LINEAR BILEVEL MULTIFOLLOWER PROBLEMS

ADDIS ABABA UNIVERSITY
COLLEGE OF COMPUTATIONAL AND NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS

"In partial fulfillment of the requirements of the degree of master of science in mathematics"

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Stream: Optimization
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Abstract

With in the framework of any bilevel decision problem, a leader′s decision is influenced by the reaction of his/her follower(s). When multiple followers who may have had a share in decision variables, objectives and constraints are involved in a bilevel decision problem, the leader′s decision will be affected, not only by the reactions of the followers, but also by the relationships among the followers. This project first identifies nine different kinds of relationships ($S_1 to S_9$) among the followers. From all these kind, the project mainly focuses on a framework for linear bilevel single follower and linear bilevel multifollower decision problems. For each of the nine relation ships a corresponding linear bilevel single follower and linear bilevel multi-follower decision model are then developed. moreover, this project particularly proposes related theories focusing on an uncooperative decision problem on which decision variables are not totally shared(i.e., S_1 model), as this model linear bilevel single follower and linear bilevel multifollower decision problems over the nine kinds of relationships are stated. The solution of such a problem will be in existence if the solution of the lower level problem is uniquely determined and the difficulty of solving such a problem is due to the complementarity condition and having many solution of the lower level problem. Two solution procedures i.e., Kuhn-Tucker approach and k^{th} best algorithm are very important to drive an optimal solution for the uncooperative decision model even if they have their own limitations.

Keywords: linear bilevel multifollower, k^{th} best algorithm, KKT reformulation, lower level and upper level objective functions and constraints, optimality conditions.

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Introduction

Bilevel decision (also called bilevel programming or bilevel optimization) techniques are mainly developed for solving decentralized management problems with decision makers in hierarchical organization. Any hierarchical organization have two decision makers. That means, the upper level and the lower level decision makers which are called the leader and the follower respectively. Each decision maker (leader or follower) tries to optimize his/her own objective function with or without considering the objective of the other level, but the decision of each level affects the objective optimization of the other level.

Therefore, the leader may be able to influence the behavior of the follower without completely controlling the follower$'s$ action(decision). At the same time the leader may be simultaneously affected by the follower$'s$ decision. Any hierarchical optimization structure appears naturally in many aspects of resource planning, management and policy making, including water resource management, financial planning, land use planning, production planning (coordination of multi- divisional firms, network facility location), and transportation planning (network design, trip demand estimation). Amouzegar and Moshirvaziri.[5], Bard [7] and Labbe.et al. [11] have already recognized that a bilevel or multilevel modeling are very important to solve these planning problems.

In general, there are two fundamental issues in both bilevel decision theory and their application. One is how to model a real-world bilevel decision problem that may have different situations at the two decision levels, and the other is how to find an optimal solution for such dicision problem. A number of researchers (e.g., [2, 13,15]) have established original forms of optimality conditions for bilevel programming. A number of bilevel decision approaches and algorithms [7, 12, 14] have been proposed to find an optimal solution. The most successful Kuhn-Tucker approach and k^{th} best algorithm are parts of this algorithms that can be used to solve linear bilevel problems.

Although most research has been carried out in this area, the existing bilevel technology has been mainly focused on a specific situation comprising one leader and one follower. However, in the case of a real world bilevel decision problem, the lower level of a bilevel decision may involve multiple decision units. The leader$'s$ decision is therefore affected by the objectives and strategies of the followers. For each possible decision of the leader, those followers may have their own, different rational reactions. The relationships between these followers can be various. They may or may not share their decision variables. They may have individual objectives and constraints, but work with others cooperatively, or may have common objectives or common constraints.

For example, as a leader, the Government$'s$ objective in land-use planning is to maximize profiles by establishing some suitable agricultural development policies. Multiple agricultural groups, involving farmers, conservationists, and regions will affect the Government$'s$ policy-making in land use. Each agricultural group, for example, a region, as a follower, has its own individual policies to optimize its objective towards different government policies in land-use. These followers may share the same decision variables, or may have the same objectives or constraints. In such a case, the decision of the Government (the leader) is partially dependent on the environmental data put forward by all these agricultural groups(the followers).

This project thus presents a definition of an optimal solution and related theories for the uncooperative bilevel models. It further extends the Kuhn-Tucker approach and k^{th} best algorithm for solving the linear bilevel multifollower uncooperative model. This project is organized as follows.

In chapter 1, preliminaries are proposed, in chapter 2, the framework for linear bilevel multifollower(LBLMF) decision problems is proposed and an extended Kuhn tucker approach and k^{th} best algorithm for solving the uncooprative model are also presented in chapter 2. Areas, where application of linear bilevel programming(LBLP) decision problems will be given which can be solved by the extended Kuhn tucker approach and k^{th} best algorithm in chapter 3. Finally summary of the chapters will be given.

Chapter 1

Preliminaries

1.1 Mathematical Models of Hierarchical Decision Making Organizations

One of the prevalent decision-making structures throughout history has been the hierarchy. Hierarchies in managements have existed for as long as people have tried to organize their work effectively. In today′s society, we have a lot of hierarchies that exist in every face of life. They occur in the government, the business world, our church, and even in our family unit. In the hierarchy decision makers′ position and individual problems are well defined. This is not to imply that the solution obtained from the hierarchy is optimal. In fact, the decisions made from organizations in the hierarchical form often seem to make no sense.

Therefore, by understanding them, we can eliminate their inherent inefficiencies and better utilize the resources which they expend with few exceptions, existing mathematical models for optimizing hierarchical systems have relied heavily on the Danzing-Wolfe Decomposition principle [16]. In such formulations, the decision space is partitioned among subunits of the decomposable system. The subunits interact through a set of corporate constraints involving decision variables of all subunits. The remaining constraints can be to each division, with each constraint a function of the decision variables under the control of a single subunit.

These multilevel decomposition models lend themselves readily to an economic interpretation of the algorithmic process. The procedure is considered as an adjustment phase with the leader (higher level decision maker) providing tentative information to the lower level decision makers(followers), observing, their reactions, and updating his/her decision based on this information. Ultimately, the leader will consider a strategy which causes the system as a whole to describe feasibly and optimally with respect to his/her objective function.

1.2 Historical Development

Decentralized planning has been long recognized as an important decision making problem. Mathematical programming methods to solve such problems trace back early in the development of linear programming. The decomposition method of Dantzig and Wolfe [16] for the solution of certain large scale linear programming problems has served well as the underpinning for much of this study. Such a formulation partitions have decision space among several planning division.

The sub problem solved by a division maximizes that portion of overall objective function controlled by the division, subject to the divisional constraints. The Dantzig and Wolfe method can then be viewed as providing inducements to the division to encourage overall optimal behavior of the corporation. These techniques were further discussed by the work of Charnes.et. al [17], who recognized that when subdivision have alternative solution for their individual optimization problem, they must receive information from the leader in order to operate coherently.

The decomposition approach has been successfully applied by Haimes and he associates to a wider range of multilevel planning problem (Haimes, Foley and Yu [19]). The decomposing approach includes coordinating mechanism of dual price preventing the various division or agencies from working against the goal of the higher decision maker(leader). Cassidy, et al [18] proposed a model and solution procedure for a specific case where such a coordinating mechanism $doesn't$ exist. Many solution approaches have been developed for the case of linear bilevel programming problems.

Candler and Townsly [20] proposed an algorithm for bilevel optimization, known as the T-set algorithm, that focuses on generating and enumerating bases from the lower level activities. The solution method involves an implicit search of all feasible behavioral optimal bases, with out reexamining any previously explored basis. But the algorithm may not stop as soon as the goal optimal is attained, it is one of the limitation of this algorithm. Narula and Nwasu [21, 22] also proposed a procedure via regular simplex pivots with modification after taking the dual of the lower level problem for two-level hierarchical programming problems.

An algorithm proposed by Bialas and Karwan [23] for bilevel programming problems use simplex method for bounded variables and finds extreme points in the set of rational reactions of the lower level problem, it then moves among the extreme points of the lower level problem, never allowing the upper level objective function to decrease. However only the local optimal solution is obtained, the intercepting algorithm by Parragar [25] suggests adding a cut to the original feasible region after a local optimal solution has been found.

The k^{th} best algorithm has been proposed by Wen [24] and Bialas and Karwan[23]. First, it solves the upper level problem over the overall solution space in order to get the first best solution. If the solution is not in the set of rational reaction of the lower level problem, then the second best solution may be found among the extreme points which are adjacent to the first best. The algorithm moves sequentially through these ordered extreme points of the overall solution space until the same, the k^{th} best is found in the rational reaction set of the lower level problems, and then terminates with a global optimal solution. Computational experience with k^{th} best algorithm has demonstrated that it finds a global solution for most linear bilevel programming problems, although it has its own weakness. i.e it takes long time to find the optimal solution of the linear bilevel programming problems.

Even if there are some weakness , this approach to the linear bilevel programming can be intuitively extended to the general n-level linear resource control problem. Bard [26] proposed an algorithm for solving the general bilevel programming problem. The algorithm is based on the grid search algorithm which exhibits the desirable property of monotonicity and the algorithm is based on two necessary optimality conditions on the paper for stationarity and local optimality. Visweswaran, Floudas, Ierapetritou and Pistikopoulos [28] offered decomposition based optimization approach to linear bilevel and quadratic programming problems.

By replacing the inner problem by its corresponding KKT optimality conditions, the bilevel problem can be transformed to a single non-convex (due to the complementarities condition) mathematical programming problem. Based on the primal-dual global optimization approach of Floudas and Visweswaran [27], the problem is decomposed into a series of primal and relaxed-dual sub problems whose solutions provide lower and upper bounds to the global optimal solution. There are a few varying approaches yet known in solving three level programming problem. One of the approaches is a hybrid method offered by Wen which combines the k^{th} best vertex enumeration algorithm proposed by Bialas and Karwan, and a complementary pivot algorithm.

Though this method works effectively for most problems, its computational load grows geometrically with the number of constraints (i.e. the size of the coefficient matrices) and the hierarchies (i.e. the number of levels in hierarchy). The second approach is offered by Bard [29]. Bard extended the idea of the grid search algorithm that is designed to solve two level hierarchies to a model of three level hierarchies. The algorithm that Bard proposed for solving the three level programming problems includes a cutting plane approach for solving a bilinear programming problem and a vertex search procedure for the third level at each iteration.

One of the main advantages of this algorithm is that, it can be extended beyond three level hierarchies and can be used for general multilevel linear programming problems. Its principal limitation seems to be the bookkeeping burden imposed by the prospect of multiple optimal solutions. With this respect, several algorithms have been developed that can find an optimal solution for the linear bi-level programming problem and some algorithms have been also developed to solve non-linear programming problem.

However, the computational efficiency of these algorithm does not consistently perform well, because of the complexity of the problem and the algorithms proposed for solving bilevel programming problems. That means the bilevel progamming problems have some limitations as we have seen above. It will be helpful to develop more efficient algorithm for solving the linear bilevel programming problem and extend it as well to the general n-level programming problem.

