_\varepsilon \cup y = \{[a, b]\}$.

Figure 3 illustrates this fact

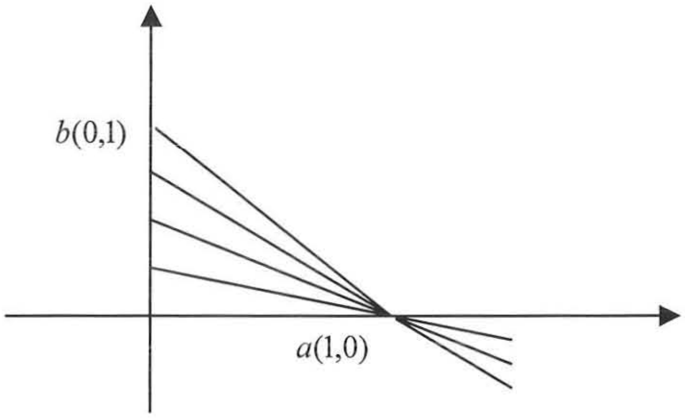

Figure 3

For $\varepsilon = 0$

$$\sup_{z',z''\in z_\varepsilon} \|z'-z''\| = \sqrt{2}$$

so $$\lim_{\varepsilon\downarrow 0} \sup_{z',z''\in z_\varepsilon} \|z'-z''\| = \lim_{\varepsilon\uparrow 0} \sup_{z^1,z''_\varepsilon} \|z'-z''\| = \sqrt{2}\,.$$

In this case over optimization problem is not correct.

For $\varepsilon > 0$

From $Y_\varepsilon \cup Y = z_\varepsilon = \{[a, b]\}$ we get in this case also

$$\lim_{\varepsilon \to 0} \sup_{z',z'' \in z_\varepsilon} \|z' - z''\| = \sqrt{2}$$

similarly it holds that for $\varepsilon < 0$ too that $\lim_{\varepsilon \to 0} \sup_{z',z'' \in z_\varepsilon} \|z' - z''\| = \sqrt{2}$

There fore in all cases the given optimization problem is not correct.

To see weakly correctness

Case 1 : suppose $\varepsilon = 0$

Then $\inf_{y \in Y} \| y_\varepsilon - y\| = 0$

There fore $\lim_{\varepsilon \to 0} \sup_{y_\varepsilon \in Y_\varepsilon} \inf_{y \in Y} \| y_\varepsilon - y\| = 0$

In the similarly way it can be show that

$$\lim_{\varepsilon \to 0} \sup_{y_\varepsilon \in Y_\varepsilon} \inf_{y \in Y} \| y_\varepsilon - y\| = 0 \text{ for } \varepsilon > 0, \varepsilon < 0$$

There fore our optimization problems is weakly correct.

Example 2

$$f(x) = x_2 \to \min, x \in S'$$

$$S: = \{x \in \Re^2 \mid x_1 + x_2 = 1, (1 + |\varepsilon| + \varepsilon)\, x_1 + (|+|\varepsilon|-\varepsilon)x_2 = 1\}$$

$$x_1 \geq 0, x_2 \geq 0$$

Case 1: Suppose $\varepsilon = 0$. Then we have from Figure 4

$$Y_\varepsilon = \{(1, 0)\}$$

$$Y = \{(1, 0)\}$$

$$z_\varepsilon = y_\varepsilon \cup Y = \{(1, 0)\}$$

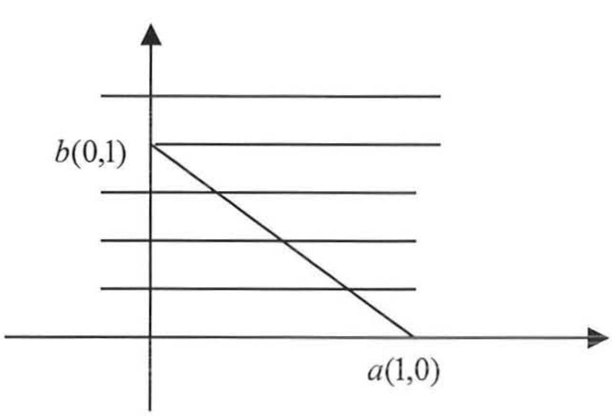

Figure 4

Case 2: $\varepsilon > 0$. Then

$x_2 = -(1+ + 2\,\varepsilon)\, x_1 + 1$ and from Figure 5 we get

$Y_\varepsilon = \{(0, 1)\}$

$Y = \{(1, 0)\}$

$z_\varepsilon = \{(0, 1), (1, 0)\}$.

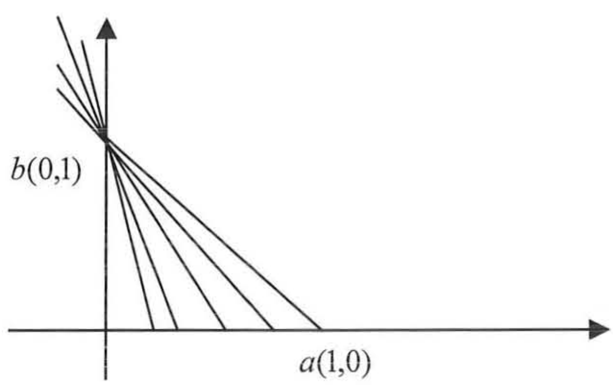

Figure 5

Case 3: $\varepsilon < 0$. Then $x_2 = -\dfrac{x_1}{1-2\varepsilon} + \dfrac{1}{1-2\varepsilon}$. From figure 6 we get

$Y_\varepsilon = \{(1, 0)\}$

$Y = \{(1, 0)\}$

$z_\varepsilon = \{(1, 0)\}$

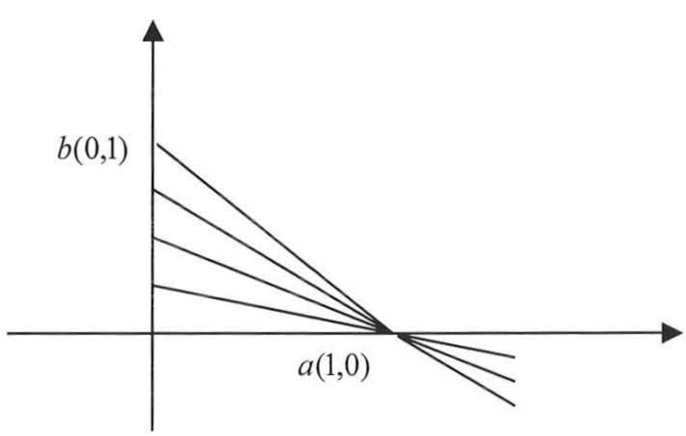

Figure 6

From the above considerations we have the following <u>generalization</u>

$z_\varepsilon = Y_\varepsilon \cup Y = \{(0, 1), (1, 0)\}$ if $\varepsilon > 0$

$z_\varepsilon = Y_\varepsilon \cup Y = \{(1, 0)\}$ if $\varepsilon \leq 0$.

