ADDIS ABABA UNIVERSITY

COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES
DEPARTMENT OF MATHEMATICS

STURM-LIOUVILLE AND PERIODIC STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS

A PROJECT SUBMITED TO DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

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Approval

This project has been examined and approved as meeting the requirements for the partial fulfillment of Master of Science in Mathematics.

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Declaration

“I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any degree, diploma, associate ship, fellowship or any other similar title to me.

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“This is certify that this project is compiled by Measho Negasi in the department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiners and eventual defense.

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Advisor’s signature”

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Abstract

In this project we discuss on, the eigenvalue problems that we have found so useful for solving the PDEs to a general class of boundary value problems that share a common set of properties. The so-called Sturm-Liouville problems define a class of eigenvalue problems, which include many of the previous problems as special cases. The S - L Problem helps to identify those assumptions that are needed to define an eigenvalue problem with the properties that we require.

Introduction

In mathematics and its applications, a classical Sturm-Liouville equation, named after JACQUES CHARLES FRANOIS STURM (1803 - 1855) and JOSEPH LIOUVILLE (1809 - 1882), is a real second-order differential equation of the form $-(py')' + q\,y = \lambda r(x)y$, where y is a function of the free variable x. In the simplest of all cases all coefficients are continuous on the finite closed interval [a, b], and p has continuous derivative. In addition, the unknown function y is typically required to satisfy some boundary conditions at a and b. The function r(x) is called the weight function or density function. The value of λ is not specified in the equation; finding the values of λ for which there exists a non-trivial solution of the above satisfying the boundary conditions is part of the problem is called the Sturm-Liouville Problem(SLP).

CHAPTER ONE

PRELIMINARIES

1.1 Basic definitions and concepts

Definition 1.1: An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a differential equation (DE).

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots \dots \dots .. + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (1.1)$$

Examples: 1. $y'' - 2y' + y = 0$

2. $\frac{\partial^2 u}{\partial x^2} = -2\frac{\partial u}{\partial t}$

1.1.1 Classification of differential equations

a. Ordinary differential equation

Definition 1.2: A differential equation which involves derivatives with respect a to single independent variable (ODE).

b. Partial differential equations:

Definition1.3: A differential equation which contains two or more independent variables and partial derivatives with respect to them (PDE).

Definition1.4: (Second Order Linear Differential Equations): A second order linear differential equation has to form:

$$p(x)\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} + r(x)y = g(x) \qquad (1.2)$$

Where p, q, r and g are continuous functions.

If g = 0, then the differential equation is said to be homogeneous.

1.1.2 Boundary value problems

Definition 1.5: A boundary-value problem (BVP) is a problem of determining a solution to a differential equation subject to conditions on the unknown function specified at two or more values of the independent variable. Such conditions are called boundary conditions.

Definition1.6: For any two functions $y_1, y_2 \in C^1$ the determinant

W (y_1, y_2) (x) = $\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$ = $y_1(x)y_2'(x)$ - $y_2(x)y_1'(x)$ is called the Wronskian of y_1 and y_2. The symbol W (y_1, y_2) (x) is sometimes abbreviated to W(x). The Wronskian is significant in the study of differential equations from the following lemmas.

Lemma1.1: If y_1, y_2 are solutions of the homogeneous equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0, \quad x \in I \tag{1.3}$$

then either W $(y_1, y_2)(x) = 0$ or W $(y_1, y_2)(x) \neq 0$ for any $x \in I$

Proof:

From definition (1.3), we have $W' = y_1y_2'' - y_2y_1''$.

Because y_1 and y_2 are solutions of equation (1.2), we have

$$y_1'' + q(x)y_1' + r(x)y_1 = 0$$

$$y_2'' + q(x)y_2' + r(x)y_2 = 0 \tag{1.4}$$

Multiply the first equation by y_2, the second by y_1, and subtracting, yields

$$y_1y_2'' + y_2y_1'' + q\,(y_1y_2' - y_2y_1') = 0$$

$$\Rightarrow W' + qW = 0$$

Integrating this last equation, we obtain

$$W(x) = ce^{-\int_a^x q(t)dt}, \quad x \in I \tag{1.5}$$

Where c is an arbitrary constant.

The exponential function does not vanish for any (real or complex) exponent, therefore W(x) = 0 if and only if c = 0.

Remark 1: The expression (1.5) implies that both W and W' are continuous.

Lemma 1.2: Any two solutions y_1 and y_2 of equation (1.3) are linearly independent if and only if W $(y_1, y_2)(x) \neq 0$ on I.

Definition 1.7: (Piecewise continuous): Given afunction f(x) on [a, b], f is piecewise continuous on [a, b] if there exists a finite number of points x_i, i = 1, 2, …, n - 1 with

$$a = x_0 < x_1 < \ldots < x_n = b$$

such that:

1. f(x) is continuous on each open subinterval (x_{n-1}, x_n).
2. f(x) has a finite limit at each end of each open interval.

Definition 1.8: (Inner product): Given two functions f(x) and g(x) that are integrable on [a, b], their inner product is $\langle f, g\rangle = \int_a^b f(x)g(x)dx,$ (1.6)

Definition 1.9: The inner product of a function with itself is called a norm squared and written

$$\langle f, f\rangle = \|f\|^2 = \int_a^b r(x)f^2(x)dx \tag{1.7}$$

We assumed the input functions f(x) and g(x) are nonzero, bounded and such that the resulting inner product integral exists. We also assume that the norm squared is nonzero unless f(x) is identically zero for all x over the interval.

