BRAKING INDEX OF ISOLATED PULSARS ACCORDING TO THE RELATIVISTIC PLASMA DIFFUSION THEORY FOR PULSAR FIELDS

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By

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God made the stars and He set them
in the expanse of the sky to give light on
the earth, to govern the day and
the night, and to separate light from darkness .
And He saw that it was good.
Genesis 1:16-18

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Abstract

The braking index of a pulsar is expected to be 3 if it slows down under ordinary magnetic dipole radiation . However, measured braking indices of young as well as old pulsars are experimentally determined to be different from the above value. Sometimes these get to large positive and negative numbers.

This is something that is not well understood. We will describe pulsar braking index variations based on Relativistic plasma diffusion theory which is currently developed by us. Using this theory we will calculate the braking indices of four-young well known pulsars and two old pulsars.

It is also important to establish whether the observed(measured) braking index n_{obs} is time dependent. We will also derive braking indices of three pulsars as a function of time and draw the graphs to show the time evolution of braking index.

INTRODUCTION

It was not after the discovery of pulsars that they were convincingly identified with isolated, rotating, magnetized neutron stars following the proposals by Gold and Pacini at 1968. The key observations were the very stable, short periods of the pulsars and the observations of polarized emission [1]. A pulsar is neutron star which emits radiation that is pulsed due to rotation and powered by rotational kinetic energy. Neutron stars are supported by neutron degeneracy pressure [2]. In theory, pulsar masses range between $0.05M_{\odot}$ and $3M_{\odot}$, where $M_{\odot}$ is solar mass. Their radii range between 6km and 100km [3]. Having those facts in mind , we can approximate the initial angular $\Omega(t=0)$ of a rotating neutron star by

$$\Omega(t=0) \approx \sqrt{\frac{GM}{R^3}}$$

This predicts pulsar rotational frequency in the range of $80s^{-1}$ and $4.3 \times 10^4 s^{-1}$. Meanwhile, realistic equations of state predict that the average values of masses and radii of neutron stars are approximated to be $1.4M_{\odot}$ and 10km respectively [4] . Using these values the average initial angular frequency of neutron star is computed to be of the order of $10^3 s^{-1}$. Hence pulsars are born with high rotational frequency. However, they subsequently slow-down with time mainly as a result of loss of rotational kinetic energy via the emission of high-energy particles, gravitational and magnetic dipole radiations. Due to these slow down mechanisms the observed angular frequencies now a days are very much smaller than the one indicated above. For instance the Crab pulsar has angular frequency of $188s^{-1}$ (pulse period of 33ms) and an initial of order of $10^3 s^{-1}$.

At the early era of neutron star evolution gravitational quadrupole radiation is dominant over other breaking mechanisms. Without considering magnetic dipole radiation, gravitational radiation is a strong decay mechanism within the first 10^5yrs after which that due to photon and neutrino emissions become predominant. It is interesting that, if the neutron star possessed a significant quadrupole moment in the early stages of formation and evolution, it would radiate gravitational radiation for which the braking index is 5, plainly inconsistent with the observed values for the Crab pulsar [6].

The pulse periods of pulsars can be measured with very high accuracy indeed, and one of the most important parameters is the rate at which the pulse period changes with time ($\dot{P}$ or $\dot{\Omega}$). For most pulsars the rate at which the pulse period increases ($\dot{\Omega} < 0$) can be measured, and this can be used to derive an age estimate. The age of neutron star is an essential parameter, which is relevant to the physics of supernova explosions and, thereafter, the evolution of stars. Using those techniques the average age of young pulsars is estimated to be 5×10^3 years [5].

The slowing down of pulsar can be described by a braking index, n, which is defined by

$$\dot{\Omega}(t) = -\Lambda(t)\Omega(t)^n$$

where $\Omega(t)$ being angular frequency of rotation of star and $\Lambda(t)$ being some function of time [7]. The braking index provides information about the energy loss mechanism. Among the most important of these is magnetic braking [8],[9],[10].

In order to produce pulsed radiation from magnetic pole of the neutron star , the magnetic dipole must be misaligned to the rotation axis [1]. This results in the radiation of electromagnetic energy from the dipole, which is extracted from the rotational energy of the neutron star. If the slow-down mechanism was due to magnetic dipole radiation alone, the braking index, n, of pulsar would be 3. The

The dynamics of the magnetic moment resulting from the magnetic and viscous couplings between the various components of the neutron star as suggested by the model has been shown to successfully address some of the current worrisome issues in pulsar astronomy today such as the lack of long period pulsars in the galaxy , delayed pulsar onset, and the problem of missing pulsars [22].

The theory of relativistic plasma diffusion as a possible source for Neutron star magnetic fields is fully treated by Kebede [47]. In this issue, the fact that pulsar fields are Neutron star internal temperature (T) and spin frequency (Ω) dependent [$B = B(\Omega, T)$] are clearly shown suggesting possible causes for field decay, which are directly related to Neutrino and photon emissions as well as various pulsar braking mechanisms.

One other area in which the new model will have contribution in is that of braking index variations. It is very well known that measured values of braking indices of pulsars show considerable variations from pulsar to pulsar and also from the canonical value of 3 expected for the standard braking due to magnetic dipole radiation [29]. Observed indices greater than 3 are believed to be due to magnetic ohmic decay [11], magnetic dipole alignment [12] even thought ohmic decay and alignment are still controversial issues [31],[40],[41]. On the other hand low values of braking indices are indicated to arise from secular increase in the dipole moment [7],[14], magnetospheric currents [13],[15], pulsar winds[18], disk-assisted spin down[16]. However, so far no direct analytic solution to the spin down low

$$\dot{\Omega} = -\Lambda\Omega^n$$

exists leading to the determination of the time evolution of the braking index. An attempt has been made by Blandford and group(1983) to solve the spin-down law for the braking index. The restriction on the time scale for field growth however limits their theory. The method demonstrated by Johnston & Galloway [39] to solve the spin-down law by integration on works if n is kept constant while of course is very doubtful.

As already indicated above the new model for neutron star surface magnetic fields indicates that pulsar fields are pulsar frequency(Ω) and internal temperature(T) dependent. Since the time evolution of Ω and T are easily derivable from the various braking rates ($\frac{dE}{dt}$) and well known cooling rates ($\frac{dT}{dt}$) this implies that exact analytic decay laws for pulsar fields at various stages in the Neutron star's life time could be derived allowing the possibility of solving the spin-down law analytically for the braking index, provided accurate values of such quantities as Ω, $\dot{\Omega}$ and t are known. Indications are that braking indices will not be restricted to values between 0 and 3 as some authors try to show [15]. Rather we should expect arbitrary values(+/-) outside of this range.

The thesis comprises four chapters. In the first chapter we develop general overview of braking index of isolated pulsars. We require this chapter to derive the relation for braking index and to distinguish between the theoretical and observed braking indices.

In chapter two the magnetic decay laws resulting from magnetic dipole and gravitational quadrupole radiations will be developed. We then derive the decay law for the surface magnetic fields as a result of the slow down based on the new model.

In chapter three, we will calculate the braking indices of four-young pulsars in accordance with relativistic plasma theory, and show that the numbers agree with the experimental value. In addition to this we will also calculate the braking indices of two more pulsars with large positive and negative values using alignment/counteralignment and magnetic field decay effects. Finally we will derive the relation for braking indices of three pulsars as a function of time and drawn the graphs showing the time evolution curve of braking indices. In the last chapter discussion and conclusion will be given.

Chapter 1

GENERAL OVERVIEW OF BRAKING INDEX OF ISOLATED PULSARS

1.1 Rate of spin-down of pulsar

Although neutron stars have now been recognized in a variety of stellar systems, radio pulsars are by far the most common observed manifestation of neutron stars. The emitted radio pulses allow us to measure the rotation rate of the underlying star. The rotation rate is, in comparison with most other astronomical measurements, exceptionally stable and easy to measure with high accuracy.

Almost every pulsar has precisely measured angular frequency, Ω and the rate of change angular frequency with time, $\dot{\Omega}$. It is an experimental fact that the pulsar spin rate, Ω , has a negative derivative, $\dot{\Omega}$. This shows that pulsars slow-down with time. Pulsar rotation frequencies are generally assumed to evolve according to the spin-down law

$$\dot{\Omega} \propto -\Omega^{n} \tag{1.1.1}$$

where Ω is angular frequency of pulsar and n is braking index.

The spin-down rate is partly quantified by the braking index. This concept originates in a particular form of theoretically predicted spin-down law, as

$$\dot{\Omega} = -\Lambda\Omega^n \tag{1.1.2}$$

where Λ is positive torque coefficient which depends in general on moment of inertia, I, and the various parameters that determine the magnitude of braking torque of Neutron stars [23].

The observed (experimental) braking index n_{obs} which had been determined for a few young pulsars, had been found to differ from the expected value for a rotating magnetic dipole model. The braking index, n, is usually assumed to be 3 as predicted by the magnetic dipole model(where n=3 corresponds to the mechanism of pure magnetic dipole braking), but actually the four young pulsars with the best determinations of braking index show values less than 3 [13],[7].

Based on the definition given above the braking index, n , can be found by differentiating the equation with respect to time ;

$$\ddot{\Omega} = -\Lambda n\dot{\Omega}\Omega^{n-1} \tag{1.1.3}$$

Dividing eq(1.1.3) by eq(1.1.2) we get

$$n = \frac{\Omega\ddot{\Omega}}{\dot{\Omega}^2} \tag{1.1.4}$$

where $\ddot{\Omega}$ is a double derivative of the angular frequency with respect to time. The braking index n may then be calculated from measured values of Ω , $\dot{\Omega}$ and $\ddot{\Omega}$.

As we can see eq(1.1.2) and eq(1.1.4) are not equivalent . This can be seen from the following two perspectives .

First, eq(1.1.4) follows from eq(1.1.2) only if Λ and n are kept constants. However, given the type of parameters contained in the torque coefficient Λ, it is likely that Λ will, in fact, be a function of time. If so, then eq(1.1.4) does not follow from eq(1.1.2) and so is not equivalent to it.

Second, eq(1.1.4) contains values that are defined ,and can be measured ,irrespective of what spin-down law the pulsar follows. In particular this formula does not presuppose a spin-down law of the special type given by eq(1.1.2).

It is, therefore, a much more general spin-down parameter than that defined by the exponent of eq(1.1.2) and is not necessarily related to any exponent in any particular spin-down law. In fact, the actual spin-down followed by a pulsar law is probably not of the type given by eq(1.1.2) at all except perhaps in some idealized approximation. If so, then the spin down law does not admit the concept of a braking index defined in the sense of eq(1.1.2), and we have no alternative to eq(1.1.4) as a definition.

The inequivalence of definitions in eq(1.1.2) and eq(1.1.4) means that we should distinguish the parameter defined by them. Since all attempts at measuring the braking index of a pulsar are based directly on eq(1.1.4) and not on eq(1.1.2), we shall call the parameter defined by eq(1.1.4) the experimental, or the observational braking index and denoted by n_{obs}.

Thus;

$$n_{obs} = \frac{\Omega\ddot{\Omega}}{\dot{\Omega}^2} \tag{1.1.5}$$

Note from eq(1.1.5) that , since $\Omega = \Omega(t)$, the observational braking index n_{obs} is in general also a function of time. Hence we have no a priori guarantee that n_{obs} will be constant. The naive derivation by which we passed from eq(1.1.2) to eq(1.1.4) shows that , in some circumstances at least, these two definitions may be equivalent. To determine those circumstances, suppose that the spin-down law has the form in eq(1.1.2) with constant Λ and constant n , then the reasoning leading to eq(1.1.4) shows that the exponent n of the braking law in eq(1.1.2) is equal in value to the observational braking index n_{obs} defined in eq(1.1.5). In particular, this makes n_{obs} constant. In this case the theoretical braking index n and the observed braking index n_{obs} coincide.

If n_{obs} is not constant but a given function of time , say n_{obs}=$n_{obs}(t)$, Λ in eq(1.1.2) is not constant in time and hence the theoretical and experimental braking indices are distinct parameters.

1.2 Time dependent parameters in the braking law

In view of the above discussion, it is important to establish whether n_{obs} is time dependent. Due to timing irregularities including glitches, post-glitch recoveries, and timing noise, the measurement of $\ddot{\Omega}$ in eq(1.1.5) requires very long data spans. So far, only four pulsars have had their braking indices reliably measured.

For the Crab pulsar, $\ddot{\Omega}$ is larger than for others and the data span is such that one can hope to obtain several measurements of n_{obs}. Lyne, Pritchard and Graham-Smith divided the data into 5-year spans and were able to determine four values of n_{obs}. Their conclusion is that n_{obs} shows no appreciable time dependence. But, given that the period covered by the data is small when compared to the pulsar's lifetime, it may perhaps be better to infer that , if there is any time dependence at all in n_{obs}, then it is probably small [26].

The inconclusiveness of the current data on the time dependence of n_{obs} makes a study of a slow time dependence in n and Λ relevant. Let us iuvestigate analytically the effects of such dependence by considering the braking law obtained from the magnetic dipole model , where we expect theoretically a braking index n=3. However, the observed braking indices have values $1 < n < 3$. Hence the deviation of experimental braking index n_{obs} from that of theoretical one, n=3 , is due to the variation of the torque coefficient Λ, which cannot be held constant with respect to time [5].

Suppose now that n is constant, and that $\Lambda = \Lambda(t)$. Then eq(1.1.5) can be rewritten as

$$n_{obs} = \frac{\Omega\ddot{\Omega}}{\dot{\Omega}^2} = n + \frac{\dot{\Lambda}(t)\Omega}{\Lambda(t)\dot{\Omega}} \tag{1.2.1}$$

Note that the difference of value in n and n_{obs} is not the result of glitching in the pulsar,but of a continuous variation in the braking coefficient Λ. We can replace the theoretical braking index n in eq(1.2.1) by 3. Where eq(1.2.1) reduces to

$$n_{obs} = 3 + \left(\frac{\dot{\Lambda}(t)}{\Lambda(t)}\right)\frac{\Omega_p(t)}{\dot{\Omega}_p(t)} \tag{1.2.2}$$

where the subscript p represents present time .