For $\varepsilon = 0$ we get

$$\sup_{z',z''\in z_\varepsilon} \|z' - z''\| = \begin{cases} \sqrt{2} & \text{if } \varepsilon > 0 \\ 0 & \text{if } \varepsilon > 0 \end{cases}$$

Therefore, $\lim_{\varepsilon\to 0} \sup_{z',z''\in z_\varepsilon} \|z' - z''\| = \sqrt{2}$ and

$$\lim_{\varepsilon\to 0} \sup_{z',z''\in z_\varepsilon} \|z' - z''\| = 0$$

Therefore the problem is not correct for $\varepsilon = 0$

Now is we consider

$$\inf_{y\in Y} \| y_\varepsilon - y\| = \begin{cases} \sqrt{2} & \text{if } \varepsilon > 0 \\ 0 & \text{if } \varepsilon \leq 0 \end{cases}$$

Therefore

$$\lim_{\varepsilon\to 0} \sup_{y_\varepsilon\in Y_\varepsilon} \inf_{y\in Y} \| y_\varepsilon - y\| = \begin{cases} \sqrt{2} & \text{if } \varepsilon > 0 \\ 0 & \text{if } \varepsilon \leq 0 \end{cases}$$

Therefore, the given optimization problem is not weakly correct.

Methods of solving mathematical optimization problems were elaborated under clear assumption that the problems under consideration were correct.

One of these methods is regularization method. In the next chapters we will consider the regularization method to approximate incorrect problems with correct ones.
We consider the following parametric linear optimization problem:

<u>Example</u>

(p_λ) $\quad 2x_1 + 3x_2 - x_3 \to \max$

subject to

$2x_1 + 3x_2 - x_3 \leq 3 + \lambda$

$-x_1 - x_2 + 2x_3 \geq 2 - \lambda$

$x_1 + x_2 + x_3 = 3,\ x_1, x_2, x_3 \geq 0, \lambda \in \Re$

The following table give the solution of our problem depending on λ.

parameter λ	x_1*	x_2*	x_3*	f(x*)
$-\infty < \lambda < -4$	No feasible solution			
$-4 \le \lambda \le 2$	0	$\frac{4}{3}+\frac{1}{3}\lambda$	$\frac{5}{3}-\frac{1}{3}\lambda$	$\frac{7}{3}+\frac{4}{3}\lambda$
$2 < \lambda \le 5$	$\frac{-2}{3}+\frac{1}{3}\lambda$	2	$\frac{5}{3}-\frac{1}{3}\lambda$	$3+\lambda$
$5 < \lambda \le 6$	$6-\lambda$	$-3+\lambda$	0	$3+\lambda$
$6 < \lambda < \infty$	0	3	0	9

Table 1

From table 1 given above it can be seen that small perturbation of λ gives changes in the solution and the value of the objective function.

3. A class of correct problems

In this chapter we consider the exact (unperturbed) problem considered in chapter 2.1 and the problem of minimizing the approximate (perturbed) problem considered in chapter 2.3 i.e. we consider the following optimization problem.

(p_ε) $g_\varepsilon(x) \to \min, x \in X.$ (Chapter 2.3)

In this consideration the feasible set X remains unperturbed, while the objective function is perturbed.

Let $Y_\varepsilon := M(g_\varepsilon, U)$

Our objective is to determine

$y_\varepsilon \in M(g_\varepsilon, X)$.

To achieve this result we assume for each $\varepsilon > 0$ there exists a point $\tilde{y}_\varepsilon \in X$ such that $g_\varepsilon(\tilde{y}_\varepsilon) \leq g_\varepsilon(y_\varepsilon) + \eta(\varepsilon)$ where η is a positive constant that depends on ε. (3.1)

We prove the following theorem under the above considerations.

<u>Theorem 3.1:</u> Let X be a convex and closed set.

Let $g: X \to \mathfrak{R}$ be a strongly convex and continuous function.

Furthermore, let the following be satisfied.

a) g_ε is continuous on X,

$g_\varepsilon \in p_\varepsilon(g) := \{g_\varepsilon \mid |g_\varepsilon(x) - g(x)| \leq \varepsilon, x \in X\}$.

b) $\lim_{\varepsilon \to 0} \eta(\varepsilon) = 0$.

Then

$$\lim_{\varepsilon \to 0} \tilde{y}_\varepsilon = y_0.$$

Proof: From $g_\varepsilon \in p_\varepsilon$ we have

$$g_\varepsilon(x) \geq g(x) - \varepsilon, \ (x \in X). \qquad (1)$$

Further more, $g(x) \to \infty$ for $\|x\| \to \infty$

(by proposition 1.3.5). (2)

consequently $g_\varepsilon(x) \to \infty$ for all $\|x\| \to \infty$.

(by (1) and (2)) (3)

Therefore, there is a constant c, such that, the set $V: =\{x \in X \mid g_\varepsilon(x) \leq c\}$ is bounded and non empty (by (3) and proposition 1.3.5). Therefore, $M(g_\varepsilon, X)$ is non empty.

