LEVEL GENERATING TREES
AND
PROPER RIORDAN ARRAYS

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A Project submitted in partial fulfillment of the requirement for the degree of master of science in mathematics

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Contents

Table of Contents **I**

Acknowledgement **II**

Abstract **III**

Introduction **1**

1 Preliminary **4**

1.1 Basic Concepts In Graphs 4
1.1.1 Definitions and Examples 4
1.1.2 Loop and Multiple Edges 5
1.1.3 Simple Graph 5
1.1.4 Adjacent Vertices 5
1.1.5 Degree of a Vertex 5
1.1.6 Path 5
1.1.7 Circuit 6
1.1.8 Connected Graph 6
1.1.9 Disjoint Graphs 6
1.2 Trees 6
1.3 Formal power series 8
1.3.1 Order of a formal power series 8
1.3.2 The Algebraic Structures Of Formal Power Series 8
1.4 Generating Functions 10
1.4.1 Definitions and Examples of generating functions 10
1.5 Coefficient Extraction 10
1.5.1 Properties of generating functions 11
1.6 Matrix representation corresponding to formal power series 13
1.7 Lagrange Inversion Formula(LIF) 13
1.8 Catalan Numbers 14
1.9 Fibonacci Numbers 17

2 Level Generating Trees And Proper Riordan Arrays **18**
2.1 Definitions and Examples of Riordan Arrays 18
2.1.1 Computation of Combinatorial Sums Using Riordan Arrays 21
2.1.2 Some Examples Related To Binomial Coefficients 23
2.1.3 Algebraic Structure Of Riordan Arrays 25
2.1.4 Subgroups Of The Riordan Group 27
2.1.5 Sequence Characterization Of Riordan Arrays 27
2.2 Level Generating Trees 32
2.2.1 Definitions and Examples Of Generating Trees 32
2.2.2 Generating Trees 32
2.2.3 Marked Generating Trees 34
2.3 Level Generating Trees and Proper Riordan Arrays 35
2.3.1 Relation Between Level Generating Trees and Proper Riordan Arrays 35
2.3.2 Matrix Associated to a Generating Tree 35

Conclusion **43**

Reference **44**

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Abstract

In this project,based on articles published by D.Merlini, R.Sprugnoli , D.G. Rogers and M.C.Verri some important relation between level generating trees and proper Riordan Arrays are discussed. This work focuses on the concepts of the theory of Riordan Arrays and a generalization of generating trees named level generating trees to describe the concept of a matrix associated to a generating tree (AGT matrix), AGT matrices, matrices associated to level generating tree, in relation to proper Riordan Arrays. Some AGT matrices which are proper Riordan Arrays and Riordan Arrays which are discussed, conditions that allow they are also given, and discussed, illustrated with examples by using classical combinatorial structures.

Introduction

The concept of Riordan Arrays and generating trees are very useful in combinatorics.The relation between Riordan Arrays with transfer matrices and level generating trees with transfer matrices are important to study the relation between a level generating trees and proper Riordan Arrays. In this work, the infinite triangles of Pascal, and Catalan triangles are important and meaningful examples.

A Riordan Array is an infinite lower triangular matrix $(d_{n,k})_{n,k\in\mathbb{N}}$ including 0 defined by two formal power series $(d(t), h(t))$,where $d(t) \in \mathcal{F}_0$, $h(t) \in \mathcal{F}$. If $d(t), h(t) \in \mathcal{F}_0$, then a Riordan array is called proper.The concept of a Riordan Array provides a remarkable characterization of many lower triangular arrays that arise in combinatorics and algorithm analysis. The theory of Riordan arrays has been introduced in 1991 by Shapiro, Getu, Woan and Woodson [6], with the aim of generalizing the concept of a renewal array defined by Rogers [11] in 1978. Their basic idea was to define a group of infinite lower triangular arrays with properties analogous to the infinite triangles such as Pascal and Catalan triangle. This concept has also been studied by Sprugnoli [3], who pointed out the relevance of these matrices from a theoretical and practical point of view.

