Addis Ababa University
School of Graduate Studies

Faculty of Computer and Mathematical Science
Department of Mathematics
Project on

Quasidifferentiable Optimization and Minimal Pairs of Compact Convex Sets

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Declaration

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship, or any other similar title to me.

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Permission

This is to certify that this project is compiled by ***Ms. Haimanot Balew*** *in the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.*

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To all my family

Abstract

In spite of the fact that the origin of non-smooth optimization as a mathematical discipline is quite recent; it is now well established as an important and very active branch of applied mathematics. Many practical problems in economics, physics, aerospace, as well as other areas of applications cannot be adequately described without the help of non-smooth functions.

In the theory of optimization several types of piecewise differentiable functions occur in quite natural way. As a typical example for such non-differentiable functions we mention the finite max-min combinations of differentiable functions. A more general class is the quasidifferentiable functions which are investigated in detail by V.F.Demyanov and A.M.Rubinov. The directional derivatives of these functions can be represented as a difference of two sublinear functions. Since a sublinear function is uniquely described by its subdifferential in the origin, there exists a natural correspondence between the directional derivatives and the set of pairs of compact convex sets. However, this representation is not unique. This nonuniqueness inspires mathematicians to find a minimal representation of the directional derivative, which is equivalent to finding a minimal pair of compact convex sets. In this project, the theory of quasidifferentiable optimization and minimal pairs of compact convex sets is discussed. In the first chapter, general introduction and description of the problem are given. In the second chapter basic definitions and concepts are mentioned and in the last chapter the detail discussion of minimal pairs of compact convex sets is given.

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List of Notations

$\underline{\partial}$	Subdifferential of a function
$\overline{\partial}$	Superdifferential of a function
$\sim$	Equivalent to
$\langle u, x \rangle$	Inner product of vectors u and x
[A, B]	An element of the Rádstrom- Hörmander lattice determined by a pair of compact convex sets (A, B)
$f'(x; d)$	Directional derivative of f in the direction of d
K(X)	Set of nonempty convex compact subsets of a vector space X
B(X)	Set of all bounded closed convex sets of X
$A \dot{+} B$	Minkowski sum of A and B
$A \dot{-} B$	Minkowski difference of A and B
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
aff A	Affine hull of a set A
conv A	Convex hull of a set A
dimK	Dimension of K
epi f	Epigraph of f
$A \vee B$	Closed convex hull of $A \cup B$
$\widetilde{X}$	Minkowski- Rádström-Hörmander space over X
H (K,.)	Supporting hyperplane
h (K,.)	Supporting function of a set K
■	End of the proof

Chapter 1

Introduction

In the theory of optimization the main objective is finding a minimize of an objective function defined on some given set. For doing this we consider a well defined function $f: \mathbb{R}^n \to \mathbb{R}$ that is locally Lipschtiz on $\mathbb{R}^n$ and which is smooth or non-smooth. If the function involved is smooth, there are different methods to solve like penalty, lagrangian, and other methods. What happens if the function is non-smooth? In this case most of the methods depend on the directional derivative of the problem functions.

If f is non-smooth, the directional derivative (or its generalization) $f'(x; d)$ of f is non linear with respect to d. To treat the non linearity of the directional derivative we use convex analysis. This may fail because the map $d \to f'(x; d)$ is not convex for general non-smooth function f. Thus to convexify the map $d \to f'(x; d)$, the generalized gradient of F. H. Clarke, which is found by regularizing the directional derivative could be the first mentioned here. However, Clarke's generalized gradient, $\partial_{Cl} f(x)$, is sometimes too large elements are also difficult to calculate even though the Clarke generalized directional derivative, $f^0(x; d)$, is convex.

The space of pairs of convex compact sets has been investigated in a number of papers [1], [4], [5]. Recently this space has found application in quasidifferential calculus. A quasidifferential is represented as a pair of convex compact sets and it is essential to find a minimal representation of this pair. In quasidifferential calculus the minimal representations of quasidifferentials (minimal pairs of compact convex sets) plays a significant role. It is known that such a minimal representation always exists see [4], [5]. However, it is not always known how to find it. In the literature, there exist several criteria which, in some cases, help to find such representations. Pairs of compact convex sets naturally arise in quasidifferential calculus as the subdifferentials and superdifferentials of the directional derivative of a quasidifferential function. A quasidifferential of a function is not uniquely determined. The nonuniqueness of quasidifferentiable function pushes expertise like D.Pallachke and R.Urbanski [5] to the study of the problem of finding minimal representation of each class.

The general framework for the investigation of minimal pairs of nonempty compact convex sets is the Raảdström-Hörmander lattice over a real topological vector space of pairs of nonempty compact convex sets. Let X be a nontrivial Hausdorff topological vector space and X* its dual space (space of all continuous linear functional on X). Let B(X) be the family of all nonempty closed bounded sets in X and K(X) the family of all nonempty compact convex sets in X. We put $B^2(X) = B(X) \times B(X)$ and $K^2(X) = K(X) \times K(X)$. Now we introduce some further notation:

A, B ∈ B(X), we call $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$ the algebraic summand and

$A \dotplus B = \overline{A + B}$ the Minkowski sum of A and B. For two pairs (A, B), (C, D) ∈ $B^2(X)$, we write (A, B) ∼ (C, D) if and only if $A \dotplus D = B \dotplus C$ and call them equivalent. Let [A, B] be the quotient class of (A, B), then $[A, B] \in B^2(X)/\sim\, = \widetilde{X}$, where $\widetilde{X}$ is the Minkowski- Rádström-Hörmander space over X.

A pair (A, B) is called minimal if it is a minimal element in [A, B] with respect to '≤ '. A minimal pair (A, B) has the property of translation if the family $\{(A + x, B + x) \mid x \in X\}$ is the family of all minimal pairs that are equivalent to (A, B). Finally, a pair (A, B) is called reduced if $[A, B] = \{(A \dotplus C,\ B \dotplus C) | C \in B(X)\}$.

For A,B ∈ B(X), we denote the Minkowski subtraction by $A \dot{-} B$, i.e. $A \dot{-} B = \{x \in X \mid x + B \subseteq A\}$. Let us define the following ordering on $B^2(X)\ \backslash_{\sim,}$ namely $[A, B] \preccurlyeq [C, D] \Leftrightarrow A \dotplus D \subseteq B \dotplus C$ this ordering is independent of the special choice of representatives, since for (A', B') ∈ [A, B] and (C', D') ∈ [C, D], the inclusion $A \dotplus D \subseteq B \dotplus C \Longrightarrow A' + D' \subseteq B' + C'$. The order space $(B^2(X)\ \backslash_{\sim,} \preccurlyeq)$ is called the Minkowski-Rådström-Hörmander lattice of bounded closed convex sets.

Define $\sup\{[A, B], [C, D]\} = [(A \dotplus D) \vee (C \dotplus B), B \dotplus D] \in B^2(X)\backslash_{\sim}$. A multiplication between elements in $B^2(X)\backslash_{\sim}$ is given by the Minkowski sum $[A, B] * [C, D] = [A \dotplus C, B \dotplus D]$ for [A, B], $[C, D] \in B^2(X)\backslash_{\sim}$.

The multiplicative inverse of $[A, B] \in B^2(X)\backslash_{\sim}$ is given by $[B, A] \in B^2(X)\backslash_{\sim}$. For all [A, B], [C, D], $[E, F] \in B^2(X)\backslash_{\sim}$ we have [A, B]*(sup {[C, D], [E, F]}) = sup {([A, B]*[C, D]), ([A, B]*[E,F])}.

Cancelation law is possible for * that is [A, B]* [C, D] = [E, F]*[C, D] implies that

[A, B] = [E, F], but for "sup" cancelation law is not true, let us see a counter example.

Example: Let X= $\mathbb{R}$ and let us choose the following compact convex sets A=B=E=D= {0}, F = {1} and C = [-1, 1] then,

> $[A, B] \leq [C, D]$ since $A \dotplus D \subseteq B \dotplus C \Longrightarrow \{0\} + \{0\} \subseteq \{0\} + [-1, 1]$
>
> $[E, F] \leq [C, D]$ since $E \dotplus D \subseteq F \dotplus C \Longrightarrow \{0\} + \{0\} \subseteq \{1\} + [-1, 1] = [0, 2]$
>
> Hence; sup {[A, B], [C, D]} = [C, D], sup {[E, F], [C, D]} = [C, D], but [A, B] ≠ [E, F] since A + F= {1} and B + E= {0}.

Chapter 2

Definitions and Preliminary Concepts

2.1. Ordered sets

An ordered set (X, ≼) is a pair consisting of a non-empty set X and a binary relation "≼ " called ordering which satisfies the following conditions:

- (Reflexivity): for every x ∈ X, x ≼ x
- (Transitivity): if x ≼ y and y ≼ z, then x ≼ z.
- (Anti-symmetric): if x ≼ y and y ≼ x, then x = y.

If the anti-symmetric condition is not postulated, then the relation is called a pre-order. If for every x, y ∈ X at least one of the relations x ≼ y or y ≼ x holds, then X is called totally ordered and if for every x, y ∈ X there exists an element z ∈ X with x ≼ z and y ≼ z, then x is called a directed set. A totally ordered subset of X is called a chain.

An element x_0 ∈ X is called minimal, if for every x ∈ X with x ≼x_0 it follows that x=x_0.

If Y ⊆ X, then x_0 ∈ X is called a lower bound of Y if for all y ∈ Y the relation x_0 ≼y holds.

Theorem 2.1.1 (Kuratowski-Zorn's Lemma) If every chain of an ordered set (X, ≼) has a lower bound (resp. upper bound), then there exists a minimal (resp. maximal) element in X.

Sublinear Functions

Let X is a vector space. A function P: X ⟶ ℝ is said to be sublinear if

i. P(λx) = λP(x) ∀x ∈X and all λ ≥0
ii. P(x + y) ≤ P(x) + P(y) ∀x, y ∈X hold.

Note:

Every sublinear function defined on a finite dimensional space is continuous and Lipschitz on the entire space because of this an isomorphism between the set of compact convex set and the set of closed sublinear functions. This was given in Demyanov, V [1].

2.2. Affine and Convex sets

Affine set: A subset M of $\mathbb{R}^n$ is an affine set if (1-λ) x + λy ∈ M, ∀ x, y ∈ M and λ ∈ ℝ.

The set of all affine combination of points in some set M ⊆ $\mathbb{R}^n$ is called affine hull of M, and denoted by aff M

$$\text{aff } M = \{\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n : x_1, x_{2,\ldots,} x_n \in M, \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\}$$

Affine hull: It is the smallest affine set that contains M. If S is any affine set with $M \subseteq S$, then $\text{aff}\, M \subseteq S$.

Linear subspace: Let X be a vector space and let $A \subseteq X$. Then A is a linear subspace of X if and only if:

i. $x, y \in A \Longrightarrow x + y \in A$

ii. $\alpha \in \mathbb{R}, x \in A \Longrightarrow \alpha x \in A$

Hyperplane: Let X be a vector space and let $u \in X \setminus \{0\}$, $\alpha \in \mathbb{R}$, then $H = \{x \in X | \langle u, x\rangle = \alpha\}$ is called a hyperplane in X. Where $\langle u, x\rangle$ is the inner product of u and x.

For $\alpha \in \mathbb{R}$ and $u \in \mathbb{R}^n \setminus \{0\}$ the sets

$H^{-} = \{x \in \mathbb{R}^n \mid \langle u, x\rangle \leq \alpha\}$ and

$H^{+} = \{x \in \mathbb{R}^n | \langle u, x\rangle \geq \alpha\}$ are called halfspaces associated with the hyperplane H. If the inequalities involved in the definitions of the above two sets are strict; the sets are called open halfspaces.

Convex set: A set $K \subseteq \mathbb{R}^n$ is convex if and only if $x, y \in K \Longrightarrow \lambda x + (1-\lambda) y \in K \ \forall \lambda \in [0,1]$.

Note that: All affine sets are convex but not the converse.