Now we have

$$g_\varepsilon(y_\varepsilon) \leq g_\varepsilon(y), \qquad (y_\varepsilon \in M(g_\varepsilon, x)),\text{and}$$

$$g(\widetilde{y}_\varepsilon) - g(y) = [g(\widetilde{y}_\varepsilon) - g_\varepsilon(\widetilde{y}_\varepsilon)] + [(g_\varepsilon(\widetilde{y}_\varepsilon) - g(y)]$$

$$\leq [g(\widetilde{y}_\varepsilon) - g_\varepsilon(\widetilde{y}_\varepsilon)] + [g_\varepsilon(y_\varepsilon) - g(y)] + \eta(\varepsilon) \qquad \text{(by 3.1)}$$

$$\leq [g(\widetilde{y}_\varepsilon) - g(\widetilde{y}_\varepsilon)] + [g_\varepsilon(y) - g(y)] + \eta(\varepsilon) \quad (\text{by } g_\varepsilon(y_\varepsilon) \leq g_\varepsilon(y)$$

$$\leq 2\varepsilon + \eta(\varepsilon) \qquad (4)$$

By the strong convexity and continuity of g on a convex and closed set X we have

$$\| \widetilde{y}_\varepsilon - y \|^2 \leq \frac{2}{\rho}[\, g(\widetilde{y}_\varepsilon) - g(y)\,] \qquad \text{(by proposition 1.3.6).}$$

(ρ is parameter of strong convexity) (5)

It follows that

$$\| \widetilde{y}_\varepsilon - y \|^2 \leq \frac{4\varepsilon + 2\eta(\varepsilon)}{\rho}, \text{ (by (4) and (5)).} \qquad (6)$$

Taking limit as $\varepsilon \to 0$ of (6) we get

$$0 \leq \lim_{\varepsilon \to 0} \| \widetilde{y}_\varepsilon - y \|^2 \leq \lim_{\varepsilon \to 0} \frac{4\varepsilon + 2\eta(\varepsilon)}{\rho} = 0$$

consequently

$$\lim_{\varepsilon \to 0} \| \widetilde{y}_\varepsilon - y \|^2 = 0$$

Therefore, we have that

$$\lim_{\varepsilon \to 0} \widetilde{y}_\varepsilon = y.$$

Remark: For any function g which satisfies Theorem 3.1, the optimization problem

(P) $g(x) \to \min, \ x \in X$, is correct.

4. Problems with Exact constraints

4.1 Regularization Method

In this chapter we will consider the regularization method to solve non-linear optimization problems.

Most of practical problems whose mathematical models are mathematical optimization problems contain some additional conditions that allow us to select an element in the solution set Y of un perturbed problem (p), (see chapter 2.1) which satisfies some addition condition.

In this connection we consider the closest element in the solution set Y to a fixed point $x_0 \in U$.

4.1.1 The Normal Solution

We consider the following un perturbed problem (p).

$$(p) \quad g(x) \to \min,\ x \in X \text{ (see chapter 2.1)}$$

$$Y := M(g,X). \tag{1}$$

Let $x_0 \in U$ be fixed and $\Omega: U \to \Re_+$ be given by

$$\Omega(x) := \|x - x_0\|^2.$$

Let

$$(P_N) \quad \Omega(y) \to \min,\ y \in Y.$$

Let $y_0 \in Y$ be a solution to (P_N), i.e.

$$\|y_0 - x_0\|^2 = \min_{y \in Y} \|y - x_0\|^2.$$ Then

a) A solution y_0 of (P_N) is called the normal solution of (p).
b) The problem (P_N) is called the problem of the normal solution.

4.1.2 Regularization

We apply the regularization method to determine the normal solution to the unperturbed problem (p) when only the objective function is perturbed. For this we consider the following:

Let $x_0 \in U$ be fixed, $g: U \rightarrow \Re$ and

$\Omega: U \rightarrow \Re_+$ be given.

Then we define a function

$N_\alpha: U \rightarrow \Re$ given by,

$N_\alpha(x) := g(x) + \alpha\Omega(x), (\alpha > 0)$. Then

a) The function N_α is called a regularization function.
b) The number α is called parameter of regularization.

Now we consider the following optimization problem:

$(P_\alpha) \qquad N_\alpha(x) \rightarrow \min, x \in X.$

Let $y_\alpha \in M(N_\alpha, X)$.

In what follows we will show how to determine the value of y_α approximately.

Let $N_\alpha^* := N_\alpha(y_\alpha) = \min_{x \in X} N_\alpha(x)$.

Further more, we assume that for each $\alpha > 0$, there exists a point $\widetilde{y}_\alpha \in X$ such that

$N_\alpha(\widetilde{y}_\alpha) \leq N_\alpha^* + \eta(\alpha), \ \eta(x) > 0.$

(η is a positive number which depends on α)

In connection with the above considerations we have the following theorem.

Theorem 4.1.1: Let X be a closed set and $g: X \rightarrow \Re$ be continuous. Let $Y = M(g,X)$ to be non-empty and $\lim_{\alpha \rightarrow 0} \frac{\eta(\alpha)}{\alpha} = 0$. Then the following conditions are satisfied:

i) $\lim_{\alpha \rightarrow 0} g(\widetilde{y}_\alpha) = g(y_0)$,

ii) $\lim_{\alpha \rightarrow 0} \rho(\widetilde{y}_\alpha, Y) = 0$.

Further more, if the set X is convex and the function g is convex, then

$$\lim_{\alpha \to 0} \tilde{y}_\alpha = y_o.$$

1 **Proof**: First we show that $N_\alpha(x) \to \infty$ for $\|x\| \to \infty$,

Obviously $g(x) \geq g(y)$, for all $x \in X$, $y \in M(g, X)$.

By $N_\alpha(x) = g(x) + \alpha\Omega(x) = g(x) + \alpha\|x - x_0\|^2$, $\alpha > 0$

we get

$$N_\alpha(x) = g(x) + \alpha\|x - x_0\|^2$$
$$\geq g(y) + \alpha\|x - x_o\|^2. \qquad (4.1)$$

Since g(y) is a constant and $\alpha > 0$ it suffices to consider $\|x - x_0\|^2$ for $\|x\| \to \infty$.

since $\|x - x_o\|^2 = \langle x - x_o, x - x_o \rangle$ (definition of norm)

$= \langle x, x\rangle - 2\langle x_o, x\rangle + \langle x_0, x_0\rangle$

$\geq \|x\|^2 - 2|\langle x_o, x\rangle| + \|x_o\|^2$

$\geq \|x\|^2 - 2\|x_o\|.\|x\| + \|x_o\|^2$, ($\langle|x_o, x\rangle| \leq \|x_o\|.\|x\|$) (Cauchy-shwarz inequality)

$= (\|x\| - \|x_0|)^2.$

By assumption $\|x\| \to \infty$, and $\|x_o\|$ is a constant gives

$$\|x\| - \|x_o\|^2 \geq \|x\| - \|x_o\| \to \infty \text{ as } \|x\| \to \infty.$$

Therefore

$$\|x - x_o\|^2 \geq (\|x\| - \|x_o\|)^2 \geq \|x\| - \|x_o\| \to \text{ as } \|x\| \to \infty.$$

Hence we conclude that

$$N_\alpha(x) = g(x) + \alpha\Omega(x) = g(x) + \alpha\|x - x_o\|^2 \to \infty \text{ as } \|x\| \to \infty.$$

2. Now we show that for arbitrary but fixed $z \in X$, the set $X_o := \{x \in X \mid N_\alpha(x) \leq N_\alpha(z) =: c\}$ is nonempty, closed and bounded.