Definition 1.10: A real-valued function f(x) is called square-integrable on the interval (a, b) with respect to the weight function r(x) when

$$\langle f, f\rangle = \|f\|^2 = \int_a^b r(x)f^2(x)dx < \infty \tag{1.8}$$

Definition 1.11 (Norm): The norm of an integrable function f(x) on [a, b] is

$$\|f\|^2 = \int_a^b |f(x)|^2 dx,$$

$$\text{or } \|f\| = \sqrt{\langle f, f\rangle} = \left(\int_a^b (f(x))^2\right)^{1/2} \tag{1.9}$$

Definition 1.12 (Orthogonal): Two piecewise continuous functions f(x) and g(x) on [a, b] are orthogonal when $\langle f, g\rangle = \int_a^b f(x)g(x)dx = 0$ and orthonormal when in addition to being orthogonal, we have $\|f\| = 1 = \|g\|$.

Definition1.13: (Orthogonal): Suppose that w(x) is a nonnegative function on [a, b]. If f(x) and g(x) are real-valued functions on [a, b] with respect to the weight function w to be

$$\langle f, g\rangle = \int_a^b f(x)g(x)w(x)dx. \tag{1.10}$$

we say f and g are orthogonal on [a, b] with respect to the weight function w if $\langle f, g\rangle = 0$.

Definition1.14 (Generalised Fourier series):

Suppose that $\{f_1, f_2, f_3, \ldots\}$ is an orthogonal set of functions on [a, b] with respect to the weight function w. If f is a function on [a, b], and $f(x) = \sum_{n=1}^{\infty} a_n f_n(x)$, then the coefficients a_n are given

By $$a_n = \frac{\langle f,, f_n\rangle}{\langle f_n, f_n\rangle} = \frac{\int_a^b f(x)f_n(x)w(x)dx}{\int_a^b f_n{}^2 w(x)dx} \tag{1.11}$$

Remark1.2: The series expansion above is called a generalized Fourier series for f, and a_n are the generalized Fourier coefficients.

Definition (Periodic function): A function f(x) is periodic with period T if, for all x,

$$f(x + T) = f(x).$$

CHAPTER TWO

STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS

2.1 STURM-LIOUVILLE DIFFERENTIAL EQUATION

Definition 2.1: A second order differential operator L is in self-adjoint form if

$$L[y] = (py')' + q(x)y. \tag{2.1}$$

Definition 2.2: A second ordered linear differential equation of the form

$$[p(x)y']' - q(x)y + \lambda r(x)y = 0, \quad x \in [a, b] \tag{2.2}$$

Where λ *is an unknown constant called the eigenvalue parameter* and p, q, r are continuous functions on interval [a, b], is called a Sturm-Liouville(SL) differential equation. The function $r(x)$ is called the weight function for the Sturm-Liouville equation.

Theorem 2.1: Any second order linear operator can be put into the form of the Sturm-Liouville operator (2.1).

Proof:

There is no loss of generality in the form of Ly $= —(py') + qy$ since it is possible to convert a general 2nd order eigenvalue problem

$$-p(x)y'' - q(x)y' + r(x)y = \lambda y \tag{2.3}$$

to this form by multiplying by an integrating factor μ(x)

$$-\mu(x)p(x)y'' - \mu q(x)y' + \mu(x)r(x)y = \lambda\mu(x)y \tag{2.4}$$

but expanding the differential operator we obtain

$$Ly = -py'' - p'y' + qy \tag{2.5}$$

Thus comparing (2.5) and (2.4) we can make the following identifications:

$$p = \mu p \text{ and } p' = \mu q \Rightarrow p' = \mu' p + \mu p' = \mu q$$

which is a linear 1st order ODE with integrating factor $e^{\int\left(\frac{p'}{p}-\frac{q}{p}\right)dx}$.

$$\mu' + \left(\frac{p'}{p} - \frac{q}{p}\right)\mu = 0 \Rightarrow [pe^{-\int\frac{q dx}{p}}\mu]' = 0$$

$$\Rightarrow \mu = \frac{e^{\int\frac{q}{p}dx}}{p} \tag{2.6}$$

Example 1: Convert $x^2y'' + xy' + 2y = 0$ into Sturm-Liouville form.

Solution: The standard form is $y'' + \frac{1}{x} y' + \frac{2}{x} y = 0$ with $p(x) = \frac{1}{x},\ q(x) = \frac{2}{x}$

We need only multiply this equation by $\frac{1}{x^2} e^{\int \frac{dx}{x}} = \frac{1}{x}$

to put the equation in Sturm-Liouville form:

$$xy'' + y' + \frac{2}{x} y = 0$$

$$(xy')' + \frac{2}{x} y = 0$$

Therefore, the Sturm-Liouville form of $x^2y'' + xy' + 2y = 0$ is $(xy')' + \frac{2}{x} y = 0$

Example 2: Convert the equation $-y" + x^4y' = \lambda y$ to Sturm-Liouville form.