Convex hull: The convex hull of a subset A of K is the smallest convex set which contains A, and denoted by convA.

i.e. $\text{conv}A = \{K | A \subseteq K \subseteq \mathbb{R}^n,\ K \text{ is convex}\}$

A convex combination of vectors $y_1, y_2, \ldots, y_m$ is their affine combination with nonnegative coefficients or which is the same as a linear combination

$$y = \sum_{i=1}^{m} \lambda_i y_i$$

with nonnegative coefficient and with unit sum:

$$\lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1$$

Relative interior of convex sets:

A point y belongs to the relative interior of a convex set $K \subseteq \mathbb{R}^n$, written $y \in \text{ri}K$, if it is an interior point of K relative to affK. The set (clK)/ (riK) is called the relative boundary of K and denoted by rbK, where clK denotes the topological closure of K with respect to the usual Euclidean topology on $\mathbb{R}^n$.

A hyperplane H supports a set $S \subseteq \mathbb{R}^n$ if either $\sup \{\langle u, x\rangle | x \in S\} = \alpha$ or $\inf\{\langle u, x\rangle \mid x \in S\} = \alpha$, where u is the normal vector of H and α is the constant determining H.

Theorem 2.2.1 (Separation Theorem) Two disjoint nonempty convex sets K, M in $\mathbb{R}^n$ can be separated by a hyperplane.

Using the above theorem we can prove:

Theorem 2.2.2 Let K be a nonempty convex set in $\mathbb{R}^n$, and x* be a boundary point of K. then there exists a nonzero vector $u \in \mathbb{R}^n$ such that $\langle x^*, u\rangle = \sup\{\langle u, x\rangle | x \in K\}$; that is, there exists a hyperplane supporting K at x*. In this case, x* is said to be a supporting point of K with normal vector (or supporting functional) u.

Proof:

Since x* is not in the relative interior of K, by separation theorem, there exists a hyperplane $H_{u,\alpha}$ such that $\langle x*, u\rangle \leq \alpha \leq \langle x, u\rangle \quad \forall x \in riK$ which implies that $\langle u, x^* - x\rangle \leq 0$ for $x \in K$. Thus the hyperplane $H=\{x \in \mathbb{R}^{\mathbf{n}} | \langle u, x^* - x\rangle = 0\}$ is a supporting hyperplane to K at x*. ■

Minkowski Addition: It is the vector sum of two nonempty sets A, B ⊆ $\mathbb{R}^n$ and is given by:

$$A + B = \{a + b | a \in A, b \in B\}$$

Note:

- If either A or B is empty, then A + B = ∅.
- If the space is infinite dimensional, Minkowski addition is defined as the closure of the above set. i.e. cl(A+B) = A ∔ B.
- For any number λ, the scalar multiplication of a set A is given by: $\lambda A = \{\lambda a | a \in A\}$

The Minkowski addition and the scalar multiplication on the set of all convex sets in $B(\mathbb{R}^n)$ defines a commutative ring.

Definition 2.2.3 A subset F of a convex set K is called a face of K if whenever a point x ∈ F is contained in a line segment joining two points y, z ∈ K, then both points are contained in F; i.e. the condition x, y ∈ K and $(\alpha x + \beta y)/(\alpha + \beta) \in F$ for $\alpha, \beta \in \mathbb{R}$ and $\alpha, \beta \geq 0$ imply x, y ∈ F.

Let X be a real topological vector space, $f \in X^*$ be a continuous linear functional, and K ⊆ X a nonempty compact convex set, then we denote by:

$$H_f(K) := \{z \in K \mid f(z) = \max_{y \in K} f(y)\} \text{ the face of K with respect to } f.$$

For the sum of the faces of two nonempty compact convex sets A, B ⊆ X with respect to $f \in X^*$ the following identity holds: $H_f(A + B) = H_f(A) + H_f(B)$.

Definition 2.2.4 A point x is called an extremal point of K, if the singleton set $\{x\}$ is an extremal set of K and the set of all points where a certain linear function achieves its maximum over K is called an exposed set (face) of K. An exposed point is a point through which there is a supporting hyperplane of K, which contains no other point of K. For A ∈ K(X) we denote by $\mathcal{E}(A)$ the set of extremal points, and by $\mathcal{E}_0(A)$ the set of exposed points.

Definition 2.2.5 The supporting function h(K, x) of a nonempty set $K \subseteq \mathbb{R}^n$, is defined for all $x \in \mathbb{R}^n$, by $h(K, x) = \sup \{\langle y, x\rangle \mid y \in K\}$.Thus for a nonempty closed convex set $K \subseteq \mathbb{R}^n$, the sets

$$H(K, u) = \{x \in \mathbb{R}^n \mid \langle x, u\rangle = h(K, u)\}$$

$$H^{-}(K, u) = \{x \in \mathbb{R}^n \mid \langle x, u\rangle \preccurlyeq h(K, u)\}$$

$$F(K,u)=\{x \in K \mid \langle x, u\rangle = h(K, u)\}$$ are respectively called the supporting hyperplane, the supporting subspace and the supporting set (or max-face) of the set K with normal vector u.

Definition 2.2.6 The dimension of a convex set K, denoted by $dimK$, is defined by the dimension of the affine hull of the set K.

Note:

- The empty set and K itself are faces of K, called the improper faces; all other faces are called proper faces.
- A face F of K is called a k-face if $dimF = k$. Thus, the 0-faces are the extermal points and a facet of K is a face F with $0 \leq dimF = dimK - 1$.

Some important examples

- The empty set $\emptyset$, any single point (i.e. singleton) $\{x_0\}$, and the whole space $\mathbb{R}^n$ are affine (hence, convex) subsets of $\mathbb{R}^n$.
- Any line is affine. If it pass through the origin, it is a subspace, and hence also a convex cone.
- A line segment is convex, but not affine (unless it reduces to a point)
- Solid ellipsoids and cubes in $\mathbb{R}^3$ are convex but not affine.
- Any subspace is affine , and a convex cone (hence convex)
- Affine set of dimensions 1 and 2 are called hyperplanes in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

2.3 Convex Function

Let $K \subseteq \mathbb{R}^n$ is a convex set. Then the function f: $K \longrightarrow \mathbb{R}^n$ is said to be convex on K if

$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)\ \forall\ x, y \in K$ and all $\lambda \in [0,1]$ (2.3.1)

whenever the right hand side is defined.

Some properties of the function f: $K \longrightarrow \mathbb{R}^n$ for $K \subseteq \mathbb{R}^n$:

1. If f is convex and strict inequality holds in (2.3.1), when $\lambda \in (0, 1)$ and $x \neq y$, then f is known to be strictly convex.
2. Concave on K if $-f$ is convex on K, and f is affine if f is finite and convex and concave on K.
3. Let $K \subseteq \mathbb{R}^n$ and f: $K \longrightarrow \mathbb{R}^n$. Then the set epi $f = \{(x, r) \in \mathbb{R}^{n+1} \mid x \in K, f(x) \leq r\}$ is called epigraph of f.
4. The set $\{x \mid f(x) < \infty\}$ is the effective domain of f and we denote it by dom f (this set is the projection of epi f on $\mathbb{R}^n$ and hence is convex).
5. A convex function f defined on a convex set K is said to be proper if $f(x) < +\infty$ for atleast one point $x \in K$, and if $f(x) > -\infty$ everywhere in K.
6. A convex function f is said to be stable at $x_0 \in$ dom f if there exists a number M with $0 \leq M < \infty$ such that

$$f(x_0) - f(x) \leq M \| x_0 - x \|_2 \text{ for all } x \in \text{dom } f$$

Theorem 2.3.1 Let X is any normed space and $K \subseteq X$ be an open set. If f: $K \longrightarrow \mathbb{R}$ be twice continuously Fréchet-differentiable, then f is convex if and only if $f''(x_0)$ is a positive semi-definite operator for all $x_0 \in K$.

Proof: ($\Longrightarrow$) Let f be convex and $x_0 \in K$ arbitrary, for $x \in X$,

define $\varphi(\lambda) = f(x_0 + \lambda x)$, where $x_0 + \lambda x \in K$, then φ is convex on [0,1]

$\Longrightarrow \varphi''(\lambda) \geq 0$, for all $\lambda \in [0, 1]$ such that $x_0 + \lambda x \in K$,

take $x = y - x_0$, thus $x_0 + \lambda x = x_0 + \lambda(y - x_0) \Longrightarrow \varphi''(0) \geq 0$

but $\varphi'(\lambda) = \frac{d}{d\lambda} f(x_0 + \lambda x) = f'(x_0 + \lambda x).x$

$= \langle f'(x_0 + \lambda x), x\rangle$ and

$\varphi''(\lambda) = \frac{d}{d\lambda}(\frac{d}{d\lambda}(f(x_0 + \lambda x)) = \langle f''(x_0 + \lambda x), x\rangle$

$\Longrightarrow \varphi''(0) = \langle f''(x_0 + 0), x\rangle = \langle f''(x_0), x\rangle \geq 0$

$\therefore f''(x_0)$ is positive semi-definite.

Since x_0 is arbitrarily chosen from K, $f''(x_0)$ is positive definite for all $x_0 \in K$.

($\Leftarrow$) Let $f''(x_0)$ is positive definite for all $x_0 \in K$ we need to show that f is convex. For $x_1, x_2 \in K$, by mean value theorem we have

$\langle f'(x_2), (x_2 - x_1)\rangle - \langle f'(x_1), x_2 - x_1\rangle = \langle f'(x_2) - f'(x_1), x_2 - x_1\rangle$

$\Rightarrow \langle f'(x_2) - f'(x_1), x_2 - x_1\rangle = \langle f'(x_2), (x_2 - x_1)\rangle - \langle f'(x_1), x_2 - x_1\rangle$

$\geq \langle f''\big(x_1 + \varepsilon(x_2 - x_1)\big), x_2 - x_1\rangle \geq 0$

$\Rightarrow \langle f'(x_2) - f'(x_1),\ x_2 - x_1\rangle \geq 0$

$\Rightarrow f'(x)$ is increasing on K

$\therefore f$ is convex. (By theorem which says that if f: K⟶ ℝ is Frechet differentiable on K, then f is convex if and only if the mapping $f'(x) = f'(x,.)$ is increasing on K) ■

Definition 2.3.2 Let $K \subseteq X$ and f: K⟶ ℝ and let $\alpha \in \mathbb{R}$, $N_\alpha(f) = \{x \in K |\ f(x) \leq \alpha\}$ is called a level set of f.

Proposition 2.3.3 Let f be a convex function, then the level set $N_\alpha(f)$ is convex set for each $\alpha \in \mathbb{R}$.

Proof: Let $x, y \in N_\alpha(f)$ we need to show that $\lambda x + (1-\lambda) y \in N_\alpha(f)$

The convexity of f implies $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$ since $x, y \in N_\alpha(f)$,

by definition of $N_\alpha(f)$ we have $\lambda f(x) + (1-\lambda) f(y) \leq \lambda\alpha + (1-\lambda)\alpha = \alpha$

$\Rightarrow f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) = \alpha$

$\Rightarrow f(\lambda x + (1-\lambda)y) \leq \alpha$ and hence $\lambda x + (1-\lambda) y \in N_\alpha(f)$

$\therefore N_\alpha(f)$ is convex set ■

Definition 2.3.4 Let f: K⟶ ℝ, where K is nonempty and convex set, then

1. f is said to be quasi-convex, if for all $x_1, x_2 \in K$ and $\lambda \in [0,1]$

   $f(\lambda x_1 + (1-\lambda)x_2) \leq \max\{f(x_1), f(x_2)\}$
2. f is quasi-concave if $-f$ is quasi-convex.
3. Quasi-monotone function is quasi-concave and quasi-convex function.

Theorem 2.3.5 Let K be a nonempty convex set and f: K⟶ ℝ be a function, f is quasi-convex if and only if $N_\alpha(f)$ is convex for each $\alpha \in \mathbb{R}$.