By the set Y is nonempty and definition of N_α we get that the set X_o is nonempty. Obviously the set X_o is level set of N_α. Moreover the function N_α is continuous which follows by the continuity of g and Ω (assumption). Hence the set X_o is closed (each level set of a continuous function is closed).

By the fact that $N_\alpha(x) \to \infty$ for $||x|| \to \infty$ and the set Y is nonempty and the definition of N_α give us that N_α has a minimum point. consequently the set X_o is bounded.

Let now $y_\alpha \in M(N_\alpha, X)$.

3. We show $\lim_{\alpha \to 0} g(\tilde{y}_\alpha) = g(y_o)$.

By $y_o \in Y$ we have

$$g(y_o) \le g(\tilde{y}_\alpha)$$

$$\le \underbrace{g(\tilde{y}_\alpha) + \alpha\Omega(\tilde{y}_\alpha)}_{N_\alpha(\tilde{y}_\alpha)} \le N_\alpha^* + \eta(\alpha) = \underbrace{g(y_\alpha) + \alpha\Omega(y_\alpha)}_{N_\alpha^*} + \eta(\alpha)$$

$$\le \underbrace{g(y_0) + \alpha\Omega(y_0)}_{N_\alpha(y_0)} + \eta(\alpha)$$

$$\le g(\tilde{y}_\alpha) + \alpha\Omega(y_0) + \eta(\alpha) \quad (g(y_0) \le g(\tilde{y}_\alpha))$$

Therefore, we have the following inequality $g(y_o) \le g(\tilde{y}_\alpha) \le g(y_0) + \alpha\Omega(y_0) + \eta(\alpha)$.

From this it follows that

$$0 \le g(\tilde{y}_\alpha) - g(y_0) \le \alpha\Omega(y_0) + \eta(\alpha).$$

Therefore

$$0 \le \lim_{\alpha \to 0} (g(\tilde{y}_\alpha) - g(y_o)) \le \lim_{\alpha \to 0} (\alpha\Omega(y_o) + \eta(\alpha)) = 0.$$

Hence we have that

$$\lim_{\alpha \to 0} g(\tilde{y}_\alpha) = g(y_0). \text{ //}$$

4. Now we show that

$$\lim_{\alpha \to 0} \rho(\tilde{y}_\alpha, Y) = 0$$

From proof (3) we get

$\alpha\Omega(\tilde{y}_\alpha) \le \alpha\Omega(y_0) + \eta(\alpha)$.

which gives $\Omega(\tilde{y}_\alpha) \le \Omega(y_0) + \dfrac{\eta(\alpha)}{\alpha}$.

Hence by $\lim_{\alpha \to 0} \dfrac{\eta(\alpha)}{\alpha} = 0$ and the definition of Ω give that

$0 \le \Omega(\tilde{y}_\alpha) = ||\tilde{y}_\alpha - x_0||^2 \le ||y_0 - x_0||^2 + \dfrac{\eta(\alpha)}{\alpha} \le c^2$. (for constant c)

which gives $0 \le ||\tilde{y}_\alpha - x_0|| \le c$.

Therefore, for sufficiently small α the family of the sets $\{\tilde{y}_\alpha\}$ is bounded.

Hence together with

$$\lim_{\alpha \to 0} g(\tilde{y}_\alpha) = g(y_o) \text{ and bounded ness of } \{\tilde{y}_\alpha\}$$

we conclude that

$$\lim_{\alpha \to 0} \rho(\tilde{y}_\alpha, Y) = 0. //$$

Let now the set X and the function g be convex, then the set Y is also convex.

5. Now we prove

$$\lim_{\alpha \to 0} \tilde{y}_\alpha = y_o.$$

By proof (3) we have that $\lim_{\alpha \to 0} g(\tilde{y}_\alpha) = g(y_o)$.

From this we have that the sequence $(g((\tilde{y}_\alpha))$ is a convergent sequence with the limit $g(y_o)$.

Let $\bar{y}$ be an arbitrary limit point of the family $\{\tilde{y}_\alpha)$. i.e. there is a subsequence $\left(\tilde{y}_{\alpha_k}\right)$ of $(\tilde{y}_\alpha)$ such that

$$\lim_{k \to \infty} \tilde{y}_{\alpha_k} = \bar{y}.$$

By the continuity of g we have

$$g(\bar{y}) = g\left(\lim_{k \to \infty} \tilde{y}_{\alpha_k}\right) = \lim_{k \to \infty} g\left(\tilde{y}_{\alpha_k}\right) = g(y_0),$$

(since $\left(g\left(\tilde{y}_{\alpha_k}\right)\right))$ is a subsequence of a convergent sequence $\left(g\left(\tilde{y}_{\alpha_k}\right)\right))$.

Hence we have that

$$\lim_{k \to \infty} g\left(\tilde{y}_{\alpha_k}\right) = g(y_o) = g(\bar{y}).$$

Consequently $\bar{y} \in Y$.

From proof (4) and continuity of Ω we have

$$\Omega\left(\lim_{k \to \infty} \tilde{y}_{\alpha_k}\right. = \lim_{k \to \infty} \Omega\left(\tilde{y}_{\alpha_k}\right) = \Omega(\bar{y}) \le \Omega(y_0).$$

But $\Omega(y_o) \le \Omega(\bar{y})$.

Hence we have

$$\min_{y \in Y} \Omega(y_o) = \Omega(\overline{y}).$$

Considering the strong convexity of Ω on a convex and closed set Y (Ω - is strictly convex), we get that Ω has a unique minimum point.

Hence we have

$$\overline{y} = y_o.$$

Therefore we have for $\alpha \to 0$ the family of the set $\{\widetilde{y}_\alpha\}$ has a single limit point: (i.e.

$$\lim_{k \to \infty} \widetilde{y}_\alpha = y_0.$$

Remark: In connection with Theorem 4.1.1 the rate of convergence of y_α to y_0 for $\alpha \to 0$ can be very low.

The following example illustrates this fact:

Example: consider a one-dimensional problem.

Let $g: \Re \to \Re$ given by $g(x) = x^{2k}$, $k \geq 1$,

and (p) $\qquad g(x) \to \min, x \in \Re.$

Clearly

$$\min_{x \in \Re} g(x) = \min_{x \in \Re} x^{2k} \text{ is attained when } x = 0 =: y_0.$$

Therefore $y_0 = 0$ is a solution to our optimization problem.

