TRIDENT PAIR PRODUCTION IN STRONG LASER PULSES

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the Degree of Master of Science in Physics

By

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Abstract

We make use of Volkov states of electron and positron for trident pair production. The periodic plane wave case shows an infinite sum over photon number in the calculations and leads to unphysical cross section. We calculate the trident pair production rate and cross section in strong laser pulses, treating nonperturbatively in strong-field QED. We then arrive at a finite and physical results through shaping of those fields into laser pulses. With the help of the optical theorem, we make the S-matrix not to diverge and precisely identify the one-step and two-step processes.

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Chapter 1

Introduction

The modification of the perturbative QED in strong electromagnetic fields was initiated soon after the invention of the laser, leading to the theory of strong-field QED, also known as laser dressed-QED. The creation of electron-positron pairs in very strong laser fields was investigated by theoretician already in the 1960s and 1970s. The interest in laser induced pair creation processes has been strongly revived in recent years due to the ongoing increase in the available laser intensities and, particularly due to the experiment E-144, which for the first time provided the laboratory proof of the multiphoton particle antiparticle pair production [1].

The program of SLAC experiment E-144 had three aspects [2];

1. Measurement of the longitudinal polarization of the electron beam via observation of an asymmetry in Compton scattering,

2. Observation of nonlinear Compton scattering,

$$e + N\omega \;\rightarrow\; e' + \omega_0, \tag{1.0.1}$$

   up to fourth order (N = 4).

3. Observation of positrons created in the collision of Compton backscattered photons with a laser beam in the multiphoton Breit-Wheeler reaction,

$$\omega_0 + N\omega \;\rightarrow\; e^+ + e^-, \tag{1.0.2}$$

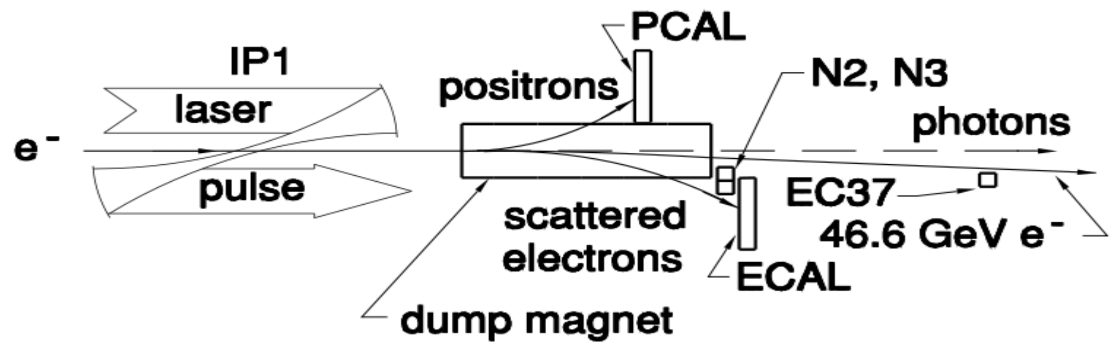


Figure 1.1: Schematic layout of the SLAC E-144 experiment [2].

where high energy photons, radiated by the electrons combined with photons in the laser to produce pairs [3,4]. In that experiment, a process called 'trident' can also have produced pairs through the Bethe-Heitler reaction

$$e \;+\; N\omega \;\rightarrow\; e' \;+\; e^+e^- \tag{1.0.3}$$

However, its contribution could only be evaluated approximately, since no exact expression for the trident amplitude was not available [4].

Experiment E-144 was performed in the Final Focus Test Beam (FFTB) at SLAC with teraWatt pulses from a frequency-doubled Nd:glass laser with a repetition rate of 0.5 Hz achieved by a final laser amplifier with slab geometry. A schematic diagram of the experiment is shown in Fig.(1.1). The apparatus was designed to detect electrons that undergo nonlinear Compton scattering, reaction (1.0.1), as well as positrons produced in electron-laser interactions by the two-step process of reaction (1.0.1) followed by reaction (1.0.2). A laser beam of $10^{18}\ W/cm^2$ optical laser pulse was focused onto electron beam of energy 46.6 GeV by an off-axis parabolic mirror of 30 cm focal length with a 17^0 crossing angle at the interaction point.

In this thesis we have done all calculations using the light cone coordinates and light cone Dirac matrices defined as follows:

$$a_+ = \frac{1}{2}(a_0 + a_3) \tag{1.0.4}$$

$$a_- = \frac{1}{2}(a_0 - a_3) \tag{1.0.5}$$

$$a_\perp = (a_1, a_2) \tag{1.0.6}$$

With this, we have put the dot product of two four-vectors as

$$a.b = 2a_+b_- + 2a_-b_+ - a_\perp b_\perp \tag{1.0.7}$$

$$\gamma_+ = \gamma^0 + \gamma^3 \tag{1.0.8}$$

$$\gamma_- = \gamma^0 - \gamma^3 \tag{1.0.9}$$

$$\gamma_\perp = (\gamma_1, \gamma_2) \tag{1.0.10}$$

The Feynman slash matrix is therefore has been used as

$$\not{a} = \gamma.a = \gamma_+a_- - \gamma_-a_+ - \gamma_\perp a_\perp \tag{1.0.11}$$

Light cone Dirac matrices satisfy the following properties

$$\gamma_\pm\gamma_\pm = 0 \tag{1.0.12}$$

$$\gamma_\pm\gamma_\mp = 2\gamma_0\gamma_\mp \tag{1.0.13}$$

$$\gamma_0\gamma_\pm = \gamma_\mp\gamma_0 \tag{1.0.14}$$

$$\gamma_\pm\gamma_\perp = -\gamma_\perp\gamma_\pm \tag{1.0.15}$$

The gradient operators in terms of the light cone coordinates have been expressed as

$$\partial_+ = \partial_0 + \partial_3 \tag{1.0.16}$$

$$\partial_- = \partial_0 - \partial_3 \tag{1.0.17}$$

$$\partial_\perp = (\partial_1, \partial_2) \tag{1.0.18}$$

Through out the thesis natural units are used, setting $\hbar = c = 1$ with $\hbar$ being the reduced Planck constant and c the speed of light in vacuum. The metric tensor $g_{\mu\nu}$ has been used with (1,-1,-1,-1) diagonal entries and μ, ν takes values (0,1,2,3). This thesis is organized as follows: Chapter 2 deals about strong-field QED and Volkov states of electron and positron. Multiphoton trident pair production in periodic plane waves is presented in chapter 3. Chapter 4 reveals the trident pair production in strong laser pulses, where we present finite laser pulses and the optical theorem. Finally, the conclusion of what we have done in this thesis is given in chapter 5.

Chapter 2

Strong Field QED and Volkov States

2.1 Strong Field QED

In QED an electric field, E, should be treated as strong if it exceeds the Schwinger limit: $E \geq E_S = m_e c^2/(e\lambda_C) = 1.3 \times 10^{16}\ V/cm$. Such field is potentially capable of separating a virtual electron-positron pair providing an energy, which exceeds the electron rest mass energy, $m_e c^2$, to a charge, $e = |e|$, over an acceleration length as small as the Compton wavelength, $\lambda_C = \frac{\hbar}{m_e c} \approx 3.9 \times 10^{11}\ cm$. The most typical effects in QED strong fields are electron-positron pair creation from high-energy photons and high-energy photon emission from electrons or positrons [3]. Strong-field QED also known as Laser-dressed QED uses Volkov functions instead of plane wave functions to represent the particles in initial and final states, as well as to construct the particle propagator [1,5]. In an external plane wave field $A(k.x)$, the free electron propagator is given by [1,4].

$$G_{free}(x_1, x_0) = \frac{1}{(2\pi)^4} \int d^4p \frac{\not{p} + m}{p^2 - m^2 + i\xi}\ exp\,(ip(x_1, x_0)) \tag{2.1.1}$$

The QED Lagrangian reads

$$L = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{2.1.2}$$

where the electromagnetic field strength tensor $F_{\mu\nu}$ is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.1.3}$$

with A_μ the photon field and the covariant derivative $D_\mu = \partial_\mu - ieA_\mu$.

An external laser field $\mathcal{A}_\mu$ can be incorporated inside the covariant derivative D_μ [6] as

$$D_\mu = \partial_\mu - ieA_\mu - ie\mathcal{A}_\mu \tag{2.1.4}$$

where A_μ is the electromagnetic photon field.

The added term modifies the electron propagator (2.1.1), which becomes dressed by the laser field, to the so-called Dirac-Volkov propagator as [1]

$$G(x_1, x_0) = \frac{1}{(2\pi)^4} \int d^4x \left(1 - \frac{e\not{k}\not{\mathcal{A}}}{2k.p}\right) exp\left(if(x_1)\right) \frac{\not{p} + m}{p^2 - m^2 + i\xi} \left(1 - \frac{e\not{\mathcal{A}}\not{k}}{2k.p}\right) exp\left(-if(x_0)\right) \tag{2.1.5}$$

with the phase term

$$f_\pm = \pm p.x - \int_{-\infty}^{y} dy' \frac{1}{2k.p} \left(2ep.\mathcal{A}(y') \pm e^2\mathcal{A}^2(y')\right) \tag{2.1.6}$$

and

$$y = k.x$$

The Dirac-Volkov propagator satisfies

$$\left(\gamma_\mu(p^\mu - e\mathcal{A}^\mu) - m\right) G(x_1, x_0) = \delta(x_1, x_0) \tag{2.1.7}$$

and can be expanded in terms of Bessel functions as

$$G(x_1, x_0) = \frac{1}{(2\pi)^4} \int d^4p \sum_{n,n'} \left(B_n(\alpha, \beta) - \frac{ea\not{k}\not{\epsilon}}{2k.p} C_n(\alpha, \beta)\right) \frac{\not{p} + m}{p^2 - m^2 + i\xi} \left(B_{n'}(\alpha, \beta) - \frac{ea\not{\epsilon}\not{k}}{2k.p} C_{n'}(\alpha, \beta)\right) exp\left(-iq(x_1, x_0) + ik.(nx_1 - n'x_0)\right) \tag{2.1.8}$$

In principle the free photon propagator, which is used as a very good approximation, should also be replaced by the full photon propagator, which contains the effects of laser-photon interactions [1]. In the next section we will introduce Volkov states.

2.2 Volkov States

In this section we want to solve the Dirac equation

$$\left(i\gamma_\mu\partial_\mu - e\gamma_\mu\mathcal{A}_\mu - m\right)\psi = 0 \tag{2.2.1}$$

in an external electromagnetic field with potential $\mathcal{A}_\mu$, moving with the velocity of light in a fixed direction, specified by the wave vector $\overrightarrow{k}$. The potential $\mathcal{A}_\mu$ is assumed to depend on the space-time coordinates x through the scalar product $\varphi = k.x$ as

$$\mathcal{A}_\mu = \mathcal{A}_\mu(\varphi) \tag{2.2.2}$$

Moreover, we will extend the general Volkov solution to the special case of linearly polarized laser field solution.

We can rewrite Eq(2.2.1), by separating the space and time parts as

$$\left[\gamma_0\partial_0 + \gamma_i\partial_i - e(\gamma_0\mathcal{A}_0 - \gamma_i\mathcal{A}_i) - m\right]\psi = 0 \tag{2.2.3}$$

We now use the technique of light cone variables, which is useful in problems involving motion at the speed of light [7].

Thus, in light cone coordinates Eq(2.2.3) becomes

$$\begin{aligned}&\left\{i\left[\frac{1}{2}(\gamma_+ + \gamma_-)\frac{1}{2}(\partial_+ + \partial_-) + \frac{1}{2}(\gamma_+ + \gamma_-)\frac{1}{2}(\partial_+ - \partial_-) + \gamma_\perp\partial_\perp\right]\right.\\ &\left. + e\left[\frac{1}{2}(\gamma_+ + \gamma_-)(\mathcal{A}_+ + \mathcal{A}_-) - \frac{1}{2}(\gamma_+ - \gamma_-)(\mathcal{A}_+ - \mathcal{A}_-) - \gamma_\perp\mathcal{A}_\perp\right] - m\right\}\psi = 0\end{aligned} \tag{2.2.4}$$

Simplifying terms we get

$$\left[i\frac{1}{2}\gamma_+\partial_+ + i\frac{1}{2}\gamma_-\partial_- + i\gamma_\perp\partial_\perp - e(\gamma_+\mathcal{A}_- + \gamma_-\mathcal{A}_+ - \gamma_\perp\mathcal{A}_\perp) - m\right]\psi = 0 \tag{2.2.5}$$

We can now assume with out loss of generality that the wave moves in the x_3 direction. Thus,

$$k_\mu = \omega(1, 0, 0, 1), \tag{2.2.6}$$

which implies that

$$k_+ = \omega \neq 0 \tag{2.2.7}$$

and

$$k_- = k_\perp = 0 \tag{2.2.8}$$

The Lorentz gauge condition is given by

$$k.\mathcal{A} = 2k_+\mathcal{A}_- + 2k_-\mathcal{A}_+ - k_\perp\mathcal{A}_\perp = 0 \tag{2.2.9}$$

which immediately implies that

$$\mathcal{A}_- = 0 \tag{2.2.10}$$

The Dirac equation then reads

$$\left[i\frac{1}{2}\gamma_+\partial_+ + i\frac{1}{2}\gamma_-\partial_- + i\gamma_\perp\partial_\perp - e\gamma_-\mathcal{A}_+ + e\gamma_\perp\mathcal{A}_\perp - m\right]\psi = 0 \tag{2.2.11}$$

The potential is assumed to move with the velocity of light in the x_3 direction and will only depend on the variable x_- [7]. Thus,

$$\mathcal{A}_+ = \mathcal{A}_+(x_-) \quad and \quad \mathcal{A}_- = \mathcal{A}_-(x_-) \tag{2.2.12}$$

The motion of the electron in the x_+ and $x_\perp$ directions can be described by a plane wave

$$\psi(x) = \psi(x_+, x_-, x_\perp) = N\phi(x_-)exp(-ip.x), \tag{2.2.13}$$

where N is the normalization constant.