1.3 Definition of Game Theory

A game is a formal representation of a situation where a number of players interact in a setting of strategic interdependence [11]. This means that the welfare of players depends upon their own decision and the decisions of the other players in the game. A game can be either cooperative, in which the players collaborate to achieve a common goal, or noncooperative, in which each of them act for their own benefit. Also, a game can be either of perfect or imperfect information, and sequential or simultaneous (the players play at the same time). A player plays a game through actions. An action is a choice or decision that a player makes, according to their own strategy. A strategy is a rule that tells the player which action(s) they should consider, according to their own information set at any particular stage of a game. Finally, a payoff function expresses the utility that a player obtains the strategy profile of all the players.

Assume that there is a finite set of players, $i = 1, ..., n$ participating in a game. Each player can consider an individual strategy represented by a vector x_i. The overall strategies taken by all players are denoted with the tuple $x = (x_1, ..., x_n)$. The rivals$'$ strategies are denoted by the n-tuple $x_{-i} = (x_1, ..., x_{i-1}, x_{k+1}, ..., x_n)$ that representes all the players strategies except for player i. X_i representes the strategy space of player i. X_i can be either continuous or integer, a convex or non convex set where the strategies can take place. For example X_i can be defined as the set

$$X_i = \{x_i \in \Re^{k_i} : h_i(x_i) = 0; g_i(x_i) \leq 0\},$$

where k_i is the number of variables, x_i controlled by player i, i.e., it is the size of vector x_i. By $u_i(x_i, x_{-i}) : X_1 \times X_2 \times \cdots \times X_k \times \cdots \times X_n \longmapsto \Re$ we define the payoff function for player i. The payoff function is considered as a cost function or benefit function. Therefore, the players are interested in minimizing or maximizing cost or benefit respectively.

1.4 Bilevel Game

Bilevel games are hierarchical games of two levels in which players make decisions turn by turn(sequentialy). The simplest bilevel game is the so called Stackelberg game [1] or single-leader single-follower game, in which a leader makes decisions before the follower$'$s decisions made. As a generalization of the two-player Stackelberg game, bilevel games have been proposed in game theory literature. In these generalizations, the lower and the upper level or one of them may have more than a single player. Thus, the players at the upper level (leaders) make decisions simultaneously competing between them and prior to the decisions of the players at the lower level (followers). After the leaders make their decisions, the followers make their decisions, also competing among themselves.

The decisions of the followers are made by considering the leaders′ and other followers′ decisions. Since a follower competes against other followers, the lower-level problem forms a Nash game parameterized in terms of the leaders′ decisions. In a similar manner, in the upper-level problem, the leaders make simultaneous decisions considering the optimal response of the followers. The leaders compete against each other in the upper-level problem in a Nash game if it exist. In bilevel games, leaders and followers can be different players or the same players at both levels, but they make different decisions.

The mathematical framework for bilevel games depend on the number of players at both levels, bilevel games generally can be classified into four categories: single-leader-single-follower, single-leader-multiple-follower, multiple-leader-single-follower and multiple-leader-multiple-follower game. In general, bilevel games can be solved as a bilevel(nested) optimization problems. A work related with bilevel optimization [2, 3] can be applied for solving bilevel games. When there are multiple players at the lower-level problem, the problem can be rewritten as a set of equilibrium constraints in the optimization problem of the leader(s). In this case, the problem is stated as a mathematical program with equilibrium constraint(MPEC) optimization problem [4,6]. If, instead, there are several players at the upper-level problem, the problem can be stated as an Equilibrium Program with Equilibrium Constraints(EPEC) optimization problem [8, 9, 10]. However, in this project, we only focused on the first two division of bilevel games.

1.4.1 Single-Leader-Single-Follower

A single-leader-single-follower game is stated as a bilevel optimization problem [2,3]. The leader′s problem is at the upper level, which chooses a decision vector, x, first. After the leader has made his/her decision, the follower chooses his/her decision vector, y that will solve the lower level optimization problem. The follower′s optimization problem is parameterized in terms of the upper level decision vector, x.

Logically, the follower selects a vector, $y(x)$, in some closed set, Y, where their objective function, $f(x, y)$ is minimized over a certain constraints which is generally nested in the upper level optimization problem. The optimal solutions set of the lower level problem is denoted by $S(x)$. Then, a vector $\bar{y}(x)$ belongs to the optimal solutions set of the lower-level problem. i.e,$\bar{y}(x) \in S(x)$ if and only if:

$$\begin{aligned} &\bar{y}(x)\ solves \\ &\qquad \min_{x} f(x, y) \\ &\qquad \text{subject to } \{y \in Y(x) : x \in X\} \end{aligned} \tag{1.1}$$

On the other hand, the leader minimizes his/her objective function, $F(x, \bar{y})$, in some closed set X, taking into account the optimal response of the follower, $\bar{y} \in S(x)$. This is generally described as follows:

$$
\begin{aligned}
&(x^e, y^e) \; solves \\
&\qquad \min_{x,\bar{y}} F(x, \bar{y}) \\
&\qquad \text{subject to} \begin{cases} x \in X \\ \bar{y} \in S(x) \end{cases}
\end{aligned} \tag{1.2}
$$

In this project we investigate the case when the lower-level and upper level constraint functions are also represented by linear functions. Therefore, the lower-level constraint set, Y, is defined as $Y = \{y : h(x, y) = 0, g(x, y) \leq 0, for some x \in X\}$ where $h(x, y)$ and $g(x, y)$ are linear. The upper-level constraints set, X, is defined as $X = \{x : H(x, y) = 0, G(x, y) \leq 0\}$ where $H(x, y)$ and $G(x, y)$ are linear. Here, we have used the superscript e to denote the optimal solution for the whole problem (upper and lower level). Additionally, we have extended the conventional definition of bilevel problems including the Lagrange multipliers from the lower-level to the upper-level objective function and constraints. In this case, the Lagrange multipliers solution at the lower-level can affect the decisions of the leader. Then, the single-leader-single-follower optimal solution is obtained by solving the problem (1.3)-(1.4).

$$
\begin{aligned}
&(x^e, y^e, \lambda^e, \mu^e) \; solves \\
&\qquad \min_{x,\bar{y},\bar{\lambda},\bar{\mu}} F(x, \bar{y}, \bar{\lambda}, \bar{\mu}) \\
&\qquad \text{subject to} \begin{cases} G(x, \bar{y}, \bar{\lambda}, \bar{\mu}) \\ H(x, \bar{y}, \bar{\lambda}, \bar{\mu}) \\ (\bar{y}, \bar{\lambda}, \bar{\mu}) \in S(x) \end{cases}
\end{aligned} \tag{1.3}
$$

Where $(\bar{y}, \bar{\lambda}, \bar{\mu}) \in S(x)$ if and only if :

$$
\begin{aligned}
&(\bar{y}, \bar{\lambda}, \bar{\mu}) \; solves \\
&\qquad \min_{y,\lambda,\mu} F(x, y) \\
&\qquad \text{subject to} \begin{cases} g(x, y) \leq 0 \\ h(x, y) = 0 \end{cases}
\end{aligned} \tag{1.4}
$$

1.4.2 Single-Leader-multiple-Followers

A single-leader-multiple-followers game is a Stackelberg problem extended with multiple followers, where the followers are competing among themselves for some common resources. In this game, a single leader makes his/her optimal decision, x, before the decision of multiple followers are made, who are competing among themselves. As soon as the optimal decision of the leader, x, is given, each i^{th} follower tries to find his/her optimal decision, y_i, taking into account their $competitors'$ optimal decisions, y_{-i}. The single-leader-multiple-follower equilibrium solution is obtained by solving the problem (1.5)-(1.6). The vector $(x^e, y^e, \bar{\lambda}^e, \bar{\mu}^e)$ represents the optimal solution of the single-leader-multi-followers problem.

Bilevel games solutions are value of the decisions of the leader and the followers, as well as the Lagrange multipliers of the lower level problem. The followers minimizes their objective function, $F(x,\cdot)$, which depends on the leader′s decision, x. The optimal decisions of the followers, $\bar{y}$, and the optimal value of the Lagrange multipliers, $\bar{\lambda}$ and $\bar{\mu}$, are obtained from the lower-level problem. The upper-level problem (1.5) is constrained by the functions $G(x,\cdot)$, $H(x,\cdot)$ and the set of the optimal solutions of the followers, $S(x)$, parameterized by the leader′s decision, x, solving a set of m problems in the lower level (1.6).

$$
\begin{aligned}
&(x^e, y^e, \lambda^e, \mu^e) \ \textit{solves} \\
&\qquad \min_{x,\bar{y},\bar{\lambda},\bar{\mu}} F(x,\bar{y},\bar{\lambda},\bar{\mu}) \\
&\qquad \text{subject to} \begin{cases} G(x,\bar{y},\bar{\lambda},\bar{\mu}) \\ H(x,\bar{y},\bar{\lambda},\bar{\mu}) \\ (\bar{y},\bar{\lambda},\bar{\mu}) \in S(x) \end{cases}
\end{aligned} \tag{1.5}
$$

$$
\begin{aligned}
&\text{Where } (\bar{y},\bar{\lambda},\bar{\mu}) \in S(x) \text{ if and only if :} \\
&\quad (\bar{y}_i,\bar{\lambda}_i,\bar{\mu}_i), \forall i = 1,...,m \ \textit{solves} \\
&\qquad \min_{y_i,\lambda_i,\mu_i} f_i(x, y_i, \bar{y}_{-i}) \\
&\qquad \text{subject to} \begin{cases} g_i(x, y_i, \bar{y}_{-i}) \le 0, & \mu_i \\ h(x, y_i, \bar{y}_{-i}) = 0, & \lambda_i \end{cases}
\end{aligned} \tag{1.6}
$$

The $\bar{y}$ tuple is the Nash equilibrium of the followers for the leader′s decision, x. The variables $\bar{\lambda}$ and $\bar{\mu}$ represent the Lagrange multipliers for the equality and the inequality constraints of the followers, respectively. The objective function, $f_i(x,\cdot)$, and the constraints, $g_i(x,\cdot)$ and $h_i(x,\cdot)$, are defined as linear functions for all the i^{th} followers′ problems. Because each i^{th} follower problem is stated as an LP, global optimality can be guaranteed for each i^{th} follower problem. But the simultaneous i^{th} followers′ problems may not have a solution, may have only one solution, or may have multiple solutions. The set of the solutions represented by $S(x)$ is rewritten sometimes as an equivalent system of constraints, e.g., KKT conditions added to the upper level problem (1.5). This system of constraints is the so-called equilibrium constraints set. The single-leader-multiple-follower problem can be stated as a mathematical program with equilibrium constraint(MPEC) optimization problem [31, 30].