Now we choose $x_0 \in \Re$ such that $x_0 \neq 0$, and define $\Omega(x) := \|x - x_0\|^2 = (x - x_0)^2$,

$$N_\alpha(x) := g(x) + \alpha(x - x_0)^2, \alpha > 0,$$

$$= x^{2k} + \alpha(x - x_0)^2, k \geq 1, \alpha > 0.$$

Then we determine the solution to the following optimization problem:

$$N_\alpha(x) \to \min, x \in \Re.$$

Since N_α is differentiable for all $x \in \Re$ we consider the derivative of N_α, i.e.

$$N_\alpha'(x) = 2kx^{2k-1} + 2\alpha(x - x_0).$$

In order to get the minimum points to our problem one has to solve:

$$N_\alpha'(x) = 0.$$

$$\text{i.e. } 2kx^{2k-1} + 2\alpha(x - x_0) = 0. \qquad (2)$$

In general the solution to (2) for some fixed k and x_0 is dependent on α and has the following form:

$$y_\alpha := y_0 + 0\left(\frac{1}{\alpha^{2k-1}}\right)$$

where $0\left(\frac{1}{\alpha^{2k-1}}\right)$ is to indicate that the solution is dependent on the α.

Theorem 4.1.2 Let X be a convex and closed set.

More over let $g: X \rightarrow \Re$ be convex and continues,

$$g_\varepsilon \in p_\varepsilon(g) := \{g_\varepsilon : |g_\varepsilon(x) - g(x)| < \varepsilon, \text{ for all } x \in X\},$$

$$Y := M(g, X).$$

If Y is nonempty and g_ε is continues, then the problem

$N_\alpha(x) \rightarrow \min$, $x \in X$ is correct for each $\alpha > 0$.

Proof: Let $N_{\alpha\varepsilon} : U \rightarrow \Re$ be given by

$$N_{\alpha\varepsilon}(x) := g_\varepsilon(x) + \alpha\Omega(x).$$

By $g_\varepsilon \in p_\varepsilon(g)$ we have

$$g_\varepsilon(x) \geq g(x) - \varepsilon \geq g(y) - \varepsilon =: c \text{ for all } x \in X, y \in Y. \qquad (1)$$

More over, the set

$\{x \in X \mid N_{\alpha\varepsilon}(x) \leq \tilde{c}\}$ for a constant $\tilde{c}$ is bounded (Theorem 4.1.1).

By (1) and (Theorem 4.1.1)

$N_{\alpha\varepsilon}(x) \rightarrow \infty$ for $\|x\| \rightarrow \infty$ and $y_{\alpha\varepsilon} \in M(N_{\alpha\varepsilon}, X)$.

Obviously the functions N_α and $N_{\alpha\varepsilon}$ satisfy (Theorem 3.1).

Therefore we have that the problem under consideration is correct.

Example 1: We consider the following non linear optimization problem

$$g_\lambda(x) = x.\lambda - e^x, \ x \in \Re.$$

Then we determine the super mum of g_λ on $\Re$.

Case 1: If $\lambda > 0$

$$g_\lambda(x) \sup_{x\in\Re} = \max_{x\in\Re} g_\lambda(x) = \lambda(\ell n\, \lambda - 1).$$

Case 2: If $\lambda = 0$, Then

$g_0(x) = -e^x < 0$ for all $x \in \Re$ and

$-e^x \to \infty$ as $x \to \infty$ and

$-e^x \to 0$ as $x \to -\infty$.

Hence in this case

$\sup_{x\in\Re} g_0(x) = 0.$

Case 3: If $\lambda < 0$, Then

$g_\lambda(x) = x.\lambda - e^x$ and

$g_\lambda(x) \to \infty$ as $x \to -\infty$ which gives

$\sup_{x\in\Re} g_\lambda(x) = \infty.$

Therefore, from above considerations we have

$$\Phi(\lambda) := \sup_{x\in\Re} g_\lambda(x) = \begin{cases} \infty \text{ if } \lambda < 0, \\ 0 \text{ if } \lambda = 0 \\ \lambda(\ell n\lambda - 1) \text{ if } \lambda > 0. \end{cases}$$

Example 2: Let $f_\lambda(x) = x.\lambda - |x - 2|, x \in \Re$,

Determine the superman of f_λ on $\Re$.

clearly

$$f_\lambda(x) = \begin{cases} x.\lambda - x + 2 \quad is \quad x \geq 2 \\ x.\lambda + x - 2 \ is \ x < 2 \end{cases}$$

$$= \begin{cases} (\lambda - 1)\, x + 2 \quad is \quad x \geq 2 \\ (\lambda + 1) - 2 \ is \ x < 2 \end{cases}$$

Case 1 : suppose $\lambda > 1$, then

$f_\lambda(x) \to \infty$ for $x \to \infty$.

Hence $\sup_{x\in\Re} f_\lambda(x) = \infty.$

Case 2: Suppose $\lambda < -1$, then

$f_\lambda(x) \to \infty$ for $x \to -\infty$.

Hence $\sup_{x\in\Re} f_\lambda(x) = \infty.$

Case 3: If $\lambda = 1$ then $\sup_{x\in\Re} f_\lambda(x) = 2 = \max_{x\in\Re} f_\lambda(x)$.

Case 4: If $\lambda = -1$, then

$$\sup_{x\in\Re} f_\lambda(x) = -2 = \max_{x\in\Re} f_\lambda(x).$$

Case 5: suppose $\lambda \in (-1, 1)$

Then $(\lambda - 1) < 0$ and $(\lambda + 1) > 0$

Consequently

$(\lambda - 1)x + 2 \to -\infty$ as $x \to \infty$.

In this case we have

$$\sup_{x\in\Re} f_\lambda(x) = 2\lambda \max_{x\in\Re} f_\lambda(x).$$

Considering

$(\lambda + 1)x + 2 \to -\infty$ as $x \to \infty$. we get again

$$\sup_{x\in\Re} f_\lambda(x) = \sup_{x\in\Re} (1 + \lambda)x + 2) = 2\lambda.$$

Hence we have the following generalization for the above considerations:

$$\Phi(\lambda) = \sup_{x\in\Re} f_\lambda(x) = \begin{cases} \infty \ is \ \lambda > 1 \ and \ \lambda < -1 \\ 2\lambda \ is \ \lambda \in [-1,1] \end{cases}$$

5. Regularization Method (the general case)

In chapters (3) and (4) we considered regularization method for problems with exact constraints.