Upon substituting Eq(2.2.13) into Eq(2.2.11), there follows

$$\left[i\frac{1}{2}\gamma_+\partial_+ + i\frac{1}{2}\gamma_-\partial_- + i\gamma_\perp\partial_\perp - e\gamma_-\mathcal{A}_+ + e\gamma_\perp\mathcal{A}_\perp - m\right]$$

$$exp\ i(2p_+x_- + 2p_-x_+ - p_\perp x_\perp)\ \phi(x_-) = 0, \tag{2.2.14}$$

which leads to

$$\Big[-\frac{1}{2}\gamma_+(2p_-) - \gamma_-(2p_+) + e\gamma_-\mathcal{A}_+ + e\gamma_\perp.\mathcal{A}_\perp - m\Big]$$

$$exp\ i(2p_+x_- + 2p_-x_+ - p_\perp x_\perp)\ \phi(x_-)$$

$$+\Big[i\frac{1}{2}\gamma_-\partial_-\phi(x_-)\ exp\ i(2p_+x_- + 2p_-x_+ - p_\perp x_\perp)\Big] = 0 \qquad (2.2.15)$$

Simplifying terms we finally obtain

$$\Big[i\frac{1}{2}\gamma_-\partial_- - \gamma_+p_- - \gamma_-p_+ - \gamma_\perp p_\perp - e\gamma_-\mathcal{A}_+ + e\gamma_\perp\mathcal{A}_\perp - m\Big]\phi(x_-) = 0 \qquad (2.2.16)$$

Splitting the wave function into its light cone projections we have

$$\phi = \frac{1}{2}(\phi_+ + \phi_-) \quad with \quad \phi_\pm = \gamma_0\gamma_\pm\phi, \qquad (2.2.17)$$

satisfying

$$\gamma_\pm\phi_\mp = 0 \quad and \quad \gamma_\pm\phi_\pm = 2\gamma_0\phi_\pm \qquad (2.2.18)$$

The Dirac equation (2.2.16) upon substitution of Eq(2.2.17) and Eq(2.2.18) becomes

$$\Big[\gamma_+p_- - \gamma_\perp(p_\perp - e\mathcal{A}_\perp) - m\Big]\phi_+$$

$$+\Big[i\frac{1}{2}\gamma_-\partial_- + \gamma_-(p_+ - e\mathcal{A}_+) - \gamma_\perp(p_\perp - e\mathcal{A}_\perp) - m\Big]\phi_- = 0 \qquad (2.2.19)$$

Multiplying Eq(2.2.19) from left by γ_- and applying the relations in Eq(1.0.12), Eq(1.0.13), Eq(1.0.14), and Eq(1.0.15) we obtain

$$\Big[2\gamma_0\gamma_+p_-\Big]\phi_+ + \Big[2\gamma_\perp\gamma_0(p_\perp - e\mathcal{A}_\perp) - 2\gamma_0 m\Big]\phi_- = 0 \qquad (2.2.20)$$

This can in turn be expressed as

$$\Big[2\gamma_0\gamma_+p_-\Big]\phi_+ = 2\gamma_0\Big[\gamma_\perp(p_\perp - e\mathcal{A}_\perp) + m\Big]\phi_- \qquad (2.2.21)$$

We can write ϕ_+ in terms of ϕ_- as

$$\phi_+ = \frac{\gamma_0}{2p_-}\Big[\gamma_\perp(p_\perp - e\mathcal{A}_\perp) + m\Big]\phi_- \qquad (2.2.22)$$

Now we multiply Eq(2.2.19) from left by γ_+ as

$$\Big[i\partial_- + 2(p_+ - e\mathcal{A}_+)\Big]\phi_- + \Big[\gamma_\perp(p_\perp - e\mathcal{A}_\perp)\gamma_0 - m\gamma_0\Big]\frac{\gamma_0}{2p_-}\Big[\gamma_\perp(p_\perp - e\mathcal{A}_\perp) + m\Big]\phi_- = 0 \quad (2.2.23)$$

This is reduced to

$$\Big[i\partial_- - \Big(- 4p_+p_- + 4e\mathcal{A}_+p_- + (p_\perp - e\mathcal{A}_\perp)^2 + m^2\Big)\Big]\phi_- = 0, \quad (2.2.24)$$

where we used the relations

$$\gamma_\perp\gamma_0 = -\gamma_0\gamma_\perp \quad and \quad (\gamma_\perp.\mathcal{A}_\perp)^2 = -\mathcal{A}_\perp^2 \quad (2.2.25)$$

Now the square of the field is given by

$$\mathcal{A}^2 = 4\mathcal{A}_+\mathcal{A}_- - \mathcal{A}_\perp^2 = -\mathcal{A}_\perp^2 \quad (2.2.26)$$

Thus,

$$\Big[i\partial_- - \frac{1}{2p_-}\Big(- 4p_+p_- + 4e\mathcal{A}_+p_- + p_\perp^2 + e^2\mathcal{A}_\perp^2 - 2ep_\perp\mathcal{A}_\perp + m^2\Big)\Big]\phi_- = 0 \quad (2.2.27)$$

Now we write

$$p^2 = 4p_+p_- - p_\perp^2 \quad (2.2.28)$$

and Eq(2.2.27) becomes

$$\Big[i\partial_- - \frac{1}{2p_-}(4e\mathcal{A}_+p_- - 2ep_\perp\mathcal{A}_\perp - e^2\mathcal{A}^2)\Big]\phi_- = 0, \quad (2.2.29)$$

where we have used the condition

$$p^2 = m^2 \quad (2.2.30)$$

Substituting

$$2\mathcal{A}.p = 4\mathcal{A}_+p_- - 2\mathcal{A}_\perp p_\perp \quad (2.2.31)$$

into Eq(2.2.29), we arrive at the most simplified Dirac equation

$$\Big[i\partial_- - \frac{1}{2p_-}(2e\mathcal{A}.p - e^2\mathcal{A}^2)\Big]\phi_- = 0 \quad (2.2.32)$$

Solution of Eq(2.2.32) is

$$\phi_-(x_-) = \phi_0 \; exp\,(-i\Phi(x_-)) \tag{2.2.33}$$

with the phase

$$\Phi_-(x_-) = \int_0^{x_-} \left(\frac{e\mathcal{A}.p}{p_-} - \frac{e^2\mathcal{A}^2}{2p_-} \right) \tag{2.2.34}$$

and ϕ_0 is a constant spinor satisfying

$$\gamma_+\phi_0 = 0 \tag{2.2.35}$$

We can choose it to be

$$\phi_0 = \gamma_0\gamma_- u(p), \tag{2.2.36}$$

$u(p)$ being a unit spinor satisfying the free Dirac equation

$$(\gamma.p - m)u(p) = 0 \tag{2.2.37}$$

We substitute Eq(2.2.22) and Eq(2.2.33) into Eq(2.2.17) and obtain

$$\begin{aligned}
\phi &= \frac{1}{2}\left[1 + \frac{1}{2p_0}\gamma_0\gamma_\perp(p_\perp - e\mathcal{A}_\perp) + m\right]\gamma_0\gamma_- u(p)exp(-i\Phi) \\
&= \frac{1}{2}\left[\gamma_0\gamma_- + \frac{1}{2p_-}\gamma_0\left(\gamma_\perp\gamma_0\gamma_-(p_\perp - e\mathcal{A}_\perp) + \gamma_0\gamma_- m\right)\right]u(p)exp(-i\Phi) \\
&= \frac{1}{2}\left[\gamma_0\gamma_- + \frac{1}{2p_-}\left(-\gamma_\perp\gamma_-(p_\perp - e\mathcal{A}_\perp) + \gamma_- m\right)\right]u(p)exp(-i\Phi) \\
&= \frac{1}{2}\left[\gamma_0\gamma_- + \frac{1}{2p_-}\left(\gamma_-\gamma_\perp(p_\perp - e\mathcal{A}_\perp) + \gamma_- m\right)\right]u(p)exp(-i\Phi)
\end{aligned} \tag{2.2.38}$$

Commuting $\gamma_0\gamma_-$ to the left and using the Dirac equation in the form

$$(\gamma_\perp p_\perp + m)u(p) = (\gamma_- p_+ + \gamma_+ p_-)u(p), \tag{2.2.39}$$

we find

$$\phi = \frac{1}{2}\left[\gamma_0\gamma_- + \frac{1}{2p_-}\gamma_-\left(\gamma_+ p_- + \gamma_- p_+ - e\gamma_\perp\mathcal{A}_\perp\right)\right]u(p)exp(-i\Phi) \tag{2.2.40}$$

which in turn be expressed as

$$\phi = \frac{1}{2}\left[\gamma_0\gamma_- + \gamma_0\gamma_+ - \frac{1}{2p_-}e\gamma_-\gamma_\perp\mathcal{A}_\perp\right]u(p)exp(-i\Phi) \tag{2.2.41}$$

Thus,

$$\phi = \frac{1}{2}\left[1 - \frac{1}{2p_-}e\gamma_-\gamma_\perp\mathcal{A}_\perp\right]u(p)exp(-i\Phi) \tag{2.2.42}$$

where $\gamma_0 = \gamma_- + \gamma_+$.

Now it is possible to show that

$$\gamma_-[\gamma.\mathcal{A}] = \gamma_-[\gamma_+\mathcal{A}_- - \gamma_-\mathcal{A}_+ - \gamma_\perp\mathcal{A}_\perp] = -\gamma_-\gamma_\perp\mathcal{A}_\perp \tag{2.2.43}$$

Therefore, we finally arrive at the solution

$$\phi = \left[1 - \frac{1}{2p_-}e\gamma_-\gamma_\perp\mathcal{A}_\perp\right]u(p)exp(-i\Phi) \tag{2.2.44}$$

Finally, the solution for the Dirac equation in an external laser field becomes

$$\psi(x) = N\left(1 - \frac{1}{4p_-}e\gamma_-\not{\mathcal{A}}\right)u(p)exp\left(-ip.x - i\Phi\right) \tag{2.2.45}$$

The solution found can be immediately generalized to an arbitrary direction of the vector k by replacing p_- and γ_- as,

$$\begin{aligned}\Phi(x_-) &= \int_0^{x_-} dx'_-\left(\frac{e\mathcal{A}.p}{p_-} - \frac{e^2\mathcal{A}^2}{2p_-}\right)\\ &= \int_0^{\varphi} d\varphi\left(\frac{e\mathcal{A}.p}{k.p} - \frac{e^2\mathcal{A}^2}{2k.p}\right),\end{aligned} \tag{2.2.46}$$

where

$$2\omega x_- = k.x \tag{2.2.47}$$

Similarly, we make a substitution on

$$\left(1 - \frac{1}{4p_-}e\gamma_-\not{\mathcal{A}}\right) \to \left(1 - \frac{1}{2k.p}e\omega\gamma_-\not{\mathcal{A}}\right) = \left(1 - \frac{1}{2k.p}e\not{k}\not{\mathcal{A}}\right), \tag{2.2.48}$$

where

$$2\omega p_- = k.p$$

$$\omega\gamma_- = \gamma.k = \not{k} \tag{2.2.49}$$

With the help of Eq(2.2.45), Eq(2.2.46) and Eq(2.2.48) we arrive at the Volkov solution

$$\psi(x) = N\left(1 - \frac{e\not{k}\not{\mathcal{A}}}{2k.p}\right) u(p)\ exp(-ip.x - i\Phi) \tag{2.2.50}$$

The electron bispinor has to be normalized by demanding that

$$\bar{u}_{p,s_1} u_{p,s_2} = 2m\delta_{s_1 s_2} \tag{2.2.51}$$

This leads to the conclusion that the normalizer above is

$$N = \sqrt{\frac{m}{EV}}, \tag{2.2.52}$$

where V is the normalization volume.