Allen and Horvath have estimated the amount of continuous variation in the torque coefficient Λ required to account for the measured value of n_{obs} for the Crab pulsar. Using eq(1.2.2) they estimate the amount of continuous variation in Λ for Crab pulsar to be

$$\frac{\dot{\Lambda}(t)}{\Lambda(t)} = \left(n_{obs} - 3\right)\frac{\Omega_p(t)}{\dot{\Omega}_p(t)} \approx 1.9 \times 10^{-4} yr^{-1}$$

It seems likely, therefore, that one will need to invoke some mechanism that leads to a continuous variation of Λ to account for the measured values of the braking index.

In rotating dipole model, Λ is a function of several parameters related to the braking dynamics. These include the surface magnetic field strength B_p, the inclination angle α of the magnetic dipole axis to the rotation axis , and the moment of inertia I of the star. We have derived equation for Λ in eq(3.1.5) as

$$\Lambda(t) = \frac{B_p^2(t) R^6 \sin^2\alpha(t)}{6Ic^3} \tag{1.2.3}$$

where R is radius of the star and c is speed of light. Time dependence in any one of the above parameters will result in Λ being time dependent and so cause theoretical & observational braking indices to differ. The effect of time dependence in these parameters(keeping I and R constant) on the measured braking index is summarized as follows.

Differentiating eq(1.2.3) with respect to time where B_p and α are both time dependent parameters , we obtain

$$\dot{\Lambda}(t) = 2\left[\frac{\dot{B}_p(t)}{B(t)} + \frac{\dot{\alpha}(t)}{\tan\alpha}\right]\frac{B_p^2(t) R^6 \sin^2\alpha(t)}{6Ic^3} \tag{1.2.4}$$

where $\dot{B}_p(t)$ and $\dot{\alpha}(t)$ are time derivative of $B_p(t)$ and $\alpha(t)$. Several papers have addressed the question of internal dynamics and its consequences for time dependence of α [27], [28],[33],[5].

Upon substituting eq(1.2.3) and eq(1.2.4) into eq(1.2.2) for $\Lambda(t)$ and $\dot{\Lambda}(t)$, eq(1.2.2) can be rewritten as

$$n_{obs} = 3 - 2\left[\frac{\dot{B}_p(t)}{B(t)} + \dot{\alpha}(t)\cot\alpha(t)\right]\left|\frac{\Omega_p(t)}{\dot{\Omega}_p(t)}\right| \tag{1.2.5}$$

The first term on the right hand side of eq(1.2.5) is that given by the 'standard' model: a perfect sphere with constant magnetic field inclined at a constant angle to the rotation axis. The first part of the second term describes magnetic field-decay , and the second part those of the effects of alignment/counteralignment .

According to eq(1.2.5) counteralignment and field growth ($\dot{B}_p(t) > 0$) decrease n from its standard value, while field decay and alignment increase it above this value [31]. Hence eq(1.2.5) gives us large positive and negative values of braking index.

Chapter 2

MAGNETIC FIELD DECAY OF NEUTRON STARS DUE TO MAGNETIC DIPOLE AND GRAVITATIONAL RADIATIONS

2.1 Introduction

As we have tried to note down in the introduction of this thesis, there is no satisfactorily self-consistent theory for the generation of Neutron stars magnetic fields at the present time. The current understanding is that it can either be a fossil remnant (the standard picture) or it may be generated by surface thermal processes soon after the formation of the Neutron star [19],[17]. Very recently, however, separated charges have been suggested as more likely source for Neutron star magnetic fields where ohmic decay is rejected[21], [2],[22].

In this chapter we are going to derive magnetic fields decay laws based on those facts. According to the relativistic plasma diffusion theory developed by L.W.Kebede the plasma density gradient which is inherent to the degenerate neutron star matter initiates some kind of stochastic diffusion on a macroscopic scale. As a result of which, charges will separate with excess negative charges accumulating on the surface (crust) and almost the same amount excess positive charge is left behind at the core of neutron star [22]. For the sake simplicity, let us assume the space charge established by the diffusion process to be spherical symmetric.

2.2 Magnetic Field Generated by Spinning Separated Charge

2.2.1 Vector potential and magnetic moment

Now, the vector potential associated with the spinning crust containing the separated negative charge, Q, will be calculated as follows[24]. Consider a thin ring of radius R with uniform surface charge density, σ. From spherical symmetry the surface current density, **J**, and resulting vector potential **A** have only ϕ-components. The surface current element of spinning spherical charge is given by

$$\mathbf{J} = -\frac{|Q|\omega}{4\pi R} P_1^1(\cos\theta')\delta(\mathbf{r}' - \mathbf{R})\delta(\cos\theta') \tag{2.2.1}$$

Where ω is the spin frequency of the crust.The current density vector **J** can be written as

$$\mathbf{J} = -J_\phi \sin\phi' \hat{\imath} + J_\phi \cos\phi' \hat{\jmath} \tag{2.2.2}$$

Assuming that the charge distribution is uniform , it should n't matter at which ϕ' observation is made. Suppose we choose $\phi = 0$. The vector potential is given by

$$\mathbf{A} = \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' \tag{2.2.3}$$

Where $dV' = r'^2 \sin\theta' d\theta' d\phi' dr'$ and

$$|\mathbf{r} - \mathbf{r}'| = [r^2 + r'^2 - 2rr'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi')]^{\frac{1}{2}}$$

Substituting eq(2.2.2) into (2.2.3) and integrating it over ϕ' we get

$$\mathbf{A}_\phi(r,\theta,\phi) = \frac{1}{c}\int \frac{J_\phi \cos\phi' r'^2 dr' \sin\theta' d\theta' d\phi'}{|\mathbf{r}-\mathbf{r}'|} \tag{2.2.4}$$

We have

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} P_l(\cos\gamma) \tag{2.2.5}$$

Where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$ and γ is the angle between $\mathbf{r}$ and $\mathbf{r}'$. Let define r' as $r_<(smaller)$ and r as $r_>(larger)$. Hence eq(2.2.5) can be rewritten as

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma) \tag{2.2.6}$$

Substituting eq(2.2.6) into (2.2.4) we get

$$\mathbf{A}_\phi(r,\theta,\phi) = \frac{1}{c}\int J_\phi \cos\phi' r'^2 \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma) dr' d\Omega' \tag{2.2.7}$$

Where $d\Omega' = \sin\theta' d\theta' d\phi' = -d(\cos\theta')d\phi'$. From addition theorem we have

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{+l} Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) \tag{2.2.8}$$

Where $Y_{lm}(\theta,\phi)$ is the spherical harmonics. Putting eq(2.2.8) into (2.2.7) the vector potential become

$$\mathbf{A}_\phi(r,\theta,\phi) = -\frac{|Q|\omega}{Rc}\int \sum_{l=0}^{\infty}\sum_{m=-l}^{l} [\frac{1}{2l+1}](\frac{r'^l}{r^{l+1}})Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)P_l^l(\cos\theta')\delta(r'-R)\cos\phi' r'^2 dr' d\phi' \tag{2.2.9}$$

Where $Y_{lm}(\theta,\phi)$ is defined as

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos\theta)e^{im\phi} \tag{2.2.10}$$

Any arbitrary function $g(\theta,\phi)$ may be expanded in terms of spherical harmonics as

$$g(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}Y_{lm}(\theta,\phi) \tag{2.2.11}$$

Where

$$A_{lm} = \int d\Omega' Y'_{lm}(\theta,\phi) g(\theta,\phi) \tag{2.2.12}$$

and the prime represents complex conjugate. Finally the vector potential inside and outside of the surface can be expressed as follows;

1) Inside the surface, $A_{\phi,in}$, $\phi' = 0$, $l = 1, r_< = r$, and $r_> = R$ .Where eq(2.2.9) reduces to

$$A_{\phi,in} = -\frac{|Q|\omega}{3Rc} \int \sum_{0}^{\infty} \delta(\phi - \phi')(\cos\theta - \cos\theta') P_1^1(\cos\theta') d(\cos\theta') \tag{2.2.13}$$

Equation(2.2.13) integrates to

$$A\phi, out = -\frac{|Q|\omega}{3c}(\frac{r}{R}) \sin\theta \tag{2.2.14}$$

2) Similarly potential outside of the surface $A_{\phi,out}$ (for $\phi' = 0, l = 1, r_< = R, andr_> = r'$) is found to be

$$A\phi, out = -\frac{|Q|\omega}{3c}(\frac{R}{r})^2 \sin\theta \tag{2.2.15}$$

The magnetic fields inside and outside the sphere of radius R carrying a uniform surface charge density spinning with angular frequency ω can be found by $\mathbf{B} = \nabla \times \mathbf{A}$ Using spherical polar coordinate. Thus

$$\begin{aligned} B_r &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) \\ B_\theta &= -\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) \\ B_\phi &= 0 \end{aligned} \tag{2.2.16}$$

Up on substituting eq(2.2.14) and (2.2.15) into (2.2.16) the magnetic fields inside and outside the spinning charge crust are found respectively to be

$$\mathrm{B}_{in} = -\frac{|Q|\omega}{3c}(\frac{2}{R})\mathbf{k} \tag{2.2.17}$$

and

$$\mathbf{B}_{out} = -\frac{|Q|\omega}{3c}(\frac{R^2}{r^3})(3\cos\theta\hat{e_r} - \mathbf{k}) \tag{2.2.18}$$

Where θ is the zenith angle and $\hat{e_r}$ and $\mathbf{k}$ are unit vectors along the radial direction and the spin axis respectively. Similar expressions may also be developed for spinning separated positive charge at the core. Equations (2.2.17) and (2.2.18) clearly show that neutron star magnetic fields generated by the spinning separated charges are dipolar. This is in good agreement with observation [22].

Next, let us also calculate expression for magnetic dipole moment of neutron star. We consider the properties of a general current distribution which is localized in a small region of space[25]. Assuming $|r| \gg |r'|$, let us expand denominator of eq(2.2.3) in power of r' measured relative to a suitable origin in the localized current distribution. Using eq(2.2.6) we can expand $\mathbf{A}(\mathbf{r})$ as

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int_{V'}\mathbf{J}(\mathbf{r}')r'P_l(\cos')dV' \tag{2.2.19}$$

Using the definition of the Legendre polynomials this may be expanded in to

$$\mathbf{A}(\mathbf{r}) = \frac{1}{cr}\int_{V'}\mathbf{J}(\mathbf{r})dV' + \frac{1}{cr^2}\int_{V'}\mathbf{J}(\mathbf{r})(\mathbf{r}\cdot\mathbf{r}')dV' + --- \tag{2.2.20}$$

Where the first term in the expansion is the monopole term with the property

$$\mathbf{A}_M = \int_{V'}\mathbf{J}(\mathbf{r}')dV' = 0$$

The second term in the expansion of the dipolar term, may be rewritten as

$$\mathbf{A}(\mathbf{r})_D = \frac{1}{cr^2}\left[\frac{1}{2}\int_{V'}\mathbf{r}'\times\mathbf{J}(\mathbf{r}')dV'\right]\times\mathbf{r} \tag{2.2.21}$$

Where the quantity in the square bracket is magnetic dipole moment $\mathbf{m}$

$$\mathbf{m} = \frac{1}{2}\int_{V'}\mathbf{r}'\times\mathbf{J}(\mathbf{r}')dV' \tag{2.2.22}$$

Hence

$$\mathbf{A}(\mathbf{r})_D = \frac{\mathbf{m} \times \mathbf{r}}{c|\mathbf{r}|^3} \tag{2.2.23}$$

For $\mathbf{m}$ along the spin axis $\hat{z}$

$$\mathbf{A}(\mathbf{r})_D = \frac{\mathrm{m}\sin\theta}{cr^2}\hat{\phi} \tag{2.2.24}$$

Hence the magnetic field can be $\mathbf{B} = \nabla \times \mathbf{A}(\mathbf{r})_D$. For a point dipole of arbitrary orientation the magnetic field can be shown to be

$$\mathbf{B}(\mathbf{r}) = \frac{1}{cr^3}[3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] \tag{2.2.25}$$

Where the magnetic field at the poles become

$$B_p = \frac{2m}{R^3}$$

or

$$m = \frac{1}{2}B_p R^3 \tag{2.2.26}$$

The net magnetic dipole moment $\mathbf{m}$ of the neutron star may be written as

$$\mathbf{m} = \mathbf{m}_{core} + \mathbf{m}_{crust} \tag{2.2.27}$$

Where $\mathbf{m}_{core}$ and $\mathbf{m}_{crust}$ are contributions from the core and the crust respectively.