In this chapter we consider problems of the form

(p_ε) $g_\varepsilon(x) \to \min, x \in X_\varepsilon$ (chapter 2.3).

In connection with this we have to determine approximate values of the normal solution to the problem

(p) $g(x) \to \min, x \in X$ (chapter 2.1)

To get this result we consider the following:

5.1 Regularization

Let $\psi_{\alpha\beta\varepsilon} : U \to \Re$ be a function defined by

$$\psi_{\alpha\beta\varepsilon}(x) = \beta[g_\varepsilon(x) + \alpha\Omega(x)] + \psi_\varepsilon(x), \quad \beta > 0,$$

$\psi_\varepsilon : U \to \Re+$ be defined by

$$\psi_\varepsilon(x) := \sum_{i=1}^{m} |\min \{ f_{\varepsilon_i}(x), 0 \}|^p, \ p > 0 \text{ be given.}$$

Then the function $\psi_{\alpha_{\beta\varepsilon}}$ is a penalty function for the problem

$$(P_{\alpha_\varepsilon}) \quad N_{\alpha_\varepsilon}(x) \to \min, x \in X_\varepsilon.$$

In the next chapter we will consider on how to determine

$$y_{\alpha\beta\varepsilon} \in M(\psi_{\alpha\beta\varepsilon}, U).$$

6. Convergence of the Regularization Method

6.1 The General Case

Let the following conditions be satisfied:

(i) the set U is convex and closed,

(ii) the function g is convex and continuous on U,

(iii) the functions f_i, $i = 1,2,\ldots,m$, are con cave and continuous on U.

(iv) the functions,

$$g_\varepsilon, f_\varepsilon \in p_\varepsilon(g,f) := \{g_\varepsilon, f_\varepsilon : |g_\varepsilon(x) - g(x)| \le \varepsilon, \|f(x) - f_\varepsilon(x)\| \le \varepsilon \text{ for all } x \in X\},$$

(v) Y: = M(g,X) be non empty,

(vi) $\psi(x) := \sum_{i=1}^{m} |\min\{f_i(x), 0\}|$,

$$\psi_\varepsilon(x) := \sum_{i=1}^{m} |\min\{f_{\varepsilon_i}(x), 0\}|,$$

Now we prove the following propositions.

Proposition 6.1.1: For any $x \in U$ the inequality

$|\psi(x) - \psi_\varepsilon(x)| \le \varepsilon.m$ holds.

Proof:- We first show

$$\|\min\{f_i(x), 0\}| - |\min\{f_{\varepsilon_i}(x), 0\}\| \le |f_i(x) - f_{\varepsilon_i}(x)|.$$

<u>**Case 1**</u> : Suppose $f_i(x) > 0$, $f_{\varepsilon_i}(x) > 0$ for all $_i=1,2,\ldots,m$, for all $x \in U$. Then

$$|\min\{f_i(x), 0\}| = 0, \quad |\min\{f_{\varepsilon_i}(x), 0\}| = 0,$$

and $\|\min\{f_i(x), 0\}| - |\min\{f_{\varepsilon_i}(x), 0\}\| = 0.$

$$\|\min\{f_i(x), 0\}| - |\min\{f_{\varepsilon_i}(x), 0\}\| \le |f_i(x) - f_{\varepsilon_i}(x)|.$$

<u>**Case 2**</u> : Suppose $f_i(x) < 0$, $f_{\varepsilon_i}(x) < 0$ for all $i = 1,2,\ldots,m$, and for all $x \in U$. Then

$|\min\{f_i(x), 0\}| = |f_i(x)|$,

$|\min\{f_{\varepsilon_i}(x), 0\}| = |f_{\varepsilon_i}(x)|$.

Therefore

$$\|\min\{f_i(x), 0\}| - |\min\{f_{\varepsilon_i}(x), 0\}\| = \|f_i(x)| - |f_{\varepsilon_i}\| \le |f_i(x) - f_{\varepsilon_i}(x)|.$$

Case 3: Suppose f_i and f_{ε_i}, for i = 1,2,…,m, have opposite sign, with out loss of generality,

Let $f_i(x) > 0$, $f_{\varepsilon_i}(x) < 0$ for all i = 1,2,…,m and for all $x \in U$.

Then

$$|\min\{f_i(x), 0\}| = 0 \text{ and } |\min\{f_{\varepsilon_i}(x), 0\}| = |f_{\varepsilon_i}(x)|;$$

Hence $||\min\{f_i(x), 0\}| - |\min\{f_{\varepsilon_i}(x), 0\}|| = |0 - |f_{\varepsilon_i}(x)||$.

But $|0 - |f_{\varepsilon_i}(x)| \leq |f_i(x) - f_{\varepsilon_i}(x)|$ as $f_i(x) < 0$ for all i, for all x

By similar considerations it can be shown that, it is in general true that

$$||\min\{f_i(x), 0\}| - |\min\{f_{\varepsilon_i}(x), 0\}|| \leq |f_i(x) - f_{\varepsilon_i}(x)|.$$

Now by $f_{\varepsilon_i} \in p_\varepsilon(g,f)$ we get

$$|f_i(x) - f_{\varepsilon_i}(x)| \leq \varepsilon \text{ for all } x \in X \text{ for all } i = 1,2,\ldots,m.$$

Therefore we have

$$|\psi(x) - \psi_\varepsilon(x)| \leq \sum_{i=1}^{m} ||\min\{f_i(x), 0\}| - |\min\{f_{\varepsilon_i}(x), 0\}||.$$

$$\leq \sum_{i=1}^{m} |f_i(x) - f_{\varepsilon_i}(x)|$$

$$\leq \sum_{i=1}^{m} \varepsilon = \varepsilon.m. \text{ //}$$

Proposition 6.1.2 : For any $x \in U$, $0 \leq \beta \leq \beta_0 < \infty$, $0 \leq \varepsilon \leq \varepsilon_0 < \infty$, and any $\alpha > 0$ there exists

$$\psi^* := \inf_{x \in U} \psi_{\alpha\beta\varepsilon}(x) > -\infty.$$

Proof: since the function N_α given by $N_\alpha(x) = g(x) + \alpha\Omega(x)$ is a strongly convex on U, ($\alpha > 0$), there is

$$N^{**} := \min_{x \in U} N_\alpha(x) > -\infty.$$

since by condition (iv) of (6.1) we have $g_\varepsilon \in p_\varepsilon(g, f)$ we have

$$g_\varepsilon(x) \geq g(x) - \varepsilon.$$

Hence

$$\psi_{\alpha\beta\varepsilon}(x) = \beta\,[g_\varepsilon(x) + \alpha\Omega(x)] + \psi_\varepsilon(x)$$

$$\geq \beta[g(x) - \varepsilon] + \beta\alpha\Omega(x) + \psi_\varepsilon(x)$$

$$\geq \beta N^{**} - \beta\varepsilon =: c > -\infty. \text{ //}$$

Now we consider the following:

Let $\psi^* = \inf_{x \in U} \psi_{\alpha\beta\varepsilon}(x)$,

$y_0 \in M(\Omega, Y)$,

$y_\alpha \in M(N_\alpha, x)$,

$N^* := N_\alpha(y_\alpha) = \min_{x \in X} N_\alpha(x)$,

$N^{**} = \min_{x \in U} N_\alpha(x) > -\infty$.