Solution: $p(x) = 1, q(x) = -x^4,\ \mu(x) = e^{-\int x^4 dx} = e^{\frac{-x^5}{5}}$

Therefore, $-e^{\frac{-x^5}{5}} y" + e^{\frac{-x^5}{5}} y' = \lambda e^{\frac{-x^5}{5}} y$

$$\Rightarrow -\left(e^{\frac{-x^5}{5}} y'\right)' = \lambda e^{\frac{-x^5}{5}} y$$

Example 3: The Bessel's equation

$$x^2y'' + xy' + (x^2 - v^2) = 0$$

Can be written in Sturm-Liouville form as

$$(xy')' + \left(x - \frac{v^2}{x}\right) y = 0.$$

Example 4: The Legendre equation:

$$(1 - x^2)y'' - 2xy' + v(v+1)y = 0$$

Can easily put into Sturm-Liouville form,

Since $(1 - x^2)' = -2x$, so, the Legendre equation is equivalent to:

$$[(1 - x^2)y']' + v(v + 1) y = 0$$

2.1.1 SOLUTION OF STURM-LIOUVILLE EQUATIONS

DEFINITION 2.3: A solution of the Sturm-Liouvilleequation (2.2) is defined to be a pair of (y, λ) with y(x) a nonzero function and λ a constant. The function y is called the eigenfunction and the Corresponding λ is called the eigenvalue.

2.1.2 Boundary conditions

Boundary conditions for a solution y of a differential equation on the interval [a, b] are classified as follows:

a. Mixed boundary conditions:

Definition 2.4: boundary conditions of the form

$$c_a y(a) + d_a y'(a) = \alpha$$
$$c_b y(b) + d_b y'(b) = \beta \quad (2.7)$$

Where c_a,d_a , c_b, d_b, α and β are constants, are mixed Dirichlet-Neumann boundary condition. When both $\alpha = \beta = 0$ the boundary conditions are said to be homogeneous. Special cases are Dirichlet BC ($d_a = d_b = 0$) and Neumann BC ($c_a = c_b = 0$)

b. Periodic Boundary conditions:

Definition 2.5: boundary conditions of the form

$y(a) = y(b)$

$y'(a) = y'(b)$ are called periodic boundary conditions.

Example: some examples are

a. Stretched vibrating string clamped at two ends: Dirichlet BC
b. Electrostatic potential on the surface of a volume: Dirichlet BC
c. Electrostatic field on the surface of a volume: Neumann BC
d. A heat-conducting rod with two ends in heat baths: Dirichlet BC

2.2 Sturm-Liouville boundary value problem

Definition 2.6: A second ordered linear differential equation of the form

$$[p(x)y']' - q(x)y + \lambda r(x)y = 0, \quad x \in [a, b] \quad (2.8)$$

Where p, q, r are continuous functions on interval [a, b]

λ is an unknown constant called the eigenvalue parameter together with boundary conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0 ,$$
$$\beta_1 y(b) + \beta_2 y'(b) = 0 \quad (2.9)$$

is called a Sturm-Liouville boundary value problem.

Definition 2.7: A second order differential operator L is in self-adjoint form if

$$L[y] = (py')' + q(x)y. \quad (2.10)$$

For the Sturm-Liouville operator,

$$L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q,$$

Sturm-Liouville equation becomes

$L(y) + \lambda r(x)y = 0$

Lagrange's and Green's identities

We have the two identities: Lagrange's identity and Green's identity

Theorem 2.2: (Lagrange's identity)

$$u\,L(v) - v\,L(u) = \left[pu\,\frac{dv}{dx} - pv\frac{du}{dx}\right]' \qquad (2.11)$$

Proof:

The proof of Lagrange's identity follows by simple manipulations of the operator:

Let u and v be two functions. Then

$$u\,L(v) = u\frac{d}{dx}\left(p\frac{dv}{dx}\right) + uqv$$

$$vL(u) = v\,\frac{d}{dx}\left(p\frac{du}{dx}\right) + vqu$$

$$u\,L(v) - v\,L(u) = u\left[\frac{d}{dx}\left(p\frac{dv}{dx}\right) + qv\right] - v\left[\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu\right]$$

$$= u\frac{d}{dx}\left(p\frac{dv}{dx}\right) - v\,\frac{d}{dx}\left(p\frac{du}{dx}\right)$$

Now let us differentiate the two expressions $\frac{d}{dx}\left[u\left(p\frac{dv}{dx}\right)\right]$ and $\frac{d}{dx}\left[v\left(p\frac{du}{dx}\right)\right]$ with respect to x. we obtain:

$$\frac{d}{dx}\left[u\left(p\frac{dv}{dx}\right)\right] = \frac{du}{dx}\left(p\frac{dv}{dx}\right) + u\frac{d}{dx}\left(p\frac{dv}{dx}\right)$$

$$\frac{d}{dx}\left[v\left(p\frac{du}{dx}\right)\right] = \frac{dv}{dx}\left(p\frac{du}{dx}\right) + v\frac{d}{dx}\left(p\frac{du}{dx}\right)$$

$$u\,L(v) - v\,L(u) = \left[\frac{d}{dx}\left[u\left(p\frac{dv}{dx}\right)\right] - \frac{du}{dx}\left(p\frac{dv}{dx}\right)\right] - \left[\frac{d}{dx}\left[v\left(p\frac{du}{dx}\right)\right] - \frac{dv}{dx}\left(p\frac{du}{dx}\right)\right]$$

$$= \frac{d}{dx}\left(u\left(p\frac{dv}{dx}\right)\right) - \frac{du}{dx}\left(p\frac{dv}{dx}\right) - \frac{d}{dx}\left(v\left(p\frac{du}{dx}\right)\right) + \frac{dv}{dx}\left(p\frac{du}{dx}\right)$$

$$= \frac{d}{dx}\left(u\left(p\frac{dv}{dx}\right)\right) - \frac{d}{dx}\left(v\left(p\frac{du}{dx}\right)\right)$$

$$= \frac{d}{dx}\left[pu\,\frac{dv}{dx} - pv\frac{du}{dx}\right] = \left[pu\,\frac{dv}{dx} - pv\frac{du}{dx}\right]'$$