2.2.2 Magnetic dipole radiation from a general localized source

Let us consider the general problem of the radiation from a localized system of harmonically oscillating charge and current densities $\rho(\mathbf{r}, t)$ and $J(\mathbf{r}, t)$ and recognize that we may make a Fourier time analysis to obtain a superposition of single frequency components[25]; we take

$$\rho(\mathbf{r}', t) = \rho(\mathbf{r}')e^{-iwt} \tag{2.2.28}$$

$$\mathbf{J}(\mathbf{r}', t) = \mathbf{J}(\mathbf{r}')e^{-iwt} \tag{2.2.29}$$

Where $\mathbf{J}$ and ρ are required to satisfy $\nabla \cdot \mathbf{J} = i\omega\rho$ by the continuity equation . The potential arising from the charge distribution is then found by using a Lorentz gauge

$$\Box^2 A^\mu = \frac{4\pi}{c} j^\mu \tag{2.2.30}$$

Where A^μ is a four -vector which is equal to $A^\mu = (\frac{\phi}{c}, \mathbf{A})$ and $j^\mu = (\rho c, \mathbf{J})$ .Then the solution to eq(2.2.30) is

$$A^\mu(\mathbf{r}, t) = \frac{1}{c} \int d^3 r' \int \frac{dt' j^\mu(\mathbf{r}', t)\delta(t' + \frac{|\mathbf{r}-\mathbf{r}|}{c} - t)}{|\mathbf{r} - \mathbf{r}'|} \tag{2.2.31}$$

Hence the vector potential would be

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c} \int_{V'} \frac{\mathbf{J}(\mathbf{r}') e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} d^3 r' \tag{2.2.32}$$

Where , $k = \frac{\omega}{c}$ and $R = |\mathbf{r} - \mathbf{r}'|$.Then magnetic field is given by $\mathbf{B} = \nabla \times \mathbf{A}$ and since outside the source we have

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\dot{E} = \frac{4\pi}{c} J$$
$$\dot{\mathbf{E}} = c^2 \nabla \times \mathbf{B} \tag{2.2.33}$$

The integral(2.2.32) is generally intractable, and various approximations must be employed. Because the radiation fields are frequently observed from a distance, larger compared to the source dimensions, it is convenient to place the source roughly at the origin and to express the vector potential in the spherical polar coordinates of the observer. We separate the problem into three special region of interest;

a) In the near zone (induction zone), where $k|\mathbf{r} - \mathbf{r}'| \ll 1 (R \ll \lambda)$ and $e^{ik|\mathbf{r}-\mathbf{r}'|} \approx 1$ and using eq(2.2.5) and $r > r'$, eq(2.2.32) can be expressed as

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c} \sum_{l,m} \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \int \mathbf{J}(\mathbf{r}') r'^l Y^*_{lm}(\theta', \phi') d^3 r' \tag{2.2.34}$$

b) In the far (radiation) zone , $r >> \lambda$ and $r >> r'$ Where

$$|\mathbf{r} - \mathbf{r}'| \approx r - \frac{\mathbf{r} \cdot \mathbf{r}'}{r} + - - -$$

Taking $\frac{1}{|\mathbf{r}-\mathbf{r}'|}$ to be constant over the region of integration ,

$$\mathbf{A}(\mathbf{r}) = \frac{e^{ikr}}{cr} \int \mathbf{J}(\mathbf{r}') e^{\frac{-ik\,\mathbf{r}\cdot\mathbf{r}'}{r}} d^3r' \tag{2.2.35}$$

If in addition, the source dimensions r' are small compared to a wavelength λ, then $kr' << 1$ and we can rewrite the vector potential as

$$\mathbf{A}(\mathbf{r}) = -\frac{i\omega e^{ikr}}{cr} \int \mathbf{r}' \rho(\mathbf{r}') d^3r'$$

or

$$\mathbf{A}(\mathbf{r}) = -\frac{i\omega e^{ikr}}{cr} \mathbf{p} \tag{2.2.36}$$

Where $\mathbf{p} = \int \mathbf{r}' \rho(\mathbf{r}') d^3r'$ is the electric dipole moment of the source.

c) In the intermediate zone,(where r,r' and λ are the same order), we must use the exact expression of

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = 4\pi ik \sum_{l,m} h_l^{(1)}(kr_>) j_l(kr_<) Y_{lm}(\theta,\phi) Y_{lm}^*(\theta,'\phi') \tag{2.2.37}$$

Where $h_l^{(1)} = j_l + in_l$ is a spherical Hankel function of the first kind . With this expansion, the general expression for the vector potential becomes

$$\mathbf{A}(\mathbf{r}) = \frac{ik}{c} \sum_{l,m} h_l^{(1)}(kr) Y_{lm}(\theta,\phi) \int \mathbf{J}(\mathbf{r}') j_l(kr') Y_{lm}^*(\theta,'\phi') d^3r' \tag{2.2.38}$$

Where $j_l(kr')$ is Bessel function. If the first term of the vector potential (2.2.38) vanishes, or the source dimensions are not overwhelmingly small, the next(l=1) term will contribute significantly to the field. This term will be seen to give rise to both electric quadruple and magnetic dipole radiation. Using eq(2.2.38), the l=1 term then yields

$$\mathbf{A}(\mathbf{r}) = \frac{3ik}{c} h_1^{(1)}(kr) \int \mathbf{J}(\mathbf{r}') j_1(kr') (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') d^3r' \tag{2.2.39}$$

For small kr', we approximate $\frac{j_l(kr')(kr')^l}{(2l+1)!!}$,(where $(2l+1)!! = (2l+1)(2l-1)(2l-3) - - - 0 or 1.$) which gives for l=1 ,$j_1(kr') \approx \frac{1}{3}kr'$. Thus ,l=1 term of the vector potential may be approximated as

$$\mathbf{A}(\mathbf{r}) = \frac{ik}{c} h_1^{(1)}(kr) \int \mathbf{J}(\mathbf{r}') kr' (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') d^3 r' \tag{2.2.40}$$

Where $h_1^{(1)}(kr) = -\frac{ie^{ikr}}{kr}(1 + \frac{i}{kr})$. Substituting this into eq(2.2.40) we get

$$\mathbf{A}(\mathbf{r}) = -\frac{ike^{ikr}}{cr^2}\left(1 - \frac{i}{ikr}\right) \int \mathbf{J}(\mathbf{r}')(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') d^3 r' \tag{2.2.41}$$

Using the identity

$$(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')\mathbf{J} = \frac{1}{2}\left[(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')\mathbf{J} + (\mathbf{r} \cdot \mathbf{J})r' + (\mathbf{r}' \times \mathbf{J}) \times \mathbf{r}\right]$$

The first monopole term gives zero when integrated and the vector potential for the magnetic dipole term is given by

$$\mathbf{A}(\mathbf{r})_m = \frac{ike^{ikr}}{cr}(1 - \frac{1}{ikr})\left(\hat{\mathbf{r}} \times \mathbf{m}\right) \tag{2.2.42}$$

Where $\hat{r} = \frac{\mathbf{r}}{r}$ Hence magnetic dipole field is

$$\mathbf{B}_m = -\frac{k^2 e^{ikr}}{cr}\hat{\mathbf{r}}(\hat{\mathbf{r}} \times \ \mathbf{m}) - \frac{1}{c}\left(1 - \frac{1}{ikr}\right)\frac{e^{ikr}}{r}\left[3\hat{\mathbf{r}}(\mathbf{m} \cdot \hat{\mathbf{r}}) - \mathbf{m}\right] \tag{2.2.43}$$

The relation $i\omega\mathbf{B} = \nabla \times \mathbf{E} = i\omega(\nabla \times A)$ suggest $\mathbf{E} = i\omega\mathbf{A}$ Hence

$$\mathbf{E}_M = -\frac{k\omega}{cr}(1 - \frac{1}{ikr})\left(\hat{\mathbf{r}} \times \mathbf{m}\right) \tag{2.2.44}$$

Finally let us find field decay associated with the magnetic dipole radiation. The well-known dipole model [8],[9],[10] demonstrates how pulsar emission is derived from the kinetic energy of rotating neutron star. In this oblique rotating version of the model, it is assumed that a neutron star rotates uniformly in vacuum at a frequency of Ω and possesses a magnetic dipole moment $\mathbf{m}$ oriented at an angle α to the rotation axis. We assumed that the rotation is sufficiently slow that the star roughly remains spherical. According to this model the external field of a neutron star is a pure dipole. Hence the magnetic dipole moment is given by

$$\mathbf{m} = m_0\left[\hat{\mathbf{e}_\parallel} \cos\alpha + \hat{\mathbf{e}_\perp} \sin\alpha \cos\Omega t + \hat{\mathbf{e}_\perp}' \sin\alpha \sin\Omega t\right] \tag{2.2.45}$$

Where $\hat{e_\perp}$ and $\hat{e_\perp}'$ are unit vectors perpendicular to spin axis and $\hat{e}_\parallel$ parallel to spin axis. The perpendicular components of m produce radiation, thus the parallel component may be neglected from here on. Hence the magnetic moment will then be given by

$$\mathbf{m} = Re\Big[m_0 \sin\alpha(\hat{e_\perp} + i\hat{e_\perp}')e^{-\Omega t}\Big] \tag{2.2.46}$$

Substituting eq(2.2.46) into (2.2.43) and (2.2.44) the electric and magnetic fields are

$$\mathbf{E} = -k^2\Big[m_0 \sin\alpha\Big(\hat{\mathbf{r}} \times ((\hat{e_\perp} + i\hat{e_\perp}')\Big)\Big]\frac{e^{i(kr-\Omega t)}}{r} \tag{2.2.47}$$

and

$$\mathbf{B} = -k^2 m_0 \sin\alpha\Big[(\hat{e_\perp} + i\hat{e_\perp}') - \hat{\mathbf{r}}\Big(\hat{\mathbf{r}} \cdot (\hat{e_\perp} + i\hat{e_\perp}')\Big)\Big]\frac{e^{i(kr-\Omega t)}}{r} \tag{2.2.48}$$

Where $k = \frac{\Omega}{c}$ is wave vector. Both of these fields are complex with an implied $e^{-i\Omega t}$ time dependent. We have

$$\hat{e_\perp} = \hat{e}_r \sin\theta\cos\phi + \hat{e}_\theta \cos\theta\cos\phi - \hat{e}_\phi \sin\phi$$

$$\hat{e_\perp}' = \hat{e}_r \sin\theta\cos\phi + \hat{e}_\theta \cos\theta\cos\phi + \hat{e}_\phi \cos\phi \tag{2.2.49}$$

Substituting eq (2.2.49) into (2.2.47) and (2.2.48) we find

$$\mathbf{E} = -\frac{k^2}{r}m_0 \sin\alpha\Big[\hat{e}_\theta(\sin\theta - i\cos\phi) + \hat{e}_\phi \cos\theta(\cos\theta + i\sin\theta)\Big]e^{i(kr-\Omega t)} \tag{2.2.50}$$

The real field that is obtained from this is

$$\mathbf{E} = -\frac{k^2}{r}m_0 \sin\alpha\Big[\cos(kr-\Omega t)\Big(\hat{e}_\theta \sin\phi + \hat{e}_\phi \cos\theta\cos\phi\Big) + \sin(kr-\Omega t)\Big(\hat{e}_\theta \cos\phi + \hat{e}_\phi \cos\theta\sin\phi\Big)\Big] \tag{2.2.51}$$

We can find B by relation $\mathbf{B} = \hat{\mathbf{r}} \times \mathbf{E}$ as

$$\mathbf{B} = -\frac{k^2}{r}m_0 \sin\alpha\Big[\cos(kr-\Omega t)\Big(-\hat{e}_\theta \cos\phi\cos\theta + \hat{e}_\phi \sin\phi\Big) + \sin(kr-\Omega t)\Big(\hat{e}_\theta \sin\phi + \hat{e}_\phi \cos\phi\Big)\Big] \tag{2.2.52}$$

When $\phi = 0$, because the emitted radiation is elliptically polarized, we have

$$\mathbf{E} = -\frac{k^2}{r}m_0 \sin\alpha\Big[\hat{e}_\theta \sin(kr - \Omega t) + \hat{e}_\phi \cos(kr - \Omega t)\cos\theta\Big] \tag{2.2.53}$$

This is circularly polarized along the polar axis ($\theta = 0, \pi$, RH and LH polarization respectively) and linearly polarized along the equator ($\theta = \frac{\pi}{2}$).

The energy flow per unit time (or energy flux or poynting vector) is given by

$$\mathrm{S} = \frac{c}{4\pi}(\mathrm{E} \times \mathrm{B}) \tag{2.2.54}$$

Substituting eq(2.2.51) and (2.2.52) into (2.2.54) and taking the radial component only we find

$$S = \frac{m_0^2 \Omega^4 \sin^2\alpha}{4\pi c^3 r^2}\left[1 - \sin^2\theta\cos^2(kr - \Omega t - \phi)\right] \tag{2.2.55}$$

Now along the poles, S is constant(appropriate for circular polarization), while along the equator S varies between 0 and a maximum with frequency 2Ω (which is appropriate for linear polarization). At a general observation angle (θ, ϕ), the observed radiation flux varies with frequency 2Ω. From the above relation follows the expression

$$S_{max} = \frac{\Omega^4 m_0^2 \sin^2\alpha}{4\pi c^3 r^2}$$

$$S_{min} = \frac{\Omega^4 m_0^2 \sin^2\alpha\cos^2\theta}{4\pi c^3 r^2} \tag{2.2.56}$$

The total power loss due to radiation is just the flux of S through a sphere centered on the neutron star.

$$\frac{dE}{dt} = \oint r^2 \langle \mathrm{S} \rangle \cdot \hat{\mathrm{r}} da \tag{2.2.57}$$

Which may be rewritten as

$$\frac{dE}{dt} = \int \frac{\Omega^4 m_0^2 \sin^2\alpha}{4\pi c^3 r^2}\left[1 - \sin^2\theta\cos^2(kr - \Omega t - \phi)\right] d\Omega'$$

Or

$$\frac{dE}{dt} = -\frac{2\Omega^4 m_0^2 \sin^2\alpha}{3c^3} \tag{2.2.58}$$

Since we have $m = \frac{B_p R^3}{2}$ the rate of energy loss may rewritten as

$$\frac{dE}{dt} = -\frac{B_p^2 R^6 \Omega^4 \sin^2\alpha}{6c^3} \tag{2.2.59}$$

2.2.3 Magnetic field decay due to Dipole radiation

The energy carried away by the radiation originates from the rotational kinetic energy of the neutron star which is given by $E = \frac{1}{2}I\Omega^2$. Where I is the moment of inertia which is constant for solo-neutron star.

According to relativistic plasma diffusion model magnetic field can be written as

$$B = \frac{2Q\Omega(t)}{3cR} \tag{2.2.60}$$

Where Q is magnitude of separated charges and R-radius of neutron star [2]. Using eq(2.2.59) and (2.2.60) we find that the magnetic field decay due to magnetic dipole to be

$$B(t) = B(0)\left[1 + \frac{8Q^2R^4\Omega^4(0)\sin^2\alpha}{27Ic^5}t\right]^{-\frac{1}{4}} \tag{2.2.61}$$

Where $B(t=0) = \frac{2Q\Omega(0)}{3cR}$ and $\Omega(0)$ are magnetic field and angular frequency at t=0 respectively[44].

2.3 Magnetic field decay law due to Gravitational radiation

2.3.1 Gravitational radiation

Einstein predicted the existence of gravitational waves by general relativity(GR) in 1918. Gravitational effects can not propagate with infinite speed. Since the speed light is the only Lorentz-invariant speed, we expect that the gravitational effects propagate in the form of wave at the speed of light. The disturbance in the gravitational field propagates outward in the speed of light. Such a propagating disturbance is a gravitational wave. The source of gravitation must therefore be the energy density. However, it is impossible to construct a Lorentz-invariant theory of gravitation in which the energy density is the only source of gravity. At each event in space-time, there exist energy momentum-tensor, $T^{\mu\nu}$. It is a machine that contain a knowledge of energy density, energy flux density and momentum flux density. But what is the energy density in one reference frame will be some combination of energy density, energy flux density and momentum flux density as seen from another reference frame. If the laws are to have the same form in all Lorentz- frames, then these quantities must be source of gravitation, that is the tensor, $\mathrm{T}^{\mu\nu}$ must be the source of gravitation.