Riordan Arrays have attracted the attention of various authors and many examples and applications can be found [1,5,9,10,14,15]. Most of them deal with the original formulation of Riordan arrays, that is, in the corresponding matrices each element $d_{n+1,k+1}$ is given by a linear combination of the elements in the previous row, starting from the previous column.
$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + ... = \sum_{i=0}^{n-k} a_i d_{n,k+i}, \forall i$,
The coefficients of this linear combination are independent of n and k, and therefore they constitute a specific sequence $(a_0, a_1, a_2, ...)$, called the A-sequence of the Riordan array. For example, in the Catalan triangle we have $A = (1, 1, 1, ...)$, in fact $C_{n+1,k+1} = \sum_{j\geq k} c_{n,j}$, as can be easily verified from Table 1.

A generating tree is a rooted, labelled and typically infinite tree such that the label of a node determine the label of its child with the property that if v_1 and v_2 are any two nodes with the same label, then for each label l, v_1 and v_2 have exactly the same number of children with label l.
In order to specify a generating tree it therefore suffices to specify:

1. the label of the root.

2. a set of rules explaining how to derive from the label of a parent the labels of all of its children.

In this project, we introduce the concept of matrix associated to a generating tree (AGT matrix, for short), this is an infinite matrix $(d_{n,k})_{n,k\in\mathbb{N}}$ where $d_{n,k}$ is the number of nodes at level n with label $k+c$, where c being the label of the root. In [1] and [13], Merlini, Sprugnoli and Verri have introduced the concept of matrix associated to a generating tree (AGT matrix, for short).

For example, the following labeled tree (this example concerns only non-marked nodes) corresponds to the generating tree, up to level 4, by the following specification is:

$$\begin{cases} root : (1), \\ rule : (k) \rightarrow (1)...(k)(k+1), \end{cases} \tag{1}$$

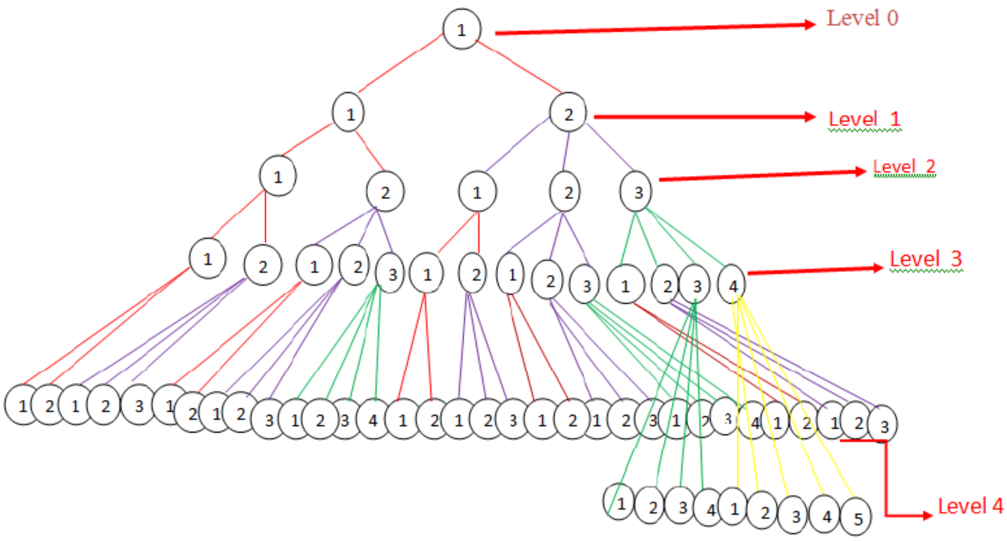

Figure 1: Generating tree of specification (1)

This is known as the Catalan generating tree: in fact, the corresponding AGT matrix, shown in Table 1, is strictly related to the generating function of Catalan numbers, $C(t) = \frac{1-\sqrt{1-4t}}{2t}$.

The generic element $C_{n,k}$ in the array is given by:

$$C_{n,k} = [t^{n-k}]C(t)^{k+1},$$

that is, $(C_{n,k})$ is the Riordan array $(C(t), tC(t)) = (\frac{1-\sqrt{1-4t}}{2t}, \frac{1-\sqrt{1-4t}}{2})$.

n/k	0	1	2	3	4	
0	1					
1	1	1				
2	2	2	1			
3	5	5	3	1		
4	14	14	9	4	1	
5	42	42	28	14	5	1

Table 1: The catalan triangle

The structure of this project is as follows. The first chapter will introduce the concept of graphs, formal power series and generating functions. Chapter two will focus on the concepts of Riordan Arrays, the concepts of generating trees named level generating trees, and the relation of level generating trees and proper Riordan Arrays.