Therefore, $u\,L(v) - v\,L(u) = \frac{d}{dx}\left[pu\,\frac{dv}{dx} - pv\frac{du}{dx}\right] = \left[pu\,\frac{dv}{dx} - pv\frac{du}{dx}\right]'$

Theorem 2.3: (Green's identity)

$$\int_a^b [uLv - vLu]dx = [\, p(x)(vu' - uv')]|_a^b \qquad (2.12)$$

Proof (Green's identity is simply proven by integrating Lagrange's Identity):

$$\int_a^b [uLv - vLu]dx = \int_a^b \left[pu\,\frac{dv}{dx} - pv\frac{du}{dx}\right]' dx$$

$$= [p(x)(vu' - uv')]|_a^b$$

Therefore, $\int_a^b [uLv - vLu]dx = [\, p(x)(vu' - uv')]|_a^b$

Theorem:(Lagrange's Identity):

Proof:

Let u and v be any sufficiently differentiable functions.

Then $\int_a^b \text{vLu}\, dx = \int_a^b \text{v}\{-(\text{p}u')' + qu\}\text{dx}$

$$= -vpu'|_a^b + \int_a^b u'p\,v'dx + \int_a^b uqvdx$$

$$= -vpu'\Big|_a^b + upv'\Big|_a^b + \int_a^b u\{-(pv')' + qv\}dx.$$

Therefore, $\int_a^b vLu\, dx = -pvu'|_a^b + v'pu|_a^b + \int_a^b uLv\, dx$

Now suppose that u and v both satisfy the SL boundary conditions.

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0, \qquad \beta_1 u(b) + \beta_2 u'(b) = 0$$

$$\alpha_1 v(a) + \alpha_2 v'(a) = 0, \qquad \beta_1 v(b) + \beta_2 v'(b) = 0$$

Then

$$\int_a^b \text{vLu}\, dx - \int_a^b \text{uLv}\, dx = -\text{p(b)u}'(\text{b})\text{v(b)} + \text{p(b)u(b)v}'(\text{b}) + \text{p(a)u}'(\text{a})\,\text{v(a)} - \text{p(a)u(a)v}'(\text{a})$$

$$= \text{p(b)}\left\{+\frac{\beta_1}{\beta_2}\text{u(b)v(b)} + \text{u(b)}\left(-\frac{\beta_1}{\beta_2}\text{v(b)}\right)\right\}$$

$$+\text{p(a)}\left\{-\frac{\alpha_1}{\alpha_2}\text{u(a)v(a)} - \text{u(a)}\left(-\frac{\alpha_1}{\alpha_2}\text{v(a)}\right)\right\}$$

$$= 0$$

$$\int_a^b (\text{vLu} - \text{uLv})\, \text{dx} = \int_a^b \text{vLu}\, dx - \int_a^b \text{uLv}\, dx = 0$$

Thus $\int_a^b vLu\,dx = \int_a^b uLv\,dx$, whenever u and v satisfy the SL boundary condition.

2.2.1 Regular Sturm-Liouville boundary value problem (RSLBVP)

Definition 2.8: A second ordered linear differential equation of the form

$[p(x)y']' - q(x)y + \lambda r(x)y = 0, \quad x \in [a, b]$

Where $p(x) > 0$, $r(x) > 0$, and p, q, r are continuous functions on interval [a, b] and λ is an unknown constant called the eigenvalue parameter together with boundary conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0\,,$$

$$\beta_1 y(b) + \beta_2 y'(b) = 0$$

is called a regular Sturm-Liouville boundary value problem (RSLBVP)

Example 5: For $\lambda \in R$, solve

$y'' + \lambda y = 0,\ y(0) = 0,\ y(\pi) = 0.$ (2.13)

For reasons that will be clear later on, it is enough to consider $\lambda \in R$

Case1: Let $\lambda < 0$. Then $\lambda = -\mu^2$ where μ is real and non-zero.

The general solution of ODE in 2.12 is given by

$$y(x) = Ae^{\mu x} + Be^{-\mu x} \tag{2.14}$$

This y satisfies boundary conditions in (2.12) if and only if A = B = 0.

That is, y = 0. Therefore, there are no negative eigenvalues.

Case 2: Let $\lambda = 0$. In this case, it easily follows that trivial solution is the only solution of

$y'' = 0, y(0) = 0, y'(\pi) = 0.$ *(2.15)*

Thus, 0 is not an eigenvalue.

Case3: Let $\lambda > 0$.Then $\lambda = \mu^2$, where μ is real and non-zero.

The general solution of ODE in (2.12) is given by

$$y(x) = A\cos(\mu x) + B\sin(\mu x) \tag{2.16}$$

This y satisfies boundary conditions in (2.12) if and only if A = 0 and $B\cos(\mu\pi) = 0$.

But $B\cos(\mu\pi) = 0$ if and only if, either B = 0 or $\cos(\mu\pi) = 0$.

The condition A = 0 and B = 0 means y = 0. This does not yield any eigenvalue. If $y \neq 0$, then b $\neq 0$. Thus $\cos(\mu\pi) = 0$ should hold. This last equation has solutions given by $\mu = \frac{2n-1}{2}$

for n = 0, ±1, ±2,… Thus eigenvalues are given by

$$\lambda_n = \frac{2n-1}{2}, n = 0, \pm 1, \pm 2, \ldots \qquad (2.17)$$

and the corresponding eigenfunctions are given by

$$\phi_n(x) = B sin(\frac{2n-1}{2}x), \quad n = 0, \pm 1, \pm 2, \ldots \qquad (2.18)$$

Note: All the eigenvalues are positive. The eigen functions corresponding to each eigenvalue form a one dimensional vector space and so the eigen functions are unique upto a constant multiple.