It will be best to explain this for a system consisting of a collection of non-interacting particle(a cloud of dust) whose energy- momentum tensor is given by

$$\mathrm{T}^{\mu\nu} = \rho\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = \rho U^{\mu}U^{\nu} \tag{2.3a}$$

Where ρ is the mass density as measured in the local rest frame of the particles and $U^{\mu} = \frac{dx^{\mu}}{d\tau}$ is four velocity. The conservation law everywhere in space -time is satisfied by the energy-momentum tensor of a closed system as

$$T^{\mu\nu}_{,\nu} = 0$$

We may just as well write as

$$T^{\mu\nu}_{;\nu} = 0 \tag{2.3b}$$

Where the semicolon denotes the covariant derivative of the tensor field. It should also be realized that the curvature of space-time and the strength of the gravitational field are two different ways of describing the same thing .

In the next few sections we are going to find the rate of loss of rotational kinetic energy of neutron star due to gravitational radiation.

2.3.2 The curvature of space-time

According to General theory of relativity, in the vicinity of massive bodies space-time is curved. The curvature is described by a fourth rank tensor as

$$R^{\lambda}_{\mu\nu\kappa} = \Gamma^{\lambda}_{\mu\nu,\kappa} - \Gamma^{\lambda}_{\mu\kappa,\upsilon} + \Gamma^{\lambda}_{\kappa\eta}\Gamma^{\eta}_{\mu\nu} - \Gamma^{\lambda}_{\nu\eta}\Gamma^{\eta}_{\mu\kappa} \tag{2.3.1}$$

which is called the Riemann curvature tensor . $\Gamma^{\lambda}_{\mu\nu}$ are known as Christoffel symbols,which are defined by the expression

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} \Big(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha} \Big) \tag{2.3.2}$$

Where $g_{\mu\nu}$ is a fundamental metric tensor and $g_{\mu\nu,\alpha}$ is its derivative with respect to x^{α} . Flatness is equivalent to zero Riemann curvature, as in special relativity where there is no gravity. For the Riemann curvature tensor, we can construct a tensor of rank 2 by the processes of contraction of indices as

$$R_{\mu\kappa} = R^{\lambda}_{\mu\lambda\kappa} = \Gamma^{\lambda}_{\mu\lambda,\kappa} - \Gamma^{\lambda}_{\mu\kappa,\lambda} + \Gamma^{\lambda}_{\kappa\eta}\Gamma^{\eta}_{\mu\lambda} - \Gamma^{\lambda}_{\lambda\eta}\Gamma^{\eta}_{\mu\kappa} \tag{2.3.3}$$

This symmetric tensor is called Ricci tensor. Further contraction of $R_{\mu\kappa}$ by $g^{\mu\nu}$ yields the scalar curvature of space-time

$$R = g^{\mu\nu} R_{\mu\nu} = R^{\lambda}_{\lambda}$$

The vanishing of the scalar curvature or even Ricci tensor is by no means sufficient for space-time to be flat.In fact, outside matter (in vacuum) the gravitational field is determined just by the equation $R_{\mu\nu} = 0$. In the next section we will see how matter determines the curvature of space-time. The Einstein field equations determine a relation between the curvature of space-time and the distribution and motion of matter and fields(excluding gravitational fields). They are written in the form of

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu} \tag{2.3.4}$$

If we contract eq(2.3.4) by $g^{\mu\nu}$ we obtain

$$R - \frac{1}{2} \delta^{\mu}_{\mu} R = -8\pi G T^{\mu}_{\mu} \tag{2.3.5}$$

by Einstein's summation convention, $\delta^{\mu}_{\mu} = 4$. Hence

$$R = 8\pi G T^{\mu}_{\mu} \tag{2.3.6}$$

By substituting eq(2.3.6) into eq(2.3.4), we can get another form of Einstein's field equations;

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(8\pi G T^{\mu}_{\mu}) = -8\pi G T_{\mu\nu} \tag{2.3.7}$$

Eq(2.3.7) clearly shows the effects of matter and its motion on the curvature of space-time. A large curvature of space-time is equivalent to strong gravitational field, and vice versa.

2.3.3 The linear equation for gravitation

Although Newton's theory is not perfect, it is in excellent agreement with observation in the limiting case of motion at low velocity in weak gravitational field. A weak gravitational field is one in which space-time is nearly flat. With small perturbation of metric about flat space-time as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ ; \ \ g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{2.3.8}$$

Where $|h_{\mu\nu}| \ll 1$ and $\eta_{\mu\nu}$ is a metric of flat space-time called the Minkowski metric tensor. The quantities $h_{\mu\nu}$ and their derivatives regarded as terms of the first order whose squares may be neglected. Linearized theory is often a weak -field approximation to general relativity. It is convenient to introduce the quantities

$$\eta^{\lambda\alpha}h_{\mu\alpha} = h^{\lambda}_{\mu} \ ; \ \ \eta^{\lambda\alpha}(h_{\mu\nu,\alpha}) = h_{\mu\nu,\lambda} \tag{2.3.9}$$

Using eq(2.3.2) we can rewrite the expression for the christoffel symbol in a weak field approximation as

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2}(\eta^{\mu\alpha} - h^{\mu\alpha})\Big[(\eta_{\alpha\lambda} + h_{\alpha\lambda})_{,\nu} + (\eta_{\alpha\nu} + h_{\alpha\nu})_{,\lambda} - (\eta_{\nu\lambda} + h_{\nu\lambda})_{,\alpha}\Big]$$

Since the values of the Minkowski metric tensor are constants, their derivatives with respect to the coordinates are zero. We can also neglect higher order terms in the derivatives of h. Hence the above expression for $\Gamma^{\mu}_{\nu\lambda}$ is reduced to

$$\Gamma^{\mu}_{\nu\lambda} \cong \frac{1}{2}\eta^{\mu\alpha}\left(h_{\alpha\lambda,\nu} + h_{\alpha\nu,\lambda} - h_{\nu\lambda,\alpha}\right) \tag{2.3.10}$$

Furthermore,eq(2.3.10) indicates that the Ricci tensor given by eq(2.3.3) can be rewritten, up to quadratic terms as

$$R_{\mu\nu} \cong \Gamma^{\lambda}_{\mu\lambda,\nu} - \Gamma^{\lambda}_{\mu\nu,\lambda}$$

Applying the relation obtained in eq(2.3.10) and the commutative property of derivative, the above equation can be reduced to

$$R_{\mu\nu} \cong \frac{1}{2}\left[\eta^{\alpha\lambda}h_{\mu\nu,\alpha\lambda} + h^{\lambda}_{\lambda,\mu\nu} - h^{\lambda}_{\nu,\mu\lambda} - h^{\alpha}_{\mu,\alpha\nu}\right]$$

or

$$R_{\mu\nu} \cong \frac{1}{2}\left[\Box^2 h_{\mu\nu,\alpha\lambda} + h^{\lambda}_{\lambda,\mu\nu} - h^{\lambda}_{\nu,\mu\lambda} - h^{\alpha}_{\mu,\alpha\nu}\right] \tag{2.3.11}$$

Where $\Box$ is the D'Alembertian operator defined by

$$\Box^2 h_{\mu\nu} = \eta^{\alpha\lambda}h_{\mu\nu,\alpha\lambda} \tag{2.3.12}$$

let us show the possibility of satisfying eq(2.3.11), by the following two equations

$$R_{\mu\nu} = \frac{1}{2}\Box^2 h_{\mu\nu} \tag{2.3.13a}$$

and

$$h^{\lambda}_{\lambda,\mu\nu} - h^{\lambda}_{\nu,\mu\lambda} - h^{\lambda}_{\mu,\lambda\nu} = 0 \tag{2.3.13b}$$

We will show the possibility of this by adopting some particular convenient choice of a coordinate system which is represented by the harmonic coordinate condition:

$$\Gamma^{\lambda} = g^{\nu\lambda}\Gamma^{\mu}_{\nu\lambda} = 0 \tag{2.3.14}$$

To see that it is always possible to choose a coordinate system in which this holds, we recall the transformation equation of the Christoffel symbol;

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial x^{\alpha}}{\partial x'^{\nu}}\Gamma^{\lambda}_{\tau\alpha} - \frac{\partial x^{\rho}}{\partial x'^{\nu}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial^2 x'^{\lambda}}{\partial x^{\rho}\partial x^{\alpha}}$$

The transformation equation of the metric tensor is also given by $g'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\tau}}\frac{\partial x'^{\nu}}{\partial x^{\alpha}}g^{\tau\alpha}$. Multiplying the left and right hand sides of the transformation equation of the Christoffel symbol by $g'^{\mu\nu}$ and $\frac{\partial x'^{\mu}}{\partial x^{\tau}}\frac{\partial x'^{\nu}}{\partial x^{\alpha}}g^{\tau\alpha}$ respectively, gives

$$\Gamma'^{\lambda} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\Gamma^{\rho} - g^{\rho\alpha}\frac{\partial^2 x'^{\lambda}}{\partial x^{\rho}\partial x^{\alpha}}$$

Hence if Γ^{ρ} does not vanish, we can always define a new coordinate system x'^{λ} by solving the second order differential equation;

$$g^{\rho\alpha}\frac{\partial^2 x'^{\lambda}}{\partial x^{\rho}\partial x^{\alpha}} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\Gamma^{\rho}$$

So that $\Gamma'^{\lambda} = 0$ in the x'^{λ} coordinate system. Now let us turn to the harmonic coordinate condition. If we substitute eq(2.3.8) and eq(2.3.10) into eq(2.3.14) and higher order terms results in

$$\frac{1}{2}\eta^{\mu\nu}\eta^{\nu\lambda}\left[h_{\alpha\nu,\lambda} + h_{\alpha\lambda,\nu} - h_{\nu\lambda,\alpha}\right] = 0$$

or

$$\frac{1}{2}\eta^{\mu\nu}\left[h^{\lambda}_{\alpha,\lambda} + h^{\lambda}_{\alpha,\lambda} - h^{\lambda}_{\lambda,\alpha}\right] = 0$$

If $\mu = \nu$,then $\eta^{\mu\nu} = 1$. In this case for the above equation to be satisfied the following harmonic coordinate condition should be fulfilled;

$$h^{\lambda}_{\alpha,\lambda} + h^{\lambda}_{\alpha,\lambda} - h^{\lambda}_{\lambda,\alpha} = 0\,, or\,, h^{\lambda}_{\alpha,\lambda} = \frac{1}{2}h^{\lambda}_{\lambda,\alpha} \qquad (2.3.15)$$

From this we can conclude that

$$h^{\lambda}_{\alpha,\lambda\nu} = \frac{1}{2}h^{\lambda}_{\lambda,\alpha\nu} \qquad (2.3.16a)$$

and

$$h^{\lambda}_{\nu,\lambda\alpha} = \frac{1}{2}h^{\lambda}_{\lambda,\alpha\nu} \qquad (2.3.16b)$$

By adding eq(2.3.16a) and eq(2.3.16b) and applying the commutative property of the derivative, we get $h^{\lambda}_{\lambda,\alpha\nu} - h^{\lambda}_{\alpha,\lambda\nu} - h^{\lambda}_{\nu,\alpha\alpha} = 0$. This also satisfies eq(2.3.13b) . Now by substituting eq(2.3.13b) into eq(2.3.11), we get

$$R_{\mu\nu} = \frac{1}{2}\Box^2 h_{\mu\nu}$$

This also satisfies eq(2.3.13a). So, the harmonic coordinate condition satisfies both conditions given by eq(2.3.13). By substituting eq(2.3.13a)and eq(2.3.13b) into eq(2.3.4), we get

$$\frac{1}{2}\Box^2 h_{\mu\nu} - \frac{1}{2}(\eta_{\mu\nu} + h_{\mu\nu})R = -8\pi G T_{\mu\nu}$$

or

$$\Box^2 h_{\mu\nu} - (\eta_{\mu\nu} + h_{\mu\nu})R = -16\pi G T_{\mu\nu}$$

Compared to $\eta_{\mu\nu}R$, $h_{\mu\nu}R$ is small and therefore may be neglected. This implies

$$\Box^2 h_{\mu\nu} - \eta_{\mu\nu}R = -16\pi G T_{\mu\nu}$$

Finally if we substitute eq(2.3.6) into this equation, we get

$$\Box^2 h_{\mu\nu} - (\eta_{\mu\nu}(8\pi G T^{\lambda}_{\lambda}) = -16\pi G T_{\mu\nu}$$

or

$$\Box^2 h_{\mu\nu} = -16\pi G S_{\mu\nu} \tag{2.3.17}$$

Where $S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}T^{\lambda}_{\lambda}$ and the possible solution to the wave equation Eq(2.3.17) is

$$h_{\mu\nu}(\mathbf{x}, t) = 4G \int \frac{S_{\mu\nu}(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|)}{|\mathbf{x} - \mathbf{x}'|} d^3x'$$

Where $\mathbf{x}'$ is the source point coordinate and $\mathbf{x}$ are the coordinates of the field points. Eq(2.3.17) are called the field equations of linearized theory, since they result from keeping terms linear in $h_{\mu\nu}$. Although it is true that the most spectacular results of gravitational theory depend in a crucial way on the non-linearity on the field equations, almost all of the results that have been the subject of experimental investigations can be described by the linear approximation.