Proof: ($\Rightarrow$) Suppose f is quasi-convex and $x_1, x_2 \in N_\alpha(f)$

$\Rightarrow f(x_1) \leq \alpha$, $f(x_2) \leq \alpha \Rightarrow \alpha \geq \max\{f(x_1), f(x_2)\}$

Let $z = \lambda x_1 + (1-\lambda)x_2$, with $\lambda \in [0, 1]$, then $z \in K$

$f(z) \leq \max\{ f(x_1), f(x_2)\} \leq \alpha \quad \Rightarrow z \in N_\alpha(f)$

$\Rightarrow N_\alpha(f)$ is convex for all $\alpha \in \mathbb{R}$.

($\Leftarrow$) Suppose $N_\alpha(f)$ is convex for $\alpha \in \mathbb{R}$, now let $x_1, x_2 \in K$ and $\lambda \in [0, 1]$,

$z = \lambda x_1 + (1-\lambda)x_2 \in K$,

for $\alpha = \max \{ f(x_1), f(x_2)\}$, $x_1, x_2 \in N_\alpha(f)$

since $N_\alpha(f)$ is convex, $z \in N_\alpha(f)$

$\Rightarrow f(z) \leq \alpha = \max\{f(x_1), f(x_2)\}$

$\Rightarrow f$ is quasi convex. ∎

Definition 2.3.6 For a subset $K \subseteq X$, a function $f: K \longrightarrow \mathbb{R}$ is called Lipschitz continuous if there exists a real number $L \in \mathbb{R}$ such that for all $u, v \in K$,

$$|f(u) + f(v)| \leq L \| u-v \|$$

2.4 Differential Properties of Convex Functions

The concept of differentiability plays an important role in the theory of optimization. When the functions involved are non-smooth, the differentiability assumption fails. To treat such problem one can need the generalization of the derivative concept. The starting point in non-smooth analysis is the consideration of convex functions. Let f be a convex function and $x \in \text{dom}\, f$. Then a vector s is said to be a subgradient of f at x if,

$$f(y) \geq f(x) + \langle s, y - x\rangle \quad \forall y \in \text{dom}\, f$$

This inequality is called the subgradient inequality. The set of subgradient vectors of f at x, denoted by $\partial f(x)$, is called the subdifferential of f at x.

Recall that let $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is convex and $x \in \text{dom}\, f$, let x and d be fixed in $\mathbb{R}^n$ and consider the difference quotient of f at x in the direction d;

$q(t) = \frac{f(x+td)-f(x)}{t}$ for $t > 0$, then the limit of q(t) as $t \longrightarrow 0^+$ is called the directional derivative of f at x.

Proposition 2.4.1 Let X be a vector space, $U \subseteq X$ be convex, $f: U \longrightarrow (-\infty, \infty]$ be a convex function and x_0 be an algebraically interior point of U, where $f(x) \in \mathbb{R}$, then

$q(t) = \frac{f(x_0+td)-f(x_0)}{t}$, $t \in I_d = \{t \in \mathbb{R} : x_0 + td \in U,\ t > 0\}$, $d \in X$ is monotone increasing function and bounded below on I_d.

<u>Proof:</u>

i. <u>Monotonicity:</u> Let $q(t) = \frac{f(x_0+td)-f(x_0)}{t}$ t>0

We need to show that q (t) is increasing, i.e. if $0 < s \leq t$, $q(s) \leq q(t)$.

Note that the function $g(t) = f(x_0+td) - f(x_0)$ is convex and

$g(0) = 0$, let $0 < s \leq t$; $s, t \in I_d$

$\Longrightarrow g(s) = g(\frac{s}{t}t + \frac{t-s}{t}*0) \leq \frac{s}{t}g(t) + \frac{t-s}{t}g(0)$ by convexity assumption of g

$\Longrightarrow g(s) \leq \frac{s}{t}g(t)$, since $g(0) = 0$

$\Longrightarrow \frac{g(s)}{s} \leq \frac{g(t)}{t} \Longrightarrow q(s) \leq q(t)$

∴ q is monotonically increasing

ii. <u>Boundedness:</u> Since f is convex, we have

$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$, for all $x, y \in \mathbb{R}^n$, $\lambda \geq 0$. Now set $\lambda = t$ and $y = x_0$. Then

$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y) \Longrightarrow f(x_0 + t(x-x_0)) \leq f(x_0) + t(f(x) - f(x_0))$

$\Longrightarrow f(x_0 + t(x-x_0)) - f(x_0) \leq t(f(x) - f(x_0)) \Longrightarrow \frac{f(x_0+td)-f(x_0)}{t} \leq f(x) - f(x_0)$ since t>0

and $x_0, x \in$ domf, $d = x-x_0$

$\Longrightarrow$ q(t) is bounded below

since q(t) is increasing and bounded below function of t, then the limit:

$\lim t \to 0^+ \frac{f(x_0 + td)-f(x_0)}{t}$ exists for all $d \in \mathbb{R}^n$. ■

Note:

1. From the proposition 2.4.1 we conclude that all convex functions are directionally differentiable.
2. From the definition, $f'(x, d)$ is positively homogonous and convex as a function of d.

   Hence for a convex function f, $f'(x, d) = \max\{\langle s, d\rangle : s \in \underline{\partial} f(x)\}$

   $$= \max_{s \in \underline{\partial} f(x)} \langle s, d\rangle$$

3. If S is a subgradient of f at $x \in$dom f, setting $y = x + td$, t>0 in $f(y) \geq f(x) + \langle s, y - x\rangle$ we obtain: $f(x + td) \geq f(x) + \langle s, td\rangle$ when we take the limit as $t \longrightarrow 0$, we get $f'(x, d) \geq \langle s, d\rangle$, for all $d \in \mathbb{R}^n$.

Proposition 2.4.2 If x* minimizes a convex function f, then $0 \in \partial f(x^*)$ and conversely, if

$0 \in \partial f(x^*)$ then x* minimizes $f(x)$.

Proof: Let y be an arbitrary point of dom f. Then $f(x^*) - f(y) \leq 0 \leq k\|x^*-y\|$, for all $k \geq 0$

Which implies $f(x)$ is stable at x*. Hence $\partial f(x^*) \neq \emptyset$. The statement $0 \in \partial f(x^*)$ if and only if x* minimizes f, follows immediately from the relation $f(x^*)- f(y) \leq 0$ when it is written as a gradient inequality form $f(y) \geq f(x^*) + k\|x^*-y\|$, for all $k \leq 0$. ■

Definition 2.4.3 Superdifferentiable functions are functions whose directional derivatives can be expressed as $f'(x, d) = \min_{s\in V}\langle s, d\rangle$, for all $d \in \mathbb{R}^n$, where V is a convex compact set.

Superdifferential of f at x, is the set of vectors $s \in \mathbb{R}^n$ which satisfy $f(y) \leq f(x) + \langle s, y - x\rangle$ for all $y \in \mathbb{R}^n$ and denoted by $\overline{\partial} f(x)$.

Remark:

i. A function f is subdifferentiable if and only if $-f$ is superdifferentiable.
ii. There is a one-to-one mapping between the family of all sublinear functions defined on $\mathbb{R}^n$ and the family of all nonempty convex compact subsets of $\mathbb{R}^n$ (Minkowski duality).

2.5 Quasidifferentiable Function

Recall that a function f is said to be quasidifferentiable at a point x if it is directionally differentiable at x and the directional derivative $f'(x; d)$ can be expressed as a difference of two sublinear functions as a function of direction d: i.e $f'(x; d) = p(d) - q(d)$, where p and q are sublinear functions for corresponding compact convex sets U and V, we have:

$$f'(x; d) = \max_{s\in U}\langle s, d\rangle - \max_{u\in V}\langle u, d\rangle \quad (2.5.1)$$

The set U and –V are called the subdifferential and superdifferentiable of f at x respectively and are denoted by $\underline{\partial} f(x)$ and $\overline{\partial} f(x)$ respectively.

Note that: The convex part of $f'(x; d)$ is taken into account by the subdifferential $\underline{\partial} f(x)$ while the concave contribution of $f'(x; d)$ is reflected in superdifferential $\overline{\partial} f(x)$.

The pair $Df(x) = (\underline{\partial} f(x), \overline{\partial} f(x))$ is called a quasidifferential of f at x. From (2.5.1) a quasidifferential of a function f is not uniquely determined since one could rewrite (2.5.1) equivalently as:

$$f'(x, d) = \left(\left(\max_{s\in U}\langle s, d\rangle + \max_{c\in C}\langle c, d\rangle\right) - \left(\max_{u\in U}\langle u, d\rangle + \max_{c\in C}\langle c, d\rangle\right)\right) \text{ for any compact convex set C.}$$

Theorem 2.5.1 (Calculus rule of quasidifferential) [1] Let functions $f_1, f_2, \ldots, f_m$ be defined on an open set $X \subseteq \mathbb{R}^n$ and quasidifferentiable at a point $x \in X$. Let $\varphi_1 = \max_{i=1,2,\ldots,m} f_i(x)$ and $\varphi_2 = \min_{i=1,2,\ldots,m} f_i(x)$, then the functions φ_1 and φ_2 are quasidifferentiable at x and

$D\varphi_1 = (\underline{\partial}\varphi_1(x), \overline{\partial}\varphi_1(x))$ and $D\varphi_2 = (\underline{\partial}\varphi_2(x), \overline{\partial}\varphi_2(x))$, where

$\underline{\partial}\varphi_1(x) = \text{conv}_{k\in R(x)} \cup (\underline{\partial} f_k(x) - \sum_{i\in R(x), i\neq k} \overline{\partial} f_i(x))$, $\quad \overline{\partial}\varphi_2(x) = \sum_{k\in Q(x)} \underline{\partial} f_k(x)$

$\overline{\partial}\varphi_1(x) = \sum_{k\in R(x)} \overline{\partial} f_k(x)$, $\quad \underline{\partial}\varphi_2(x) = \text{conv}_{k\in Q(x)} \cup (\overline{\partial} f_k(x) - \sum_{i\in Q(x), i\neq k} \underline{\partial} f_i(x))$ and

$R(x) = \{i \in I: f_i(x) = \varphi_1(x)\}$, $\quad Q(x) = \{i \in I: f_i(x) = \varphi_2(x)\}$, $I = \{1, 2, \ldots, n\}$.

Here $(\underline{\partial} f_k(x), \overline{\partial} f_k(x))$ is a quasidifferential of the function f_k at the point x.

Example 1) Consider $f(x) = |x|$, $x \in \mathbb{R}$, calculate the quasidifferential of f.

<u>Solution:</u> $f(x) = \max\{f_1(x), f_2(x)\} = \max f_i(x)$, $i \in I = \{1, 2\}$ where $f_1(x) = x$, $f_2(x) = -x$

We consider two cases:

<u>Case1:</u> For x = 0, R(x) = {1, 2} and the functions f_1 and f_2 are smooth. Using the above theorem we have; $\underline{\partial} f_1(x) = \{f'_1(x)\} = \{1\}$, $\quad \overline{\partial} f_1(x) = \{0\}$

$\underline{\partial} f_2(x) = \{f'_2(x\} = \{-1\}$, $\quad \overline{\partial} f_2(x) = \{0\}$

Since $Df(x) = (\underline{\partial} f(x), \overline{\partial} f(x))$, where $\underline{\partial} f(x) = \text{conv}\{\underline{\partial} f_1(x) - \overline{\partial} f_2(x), \underline{\partial} f_2(x) - \overline{\partial} f_1(x)\}$

$= \text{conv}\{1\text{-}0, -1\text{-}0\} = \text{conv}\{1,-1\} = [-1, 1]$

$$\overline{\partial} f(x) = \overline{\partial} f_1(x) + \overline{\partial} f_2(x) = \{0\} + \{0\} = \{0\}$$

Hence for x=0, $Df(x) = (\underline{\partial} f(x), \overline{\partial} f(x)) = [-1, 1]$

Case2: $x \neq 0$, $f(x) = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$

$\Longrightarrow Df(x) = (\underline{\partial} f(x), \overline{\partial} f(x))$, where $\underline{\partial} f(x) = \begin{cases} \{1\}, & x > 0 \\ \{-1\}, & x < 0 \end{cases}$

$$\overline{\partial} f(x) = \{0\}, \forall x \neq 0$$

From case1 and case2, we have $Df(x) = (\underline{\partial} f(x), \overline{\partial} f(x))$, where

$$\underline{\partial} f(x) = \begin{cases} \{1\}, & x > 0 \\ \{-1\}, & x < 0 \\ [-1, 1], & x = 0 \end{cases}$$

$$\overline{\partial} f(x) = \{0\}, \forall x \in \mathbb{R}$$

Example 2): Let $f(x) = -|x|$, $x \in \mathbb{R}$, calculate the quasidifferential of f.