2.2.1.1 Properties of Regular Sturm-Liouville boundary value problem (RSLBVP)

Theorem 2.4: The eigenvalues, if any, of a regular SL-BVP are real.

PROOF:

Suppose $\lambda \in \mathrm{C}$ is an eigenvalue of a regular SL-BVP and let y be corresponding eigen function. That is,

$$\mathrm{L[y]} + \lambda \mathrm{r(x)y} = 0,$$

$$\alpha_1 \mathrm{y(a)} + \alpha_2 \mathrm{y'(a)} = 0 ,$$

$$\beta_1 y(b) + \beta_2 y'(b) = 0 \qquad (2.19)$$

Taking the complex conjugates, we get $\mathrm{L[\bar{y}]} + \lambda \mathrm{r(x)\bar{y}} = 0$

$$\alpha_1 \mathrm{\bar{y}(a)\bar{y}'(a)} + \alpha_2 \mathrm{p(a)\bar{y}'(a)} = 0,$$

$$\beta_1 \bar{y}(b) + \beta_2 p(b)\bar{y}'(b) = 0. \qquad (2.20)$$

Multiply the ODE in (2.18) with $\bar{y}$ and multiply that of (2.18) with y, and subtracting one from the other yields

$$[p(y'\bar{y} - \bar{y}'y)]' + (\lambda - \bar{\lambda})\mathrm{ry\bar{y}} = 0 \,)$$

$$[p(y'\bar{y} - \bar{y}'y)]' = -(\lambda - \bar{\lambda})\mathrm{ry\bar{y}} \qquad (2.20)$$

Integrating the last equality yields

$$[p(y'\bar{y} - \bar{y}'y)]|_a^b = -(\lambda - \bar{\lambda}) \int_a^b r\,(x)|(yx)|^2 dx \qquad (2.21)$$

But LHS of the last equation is zero, since we have both

$$\beta_1 \mathrm{y(b)} + \beta_2 \mathrm{p(b)y'(b)} = 0 \text{ and } \beta_1 \mathrm{\bar{y}(b)} + \beta_2 \mathrm{p(b)\bar{y}'(b)} = 0,$$

We also know that $\beta_1^2 + \beta_2^2 \neq 0$, and hence a certain determinant associated is zero.

Thus we have

$$\left(\lambda-\bar{\lambda}\right)\int_a^b r|y|^2 dy = 0 \qquad (2.22)$$

Since y, being an eigenfunction, is not identically equal to zero, and integral of nonnegative function (since r > 0) is not zero, the only possibility is that $\lambda = \bar{\lambda}$.

That is, λ is real.

Theorem 2.5: The eigenfunctions of a regular SL-BVP corresponding to distinct eigenvalues are orthogonal w.r.t. weight function r on [a, b], that is, if u and v are eigenfunctions corresponding to distinct eigenvalues λ and μ respectively, then

$$\int_a^b r(x)u(x)v(x)\,dx = 0 \qquad (2.23)$$

PROOF:

As in the previous proof, writing down the equations satisfied by u and v, and multiplying the equation for u with v and vice versa, finally subtracting one from another, we get

$$\left[p\left(u'v - v'u\right)\right]' + (\lambda - \mu)ruv = 0$$

$$\left[p\left(u'v - v'u\right)\right]' = -(\lambda - \mu)ruv \qquad (2.24)$$

Integrating the last equality yields

$$p\left(u'v - v'u\right)\Big|_a^b = -(\lambda - \mu)\int_a^b r(x)u(x)v(x)\,dx = 0 \qquad (2.25)$$

Reasoning exactly as in the previous proof, LHS of the above equality is zero.

Since $\lambda \neq \mu$ we get the desired (2.23)

Theorem 2.6: The eigenvalues of a regular SL-BVP are simple. Thus an eigenfunction corresponding to an eigenvalue is unique up to a constant multiple.

PROOF:

Let ϕ_1 and ϕ_2 be two eigenfunctions corresponding to the same eigenvalue

λ. We recall from the section on Green's functions (the identity) here: By Lagrange's identity, we get $\frac{d}{dx}[p(\phi_1{}'\phi_2 - \phi_1\phi_2')] = 0$. This implies

$$p(\phi_1{}'\phi_2 - \phi_1\phi_2') = c, \text{ is a constant} \qquad (2.26)$$

Since ϕ_1 and ϕ_2 satisfy the boundary conditions we get the following

$$\begin{vmatrix} \phi_1(a) & \phi_2(a) \\ \phi_1{}'(a) & \phi_2'(a) \end{vmatrix} = 0 \quad (2.27)$$

Since $(\phi_1\phi_2{}' - \phi_2\phi_1')$ Wronskian of two solutions of a second order ODE, it is identically equal to zero. From here, it follows that ϕ_1 and ϕ_2 differ by a constant multiple.

Theorem 2.7: A self-adjoint regular SL-BVP has an infinite sequence of real eigenvalues $(\lambda_n)_{n\in N}$ that are simple satisfying

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \dots \quad (2.28)$$

With $\lim_{n\to\infty} \lambda_n \to \infty$

2.2.2 Singular Sturm-Liouville Boundary Value Problems

Definition 2.9: A Sturm-Liouville boundary value problem a second ordered linear differential equation of the form

$$[p(x)y']' - q(x)y + \lambda r(x)y = 0, \quad x \in [a, b] \quad (2.29)$$

Where p, q, r are continuous functions on interval [a, b] and

λ is an unknown constant called the eigenvalue parameter together with boundary conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0,$$

$$\beta_1 y(b) + \beta_2 y'(b) = 0,$$

is called a singular SL-BVP if $p > 0$ on (a, b) and $p(a) = 0 = p(b)$, and $r \geq 0$ on [a, b].