2.3.4 Plane waves

To the retarded solution given in the preceding section we can add any solution of the homogenous equation

$$\Box^2 h_{\mu\nu} = 0 \qquad (2.3.18)$$

All solutions that satisfy these homogenous equations represent the gravitational radiation coming from infinity ($r \longmapsto \infty$) and the equations are valid under the harmonic conditions given by eq(2.3.15). The general solution of the homogenous equation is a linear superposition of the solution of the form

$$h_{\mu\nu}(x) = e_{\mu\nu} exp(ik_\lambda x^\lambda) + e^*_{\mu\nu} exp(-ik_\lambda x^\lambda) \qquad (2.3.19)$$

Where the matrix $e_{\mu\nu}$ and k_λ are the polarization tensor and the wave vector respectively. Substituting eq(2.3.19) into eq(2.3.18) we obtain

$$k^\alpha k_\alpha h_{\mu\nu} = 0$$

Since $h_{\mu\nu} \neq 0$ the field equations(2.3.18) are satisfied if

$$k^\mu k_\mu = 0 \qquad (2.3.20)$$

We can also show that

$$h^\mu_{\nu,\mu} = ik_\mu e^\mu_\nu exp(ik_\lambda x^\lambda) - ik_\mu e^{\mu^*}_\nu exp(-ik_\lambda x^\lambda)$$

and

$$h^\mu_{\mu,\nu} = ik_\nu \left[e^\mu_\mu exp(ik_\lambda x^\lambda) - e^{\mu^*}_\mu exp(-ik_\lambda x^\lambda) \right]$$

So, the harmonic conditions in eq(2.3.15) are satisfied if

$$k_\mu e^\mu_\nu = \frac{1}{2} k_\nu e^\mu_\mu \qquad (2.3.21)$$

Generally a symmetric 4×4 matrix such as $e_{\mu\nu}$ would have ten independent components and the four relations given by eq(2.3.21) would lower the number to six. We shall now prove that out of these only two of them are physically significant. Suppose we change the coordinate using the coordinate transformation

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x) \tag{2.3.22}$$

Where ε^{μ} are functions of the coordinate. The corresponding transformation equation of the metric tensor gives

$$g'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}} g^{\alpha\beta} = \left[\delta^{\mu}_{\alpha} + \frac{\partial \varepsilon^{\mu}}{\partial x^{\alpha}}\right]\left[\delta^{\nu}_{\beta} + \frac{\partial \varepsilon^{\nu}}{\partial x^{\beta}}\right] g^{\alpha\beta}$$

As a result

$$h'^{\mu\nu} = g'^{\mu\nu} - \eta^{\mu\nu} = h^{\mu\nu} - \frac{\partial \varepsilon^{\mu}(x)}{\partial x^{\nu}} - \frac{\partial \varepsilon^{\nu}(x)}{\partial x^{\mu}} \tag{2.3.23}$$

Suppose we choose

$$\varepsilon^{\mu}(x) = i\varepsilon^{\mu} exp(ik_{\lambda}x^{\lambda}) - i\varepsilon^{\mu^*} exp(-ik_{\lambda}x^{\lambda}) \tag{2.3.24}$$

This implies

$$\frac{\partial \varepsilon^{\nu}(x)}{\partial x^{\mu}} = -\varepsilon^{\nu} k^{\mu} exp(ik_{\lambda}x^{\lambda}) - \varepsilon^{\nu^*} k^{\mu} exp(-ik_{\lambda}x^{\lambda}) \tag{2.3.25}$$

and

$$\frac{\partial \varepsilon^{\mu}(x)}{\partial x^{\nu}} = -\varepsilon^{\mu} k^{\nu} exp(ik_{\lambda}x^{\lambda}) - \varepsilon^{\mu^*} k^{\nu} exp(-ik_{\lambda}x^{\lambda}) \quad (2.3.26$$

By substituting eq(2.3.19),(2.3.25) and (2.3.26) into eq(2.3.23),we get

$$h'^{\mu\nu}(x) = e'^{\mu\nu} exp(ik_{\lambda}x^{\lambda}) + e'^{\mu\nu^*} exp(-ik_{\lambda}x^{\lambda}) \tag{2.3.27}$$

Where

$$e'^{\mu\nu} = e^{\mu\nu} + k^{\mu}\varepsilon^{\nu} + k^{\nu}\varepsilon^{\mu} \tag{2.3.28}$$

Harmonic coordinate condition is still satisfied if

$$k_{\mu} e'^{\mu}_{\nu} = \frac{1}{2} k_{\nu} e'^{\mu}_{\mu} \tag{2.3.29}$$

We then conclude that $e'^{\mu\nu}$ and $e^{\mu\nu}$ represent the same physical situation for arbitrary value of the four parameters, ε_{μ} . In addition, due to the gauge condition given by eq(2.3.28) out of the six independent $e^{\mu\nu'}s$ satisfying the harmonic coordinate condition only 6-4=2 are physically significant. As an example, consider a wave which propagates in the positive z-direction. So in this case $k^1 = k^2 = 0$. So that

$$k^0 = k^3 = k = k_0 = -k_3 \tag{2.3.30}$$

Finally using harmonic coordinate condition we get

$$k_0 e^0_\nu + k_1 e^1_\nu + k_2 e^2_\nu + k_3 e^3_\nu = \frac{1}{2} k_\nu (e^0_0 + e^1_1 + e^2_2 + e^3_3)$$

Using eq(2.3.30), this may be reduced to

$$k(e^3_\nu - e^0_\nu) = \frac{1}{2} k_\nu (e^0_0 + e^1_1 + e^2_2 + e^3_3) \tag{2.3.31}$$

But

$$e^\mu_\nu = \eta^{\mu\beta} e_{\nu\beta} \tag{2.3.32}$$

Substituting eq(2.3.32) into eq(2.3.31) we get

$$k(e_{\nu 3} - e_{\nu 0}) = \frac{1}{2} k_\nu (e_{11} + e_{22} + e_{33} - e_{00}) \tag{2.3.33}$$

For $\nu = 0, 3$ eq(2.3.33) is reduced to

$$(e_{03} + e_{00}) = -\frac{1}{2}(e_{11} + e_{22} + e_{33} - e_{00}) \tag{2.3.34}$$

$$(e_{33} + e_{30}) = -\frac{1}{2}(e_{11} + e_{22} + e_{33} - e_{00}) \tag{2.3.35}$$

Combining eq(2.3.34) and (2.3.35),with $e^{\mu\nu}$ is symmetric

$$e_{03} = -\frac{1}{2}(e_{33} + e_{00}) \tag{2.3.36}$$

By subtracting eq(2.3.34) from eq(2.3.35),we get

$$e_{11} = -e_{22} \tag{2.3.37}$$

$$e'_{12} = -\sin\theta\cos\theta e_{11} + \cos^2\theta e_{12} - \sin\theta\cos\theta e_{11} - \sin^2\theta e_{21} \qquad (2.3.41)$$

From which follows

$$e'_{\pm} = e'_{11} \mp ie'_{12} = \cos^2\theta e_{11} + \sin\theta\cos\theta e_{12} - \sin^2\theta e_{11}$$

$$+\sin\theta\cos\theta e_{21} \pm i\sin\theta\cos\theta e_{11} \mp i\cos^2\theta e_{12}$$

$$\pm i\sin\theta\cos\theta e_{11} \pm i\sin^2\theta e_{21} \qquad (2.3.42)$$

Using relations $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta = \frac{e^{i\theta}+e^{-i\theta}}{2i}$ and the symmetric property of $e_{\mu\nu}$,eq(2.3.42) gives

$$e'_{\pm} = exp(\pm i2\theta)e_{\pm}$$

Where $e_{\pm} = e_{11} \mp ie_{12}$. Any plane wave ψ, which is transformed by a rotation of an angle θ about the direction of propagation into $\psi' = e^{ik\theta}\psi$ is said to have helicity h. Thus the physically significant components of the plane wave $e_{\pm}$ are those with helicity ± 2. So gravitation is carried by a wave of spin 2.

2.3.5 Generation of gravitational wave

Since gravitational waves can exert forces and do work, they carry energy and momentum. In a quasi- Minkowskian coordinate system in the sense that the metric $g_{\mu\nu}$ be approximated by $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and the part of Ricci tensor that is linear in $h_{\mu\nu}$ is given by

$$R^{(1)}_{\mu\kappa} \simeq \frac{1}{2}\left[\frac{\partial^2 h^\lambda_\lambda}{\partial x^\mu \partial x^\kappa} - \frac{\partial^2 h^\lambda_\mu}{\partial x^\lambda \partial x^\kappa} - \frac{\partial^2 h^\lambda_\kappa}{\partial x^\lambda \partial x^\mu} + \frac{\partial^2 h_{\mu\kappa}}{\partial x^\lambda \partial x^\lambda}\right] \qquad (2.3.43)$$

So the exact Einstein equation in this approximation can be rewritten as

$$R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)\lambda}_{\lambda} = -8\pi G(T_{\mu\nu} + t_{\mu\nu}) \qquad (2.3.44)$$

Where

$$t_{\mu\nu} = \left(\frac{1}{8\pi G}\right)\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^\lambda_\lambda - R^{(1)}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}R^{(1)\lambda}_\lambda\right] \qquad (2.3.45)$$

Where tensor $t_{\mu\nu}$ is simply the energy momentum tensor of gravitational field itself. Like any other tensor it is divergence free in vacuum

$$t^{\mu\nu}_{,\nu} = 0 \tag{2.3.46}$$

We can compute $t_{\mu\nu}$ as a power series in h and find that the highest term is quadratic ;

$$t_{\mu\nu} \cong \frac{1}{8\pi G}\left[-\frac{1}{2}h_{\mu\nu}R^{(1)\lambda}_{\lambda} + \frac{1}{2}\eta_{\mu\nu}h^{\rho\alpha}R^{(1)}_{\rho\alpha} + R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{\rho\alpha}R^{(21)}_{\rho\alpha}\right] \tag{2.3.47}$$

Where

$$\begin{aligned} R^{(2)}_{\mu\nu} = -\frac{1}{2}h^{\nu\lambda}\Big(h_{\mu\lambda,\mu\kappa} - h_{\kappa\lambda,\mu\nu} - h_{\mu\nu,\kappa\lambda} + h_{\mu\kappa,\lambda\nu}\Big) \\ +\frac{1}{4}\Big(2h^{\nu}_{\alpha,\nu} - h^{\nu}_{\nu,\alpha}\Big)\Big(h^{\alpha}_{\mu,\kappa} - h_{\mu\kappa,\alpha}\Big) \\ -\frac{1}{4}\Big(h_{\alpha\lambda,\kappa} + h_{\alpha\kappa,\lambda} - h_{\kappa\lambda,\alpha}\Big)\Big(h^{\alpha}_{\mu,\lambda} + h^{\alpha\alpha}_{,\mu} - h^{\alpha}_{\mu,\alpha}\Big) \end{aligned} \tag{2.3.48}$$

Einstein made the assumption that in the empty space $R^{(1)}_{\mu\nu} = 0$. And then we can drop those terms iu $t_{\mu\nu}$ to get

$$t_{\mu\nu} \cong \frac{1}{8\pi G}\left[R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\lambda\rho}R^{(2)}_{\lambda\rho}\right] \tag{2.3.49}$$

We now have to calculate $\langle R^{(2)}_{\mu\nu}\rangle$. Finally we have to insert the plane wave solution eq(2.3.19) iuto $R^{(2)}_{\mu\nu}$ to find

$$\langle R^{(2)}_{\mu\nu}\rangle = Re\left(\begin{array}{c} e^{\lambda\rho*}(k_\mu k_\nu e_{\lambda\rho} - k_\mu k_\lambda e_{\nu\rho} - k_\lambda k_\rho e_{\mu\nu}) \\ +(k_\lambda e^{\lambda}_{\rho} - \frac{1}{2}k_\rho e^{\lambda}_{\lambda})^*(k_\mu e^{\rho}_{\nu} + k_\nu e^{\rho}_{\mu} - k^\rho e_{\mu\nu}) \\ -\frac{1}{2}(k_\lambda e_{\nu\rho} + k_\nu e_{\rho\lambda} - k_\rho e_{\lambda\nu})^*(k^\lambda e^{\rho}_{\mu} + k_\mu e^{\rho\lambda} - k^\rho e^{\lambda}_{\mu}) \end{array}\right) \tag{2.3.50}$$

Using the harmonic condition the above relation will be reduced to

$$\langle R^{(2)}_{\mu\nu}\rangle = \frac{k_\mu k_\nu}{2}\left[e^{\lambda\rho^*} - \frac{1}{2}|e^{\lambda}_{\lambda}|^2\right] \tag{2.3.51}$$

But $\eta^{\mu\nu}\langle R^{(2)}_{\mu\nu}\rangle = k^\nu k_\nu[e^{\lambda\rho^*} - \frac{1}{2}|e^{\lambda}_{\lambda}|^2]$, Siuce $k^\nu k_\nu = 0$

Hence ,the average energy -momentum tensor of plane wave will be

$$t_{\mu\nu} \cong \frac{1}{8\pi G}R^{(2)}_{\mu\nu} \tag{2.3.52}$$

or

$$\langle t_{\mu\nu}\rangle = \frac{k_\mu k_\nu}{16\pi G}\left[e^{\lambda\rho^*} - \frac{1}{2}|e^\lambda_\lambda|^2\right] \tag{2.3.53}$$

In particular for the wave travelling along z-axis as described in the preceding section the average energy -momentum tensor takes the form

$$\langle t_{\mu\nu}\rangle = \frac{k_\mu k_\nu}{8\pi G}\left[|e_{11}|^2 + |e_{12}|^2\right] \tag{2.3.54}$$

or in terms of helicity amplitudes as

$$\langle t_{\mu\nu}\rangle = \frac{k_\mu k_\nu}{8\pi G}\left[|e_+|^2 + |e_-|^2\right]$$

We wish to calculate the energy emitted in the form of gravitational radiation by a system whose energy-momentum tensor can be expressed as a Fourier integral

$$T_{\mu\nu}(\mathbf{x}, t) = \int_0^\infty d\omega T_{\mu\nu}(\mathbf{x}, \omega)e^{-i\omega t} + c.c \tag{2.3.55}$$

or as a sum of Fourier components

$$T_{\mu\nu}(\mathbf{x}, t) = \sum_\omega e^{-i\omega t}T_{\mu\nu}(\mathbf{x}, \omega) + c.c$$

Considering a single Fourier components the field emitted by the source is

$$h_{\mu\nu}(\mathbf{x}, t) = 4G\int \frac{d^3x' S_{\mu\nu}(\mathbf{x}', \omega)}{|\mathbf{x}\text{-}\mathbf{x}'|}e^{(-i\omega t + i\omega|\mathbf{x}-\mathbf{x}'|)} + c.c \tag{2.3.56}$$

Where

$$S_{\mu\nu}(\mathbf{x}, \omega) = T_{\mu\nu}(\mathbf{x}, \omega) - \frac{1}{2}\eta_{\mu\nu}T^\lambda_\lambda(\mathbf{x}, \omega) \tag{2.3.57}$$