Solution: $f(x) = \min\{f_1(x), f_2(x)\}$, where $f_1(x) = -x$ and $f_2(x) = x$ since f is concave.

$Q(x) = \{1, 2\}$ and the functions f_1 and f_2 are smooth.

Case-1: $x = 0$

$Df_1(x) = (\underline{\partial} f_1(x), \overline{\partial} f_1(x))$ and $Df_2(x) = (\underline{\partial} f_2(x), \overline{\partial} f_2(x))$, where

$\underline{\partial} f_1(x) = \{0\}$, $\quad\quad \underline{\partial} f_2(x) = \{0\}$

$\overline{\partial} f_1(x) = \{ f'_1(x)\} = \{-1\}$, $\quad\quad \overline{\partial} f_2(x) = \{ f'_2(x)\} = \{1\}$

By calculus rule of quasidifferentials we have:

$\underline{\partial} f(x) = \underline{\partial} f_1(x) + \underline{\partial} f_2(x) = \{0\} + \{0\} = \{0\}$

$\overline{\partial} f(x) = \text{conv}\{\overline{\partial} f_1(x) - \underline{\partial} f_2(x), \overline{\partial} f_2(x) - \underline{\partial} f_1(x)\} = \text{conv}\{-1-0, 1-0\}$

$= \text{conv}\{-1, 1\} = [-1, 1]$

<u>Case-2:</u> $x \neq 0$

$f(x) = \begin{cases} -x, & x < 0 \\ x, & x < 0 \end{cases}$, thus $Df(x) = (\underline{\partial} f(x), \overline{\partial} f(x))$, where $\overline{\partial} f(x) = \begin{cases} \{-1\}, & x > 0 \\ \{1\}, & x < 0 \end{cases}$

$\underline{\partial} f(x) = \{0\}, \forall\, x \neq 0$

From above cases, we can have: $Df(x) = (\underline{\partial} f(x), \overline{\partial} f(x))$, where $\underline{\partial} f(x) = \{0\}, \forall\, x \in \mathbb{R}$

$$\overline{\partial} f(x) = \begin{cases} \{-1\}, & x > 0 \\ \{1\}, & x < 0 \\ [-1, 1], & x = 0 \end{cases}$$

By using other cases, we can see the equivalence between pairs of quasidifferentials,

<u>Case-3:</u> $Df_1(x) = (\underline{\partial} f_1(x), \overline{\partial} f_1(x))$, $Df_2(x) = (\underline{\partial} f_2(x), \overline{\partial} f_2(x))$,

where $\underline{\partial} f_1(x) = \{ f'_1(x)\} = \{-1\}$, $\overline{\partial} f_1(x) = \{0\}$

$\underline{\partial} f_2(x) = \{0\}$ $\overline{\partial} f_2(x) = \{ f'_2(x)\} = \{1\}$

Then we get $Df(x) = (\underline{\partial} f(x), \overline{\partial} f(x))$, where

$\underline{\partial} f(x) = \underline{\partial} f_1(x) + \underline{\partial} f_2(x)$

$= \{-1\} + \{0\} = \{-1\}$

$\overline{\partial} f(x) = \text{conv}\{\overline{\partial} f_1(x) - \underline{\partial} f_2(x), \overline{\partial} f_2(x) - \underline{\partial} f_1(x)\}$

$= \text{conv}\,\{0-0, 1- -1\} = \text{conv}\{0, 2\} = [0, 2]$

Example 3) Let $x = (x_1, x_2)$, $x \in \mathbb{R}^n$ and $f(x) = |x_1| - |x_2|$. Find the quasidifferential of f.

<u>Solution:</u> put $f_1(x) = |x_1|$ and $f_2(x) = -|x_2|$

$Df_1(x) = Df_1(x) = (\underline{\partial} f_1(x), \overline{\partial} f_1(x))$ and $Df_2(x) = (\underline{\partial} f_2(x), \overline{\partial} f_2(x))$, From example 1 and example 2, we obtain:

$$\underline{\partial} f_1(x) = \begin{cases} \{(1,0)\} & , x_1 > 0 \\ \{(-1,0)\} & , x_1 < 0 \\ \text{conv}\{(-1,0), (1,0), & x_1 = 0 \end{cases}$$

$\overline{\partial} f_1(x) = \{(0, 0)\}$, for all $x \in \mathbb{R}^n$

$$\overline{\partial} f_2(x) = \begin{cases} \{(0,-1)\}, & x_2 > 0 \\ \{(0,1)\}, & x_2 < 0 \\ \text{conv}\{(0,-1), (0,1)\} & , x_2 = 0 \end{cases}$$

$\underline{\partial} f_2(x) = \{(0, 0)\}$, for all $x \in \mathbb{R}^n$

For rewrite this in the form of $Df(x) = (\underline{\partial} f(x), \overline{\partial} f(x))$, let us see the following theorem:

Theorem 2.5.2 Let the functions f_1 and f_2 be quasidifferentiable at a point x. Then their sum and their product of these functions are quasidifferentiable at x, which can written as:

$$D(f_1 + f_2)(x) = Df_1(x) + Df_2(x)$$

$$D(f_1 . f_2)(x) = f_1(x)Df_2(x) + f_2(x)Df_1(x)$$

Appling this theorem to the above example it gives:

$Df(x) = (\underline{\partial} f(x), \overline{\partial} f(x))$, where $\underline{\partial} f(x) = \underline{\partial} f_1(x) + \underline{\partial} f_2(x)$

$$\overline{\partial} f(x) = \overline{\partial} f_1(x) + \overline{\partial} f_2(x)$$

A quasidifferential of a function is not uniquely expressed. Actually, suppose that the pair (U, V) is a quasidifferential of f at x, then for any convex compact set $S \subseteq \mathbb{R}^n$, (U + S, V- S) is still a quasidifferential of f at x. Pallashcke et al [5] introduced an important notion, the minimal quasidifferential, $(\underline{\partial}^m f(x), \overline{\partial}^m f(x)) \in Df(x)$ is called minimal provided that any pair

$(\underline{\partial} f(x), \overline{\partial} f(x)) \in Df(x)$ satisfying that the relation $\underline{\partial} f(x) \subseteq \underline{\partial}^m f(x)$; $\overline{\partial} f(x) \subseteq \overline{\partial}^m f(x)$ implies that $(\underline{\partial} f(x), \overline{\partial} f(x)) = (\underline{\partial}^m f(x), \overline{\partial}^m f(x))$. Furthermore, Pallashcke et al [5] proved the existence of the minimal quasidifferentials, which leads, if $(\underline{\partial} f(x), \overline{\partial} f(x)) \in Df(x)$, such that

$\underline{\partial}^m f(x) \subseteq \underline{\partial} f(x)$; $\overline{\partial}^m f(x) \subseteq \overline{\partial} f(x)$. However a minimal quasidifferential is not a unique expression. Actually, any translation of a minimal quasidifferential is still minimal, say, if
[A, B] is a minimal quasidifferential, then for any singleton {c}, [A+{c}, B+{c}], a translation of [A, B], is still a minimal quasidifferential.

Grzybowski [2] and Scholtes [6] proved independently the fact that equivalent minimal quasidifferentials, in the two-dimensional case are uniquely determined up to translation, i.e. let f be quasidifferentiable on $\mathbb{R}^2$, given the two minimal quasidifferentials $(\underline{\partial}_1^m f(x), \overline{\partial}_1^m f(x))$ and $(\underline{\partial}_2^m f(x), \overline{\partial}_2^m f(x))$, there exist $c \in \mathbb{R}^2$ (where c is depend on $(\underline{\partial}_1^m f(x), \overline{\partial}_1^m f(x))$ and $(\underline{\partial}_2^m f(x), \overline{\partial}_2^m f(x))$ satisfying $(\underline{\partial}_2^m f(x), \overline{\partial}_2^m f(x)) = (\underline{\partial}_1^m f(x) + \{c\}, \overline{\partial}_1^m f(x) - \{c\})$.

Hence defining and determining ‘minimal’ representative, from the class of pairs of compact convex sets corresponding to the quasidifferentials of the functions, in quasidifferentiable optimization problems is necessary, hence the next chapter is about minimal pairs of compact convex sets.

Chapter 3

Minimal Pairs of Compact Convex Sets

3.1 Basic Notions and Properties

Let X = (X, τ) be a topological Hausdorff vector space, B(X) be the set of all nonempty bounded closed convex subsets of X, and K(X) denote the set of nonempty convex compact subsets of X endowed with the usual Minkowski addition and scalar multiplication. The set K(X) is a commutative semi group with cancellation property (Urbanski, R, and Generalization of the Minkowski-Rάdströ́m-Hö́rmander Theorem).Let $K^2(X)=K(X)\times K(X)$, then we define minimality of pairs of compact convex sets in $K^2(X)$ which was given in Pallaschke, D., Scholtes, S and Urbanski, r.,[4], [6] On minimal pairs of convex compact sets.

Definition 3.1.1 Two pairs (A, B) and (C, D) from $K^2(X)$ are said to be equivalent, written

(A, B) ~ (C, D) if, and only if A + D = B + C with the Minkowski addition.

Definition 3.1.2

1) A subset τ of the set of all subsets of X which satisfies the axioms:

1. ∅ and X are open sets
2. The intersection of finitely many open sets is an open set
3. The union of arbitrary many open sets is an open set is called a topology for X and the pair (X, τ) a topological space. A subset β ⊆ τ is called a basis of τ if every U ∈ τ is the union of elements from β.

2) Let M ⊆ X, the smallest closed subset of X which contains M, is called the closure of M, denoted by cl(M) or $\overline{M}$. If M is closed M = $\overline{M}$.

3) A topology τ for X is called Hausdorff if for every two points x, y ∈ X with x ≠ y there exist open sets U, V ∈ τ with x ∈ U and y ∈ V such that U ∩ V = ∅ . Every U ∈ τ with x ∈ U is called a neighborhood of X.

4) Let X be a real vector space endowed with a Hausdorff topology τ, then the pair (X, τ) is called a topological vector space if:

i. For every x, y ∈ X and any neighborhoods U_{x+y} of x + y, there exist neighborhoods U_x of x and U_y of y such that $U_x + U_y \subseteq U_{x+y}$

ii. For every x ∈ X, $t_0 \in \mathbb{R}$ and every negibohrhood $U_{t_{ox}}$ of t_o x, there exists a neighborhood U_x of x and ε > 0 such that for all t ∈ ℝ with $|t - t_0| < \varepsilon$, the inclusion $tU_x \subseteq U_{t_{ox}}$ holds.

5) A topological vector space (X, τ) is called locally convex if there exists a basis β ⊆ τ of neighborhoods zero that consists of convex sets. Observe that a locally convex space has always a basis β ⊆ τ of neighborhoods of zero which consists of absolutely convex sets i.e. convex sets U with U = -U.

6) A nonempty set $P \subseteq \mathbb{R}^n$ is called a polyhedron if there exists an $m \times n$-matrix A and an n-vector b such that $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. A bounded polyhedral set is called a polytope.

3.2 Algebraic Properties Compact Convex Sets in Two Dimensions

Recall that for A, B ∈ A(X) [the set of nonempty subsets of X], the algebraic sum is defined by A + B={x= a + b| a ∈ A, b ∈ B} and for λ ∈ ℝ and A ∈ A(X) the multiplication is defined by

$\lambda A = \{x \mid x = \lambda a, a \in A\}$. The Minkowski sum for A, B ∈ A(X) is defined as: $A \dotplus B = cl(\{x = a + b \mid a \in A, b \in B\})$, where the closurity is with respect to τ. If A and B are compact convex sets, the Minkowski sum coincide with the algebraic sum, i.e. $A \dotplus B = A + B$. For two compact convex sets A, B ⊆ X, we will use the notation: $A \vee B := conv(A \cup B)$, and for two elements a, b ∈ X the interval with end points a and b denoted by $[a, b] := \{a\} \vee \{b\}$.