2.2.3 Periodic Sturm-Liouville Boundary Value Problem

Definition 2.10: A second ordered linear differential equation of the form

$$[p(x)y']' - q(x)y + \lambda r(x)y = 0, \quad x \in [a, b]$$

Where p, q and r are continuous functions on interval [a, b] and

λ is an unknown constant called the eigenvalue parameter together with

boundary conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0,$$
$$\beta_1 y(b) + \beta_2 y'(b) = 0$$

is called a periodic Sturm-Liouville boundary value problem if $p(a) = p(b)$, $p > 0$ and $r > 0$ on $[a, b]$, coupled with boundary conditions

$$y(a) = y(b), \quad y'(a) = y'(b), \tag{2.30}$$

Example 6: For $\lambda \in R$, solve

$$y'' + \lambda y = 0, \ y(0) - y(\pi) = 0, \ y'(0) - y'(\pi) = 0. \tag{2.31}$$

This is not a SL-BVP. It is a mixed boundary condition unlike the separated BC above. These boundary conditions are called periodic boundary conditions.

Case 1: Let $\lambda < 0$.Then $\lambda = -\mu^2$, where μ is real and non-zero. In this case, it can be easily verified that trivial solution is the only solution of the BVP (2.31).

Case 2: Let $\lambda = 0$. In this case, general solution of ODE in (2.31) is given by

$$y(x) = A + Bx \tag{2.32}$$

This y satisfies the BCs in (2.31) if and only if $B = 0$. Thus A remains arbitrary.

Thus 0 is an eigenvalue with eigenfunction being any non-zero constant.

Note that eigenvalue is simple. An eigenvalue is called simple eigenvalue if the corresponding eigenspace is of dimension one, otherwise eigenvalue is called multiple eigenvalue.

Case 3: Let $\lambda > 0$. Then $\lambda = \mu^2$, where p is real and non-zero. The general solution of ODE in (2.31) is given by

$$y(x) = A\cos(\mu x) + B\sin(\mu x) \tag{2.33}$$

This y satisfies boundary conditions in (2.30) if and only if

$$A\sin(\mu x) + B(1 - \cos \mu x)) = 0,$$

$$A(1 - \cos(\mu x)) - B\sin(\mu x) = 0.$$

This has non-trivial solution for the pair (A, B) if and only if

$$\begin{vmatrix} \sin(\mu\pi) & 1 - \cos(\mu\pi) \\ 1 - \cos(\mu\pi) & -\sin(\mu\pi) \end{vmatrix} = 0. \tag{2.34}$$

That is, $\cos(\mu\pi) = 1$.

This further implies that $\mu = \pm 2n$ with $n \epsilon N$, and hence $\lambda = 4n^2$ with $n \epsilon N$.

Thus positive eigenvalues are given by

$$\lambda_n = 4n^2, \quad n \in N. \tag{2.35}$$

and the eigen functions corresponding to λ_n are given by

$$\phi_n(x) = \cos(2nx),$$

$$\psi_n(x) = \sin(2nx), \quad n\epsilon N. \tag{2.36}$$

Note: All the eigenvalues are non-negative. There are two linearly independent eigenfunctions, namely $cos(2nx)$, and $sin(2nx)$ corresponding to each positive eigenvalue

$\lambda_n = 4n^2$.

Therefore, the general solution of differential equation is $y(x) = c_1 \cos(2nx) + c_2 \sin(2nx)$.

Properties of periodic Sturm-Liuville problem

Theorem 2.8: The eigenvalues, if any, of a regular SL-BVP are real.

PROOF:

Suppose $\lambda \in C$ is an eigenvalue of a regular SL-BVP and let y be corresponding eigenfunction. That is, $L[y] + \lambda r(x)y = 0$

$$\alpha_1 y(a) + \alpha_2 p(a)y'(a) = 0,$$

$$\beta_1 y(b) + \beta_2 p(b)y'(b) = 0. \tag{2.37}$$

Taking the complex conjugates, we get

$$L[\bar{y}] + \lambda r(x)\bar{y} = 0$$

$$\alpha_1 \bar{y}(a) + \alpha_2 p(a)\bar{y}'(a) = 0,$$

$$\beta_1 \bar{y}(b) + \beta_2 p(b)\bar{y}'(b) = 0. \tag{2.38}$$

Multiply the ODE in (2.37) with $\bar{y}$, and multiply that of (2.37) with y, and subtracting one from the other yields

$$[p(y'\bar{y} - \bar{y}'y)]' + (\lambda - \bar{\lambda})ry\bar{y} = 0 \tag{2.39}$$

Integrating the last equality yields

$$[p[y'y - y'y)]|_a^b = -(\lambda - \bar{\lambda}) \int_a^b r(x)|y|^2\, dx. \tag{2.40}$$

But, LHS of the last equation is zero, since both y and $\bar{y}$ satisfy the periodic boundary conditions. Thus we have

$$(\lambda - \bar{\lambda}) \int_a^b r|y|^2 dy = 0$$

Since y, being an eigenfunction, is not identically equal to zero, and integral of non-negative function (since r > 0) is not zero, the only possibility is that $\lambda = \bar{\lambda}$.That is, λ is real.