Suppose that we observe this radiation in the far radiation zone ,that is at distance $r = |\mathbf{x}|$, much larger than the dimension of the source ($r \gg \lambda$ and $\mathbf{x} \gg \mathbf{x}'$), and much larger than ωR^2 and $\frac{1}{\omega}$. So in the denominator $|\mathbf{x}\text{-}\mathbf{x}'| \sim r$,while in the exponent we may approximate $|\mathbf{x}\text{-}\mathbf{x}'| \simeq r - \mathbf{x}' \cdot \hat{\mathbf{x}}$. So in the radiation zone the field becomes

$$h_{\mu\nu}(\mathbf{x}, t) = \frac{4G}{r}e^{(i\omega r - i\omega t)}\int S_{\mu\nu}(\mathbf{x}', \omega)e^{-i\omega\hat{\mathbf{x}}\cdot\mathbf{x}}d^3x' \tag{2.3.58}$$

Since $r\omega$ is large $h_{\mu\nu}(\mathbf{x}, t)$ looks like a plane wave i.e

$$h_{\mu\nu}(\mathbf{x}, t) = e_{\mu\nu}(\mathbf{x}, \omega)e^{-i\omega(t-r)} \tag{2.3.59}$$

Where

$$e_{\mu\nu}(\mathbf{x}, \omega) = \frac{4G}{r}\int S_{\mu\nu}(\mathbf{x'}, \omega)e^{-i\omega\hat{\mathbf{x}}\cdot\mathbf{x}}d^3x' \tag{2.3.60}$$

The frequency of the wave is

$$\omega = k^0 = k_0 \; ; \; \omega t = k_o x^0$$

$$t = x^0 \; ; \; \omega = k^0 = -\mathbf{k}\cdot\hat{\mathbf{x}} \tag{2.3.61}$$

It will be convenient to write $e_{\mu\nu}$ explicitly in terms of the Fourier transform of $T_{\mu\nu}$

$$e_{\mu\nu}(\mathbf{x}, \omega) = \frac{4G}{r}\left[T_{\mu\nu}(\mathbf{k}, \omega) - \frac{1}{2}\eta_{\mu\nu}T^{\lambda}_{\lambda}(\mathbf{k}, \omega)\right] \tag{2.3.62}$$

Where

$$T_{\mu\nu}(\mathbf{k}, \omega) = \int T_{\mu\nu}(\mathbf{x'}, \omega)e^{-i\mathbf{k}\cdot\,\mathbf{x'}}d^3x'$$

Applying the conservative equation

$$\frac{\partial T^{\mu}_{\nu}(\mathbf{x}, t)}{\partial x^{\mu}} = 0 \tag{2.3.63}$$

to the single Fourier component,we are considering $T^{\mu}_{\nu}(\mathbf{x}, t) = T^{\mu}_{\nu}(\mathbf{x}, \omega)e^{-i\omega t} + c.c$ We find

$$\frac{\partial T^{i}_{\nu}(\mathbf{x}, \omega)}{\partial x^{i}} - i\omega T^{0}_{\nu}(\mathbf{x}, \omega) = 0 \tag{2.3.64}$$

Multiplying this with $e^{-i\mathbf{k}\cdot\mathbf{x}}$ and integrating over $\mathbf{x}$ we find that $T_{\mu\nu}(\mathbf{k}, \omega)$ is subjected to the algebraic relations;

$$k_{\mu}T^{\mu}_{\nu}(\mathbf{k}, \omega) = 0 \tag{2.3.65}$$

Now let us calculate the power radiated per unit solid angle emitted in a direction $\hat{\mathbf{x}}$. Since $r \gg \frac{1}{\omega} = \lambda$, we can use for the energy flux vector of the value $\langle t^{io}\rangle$ obtained by averaging over space-time dimensions larger compared with $\frac{1}{\omega}$ so that the power radiated per unit solid angle is

$$\frac{dP}{d\Omega} = r^2 \langle t^{io} \rangle \cdot \hat{x}^i \tag{2.3.66}$$

But using eq(2.3.52), eq(2.3.65) may be expanded as

$$\frac{dP}{d\Omega} = \frac{r^2}{16\pi G}(\mathbf{k} \cdot \hat{\mathbf{x}})k^0 \left[e^{\lambda\nu *}(\mathbf{x}, \omega) e_{\lambda\nu}(\mathbf{x}, \omega) - \frac{1}{2} |e^{\lambda}_{\lambda}(\mathbf{x}, \omega)|^2 \right] \tag{2.3.67}$$

Using eq(2.3.65), eq(2.3.67) can be rewritten as

$$\frac{dP}{d\Omega} = \frac{G\omega^2}{\pi} \left[T^{\lambda\nu *}(\mathbf{k}, \omega) T_{\lambda\nu}(\mathbf{k}, \omega) - \frac{1}{2} |T^{\lambda}_{\lambda}(\mathbf{k}, \omega)|^2 \right] \tag{2.3.68}$$

$$\frac{dP}{d\Omega} = \frac{G\omega^2}{\pi} \Lambda_{ij,lm}(\mathbf{k}) T^{ij *}(\mathbf{k}, \omega) T^{lm}(\mathbf{k}, \omega) \tag{2.3.69}$$

Where

$$\Lambda_{ij,lm}(\mathbf{k}) = \delta_{il}\delta_{jm} - 2\hat{k}_i \hat{k}_m \delta_{il} + \frac{1}{2}\hat{k}_i \hat{k}_l \hat{k}_j \hat{k}_m - \frac{1}{2}\delta_{ij}\delta_{lm} + \frac{1}{2}\delta_{ij}\hat{k}_i \hat{k}_m + \frac{1}{2}\delta_{lm}\hat{k}_i \hat{k}_j \tag{2.3.70}$$

is a projection tensor for gravitational wave. We now derive $\frac{dE}{d\Omega}$ from

$$\frac{dE}{d\Omega} = \int \frac{dP}{d\Omega} dt = 2G \int_0^\infty \omega^2 \left[T^{\lambda\nu *}(\mathbf{k}, \omega) T_{\lambda\nu}(\mathbf{k}, \omega) - \frac{1}{2} |T^{\lambda}_{\lambda}(\mathbf{k}, \omega)|^2 \right] d\omega$$

or in terms of space-space component;

$$\frac{dE}{d\Omega} = 2G\Lambda_{ij,lm} \int_0^\infty \omega^2 T^{ij *}(\mathbf{k}, \omega) T^{lm}(\mathbf{k}, \omega) d\omega \tag{2.3.71}$$

2.3.6 Radiation by Rotating Neutron Star

Up to this point we made an approximation beyond the basic assumption that the fields are weak. We now make a further approximation and assume that the source radius R is much smaller than the wavelength ($\frac{1}{\omega}$) or $\omega R \ll 1$. Since most frequencies emitted are the order ($\frac{\bar{v}}{R}$) ($\bar{v}$ is some typical velocity of the system). Our assumption

The Fourier transform of $D_{ij}(t)$ is

$$D_{ij}(t) = \sum_{\omega} e^{-i\omega t} D_{ij}(\omega) \tag{2.3.84}$$

Since the quadrupole moment repeats when particles in the star move through one -half of their orbit, frequency of the emitted wave is twice the orbital frequency, ie $\omega = 2\Omega$. For single Fourier component eq(2.3.84) becomes $D_{ij}(2\Omega) = e^{2i\Omega t} D_{ij}(t)$. Hence the time average values of D_{ij} will be

$$\langle D_{11}(2\Omega)\rangle = -\langle D_{22}(2\Omega)\rangle = i\langle D_{12}(2\Omega)\rangle = \frac{1}{4}(I_{11} - I_{22}) \tag{2.3.85}$$

Whereas the time-average of other components are zero. Finally substituting eq(2.3.85) into eq(2.3.80), the power radiated by this system is given by

$$P(2\Omega) = -\frac{32GI^2\epsilon^2}{5c^5}\Omega^6 \tag{2.3.86}$$

Where I and ϵ are the moment of inertia and the ellipticity $\epsilon = ((I_{11} - I_{22})/I)$.

A neutron star emits gravitational radiation if its configuration is not axisymmetric with respect to the rotation axis. The time evolution of angular frequency as a result of gravitational quadrupole radiation can be computed by assuming that the magnetic dipole is aligned in the direction of spin axis of the star, so that there is no magnetic dipole radiation. Using eq(2.3.86); the rate of loss of rotational kinetic energy is given by

$$\frac{dE}{dt} = I\Omega\dot{\Omega} = -\frac{32GI^2\epsilon^2}{5c^5}\Omega^6$$

Hence the rate of slow-down will be

$$\dot{\Omega} = -\lambda\Omega^5 \tag{2.3.87}$$

Where $\lambda = \frac{32GI^2\epsilon^2}{5c^5}$. Integrating this we derive the time evolution equation as

$$B(t) = B(0)\left[1 + \frac{4\lambda\Omega^4(0)}{I}t\right]^{\frac{-1}{4}} \tag{2.3.89}$$

Where $B(t = 0) = B(0) = \frac{2Q\Omega(0)}{3cR}$ is the same as the field due to magnetic dipole radiation[43].

is expressed as

$$E = \frac{1}{2} I \Omega^2 \tag{3.1.1}$$

where I is the moment of inertia of Neutron star.

Using eq(2.2.59) the rate of loss of energy can be written as

$$\frac{dE}{dt} = -\frac{B_p^2(t) R^6 \sin^2\alpha}{6c^3} \Omega^4 \tag{3.1.2}$$

Where α is the angle between B and Ω, B and R are magnetic field and radius of the star respectively, and c is speed of light. Differentiating eq(3.1.1) with respect to time (keeping I constant) and substituting the result into eq(3.1.2) we obtain the relation

$$I\Omega\dot{\Omega} = -\frac{B_p^2(t) R^6 \sin^2\alpha}{6c^3} \Omega^4$$

or

$$\dot{\Omega} = -\left(\frac{B_p^2(t) R^6 \sin^2\alpha}{6Ic^3}\right) \Omega^3 \tag{3.1.3}$$

where $\dot{\Omega} = \frac{d\Omega}{dt}$ or derivative of Ω with respect to time.

The power of Ω in the eq(3.1.3) is exactly 3. However, the observed values of braking indices of most of known-pulsars are different from value 3. Hence we can replace 3 by n in the above relation as

$$\dot{\Omega}(t) = -\Lambda(t)\Omega^n(t) \tag{3.1.4}$$

Where $\Lambda(t)$ is torque coefficient which is function of time and expressed as

$$\Lambda(t) = \frac{B_p^2(t) R^6 \sin^2\alpha}{6Ic^3} \tag{3.1.5}$$

The braking index, n(t) , of Neutron star can be found using eq(3.1.4). By putting this equation under absolute value eq(3.1.4)can be rewritten as

$$\Omega^n(t) = |\dot{\Omega}(t)|\Lambda^{-1}(t) \tag{3.1.6}$$

Substituting both sides of eq(3.1.6) into natural logarithm we obtain an expression for n(t) as

$$n(t) = \ln\left[\Lambda^{-1}(t)|\dot{\Omega}(t)|\right]\left[\ln\Omega(t)\right]^{-1} \tag{3.1.7}$$

Finally substituting eq(3.1.5) for $\Lambda(t)$ into eq(3.1.7) , eq(3.1.7) can be rewritten as

$$n(t) = \ln\left[\frac{6Ic^3|\dot{\Omega}(t)|}{B_p^2(t)R^6\sin^2\alpha}\right]\left[\ln\Omega(t)\right]^{-1} \tag{3.1.8}$$

The above equation shows that n(t) is a function of time since both $B_p(t)$, $\Omega(t)$ and $\dot{\Omega}(t)$ are dependent on time .

3.2 Braking index of pulsar due to magnetic dipole radiation ,Neutrino emission and Gravitational radiation .

As we have seen in eq(3.1.8) above the braking index, n(t), is expressed in terms of $B_p(t)$ and other physical quantities .

Therefore, we have to express the total surface magnetic field , $B_p(t)$, due to magnetic dipole radiation , Neutrino emission and Gravitational radiation.

The expression for field decay laws due to magnetic dipole radiation, gravitational radiation and Neutrino emission are given bellow.

We have magnetic field decay law due to magnetic dipole radiation from eq(2.2.61) as

$$B_M(t) = B(0)\left[1+\gamma t\right]^{-\frac{1}{4}} \tag{3.2.1}$$

where

$$\gamma = \frac{8Q^2R^4\Omega^4(0)\sin^2\alpha}{27Ic^5} \tag{3.2.2}$$

And where Q and $\Omega(0)$ are separated charge and initial angular frequency of rotating Neutron star, respectively and t is time is time in year.

The magnetic field decay due to Neutrino emission as calculate by Kebede(2004) is expressed as

$$B_N(t) = B(0)\left[1 + yr^{-1}(\frac{\rho}{\rho_{necl}})t\right]^{-\frac{1}{6}} \tag{3.2.3}$$

Where ρ and ρ_{necl} are densities of Neutron star and nucleus respectively.

Using the fact that Neutron stars are compact as nucleus, we can approximate the ratio, $\frac{\rho}{\rho_{necl}} \approx 1$. Hence eq(3.2.3) can be rewritten as

$$B_N(t) = B(0)\left[1 + yr^{-1}t\right]^{-\frac{1}{6}} \tag{3.2.4}$$

Similarly the magnetic field decay due to Gravitational radiation given in eq(2.3.89) as

$$B_G(t) = B(0)\left[1 + \beta t\right]^{-\frac{1}{4}} \tag{3.2.5}$$

where

$$\beta = \frac{4\lambda\Omega^4(0)}{I} \tag{3.2.6}$$

and

$$\lambda = \frac{32G\epsilon^2 I^2}{5c^5} \tag{3.2.7}$$

where G is universal gravitational constant ($G = 6.67 \times 10^{-8}cm^3/g.sec^2$) and ϵ is ellipticity of Neutron star.