Further more,

Let $\widetilde{y}_{\alpha\beta\varepsilon} \in U$ satisfies the following

$$\psi_{\alpha\beta\varepsilon}\left(\widetilde{y}_{\alpha\beta\varepsilon}\right) \le \psi^* + \eta(\varepsilon),\ \eta(\varepsilon) > 0.$$

In connection with the above considerations we have the following theorem.

Theorem 6.1.1 : Suppose the conditions (i – vi) of chapter (6.1) are satisfied. Further more let $\lim_{\varepsilon \to 0} \beta(\varepsilon) = 0$, $\lim_{\varepsilon \to 0} \frac{\varepsilon}{\beta(\varepsilon)} = 0$, $\lim_{\varepsilon \to 0} \frac{\eta(\varepsilon)}{\beta(\varepsilon)} = 0$. Then

$$\lim_{\alpha \to 0} \lim_{\varepsilon \to 0} \widetilde{y}_{\alpha\beta\varepsilon} = y_0.$$

<u>**Proof:**</u> Obviously

$\psi_\varepsilon(x) \ge 0$ for all $x \in U$ (definition of ψ_ε).

$g(x) \le g_\varepsilon(x) + \varepsilon$ ($|g(x) - g_\varepsilon(x)| \le \varepsilon$ for all $x \in U$

$$g_\varepsilon(x) \le g(x) + \varepsilon \tag{1}$$

More over

$$\psi^* = \inf_{x \in U} \psi_{\alpha\beta\varepsilon}(x) \le \psi_{\alpha\beta\varepsilon}(x) \text{ for all } x \in U. \tag{2}$$

By definition of $\widetilde{y}_{\alpha\beta\varepsilon}$ we get

$$\psi_{\alpha\beta\in}\left(\widetilde{y}_{\alpha\beta\varepsilon}\right) \le \psi^* + \eta(\varepsilon). \tag{3}$$

All these imply

$$\begin{aligned} N^{**} = \min_{x \in U} N_\alpha(x) &\le N_\alpha\left(\widetilde{y}_{\alpha\beta\varepsilon}\right) \le N_\alpha\left(\widetilde{y}_{\alpha\beta\varepsilon}\right) + \frac{1}{\beta}\psi_\varepsilon\left(\widetilde{y}_{\alpha\beta\varepsilon}\right) \\ &= g\left(\widetilde{y}_{\alpha\beta\varepsilon}\right) + \alpha\Omega\left(\widetilde{y}_{\alpha\beta\varepsilon}\right) + \frac{1}{\beta}\psi_\varepsilon\left(\widetilde{y}_{\alpha\beta\varepsilon}\right) \\ &\le g_\varepsilon\left(\widetilde{y}_{\alpha\beta\varepsilon}\right) + \alpha\Omega\left(\widetilde{y}_{\alpha\beta\varepsilon}\right) + \frac{1}{\beta}\psi_\varepsilon\left(\widetilde{y}_{\alpha\beta\varepsilon}\right) + \varepsilon \end{aligned}$$

$$= \frac{1}{\beta}[\ \beta(g_\varepsilon(\tilde{y}_{\alpha\beta\varepsilon}) + \alpha\Omega(\tilde{y}_{\alpha\beta\varepsilon})] + \frac{1}{\beta}\psi_\varepsilon(\tilde{y}_{\alpha\beta\varepsilon}) + \varepsilon$$

$$= \frac{1}{\beta}\ \psi_{\alpha\beta\varepsilon}(\tilde{y}_{\alpha\beta\varepsilon}) + \varepsilon$$

$$\leq \frac{1}{\beta}(\psi^* + \eta(\varepsilon)) + \varepsilon \qquad \text{(by 3)}$$

$$\leq \frac{1}{\beta}\psi_{\alpha\beta\varepsilon}(y_\alpha) + \frac{1}{\beta}\eta(\varepsilon) + \varepsilon \qquad \text{(by 2)}$$

$$= g_\varepsilon(y_\alpha) + \alpha\Omega(y_\alpha) + \frac{1}{\beta}\psi_\varepsilon(y_\alpha) + \frac{\eta(\varepsilon)}{\beta} + \varepsilon \qquad (4)$$

Since $y_\alpha \in X = \{x \in U \mid f_i(x) \geq 0 \text{ for all } i = 1,2,\ldots,m\}$, from the definition of ψ if follows that $\psi(y_\alpha) = 0$.

From proposition (6.1.1)

$|\psi(y_\alpha) - \psi_\varepsilon(y_\alpha)| \leq \varepsilon.m$, $\psi(y_\alpha) = 0$, gives $\psi_\varepsilon(y_\alpha) \leq \varepsilon.m$ as $\psi_\varepsilon(y_\alpha) \geq 0$ (5)

From $g_\varepsilon(y_\alpha) \leq g(y_\alpha) + \varepsilon$ and $\psi_\varepsilon(y_\alpha) \leq \varepsilon.m$ we get

$$N^{**} \leq N_\alpha(\tilde{y}_{\alpha\beta\varepsilon}) \leq g_\varepsilon(y_\alpha) + \alpha\Omega(y_\alpha) + \frac{1}{\beta}\psi_\varepsilon(y_\alpha) + \frac{\eta(\varepsilon)}{\beta} + \varepsilon$$

$$\leq g_\varepsilon(y_\alpha) + \alpha\Omega(y_\alpha) + \frac{1}{\beta}\varepsilon m + \frac{\eta(\varepsilon)}{\beta} + \varepsilon$$

$$= N^* + \frac{\varepsilon m}{\beta} + \frac{\eta(\varepsilon)}{\beta} + 2\varepsilon$$

$$= N^* + \frac{\varepsilon m}{\beta(\varepsilon)} + \frac{\eta(\varepsilon)}{\beta(\varepsilon)} + 2\varepsilon \ \leq C. \qquad (6)$$

Since by supposition $\lim_{\varepsilon \to 0} \frac{\varepsilon.m}{\beta(\varepsilon)} = m \lim_{\varepsilon \to 0} \frac{\varepsilon}{\beta(\varepsilon)} = 0$ and $\lim_{\varepsilon \to 0} \frac{\eta(\varepsilon)}{\beta} = 0$.

we conclude from (6) that

$N_\alpha(\tilde{y}_{\alpha\beta\varepsilon}) = g(\tilde{y}_{\alpha\beta\varepsilon}) + \alpha\Omega(\tilde{y}_{\alpha\beta\varepsilon})$ is bounded.