Chapter 2

A Framework for Bilevel Multi-follower Decision Problems

Different reactions could be generated at the lower level towards each possible action(choice) conducted at the upper level when multiple followers are involved in a bilevel decision making. The different relationships among the followers could cause multiple different techniques to find an optimal solution for the upper level′s decision maker. Therefore, the leader′s decision will be affected by the reactions of the followers and the relationships among the followers.

Basically, there are three different kinds of relationships among the followers determined by the form of a share in decision variables which is the first relationship factor. The first kind of these relationships is the uncooperative situation in which there is no sharing of decision variables among the followers. At this time, there are obviously neither shared objectives nor shared constraints among the followers.

The second case is the cooperative situation where the followers totally share the decision variables in their objectives and constraints together. However, there are several different sub-cases with in the cooperative situation which are determined by the relationships among the objectives (the second relationship factor) and constrains (the third relationship fators) of the followers.

Each follower may have an individual objective whatever variables they share in their constraints with other followers. For example, one agricultural group has its objective to maximize its profile of agriculture, and another agricultural group′s objective is to maximize its land sustainability, towards the Government′s policy in land use. The two followers share all other decision variables, but have different objectives.

Another pair of sub-cases is that the followers have their common objectives whatever sharing their constraints. For example, for any governmental agricultural policy, the two agriculture groups have their common objective to maximize its profile of agriculture. But they may or may not share constraints in financial, environment protection, and cultural parameters in the context of attempting to achieve an optimal solution.

The last case is the partial cooperative situation where the followers partially share decision variables in their objectives or constraints or both. Similar to the second case, four sub cases are involved with in this one as well. Based on the three cases and their various sub-cases determined by the three relationship factors, decision variables, objectives and constraints, totally nine different kinds of situations among the followers can be identified.

A framework can be established to describe these situations. For a linear bilevel decision problem, if some followers share their decision variables or some not, it will be dealt with as a variable sharing situation. Similarly, if some followers share their objective (or) functions or some not, it will be dealt with as an objective (or constraint) sharing situation. Therefore,this project mainly focuses on the uncooperative cases of linear bilevel multifollower problems(LBLMFP) which is modeled as given below.

Among the different frameworks,the uncooperative situations is given here:

For $x \in X \subset \Re^n \quad y_i \in Y_i \subset \Re^{m_i} \quad Y = (y_1, y_2, ..., y_k)^T$
$F : X \times Y_i \to \Re^1 \quad f_i : X \times Y_i \to \Re^1$ and $i = 1, 2, ..., k$ and a linear BLMF decision problem in which K followers are involved and there is no shared decision variable, objective function or constraint function among them is defined as follows (called an uncooperative decision model).

$$\begin{aligned} \min_{x \in X} \qquad & F(x, y_1, ..., y_k) = cx + \sum_{s=1}^{k} d_s y_s \\ \text{subject to} \qquad & Ax + \sum_{t=1}^{k} B_t y_t \leq b \end{aligned} \tag{2.1}$$

Where Y_i $(i = 1, 2, ..., k)$ for each value of x, is the solution of the lower level problem:

$$\begin{aligned} \min_{y_i \in Y_i} \qquad & f_i(x, y_i) = c_i x + e_i y_i \\ \text{subject to} \qquad & A_i x + C_i y_i \leq b_i \end{aligned}$$

where $c \in \Re^n, c_i \in \Re^n, d_i \in \Re^{m_i}, e_i \in \Re^{m_i}, b \in \Re^p, b_i \in \Re^{q_i}, B_i \in \Re^{p \times m_i}, A_i \in \Re^{q_i \times n}, C_i \in \Re^{q_i \times m_i}, \ i = 1, 2, ..., k$

2.1 General Definition of Bilevel Programming

In any hierarchical system involving two levels of decision makers with different objectives, the decision makers are divided into two categories: the upper level (leader) and the lower level (follower). The upper level controls a subset of decision variables, while the lower level controls remaining decision variables. To each decision made by the upper level, the lower level responds by a decision optimizing its objective function over a constraint set which depends on the decision of the upper level.

Bilevel programming problem(BLPP), as a hierarchical optimization problem, has been proposed for dealing with this kind of decision processes, which is characterized by the existence of two optimization problems, the upper level and the lower level problems. The distinguished feature of this kind of problems is that the constraint region of the upper level problem is implicitly determined by optimization problem of the lower level problem, and any feasible solution must satisfy the optimality of the lower level problem as well as all constraints. The general bilevel programming problem can be formulated as follows:

$$\begin{aligned} &\min_{x \in X} F(x,y) \\ &\text{subject to} \quad G(x,y) \leq 0 \\ &\textit{where } y \textit{ solves} \\ &\begin{cases} \min\limits_{y \in Y} f(x,y) \\ \text{subject to} \\ g(x,y) \leq 0 \end{cases} \end{aligned} \tag{2.2}$$

If all the functions which are involved in the above general bilevel programming problem(GBLP)are linear then we call it a linear bilevel programming problem which is defined as:

$$\begin{aligned} &\min_{x \in X} F(x,y) = c_1^T x + d_1^T y \\ &\text{subject to } A_1 x + B_1 y \leq b_1 \\ &\textit{where } y \textit{ solves} \\ &\begin{cases} \min\limits_{y \in Y} f(x,y) = c_2^T x + d_2^T y \\ \text{subject to } \; A_2 x + B_2 y \leq b_2 \end{cases} \end{aligned} \tag{2.3}$$

Where $x \in \Re^n,\ y \in \Re^m, c_1,\ c_2 \in \Re^n,\ d_1, d_2 \in \Re^m,\ b_1 \in \Re^p,\ b_2 \in \Re^q$.

It is very difficult to find the solution of linear bilevel programming problem if the solution of lower level problem is not uniquely determined. Based on these, we get the following definitions (see [7]):

1. Constraint region of the linear bilevel programming problem:

$S = \{(x,y) \in X \times Y : A_1 x + B_1 y \leq b_1, A_2 x + B_2 y \leq b_2, \}$

The linear BLPP problem constraint region refers to all possible combinations of choices that the leader and follower(s) may make.

2. Projection of S onto the *leader's* decision space:

$S(X) = \{x : \exists y, A_1 x + B_1 y \leq b_1, A_2 x + B_2 y \leq b_2\}$

3. Feasible set for the follower for each x:

$S(x) = \{y : B_2y \leq b_2 - A_2x\}$

4. Follower rational reaction set for $x \in S(X)$

$\Psi_L(x) = \{\underset{y}{Arg} \min c_2^T x + d_2^T y : y \in S(x)\}$

5. Inducible region:

$IR = \{(x, y) : (x, y) \in S, y \in \Psi_L(x)\}.$

The inducible region is the most important area that helps to define the feasible set of the leaders problem at least in the case when the lower level problem has unique optimal solution for all values of x. Note that the constraints $A_2x + B_2y \leq b_2$ can be dropped in the set S since they are satisfied necessarily by an optimal solution $y \in \Psi_L(x)$ of the lower level problem(2.3).

Definition 2.1. *A point-to-set mapping $\Psi_L : \Re^p \to 2^{\Re^q}$ is called polyhedral if its graph $\Psi_L = \{(x, y) \in \Re^p \times \Re^q, y \in \Psi_L(x)\}$ is equal to the union of a finite number of polyhedral sets. Here a polyhedral set is the intersection of a finite number of half spaces; it is a closed and convex set.*

Definition 2.2. *Consider problem (2.3) and assume that the rational reaction set of the follower consists of at most one point for all $x \in S(X)$: $|\Psi_L(x)| \leq 1, for, all, x \in S(X)$. Then, a point $(\bar{x}, \bar{y}) \in IR$ is an optimal solution of problem (2.3). if*

$$c_1\bar{x} + d_1\bar{y} \leq c_1x + d_1y, \forall(x, y) \in IR.$$

The situation is a little bit difficult in the case when the lower level problem can have multiple optimal solutions. The difficulty is due to an unclear value of the leader′s objective function (and feasibility of the point (x, y)) prior to the publication of the optimal solution taken by the follower. If the leader has control over x, only he/she is not able to predict this value and hence he/she cannot find a best decision.

There are mainly two ways out of this situation discussed in the literature: the optimistic (or weak) and the pessimistic(or strong) formulation of the bilevel programming problem, see e.g. [2] and [33]. The pessimistic formulation needs to be selected if the leader is not able to influence the follower in the sense that he/she can ask the follower to select one preferable solution out of $\Psi_L(x)$. Then, the leader is forced to protect him/herself against bad selections of the follower. But, if the leader can influence the follower′s selection he/she is able to motivate the follower to select a point and such a situation is considered as optimistic formulation of a bilevel programming problem.

$$\underset{y}{Arg} \min\{c_1x + d_1y : y \in \Psi_L(x)\}$$

i.e. a best solution in $\Psi_L(x)$ from the leader's point of view. The resulting problem is equivalent to the following one, provided, this has an optimal solution [34]:

$$\underset{x,y}{Arg}\min\{c_1x + d_1y : (x, y) \in IR\} \tag{2.4}$$

Hence, we get the following definition of an optimal solution:

Definition 2.3. *Consider problem (2.3) in the optimistic formulation, i.e. consider problem* $\underset{x,y}{Arg}\min\ \{c_1x + d_1y : (x, y) \in IR\}$*. Then, a point* $(\bar{x}, \bar{y}) \in IR$ *is an optimal solution if*

$$c_1\bar{x} + d_1\bar{y} \leq c_1x + d_1y, \forall (x, y) \in IR.$$

The point $(\bar{x}, \bar{y}) \in IR$ *is a local optimal solution for* $\underset{x,y}{Arg}\min\ \{c_1x + d_1y : (x, y) \in IR\}$ *provided that this inequality is valid for all* $(x, y) \in IR$ *sufficiently close to* $(\bar{x}, \bar{y})$*.*

This implies that the problem with uniquely solvable lower level problems is a special case of the optimistic formulation and we need only to consider the above case when we find sufficient conditions for the existence of an optimal solution.

2.2 Linear Bilevel Multifollower Decision Problem for Uncooperative Case

(LBLMFDP) The uncooperative situation (2.1) is the most basic form of Linear BLMF decision problems over the different kinds of decision models. This section, therefore, focuses on this model by giving the definition for an optimal solution and related theorems for solving the uncooperative situation of (2.1) decision model.