Therefore, the complete solution of the Dirac equation for an electron in an electromagnetic field is given by:

$$\psi = \sqrt{\frac{m}{EV}}\left(1 - \frac{e\not{k}\not{\mathcal{A}}}{2k.p}\right) u(p)\ exp(if_-) \tag{2.2.53}$$

with f_- the phase term

$$f_- = -px - \int_0^{\varphi} d\varphi \left(\frac{e(p\mathcal{A})}{(kp)} - \frac{e^2\mathcal{A}^2}{2(kp)}\right) \tag{2.2.54}$$

and $\varphi = k.x$ and E denotes the energy of the particle in the field.

The corresponding equation for positron in an electromagnetic field can be written using the Feynman-Stuckelberg interpretation; the negative-energy particle solutions going backward in time describe positive-energy antiparticle solutions going forward in time as

$$\psi = \sqrt{\frac{m}{EV}}\left(1 + \frac{e\not{k}\not{\mathcal{A}}}{2k.p}\right)\ exp(if_+) \tag{2.2.55}$$

with the phase f_+ given by

$$f_+ = px - \int_0^{\varphi} d\varphi \left(\frac{e(p\mathcal{A})}{(kp)} + \frac{e^2\mathcal{A}^2}{2(kp)}\right) \tag{2.2.56}$$

Here we note that the exact solutions of the Dirac equation for an electron Eq(2.2.53) and positron Eq(2.2.55) in the absence of an electromagnetic field $\mathcal{A}^\mu = 0$ reduces to the free-field solutions,

$$\psi = \sqrt{\frac{m}{EV}} u(p)\ e^{-ip.x}$$

and

$$\psi = \sqrt{\frac{m}{EV}} v(p)\ e^{ip.x} \tag{2.2.57}$$

for an electron and positron respectively. The the effective mass m^*, which the particles acquired additional mass due to the external laser field also reduces to the electron(positron) mass m.

For a linearly polarized plane wave laser field we write the potential in the form

$$\mathcal{A}(x) = \epsilon a\ cos(\varphi), \tag{2.2.58}$$

where a indicates the amplitude of the vector potential and the polarization is chosen as $\epsilon = (0, 1, 0, 0)$.

Substituting the $\mathcal{A}(x)$ into the phase term we obtain

$$f_{\pm} = \pm px - \int_0^{\varphi} d\varphi \left(\frac{e\epsilon ap\ cos(\varphi)}{(kp)} \pm \frac{e^2\epsilon^2 a^2\ cos^2(\varphi)}{2(kp)} \right) \tag{2.2.59}$$

Rearranging terms and putting $\epsilon^2 = 1$ we have

$$f_{\pm} = \pm px - \frac{1}{2kp} \int_0^{\varphi} d\varphi \Big(2e\epsilon ap\ cos(\varphi) \pm e^2 a^2\ cos^2(\varphi) \Big) \tag{2.2.60}$$

$$= \pm px - \frac{1}{2kp} \left[2e\epsilon ap\ sin(\varphi) \pm e^2 a^2 \left(\frac{1}{4} sin(2\varphi) + \frac{1}{2}\varphi \right) \right], \tag{2.2.61}$$

where we have used the integrals

$$\int_0^{\varphi} d\varphi\ cos(\varphi) = sin(\varphi)$$

$$\int_0^{\varphi} d\varphi\ cos^2(\varphi) = \frac{1}{4} sin(2\varphi) + \frac{1}{2}\varphi \tag{2.2.62}$$

Rearranging terms we obtain

$$f_{\pm} = \pm qx - \frac{e\epsilon ap\ sin(\varphi)}{2(kp)} + \frac{e^2 a^2\ sin(2\varphi)}{(8kp)}, \tag{2.2.63}$$

where

$$q = p + \frac{e^2 a^2}{(2kp)} k \tag{2.2.64}$$

is the effective four-momentum.

Now the full Volkov solution for a linearly polarized plane wave laser field is written as

$$\psi(\varphi) = \sqrt{\frac{m}{EV}} \left(1 \pm \frac{e\not{k}\not{\mathcal{A}}}{2(kp)}\right) \; exp\left[\pm qx - \frac{e\epsilon ap\ sin(\varphi)}{(kp)} \pm \frac{e^2 a^2\ sin(2\varphi)}{(8kp)}\right], \tag{2.2.65}$$

where the $+$ and $-$ signs indicate that they are for positron and electron state respectively. The oscillating part of the phase term in Eq(2.2.65),

$$f_{osci} = \frac{ea\epsilon p}{(kp)}\ sin(\varphi) \pm \frac{e^2 a^2}{(8k.p)}\ sin(2\varphi) \tag{2.2.66}$$

can be put in Fourier series form

$$f_{osci} = \sum_{n=-\infty}^{\infty} J_n \left(\frac{ea\epsilon p}{(kp)}\ ,\ \ \pm\frac{e^2 a^2}{(8kp)}\right) \; exp(in\varphi) \tag{2.2.67}$$

Thus, Eq(2.2.65) becomes

$$\psi(\varphi) = \sqrt{\frac{m}{EV}} \left(1 \pm \frac{e\not{k}\not{\mathcal{A}}}{2(kp)}\right) \; exp\left(\pm qx\right) \sum_{n=-\infty}^{\infty} J_n(\alpha, \beta)\ exp(in\varphi), \tag{2.2.68}$$

which in turn can be expressed as

$$\psi(\varphi) = \sqrt{\frac{m}{EV}}\ u_{\pm}\ exp(iq_{\pm}.x) \sum_{n=-\infty}^{\infty} \left(B_n(\alpha,\beta) \pm \frac{ea\not{k}\not{\epsilon}}{2k.p_{\pm}} C_n(\alpha,\beta)\right) \; exp(in\varphi), \tag{2.2.69}$$

with $\alpha = \frac{ea\epsilon p}{(kp)}$, $\beta = \pm\frac{e^2a^2}{(8kp)}$, $B_n(\alpha,\beta) = J_n(\alpha,\beta)$ and

$$C_n(\alpha,\beta) = \tfrac{1}{2}(J_{n-1}(\alpha,\beta) + J_{n+1}(\alpha,\beta)),$$

where $J_n(\alpha,\beta)$ are the generalized Bessel functions.

Due to the use of the Volkov states, the theory of laser-dressed QED takes into account to all orders the particle-laser interaction in a scattering process, and still adopts a similar formalism like that of the ordinary QED. The remaining interaction between their laser-dressed particle and the QED vacuum is weak, and the perturbative expansion in the fine structure constant α is resorted to as in the ordinary QED. Feynman techniques can be used to picturise the theory, and laser-dressed Furry-Feynman diagrams follow almost the rules of the conventional Feynman diagrams, except that the appearing particles are in Volkov states and the particle propagators are Dirac-Volov propagators [1]. In chapter 3 and chapter 4 we make use of the exact, Volkov wave functions derived in this chapter to study electron-positron pair production in periodic(infinite) plane waves and finite laser pulses respectively.

Chapter 3

Multiphoton Trident Pair Production in Periodic Plane Waves

In laser-electron collisions, two pair creation processes are usually distinguished. The first is of Bethe-Heitler type (one-step process); the pair is produced by the absorption of N laser photons in the Coulomb field of the incoming electron [1].

$$e \ + \ N\omega \ \rightarrow \ e' \ + \ e^+e^-, \tag{3.0.1}$$

where one electron scatters by colliding photon and emits a virtual photon, which subsequently transforms into an e^+e^- pair. When the photon source is a laser field, the electron may couple nonlinearly to the field, giving rise to multi-photon processes [1].

The second is of Breit-Wheeler type (two-step process), where first a high-energy γ-photon is generated by Compton backscattering off the electron beam, which afterwards creates the pair in a photon-multiphoton collision.

$$\gamma \ + \ N\omega \ \rightarrow \ e^+e^- \tag{3.0.2}$$

The reaction in Eq (3.0.2) represents the strong-field generalization of the process

$$2\gamma \ \rightarrow \ e^+e^- \tag{3.0.3}$$

and exhibits a nonperturbative nature at very high fields. The laser-dressed Furry-Feynman diagrams of the multiphoton pair production are given in Fig (3.1). The lepton lines in the Furry-Feynman diagrams are described by Dirac-Volkov states, which fully account for their interaction with the external plane wave laser field (Furry picture) [1].

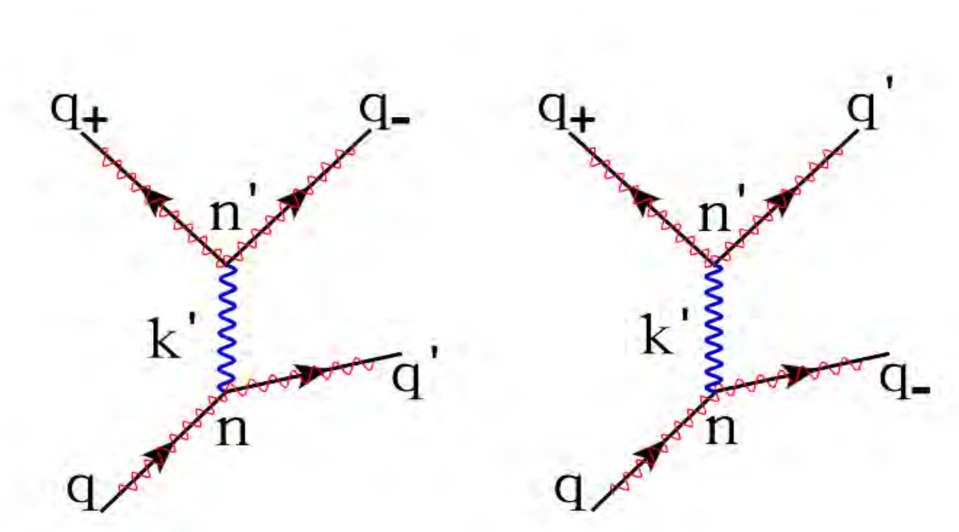


Figure 3.1: Furry-Feynman diagrams of multi-photon trident pair production in electron-laser collisions.

The zigzag-lines represent the exact lepton wave functions in the laser field (Dirac-Volkov states) and are labeled by the laser-dressed particle momenta. In the left diagram, the incoming electron scatters from a state of dressed momentum q to q' by absorbing n laser photons and emitting an intermediate photon, which afterwards decays into an e^+e^- pair upon absorption of n' laser photons [1,5]. The corresponding exchange diagram is given on the right.

The amplitude for the process reads

$$S_{fi} = \mathcal{M}(q, q', q_+, q_-) - \mathcal{M}(q, q_-, q_+, q'), \tag{3.0.4}$$

where for the direct process

$$\mathcal{M}(q, q', q_+, q_-) = e^2 \int d^4x \int d^4y \ \bar{\psi}_{q'}(x)\gamma_\mu\psi_q(x)G^{\mu\nu}(x-y)\bar{\psi}_{q_-}(y)\gamma_\nu\psi_{q_+}(y) \tag{3.0.5}$$

and for the exchange process

$$\mathcal{M}(q, , q_-, q_+, q') = e^2 \int d^4x \int d^4y \ \bar{\psi}_{q_-}(x)\gamma_\mu\psi_q(x)G^{\mu\nu}(x-y)\bar{\psi}_{q'}(y)\gamma_\nu\psi_{q_+}(y) \tag{3.0.6}$$

Here $\psi_q,\ \psi_{q'},\ \psi_{q+},\ \psi_{q-}$ denote the laser-dressed lepton states and

$$G^{\mu\nu}(x-y) = \int \frac{d^4k'}{(2\pi)^4}\frac{-4\pi g^{\mu\nu}}{k'^2+i\xi}\ exp(ik'.(x-y)) \tag{3.0.7}$$

is the usual free photon propagator.