Pinsker's Identity 3.2.1 For A, B, C ∈ K(X) in a topological vector space X we have:

$$(A \vee B) + C = (A + C) \vee (B + C).$$

Proof: For every x ∈ (A ∨ B) + C can be represented as $x = \alpha a + (1 - \alpha)b + c$ with $a \in A$, $b \in B$, $c \in C$ and $0 \leq \alpha \leq 1$.

Now $x = \alpha a + (1 - \alpha)b + c = \alpha(a + b) + (1 - \alpha)(b + c)$

and hence $C + (A \vee B) \subseteq (A + C) \vee (B + C)$.

For the converse: Let $x \in (A + C) \vee (B + C)$, then we have: $x = \alpha(a + c_1) + (1 - \alpha)(a + c_2)$ with

$a \in A$, $b \in B$, $c_1, c_2 \in C$ and $0 \le \alpha \le 1$.

Now $x = \alpha(a + c_1) + (1 - \alpha)(b + c_2) = \alpha a + (1 - \alpha)b + \alpha c_1 + (1 - \alpha)\, c_2$. Hence the converse inclusion $(A + C) \vee (B + C) \subseteq C + (A \vee B)$ also holds. ■

We will use the abbreviation $A + B \vee C$ for $A + (B \vee C)$ and $C + d$ for $C + \{d\}$ for compact convex sets A, B, C and a point d.

Theorem 3.2.2 (Minkowski-Ràdström-Hörmander-Theorem) Let X is a topological space**;**

i. For $A, B, C \in B(X)$ the implication $A \dotplus B \subseteq C \dotplus B \Longrightarrow A \subseteq C$ holds.
ii. For $A, B, C \in K(X)$ the implication $A + C \subseteq C + B \Longrightarrow A \subseteq B$ holds.
    The implication $A + C \subseteq C + B \Longrightarrow A \subseteq B$ is called the order cancelation law and the weaker implication $A + B = C + B \Longrightarrow A = C$ is called the cancelation law.

Proof: Let β be a base of neighborhoods of zero in the topological vector space X. Given any neighborhood $U \in \beta$ we define a sequence $(V_n)n \in \mathbb{N}$ such that:

$V_0 + V_0 \subseteq U$ and $V_{n+1} + V_{n+1} \subseteq V_n$. From $A + B \subseteq C \dotplus B$ it follows that for every

$V \in \beta$ we have $A + B \subseteq C + B + V$, and hence for every $n \in \mathbb{N}$ we have:

$A + B \subseteq C + B + V_n$. Now let $a \in A$ and $b_1 \in B$.

Then $a + b_1 = c_1 + b_2 + u_1$ for some $c_1 \in C$, $b_2 \in B$, $u_1 \in V_1$,

$a + b_2 = c_2 + b_3 + v_2$ for some $c_2 \in C$, $b_3 \in B$, $v_2 \in V_2$, and in general, for every

$n \in \mathbb{N}$: $a + b_n = c_n + b_{n+1} + u_n$ for some $c_n \in C$, $b_{n+1} \in B$, $u_n \in V_n$.

Hence $a = \frac{1}{n}(c_1 + \cdots + c_n) + \frac{1}{n}(b_{n+1} - b_1) + \frac{1}{n}(u_1 + \cdots + u_n)$, $n \in \mathbb{N}$ and thus by the convexity of C and the boundedness of B we get for sufficiently large $n \in \mathbb{N}$ that

$$a \in C + V_0 + V_1 + \cdots + V_n \subseteq C + U.$$

Thus $A \subseteq C + U$ for every $U \in \beta$, and therefore, $A \subseteq C$. ■

Let X = (X, τ) be a topological vector space and B(X) be the set of all nonempty bounded closed convex subsets of X, on $B^2(X) = B(X) \times B(X)$, the equivalence relation

(A, B) ~ (C, D) ⇔ A $\dotplus$ D = B $\dotplus$ C introduced. Such pairs (A, B) and (C, D) are called equivalent and [A, B] ∈ $B^2(X)/\sim$ denotes equivalence class which contains (A, B) ∈ $B^2(X)$.

The ordered space ($B^2(X)/\sim, \leq$) is called Pinsker-Minkowski-R$\dot{a}$dstr$\ddot{o}$m-H$\ddot{o}$rmander lattice [4] of bounded closed convex sets.

Definition 3.2.3 Let X be a real topological vector space and A, B, S ∈ K(X). Then S is separating the sets A and B if and only if for every a ∈ A and b ∈ B we have [a, b] ∩ S≠ ∅.

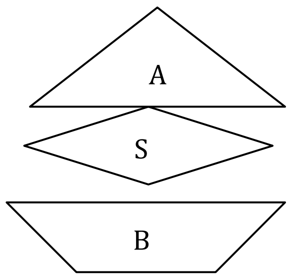


Figure 1: Separating by hyperplane

Definition 3.2.4 A pair (A, B) ∈ $K^2(X)$ is called convex if A ∪ B is a convex set. We call (A, B) convexly reduced if for any convex pair (C, D) in [A, B] there exists M ∈ K(X) such that

C = A $\dotplus$ M and D = B $\dotplus$ M.

Lemma 3.2.5 Let X is a real topological vector space and A, B, C ⊆ X nonempty subsets. Then

A ∪ B + C = (A + C) ∪ (B + C).

Proof: For x ∈ A ∪ B + C, there exists c ∈ C and d ∈ A ∪ B such that x = c + d.

Hence x ∈ (A + C) ∪ (B + C), i.e. A ∪ B + C ⊆ (A + C) ∪ (B + C).

Conversely, for x ∈ (A + C) ∪ (B + C) there exist elements c ∈ C and d ∈ A or d ∈ B such that

x = c + d. Hence x ∈ A ∪ B + C, i.e. (A + C) ∪ (B + C) ⊆ A ∪ B + C. ■

3.3 Minimal Pairs of Compact Convex Sets in Two Dimensions

Definition 3.3.1 Let (X, τ) be a topological vector space. For pair of compact convex sets

$(A, B), (C, D) \in K^2(X)$, let us define an ordering: $(A, B) \preccurlyeq (C, D)$ iff $A \subseteq C$ and $B \subseteq D$. Here, showing that the relation $'' \preccurlyeq ''$ is an ordering in $K^2(X)$. A pair $(A, B) \in K^2(X)$ is called minimal if it is minimal in the class $[A, B]$ with respect to the above ordering relation, i.e. (A, B) is minimal if for any pair $(C, D) \in [A, B]$ the relation $(C, D) \preccurlyeq (A, B) \Rightarrow (C, D) = (A, B)$.

The existence of minimal pair in the class $[A, B] \subseteq K^2(X)$ can be proved by using Kuratowski Zorn's theorem.

Theorem 3.3.2 Let (X, τ) be a topological vector space, then for any pair $(A, B) \in K^2(X)$ there exists a pair $(C, D) \in [A, B]$ which is minimal.

<u>Proof</u>: Using (2.1.1) Kuratowski-Zorn's Lemma it is sufficient to show that for any totally ordered subset $\Sigma = \{(C, D) \in [A, B]: (C, D) \preccurlyeq (A, B)\}$ of $[A, B]$ there exists an element $(A^*, B^*) \in [A, B]$ such that for any $(C, D) \in \Sigma$ the relation $(A^*, B^*) \preccurlyeq (C, D)$ holds.

For any $\sigma = (C, D) \in \Sigma$ we denote by A_σ the set C and by B_σ the set D. The ordering on Σ yields that $\sigma_1 \preccurlyeq \sigma_2$ if and only if $A_{\sigma 1} \subseteq A_{\sigma 2}$ and $B_{\sigma 1} \subseteq B_{\sigma 2}$. Now we fix $\sigma_0 \in \Sigma$ and define the sets

$A^* = \bigcap_{\sigma_0 \in \Sigma_0} A_\sigma$ and $B^* = \bigcap_{\sigma_0 \in \Sigma_0} B_\sigma$, where $\Sigma_0 = \{\sigma \in \Sigma: \sigma \preccurlyeq \sigma_0\}$.

By Cantor intersection property [4] the set A^* is nonempty. Moreover, A^* is a closed subset of $A_{\sigma 0}$ and hence it is compact. The convexity of A^* follows immediately from the convexity of A_σ for $\sigma \in \Sigma_0$. Since the same arguments hold for B^* it follows that $(A^*, B^*) \in K^2(X)$.

$$A^* + B = B + \bigcap_{\sigma \in \Sigma_0} A_\sigma = \bigcap_{\sigma \in \Sigma_0}(B + A_\sigma) = \bigcap_{\sigma \in \Sigma_0}(B_\sigma + A)$$

$$= \bigcap_{\sigma \in \Sigma_0} B_\sigma + A = B^* + A \qquad \blacksquare$$

To make the above theorem clear, let's associate it with the concept of fraction. Let (S,.) be a commutative semi group ordered by the relation $\preccurlyeq$. Moreover, let S satisfy the order law of cancellation i.e. If $as \preccurlyeq bs$, for some s in S, then $a \preccurlyeq b$. We say that $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent fractions and write $\frac{a}{b} \sim \frac{c}{d}$ if ad = bc. This is an equivalence relation on the set of S^2 and we denote by $[\frac{a}{b}] = \{\frac{c}{d} \in S^2 | \frac{a}{b} \sim \frac{c}{d}\} \subseteq S^2$ the equivalence class which contains $\frac{a}{b}$.

The pair $\frac{a}{b} \in S^2$ is called a minimal fraction, if for any fraction $\frac{c}{d}$ such that $\frac{c}{d} \sim \frac{a}{b}$ and $\frac{c}{d} \preccurlyeq \frac{a}{b}$ it follows that a = c and b = d.

Proposition 3.3.3 If (A, B) $\in K^2(X)$ is minimal, then

i. For all $\alpha \in \mathbb{R}$, α(A, B) = (α A, αB) is also minimal.
ii. For every $\beta \in [0, 1]$, the pair (A, βB) is also minimal.

Proof: (i) For simplicity let's decompose the statement into two parts:

(a) If (A, B) $\in K^2(X)$ is minimal, then (-A, -B) is minimal.
(b) If (A, B) $\in K^2(X)$ is minimal, then α(A, B) = (α A, αB) is minimal for all $\alpha \geq 0$.

Now let us prove the (a): Let (A, B) $\in K^2(X)$ be minimal, we need to show that (-A, -B) is minimal. Let (A′, B′) $\preccurlyeq$ (-A, -B) i.e. (A′, B′) ~ (-A, -B) and A′ ⊆ –A and B′ ⊆ –B.

Since (A′, B′) ~ (-A, -B), which implies –A + B′ = -B + A′.

Thus A′ ⊆ -A and B′ ⊆ -B, -A′ ⊆ A and –B′ ⊆ B.

$\Rightarrow$ A – B′ = B – A′ since (-A′, -B′) ~ (A, B) and hence (-A′, -B′) $\preccurlyeq$ (A, B) follows. By the minimality of (A, B) we have –B′ = B and –A′ = A which implies B′ = -B and A′ = -A

(A′, B′) $\preccurlyeq$ (-A, -B) $\Rightarrow$ (A′, B′) = (-A, -B) therefore, (-A, -B) is minimal.

By similar form we can prove (b).

(ii) For showing the second claim let us take two cases:

Case 1: If β = 0, thus (A, βB) = (A, 0B) = (A, 0) which is minimal for (A, B) minimal.

Case 2: If $\beta \neq 0$, i.e. for $\beta \in (0, 1]$. Let (A′, B′) $\preccurlyeq$ (A, βB) i.e. A′ ⊆ A, B′ ⊆ βB and A + B′ = βB + A′. We need to show that A′=A and B′ = βB.