Definition 2.4. *A topological space is compact if every open cover of the entire space has a finite subcover. (A subset S of* $\Re^n$ *is compact if and only if S is closed and bounded. Heine-Borel theorem) For example* $[a, b]$ *is compact in* $\Re$*.[32].*

Definition 2.5. *a) Constraint region of a linear bilevel multifollower programming(LBLMFP) problem:*

$$S = \{(x, y_i) \in X \times Y_i, Ax + \textstyle\sum_{t=1}^{k} B_t y_t \leq b, A_i x + C_i y_i \leq b_i, i = 1, 2, ..., k\}$$

The linear BLFMP problem constraint region refers to all possible combinations of choices that the leader and followers may make.
b) Projection of S onto the leader's decision space:

$$S(X) = \{x \in X, \exists y_i \in Y_i, Ax + \textstyle\sum_{t=1}^{k} B_t Y_t \leq b, A_i x + C_i y_i \leq b_i, i = 1, 2, ..., k\}$$

Unlike the rules in an uncooperative game theory, each player must choose a strategy simultaneously. The definition of the BLMF model requires that the leader move first by selecting an x in attempt to minimize his/her objective function subjecting to the constraints of both the upper and the lower level problems.

c) Feasible set for each follower $\forall x \in S(x)$*:*

$$S(x) = \{y_i \in Y_i, (x, y_i) \in S, i = 1, 2, ..., k\}$$

The feasible region for the follower is affected by the leader's choice of ,x and allowable choices of each follower are the elements of S.

(d) Each follower's rational reaction set for $x \in S(X)$:

$$p_i(x) = \{y_i \in Y_i, y_i \in arg\min[f_i(x, \bar{y}_i), \bar{y}_i \in S_i(x)]\}, \text{ where}$$
$$arg\min[f_i(x, \bar{y}_i) : \bar{y}_i \in S_i(x)] = \{y_i \in S_i(x) : f_i(x, y_i) \leq f_i(x, \bar{y}_i), \bar{y}_i \in S_i(x)\}$$

The followers observe the leader's action(decision) and simultaneously react by selecting y_i from his/her feasible set to minimize his/her objective function.

e) Inducible region:

$$IR = \{(x, y_i) : (x, y_i) \in S, y_i \in p_i(x), i = 1, 2, ..., k\}$$

2.3 Theoretical properties of LBLMFP problem

Bilevel programming problems are characterized by two levels of hierarchical decision making. The top planner(leader) makes his/her decision in full knowladge of the bottom planner(follower). Each planner attempts to optimize his/her objective function and is affected by the choice of the other planner. The properties of linear bilevel programming problems are summarized as follows:

1. The system has interacting decision making units with in a hierarchical structure.

2. The follower executes his/her policies after, and in view of, the decisions of the leader.

3. Each level maximizes net benefits independently, no compromise is possible.

4. The effect of the upper level decision maker on the lower level problem is shown in both his/her objective function and set of feasible decisions.

Let x be a vector variable representing the higher decision level's choice and y be a vector variable denoting the lower decision level's choice. Let S be the set of feasible choices (x, y), for any fixed choice of x, level two will choose a value of y that optimizes the level two objective function $f(x, y)$. Hence, for each feasible value of x, the follower will react with a corresponding value of y. This shows that the functional relationship between the decisions of leader and the reactions of follower, say $y = \Psi_L(x)$. Before introducing some of the prominent algorithms that used for solving linear BLPPS, we need to introduce the following theorems:

Theorem 2.1. *[31] IR can be written equivalently as a piecewise linear equality constraint comprised of supporting hyperplanes of S.*

Solving the linear BLLP is equivalent to minimizing the upper level objective function over a piecewise linear equality constraint.

2.4 Existence of an Optimal Solution for a linear bilevel multifollower problems

Consider the assumption

A_1: S is a nonempty compact set.

Then we have the following lemma.

Lemma 2.1. *If A_1 holds then the set IR is closed.*

Proof: Since S is not empty there is at least one parameter value $\bar{x} \in S(X)$ with $S(\bar{x}) \neq \emptyset$. By compactness, $\Psi_L(\bar{x}) \neq \emptyset$. Consider a sequence $(x^k, y^k)_{k=1}^{\infty} \subset IR$ converging to $(\bar{x}, \bar{y})$. Then, by well-known results of linear parametric optimization [2], $\bar{y} \in \Psi_L(\bar{x})$.
Hence,$(\bar{x}, \bar{y}) \in IR$. The same result has been shown e.g. in [7] under the additional assumption that the lower level problem has at most one optimal solution for all parameter values.

Example 2.1. *Consider the following example*

$$
\begin{aligned}
&\min_{x\geq 0} F(x,y) = x - 8y \\
&\textit{subject to } \{-5x + 2y \geq -33, x + 2y \geq 0\} \\
\textit{where } y \textit{ solves}&
\end{aligned} \tag{2.5}
$$

$$
\begin{cases}
\min_{y\geq 0} f(x,y) = y \\
\textit{subject to } \{-7x + 3y \leq 5, x + y \leq 15\}
\end{cases}
$$

Clearly S is a non-empty, compact set. But the bilevel problem has no solution since $IR = \emptyset$. If we interchange the position of the upper level constraints and the lower level constraints of the above example then S remains closed and bounded and the resulting problem has solution.

Therefore, from this example we can see that one needs more assumptions besides non-emptiness and compactness of S to guarantee the existence of a solution. Namely, Lemma (2.1) does not imply that the set $IR \neq \emptyset$.
A_2: IR is non-empty.

Lemma 2.2. *If assumptions A_1 and A_2 are satisfied then IR is the finite union of compact polyhedral sets.*

The proof follows since the graph of $\Psi_L(\cdot)$ is polyhedral [7] and IR is equal to the intersection of two polyhedral sets.

Theorem 2.2. *If A_1 and A_2 are satisfied the problem* $\underset{x,y}{Arg}\min \{c_1x + d_1y : (x,y) \in IR\}$ *has a solution.*

Proof: By Lemma (2.1), IR is closed and due to $IR \subset S$ it is also bounded. By A_2 it is also non-empty. Hence, problem $\underset{x,y}{Arg} \min \{c_1x + d_1y : (x, y) \in IR\}$ consists in minimizing a continuous function over a compact non-empty set, which implies the proof by Weierstra′s Theorem. However, unlike in the optimistic case the existence of a solution in the pessimistic case is generally not guaranteed even under the above assumptions [2]. The proof follows from the Weierstra′s Theorem since IR is a nonempty compact set in this case. The main idea in the project [33] reduces to shifting connecting constraints to the lower level to get the existence of an optimal solution.

Corollary 2.1. *If assumption A_1 is satisfied and there do not exist connecting constraints, then problem (2.3) has an optimal solution.*

Definition 2.6. *A pair (x, y) is said to be feasible to the linear bilevel programming problem if it satisfies $x \in X andy \in S(x)$.*

Definition 2.7. *A feasible pair (x^*, y^*) is called an optimal solution to the bilevel programming problem if $F(x^*, y^*) \leq F(x, y)$ for all the feasible solutions (x, y).*

Thus in terms of the above notations, the linear BLMF problem can be written as:

$$\min\{F(x, y_i) : (x, y_i) \in IR, i = 1, ..., k\} \tag{2.6}$$

We present the following theorem to characterize the condition under which there is an optimal solution for a linear BLMF problem:

Theorem 2.3. *The solution $(x^*, y_1^*, ..., y_k^*)$ of the linear BLMFP problem occurs at a vertex of S.*

Proof. Let $(x^1, y_1^1, ..., y_k^1)$, ..., $(x^r, y_1^r, ..., y_k^r)$ be the distinct vertices of S. Since any point in S can be written as convex combination of these vertices.

Let $(x^*, y_1^*, ..., y_k^*) = \sum_{j=1}^{r} \alpha_j(x^j, y_1^j, ..., y_k^j)$ where $\sum_{j=1}^{r} \alpha_j = 1$, $\alpha_j \geq 0$, $j = 1, 2, ..., \bar{r}$ and $\bar{r} \leq r$. It must be shown that $\bar{r} = 1$. To see this let us write the constraints to (2.1) at $(x^*, y_1^*, ..., y_k^*)$ in their piecewise linear form (5).

$0 = Q_i(x, y_l^*) - e_{ii}y_i^*$, where, Q_i is a linear program parameterized in x, y_j, $j = 1, 2, ..., k$, $j \neq i$ $l \neq i$ and $l = 1, 2, ..., k$, $i = 1, 2, ..., k$

Rewrite it as:
$0 = Q_i(\sum_j \alpha_j(x^j, y_l^j)) - e_{ii}(\sum_j \alpha_j y_i^j)$
$\leq \sum_j \alpha_i Q_i(x^j, y_l^j) - \sum_j \alpha_j e_{ii} y_i^j$, $i \neq l$, where i=1, 2, ..., k

But by definition of optimalityconditions

$$Q_i(x^j, y_l^j, l = 1, 2, ..., k, l \neq i) = \min_{y_i \in S(x^j)} \{e_{ii}y_i \leq e_{ii}y_i^j, l \neq i, i = 1, 2, ..., k\}$$

Therefore, $Q_i(x^j, y_l^j) - e_{ii}y_l^j \leq 0$, $j = 1, 2, ..., \bar{r}$ and $i = 1, 2, ..., k$

Noting that $\alpha_j \geq 0$ $j = 1, 2, ..., \bar{r}$ the equality in the preceding expression must hold or else a contradiction would result in the sequence above. Consequently,

$Q_i(x^j, y_l^j) - e_{ii} y_i^j = 0$, $j = 1, 2, ..., \bar{r}$ and i=1, 2, ..., k, $l \neq i$ This implies that $(x^j, y_1^j, ..., y_k^j) \in IR$, $j = 1, 2, ..., \bar{r}$ and $(x^*, y_1^*, ..., y_k^*)$ can be written as a convex combination of points in IR.

Because $(x^*, y_1^*, ..., y_k^*)$ is a vertex of IR , a contradiction results unless $\bar{r} = 1$.

This means that $(x^*, y_1^*, ..., y_k^*)$ is an extreme point of S.The proof is completed. □

Theorem 2.4. *If S is nonempty and compact then a linear BLMF problem has an optimal solution.*

Proof. Since S is nonempty, there exists a point $(x^*, y_i^*) \in S$. Then, we have $x^* \in S(X) \neq \emptyset$ by Definition 2b. Consequently, we have
$S_i(x^*) \neq \emptyset$ $i = 1, 2, ..., k$ by Definition 2c. Because S is compact and from Definition 2d, we have

$$\begin{aligned} p_i(x^*) &= \{y_i \in Y_i, y_i \in arg \min[f_i(x^*, \bar{y}_i), \bar{y}_i \in S_i(x^*)]\} \\ &= \{y_i \in Y_i : y_i \in y_i \in S_i(x^*) : f_i(x^*, y_i) \leq f_i(x^*, \bar{y}_i), \bar{y}_i \in S_i(x^*)\} \\ &\neq \emptyset \end{aligned}$$

where $i = 1, 2, ..., k$, Hence, there exists $y_i^o \in p_i(x^*)$, $i = 1, 2, ..., k$ such that $(x^*, y_1^o, ..., y_k^o) \in S$. Therefore, we have $IR = \{(x, y_1, ..., y_k) : (x, y_1, ..., y_k) \in S, y_i \in p_i(x), i = 1, 2, ..., k\} \neq \emptyset$ by Definition 2e. Because we are minimizing a linear function over a set which is nonempty and bounded $\in \Re^n$, an optimal solution to the linear BLMF problem must exist. The proof is completed. □

2.5 Optimality Conditions for (LBLMFP)

Knowing the properties of an optimal condition is very important so that the following lemma gives the necessary condition:

Lemma 2.3. *[33] A necessary condition that (x^*, y^*) solves the optimistic linear bilevel programming problem (2.4) then there exists a vector λ^* such that (x^*, y^*, λ^*) locally solves:*

$$\begin{aligned} &\min c_1 x + d_1 y \\ &\text{subject to} \\ &A_1 x + B_1 y \leq b_1 \\ &A_2 x + B_2 y \leq b_2 \\ &B_2^T \lambda + d_2 = 0 \\ &\lambda^T (b_2 - A_2 x - B_2 y) = 0 \\ &\lambda \geq 0 \end{aligned} \tag{2.7}$$

This result is a simple consequence from duality of linear programming (i.e the Karush-Kuhn-Tucker conditions are sufficient and necessary optimality conditions in linear programming). Moreover, this result is valid for all optimal solutions of the dual linear programming problem of the lower level problem(2.3).