Theorem 2.9: The eigenfunctions of a periodic SL-BVP corresponding to distinct eigenvalues are orthogonal w.r.t. weight function r on $[a, b]$, that is, if u and v are eigenfunctions corresponding to distinct eigenvalues λ and μ, respectively, then

$$\int_a^b r(x)u(x)v(x)dx = 0. \qquad (2.41)$$

PROOF:

As in the previous proof, writing down the equations satisfied by u and v, and multiplying the equation for u with v and vice versa, finally subtracting one from another, we get

$$[p(u'v - v'u)]' + (\lambda - \mu)ruv = 0 \qquad (2.42)$$

Integrating the last equality yields

$$[p(u'v - v'u)]|_a^b = -(\lambda - \mu)\int_a^b r(x)u(x)v(x)dx \qquad (2.43)$$

Reasoning exactly as in the previous proof, LHS of the above equality is zero. Since $\lambda \neq \mu$, we get the desire (2.41)

Definition 2.10: y(x) is said to be a normalized eigenfunction of a Sturm-Liouville problem if $\|y(x)\|_r = 1$,

so that $\int_a^b |y(x)|^2 r(x)dx = 1 \qquad (2.44)$

Equation (2.44) is called a normalization condition, and eigenfunctions satisfying this condition are said to be normalized. Indeed in this case, the eigenfunctions are said to form an orthonormal set (with respect to the weight function r).

Since they already satisfy the orthogonality relation (2.41)

Example: determine the normalized eigenfunctions of the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) = 0$$

Solution: The eigenvalues are $\lambda_n = n^2 \pi^2$, with corresponding eigenfunctions $y_n = \sin n\pi x$.

The eigenvalues of this problem are $\lambda_1 = \pi^2$, $\lambda_2 = 4\pi^2$, … , $\lambda_n = n^2\pi^2$, … , the corresponding eigenfunctions are $k_1 \sin \pi x$, $k_2 \sin 2\pi x$, …, $k_n \sin n\pi x$, …, respectively. In this case the weight function is r(x) = 1. To satisfy equation (2.44) we must choose k_n so that

$$\int_0^1 (k_n \sin n\pi x\,,)^2 dx = 1 \qquad (2.44^{**})$$

for each value of n. Since

$$k_n{}^2 \int_0^1 \sin^2(n\pi x)dx = k_n{}^2 \int_0^1 \left(\frac{1}{2} - \frac{1}{2}\cos 2n\pi x\right)dx = \frac{1}{2}k_n{}^2$$

Equation (2.44**) is satisfied if k_n is choose $\sqrt{2}$ for each value of n. hence the normalized eigenfunctions of the given boundary value problem are

$$y_n = \sqrt{2}\sin(n\pi x), \quad n = 1, 2, 3, \ldots \qquad (2.44^*)$$

Theorem 2.10: The eigenvalues of a periodic SL-BVP are not simple.

Theorem 2.11: A self-adjoint periodic SL-BVP has an infinite sequence of real eigenvalues satisfying

$$-\infty < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \ldots. \qquad (2.45)$$

The first eigenvalue λ_1 is simple. The number of linearly independent eigenfunctions corresponding to any eigenvalue μ is equal to the number of times μ is repeated in the above listing.

Definition 2.11 (Symmetric operators): Let L be a regular or periodic Sturm-Liouville operator, that is

$$\alpha y(a) + \beta y'(a) = 0,$$

$$\mu y(b) + v y'(a) = 0,\ \alpha^2 + \beta^2 > 0 \text{ and } \alpha^2 + v^2 > 0, \qquad (2.46)$$

or, in the periodic case, $y(a) = y(b), \quad y'(a) = y'(b)$ (2.47)

Define the domain of L as follow

Dom L = $\{u \in C^2(I)$: 2.46 or 2.47 holds$\}$

Definition 2.12: L: dom L → C (I) is said to be symmetric if <u, Lv> = <Lu, v>, $\forall_{u,v} \in$ dom of L.

2.3. Properties of Sturm-Liouville problems

Theorem 2.12: All the eigenvalues of Sturm-Liouville problem are real.

Proof: Let $Ly = \lambda ry$ (2.48)

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0,$$

$$\beta_1 y(b) + \beta_2 y'(b) = 0$$

Let us take the conjugate of (2.47) $L\bar{y} = \bar{\lambda} r\bar{y}$. By Lagrange's Identity:

$$0 = (\bar{y}, Ly) - (y, L\bar{y})$$

$$= (\bar{y}, \lambda ry) - (y, r\bar{\lambda}\bar{y})$$

$$= \int_a^b \bar{y}(x) r\lambda y(x)dx - \int_a^b y(x) r\bar{\lambda}\bar{y}(x)dx$$

$$= (\lambda - \bar{\lambda}) \int_a^b r(x)|y(x)|^2 dx$$

Since $r(x)|y(x)|^2 \geq 0$ it follows that $\lambda = \bar{\lambda} \Rightarrow \lambda$ is real.

2.3.1 The Rayleigh Quotient

We now develop a result technique for estimating eigenvalues, which is very useful for numerical computation.

Definition 2.13: (The Rayleigh Quotient):

Let L be the differential operator of the Sturm-Liouville boundary value problem

$$[p(x)y']' - q(x)y + \lambda r(x)y = 0,$$

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0,$$

$$\beta_1 y(b) + \beta_2 y'(b) = 0,$$

The expression

$$R(u) = \frac{\langle u, L[u]\rangle}{\langle u, u\rangle} = \frac{\int_a^b uL[u]dx}{\int_a^b u^2 r dx}$$ is called the Rayleigh quotient of u.