Since $B' \subseteq \beta B$, we have $\frac{1}{\beta}B' \subseteq B$ and let us denote $\frac{1}{\beta}B'$ by B''. From $A + B' = \beta B + B'$ and

$B'' = \frac{1}{\beta}B'$, $A + \beta B'' = \beta B + A'$ follows. $\Rightarrow A + B - \beta B + B'' = B + A'$, by adding B in both sides

$$\Rightarrow A + (1- \beta)B + \beta B'' = B + A'$$

Since $A' \subseteq A$, by the order cancellation law we have that $(1- \beta)B + \beta B'' \subseteq B$

$$\Rightarrow B - \beta B + \beta B'' \subseteq B \Rightarrow \beta B = \beta B''$$

Since (A, B) is minimal, we obtain $B = B''$ and it implies $B = \frac{1}{\beta}B'$, i.e. $B' = \beta B$ and $A' = A$

$\therefore$ $(A, \beta B)$ is minimal. ■

Proposition 3.3.4 If $(A, B) \in K^2(X)$ is minimal, then

i. $(A + x, B + y)$ is also minimal for all $x, y \in X$
ii. $(\alpha A, \beta B)$ is also minimal for all $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta \geq 0$.
iii. If moreover, $A = A' + V$ and $B = B' + V$ for some $A', B', V \in K(X)$ then V is a singleton.
iv. If, $A = B + V$ for some $V \in K(X)$ then B is singleton.

From above proposition we can deduce that if a pair of convex compact sets is not minimal then the pair of their translation is also not minimal.

Proposition 3.3.5 If A is a point set and B is an arbitrary convex compact set, then (A, B) is always minimal.

Since minimal pairs are defined on the class of pairs of compact convex sets, these pairs are not unique in general. However, in the case of $\mathbb{R}^2$ it was proved by S. Scholtes [6] independently that minimal pairs in the plane are unique up to translation.

For $X = \mathbb{R}$ (in the case of one dimensional space), all convex compact sets are finite closed intervals and hence the minimal pairs are characterized as follows:

Proposition 3.3.6 The pair (A, B) $\in K^2(\mathbb{R})$ is minimal if and only if one of the sets A or B is singleton.

Proof: Let A = $[a, b]$ and B = [c, d]. Then either $d-c \geq b-a$ or $b-a \geq d-c$

Without loss of generality, assume that $d-c \geq b-a$. If A is not singleton set, then $b-a > 0$ and

$(d-c) - (b-a) \geq 0$. Then the set $U := [c-a, d-b]$ is nonempty.

Therefore, B = [c, d] = $[c-a, d-b] + [a, b]$ = U + A and hence from proposition 3.3.4 it implies that (A, B) is not minimal. The converse follows from proposition 3.3.5. ■

3.4 Reduction Algorithm in Two Dimensional Spaces

In this section, we are going to see Handschug's algorithm that enable us to construct a quasidifferential, which is unique up to translation in a plane. Recall that a quasidifferential of f at x is given by: $Df(x) = (\underline{\partial}f(x), \overline{\partial}f(x))$ and can be also expressed as:

$Df(x) = (\underline{\partial}f(x) + A, \overline{\partial}f(x) - A)$ for every convex compact set $A \subseteq \mathbb{R}^n$, which shows clearly its nonuniqueness. Thus in the process of solving optimization problems is unquestionable to rewrite the quasidifferential in a simpler form so as to reduce the extent of calculations. In particular, in the case of a pair of convex polytopes in the plane we will present here an algorithm to compute an equivalent pair with a minimal number of corners (vertices).

Definition 3.4.1 For a polytope A in $\mathbb{R}^n$ we denote by $\mathcal{F}_1(A)$ the set of all edges (which are one dimensional edges of A). Let A, B be polytopes in $\mathbb{R}^n$, we say that $F \in \mathcal{F}_1(A)$ and $G \in \mathcal{F}_1(B)$ are equiparallel if F and G are parallel and if there is $u \in S^{n-1}$ with F = F(A, u) and G = F(B, u).

The following theorem gives us the necessary and sufficient condition for a pair of polytopes in the plane to be minimal.

Theorem 3.4.2 Let A, B be polytopes in $\mathbb{R}^2$. The pair (A, B) $\in K^2(\mathbb{R}^2)$ is minimal if and only if A and B have at most one pair of equiparallel edges.

This theorem is fundamental characterization of minimal pairs of polytopes in the plane.

3.4.1 Handschug's Algorithm

In this section we present Handschug's Algorithm which was first stated in Handschug's**,** M. [3], on Equivalent Quasidifferentials in the Two-dimensional case, and we give a reduction algorithm for pairs of polytopes to find a minimal pair. The algorithm is proved to be very useful in investigating minimal pairs of polytopes in the plane.

Let a polytope $A \in K(\mathbb{R}^2)$ be given. Choose an orthonormal basis e_1, e_2 of $\mathbb{R}^2$ and let a_0 be the vertex of A with smallest e_2-coordenate and, if there are more vertices with this property, also with smallest e_1-coordenate. Let $a_0, a_1, \ldots, a_m, a_{m+1} = a_0$ be the vertices of A in cyclic order in counterclockwise direction, in such a way that the vectors, called edge vectors of A,

$$\upsilon_i = a_{i+1} - a_i,\ i = 1, 2, \ldots, m+1$$

Satisfy the condition, $\upsilon_i = r_i(e_1\cos\alpha_i + e_2\sin\alpha_i)$ with $r_i > 0$ and $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{k+1} < 2\pi$. Then

$$\sum_{i=1}^{m+1} \upsilon_i = 0 \qquad (3.4.1)$$

Conversely, if vectors $\upsilon_1, \ldots, \upsilon_{m+1}$ satisfying the above conditions are given, then

$$a_0(\text{arbitrary}),\ a_j := a_0 + \sum_{i=1}^{j} \upsilon_i,\ j = 1, \ldots, m \qquad (3.4.2)$$

are the vertices of polytope A. This describes polytope A uniquely (up to translation). From above discussion, we can associate to every polytope $A \in K(\mathbb{R}^2)$ with m vertices the set of its edge vectors by setting $V_A := \{ \upsilon \mid \upsilon = a_{i+1} - a_i,\ i = 1, 2, \ldots, m+1\}$ with the convention that

$a_{m+1} := a_1$. For m = 1, we define $V_A = \emptyset$.

The next theorem (see [7]), which expresses the sum of two polytopes in terms of the sets of their edge vectors, is fundamental in the reduction technique of a pair of two polytopes.

Theorem 3.4.1.1 If A and B are polytopes in $\mathbb{R}^2$, V_A denote the set of all edge vectors of A and α_u denotes the polar angle of an edge vector u, then

$$V_{A+B} = \{w \in V_A \mid \nexists u \in V_B \text{ with } \alpha_u = \alpha_w\}$$

$$\cup \{u \in V_B \mid \nexists w \in V_A \text{ with } \alpha_w = \alpha_u\} \qquad (3.4.3)$$

$\cup \{\upsilon \mid \exists w \in V_A, \exists u \in V_B \text{ with } \alpha_w = \alpha_u, \text{ and } \upsilon = u + w\}$

$:= V_A \oplus V_B$

Proof: Since A is a convex set, υ is an edge vector of A iff for an outer normal g of υ, we have $\mathcal{F}_1(A, g) = \text{conv}\{ a_i, a_{i+1} \}$, where a_i and a_{i+1} are two consecutive vertices of A and $\upsilon = a_{i+1} - a_i$.

Therefore, $\mathcal{F}_1(A + B, g) = \mathcal{F}_1(A, g) + \mathcal{F}_1(B, g)$

$= \text{conv}\{ a_i, a_{i+1}\} + \text{conv}\{ b_j, b_{j+1} \}$

Here, each i, i+1, j, j+1 should not necessarily be distinct. From this equation it holds true that $\upsilon \in V_{A+B}$ iff $\upsilon = a_{i+1} - a_i$ and $b_{j+1} - b_j = 0$, or $a_{i+1} - a_i = 0$ and $\upsilon = b_{j+1} - b_j$ or

$\upsilon = (a_{i+1} - a_i) + (b_{j+1} - b_j)$ (with none of the two are zero) whenever g is an outer normal vector for both A and B. ■

From theorem 3.4.1.1, the following corollary follows.

Corollary 3.4.1.2 Given two polytope A and B in $\mathbb{R}^2$. Then there exists a polytope D such that

$A + D = B$ iff $\forall w \in V_A, \exists u \in V_B$ with w and u having the same polar angles and $|w| \leq |u|$.

In this case then, $V_D = \{ \upsilon \mid \upsilon = u \in V_B, \nexists\, w \in V_A \text{ with } \alpha_w = \alpha_u\}$ (3.4.4)

$\cup \{\upsilon \mid \exists w \in V_A, \exists u \in V_B, \alpha_\upsilon = \alpha_w = \alpha_u, |w| \leq |u|, \upsilon = u - w\}$

$:= V_B \ominus V_A$

Note that, $\ominus$ is the reverse operation to $\oplus$. However these operations give us the way to restore back the poytopes in question.

Now to carry out the reduction technique for a pair of polytopes, we first rewrite relation

$A_1 + C_1 = A_2 + C_2$ and $B_1 + C_1 = B_2 + C_2$ in terms of the operation $\oplus$ and $\ominus$, defined above, as follows.

$$V_{A2} = (V_{A1} \oplus V_{C1}) \ominus V_{C2}, \qquad V_{B2} = (V_{B1} \oplus V_{C1}) \ominus V_{C2} \qquad (3.4.5)$$

From corollary 3.4.1.2 it follows that

$$\forall w \in V_{C2},\ \exists u \in (V_{A1} \oplus V_{C1}) \text{ with } \alpha_w = \alpha_u \text{ and } |w| \leq |u|$$

and $$\forall w \in V_{C2},\ \exists \upsilon \in (V_{B1} \oplus V_{C1}) \text{ with } \alpha_w = \alpha_\upsilon \text{ and } |w| \leq |\upsilon|$$

That means, the elements of V_{C2} are vectors which have identical polar angles in both sets $V_{A1} \oplus V_{C1}$ and $V_{B1} \oplus V_{C1}$. Our aim is to find a polytope C_2 as large as possible and C_1 as small as possible. To this end, define the set

$$V_0 := \{w \mid \exists u \in V_{A1}, \exists \upsilon \in V_{B1} \text{ with } \alpha_\upsilon = \alpha_w = \alpha_u,\ |w| = \min\{|u|, |\upsilon|\} \qquad (3.4.6)$$

and the vector $$\upsilon_0 := \sum_{w \in V_0} w \qquad \text{if } V_0 \neq \emptyset \text{ and } \upsilon_0 := 0 \text{ otherwise.} \qquad (3.4.7)$$

Then put $$V_{C1} = \{\upsilon_0, -\upsilon_0\} \text{ and } V_{C2} = V_0 \oplus \{-\upsilon_0\} \text{ if } \upsilon_0 \neq 0$$

$$V_{C1} = \emptyset \text{ and } V_{C2} = V_0 \quad \text{if } \upsilon_0 = 0. \qquad (3.4.8)$$

It is clear from equation (3.4.8) that the sets V_{C1} and V_{C2} represent the edge vectors of convex polytopes C_1 and C_2 in $\mathbb{R}^2$.Therefore, equation (3.4.5) will be reduced to

$$V_{A2} = (V_{A1} \oplus \{\upsilon_0\}) \ominus V_0, \text{ and } V_{B2} = (V_{B1} \oplus \{\upsilon_0\}) \ominus V_0 \text{ if } \upsilon_0 \neq 0 \qquad (3.4.9)$$

and $$V_{A2} = V_{A1} \ominus V_0, \text{ and } V_{B2} = V_{B1} \ominus V_0 \qquad \text{if } \upsilon_0 = 0 \qquad (3.4.10)$$

To summarize the above discussion , if a pair of polytopes (A_1, B_1) from the class [A, B] is given, then we can reduce this pair to an equivalent and minimal pair of polytope (A_2, B_2) (which is unique up to translation) using the following algorithm.

Step 1: Choose an orthonormal basis e_1, e_2 of $\mathbb{R}^2$ and determine the set of vertices of A_1 and B_1 in a counterclockwise direction starting the vertex with the smallest e_2-coordinate (if there are more with this property also with the smallest e_1-coordinate) say a_0 and b_0 respectively. Then compute V_{A1}, V_{B1} and the polar coordinates of their elements.