However, the opposite implication is not true in general if the optimal solution of the dual of the lower level problem(2.3) is not uniquely determined. Because of the addition of new variables to the optimistic bilevel programming problem (2.4. The above problem can be described by decomposing it into a system which can be handled by linear programming. For this case, take any index set $\tilde{I}$ with (i: λ_i^*)$\subseteq \tilde{I} \subseteq \{i : (A_2x^* + B_2y^* - b_2)_i = 0\}$ and consider

$$\begin{aligned}
&\min c_1x + d_1y \\
&\text{subject to} \\
&A_1x + B_1y \leq b_1 \\
&A_2x + B_2y \leq b_2 \\
&B_2^T\lambda + d_2 = 0 \\
&(A_2x + B_2y - b_2)_i = 0, i \in \tilde{I} \\
&\lambda_i = 0, i \notin \tilde{I} \\
&\lambda \geq 0
\end{aligned} \tag{2.8}$$

Here, the matrix $(B_2)_{i\in\tilde{I}}$ has full row rank. Then, if (x^*, y^*) is a local optimal solution of (2.4), (x^*, y^*, λ^*) is a global optimal solution of (2.8). Now consider

$$\begin{aligned}
&\min c_1x + d_1y \\
&\text{subject to} \\
&A_1x + B_1y \leq b_1 \\
&A_2x + B_2y \leq b_2 \\
&(A_2x + B_2y - b_2)_i = 0, i \in \tilde{I}
\end{aligned} \tag{2.9}$$

where the dual variable λ is a solution of

$$\begin{aligned}
B_2^T + d_2 &= 0 \\
\lambda_i = 0,\ i &\notin \tilde{I} \\
\lambda &\geq 0
\end{aligned} \tag{2.10}$$

with sets

$\bar{I} \in D(x, y) := \{\bar{I} : \exists\lambda \geq 0,\ B_2^T + d_2 = 0,\ \lambda_i = 0,\ \forall i \notin I,\ (A_2x + B_2y - b_2)_i = 0,\ \forall i \in I,\ (B_2)_{i\in\tilde{I}},\ hasfullrank\}$.

Then, if (x^*, y^*) is a local optimal solution of (2.4) then (x^*, y^*) is an optimal solution of problem (2.9) for all $\bar{I} \in D(x, y)$. In addition, if (x^*, y^*) is an optimal solution of (2.9) for all $\bar{I} \in D(x, y)$ then (x^*, y^*) is a local optimal solution of (2.4).

if (x^*, y^*, λ^*) is a local optimal solution of (2.7) then each point (x, y, λ) in a sufficient small neighborhood of (x^*, y^*, λ^*) can not have a smaller objective function value. By duality this neighborhood is associated to a facet of M corresponding to some part of IR.

Theorem 2.5. *The point* (x^*, y^*) *is an optimal solution of the problem* (2.4) *if and only if it is an optimal solution of the problem (2.9) for all* $\tilde{I} \in D(x^*, y^*)$.

Proof. Since the Karush-Kuhn-Tucker conditions are necessary and sufficient optimality conditions for the lower level problem, local optimality of (x^*, y^*) for the bilevel programming problem implies that (x^*, y^*, λ^*) is an optimal solution for the problems (2.9) for each solution λ for (2.10). To show the if part of the theorem assume that $(x^*, y^*) \in IR$ is not an optimal solution of (2.4). Then there exists a sequence (x_n, y_n) converging to (x^*, y^*) with

$(x_n, y_n) \in IR$, $c_1x_n + d_1y_n < c_1x^* + d_1y^*$ for all n. Since $(x_n, y_n) \in IR$, $A_1x_n + B_1y_n \leq b_1$, $y_n \in \Psi(x_n)$, i.e
$\exists\lambda_n \geq 0$, $B_2^T\lambda_n + d_2 = 0$, $\lambda_n^T(b_2 - A_2x_n - B_2y_n) = 0$. Here, λ_n can be taken as vertex of

$\{\lambda \geq 0 : B_2^T\lambda_n + d_2 = 0, \lambda^T(b_2 - A_2x_n - B_2y_n) = 0.\}$

Hence also as a vertex of

$\{\lambda \geq 0 : B_2^T\lambda_n + d_2 = 0, \lambda_i = 0, fori \notin I_n\}$

for some set I_n satisfying $(b_2 - A_2x_n - B_2y_n)_i = 0$, $i \in I_n$. Since the number of different sets $I \subseteq \{1, 2, ...q\}$ is finite there exists an infinite subsequence of $\{(x_n, y_n, \lambda_n)\}$ with λ_n is a vertex of

$\{\lambda \geq 0 : B_2^T\lambda_n + d_2 = 0, \lambda_i = 0, for\ i \notin \hat{I}\}$ for a fixed set $\hat{I}$ satisfying $(b_2 - A_2x^* - B_2y^*)_i = 0$, $i \in I$ for all n. This means that $\hat{I} \in D(x^*, y^*)$.

This shows that (x^*, y^*) can not an optimal solution of (2.9) for this set $\hat{I}$. □

2.6 Solution Procedure for single leader single follower problems

The possible existence of local optimal solution, even for linear multilevel programming problem aggravates the general task for algorithmic development. Many solution approaches to solve multilevel programming problem have been developed to date. Most of these algorithmic developments have been devoted to the linear bilevel programming problem which can be defined as:

$$\begin{aligned} &\min_{x\in X} F(x,y) = c_1^T x + d_1^T y \\ &\text{subject to } A_1 x + B_1 y \le b_1 \\ \textit{where } y \textit{ solves} & \\ \begin{cases} \min_{y\in Y} f(x,y) = c_2^T x + d_2^T y \\ \text{subject to } \; A_2 x + B_2 y \le b_2 \end{cases} & \end{aligned} \tag{2.11}$$

Here, since x is fixed when the lower level problem is solved by the level two decision maker, we can assume that $c_2 = 0$ and latter on, add the term $c_2 x$, if necessary.

There are different solution method that can be used to solve linear BLMFP problems. However, it is enough to show how the two methods are used to solve linear bilevel multifollower programming problems. These are :

1. the Kuhn-Tucker approach and

2. k^{th} best algorithm

These have been described in specific paper in the literature (see the paper by Bialas and Karwan [36]). Some mathematicians have worked on the extension of these solution procedures to three level programming problems and the general n-level programming problems.

2.6.1 The k^{th} Best Algorithm

The k^{th} best algorthm proposed by Bialas and Karwan (1984), [36] and Wen [37] can be thought of as the vertex enumeration approach, and it is based on a very simple idea. The solution search procedure of the method starts from a point which is an optimal solution to the problem of the leader and checks whether it is also an optimal solution to the problem of the follower or not. If the first point is not the Stackelberg solution, the procedure continues to examine the second best solution to the problem of the leader, and so forth.

It means that by searching extreme points on the constraint region S, we can efficiently find an optimal solution for a linear BLMFP problem. The basic idea of this algorithm is that according to the objective function of the upper level, we arrange all the extreme points in S in descending order, and select the first extreme point to check if it is on the inducible region IR or not.

If yes, the current extreme point is the optimal solution. Otherwise, the next one will be selected and checked. Let $(x_{[1]}, y_{[1]})$, $(x_{[2]}, y_{[2]})$, ..., $(x_{[N]}, y_{[N]})$ denote the N ordered basic feasible solutions for the LP(called LPR).

$$\min F(x,y) = c_1^T x + d_1^T y$$
$$\text{subject to} \begin{cases} A_1 x + B_1 y \leq b_1 \\ A_2 x + B_2 y \leq b_2 \\ x, y \geq 0 \end{cases} \tag{2.12}$$

and suppose that these solutions are such that

$$c_1 x_{[i]} + d_1 y_{[i]} \leq c_1 x_{[i+1]} + d_1 y_{[i+1]}, i = 1, 2, ..., N-1$$

then solving the linear BLMFPP is equivalent to finding the index $k^* = \min\{i \in i = 1, 2, ..., N : (x_{[i]}, y_{[i]}) \in IR\}$. Let us also define the lower level problem for a given value $\tilde{x}$, $LL(\tilde{x})$ as:

$$\min_y f(x,y) = c_2^T x + d_2^T y$$
$$\text{subject to} \begin{cases} A_2 x + B_2 y \leq b_2 \\ y \geq 0 \end{cases} \tag{2.13}$$

Let W be the set of basic solutions to be investigated, $W_{[i]}$ the set of the basic solutions adjacent to the incumbent, T is the set of the basic solutions which have already been tested. The algorithm is the following:

Algorithm

Step 1. Set i=1. Solve problem (2.6) to get optimal solution $(x_{[1]}, y_{[1]})$ by simplex method.

Let $W = \{(x_{[1]}, y_{[1]}\}$ and $T = \emptyset$. Go to step 2.

Step 2. Solve the problem (2.7) with the bounded simplex method. Let $\tilde{y}$ denote the optimal solution to (2.7)

If $\tilde{y} = y_{[i]}$
STOP
therefore $(x_{[i]}, y_{[i]})$ is the optimal solution of the linear BLPP.
else
go to the next step

Step 3. Let $W_{[i]}$ be the set of extreme points of S, which are adjacent to $(x_{[i]}, y_{[i]})$.

Let $T = T \bigcup \{(x_{[i]}, y_{[i]})\}$ and $W = (W \bigcup W_{[i]}) \setminus T$. Go to step 4

Step 4. Set k=k+1.

choose $(x_{[i]}, y_{[i]})$ such that
$c_1 x_{[i]} + d_1 y_{[i]} = min\{c_1 x + d_1 y : (x,y) \in W\}$
go to step (2).