Since N_α is strongly convex on U (U is closed and convex by assumption 6.1.(1)) N_α is trivially convex we have that the set

Q:= $\{\tilde{y}_{\alpha\beta\varepsilon} \subseteq Nc(N_\alpha)$ is the level set of N_α with the levels c.

Each level set of a continuous function is closed. By Theorem (1.3.1.) we get that each level set is bounded. Therefore, we have that the set Q is closed and bounded i.e. it is compact.

Let now $\widetilde{y}$ be a limit point of Q, i.e. there is a sequence $\left(\widetilde{y}_{\alpha\beta_k\varepsilon_k}\right)$ such that

$$\widetilde{y} = \lim_{\substack{k\to\infty \\ \varepsilon_k\to 0}} \widetilde{y}_{\alpha\beta_k\varepsilon_k} .$$

Then from the inequality (4) i.e. from

$$N^{**} \le N_\alpha\left(\widetilde{y}_{\alpha\beta\varepsilon}\right) \le N_\alpha\left(\widetilde{y}_{\alpha\beta\varepsilon}\right) + \tfrac{1}{\beta}\psi_\varepsilon\left(\widetilde{y}_{\alpha\beta\varepsilon}\right) \le N^* + \frac{\varepsilon m}{\beta} + \frac{\eta(\varepsilon)}{\beta} + 2\varepsilon$$

we get

$$0 \le \frac{1}{\beta}\psi_\varepsilon\left(\widetilde{y}_{\alpha\beta\varepsilon}\right) \le N^* + \frac{\varepsilon m}{\beta} + \frac{\eta(\varepsilon)}{\beta} + 2\varepsilon - N_\alpha\left(\widetilde{y}_{\alpha\beta\varepsilon}\right)$$

or $$0 \le \psi_\varepsilon\left(\widetilde{y}_{\alpha\beta\varepsilon}\right) \le \beta\, [N^* + \frac{\varepsilon m}{\beta} + \frac{\eta(\varepsilon)}{\beta}] + 2\beta\varepsilon - \beta N_\alpha\left(\widetilde{y}_{\alpha\beta\varepsilon}\right)$$

Therefore

$$0 \le \psi_{\varepsilon_k}\left(\widetilde{y}_{\alpha\beta_k\varepsilon_k}\right) \le \beta_k \left[N^* + \underset{\substack{\to 0\\ \to 0 \quad \varepsilon_k}}{\frac{\varepsilon_k m}{\beta_k(\varepsilon_k)}} + \underset{\substack{\to 0\\ \to 0}}{\frac{\eta(\varepsilon_k)}{\beta_k(\varepsilon_k)}} \right] + 2\underset{\to 0}{\beta_k\varepsilon_k} - \beta_k \underset{\le : c}{N_\alpha\left(\widetilde{y}_{\alpha\beta_k\varepsilon_k}\right)}$$

Now we have

$$\psi(\widetilde{y}) = \psi\left(\lim_{\substack{k\to\infty \\ \varepsilon_k\to 0}} \widetilde{y}_{\alpha\beta_k\varepsilon_k}\right) = \lim_{k\to\infty} \psi\left(\widetilde{y}_{\alpha\beta_k\varepsilon_k}\right)$$
$$\le \lim_{k\to\infty}\left[\psi_{\varepsilon_k}\left(\widetilde{y}_{\alpha\beta_k\varepsilon_k}\right) + \varepsilon_k m\right] = 0$$

(since $\psi(x) \le \psi_\varepsilon(x) + \varepsilon.m$ proposition 6.1.1)

which implies $\psi(\widetilde{y}) = 0$. Then by the definition of ψ, $f_i(\widetilde{y}) \ge 0$ for all $i = 1,2,\ldots, m$ is gives that $\widetilde{y} \in X$.

From (6) we get

$$N_\alpha(\widetilde{y}) = \lim_{\substack{k\to\infty \\ \varepsilon_k\to 0}} N_\alpha\left(\widetilde{y}_{\alpha\beta_k\varepsilon_k}\right) \le N^* + \lim_{\substack{k\to\infty \\ \varepsilon_k\to 0}}\left[\frac{\varepsilon_k m}{\beta_k} + \frac{\eta(\varepsilon_k)}{\beta_k} + 2\varepsilon_k\right] = N^*$$

Therefore we have

$$N_\alpha(\widetilde{y}) \le N_\alpha(y_\alpha) = N^* = \min_{x\in X} N_\alpha(x).$$

On the other hand we have

$$N_\alpha(y_\alpha) = \min_{x\in X} N_\alpha(x) \le N_\alpha(\widetilde{y}), \text{ since } \widetilde{y} \in X.$$

Hence we have $N_\alpha(y_\alpha) = N_\alpha(\widetilde{y})$.

But since the function N_α is strongly convex on U it has a unique minimum point on $X \subseteq U$.

consequently we have $y_\alpha = \widetilde{y}$.

So we have for any limit point $\widetilde{y}$ of the compact set

$$Q := \{\widetilde{y}_{\alpha\beta\varepsilon} \text{ for } \varepsilon \to 0 \text{ we get } \widetilde{y} = y_\alpha.$$

Therefore $\lim_{\varepsilon \to 0} \widetilde{y}_{\alpha\beta\varepsilon} = y_\alpha$

From Theorem 4.1.1 of chapter 4 we get

$$\lim_{\alpha \to 0} y_\alpha = y_0.$$

i.e $\lim_{\alpha \to 0} \lim_{\varepsilon \to 0} \widetilde{y}_{\alpha\beta\varepsilon} = y_0$. //

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