Substituting the Fourier-expanded form of the Dirac-Volkov states for linearly polarized plane waves,

$$\psi = \sqrt{\frac{m}{EV}}\ u_\pm\ exp(iq_\pm.x)\sum_{n=-\infty}^{\infty}\left(B_n(\alpha,\beta) \pm \frac{ea\not{k}\not{\epsilon}}{2k.p_\pm}C_n(\alpha,\beta)\right)\ exp(in\varphi) \tag{3.0.8}$$

The space-time integrations can be performed and the amplitude adopts the form

$$\begin{aligned}\mathcal{M}(q,q',q_+,q_-) = -\alpha\frac{2(2\pi)^5m^2}{V^2\sqrt{EE'E_+E_-}}\sum_{N>N_0}\sum_{n=-\infty}^{\infty}\frac{\delta^4(q+Nk-q'-q_+-q_-)}{(q-q'+nk)^2}\\ M^\mu(q,q'|n)\ M_\mu(q_+q_-|N-n),\end{aligned} \tag{3.0.9}$$

where $\alpha = \frac{e^2}{4\pi}$, is the fine-structure constant. N is the total number photons absorbed, n is the number of photons absorbed at the first vertex with a negative n meaning $|n|$ photons emitted, and $\mathcal{M}^\mu(q,q'|n)$ and $\mathcal{M}_\mu(q_+q_-|N-n)$ are complex functions of the particle momenta and laser parameters, containing spinor-matrix products and the coefficients of terms being the generalized Bessel functions [1,5],

$$\mathcal{M}^\mu(q,q'|n) = \bar{u}_{p'}\left\{b_n\gamma^\mu - \left(\frac{ea\not{\epsilon}\not{k}\gamma^\mu}{2k.p'}+\frac{ea\gamma^\mu\not{k}\not{\epsilon}}{2k.p}\right)c_n + \frac{e^2a^2k^\mu\not{k}}{2(k.p')(k.p)}d_n\right\}u_p \tag{3.0.10}$$

$$\mathcal{M}^\mu(q_+,q_-|n') = \bar{u}_{p_-}\left\{B_{n'}\gamma_\mu - \left(\frac{ea\not{\epsilon}\not{k}\gamma_\mu}{2k.p_-}-\frac{ea\gamma^\mu\not{k}\not{\epsilon}}{2k.p_+}\right)C_{n'} - \frac{e^2a^2k_\mu\not{k}}{2(k.p_+)(k.p_-)}D_{n'}\right\}v_{p_+}, \tag{3.0.11}$$

where $n' = N-n$ and

$$b_n = J_n(\theta_1,\theta_2) \tag{3.0.12}$$

$$c_n = \frac{1}{2}\Big[J_{n-1}(\theta_1,\theta_2)+J_{n+1}(\theta_1,\theta_2)\Big] \tag{3.0.13}$$

$$d_n = \frac{1}{2}\Big[J_{n-2}(\theta_1,\theta_2)+2J_n(\theta_1,\theta_2)+J_{n+2}(\theta_1,\theta_2)\Big] \tag{3.0.14}$$

with the arguments

$$\theta_1 = \frac{ea(\epsilon.p)}{k.p'} - \frac{ea(e.p)}{k.p} \tag{3.0.15}$$

$$\theta_2 = \frac{e^2a^2}{8}\left[\frac{1}{k.p} - \frac{1}{k.p'}\right] \tag{3.0.16}$$

Similarly,

$$B_{n'} = J_{n'}(\Theta_1, \Theta_2) \tag{3.0.17}$$

$$C_{n'} = \frac{1}{2}\left[J_{n'-1}(\Theta_1, \Theta_2) + J_{n'+1}(\Theta_1, \Theta_2)\right] \tag{3.0.18}$$

$$D_{n'} = \frac{1}{2}\left[J_{n'-2}(\Theta_1, \Theta_2) + 2J_{n'}(\Theta_1, \Theta_2) + J_{n'+2}(\Theta_1, \Theta_2)\right] \tag{3.0.19}$$

with the arguments

$$\Theta_1 = \frac{ea(\epsilon.p_-)}{k.p_-} - \frac{ea(\epsilon.p_+)}{k.p_+} \tag{3.0.20}$$

$$\Theta_2 = \frac{e^2a^2}{8}\left[\frac{1}{k.p_-} - \frac{1}{k.p_+}\right] \tag{3.0.21}$$

The Dirac delta function

$$\delta^4(q + Nk - q' - q_+ - q_-) \tag{3.0.22}$$

guarantees the four-momentum conservation of the process in which $|N|$ laser photons are absorbed ($N > 0$), or emitted ($N < 0$). This leads to the threshold condition for the participating photon number

$$N\omega' \geq 4m^* \tag{3.0.23}$$

with the intensity dependent effective mass

$$m^* = m\sqrt{1 + a_0^2}, \tag{3.0.24}$$

Thus,

$$m^* > m, \tag{3.0.25}$$

where additional mass is acquired by the particles, dressed by the background field. The dimensionless intensity parameter a_0 is introduced as,

$$a_0 = \frac{e\bar{\mathcal{A}}}{m}, \tag{3.0.26}$$

where $\bar{\mathcal{A}}$ is the root-mean-square value of the vector potential of the laser field. For a relativistically strong pulsed field $a_0 \gg 1$ [3] and measures the strength of the electron-laser interaction.

The total rate of production is obtained by

$$R = \frac{1}{T}\int \frac{Vd^3q_+}{(2\pi)^3}\int \frac{Vd^3q_-}{(2\pi)^3}\int \frac{Vd^3q'}{(2\pi)^3}\frac{1}{4}\sum_{spins}|S_{fi}|^2 \tag{3.0.27}$$

with the interaction time T, and a statistical factor $\frac{1}{4}$ due to initial spin averaging and the two final-state electrons [2].

It is possible to show that

$$\left[\delta^4(q + Nk - q' - q_+ - q_-)\right]^2 = \frac{VT}{(2\pi)^4}\delta^4(q + Nk - q' - q_+ - q_-) \tag{3.0.28}$$

Upon substitution of Eq(3.0.4), Eq(3.0.27) becomes

$$R = \frac{1}{T}\int \frac{Vd^3q_+}{(2\pi)^3}\int \frac{Vd^3q_-}{(2\pi)^3}\int \frac{Vd^3q'}{(2\pi)^3}\left(\alpha\frac{2(2\pi)^5m^2}{V^2\sqrt{EE'E_+E_-}}\right)^2$$
$$\frac{1}{4}\sum_{spins}\frac{VT}{(2\pi)^3}\sum_{N}\delta^4(q + Nk - q' - q_+ - q_-)$$
$$\sum_{n_1}\sum_{n_2}\left(M^{qq'}_{n_1n_2} + M^{qq_-}_{n_1n_2} - 2M^{ext}_{n_1n_2}\right) \tag{3.0.29}$$

Thus,

$$R = \frac{\alpha^2m^4}{(2\pi)^3E}\int \frac{d^3q'}{E'}\int \frac{d^3q_+}{E_+}\int \frac{d^3q_-}{E_-}\sum_{N}\delta^4(q + Nk - q' - q_+ - q_-)$$
$$\sum_{n_1}\sum_{n_2}\sum_{spins}\left(M^{qq'}_{n_1n_2} + M^{qq_-}_{n_1n_2} - 2M^{ext}_{n_1n_2}\right), \tag{3.0.30}$$

where $\sum_{spins} M^{qq'}_{n_1n_2}$ and $\sum_{spins} M^{qq_-}_{n_1n_2}$ are trace products:

$$\sum_{spins} M^{qq'}_{n_1n_2} = \frac{1}{(q - q' + n_1k)^2}\frac{1}{(q - q' + n_2k)^2}$$
$$Tr\left[\frac{\not{p}_- + m}{2m}\Gamma_{\mu n_1}(q_-, q_+)\frac{\not{p}_+ - m}{2m}\Gamma_{\nu n_2(q-q_+)}\right]$$
$$Tr\left[\frac{\not{p}' + m}{2m}\Gamma^{\mu}_{n_1}(q, q')\frac{\not{p} + m}{2m}\Gamma^{\nu}_{n_2}(q, q')\right], \tag{3.0.31}$$

with

$$\Gamma^{\mu}_{n_1}(q,q') = b_{n_1}\gamma^{\mu} - \left(\frac{ea\not{\epsilon}\not{k}\gamma^{\mu}}{2kp'} + \frac{ea\gamma^{\mu}\not{k}\not{\epsilon}}{2kp}\right)c_{n_1} + \frac{e^2a^2k^{\mu}\not{k}}{2(kp')(k.p)}d_{n_1} \tag{3.0.32}$$

$$\Gamma_{\mu n_1}(q_-,q_+) = B_{N-n_1}\gamma_{\mu} - \left(\frac{ea\not{\epsilon}\not{k}\gamma_{\mu}}{2kp_-} - \frac{ea\gamma^{\mu}\not{k}\not{\epsilon}}{2kp_+}\right)C_{N-n_1} - \frac{e^2a^2k_{\mu}\not{k}}{2(kp_+)(k.p_-)}D_{N-n_1} \tag{3.0.33}$$

and

$$\bar{\Gamma} = \gamma^0\Gamma^+\gamma^0 \tag{3.0.34}$$

$\sum_{spins} M^{qq_-}_{n_1n_2}$ can be obtained from $\sum_{spins} M^{qq'}_{n_1n_2}$ by the exchanges of $q' \leftrightarrows q_-$ as well as $p' \leftrightarrows p_-$ in Eq(3.0.31). $\sum_{spins} M^{ext}_{n_1n_2}$ accounts for the interference of diagrams with exchanged electrons in the final states.

$$\sum_{spins} M^{ext}_{n_1n_2} = \frac{1}{(q-q'+n_1k)^2}\frac{1}{(q-q_-+n_2k)^2} \tag{3.0.35}$$

$$Tr\left[\frac{\not{p}_-+m}{2m}\Gamma_{\mu n_1}(q,q_+)\frac{\not{p}_+-m}{2m}\Gamma_{\nu n_2}(q',q_+)\frac{\not{p}'+m}{2m}\Gamma^{\mu}_{n_2}(q,q')\frac{\not{p}+m}{2m}\Gamma^{n_2}_{\nu}(q,q_-)\right]$$

The S-matrix in Eq(3.0.4) shows divergence due to the pole in the photon propagator (3.0.7), when enough energy is taken from the background to put the photon on shell and $k'^2 = 0$. It was suggested in [1,5] that the divergence could be dealt with by modifying the photon propagator as

$$\frac{1}{k'^2} \quad\longrightarrow\quad \frac{1}{k'^2+i\xi} \tag{3.0.36}$$

In the next chapter we will briefly see how the divergence is dealt with the help of optical theorem and how it helps us to identify precisely the two contributions, from the one-step and two-step processes. We will also do the detailed calculation of the S-matrix by taking the two Feynman diagrams, for direct- and exchange-processes, into account.

The 9-dimensional integral for the rate (3.0.30) [1,5] is first reduced to a 5-dimensional one by the δ-function. Still, the remaining multi-dimensional integration is time consuming [4], and they apply an appropriate Monte Carlo method to speed up the computation. However, performing the remaining integrals, summing over spins, dividing over the pulse volume V, duration T, incoming flux I, and jumping over the details, arrive at a cross section of the form

$$\sigma = \sum_{N>N_0} \sum_{n} M_n \, \delta^4(q + Nk - q' - q_+ - q_-) \tag{3.0.37}$$

for some unspecified elements M_n. This is then an infinite sum, each term describing momentum conservation, and is unphysical. We expect a physical result not to take the appearance of delta functions. These delta functions are an artifact from considering the back ground laser as an infinite plane wave [6]. We know that cross section is a finite and measurable physical quantity. Therefore, to arrive at a finite cross section, then we need to shape our laser background into finite pulses. This will be shown in the next chapter where we study trident pair production in strong laser pulses.

Chapter 4

Trident Pair Production in Strong Laser Pulses

4.1 Finite Laser Pulses

As seen in the previous chapter a laser background modeled by an infinite plane wave is problematic. A more realistic approach which as it turns out solves these problems is taking pulse shapes into account. The presented theory in [6] for laser pulses is built around intense laser fields with neglected transverse size effects, again approximated as plane waves only dependent on the phase $\varphi = k.x$. We direct the laser beam along the x_3 axis as usual, with photon momenta k_μ as

$$k_\mu = \omega(1, 0, 0, 1), \tag{4.1.1}$$

of which only k_+ is non zero.

In general, the background field takes the form

$$\mathcal{A}(k.x) = a_\mu^j f_j(k.x), \tag{4.1.2}$$

where a_μ^j are the polarization vectors and $f_j(k.x)$ are functions shaping our laser pulses. The laser field tensor is then given by

$$F_{\mu\nu} = f_j'(\varphi) \left(k_\mu a_\nu^j - k_\nu a_\mu^j \right) \tag{4.1.3}$$

with the prime denoting derivation with respect to φ.

We use the background field as circularly polarized, given by the four-potential $\mathcal{A}(k.x)$ as,

$$\mathcal{A}(k.x) = a^1_\mu f_1(k.x) + a^2_\mu f_2(k.x), \tag{4.1.4}$$

where

$$a^1_\mu = \frac{ma_0}{e}(0,1,0,0) \tag{4.1.5}$$

$$a^2_\mu = \frac{ma_0}{e}(0,0,1,0), \tag{4.1.6}$$

are polarization vectors and a_0 is a dimensionless laser parameter. The laser pulses are shaped by setting [6]

$$f_1(k.x) = cos(\varphi)sin^4(\frac{\varphi}{2N}) \tag{4.1.7}$$

$$f_2(k.x) = sin(\varphi)sin^4(\frac{\varphi}{2N}), \tag{4.1.8}$$

for $\varphi \in \{0, 2N\}$, 0 otherwise, with N then defining the number of oscillations of the field in a pulse. For convenience we also define

$$f_3(\varphi) = f_1^2(\varphi) + f_2^2(\varphi) \tag{4.1.9}$$

since it will appear in our calculations, in section 4.3. In the next section we will discuss the optical theorem.