2.6.2 Numeric Example for k^{th} Best Algorithm to Solve LBLMFP

Example 2.2. *Consider a following linear BLMFP problem with* $x \in \Re^1$, $y_1, y_2 \in \Re^1$, $z \in \Re^1$, *and* $X = \{x \geq 0\}$, $Y = \{y_1 \geq 0, y_2 \geq 0\}$, $Z = \{z \geq 0\}$

$$\min_{x \in X} F(x, y_1, y_2, z) = -8x + y_1 + 2y_2 - z$$
$$\text{subject to } x \leq 1$$

$$\left\{ \begin{array}{ll} \min_{y_1 \in Y, z \in Z} f_1(x, y_1, y_2, z) = x - 2y_1 + y_2 + z & \min_{y_2 \in Y, z \in Z} f_2(x, y_1, y_2, z) = x + y_1 - 2y_2 + z \\ \text{subject to} & \text{subject to} \\ y_1 \leq 1 & y_2 \leq 1 \\ & z \leq 1 \end{array} \right.$$

Since the followers share the variable z. we have to transform the above problem as given below:

$$\min_{x \in X} F(x, y_1, y_2, z) = -8x + y_1 + 2y_2 - z$$
$$\text{subject to } x \leq 1$$
$$\min_{y_1 \in Y} f_1(x, y_1, y_2, z) = x - 2y_1 + y_2 + z$$
$$\min_{y_2 \in Y} f_2(x, y_1, y_2, z) = x + y_1 - 2y_2 + z$$
$$\min_{z \in Z} f_3(x, y_1, y_2, z) = 2x - y_1 - y_2 + 2z$$
$$\text{subject to}$$
$$y_1 \leq 1$$
$$y_2 \leq 1$$
$$z \leq 1$$

According to the extended $K^{th} - best$ *approach, the transferred form of the above example can be rewritten as follows in the form of:*

$$\min_{x \in X} F(x, y_1, y_2, z) = -8x + y_1 + 2y_2 - z$$
$$\text{subject to } x \leq 1$$
$$y_1 \leq 1$$
$$y_2 \leq 1$$
$$z \leq 1$$
$$x \geq 0, y_1 \geq 0, y_2 \geq 0, z \geq 0$$

Step 1, set $i = 1$, *and solve the above problem with the simplex method to obtain the optimal solution* $(x_{[i]}, y_{1[i]}, y_{2[i]}, z_i)$= *(1,0,0,1). Let* $W = \{(1, 0, 0, 1)\}$ *and* $T = \emptyset$. *Go to*

Step 2.

Loop 1.
Setting $i = 1$*, we have the following:*

$$\min_{y_1 \in Y} f_1(x, y_1, y_2, z) = x - 2y_1 + y_2 + z$$
subject to
$$x \leq 1$$
$$y_1 \leq 1$$
$$y_2 \leq 1$$
$$z \leq 1$$
$$x = 1$$
$$y_1 \geq 0$$
$$y_2 \geq 0$$
$$z \geq 0$$

Using the bounded simplex method, we have $y_{1i} = 1$*. Because of* $\tilde{y}_{1i} \neq y_{1[i]}$*, we go to Step 3. We have* $W_{[i]} = \{(0,1,0,1),(1,1,1,1),(1,0,0,0),(1,1,0,0)\}$*,* $T = \{(1,0,0,1)\}$ *and* $W = \{(1,0,1,1),(1,1,0,1),(1,0,0,0),(0,00,1)\}$ *, then go to Step 4. Update* $i = 2$ *and choose* $(x_{[i]}, y_{1[i]}, y_{2[i]}, z_i) = (1,1,0,1)$ *then go to Step 2.*

Loop 2:

Setting $i = 1$*, we have*

$$\min_{y_1 \in Y} f_1(x, y_1, y_2, z) = x - 2y_1 + y_2 + z$$
subject to
$$x \leq 1$$
$$y_1 \leq 1$$
$$y_2 \leq 1$$
$$z \leq 1$$
$$x = 1$$
$$y_1 \geq 0$$
$$y_2 \geq 0$$
$$z \geq 0$$

Using the bounded simplex method, we have $y_{1i} = 1$ *and* $\tilde{y}_{1i} \neq y_{1[i]}$*. Setting i=i +1 we have*

$$\min_{y_2 \in Y} f_2(x, y_1, y_2, z) = x + y_1 - 2y_2 + z$$

subject to

$$x \leq 1$$
$$y_1 \leq 1$$
$$y_2 \leq 1$$
$$z \leq 1$$
$$x = 1$$
$$y_1 \geq 0$$
$$y_2 \geq 0$$
$$z \geq 0$$

Using the bounded simplex method, we have $y_{2i} = 1$. *Because of* $\tilde{y}_{2i} \neq y_{2[i]}$, *we go*

to Step 3. We have $W_{[i]} = \{(0,1,0,1),(1,1,1,1),(1,0,0,0),(1,1,0,0)\}$, $T = \{(1,0,0,1),(1,1,0,1)\}$

and $W = \{(0,0,0,0),(1,0,0,0),(1,0,1,1),(0,1,0,1),(1,1,1,1),(1,1,0,0)\}$ *, then go to Step 4. Update* $i = i + 1$

and choose $(x_{[i]}, y_{1[i]}, y_{2[i]}, z_i) = (1,0,0,0$ *then go to Step 2.*

Loop 3:

Setting $i = 1$, *we have*

$$\min_{y_1 \in Y} f_1(x, y_1, y_2, z) = x - 2y_1 + y_2 + z$$

subject to

$$x \leq 1$$
$$y_1 \leq 1$$
$$y_2 \leq 1$$
$$z \leq 1$$
$$x = 1$$
$$y_1 \geq 0$$
$$y_2 \geq 0$$
$$z \geq 0$$

Using the bounded simplex method, we have $y_{1i} = 1$. *Because of* $\tilde{y}_{1i} \neq y_{1[i]}$, *we go*

to Step 3. We have $W_{[i]} = \{(1,0,1,0),(0,0,0,0)\}$, $T = \{(1,0,0,1),(1,1,0,1),(1,0,0,0)\}$

and $W = \{(0,0,0,1),(1,0,1,1),(0,1,0,1),(1,1,1,1),(1,1,0,0),(1,0,1,0),(0,0,0,0)\}$, *then go to Step 4. Update* $i = i + 1$

and choose $(x_{[i]}, y_{1[i]}, y_{2[i]}, z_i) = (1,0,1,1)$ *then go to Step 2.*

Loop 4:

Setting $i = 1$, *we have*

$$\begin{aligned}
&\min_{y_1 \in Y} f_1(x, y_1, y_2, z) = x - 2y_1 + y_2 + z \\
&\textit{subject to} \\
&x \le 1 \\
&y_1 \le 1 \\
&y_2 \le 1 \\
&z \le 1 \\
&x = 1 \\
&y_1 \ge 0 \\
&y_2 \ge 0 \\
&z \ge 0
\end{aligned}$$

Using the bounded simplex method, we have $y_{1i} = 1$. *Because of* $\tilde{y}_{1i} \ne y_{1[i]}$, *we go*

to Step 3. We have $W_{[i]} = \{(0,0,1,1)\}$, $T = \{(1,0,0,1),(1,1,0,1),(1,0,0,0),(1,0,1,1)\}$

and $W = \{(0,0,0,1),(0,1,0,1),(1,1,1,1),(1,1,0,0),(1,0,1,0),(0,0,0,0),(0,0,1,1)\}$, *then go to Step 4. Update* $i = i + 1$

and choose $(x_{[i]}, y_{1[i]}, y_{2[i]}, z_i) = (1,1,0,0)$ *then go to Step 2.*

Loop 5:

Setting $i = 1$, *we have*

$$\min_{y_1 \in Y} f_1(x, y_1, y_2, z) = x - 2y_1 + y_2 + z$$

subject to

$x \leq 1$

$y_1 \leq 1$

$y_2 \leq 1$

$z \leq 1$

$x = 1$

$y_1 \geq 0$

$y_2 \geq 0$

$z \geq 0$

Using the bounded simplex method, we have $\tilde{y}_{1i} = 1$ *and* $\tilde{y}_{1i} = y_{1[i]}$ *Setting* $i \leftarrow i + 1$, *we have:*

$$\min_{y_2 \in Y} f_2(x, y_1, y_2, z) = x + y_1 - 2y_2 + z$$

subject to

$x \leq 1$

$y_1 \leq 1$

$y_2 \leq 1$

$z \leq 1$

$x = 1$

$y_1 \geq 0$

$y_2 \geq 0$

$z \geq 0$

Using the bounded simplex method, we have $\tilde{y}_{2i} = 1$, *because of* $\tilde{y}_{2i} \neq \tilde{y}_{2[i]}$ *we go to step 3, we have* $W_{[i]} = \{(0,1,0,0),(1,1,1,0)\}$, $T = \{(1,0,0,1),(1,1,0,1),(1,0,0,0),(1,0,1,1),(1,1,0,0)\}$ *and* $W = \{(0,0,0,1),(0,1,0,1),(1,1,1,1),(1,0,1,0),(0,0,0,0),(0,0,1,1),(0,1,0,0),(1,1,1,0)\}$, *then go to*

Step 4. Update $i = i + 1$, *and choose* $(x_{[i]}, y_{1[i]}, y_{2[i]}, z_{[i]}) = (1,1,1,1)$, *then go to Step 2. Loop 6:*

Setting $i \leftarrow 1$, *we have*

$$\min f_1(x, y_1, y_2, z) = x - 2y_1 + y_2 + z$$
$$subject\ to$$
$$x \leq 1$$
$$y_1 \leq 1$$
$$y_2 \leq 1$$
$$z \leq 1$$
$$x = 1$$
$$y_1 \geq 0$$
$$y_2 \geq 0$$
$$z \geq 0$$

Using the bounded simplex method, we have, $\tilde{y}_{1i} = 1$ *and* $\tilde{y}_{1i} = y_{1[i]}$ *setting* $i \leftarrow i + 1$ *we have*

$$\min_{y_2 \in Y} f_2(x, y_1, y_2, z) = x + y_1 - 2y_2 + z$$
$$subject\ to$$
$$x \leq 1$$
$$y_1 \leq 1$$
$$y_2 \leq 1$$
$$z \leq 1$$
$$x = 1$$
$$y_1 \geq 0$$
$$y_2 \geq 0$$
$$z \geq 0$$

Using the bounded simplex method, we have, $\tilde{y}_{2i} = 1$ *and* $\tilde{y}_{2i} = \tilde{y}_{2[i]}$ *setting* $i \leftarrow i + 1$ *we have*

$$\min_{z \in Z} f_3(x, y_1, y_2, z) = 2x - y_1 - y_2 + 2z$$
$$subject\ to$$
$$x \leq 1$$
$$y_1 \leq 1$$
$$y_2 \leq 1$$
$$z \leq 1$$
$$x = 1$$
$$y_1 \geq 0$$
$$y_2 \geq 0$$
$$z \geq 0$$