Step 2: Compute V_0 from (3.4.6) and υ_0 from (3.4.7). If V_0 contains at most one element, then the pair (A_1, B_1) is already in its simplest form and hence stop the algorithm. Otherwise go to Step 3.

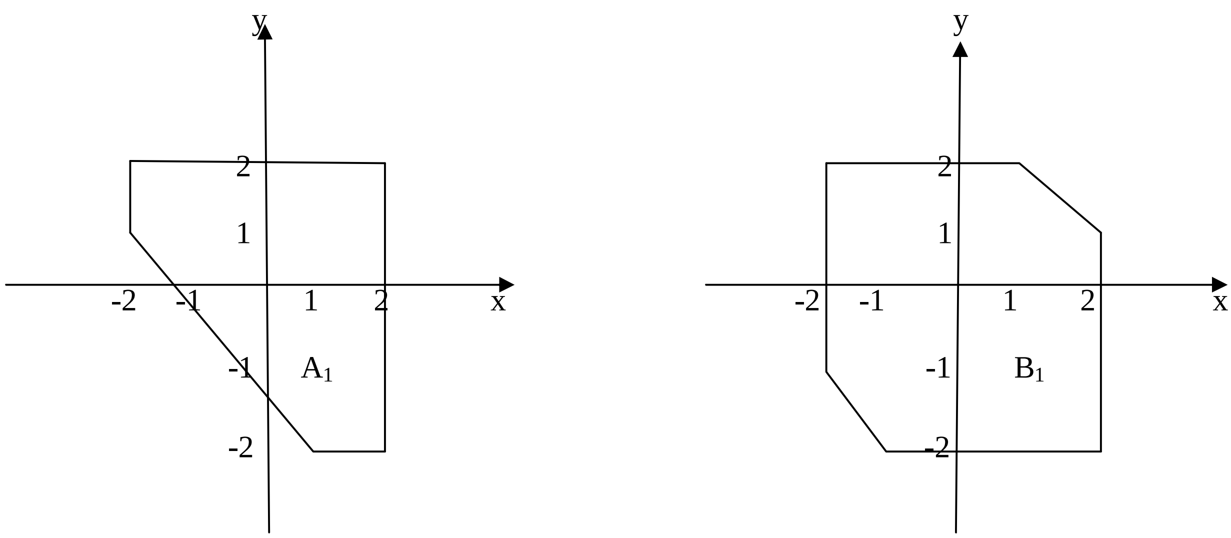
Step 3: If $\upsilon_0 \neq 0$, then compute V_{A2} and V_{B2} from (3.4.9). Otherwise from (3.4.10) and arrange their element in the increasing order of the polar angles of the vectors.

Step 4: Determine A_2 and B_2 from V_{A2} and V_{B2} starting from a_0 and b_0 using equation (3.4.2).

Note that the set of edge vectors of every polytope in the plane should be described in such a way that the elements are arranged in an increasing order of their polar angles so that we can determine the polytopes from the sets of edge vectors in Step 4.

Example: Let A_1 = conv{(1, -2), (2, -2), (2, 2), (-2, 2), (-2, 1)} and

B_1 = cpnv{(-1, -2), (2, -2), (2, 1), (1, 2), (-2, 2), (-2, -1)}



To find its minimal pairs, we have to pass through the following steps:

Step 1: By using the relation $\upsilon_i = a_{i+1} - a_i$, V_{A1} and V_{B1} can be computed as follows:

V_{A1} = {(1, 0), (0, 4), (-4, 0), (0, -1), (3, -3)}

= {(1, 0), $(4, \frac{\pi}{2})$, $(4, \pi)$, $(1, \frac{3\pi}{2})$, $(3\sqrt{2}, \frac{7\pi}{4})$} in polar form

and V_{B1} = {(3, 0), (0, 3), (-1, 1), (-3, 0), (0, -3), (1, -1)}

= {(3, 0), $(3, \frac{\pi}{2})$, $(\sqrt{2}, \frac{3\pi}{4})$, $(3, \pi)$, $(3, \frac{3\pi}{2})$, $(\sqrt{2}, \frac{7\pi}{4})$} in polar form

Step 2: By computing V_0 from (3.4.6) and υ_0 from (3.4.7) we get:

$$V_0 = \{(1, 0), (3, \tfrac{\pi}{2}), (3, \pi), (1, \tfrac{3\pi}{2}), (\sqrt{2}, \tfrac{7\pi}{4})\}$$

$$\upsilon_0 = \{(\sqrt{2}, \tfrac{3\pi}{4})\}$$

Step 3: Since $\upsilon_0 \neq 0$, then compute V_{A2} and V_{B2} from (3.4.9) we have:

$$V_{A2} = \{(0, 1), (-1, 1), (-1, 0), (2, -2)\} = \{(1, \tfrac{\pi}{2}), (\sqrt{2}, \tfrac{3\pi}{4}), (1, \pi), (2\sqrt{2}, \tfrac{7\pi}{4})\}$$

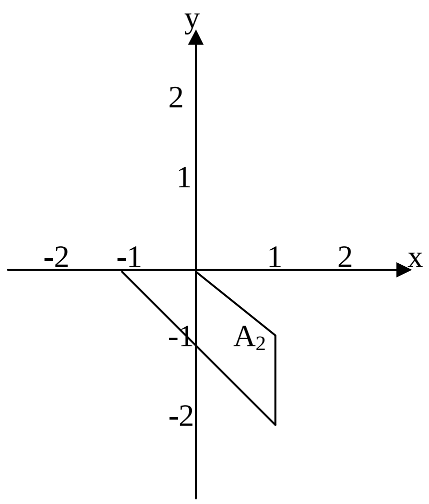
$$V_{B2} = \{(2, 0), (-2, 2), (0, -2)\} = \{(2, 0), (2\sqrt{2}, \tfrac{3\pi}{4}), (2, \tfrac{3\pi}{2})\}$$

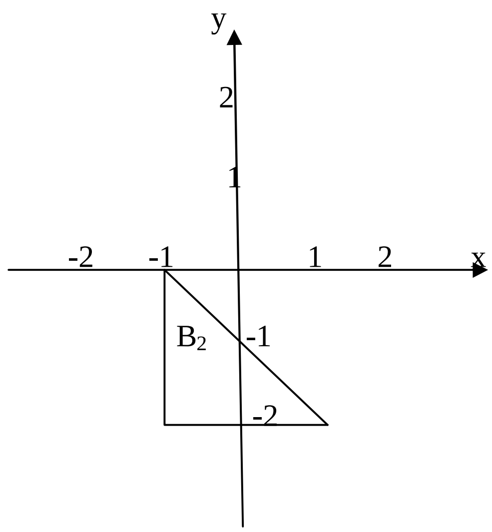
Step 4: Compute A_2 and B_2 from V_{A2} and V_{B2} starting from a_0 and b_0 using equation (3.4.2).

$$A_2 = \text{conv}\{(1, -2), (1, -1), (0, 0), (-1, 0)\}$$

$$B_2 = \text{conv}\{(-1, -2), (1, -2), (-1, 0)\}$$

Graphically, it looks like as indicated below;





Remark: Equation (3.4.9) and (3.4.10) eliminate all edge vectors which are common to both V_{A1} and V_{B1} and those which have identical polar angles. The only edge vector which may remain in both sets of edge vectors of A_2 and B_2 having the same polar angle, will be the one found because of the introduction of υ_0 in both sets in (3.4.9). Therefore, the algorithm certainly produces a minimal pair after exactly one iteration.

After carrying out above algorithm over a pair (A_1, B_1) we can see that some of the edge vectors contributed from A_1 will be reduced in A_2 and likewise those from B_1 will be reduced in size or eliminated from B_2. That means, A_2 will have a number of edge vectors at most equal to that of A_1 and B_2 will have a number of edge vectors at most equal to that of B_1.

3.4.2 Reduction Techniques for Pairs of Compact Convex Sets

Let $A, B \subseteq \mathbb{R}$ be convex compact sets. The reduction pairs of compact convex sets by cutting hyperplanes or by excision:

Lemma 3.4.2.1 Let X be a real topological vector space and $A, B, S \in K(X)$. Then

$A \vee B + S \subseteq (A \vee S) + (B \vee S)$.

<u>Proof:</u> Given any $x \in A \vee B$ and $s \in S$, then $x = \alpha a + \beta b$ for some elements $a \in A$, $b \in B$ and numbers $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Hence $x + s = \alpha a + \beta b + s = \alpha a + \beta b + (\alpha + \beta)s$

$$= (\alpha a + \beta s) + (\alpha s + \beta b) \in (A \vee S) + (B \vee S). \qquad \blacksquare$$

Remark:

i. If $S = A \cap B \neq \emptyset$ then $A \vee B + A \cap B \subseteq A+B$.
ii. If for a compact convex set $S \neq \emptyset$ the equation $A + B = A \vee B + S$ holds then $S = A \cap B$.

<u>Proof</u>: Part (i) is obvious. To prove (ii) observe that $A + B = A \vee B+S = (A + S) \vee (B + S)$ implies both $B + S \subseteq A +B$ and $A +S \subseteq A+ B$ and hence $S \subseteq A \cap B$.
From Lemma 3.4.2.1 follows; $A \vee B + A \cap B \subseteq A + B \subseteq A \vee B + S$. Hence $A \cap B \subseteq S$ and therefore $S = A \cap B$. $\blacksquare$

Lemma 3.4.2.2 Let X be a real topological vector space $A, B, S \in K(X)$ such that S is separating A and B. Then $A + B \subseteq A \vee B + S$.

<u>Proof</u>: Let $a \in A$, $b \in B$, then there exists $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ such that $\alpha a + \beta b \in S$.

Hence $a + b = (\alpha a + \beta a) + (\alpha b + \beta b)$

$$= (\alpha b + \beta a) + (\alpha a + \beta b) \in A \vee B + S. \qquad \blacksquare$$

Proposition 3.4.2.3 Let X be a real topological vector space A, B $\in$ K(X) such that

$A \cap B \neq \emptyset$. Then

i. If $A \cap B$ separates the sets A and B, then $A \vee B + A \cap B = A + B$.
ii. If X is a locally convex then $A \vee B + A \cap B = A + B$ implies that $A \cap B$ separates the sets A and B.

Proof: (i) Let $S = A \cap B$, then from Remark (i) it follows that $A \vee B + A \cap B \subseteq A+B$,

Moreover from Lemma 3.4.2.2 we have that $A + B \subseteq A \vee B + A \cap B$.

Hence $A + B = A \vee B + A \cap B$.

(ii) Now let us assume that X is locally convex and that $A + B = A \vee B + A \cap B$ holds.

If $A \cap B \neq \emptyset$ does not separate the sets A and B such that $[a, b] \cap (A \cap B) = \emptyset$.
Since X is locally convex there exists a continuous linear functional $f \in X^*$ such that

$$\max(f(a), f(b)) \preccurlyeq \min_{z \in A \cap B} f(z)$$

Now choose elements $a_0 \in A$, $b_0 \in B$ such that $f(a_0) = \min_{a \in A} f(a)$, $f(b_0) = \min_{b \in B} f(b)$.
Since $\max(f(a_0), f(b_0)) \preccurlyeq \max(f(a), f(b)) \leq_{z \in A \cap B} \min f(z)$
It follows that $[a_0, b_0] \cap A \cap B = \emptyset$. Since by assumption $A + B = A \vee B + A \cap B$, there exist elements $a_1 \in A$, $b_1 \in B$, $z_1 \in A \cap B$, and numbers $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, such that $a_0 + b_0 = \alpha\, a_1 + \beta b_1 + z_1$. Hence $f(a_0) + f(b_0) = \alpha\, f(a_1) + \beta\, f(b_1) + f(z_1)$. Since max $(f(a_0) - f(z_1), f(b_0) - f(z_1)) < 0$, this implies $f(a_0) > \alpha\, f(a_1) + \beta\, f(b_1)$ and $f(b_0) > \alpha\, f(f_1) + \beta\, f(b_1)$.
Hence $f(a_0) > f(b_1)$ and $f(b_0) > f(a_1)$ which leads to the contradiction

$$f(a_0) > f(b_1) \geq f(b_0) > f(a_1).$$

■

Theorem 3.4.2.4 Let X is a real topological vector space, $A \in K(X)$ a nonempty compact convex set. Moreover let us assume that there exists a nonempty compact convex subset $C \subseteq A$ such that $A \backslash C$ is nonempty and convex. Then the pairs (A, C), $(\overline{A \setminus C}, C \cap (\overline{A \setminus C})) \in K^2(X)$ are equivalent.