4.2 The Optical Theorem

The two-point correlation of quantum fields, viewed as analytic function of the momentum p^2, has branch cut singularities associated with multi-particle intermediate states [8]. This conclusion is similar to non-relativistic scattering theory, since it is already true there that the scattering amplitude, as a function of energy, has a branch cut on the positive real axis. The imaginary part of the scattering amplitude appears as a discontinuity across this branch cut. It arises from a sum of contributions from all possible intermediate state particles. By the optical theorem, the imaginary part of the forward amplitude is

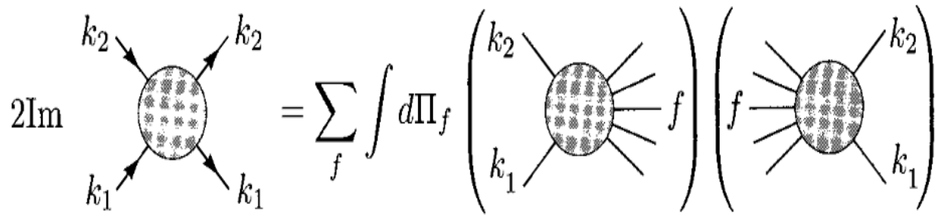


Figure 4.1: The optical theorem: Contributions to the imaginary part of a forward scattering amplitude

proportional to the cross section [8,9]. We will now prove the field-theoretic version of the optical theorem and illustrate how it arises in Feynman diagram calculations.

The optical theorem is a direct consequence of the unitarity of the S-matrix.

$$S^{+}S = 1 \tag{4.2.1}$$

The T-matrix is defined by

$$S = 1 + iT, \tag{4.2.2}$$

which is the interaction term of the S-matrix.

Substituting Eq(4.2.2) into Eq(4.2.1) gives

$$-i(T - T^{+}) = T^{+}T, \tag{4.2.3}$$

Let us take the matrix element of this equation between two-particle states $|p_1p_2\rangle$ and $|k_1k_2\rangle$. We evaluate the right-hand side of Eq(4.2.3), by inserting a complete set of intermediate states as

$$\langle p_1p_2|T^{+}T|k_1k_2\rangle = \sum_n \left(\prod_{i=1}^{n} \int \frac{d^3q_i}{(2\pi)^3} \frac{1}{2E_i} \right) \langle p_1p_2|T^{+}|\{q_i\}\rangle \ \langle \{q_i\}|T|k_1k_2\rangle \tag{4.2.4}$$

Now we express the T-matrix elements as invariant matrix elements $\mathcal{M}$ times 4-momentum-conserving delta functions.

The identity in Eq(4.2.3) becomes

$$-i[\mathcal{M}(k_1k_2 \to p_1p_2) - \mathcal{M}^*(p_1p_2 \to k_1k_2)]$$
$$= \sum_n \Big(\prod_{i=1}^n \int \frac{d^3q_i}{(2\pi)^3}\frac{1}{2E_i} \Big) \mathcal{M}^*(p_1p_2 \to \{q_i\})\mathcal{M}(k_1k_2 \to \{q_i\})(2\pi)^4$$
$$\delta^4(k_1 + k_2 - \sum_i q_i)\delta^4(k_1 + k_2 - p_1 - p_2). \quad (4.2.5)$$

We abbreviate this identity as

$$-i[\mathcal{M}(a \to b) - \mathcal{M}^*(b \to a)] = \sum_f \int d\pi_f \mathcal{M}^*(b \to f)\mathcal{M}(a \to f), \quad (4.2.6)$$

where the sum runs over all possible sets f of final-state particles. Although we have so far assumed that a and b are two-particle states, they could equally well be one-particle or multi-particle asymptotic states.

For the important special case of forward scattering, we can put $p_i = k_i$ to obtain a simpler identity, shown in fig(4.1). Supplying the kinematic factors required by (4.2.6) to build a cross section, we obtain the standard form [8,9] of the optical theorem,

$$Im\ \mathcal{M}(k_1k_2 \to k_1, k_2) = 2E_{cm}p_{cm}\sigma_{tot}(k_1k_2 \to anything), \quad (4.2.7)$$

where E_{cm} is the total center-of-mass energy and p_{cm} is the momentum of either particle in center-of-mass frame. This equation relates the forward amplitude to the cross section for production of all final states. Since the imaginary part of the forward scattering amplitude gives the attenuation of the forward-going wave as the beam passes through the target, it is natural that quantity should be proportional to the probability of scattering. The optical theorem clearly shows that in general the amplitude can not be purely real and that it has a positive imaginary part near the forward direction [9].

4.2.1 The Optical Theorem for Feynman Diagrams

The Feynman diagram yields an imaginary part $i\xi$ for $\mathcal{M}$ only when the virtual particles in the diagram go on-shell [9,10]. We will now show how to isolate and compute this imaginary part. For our present purposes, let us define $\mathcal{M}$ by the Feynman rules for perturbation theory. This allows us to consider $\mathcal{M}(s)$ as an analytic function of the complex variable $s = E_{cm}^2$, even though S-matrix elements are defined only for external particles with real momenta.

We first demonstrate that the appearance of the imaginary part of $\mathcal{M}(s)$ always requires a branch cut singularity. Let s_0 be the threshold energy for production of the lightest multi-particle state. For real s below s_0 the intermediate state cannot go on-shell, so $\mathcal{M}(s)$ is real. Thus, for real $s < s_0$, we have the identity

$$\mathcal{M}(s) = [\mathcal{M}(s^*)]^* \tag{4.2.8}$$

Each side of this equation is an analytic function of s. So it can be analytically continued to the entire complex s plane. In particular, near the real axis for $s > s_0$, Eq(4.2.8) implies

$$Re\ \mathcal{M}(s + i\xi) = Re\ \mathcal{M}(s - i\xi) \tag{4.2.9}$$

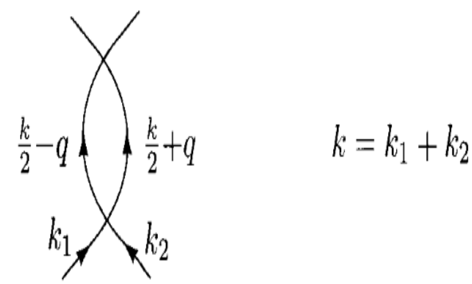
$$Im\ \mathcal{M}(s + i\xi) = -Im\ \mathcal{M}(s - i\xi) \tag{4.2.10}$$

There is a branch cut across the real axis, starting at threshold energy s_0; the discontinuity across the cut is

$$Disc\ \mathcal{M}(s) = 2i\ Im\ \mathcal{M}(s + i\xi) \tag{4.2.11}$$

Usually it is easier to compute the discontinuity of the diagram than to compute the imaginary part directly. The $i\xi$ prescription in the Feynman propagator indicates that physical scattering amplitudes should be evaluated above the cut, at $s + i\xi$. We then study more general one-loop diagrams, and show that their discontinuities give precisely the imaginary parts required by Eq(4.2.6). The generalization of this result to multi-loop diagrams has been proven by Cutkosky, who showed in the process that the discontinuity of a Feynman diagram across its branch cut is always given by a simple set of cutting rules.

We begin by checking Eq(4.2.6) in ϕ^4 theory. Since the right-hand side of Eq(4.2.6) begins in order λ^2, we expect that $Im\ \mathcal{M}$ should also receive its first contribution from higher-order diagrams. Consider, then, the order-λ^2 diagram



with a loop in the s-channel. (It is easy to check that the corresponding t- and u- channel diagrams have no branch cut singularities for s above threshold.) The total momentum is $k = k_1 + k_2$, and for simplicity we have chosen the symmetrical routing of momenta shown above. The amplitude of this Feynman diagram is

$$i\delta\mathcal{M} = \frac{\lambda^2}{2}\int \frac{d^4q}{(2\pi)^4}\frac{1}{(\frac{k}{2}-q)^2 - m^2 + i\xi}\,\frac{1}{(\frac{k}{2}+q)^2 - m^2 + i\xi} \tag{4.2.12}$$

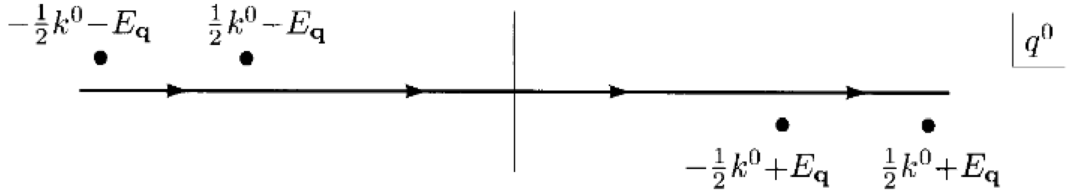
We would like to verify that the integral in Eq(4.2.12) has a discontinuity across the real axis in the physical region $k^0 > 2m$. It is easiest to identify this discontinuity by computing the integral for $k^0 < 2m$, then increasing k^0 by analytic continuation. It is not difficult to compute the integral directly using Feynman parameters [8]. However, it is illuminating to use a less direct approach, as follows.

Let us work in the center-of-mass frame, where $k = (k^0, 0)$. Then the integrand of Eq(4.2.12) has four poles in the integration variable q^0, at the locations

$$q^0 = \frac{1}{2}k^0 \pm (E_q - i\xi)$$

$$q^0 = -\frac{1}{2}k^0 \pm (E_q - i\xi) \tag{4.2.13}$$

Two of these poles lie above the real q^0 axis and two lie below, as shown below:



We will close the integration contour downward and pick up the residues of the poles in the lower half plane. Of these, only the pole at $-\frac{1}{2}k^0 + E_q$ will contribute to the discontinuity. Note that picking up the residue of this pole is equivalent to replacing

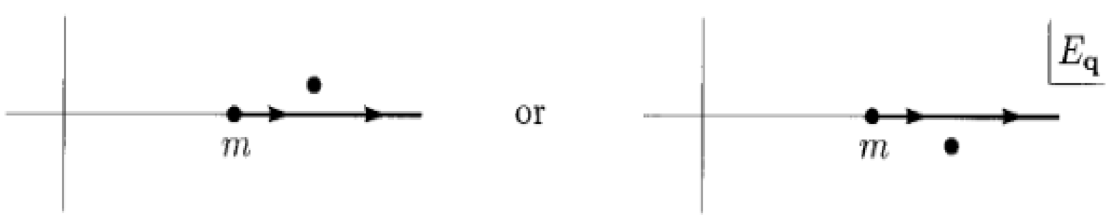
$$\frac{1}{(\frac{k}{2}+q)^2 - m^2 + i\xi} \quad \rightarrow \quad -2\pi i \delta((\frac{k}{2}+q)^2 - m^2) \tag{4.2.14}$$

under the dq^0 integral.