The total surface magnetic field due to magnetic dipole radiation ,Neutrino emission and gravitational radiation can be approximated as

$$B_p(t) = \left[B_M(t)B_N(t)B_G(t)\right]^{\frac{1}{3}} \tag{3.2.8}$$

Hence upon substituting eq(3.2.1), eq(3.2.4) and eq(3.2.5)into eq(3.2.8), the surface magnetic field of Neutron star can be expressed as

$$B_p(t) = B(0)\left[(1+\gamma t)^{\frac{-1}{4}}(1+yr^{-1}t)^{\frac{-1}{6}}(1+\beta t)^{\frac{-1}{4}}\right]^{\frac{1}{3}} \tag{3.2.9}$$

The final expression for the braking index of the three processes can be expressed in terms of time by substituting eq(3.2.9) into eq(3.1.8) as

$$n(t) = \ln\left[\frac{6Ic^3|\dot{\Omega}(t)|}{B^2(0)\left[(1+\gamma t)^{\frac{-1}{4}}(1+yr^{-1}t)^{\frac{-1}{6}}(1+\beta t)^{\frac{-1}{4}}\right]^{\frac{2}{3}}R^6\sin^2\alpha}\right]\left[\ln\Omega(t)\right]^{-1} \tag{3.2.10}$$

Hence the original magnetic field B(0) can be approximated as $B(t=0) = 10^{14-15}$G which is free parameter .

3.3 Braking indices of four-known pulsars.

In this section we try to calculate the brakiug indices of four well-known pulsars , such as ;
the Crab, Vela, PSR 0540-69 aud PSR 1509-58.

The calculation is carried out by using eq(3.2.10) and based on data given for each physical quantities in the equation for each pulsar.
The braking index of each pulsar can be calculated as follows.

3.3.1 Crab Pulsar(PSR B0531+21)

We have the following Crab-pulsar parameters.

$$\Omega(t) = 188.12s^{-1}$$

$$\dot{\Omega}(t) = -2.42\times10^{-9}s^{-2}$$

$$t_p = 951yrs = 2.9991\times10^{10}sec \tag{3.3.1}$$

$$I = 10^{45} gcm^2$$

$$\epsilon \approx 3 \times 10^{-4}$$

$$\alpha \approx 80^0 \text{ , } and \text{ , } R = 10^6 cm$$

$$\Omega(t = 0) \approx 10^3 sec^{-1}$$

Using eq(3.2.10) and the above parameters we can compute braking index ,n, as follows.

From eq(3.2.2) ,eq(3.2.7) and eq(3.2.6) the values of both γ , λ , and β for Crab pulsar respectively are

$$\gamma = 1.18275 \times 10^{-8} sec^{-1} = 0.37311 yr^{-1}$$

$$\lambda = 1.581 \times 10^{24} g.cm^2.sec^3 \qquad (3.3.2)$$

and

$$\beta = 6.324 \times 10^{-9} sec^{-1} = 0.1994953 yr^{-1}$$

Hence the expressions in eq(3.2.10) can be calculated using eq(3.3.2) as

$$1 + \gamma t_p = 355.72$$

$$1 + \beta t_p = 190.66 \qquad (3.3.3)$$

$$1 + yr^{-1} t_p = 951$$

Upon substituting values in eq(3.3.1) and eq(3.3.3) into eq(3.2.10) with $B(t = 0) \approx 10^{14} G$ for Crab, the value of braking index is given to be

$$n(t) = 2.525 \qquad (3.3.4)$$

where the measured value of braking index of Crab pulsar is found to be $n = 2.51 \pm 0.01$ [33], [26].

3.3.2 Vela Pulsar(PSR B0833-45)

We have the following Vela pulsar parameter

$$\Omega(t) = 70.36s^{-1}$$

$$\dot{\Omega}(t) = -9.85 \times 10^{-11}s^{-2}$$

$$t_p = 1.8 \times 10^4 yrs = 5.67648 \times 10^{11} sec \tag{3.3.5}$$

$$I = 10^{45} gcm^2$$

$$\epsilon \approx 3.3 \times 10^{-4}$$

$$\alpha \approx 65^0 \ , and \ , R = 10^6 cm$$

$$\Omega(t = 0) \approx 10^3 sec^{-1}$$

Using eq(3.2.10) and the above parameters we can compute braking index, n , as follows.

From eq(3.2.2) ,eq(3.2.7) and eq(3.2.6) the values of both γ , λ , and β for Vela pulsar respectively are

$$\gamma = 1.00155 \times 10^{-8} sec^{-1} = 0.31595 yr^{-1}$$

$$\lambda = 1.9130 \times 10^{24} g.cm^2.sec^3 \tag{3.3.6}$$

and

$$\beta = 7.652 \times 10^{-9} sec^{-1} = 0.241388 yr^{-1}$$

where

$$1 + \gamma t_p = 5688.81$$

$$1 + \beta t_p = 4344.6 \tag{3.3.7}$$

$$1 + yr^{-1} t_p = 18001$$

Upon substituting eq(3.3.5) and eq(3.3.7) into eq(3.2.10) with $B(t=0) \approx 10^{15}G$ for Vela pulsar , the value of braking index is given to be

$$n(t) = 1.62 \tag{3.3.8}$$

where the measured value of braking index of vela pulsar is found to be $n = 1.40 \pm 0.01$ [34].

3.3.3 PSR B1509-58

We have the following parameters for this pulsar.

$$\begin{aligned} \Omega(t) &= 41.7 s^{-1} \\ \dot{\Omega}(t) &= -4.25 \times 10^{-10} s^{-2} \\ t_p &= 1800 yrs = 5.67648 \times 10^{10} sec \\ I &= 10^{45} gcm^2 \\ \epsilon &\approx 10^{-4} \\ \alpha &\approx 60^0 \text{ , } and \text{ , } R = 10^6 cm \\ \Omega(t=0) &\approx 10^3 sec^{-1} \end{aligned} \tag{3.3.9}$$

Similarly using eq(3.2.2), eq(3.2.7) and eq(3.2.6) the values of γ , λ, and β for this pulsar respectively are

$$\begin{aligned} \gamma &= 9.145 \times 10^{-9} sec^{-1} = 0.288486 yr^{-1} \\ \lambda &= 1.7567 \times 10^{23} g.cm^2.sec^3 \end{aligned} \tag{3.3.10}$$

and

$$\beta = 7.0268 \times 10^{-10} sec^{-1} = 0.0221665 yr^{-1}$$

where

$$1 + \gamma t_p = 520.12$$
$$1 + \beta t_p = 40.89 \qquad (3.3.11)$$
$$1 + yr^{-1} t_p = 1801$$

Upon substituting eq(3.3.9) and eq(3.3.11) into eq(3.2.10) with $B(t=0) \approx 1.5 \times 10^{14} G$ for this pulsar , the value of braking index is given to be

$$n(t) = 2.897 \qquad (3.3.12)$$

where the measured value of braking index of PSR B1509-58 is found to be $n = 2.837 \pm 0.001$ [35].

3.3.4 PSR B0540-69

We have the following parameters for this pulsar.

$$\Omega(t) = 124.66 s^{-1}$$
$$\dot{\Omega}(t) = -1.185 \times 10^{-9} s^{-2}$$
$$t_p = 900yrs = 2.83824 \times 10^{10} sec \qquad (3.3.13)$$
$$I = 10^{45} gcm^2$$
$$\epsilon \approx 2.4 \times 10^{-4}$$
$$\alpha \approx 14^0 \text{ , } and \text{ , } R = 10^6 cm$$

$$\Omega(t=0) \approx 10^3 sec^{-1}$$

Similarly using eq(3.2.2), eq(3.2.7) and eq(3.2.6) the values of γ , λ , and β for PSR B0540-69 respectively are

$$\gamma = 7.133 \times 10^{-10} sec^{-1} = 0.0225016 yr^{-1}$$

$$\lambda = 1.0119 \times 10^{24} g.cm^2.sec^3 \quad (3.3.14)$$

and

$$\beta = 4.0476 \times 10^{-9} sec^{-1} = 0.1276845 yr^{-1}$$

where

$$1 + \gamma t_p = 21.245$$

$$1 + \beta t_p = 115.88 \quad (3.3.15)$$

$$1 + yr^{-1} t_p = 901$$

Upon substituting eq(3.3.13) and eq(3.3.15) into eq(3.2.10) with free parameter $B(t = 0) \approx 6.5 \times 10^{14} G$ for this pulsar , the value of braking index is given to be

$$n(t) = 2.281 \quad (3.3.16)$$

where the measured value of braking index of PSR B0540-69 is found to be $n = 2.01 \pm 0.01$ [36], [37].

3.4 Large positive and negative values of braking indices.

As we have tried to note down in the chapter one there are pulsars with large positive and negative values of braking indices different from those values of four-young pulsars above. The reason for this might be the time evolution of angle of inclination $\alpha(t)$ of magnetic dipole moment with rotation axis.

At this point it must be noted that there are two ways of time evolution of angle $\alpha(t)$ of pulsars.

1) A counter-aligning :- in which $\alpha(t)$ increases with time.

2) An aligning:- in which $\alpha(t)$ decreases with time.

Using eq(1.2.5) derived in chapter one, let us find the expressions for observed braking indices of alignment and couteralignments of α.

Eq(1.2.5) is expressed as

$$n_{obs} = 3 - 2\left[\frac{\dot{B}_p(t)}{B_p(t)} + \dot{\alpha}(t)\cot\alpha(t)\right]\left|\frac{\Omega_p(t)}{\dot{\Omega}_p(t)}\right| \tag{3.4}$$

3.4.1 Braking index due to alignment and field decay.

We can rewrite $\dot{\alpha}(t)\cot\alpha(t)$ in the above equation as

$$\dot{\alpha}(t)\cot\alpha(t) = \frac{1}{2}\frac{d}{dt}(\ln[sin^2\alpha(t)]) \tag{3.4.1}$$

For alignment , we adopt Jones' model [38],[27], which gives

$$\sin\alpha(t) = \sin\alpha_0\, e^{-t/\tau_a} \tag{3.4.2}$$

Where τ_c is the alignment timescale , which can be approximated as [31]

$$\tau_a \approx \left|\frac{\Omega_p(t)}{\dot{\Omega}_p(t)}\right| \tag{3.4.3}$$

Hence squaring both sides of eq(3.4.2) and putting the result in natural logarithm we get

$$\ln[sin^2\alpha(t)] = \ln[sin^2\alpha_0] - 2t/\tau_a \tag{3.4.4}$$

Differentiating eq(3.4.4) with respect to time we obtain

$$\frac{d}{dt}(\ln[sin^2\alpha(t)]) = -2/\tau_a \tag{3.4.5}$$

Substituting eq(3.4.5) into eq(3.4.1), then eq(3.4.1) can be rewritten as

$$\dot{\alpha}(t)\cot\alpha(t) = -1/\tau_a \tag{3.4.6}$$

For the case of field decay using the equation of the surface magnetic field approximation of eq(3.2.8) we obtain

$$\frac{\dot{B}_p(t)}{B_p(t)} = \frac{1}{3}\left[\frac{\dot{B}_M(t)}{B_M(t)} + \frac{\dot{B}_G(t)}{B_G(t)} + \frac{\dot{B}_N(t)}{B_N(t)}\right] \tag{3.4.7}$$

Where the expression for $\dot{B}_M(t)$,$\dot{B}_G(t)$ and $\dot{B}_N(t)$ can be obtained from eq(3.2.1),eq(3.2.5) and eq(3.2.4) respectively. Hence eq(3.4.7) can be rewritten as

$$\frac{\dot{B}_p(t)}{B_p(t)} = -\frac{1}{3}\left[\frac{\gamma}{4(1+\gamma t)} + \frac{\beta}{4(1+\beta t)} + \frac{yr^{-1}}{6(1+yr^{-1}t)}\right] \tag{3.4.8}$$

Where γ and β given by eq(3.2.2) and eq(3.2.6) respectively.

Snbstituting eq(3.4.6) and eq(3.4.8) into eq(3.4) the expression for observed braking index of alignment and field decay will be expressed as

$$n_{obs} = 3 + \frac{2}{3}\left[\frac{\gamma}{4(1+\gamma t)} + \frac{\beta}{4(1+\beta t)} + \frac{yr^{-1}}{6(1+yr^{-1}t)} + \frac{3}{\tau_a}\right]\left|\frac{\Omega_p(t)}{\dot{\Omega}_p(t)}\right| \tag{3.4.9}$$

Let us calculate the braking index of PSR 0540 + 23 ,with following parameters [39]

$\Omega_p(t) = 4.066 sec^{-1}$, $\dot{\Omega}_p(t) = -2.55 \times 10^{-13} sec^{-2}$, $\Omega(0) \approx 10^3 sec^{-1}$, $R = 10^6 cm$, $\epsilon \approx 10^{-4}$, $Q = 10^{27} esu$, $t_p = 10^5 yr = 3.1536\times10^{12} sec$, $\tau_a = 1.59451\times10^{13} sec$=505,615.8yr and $\alpha = 10^0$

Using those parameters the values of γ, β and λ will be

$$\gamma = 3.6788 \times 10^{-10} sec^{-1} = 1.16015 \times 10^{-2} yr^{-1}$$
$$\lambda = 1.756 \times 10^{23} g.cm^2.sec^3 \tag{3.4.10}$$
$$\beta = 7.0268 \times 10^{-10} sec^{-1} = 2.216 \times 10^{-2} yr^{-1}$$

Finally substituting values of eq(3.4.10) , τ_a, and t_p we obtain the value of braking index

$$n_{obs} = 7.25 \tag{3.4.11}$$

where the value of braking index of PSR 0540+23 is 11.81 ± 0.12 [40].

3.4.2 Braking index due to counteralignment and field-decays.

We can see that, due to counteralignment n reduces bellow its standard value. For counteralignment ,we adopt [42] the form

$$\alpha(t) = \frac{\pi}{2} - \left(\frac{\pi}{2} - \alpha_0\right)e^{-t/\tau_c} \tag{3.4.12}$$

Where τ_c being the counteralignment timescale.