Proof: Let $S = C \cap (\overline{A \setminus C})$, then S separates $\overline{A \setminus C}$ and C. Hence by proposition 3.4.2.3 we have $(\overline{A \setminus C}) \vee C + S = \overline{A \setminus C} + C$. Since $(\overline{A \setminus C}) \vee C = A$, we get $A + S = (\overline{A \setminus C}) + C$, which means $(A, C) \sim (\overline{A \setminus C}, C \cap (\overline{A \setminus C}))$. ■

In the case where X is a real locally convex topological vector space, the assumption that the sets C and A\C are convex is equivalent to the existence of a point $z \in A$ and a continuous linear functional $f \in X^*$ such that:

$$\overline{A \setminus C} = A^+_{f,z} := \{x \in A: f(x) \geq f(z)\}, \text{ and}$$

$$C = A^-_{f,z} := \{x \in A: f(x) \leq f(z)\}.$$

$$\overline{A \setminus C} \cap C = A_{f,z} := \{ x \in A: f(x) = f(z)\}.$$

The above result now leads to a theorem on the reduction of pairs of nonempty compact convex sets by cutting hyperplanes.

Theorem 3.4.2.5 Let X be a real locally convex topological vector space, $A, B \in K(X)$ nonempty compact convex sets and let us assume that there exists an element $z \in A \cap B$ and a continuous linear functional $f \in X^*$ such that $A^+_{f,z} = B^+_{f,z}$ and $A_{f,z} = B_{f,z}$. Then the pairs

$(A, A^-_{f,z}), (B, B^-_{f,z}) \in K(X)$ are equivalent.

Proof: By Theorem 3.4.2.4 we have $(A_{f,z}, A^-_{f,z}) \sim (A^+_{f,z}, A_{f,z})$. Since by assumption

$A^+_{f,z} = B^+_{f,z}$ and $A_{f,z} = B_{f,z}$, it follows that $(A, A^-_{f,z}) \sim (B, B^-_{f,z})$. ■

The idea of Theorem 3.4.2.5 can be illustrated in figure2.

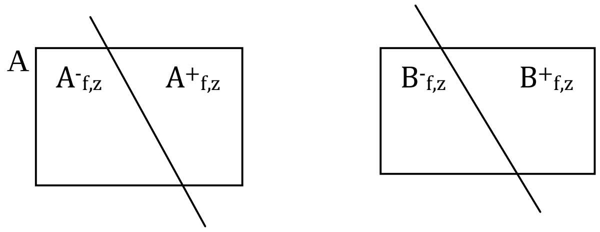


Figure 2: Reduction by cutting hyperplane method.

For a real topological vector spaces reduction by excision method can be proved by the following theorem.

Theorem 3.4.2.6 Let X be a real topological vector space, $(A, B) \in K^2(X)$ such that $B \subsetneq A$. Then

$(A, B) \sim (\overline{\mathrm{conv}(A \setminus B)}, B \cap \overline{\mathrm{conv}(A \setminus B)})$.

<u>Proof</u>: Put $A_1 := \overline{\mathrm{conv}(A \setminus B)}$ and $B_1 := B$. We need to show that $A_1 \cap B_1$ separates A_1 and B_1. Since A can be decomposed as a disjoint union of the form

$$A = (B \setminus \overline{\mathrm{conv}(A \setminus B)}) \cup (B \cap \overline{\mathrm{conv}(A \setminus B)}) \cup (A \setminus B).$$

That is, $A_1 \cup B_1 = (B_1 \setminus A_1) \cup (A_1 \cap B_1) \cup ((A_1 \cup B_1) \setminus B_1)$, it follows from the convexity of A that for every $a \in A_1 \setminus B_1$ and $b \in B_1 \setminus A_1$.

The line segment $[a, b]$ intersects $A_1 \cap B_1$ separates A_1 and B_1.

Now from proposition 3.4.2.3 (i) we have $A_1 \vee B_1 + A_1 \cap B_1 = A_1 + B_1$, which means

$$A + B \cap \overline{\mathrm{conv}(A \setminus B)} = \overline{\mathrm{conv}(A \setminus B)} + B; \text{ since } A_1 \vee B_1 = A_1 \cup B_1 = A.$$

Hence $(A, B) \sim (\overline{\mathrm{conv}(A \setminus B)}, B \cap \overline{\mathrm{conv}(A \setminus B)})$. ■

The idea of Theorem 3.4.2.6 can be illustrated in figure 3;

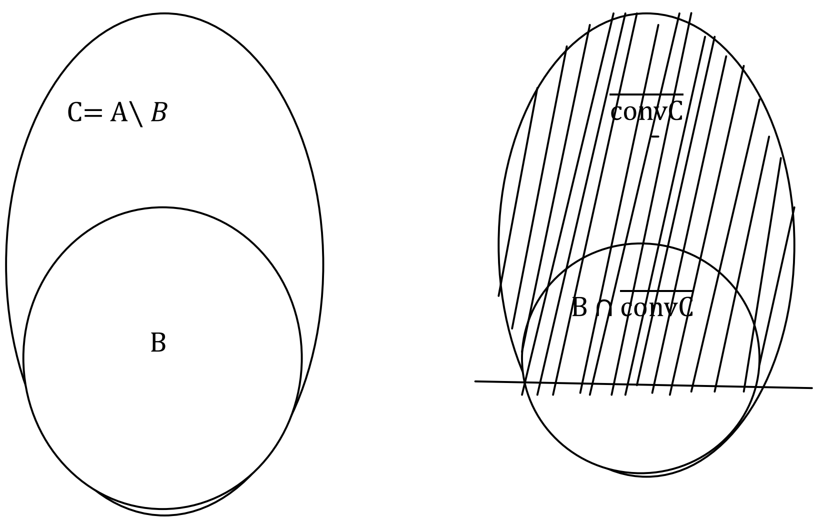


Figure 3: Reduction by excision method.

Theorem 3.4.2.7 The convex pair $(A, B) \in K^2(X)$ is convexly reduced if and only if

$(A \cap B, A \cup B)$ is reduced.

Proof: ($\Rightarrow$) Let the pair (A, B) be convexly reduced and $(F, G) \in [A \cap B, A \cup B]$, there exists

$(A_0, B_0) \in [A, B]$ such that $A_0 \cap B_0 = F$ and $A_0 \cup B_0 = G$. From the assumption, $A_0 = A \dotplus M$ and $B_0 = B \dotplus M$ for some $M \in K(X)$. Then $F = A_0 \cap B_0 = A \cap B \dotplus M$ and

$G = A_0 \cup B_0 = A \cup B \dotplus M$. Therefore, the pair $(A \cap B, A \cup B)$ is reduced.

($\Leftarrow$) Let $(A \cap B, A \cup B)$ be reduced, $(C, D) \in [A, B]$ and $C \cup D$ be convex, then

$A \dotplus D = B \dotplus C = A \cap B \dotplus C \cup D = C \cap D \dotplus A \cup B$, hence $C \cap D = A \cap B \dotplus M$ and

$C \cup D = A \cup B \dotplus M$ for some $M \in K(X)$.

From the law of cancellation, we obtain $C = A \dotplus M$ and $D = B \dotplus M$.

Hence (A, B) is convexly reduced. ∎

Definition 3.4.2.8 Let X be a real topological vector space and $C \in K(X)$, then the pair $(A, B) \in K^2(X)$ is called C-minimal if the pair $(A + C, B + C)$ is convex, and if for every $C_1 \in K(X)$ with

$C_1 \subseteq C$ and such that $(A+ C_1, B+ C_1)$ is a convex pair it follows that $C_1= C$.

Theorem 3.4.2.9 Let X be a real topological vector space and $C \in K(X)$, then the pair

$(A, B) \in K^2(X)$ is called C-minimal if and only if there exists $D \in K(X)$ such that that pair

(C, D) is minimal and equivalent to $(A \vee B, A + B)$.

Proof: ($\Rightarrow$) Since the pair $(A, B) \in K^2(X)$ is C-minimal; thus

$$A + C + B + C = (A + C) \cup (B + C) + (A + C) \cap (B + C)$$

Since $(A + C) \cup (B + C) = (A + C) \vee (B + C) = (A \vee B) + C$

We obtain that $A + B + C = A \vee B + (A + C) \cap (B + C)$.

If we put $D := (A + C) \cap (B + C)$, then it follows that $(A \vee B, A + B) \sim (C, D)$.

(⇒) Suppose that there exists a $C_1 \subseteq C$ and $D_1 \subseteq D$ such that $(C, D) \sim (C_1, D_1)$, then

$(A \vee B, A + B) \sim (C_1, D_1)$. Hence we have $A \vee B + D_1 = A + B + C_1$, which implies

$(A + C_1) \vee (B + C_1) + D_1 = (A + C_1) + (B + C_1)$. Thus $(A + C_1) \vee (B + C_1)$ is a summand of

$(A + C_1) + (B + C_1)$, which means that the set $(A + C_1) \cup (B + C_1)$ is convex.
Hence by the C-minimality of (A, B) it follows that $C = C_1$ and $D = D_1$.

If $(A \vee B, A + B) \sim (C, D)$ and (C, D) minimal, then (A + C, B + C) is a convex pair.
Then for every $(C_1, D_1) \leq (C, D)$ which is equivalent to (C, D).

We have $C = C_1$ and $D = D_1$ ■

3.4.3 Criteria for Non-Minimality

From the reduction by cutting hyperplanes we receive the following criterion for minimality.

Theorem 3.4.3.1 Let X be a real locally convex topological vector space, $(A, B) \in K^2(X)$, and let us assume that there exists an element $z \in A \cap B$ and a continuous linear functional $f \in X^*$ such that $A_{f,z} = B_{f,z}$ and $A^+_{f,z} = B^+_{f,z}$. Then the pair (A, B) is not minimal.

Proof: From theorem 3.4.2.7 it follows that the pairs $(A, A^-_{f,z}), (B, B^-_{f,z}) \in K^2(X)$ are equivalent. Hence we have $A + B^-_{f,z} = B + A^-_{f,z}$, which implies $(A, B) \sim (A^-_{f,z}, B^-_{f,z})$.

Since $A^-_{f,z} \subseteq A$ and $B^-_{f,z} \subseteq B$ the pair $(A, B) \in K^2(X)$ is not minimal. ■

Lemma 3.4.3.2 Let X be a real locally convex topological vector space and let $(A, B), (C, D) \in K^2(X)$ be two equivalent pairs. Then $(A \vee B, B), (C \vee D, D) \in K^2(X)$ are equivalent.

Proof: This follows from Pinsker identity [4], [5], since

$$A \vee B + D = (A + D) \vee (B + D)$$

$$= (B + C) \vee (B + D)$$

$$= B + C \vee D.$$ ■

Proposition 3.4.3.3 Let X is a real locally convex topological vector space and let (A, B) ∈ $K^2(X)$. If (A, B) ∈ $K^2(X)$ is not minimal, then the pairs (A ∨ B, B) and (A + B, A ∨ B) are also not minimal.

Proof: Since (A, B) ∈ $K^2(X)$ is not minimal there exists an equivalence pair (A′, B′) ∈ $K^2(X)$ with A′ ⊆ A, B′ ⊆ B. From lemma 3.4.3.2 follows that (A ∨ B, B) is not minimal. Moreover the equivalence (A ∨ B, B) ~ (A′ ∨ B′, B) implies that

A + (A ∨ B + B′) = A + (B + A′ ∨ B′) which leads to the equivalence

(A + B, A ∨ B) ~ (A + B′, A′ ∨ B′).

Therefore the pair (A + B, A ∨ B) ∈ $K^2(X)$ is not minimal. ■

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