The contribution of this pole yields the integral

$$\begin{aligned} i\delta\mathcal{M} &= -2\pi i \frac{\lambda^2}{2} \int \frac{d^3 q}{(2\pi)^4} \frac{1}{2E_q} \frac{1}{(k^0 - E_q)^2 - E_q^2} \\ &= -2\pi i \frac{\lambda^2}{2} \frac{4\pi}{(2\pi)^4} \int_m^\infty dE_q E_q |q| \frac{1}{2E_q} \frac{1}{k^0(k^0 - 2E_q)} \end{aligned} \tag{4.2.15}$$

The integrand in the second line has a pole at $E_q = \frac{k^0}{2}$. When $k^0 < 2m$, this pole does not lie on the integration contour, so $\delta\mathcal{M}$ is manifestly real. When $k^0 > 2m$, however, the poles lies just above or below the contour of integration, depending upon whether k^0 is given a small positive or negative imaginary part



Thus, the integral acquires a discontinuity between $k^2 + i\xi$ and $k^2 + i\xi$. To compute this discontinuity, we apply

$$\frac{1}{k^0 - 2E_q \pm i\xi} = \mathcal{P}\frac{1}{k^0 - 2E_q} \mp i\pi\delta(k^0 - 2E_q), \tag{4.2.16}$$

where $\mathcal{P}$ denotes the principal value, the discontinuity is given by replacing the pole with a delta function. This in turn is equivalent to replacing the original propagator by a delta function

$$\frac{1}{(\frac{k}{2}-q)^2 - m^2 + i\xi} \quad \rightarrow \quad -2\pi i \delta\bigg((\frac{k}{2}-q)^2 \frac{1}{2E_1} - m^2 \bigg) \tag{4.2.17}$$

Let us now retrace our steps and see what we have proved. We go back to the original integral Eq(4.2.12), relabel the momenta on the two propagators as $p_1,\ p_2$ and substitute

$$\int \frac{d^4q}{(2\pi)^4} = \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} (2\pi)^4 \delta^4(p_1+p_2-k) \tag{4.2.18}$$

We have shown that the discontinuity of the integral is computed by replacing each of the two propagators by a delta function

$$\frac{1}{p_i^2 - m^2 + i\xi} \quad \rightarrow \quad -2\pi i \delta(p_i^2 - m^2) \tag{4.2.19}$$

The discontinuity of $\mathcal{M}$ comes only from the region of the d^4q integral in which the two delta functions are simultaneously satisfied. By integrating over the delta functions, we put the momenta p_i on-shell and convert the integrals d^4p into integrals over relativistic phase space. What is left over in expression Eq(4.2.12) is just the factor λ^2, the square of the leading-order scattering amplitude, and the symmetry factor $(\frac{1}{2})$, which can be reinterpreted as the symmetry factor for identical bosons in the final state. Thus, we have shown that, to order λ^2 on each side of the equation,

$$Disc\ \mathcal{M}(k) = 2iIm\ \mathcal{M}(k)$$

$$= \frac{i}{2} \int \frac{d^3p_1}{(2\pi)^3} \frac{1}{2E_1} \int \frac{d^3p_2}{(2\pi)^3} \frac{1}{2E_2} |\mathcal{M}(k)|^2 (2\pi)^4 \delta^4(p_1+p_2-k) \tag{4.2.20}$$

This explicitly verifies Eq(4.2.6) to order λ^2 in ϕ^4 theory.

The preceding argument made no essential use of the fact that the two propagators in the diagram had equal masses, or of the fact that these propagators connected to a simple point vertex. Indeed, the analysis can be applied to an arbitrary one-loop diagram. Whenever, in the region of momentum integration of the diagram, two propagators can simultaneously go on-shell, we can follow the argument above to compute a nonzero discontinuity is given by making the substitution Eq(4.2.19) for each of the two propagators. The poles of the additional propagators in the diagrams do not contribute to the discontinuities. By integrating over the delta functions as in the previous paragraph, we derive the indicated relations between the imaginary parts of these diagrams and contributions to the total cross section. Cutkosky proved that this method of computing discontinuities is completely general. The physical discontinuity of any Feynman diagram is given by the following algorithm [8].

1. Cut through the diagram in all possible ways such that the cut propagator, can simultaneously be put on-shell.

2. For each cut, replace $\frac{1}{(p^2-m^2+i\xi)} \;\rightarrow\; -2\pi i\delta(p^2-m^2)$ in each cut propagator, then perform the loop integrals.

3. Sum the contributions of all possible cuts.

In the next section we will reveal the theoretical approach to trident pair production.

4.3 The Theoretical Approach to Trident Pair Production

We consider an electron, incident upon a laser field, emitting a γ photon. This photon then combines with photons in the laser to produce an electron-positron pair [4]. In the SLAC experiment two processes are distinguished: The first is 'one-step', traditionally referred to as trident, in which the intermediate photon is virtual. The second process is 'two-step', in which a real photon is scattered from the incoming electron and then creates a pair via stimulated pair production [4]. Since the Feynman diagram in Fig(4.2) makes no distinction between these processes, we will refer the full diagram as 'trident', and use the notions of one-step and two-step processes to distinguish the two contributions as usual.

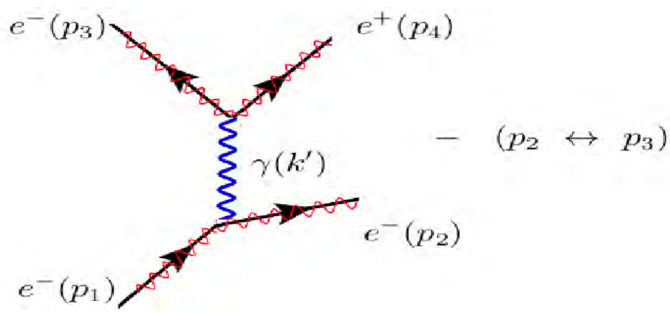


Figure 4.2: Furry-Feynman diagram for the trident process: The second term accounts for the exchange of indistinguishable fermions.

The trident S-matrix element is given by

$$S_{fi} = e^2 \int d^4x \int d^4y \; \bar{\psi}_2(x)\gamma^\mu \psi_1(x) G_{\mu\nu}(x-y)\bar{\psi}_3(y)\gamma^\nu \psi_4(y) - (p_2 \leftrightarrow p_3), \qquad (4.3.1)$$

where $(p_2 \leftrightarrow p_3)$ accounts for the exchange process in the Feynman diagram in Fig(4.2) and $G_{\mu\nu}(x-y)$ is the photon propagator given by

$$G_{\mu\nu}(x-y) = \int \frac{d^4k'}{(2\pi)^4} \frac{-4\pi g^{\mu\nu}}{k'^2 + i\xi} \; exp(ik'.(x-y)) \qquad (4.3.2)$$

The Volkov states for electron and positron, as derived in chapter(2), in Eq(2.2.53) and Eq(2.2.55) given respectively as

$$\psi_-(x) = \sqrt{\frac{m}{EV}} \left(1 - \frac{e\not{k}\not{\mathcal{A}}}{2k.p}\right) u^{(s)}(p) \; exp(if_-(\varphi))$$

$$\psi_+(x) = \sqrt{\frac{m}{EV}} \left(1 + \frac{e\not{k}\not{\mathcal{A}}}{2k.p}\right) v^{(s)}(p) \; exp(if_+(\varphi)), \qquad (4.3.3)$$

with the phase term

$$f_\mp(\varphi) = \mp p.x - \int_{-\infty}^{\varphi} \left(\frac{ep.\mathcal{A}(\varphi)}{k.p} \mp \frac{e^2\mathcal{A}^2(\varphi)}{2k.p} \right)$$

The x and y integrals in Eq(4.3.1) simply reduced to

$$\int d^4x \int d^4y \; exp(p_2 - p_1 + k').x \; exp(p_3 + p_4 - k').y$$

$$= (2\pi)^8 \delta^4(p_2 - p_1 + k' - rk)\delta^4(p_3 + p_4 - k' - sk) \qquad (4.3.4)$$

and integration over k' combines the two delta functions as

$$(2\pi)^8 \int \frac{d^4k'}{(2\pi)^4} \delta^4(p_2 - p_1 + k' - rk)\delta^4(p_3 + p_4 - k' - sk)\left[\frac{-4\pi}{k'^2 + i\xi}\right]$$

$$= (2\pi)^4 \delta^4(p_{out} - p_1 - rk - sk) \frac{-4\pi}{k'^2 + i\xi}\bigg|_{k'=\delta p + rk} \qquad (4.3.5)$$

with the laser four-momentum balance

$$k' = \delta p + rk \qquad (for\ nonlinear\ Compton\ scattering) \qquad (4.3.6)$$

or

$$k' = p_3 + p_4 - sk \qquad (for\ stimulated\ pair\ production) \qquad (4.3.7)$$

We have also put, for simplicity

$$\delta p = p_2 - p_1 \qquad (4.3.8)$$

$$p_{out} = p_2 + p_3 + p_4 \qquad (4.3.9)$$

The S-matrix now takes the form

$$S_{fi} = -4\pi e^2 (2\pi)^4 \frac{1}{V^2} \sqrt{\frac{m^4}{E_1 E_2 E_3 E_4}} \int d^4r \int d^4s\ M(r,s) \delta^4(p_{out} - p_1 - rk - sk)$$

$$\frac{1}{k'^2 + i\xi}\bigg|_{k'=\delta p + rk} - (p_2 \leftrightarrow p_3), \qquad (4.3.10)$$

where Fourier transformation has been made to trade the φ dependence on the Volkov states, introducing the variables r and s, which are the analogues of photon number for pulsed plane waves [4].

The delta function in Eq(4.3.10) guarantees the momentum and energy conservation of the process and can be expanded in light cone coordinates as.

$$\delta^4(p_{out} - p_1 - rk - sk) =$$

$$\frac{1}{2}\delta^{\perp}(p_{out} - p_1)\delta^{-}(p_{out} - p_1)\delta^{+}(p_{out} - p_1 - rk - sk) \qquad (4.3.11)$$

The pole prescription in the photon propagator can be treated as

$$\frac{1}{k'^2 + i\xi} = -i\pi\delta(k'^2) + \mathcal{P}\frac{1}{k'^2} \qquad (for\ \xi \to 0) \tag{4.3.12}$$

which corresponds to a split into real and imaginary parts. From the optical theorem for scattering amplitudes, one expects the appearance of imaginary, or absorptive, parts to correspond to the excitation of real rather than virtual intermediate states. Indeed, we have seen that the imaginary part of Eq(4.3.12) correspond to the intermediate photon becoming real. The existence of this imaginary part is entirely due to the dressing of the fermions by the background, which allows one to obtain a physical, nonzero amplitudes, those for nonlinear Compton scattering and stimulated pair production [4].

Direct substitution of Eq(4.3.11) and Eq(4.3.12) into Eq(4.3.10) we obtain

$$S_{fi} = e^2(2\pi)^5 \frac{1}{V^2}\sqrt{\frac{m^4}{E_1E_2E_3E_4}}\delta^{\perp}(p_{out} - p_1)\delta^{-}(p_{out} - p_1)$$
$$\Big[\int dr \int dsM(r, s_r)\Big(i\pi\delta^{+}(p_{out} - p_1 - rk - sk)\delta^{+}(2rk.\delta p + \delta p^2)\Big)$$
$$-\int dr \int dsM(r, s)\delta^{+}(p_{out} - p_1 - rk - sk)\mathcal{P}\frac{1}{(\delta p + rk)^2} - (p_2 \leftrightarrow p_3)\Big] \tag{4.3.13}$$

Since s appears only in the q_+ component of the delta function, it eliminates the s integral by fixing

$$s = \frac{p_{out} - p_1}{k_+} - r \equiv s_r \tag{4.3.14}$$

In the first term of Eq(4.3.13), the delta function is

$$\delta(k'^2) = \delta(2rk.\delta p + \delta p^2) \tag{4.3.15}$$

and is therefore eliminated by performing the r integral by fixing

$$r = -\frac{\delta p^2}{2k.\delta p^2} \equiv r' \tag{4.3.16}$$

Thus, Eq(4.3.13) now becomes

$$S_{fi} = e^2(2\pi)^5 \frac{1}{V^2}\sqrt{\frac{m^4}{E_1E_2E_3E_4}}\delta^{if}(p_{out}-p_1)\left[\frac{i\pi}{2k.\delta p}M(r', s_{r'})\right.$$
$$\left. - \int dr M(r,s_r)\mathcal{P}\frac{1}{(\delta p + rk)^2} - (p_2 \leftrightarrow p_3)\right], \tag{4.3.17}$$

where we have introduced in Eq (4.3.17) that

$$\delta^{if}(p_{out}-p_1) = \frac{1}{k_+}\delta^{\perp}(p_{out}-p_1)\delta^{-}(p_{out}-p_1) \tag{4.3.18}$$

and the integrals

$$\int dr\ \delta^+(2rk.\delta p + \delta p^2) = \frac{1}{2k.\delta p}$$
$$\int ds\ \delta^+(p_{out} - p_1 - rk - sk) = \frac{1}{k_+} \tag{4.3.19}$$

From Eq(4.3.17) we see that the off- and on-shell parts of Eq(4.3.12) correspond precisely to the one- and two- step processes [4]. It can be shown by using the momentum conservation given by the explicit delta function in Eq(4.3.10) for the one-step process and the two momentum conservation relations given in Eq(4.3.6) and Eq(4.3.7) for the two-step process. Squaring the argument of the delta function in the one-step process, one finds that

$$r + s > \frac{4m^2}{k.p_1}, \tag{4.3.20}$$

which states that the incoming energy (that of the initial electron and that taken from the laser), must be sufficient to produce three particles of rest mass m. This is the only constraint in S_{fi} if the photon is off-shell, $k^2 \neq 0$, and we pick up the principal value term. This is the 'trident' contribution in the old nomenclature [4].

Squaring Equations (4.3.6) and (4.3.7), for nonlinear Compton scattering and stimulated pair production respectively, we find in the two step process that for the individual processes to occur,

$$r > 0$$

and

$$s > \frac{2m^2}{k.k'} \tag{4.3.21}$$

Using the over all delta function in Eq(4.3.10), it is possible to show that the fixed parameters r' and $s_{r'}$, which appear in our on-shell part obey

$$r' > 0$$

and

$$s_{r'} > \frac{2m^2}{k.\delta p} \tag{4.3.22}$$

Recalling that $k.\delta p = k.k'$ evaluated on-shell, we recover the previous constraints. The solution thus contains the correct kinematics of both the one-step and two-step processes, which are described precisely by the off- and on-shell parts of Eq(4.3.12) [4].