Chapter 1

Preliminary

1.1 Basic Concepts In Graphs

Graphs are among the most basic of all mathematical structures that have many different versions and representations.

1.1.1 Definitions and Examples

Definition 1.1.1. *A graph is an ordered pair of the sets V, E denoted by G=(V,E), where V is the set of vertices or nodes or points, E is the set(possibly empty) of edges or arcs or lines, and each $e \in E$* ***connects*** *two vertices $v, w \in V$.*

Example 1.1.1. *The following figures are examples of graph*

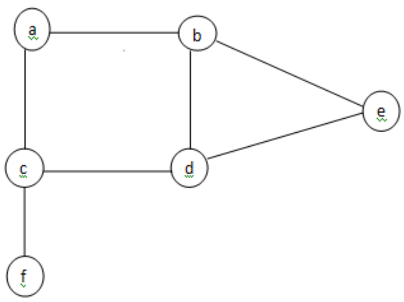

Figure 1.1:

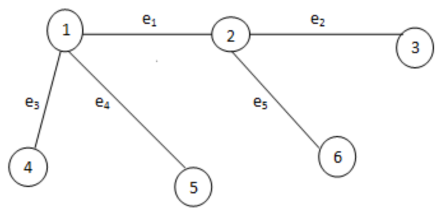

Figure 1.2:

From the above **figure 1.1,** V={a,b,c,d,e,f} and $E = \{ab, ac, bd, be, cd, cf, de\}$ and from the above **figure 1.2,** V={1,2,3,4,5,6} and $E = \{e_1, e_2, e_3, e_4, e_5\}$, where V is Vertices of the graph and E is Edges of the graph.
The lines are called **edges** if they are **undirected**, and or **arcs** if they are directed.

Some Other Basic Definitions In Graph

1.1.2 Loop and Multiple Edges

A loop is an edge that connects a vertex to itself. If a graph has more than one edge joining some pair of vertices then these edges are called **multiple (parallel) edges**.

1.1.3 Simple Graph

A graph with no loops and no multiple(parallel) edges is known as a **simple graph**.

1.1.4 Adjacent Vertices

Two vertices are said to be **adjacent** if there is an **edge (arc)** connecting them. **Adjacent edges** are edges that share a common vertex.

1.1.5 Degree of a Vertex

The Degree of a vertex is the number of edges incident with that vertex.

1.1.6 Path

A path is a sequence of vertices with the property that each vertex in the sequence is adjacent to the vertex next to it. A path that does not repeat vertices is called a simple path.

1.1.7 Circuit

A circuit is path that begins and ends at the same vertex. A Circuit that doesn't repeat vertices is called **a cycle**.

1.1.8 Connected Graph

A graph is said to be **connected** if any two of its vertices are joined by a path. A graph that is not connected is **a disconnected** graph. A disconnected graph is made up of connected subgraphs that are called components.

1.1.9 Disjoint Graphs

Two or more graphs are said to be **disjoint graphs** if they share no edges and no vertices.

1.2 Trees

Definition 1.2.1. *A tree is a connected acyclic graph. A tree in which one vertex is designated as the root distinguished from all the others is called a rooted tree.*

Example 1.2.1. *The following are examples of trees with roots* r_1 *and* r_2 *respectively*

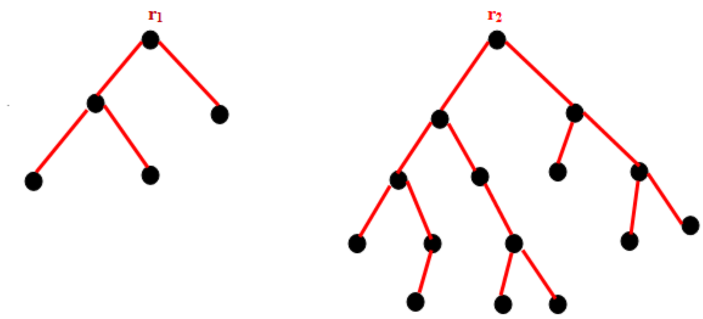

Figure 1.3: :Two examples of tree

A parent under a tree: *A parent of a node is the immediate predecessor of a node.* ***Children under a tree:*** *All the immediate successor of a parent node are known as children.*