Differentiating eq(3.4.12) with respect to time we obtain

$$\dot{\alpha}(t) = \frac{1}{\tau_c}\left(\frac{\pi}{2} - \alpha_0\right)e^{-t/\tau_c}$$

We can maximize the counteralignment effect by putting $\alpha_0 = 0$ at t=0 ,where the above relation reduces to

$$\dot{\alpha}(t) = \frac{1}{\tau_c}\left(\frac{\pi}{2}\right)e^{-t/\tau_c} \tag{3.4.13}$$

Substituting eq(3.4.8) and eq(3.4.13) into eq(3.4) the expression for observed braking index of counteralignment and field-decay will be expressed as

$$n_{obs} = 3+\frac{2}{3}\left[\frac{\gamma}{4(1+\gamma t)}+\frac{\beta}{4(1+\beta t)}+\frac{yr^{-1}}{6(1+yr^{-1}t)}-\frac{3\pi e^{-t/\tau_c}}{2\tau_c}\cot\alpha(t)\right]\left|\frac{\Omega_p(t)}{\dot{\Omega}_p(t)}\right| \tag{3.4.14}$$

Let us calculate the braking index of PSR 0950+08 , nsing the following parameters ,[31]

$\Omega_p(t) = 3.951 sec^{-1}$, $\dot{\Omega}_p(t) = -3.578 \times 10^{-15} sec^{-2}$, $\Omega(0) \approx 10^3 sec^{-1}$, $R = 10^6 cm$, $\epsilon \approx 10^{-4}$, $Q = 10^{27} esu$, $t_p = 5 \times 10^6 = 1.5768 \times 10^{14} sec$, $\tau_c = 1.10425 \times 10^{15} sec = 35,015,537.8 yr$ and $\alpha = 12.3^0$.

Using those parameters the values of γ, λ, and β for this pulsar will be

$$\gamma = 7.13625 \times 10^{-10} sec^{-1} = 2.2505 \times 10^{-2} yr^{-1}$$

$$\lambda = 1.725 \times 10^{23} g.cm^2.sec^3 \tag{3.4.15}$$

$$\beta = 7.0268 \times 10^{-10} sec^{-1} = 2.216 \times 10^{-2} yr^{-1}$$

Finally substituting the values in eq(3.4.15), t_p ,and τ_c from above into eq(3.4.14) we obtain

$$n_{obs} = -6.35$$

3.5 Time dependence of Braking index.

In previous three section we have calculated the braking indices of six-pulsars. As we have seen the results those section there are pulsars with braking indices lying between 1 and 3 and with large positive and negative values.

In this section let us see whether braking index is time dependent or constant in time. According to the study carried out by Lyne, Pritchard and Graham-Smith(1993) over 23-years of Crab pulsar rotational history, there is no as such appreciable time dependence of braking index. But this is not an implication that braking index of pulsar appears constant over a long period of time in the order of 100 years.

In the next subsections we will see the behavior of graph of braking index-verse-time for three pulsars.

3.5.1 Time evolution of Braking index of the Crab pulsar.

According to the relativistic plasma diffusion model for Neutron stars developed by Kebede, 2002, the angular frequency of spinning down solo Neutron star can easily be found from eq(2.2.60) and eq(3.2.9) as

$$\Omega_p(t) = \Omega(0)\left[(1+\gamma t)^{-\frac{1}{4}}(1+yr^{-1}t)^{-\frac{1}{6}}(1+\beta t)^{-\frac{1}{4}}\right]^{\frac{1}{3}} \tag{3.5.1}$$

Using this equation we can obtain the expression for $\dot{\Omega}_p(t)$ (time derivative of $\Omega(t)_p$) as

$$\dot{\Omega}_p(t) = -\frac{\Omega(0)}{12}\left[\frac{\gamma}{(1+\gamma t)}+\frac{\beta}{(1+\beta t)}+\frac{2yr^{-1}}{3(1+yr^{-1}t)}\right]\left[(1+\gamma t)^{-\frac{1}{4}}(1+yr^{-1}t)^{-\frac{1}{6}}(1+\beta t)^{-\frac{1}{4}}\right]^{\frac{1}{3}} \tag{3.5.2}$$

Hence substituting eq(3.5.1) and eq(3.5.2) into eq(3.2.10) with $\Omega(0) = 330.7sec^{-1}$ and $B(0) = 5\times10^{13}G$ for the Crab pulsar eq(3.2.10) can be rewritten as a function

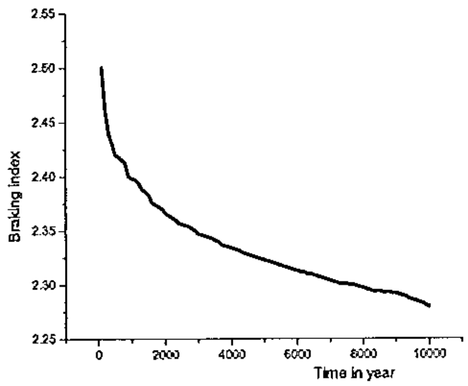

Figure 3.1: The graph of braking index of Crab pulsar verse time.

of time as

$$n(t) = \frac{\ln\left\{\frac{4.412\times10^{7}\left[\frac{\gamma}{(1+\gamma t)}+\frac{\beta}{(1+\beta t)}+\frac{2yr^{-1}}{3(1+yr^{-1}t)}\right]}{\left[(1+\gamma t)^{-\frac{1}{4}}(1+yr^{-1}t)^{-\frac{1}{6}}(1+\beta t)^{-\frac{1}{4}}\right]^{\frac{1}{3}}}\right\}}{\ln\left\{10^{3}\left[(1+\gamma t)^{-\frac{1}{4}}(1+yr^{-1}t)^{-\frac{1}{6}}(1+\beta t)^{-\frac{1}{4}}\right]^{\frac{1}{3}}\right\}} \tag{3.5.3}$$

where the values of γ and β are given in eq(3.3.2) and t is time in years.

Using eq(3.5.3) the graph for braking index of the Crab pulsar verse time is shown in Figure- 3.1.

3.5.2 Time evolution of Braking index of PSR 0540 +23 pulsar.

The braking index of PSR 0540 +23 was calculated in the previous section . In this subsection we are going to write the braking index of this particular pulsar as a function of time and draw the graph showing the time evolution of its braking index.

Upon substituting eq(3.5.1) and eq(3.5.2) into eq(3.4.9) for $\left|\frac{\dot{\Omega}_p(t)}{\Omega_p(t)}\right|$, eq(3.4.9) will

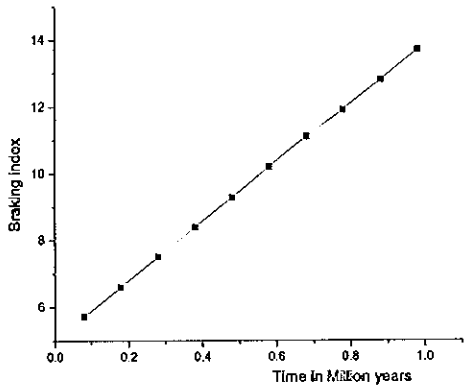

Figure 3.2: The graph of braking index of PSR 0540+23 verse time.

be reduced to

$$n_{obs} = 5 + \frac{12}{\tau_a \left[\frac{\gamma}{(1+\gamma t)} + \frac{\beta}{(1+\beta t)} + \frac{2yr^{-1}}{3(1+yr^{-1}t)} \right]} \tag{3.5.4}$$

Where the values of γ, β and τ_a are given in eq(3.4.10) and t time in years.

Using eq(3.5.4) the graph for braking index of PSR 0540+23 verse time is shown in Figure-3.2.

3.5.3 Time evolution of Braking index of PSR 0950 +08 pulsar.

The braking index of PSR 0950 +08 was calculated in the previous section. In this subsection we are going to write the braking index of this particular pulsar as a function of time and draw the graph showing the time evolution of its braking index.

Upon substituting eq(3.5.1) and eq(3.5.2) into eq(3.4.14) for $\left|\frac{\ddot{\Omega}_p(t)}{\dot{\Omega}_p(t)}\right|$, eq(3.4.14) will be reduced to

$$n_{obs} = 5 - \frac{12\pi e^{-t/\tau_c} \cot \alpha(t)}{\tau_c \left[\frac{\gamma}{(1+\gamma t)} + \frac{\beta}{(1+\beta t)} + \frac{2yr^{-1}}{3(1+yr^{-1}t)} \right]} \tag{3.5.5}$$

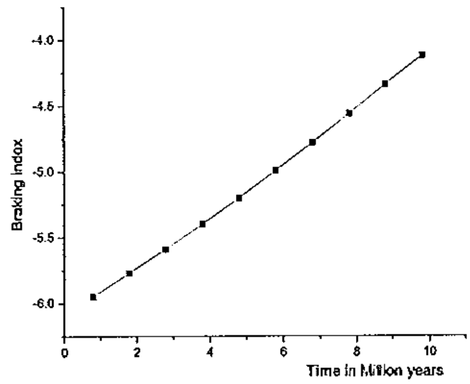

Figure 3.3: The graph of braking index of PSR 0950+08 verse time

Where the values of γ, β and τ_a are given in eq(3.4.15) and $\alpha(t)$ is given in eq(3.4.12).

Using eq(3.5.5) the graph for braking index of PSR 0950+08 verse time is shown in Figure-3.3.

Chapter 4

SUMMERY AND CONCLUSION

4.1 Summary

As we have discussed in the introduction of this thesis Neutron stars are powered by rotational kinetic energy and lose energy by accelerating particle winds and by emitting electromagnetic radiation at their rotational frequency[39]. As a result, their angular frequencies slow-down in time. This particular behavior has been interpreted as evidence for evolution of the torque acting up on the star. The time dependent behavior of the torque is closely linked to the important question of magnetic decay and alignment/counteralignment of the magnetic and rotational axes of a neutron star[45]. The spin-down rate is quantified by the braking index as indicated by the braking law[7].

It is an experimental fact that the braking index of a few well known young pulsars had been found to differ from the expected canonical value 3 [7],[13]. As we have shown in chapter 1 of this thesis, the torque coefficient, Λ, is time dependent through surface magnetic field $B_p(t)$ and the inclination angle $\alpha(t)$ of the magnetic dipole axis to the rotation axis. Hence magnetic field decay and alignment/counteralignment are responsible for this difference in theoretical & observational braking indices[31],[45].

However, since the early days of pulsar astronomy the question of magnetic field decay has coutinued to be rather controversial. Soon after the discovery of pulsars

Gnnn & Ostriker(1969) suggested that magnetic field of a neutron star should decay exponentially on a time scale of 4×10^6 years, because of ohmic dissipation of the supporting currents. The same authors also presented observational evidence in favor of field decay[11]. However, it was soon argued by others that ohmic decay cannot be important since the interior of a neutron star is likely to be superconducting[46].

So far we have pointed out the weakness of existing theories for the sources of neutron star magnetic fields and as well as on the models for magnetic decay laws. Very recently, however, the relativistic plasma diffusion has been identified as a possible source for neutron star magnetic fields [22]. The theory predicts that pulsars magnetic fields decay as a result of various non-conventional processes which mainly include, magnetic dipole radiation, Gravitational quadrupole radiation and neutrino and photon emissions. Based on this theory we have been able to address the worrisome issue of braking index variations.

It is also important to note that the braking torque evolves through alignment/counteralignme processes. The idea that the magnetic axis aligns/counteraligns with the rotation axis was first analyzed analytically by Jones(1976). He suggested a simple exponential decay of the inclination of angle $\alpha(t)$ for both processes. Hence we have also considered the time evolution of the angle of inclination $\alpha(t)$ of the magnetic dipole moment with the rotation axis as a possible source of braking index variations for older pulsars.

4.2 Conclusion

As we have shown in the previous chapter, the calculated braking indices of the four well known young pulsars are in agreement with experimentally observed values. This shows us that the relativistic plasma diffusion theory for pulsar fields properly addresses dynamics of pulsar magnetic fields. Large positive indices($n_{obs} > 3$) are obtained as a result of alignment($\dot{\alpha}(t) < 0$ or angle decay) and field decay($\dot{B}_p(t) < 0$).

Whereas large negative braking indices are obtained as a result of counteralignment($\dot{\alpha}(t) > 0$) and field decay($\dot{B}_p(t) < 0$).

According to the study carried out by Lyne, Pritchard and Graham-Smith(1993) over 23-years of Crab pulsar rotational history, there is no as such appreciable time variations of the braking index. But this is not an implication that braking index of pulsars are constants over a long period of time in the order of 100 years and over(Fig-3.1). By considering the magnetic field decay($\dot{B}_p(t) < 0$), the braking indices of young pulsars decrease with time. Whereas, for old pulsars with both field decay and alignment/counteralignment in action braking indices increase with time(Fig-3.2 and Fig-3.3).

APPENDIX

Torque decay of isolated pulsar

In the magnetic dipole model[8],[9],[10] the spin-down energy is carried away by

$$\frac{dE_{rot}}{dt} = I\Omega\dot{\Omega} = -\frac{2}{3c^3}|\ddot{\mathrm{m}}|^2 \qquad (App.1)$$

Where the dipole moment m is given in(2.2.45) as

$$\mathrm{m} = m_0\left[\hat{\mathrm{e}}_{\parallel}\cos\alpha + \hat{\mathrm{e}_{\perp}}\sin\alpha\cos\Omega t + \hat{\mathrm{e}_{\perp}}'\sin\alpha\sin\Omega t\right] \qquad (App.2)$$

with α being the angle of inclination of magnetic axis w.r.t the rotation axis and Ω is angular frequency. We have

$$|\mathrm{m}_0| = \frac{B_pR^3}{2} \qquad (App.3)$$

where B_p is surface magnetic field and R is the radius of neutron star. Hence

$$\ddot{m} = -\frac{B_pR^3}{2}\left[\hat{\mathrm{e}_{\perp}}\Omega^2\sin\alpha\cos\Omega t + \hat{\mathrm{e}_{\perp}}'\sin\alpha\sin\Omega t\right] \qquad (App.4)$$

$$|\ddot{m}|^2 = \frac{1}{4}B_pR^6\Omega^4\sin^2\alpha \qquad (App.5)$$

Then substituting (App.5) into (App.1)

$$\frac{dE_{rot}}{dt} = -\frac{B_pR^6\Omega^4\sin^2\alpha}{6c^3} \qquad (App.6)$$

But the rotational angular momentum of from star is

$$L = I\Omega \qquad (App.7)$$

The braking torque (i.e the radiation-reaction torque) acting on the star is thus:

$$N = \frac{dL}{dt} = I\dot{\Omega} = \frac{1}{\Omega}\frac{dE_{rot}}{dt} \qquad (App.8)$$

From an isolated neutron star $\frac{dE_{rot}}{dt} < 0$, and hence $N < 0$. The change in magnitude of the torque as a function of time is given by

$$\frac{d|N|}{dt} = -I\ddot{\Omega} \qquad (App.9)$$

The term 'torque decay' thus refers to a decrease in the magnitude of the torque.

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