From the corresponding Volkov states in Eq(4.3.3) we define

$$J(p, b, c) = -\frac{1}{2k.p}\int_c^b d\varphi\Big(2e\mathcal{A}(\varphi).p - e^2\mathcal{A}^2(\varphi)\Big)$$

$$S(p, k.x) = 1_4 + \frac{e}{2k.p}\mathcal{A}\!\!\!/(\varphi)k\!\!\!/ \tag{4.3.23}$$

Upon substitution of Eq(4.3.23) one can write the amplitude for "nonlinear Compton scattering" Γ^μ as,

$$\Gamma^\mu(k.x) = \bar{u}_{p_2}\hat{S}(p_2, k.x)\gamma^\mu S(p_1, k.x)u_{p_1}$$
$$exp\Big(iJ(p_2, k.x, \infty) + iJ(p_1, -\infty, k.x)\Big) \tag{4.3.24}$$

and the amplitude for "pair production" Δ^μ as,

$$\Delta^\mu(k.y) = \bar{u}_{p_3}\hat{S}(p_3, k.x)\gamma^\mu S(-p_4, k.x)v_{p_4}$$
$$exp\Big(iJ(p_3, k.y, \infty) + iJ(-p_4, \infty, k.y)\Big) \tag{4.3.25}$$

The amplitude $M(r,s)$ now becomes

$$M(r,s) = \int d\varphi \Gamma^{\mu}(\varphi)\ exp(ir\varphi) \int d\varphi \Delta_{\mu}(\varphi)\ exp(is\varphi)$$

$$\equiv \Gamma^{\mu}(r)\Delta_{\mu}(s), \tag{4.3.26}$$

where the Fourier transforms are given by

$$\Gamma^{\mu}(r) = \int d\varphi\ \bar{u}_{p_2}\left(1_4 - \frac{e\not{\mathcal{A}}(\varphi)\not{k}}{2k.p_2}\right)\gamma_{\mu}\left(1_4 + \frac{e\not{k}\not{\mathcal{A}}(\varphi)}{2k.p_1}\right)\ u_{p_1}$$

$$exp\left(ir\varphi - i\alpha_j \int_0^{\varphi} d\varphi f_j(\varphi)\right) \tag{4.3.27}$$

and

$$\Delta^{\mu}(s) = \int d\varphi\ \bar{u}_{p_3}\left(1_4 + \frac{e\not{\mathcal{A}}(\varphi)\not{k}}{2k.p_3}\right)\gamma^{\mu}\left(1_4 + \frac{e\not{\mathcal{A}}\not{k}}{2k.p_4}\right)\ v_{p_4}$$

$$exp\left(is\varphi - i\alpha_j \int_0^{\varphi} d\varphi f_j(\varphi)\right) \tag{4.3.28}$$

The coefficients can be read of from Eq(4.3.23) and Eq(4.3.26) as

$$\alpha_j = ea^j.\left(\frac{p_4}{k.p_4} - \frac{p_3}{k.p_3}\right) \qquad\qquad j = 1,2$$

$$\alpha_3 = -\frac{m^2 a_0^2}{2}\left(\frac{1}{k.p_4} - \frac{1}{k.p_3}\right) \tag{4.3.29}$$

Manipulation leads to

$$\Gamma^{\mu}(r) = \int d\varphi\ \bar{u}_{p_2}\left[\gamma_{\mu} + \frac{e\gamma_{\mu}\not{k}\not{\mathcal{A}}}{2k.p_1} - \frac{e\not{\mathcal{A}}\not{k}\gamma_{\mu}}{2k.p_2} - \frac{e^2\not{\mathcal{A}}\not{k}\gamma_{\mu}\not{k}\not{\mathcal{A}}}{2k.p_1 2k.p_2}\right]\ u_{p_1}$$

$$exp\left(ir\varphi - i\alpha_j \int_0^{\varphi} d\varphi f_j(\varphi)\right), \tag{4.3.30}$$

where we have expanded

$$\left(1_4 - \frac{e\not{k}\not{\mathcal{A}}}{2k.p_2}\right)\gamma_{\mu}\left(1_4 + \frac{e\not{k}\not{\mathcal{A}}}{2k.p_1}\right) = \gamma_{\mu} + \frac{e\gamma_{\mu}\not{k}\not{\mathcal{A}}}{2k.p_1} - \frac{e\not{\mathcal{A}}\not{k}\gamma_{\mu}}{2k.p_2} - \frac{e^2\not{\mathcal{A}}\not{k}\gamma_{\mu}\not{k}\not{\mathcal{A}}}{2k.p_1 2k.p_2} \tag{4.3.31}$$

The amplitude for "nonlinear Compton scattering" now becomes

$$\Gamma^\mu(r) = \int d\varphi\ \bar{u}_{p_2}\Big[\gamma_\mu + \Big(\frac{e\gamma_\mu \not{k}\not{a}_j}{2k.p_1} - \frac{e\not{a}_j\not{k}\gamma_\mu}{2k.p_2}(\varphi)\Big) f_1(\varphi) + \Big(\gamma_\mu + \frac{e\gamma_\mu\not{k}\not{a}_j}{2k.p_1}$$

$$-\frac{e\not{a}_j\not{k}\gamma_\mu}{2k.p_2}(\varphi)\Big) f_2(\varphi) - \frac{a^2m^2k_\mu}{2k.p_1k.p_2}\not{k}f_3(\varphi)\Big)\Big]\ u_{p_1}\ exp\Big(ir\varphi - i\alpha_j\int_0^\varphi d\varphi f_j(\varphi)\Big) \quad (4.3.32)$$

with f_3 given in Eq(4.1.9) and that

$$\mathcal{A}_j^2(k.x) = -\frac{m^2a^2}{e^2}f_3(\varphi) \quad (4.3.33)$$

One can rewrite Eq(4.3.32) as,

$$\Gamma^\mu(r) = \int d\varphi\ \bar{u}_{p_2}\Big[\gamma_\mu\ exp\Big(ir\varphi - i\alpha_j\int_0^\varphi d\varphi f_j(\varphi)\Big)$$

$$+\sum_{j=1}^{2}\Big(\frac{e\gamma_\mu\not{k}\not{a}_j}{2k.p_1} - \frac{e\not{a}_j\not{k}\gamma_\mu}{2k.p_2}\Big) f_j(\varphi)\ exp\Big(ir\varphi - i\alpha_j\int_0^\varphi d\varphi f_j(\varphi)\Big)$$

$$-\frac{a^2m^2k_\mu}{2k.p_1k.p_2}\not{k}f_3(\varphi)\ exp\Big(ir\varphi - i\alpha_j\int_0^\varphi d\varphi f_j(\varphi)\Big)\Big]\ u_{p_1} \quad (4.3.34)$$

Now we define

$$B_0(r) = \int d\varphi\ exp\Big(ir\varphi - i\alpha_k\int_0^\varphi d\varphi f_k(\varphi)\Big) \quad (4.3.35)$$

and

$$B_j(r) = \int d\varphi\ f_j(\varphi)\ exp\Big(ir\varphi - i\alpha_k\int_0^\varphi d\varphi f_k(\varphi)\Big), \quad (4.3.36)$$

where $j,\ k = 1, 2, 3$.

We have then the amplitude for "nonlinear Compton scattering" in reduced form as,

$$\Gamma^\mu(r) = \bar{u}_{p_2}\Big[\gamma_\mu B_0(r) + \sum_{j=1}^{2}\Big(\frac{e\gamma_\mu\not{k}\not{a}_j}{2k.p_1} - \frac{e\not{a}_j\not{k}\gamma_\mu}{2k.p_2}\Big)B_j(r) - \frac{a^2m^2k_\mu}{2k.p_1k.p_2}\not{k}B_3(r)\Big]\ u_{p_1} \quad (4.3.37)$$

Similarly, we have the amplitude for "pair production" in reduced form as,

$$\Delta^\mu(s) = \bar{u}_{p_3}\Big[\gamma_\mu B_0(s) + \sum_{j=1}^{2}\Big(\frac{e\gamma_\mu\not{a}_j\not{k}}{2k.p_4} - \frac{e\not{a}_j\not{k}\gamma_\mu}{2k.p_3}\Big)B_j(s) - \frac{a^2m^2k_\mu}{2k.p_3k.p_4}\not{k}B_3(s)\Big]\ v_{p_4}, \quad (4.3.38)$$

where we have already defined

$$B_0(s) = \int d\varphi\ exp\Big(is\varphi - i\alpha_k \int_0^{\varphi} d\varphi f_k(\varphi)\Big) \tag{4.3.39}$$

and

$$B_j(s) = \int d\varphi\ f_j(\varphi)\ exp\Big(is\varphi - i\alpha_k \int_0^{\varphi} d\varphi f_k(\varphi)\Big) \tag{4.3.40}$$

All dependence on r and s is contained in the above four functions B_0 and B_j, for $j = 1, 2, 3$ and all the functions $f_k(\varphi)$ are defined by the sum over $k = 1, 2, 3$.

All the functions given by Equations (4.3.36) and (4.3.40) are finite due to the finite supports of $f_k(\varphi)$. The fourth function B_0, given by Equations (4.3.35) and (4.3.39), is the Fourier transform of a pure pulse and is put with out any damping factor of $f_k(\varphi)$ in the integral. Therefore, it seems to diverge since there is no shaping envelope in the integrand. However, the S-matrix in Eq(4.3.10) is gauge invariant [4,6] provided that

$$B_0(r) = \frac{\alpha_j B_j(r)}{r} \tag{4.3.41}$$

with a sum as usual going over $j = 1, 2, 3$, giving a convergent expression for this and thus all terms B.

Plugging Eq(4.3.26) into Eq(4.3.17) we obtain

$$S_{fi} = e^2(2\pi)^5 \frac{1}{V^2}\sqrt{\frac{m^4}{E_1E_2E_3E_4}}\delta^{if}(p_{out} - p_1)\Big[\frac{i\pi}{2k.\delta p}\Gamma^\mu(r')\Delta^\mu(s_{r'}) - \int dr \Gamma^\mu(r)\Delta^\mu(s_r)\mathcal{P}\frac{1}{(\delta p + rk)^2} - (p_2 \leftrightarrow p_3)\Big] \tag{4.3.42}$$

Now we introduce the bracketed term in Eq(4.3.42), containing all the dependence on the pulse profile as

$$K = \frac{i\pi}{2k.\delta p}\Gamma^\mu(r')\Delta^\mu(s_{r'}) - \int dr \Gamma^\mu(r)\Delta^\mu(s_r)\mathcal{P}\frac{1}{(\delta p + rk)^2} - (p_2 \leftrightarrow p_3) \tag{4.3.43}$$

We can easily write the exchange term, in the first term of Eq(4.3.43) by replacing the four-momenta p_2 with p_3 as

$$(p_2 \leftrightarrow p_3) = \frac{i\pi}{2k.\delta p}\mathit{\Gamma}^\mu(r')\mathit{\Delta}^\mu(s_{r'}) - \int dr \mathit{\Gamma}^\mu(r)\mathit{\Delta}^\mu(s_r)\mathcal{P}\frac{1}{(\delta p + rk)^2} \tag{4.3.44}$$

The S-matrix then becomes

$$S_{fi} = e^2(2\pi)^5 \frac{1}{V^2}\sqrt{\frac{m^4}{E_1E_2E_3E_4}}\delta^{if}(p_{out} - p_1)K \tag{4.3.45}$$

One now take $|S_{fi}|^2$ to calculate the rate of production R as

$$|S_{fi}|^2 = e^4(2\pi)^{10}\frac{1}{V^4}\frac{m^4}{E_1E_2E_3E_4}\Big(\delta^{if}(p_{out} - p_1)\delta^{if}(0)\Big)|K|^2 \tag{4.3.46}$$

The divergent delta function $\delta^{if}(0)$ is removed by calculating the pulse volume and duration as in [6,7]. Thus,

$$\begin{aligned} VT = \int_{pulse} d^4x &= \frac{1}{2}\int_{-\infty}^{\infty} d^3x^{\perp,-}\int_{pulse} dx^+ \\ &= \frac{1}{2}\int_{-\infty}^{\infty} d^3x^{\perp,-}\int_0^{2\pi N}\frac{d(k.x)}{k_+} \\ &= \frac{2\pi N}{2k_+}(2\pi)^3\delta^{\perp,-}(0), \end{aligned} \tag{4.3.47}$$

where $2\pi N$ is the Lorentz invariant duration and N is the number of oscillations of the field of the pulse in $k.x$ [6].