Using the bounded simplex method, we have $\tilde{z}_i = 0$ *because* $\tilde{z}_i \neq z_{[i]}$, *we go to step 3, we have* $W_{[i]} = \{(0, 1, 1, 1)\}, \quad T = \{(1, 0, 0, 1), (1, 1, 0, 1), (1, 0, 0, 0), (1, 0, 1, 1), (1, 1, 0, 0), (1, 1, 1, 1)\}$

and $W = \{(0,0,0,1),(0,1,0,1),(1,0,1,0),(0,0,0,0),(0,0,1,1),(0,1,0,0),(1,1,1,0),(0,1,1,1)\}$ *, then go to*

Step 4. Update $i = i+1$*, and choose* $(x_{[i]}, y_{1[i]}, y_{2[i]}, z_{[i]}) = (1,0,1,0)$ *then go to step 2*

Loop 7:

Setting $i \leftarrow 1$ *we have*

$$\min f_1(x,y_1,y_2,z) = x - 2y_1 + y_2 + z$$
$$\textit{subject to}$$
$$x \le 1$$
$$y_1 \le 1$$
$$y_2 \le 1$$
$$z \le 1$$
$$x = 1$$
$$y_1 \ge 0$$
$$y_2 \ge 0$$
$$z \ge 0$$

Using the bounded simplex method, we have $\tilde{y}_{1i} \ne y_{1[i]}$ *we go to step 3 we have* $W_{[i]} = \{(0,0,1,0)\}$, $T = \{(1,0,0,1),(1,1,0,1),(1,0,0,0),(1,0,1,1),(1,1,0,0),(1,1,1,1),(1,0,1,0)\}$ *and* $W = \{(0,0,0,1),(0,1,0,1),(0,0,0,0),(0,0,1,1),(0,1,0,0),(1,1,1,0),(0,1,1,1),(0,0,1,0)\}$*, then go to*

Step 4. Update $i = i+1$*, and choose* $(x_{[i]}, y_{1[i]}, y_{2[i]}, z_{[i]}) = (1,1,1,0)$ *then go to Step 2.*

Loop 8:

Setting $i \leftarrow 1$ *we have*

$$\min f_1(x,y_1,y_2,z) = x - 2y_1 + y_2 + z$$
$$\textit{subject to}$$
$$x \le 1$$
$$y_1 \le 1$$
$$y_2 \le 1$$
$$z \le 1$$
$$x = 1$$
$$y_1 \ge 0$$
$$y_2 \ge 0$$
$$z \ge 0$$

Using the bounded simplex method, we have $\tilde{y}_{1i} = 1$ *and* $\tilde{y}_{1i} = y_{1[i]}$.

Setting $i \leftarrow i+1$, we have

$$\min_{y_2 \in Y} f_2(x, y_1, y_2, z) = x + y_1 - 2y_2 + z$$
subject to
$$x \leq 1$$
$$y_1 \leq 1$$
$$y_2 \leq 1$$
$$z \leq 1$$
$$x = 1$$
$$y_1 \geq 0$$
$$y_2 \geq 0$$
$$z \geq 0$$

Using the bounded simplex method, we have $\tilde{y}_{2i} = 1$ and $\tilde{y}_{2i} = y_{2[i]}$.
Setting $i \leftarrow i+1$, we have

$$\min_{z \in Z} f_3(x, y_1, y_2, z) = 2x - y_1 - y_2 + 2z$$
subject to
$$x \leq 1$$
$$y_1 \leq 1$$
$$y_2 \leq 1$$
$$z \leq 1$$
$$x = 1$$
$$y_1 \geq 0$$
$$y_2 \geq 0$$
$$z \geq 0$$

Using the bounded simplex method, we have $\tilde{z}_i = 1$ and $\tilde{z}_i = z_{[i]}$. Therefore the extreme point $(x_{[i]}, y_{1[i]}, y_{2[i]}, z_{[i]}) = (1, 1, 1, 0)$ is the global solution of example given above.
By examining above procedure, we found that the solution occurs at the point $(x^*, y_1^*, y_2^*, z^*) = (1, 1, 1, 0)$ with $F^* = -5$, $f_1^* = 0$, $f_2^* = 0$ for the given example.

2.6.3 An Extended Kuhn-Tucker Approach for the Uncooperative Linear Bilevel Multifollower Decision Problem

The basic concept to deal with the uncooperative linear BLMF decision problems is that it replaces each follower′s problem with its Kuhn-Tucker conditions and include the KKT conditions to the leader′s problem as constraint. The reformulation of the linear BLMF problem is a standard mathematical program and relatively easy to solve because all but complementary constraints are linear. Omitting or relaxing the constraints leaves a standard linear program that can be solved by using a simplex method [7].

In an uncooperative situation, the leader will be required to first select an $x \in X \subset \Re^n$ in attempting to minimize his/her objective subject to constraints of both the upper and each follower at the lower level. It then defines each follower′s rational reaction set simultaneously by selecting the individual variable $y_i \in Y_i \subset \Re^{m_i}$ from his/her feasible set to minimize his/her objective function for the leader′s choice.

The Kuhn-Tucker approach is the most popular procedure for solving single-leader-single-follower decision problems. Based on the definition of an optimal solution [38], an extended Kuhn-Tucker approach for the uncooperative BLMF decision problem is proven and described as follows.

Let us first consider a linear programming (LP) problems written as:

$$\min f(x) = cx$$
$$\text{subject to } \begin{cases} Ax \geq b \\ x \geq 0 \end{cases} \tag{2.14}$$

where c is an n-dimensional row vector, b an m-dimensional column vector, A an $m \times n$ matrix with $m \leq n$ and $x \in \Re^n$.

Let $\lambda \in \Re^m$ and $\mu \in \Re^n$ be the dual variables associated with constraints $Ax \geq b$ and $x \geq 0$ respectively. Bard (Bard 1998) gave the following proposition.

Proposition 2.1. *A necessary and sufficient condition that(x^*) solves the above-mentioned LP is that there exist row vectors λ^*, μ^* such that (x^*, λ^*, μ^*) solves:*

$$\begin{aligned} \lambda A - \mu &= -c \\ Ax - b &\geq 0 \\ \lambda(Ax - b) &= 0 \\ \mu x &= 0 \\ x \geq 0, \lambda \geq 0, \mu &\geq 0 \end{aligned} \tag{2.15}$$

Proof: See (Bard 1998, PP. 59-60)
Let $u_i \in \Re^p$, $v_i \in \Re^{q_i}$ and $w_i \in \Re^{m_i}$ (i=1, 2, ..., k) be the dual variables associated with constraints $Ax + \sum_{t=1}^{k} B_t Y_t \leq b, A_i x + C_i y_i \leq b_i, i = 1, 2, ..., k$ and $y_i \geq 0$ (i=1, ..., k)respectively. We have the following theorem.

Theorem 2.6. *A necessary and sufficient condition that $(x^*, y_1^*, ..., y_k^*)$ solves the linear BLMF problem (2.1) is that there exist (row) vectors, $u_1^*, u_2^*, ..., u_k^*$, $v_1^*, v_2^*, ..., v_k^*$ and $w_1^*, w_2^*, ..., w_k^*$ such that $(x^*, y_1^*, ..., y_k^*, u_1^*, ..., u_k^*, v_1^*, ..., v_k^*, w_1^*, ..., w_k^*)$ solve:*

$$\min F(x, y_1, ..., y_k) = cx + \sum_{s=1}^{k} d_s y_s \tag{2.16}$$

subject to

$$Ax + \sum_{t=1}^{k} B_t y_t \le b \tag{2.17}$$

$$A_i x + C_i y_i \le b_i \tag{2.18}$$

$$u_i B_i + v_i C_i - w_i = -e_i \tag{2.19}$$

$$v_i(b - Ax - \sum_{t=1}^{k} B_t y_t) + v_i(b_i - A_i x - C_i y_i) + w_i y_i = 0 \tag{2.20}$$

$$x \ge 0, y_i \ge 0, u_i \ge 0, v_i \ge 0, w_i \ge 0, i = 1, 2, ..., k \tag{2.21}$$

Proof. Let us get an explicit expression of (2.4)
Rewrite (2.4) as follows: □

$$\min F(x, y_1, ..., y_k)$$
$$\text{subject to } (x, y_1, ..., y_k) \in IR$$

We have

$$\min F(x, y_1, ..., y_k)$$
$$\text{subject to } (x, y_1, ..., y_k) \in S$$
$$y_i = p_i(x)$$

where $i = 1, 2, ..., k$ by Definition $(2.5e)$. Then, we have

$$\min F(x, y_1, ..., y_k)$$
$$\text{subject to } (x, y_1, ..., y_k) \in S$$
$$y_i \in arg \min[f_i(x, y_i) : y_i \in S_i(x)]$$

where $i = 1, 2, ..., k$ by Definition $(2.5d)$. We rewrite it as

$$\min F(x, y_1, ..., y_k)$$
$$\text{subject to } (x, y_1, ..., y_k) \in S$$

$$\min f_i(x, y_i)$$
$$\text{subject to } y_i \in S_i(x),\ \forall i = 1, 2, ..., k$$

Where $i = 1, 2, ..., k$, we have

$$\min F(x, y_1, ..., y_k)$$
$$\text{subject to } (x, y_1, ..., y_k) \in S$$
$$\min_{x, y_i \in Y_i} f_i(x, y_i)$$
$$\text{subject to } (x, y_1, ..., y_k) \in S$$

where $i = 1, 2, ..., k$ by Definition $(2.5c)$. Consequently, we can have

$$\min F(x, y_1, ..., y_k) = cx + \sum_{s=1}^{k} d_s y_s \tag{2.22}$$

subject to

$$Ax + \sum_{t=1}^{k} B_t y_t \leq b \tag{2.23}$$

$$A_j x + C_j y_j \leq b_j, j = 1, 2, ..., k \tag{2.24}$$

$$\min_{x, y_i \in Y_i} f_i(x, y_i) = c_i x + e_i y_i \tag{2.25}$$

subject to

$$Ax + \sum_{t=1}^{k} B_t y_t \leq b \tag{2.26}$$

$$A_j x + C_j y_j \leq b_j, j = 1, 2, ..., k \tag{2.27}$$

where $i = 1, 2, ..., k$ by Definition $(2.5a)$

This simple transformation has shown that to solve the linear BLMFP problem (2.4) is equivalent to solve the problem $(2.17 - 2.22)$.

1). Necessity is obvious from $(2.17 - 2.22)$.

2). Sufficiency.

If $(x^*, y_1^*, ..., y_k^*)$ is an optimal solution of (2.4), we need to show that there exist (row) vectors
$u_1^*, u_2^*, ..., u_k^*, v_1^*, v_2^*, ..., v_k^*$ and $w_1^*, w_2^*, ..., w_k^*$ such that$(x^*, y_1^*, ..., y_k^*, u_1^*, ..., u_k^*, v_1^*, ..., v_k^*, w_1^*, ..., w_k^*)$ solve (2.11-2.16).