Theorem 2.13: $\lambda_j > 0$ provided $\frac{\alpha_1}{\alpha_2} < 0$, $\frac{\beta_1}{\beta_2} > 0$ and $q(x) > 0$.

Proof:

$\lambda_j > 0$ provided $\frac{\alpha_1}{\alpha_2} < 0$, $\frac{\beta_1}{\beta_2} > 0$ and $q(x) > 0$.

Consider $Ly = -(py')' + qy = \lambda ry$ (SL) and multiply (SL) by y and integrate from a to b:

$(y, Ly) = \int_a^b -(py')'y + qy^2 dx = \lambda \int_a^b r(x)[y(x)]^2 dx$

Therefore, $\lambda = \frac{\int_a^b -(py')'y + qy^2 dx}{\int_a^b ry^2 dx}$ is known as Rayleigh's Quotient.

$$= \frac{[-py'y]_a^b + \int_a^b (p\,(y')^2 + qy^2)dx}{\int_a^b ry^2\, dx}$$

$$= \frac{p(b)\frac{\beta_1}{\beta_2}[y(b)]^2 - p(a)\frac{\alpha_1}{\alpha_2}[y(a)]^2 + \int_a^b (p(y')^2 + qy^2)dx}{\int_a^b ry^2}$$

Therefore $\lambda > 0$ since the RHS is all positive.

Note:

If $q(x) = 0$ and $\alpha_1 = 0 = \beta_1$, then with $y'(a) = 0 = y(b)$ we have nontrivial eigenfunction $y(x) = 1$ and eigenvalue $\lambda = 0$.

Theorem 2.14: Eigenfunctions: For each λ_j there is an eigenfunction ϕ_j (x) that is unique up to a multiplicative constant, and which satisfy:

(a) $\phi_j(x)$ are real and can be normalized so that $\int_a^b r(x)\, \phi_j^2(x)dx = 1$

(b) The eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the weight function r(x). That is,

$$\int_a^b r(x)\phi_j\phi_k(x)dx = 0, \;\; j \neq k$$

Proof:

Eigen functions corresponding to different eigenvalues are orthogonal.

Consider two distinct eigen values $\lambda_j \neq \lambda_k$, $\lambda_j : L\phi_j = r\lambda_j\phi_j \; and \lambda_k : L\phi_k = r\lambda_k\phi_k$. Then,

$$0 = (\phi_k, L\phi_j) - (\phi_j, L\phi_k) \text{ by Lagrange's Identity}$$

$$= (\phi_k, r\lambda_j\phi_j) - (\phi_j, r\lambda_k\phi_k)$$

$$= (\lambda_j - \lambda_k)\int_a^b r(x)\phi_k(x)\phi_j(x)\, dx$$

Now $\lambda_j \neq \lambda_k$ implies that

$$\int_a^b r(x)\phi_k(x)\phi_j(x)\, dx = 0$$

2.3.2 Generalized Fourier series

Consider the regular SL-problem

$$[p(x)y']' - q(x)y + \lambda r(x)y = 0,$$

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0,$$

$$\beta_1 y(b) + \beta_2 y'(b) = 0$$

We know that the eigenvalues λ_n form an increasing sequence and $lim_{n\to\infty} \lambda_n = \infty$ and for each n, the eigenspace has dimension 1 (Theorem 2.7). Let y_n(x) be an eigenfunction corresponding to the eigenvalue λ_n. By analogy with Fourier series, we are going to associates to each piecewise continuous function f on [a, b] a series in the eigen functions λ_n:

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$$

Where the coefficients c_n are given by $c_n = \frac{\langle f, y_n\rangle_r}{\|y_n\|_r^2} = \frac{\int_a^b r(x)f(x)y_n(x)dx}{\int_a^b r(x)y_n^2(x)dx}$

2.3.3 Completeness

Definition 2.13: An orthogonal sequence $\{\phi_n\}$ of a nonzero functions in C[a, b] is called a complete system for C[a, b] with respect to weight function r if every f ∈ C[a, b] can be written as

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

Where the equality above is in the sense that

$\int_a^b |f(x) - \sum_{n=1}^{N} c_n \phi_n(x)|^2 r(x)dx \rightarrow 0$ as $N \rightarrow \infty$

It can seen that $c_n = \frac{\langle f, \ \phi_n \ \rangle_r}{\langle f_n, \phi_n \rangle_r}$

The eigenfunctions of a Sturm-Liouville problem can be used to describe piecewise continuous functions, which is very useful for solving time dependent PDE for which separation of variables yields a Sturm-Liouville problem.

Theorem 2.15: (Completeness Theorem): The set of eigenfunctions is complete; i.e ,.any piecewise smooth function can be represented by a generalized Fourier series expansion of the eigenfunctions,

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

Where, $c_n = \frac{\int_a^b r(x) f(x) \phi_n(x) dx}{\int_a^b r(x) \phi_n^2(x) dx}$

Actually, one needs f(x) ∈ $L_r^2(a, b)$, the set of square integrable functions over [a, b] with weight function r(x). By square integrable, we mean that $\langle f, f \rangle < \infty$. One can show that such a space is isomorphic to a Hilbert space, a compete inner product space.

CONCLUSION

The theory developed here may be summarized as follows. For a regular Sturm-Liouville problem with separable boundary conditions, their exist in finitely many real eigen values. The eigen functions corresponding to distinct eigenvalues are orthogonal, and the set of all eigen functions is complete in the sense that every square-integrable function f can be expanded in terms of the eigen functions.

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