Thus, we find that

$$\delta^{if}(0) = \frac{2}{(2\pi)^4}\frac{VT}{N} \tag{4.3.48}$$

Substituting Eq(4.3.48) into Eq(4.3.49) we get

$$|S_{fi}|^2 = 2e^4(2\pi)^6\frac{T}{N}\frac{1}{V^3}\frac{m^4}{E_1E_2E_3E_4}\delta^{if}(p_{out} - p_1)|K|^2 \tag{4.3.49}$$

The rate of production is given by [4,7,11]

$$R = \frac{1}{T}\int\frac{Vd^3p_4}{(2\pi)^3}\int\frac{Vd^3p_3}{(2\pi)^3}\int\frac{Vd^3p_2}{(2\pi)^3}\frac{1}{4}\sum_{spins}|S_{fi}|^2 \tag{4.3.50}$$

Plugging Eq(4.3.49) into Eq(4.3.50) reveals

$$\begin{aligned} R = e^4(2\pi)^6\frac{1}{2N}\frac{m^4}{E_1E_2E_3E_4}\int\frac{d^3p_4}{(2\pi)^3}\int\frac{d^3p_3}{(2\pi)^3}\int\frac{d^3p_2}{(2\pi)^3}\delta^{if}(p_{out} - p_1) \\ \sum_{spins}|K|^2\Big|_{shell}, \end{aligned} \tag{4.3.51}$$

Here there are nine integrals corresponding to three momentum components for three outgoing particles, and again it is natural to use light cone variables. Three of these are eliminated by the remaining δ^{if}. The instruction 'shell' indicates that each p_+ is evaluated on shell, i.e

$$p_+ = \frac{p_\perp^2 + m^2}{4p_-} \tag{4.3.52}$$

and p_2, which is eliminated by momentum conservation with the momentum balance

$$p_{2\perp} = (p_1 - p_3 - p_4)_\perp$$

$$p_{2-} = (p_1 - p_3 - p_4)_- \tag{4.3.53}$$

The remaining integrals are over the momenta of the produced electron-positron pair. Thus, we write the rate of production in reduced form as

$$R = e^4 \frac{1}{2N} \frac{m^4}{E_1 E_2 E_3 E_4} \int d^3 p_{4\perp,-} \int d^3 p_{3\perp,-} \ \mathcal{J}, \tag{4.3.54}$$

with

$$\mathcal{J} = \sum_{spins} |K|^2, \tag{4.3.55}$$

which contains all the pulse profile.

The spin summation in Eq(4.3.55) can be evaluated as

$$\mathcal{J} = \sum_{spins} \Bigg[\frac{\pi^2}{(2k.\delta p)^2} \Big| \Gamma^\mu(r') \Delta^\mu(s_{r'}) + \mathit{\Gamma}^\mu(r') \mathit{\Delta}^\mu(s_{r'}) \Big|^2$$

$$- \int dr \ \mathcal{P}^2 \frac{1}{(\delta p + rk)^4} \Big| \Gamma^\mu(r) \Delta^\mu(s_r) + \mathit{\Gamma}^\mu(r) \mathit{\Delta}^\mu(s_r) \Big|^2 \Bigg] \tag{4.3.56}$$

We expand terms and abbreviate as

$$\mathcal{J} = \sum_{spins} \Bigg[\frac{\pi^2}{(2k.\delta p)^2} \bigg(M^{p_1 p_2}(r', s_{r'}) + M^{p_2 p_3}(r', s_{r'}) + M^{ext}(r', s_{r'}) \bigg)$$

$$- \int dr \ \mathcal{P}^2 \frac{1}{(\delta p + rk)^4} \bigg(M^{p_1 p_2}(r, s_r) + M^{p_2 p_3}(r, s_r) + M^{ext}(r, s_r) \bigg) \Bigg], \tag{4.3.57}$$

where

$$M^{p_1p_2}(r, s_r) = \Delta^{\mu*}(s_r)\Gamma^*_\mu(r)\Gamma^\mu(r)\Delta_\mu(s_r) \tag{4.3.58}$$

$$M^{p_2p_3}(r, s_r) = \mathit{\Delta}^{\mu*}(s_r)\mathit{\Gamma}^*_\mu(r)\mathit{\Gamma}^\mu(r)\mathit{\Delta}_\mu(s_r) \tag{4.3.59}$$

and the third term accounts for the interference terms of diagrams with exchanged electrons in the final state.

$$\begin{aligned} M^{ext}(r, s_r) = {} & \Delta^{\mu*}(s_r)\Gamma^*_\mu(r)\mathit{\Gamma}^\mu(r)\mathit{\Delta}_\mu(s_r) \\ & + \mathit{\Delta}^{\mu*}(s_r)\mathit{\Gamma}^*_\mu(r)\Gamma^\mu(r)\Delta_\mu(s_r) \end{aligned} \tag{4.3.60}$$

The "nonlinear Compton scattering" and the "pair production" amplitudes in Eq(4.3.37) and (4.3.38) can also be put respectively as,

$$\Gamma^\mu(r) = \bar{u}_{p_2} T_\mu u_{p_1} \tag{4.3.61}$$

$$\Delta^\mu(s_r) = \bar{u}_{p_3} T'_\mu v_{p_4}, \tag{4.3.62}$$

where T_μ and T'_μ are introduced as a collection of the individual terms in the functions $\Gamma^\mu(r)$ and $\Delta^\mu(s)$ and given respectively as,

$$T_\mu = \gamma_\mu B_0(r) + \sum_{j=1}^{2} \left(\frac{e\gamma_\mu \not{k}\not{a}_j}{2k.p_1} - \frac{e\not{a}_j\not{k}\gamma_\mu}{2k.p_2} \right) B_j(r) - \frac{a^2m^2k_\mu}{2k.p_1k.p_2}\not{k}B_3(r) \tag{4.3.63}$$

$$T'_\mu = \gamma_\mu B_0(s_r) + \sum_{j=1}^{2} \left(\frac{e\gamma_\mu \not{a}_j\not{k}}{2k.p_4} - \frac{e\not{a}_j\not{k}\gamma_\mu}{2k.p_3} \right) B_j(s_r) - \frac{a^2m^2k_\mu}{2k.p_3k.p_4}\not{k}B_3(s_r) \tag{4.3.64}$$

Here we note that

$$T_\mu = T_\mu(r) \qquad \text{and} \qquad T'_\mu = T'_\mu(s_r)$$

Upon substitution of Eq(4.3.61) and Eq(4.3.62), Eq(4.3.58) becomes

$$\sum_{spins} M^{p_1p_2}(r, s_r) = \sum_{spins} \left(\bar{v}_{p_4} T'^*_\mu u_{p_3}\right)\left(\bar{u}_{p_1} T^*_\mu u_{p_2}\right)\left(\bar{u}_{p_2} T_\mu u_{p_1}\right)\left(\bar{u}_{p_3} T'_\mu v_{p_4}\right) \tag{4.3.65}$$

Now using the spin sum formulas as given in [7,11]

$$\sum_{s=1}^{2} u_p \bar{u}_p = \frac{\not{p} + m}{2m}$$

$$\sum_{s=1}^{2} v_p \bar{v}_p = \frac{\not{p} - m}{2m}, \tag{4.3.66}$$

we write Eq(4.3.65) in trace form as

$$\sum_{spins} M^{p_1p_2}(r, s_r) = \frac{1}{16m^4} Tr\Big[(\not{p}_2 + m)T_\mu(\not{p}_1 + m)T^*_\mu\Big] Tr\Big[(\not{p}_3 + m)T'_\mu(\not{p}_4 - m)T'^*_\mu\Big] \tag{4.3.67}$$

while the spin summations of Eq(4.3.59) and Eq(4.3.60) are

$$\sum_{spins} M^{p_2p_3}(r, s_r) = \frac{1}{16m^4} Tr\Big[(\not{p}_3 + m)T_\mu(\not{p}_1 + m)T^*_\nu\Big] Tr\Big[(\not{p}_2 + m)T'^\mu(\not{p}_4 - m)T'^{\nu *}\Big] \tag{4.3.68}$$

and

$$\sum_{spins} M^{ext}(r, s_r) = \frac{1}{16m^4} \Big\{ Tr\Big[(\not{p}_4 - m)T'^*_\mu(\not{p}_3 + m)T_\nu(\not{p}_1 + m)T^{\mu *}(\not{p}_2 + m)T'^\nu\Big]$$

$$+ \; Tr\Big[(\not{p}_4 - m)T'_{\mu *}(\not{p}_2 + m)T_\nu(\not{p}_1 + m)T^{\mu *}(\not{p}_3 + m)T'^\nu\Big]\Big\} \tag{4.3.69}$$

We divide the rate of production in Eq(4.3.54) by the incident energy flux of the electron,

$$I = \frac{1}{E_1 V}\sqrt{p_1^2} \tag{4.3.70}$$

to arrive at the cross section [7,11].

Thus, the cross section for our considered process becomes

$$\sigma = e^4 \frac{1}{2IN} \frac{1}{E_1E_2E_3E_4} \int d^3p_{4\perp,-} \int d^3p_{3\perp,-} \; \mathcal{J} \tag{4.3.71}$$

with $\mathcal{J}$ now becomes

$$\mathcal{J} = \frac{\pi^2}{(2k.\delta p)^2}\Big[Z^{p_1p_2}(r', s_{r'}) + Z^{p_2p_3}(r', s_{r'}) + Z^{ext}(r', s_{r'})\Big]$$

$$+ \int dr \; \mathcal{P}^2 \frac{1}{(\delta p + rk)^4}\Big[Z^{p_1p_2}(r, s_r) + Z^{p_2p_3}(r, s_r) + Z^{ext}(r, s_r)\Big], \tag{4.3.72}$$

where

$$Z^{p_1p_2}(r,s_r) = Tr\Big[(\not{p}_2+m)T_\mu(r)(\not{p}_1+m)T^*_\mu(r)\Big]Tr\Big[(\not{p}_3+m)T'_\mu(s_r)(\not{p}_4-m)T'^*_\mu(s_r)\Big] \quad (4.3.73)$$

$$Z^{p_2p_3}(r,s_r) = Tr\Big[(\not{p}_3+m)T_\mu(r)(\not{p}_1+m)T^*_\nu(r)\Big]Tr\Big[(\not{p}_2+m)T'^\mu(s_r)(\not{p}_4-m)T'^{\nu *}(s_r)\Big] \quad (4.3.74)$$

$$Z^{ext}(r,s_r) = Tr\Big[(\not{p}_4-m)T'^*_\mu(s_r)(\not{p}_3+m)T_\nu(r)(\not{p}_1+m)T^{\mu *}(r)(\not{p}_2+m)T'^\nu(s_r)\Big]$$
$$+ \; Tr\Big[(\not{p}_4-m)T'^*_\mu(s_r)(\not{p}_2+m)T_\nu(r)(\not{p}_1+m)T^{\mu *}(r)(\not{p}_3+m)T'^\nu(s_r)\Big] \quad (4.3.75)$$

An important step in the calculation of the rate and cross section is the evaluation of the functions B_j, given by Eq(4.3.36), Eq(4.3.40) and B_0, given by Eq(4.3.35), Eq(4.3.39). The integrals in these functions will converge due to the finite supports of $f_k(\varphi)$ and we have escaped the infinite sum of the cross section in Eq(3.0.37) with an infinite plane wave background. However, solving the integrals B_j is still a difficult task and hence approximation methods can be used to estimate them [6]. In general, these are finite and must be calculated numerically [4].

Chapter 5

Conclusion

We have shown how we can include a background field in the theory. We have shaped our laser field into finite laser pulse. Finite size effects has been included due to the ultrashort duration of modern pulses. The first full calculation of trident pair production in a laser field was given in Ref. [4], using strong-field QED. For the first time we have calculated the rate and cross section of the considered process in chapter 4. The results we obtained are finite and physical. The "nonlinear Compton scattering" and "pair production" amplitudes are calculated. It has been shown that there is no divergence in the trident amplitude that had been identified in earlier approaches through the optical theorem.

Despite the length of expressions involved, the final result for the rate and cross section is not more complicated than that of the periodic plane wave case in Ref. [1]. Three real differences had been identified in Ref. [4] by comparing the expressions of the periodic plane wave case, in chapter 3 and of finite laser pulses, in chapter 4. The first was that the discrete sums over photon number are replaced by Fourier integrals over r and s. The second was that the shifted mass in Eq. (3.0.24) plays, in general, no role in all expressions. Finally, the S-matrix contains two distinct terms corresponding to the one- and two-step processes. It was also suggested in Ref. [4] that the numerical methods previously employed to calculate the amplitude are equally appropriate in Ref. [4].

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Declaration

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