Going one step further, we only need to prove that there exist (row) vectors $u_1^*, ..., u_k^*$, $v_1^*, ..., v_k^*$ and $w_1^*, ..., w_k^*$ such that $(x^*, y_1^*, ..., y_k^*, u_1^*, ..., u_k^*, v_1^*, ..., v_k^*, w_1^*, ..., w_k^*)$ satisfies the following conditions:

$$u_i B_i + v_i C_i - w_i = -e_i \tag{2.28}$$

$$u_i(b - Ax - \sum_{t=1}^{k} B_t y_t) = 0 \tag{2.29}$$

$$v_i(b_i - A_i x - C_i y_i) + w_i y_i = 0 \tag{2.30}$$

$$w_i y_i = 0 \tag{2.31}$$

where $u_i \in \Re\Re^p$, $v_i \in \Re^{q_i}$, and $w_i \in \Re^{m_i}$, $i = 1, 2, ..., k$ they are not negative variables.

Because $(x^*, y_1^*, ..., y_k^*)$is an optimal solution of (2.4), we have

$(x^*, y_1^*, ..., y_k^*) \in IR$

by (2.5). Thus we have
$(x^*, y_1^*, ..., y_k^*)$ where $i = 1, 2, ..., k$ by Definition (2.5e) Consequently, $(x^*, y_1^*, ..., y_k^*)$ is an optimal solution to the following problem:

$$\min(f_i(x, y_i) : y_i \in S_i(x^*)) \tag{2.32}$$

where $i = 1, 2, ..., k$ by Definition (2.5d). Rewrite (2.27) as follows

$$\begin{aligned} \min f(x, y_i) \\ \text{subject to } y \in S_i(x) \\ x = x^* \\ y_j = y_j^*(j = 1, 2, ..., k, i \neq j) \end{aligned} \tag{2.33}$$

where $i = 1, 2, ..., k$ From Definition (2.5c), we have

$$\min f(x, y_i) = c_i x + e_i y_i \text{subject to} \tag{2.34}$$

$$Ax + \sum_{t=1}^{k} B_t y_t \leq b \tag{2.35}$$

$$A_j x + C_j y_j \leq b_j, j = 1, 2, ..., k \tag{2.36}$$

$$x = x^* \tag{2.37}$$

$$y_i \geq 0 \tag{2.38}$$

$$y_j = y_j^*(j = 1, 2, ..., k, i \neq j) \tag{2.39}$$

where $i = 1, 2, ..., k$ Thus simplifying $(2.29 - 2.34)$, we can have

$$\begin{aligned} \min f_i(y_i) = e_i y_i \\ \text{subject to} \end{aligned} \tag{2.40}$$

$$\begin{pmatrix} B_i \\ C_i \end{pmatrix} y_i \leq \begin{pmatrix} b - Ax^* - \sum_{j=1, j\neq 1}^{k} B_j y_j^* \\ b_i - A_i x^* \end{pmatrix} \tag{2.41}$$

$$y_i \geq 0 \tag{2.42}$$

where $i, 2, ..., k$. Now we see that y_i^* is an optimal solution of (2.40-2-42) which is a LP problem. By Proposition(2.1), there exist vectors $\lambda_i^*, \mu_i^*,\ i = 1, 2, ..., k$ that satisfy the following system:

$$\begin{pmatrix} B_i \\ C_i \end{pmatrix} \lambda_i - \mu_i = -e_i \tag{2.43}$$

$$\begin{pmatrix} B_i \\ C_i \end{pmatrix} (-\lambda_i y_i) + \begin{pmatrix} b - Ax^* - \sum_{j=1, j\neq 1}^{k} B_j y_j^* \\ b_i - A_i x^* \end{pmatrix} \lambda_i = 0 \tag{2.44}$$

$$\mu_i y_i = 0 \tag{2.45}$$

Where $\lambda_i \in \Re^{p+q_i}$, $\mu_i \in \Re^{m_i}$, i = 1, 2, ..., k

Let $u_i \in \Re^p$, $v_i \in \Re^{q_i}$, $w_i \in \Re^{m_i}$ and define

$\lambda_i = (u_i, v_i)$

$w_i = \lambda_i$

where $i = 1, 2, ..., k$ Thus we have $(x^*, y_1^*, ..., y_k^*, u_1^*, ..., u_k^*, v_1^*, ..., v_k^*, w_1^*, ..., w_k^*)$ that satisfy (2.23-2.26). Our proof is completed.

Theorem (2.6) indicates that the most direct approach for solving (2.4) is to solve the equivalent mathematical program given in (2.11-2.16). One of its advantages is that it allows a more robust model to be solved without introducing any new computational difficulties.

2.6.4 Numeric Example for KKT Approach to Solve Single-Leader-Single-Follower Problem

Example 2.3. *Consider the following linear BLP problem with $x \in \Re^1$, $y \in \Re^1$, and $X = \{x \geq 0\}$, $Y = \{y \geq 0\}$*

$$\min_{x \in X} F(x, y) = x - 4y$$

$$\textit{subject to} \begin{cases} x + y \geq 3 \\ -3x + 2y \geq -4 \end{cases} \quad \textit{where } y \textit{ solves} \begin{cases} \min_{y \in Y} f(x, y) = x + y \\ \textit{subject to} \begin{cases} -2x + y \leq 0 \\ 2x + y \leq 12 \end{cases} \end{cases} \tag{2.46}$$

According to our approach, let us write all the inequalities but $x \geq 0$ of Example (2.3)as follows

$$g_1(x, y) = x + y - 3 \geq 0 \tag{2.47}$$

$$g_2(x, y) = -3x + 2y + 4 \geq 0 \tag{2.48}$$

$$g_3(x, y) = 2x - y \geq 0 \tag{2.49}$$

$$g_4(x, y) = -2x - y + 12 \geq 0 \tag{2.50}$$

$$g_5(x, y) = y \geq 0 \tag{2.51}$$

From theorem (2.6) *we have the following:*

$$\begin{aligned} &\min F(x, y) = x - 4y \\ &\textit{subject to} \end{aligned} \tag{2.52}$$

$$-x - y \leq -3 \tag{2.53}$$

$$-3x + 2y \geq -4 \tag{2.54}$$

$$-2x + y \leq 0 \tag{2.55}$$

$$2x + y \leq 12 \tag{2.56}$$

$$-u_1 - 2u_2 + u_3 + u_4 - u_5 = -1 \tag{2.57}$$

$$u_1 g_1(x,y) + u_2 g_2(x,y) + u_3 g_3(x,y) + u_4 g_4(x,y) + u_5 g_5(x,y) = 0 \tag{2.58}$$

$$x \geq 0, y \geq 0, u_1 \geq 0, u_2 \geq 0, u_3 \geq 0, u_4 \geq 0, u_5 \geq 0 \tag{2.59}$$

From (2.57) and (2.59) we can have the following three possibilities

Case 1$(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*) = (1, 0, 0, 0, 0)$

Case 2$(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*) = (0, 0.5, 0, 0, 0)$

Case 3$(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*) = (0, 0, 0, 0, 1)$

From case1, (2.58) and (2.47) we have $g_1(x,y) = x + y - 3 = 0$

Consequently, (2.52)can be rewritten as follows:

$$\begin{aligned} &\min F(x,y) = x - 4y \\ &\textit{subject to } -x - y = -3 \\ &-3x + 2y \geq -4 \\ &-2x + y \leq 0 \\ &2x + y \leq 12 \\ &x \geq 0, y \geq 0 \end{aligned} \tag{2.60}$$

Using simplex algorithm, a solution occurs at the point $(x^*, y^*) = (1, 2)$ *with* $F^* = -7$ *and* $f^* = 3$*. By using the same way as that of Case 1, we found that a solution occurs at the point* $(x^*, y^*) = (4, 4)$ *with* $F^* = 12$ *and* $f^* = 8$ *for Case 2; it is infeasible for Case 3. By examining above procedure, we found that the optimal solution occurs at the point* $(x^*, y^*) = (4, 4)$ *with* $F^* = -12$ *and* $f^* = 8$*.*

Chapter 3

Summary And Conclusion

The bilevel optimization problems belong to the class of NP-hard problems, which mean that no polynomial time algorithm exist for solving it unless P = NP[39]. They can be described as an optimization problem constrained by another optimization problem is called a bilevel programming problem (BLPP). From the mathematical point of view it is a problem with hierarchical structure where two independent decision-makers are involved. Nowadays, this problem becomes useful to define problems in an engineering, automatic control, maximizing production in chemical industries where reactions occur under equilibrium and structure design. Other applications also include, optimal shape design, optimal operating configuration, optimal facility location and solving road network problem. Unfortunately, these problems are known to often be very hard to solve in practice. Wide applications of LBLP Problems are practicable in economics like Principal-Agency problems, Taxation policies etc. In general some of the basic particular areas that can solved by linear bilevel programming problems(LBLPP) are:

Economic Planning at the Regional or National Level, which mean the leader will control policy variables e.g. tax rates, import quotas in economic planning in order to maximize employment/minimize the use of a resource and the follower also tries to optimize the net income of economic and governmental constraints. Determining Price Support Levels for Biofuel Crops: At this time the leader will determine the level of tax credits for each biofuel product that means he/she minimizes total cost and the follower which is the Petro-chemical industry minimize costs. Here the leader(Central resource supplier) allocates products to manufacturers and maximize the profit of firm as a whole. The follower(Manufacturing facilities at different locations) also optimizes his/her own production mix/output and maximize performance of his/her own unit.

Transportation System Network Design: The leader(Central planner) Controls investment costs and influence users′ preferences to minimize total costs and the follower(Individual users) determines the traffic flows, operational costs and Seek to minimize cost of his/her own route. Generally, for all these problems we can find an optimal solution even if there is a difficulty of solving such a problem due to the non uniqueness solution of the lower level problem and non convexity of the complementary slackness condition. In this project, two basic solution procedueres are used to find the optimal solution of (LBLMFP) problems. Therefore bilevel programming problem (BLPP) is valid only in the case when the lower level solution is uniquely determined.

If the lower level problem has at most one solution then this leads to the uncertainty definition of bilevel programming problems. The optimal solution of the lower level problems will be the feasible solution of the upper level problem. An optimal solution of the linear bilevel programming problem can be determined from the vertex point of the polyhedral graph which is the intersection of the constraints of the leader and the follower if the lower level problem optimal solution is uniquely determine. Moreover if the order of the leader and the follower objectives and constraints are symmetric, the solution of the (LBLMFP) problem may or may not be in existence.

Additionally, if the lower level problem is not convex then the optimization problem after replacing the lower level problem by KKT has a larger feasible set. And this makes the problem to be difficult to be solved. Because the original problem before we replace the lower level problem by KKT and after replacing the lower level problem by KKT may not be equivalent. And hence to get an optimal solution for (LBLMFP) the order in which decisions are made is important. That means, the roles of the leader and the follower are NOT interchangeable(problem should not be symmetric).

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