If $T_1, T_2, ..., T_t$ *are disjoint trees with roots* $r_1, r_2, ..., r_t$ *respectively the graph formed by attaching a new vertex* r *by a single edge to each* $r_1, r_2, ..., r_t$ *is a tree with root* r*. The roots* $r_1, r_2, ..., r_t$ *are* ***children*** *of* r*, and* r *is* ***a parent*** *of* $r_1, r_2, ..., r_t$*.*

Example 1.2.2.

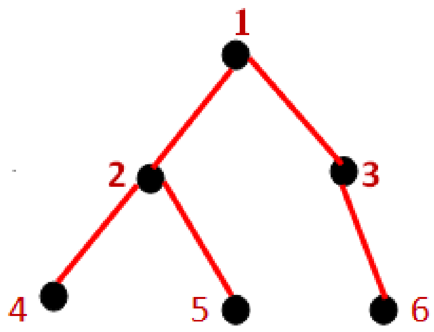

Figure 1.4:

*From **figure 1.4** above, 1 is a parent of 2 and 3;*
2 and 3 are children of 1, 2 is a parent of 4 and5,
4 and 5 are children of 2.

Leaf: *A vertex(node) with no children is called a **leaf** of the tree.*

*For example,from **figure 1.4** above,nodes 4,5 and 6 are leaves of the given tree.*

Internal nodes: *In a tree all non-leaf nodes are called internal nodes.*

Path of tree: *A sequence of edges, each edge starts from with the node where the previous edge ends.*

Length of path: *The length of path is the number of edges in the path.*

The height of tree: *The height of tree is the height of root.*

Depth(or level) of a node: *The depth of a node in a tree is the length of the path from the root to the node, the root itself has depth zero.*

Depth of a tree: *The depth of tree is the length of longest path from root to a leaf.*

1.3 Formal power series

Definition 1.3.1. *Let $\mathbb{F}$ be a field of characteristic 0 and let t be any indeterminate over $\mathbb{F}$(i.e a symbol different from any element in $\mathbb{F}$), Then,a formal power series over $\mathbb{F}$ in the indeterminate t is an expression:*
$f(t) = f_0 + f_1t + f_2t^2 + ... + f_nt^n + ... = \sum_{k=0}^{\infty} f_kt^k$,$\forall f_k \in \mathbb{F}$, $k = 0, 1, 2, ...$
Notation:*The set of all formal power series on $\mathbb{F}$ in the indeterminate t is denoted by $\mathbb{F}[t]$, where $\mathbb{F}$ can be real numbers($\mathbb{R}$) or complex numbers($\mathbb{C}$). In fact,in Combinatorial analysis the coefficients $f_0, f_1, f_2, ...$ of a formal power series are mostly used to count objects and therefore they are positive integers,or in some case positive rational numbers(for example when they are the coefficients of an exponential generating functions).*

1.3.1 Order of a formal power series

Definition 1.3.2. *The order of a formal power series $f(t)$ is the smallest index 'r' for which $f_r \neq 0$.*

Example 1.3.1. *If $f(t) = 1 + t + t^2 + t^3 ...$,then order of $f(t)$ is 0.*

Notation:

1. The set of all formal power series of order r is denoted by $\mathbb{F}_r[t]$ or $\mathbb{F}_r$,where $r = ord(f)$,$f \in \mathbb{F}_r$.
2. The set of all formal power series of order zero is denoted by $\mathbb{F}_0[t]$ or $\mathbb{F}_0$,where $ord(f) = 0$,$f \in \mathbb{F}_r$.

1.3.2 The Algebraic Structures Of Formal Power Series

The set $\mathcal{F}$ of formal power series can be embedded into several algebraic structures.
Let us define some operations on the set of all formal power series $\mathcal{F}$ as follows:
Let $f(t) = \sum_{k=0}^{\infty} f_kt^k \in \mathcal{F}$,$g(t) = \sum_{k=0}^{\infty} g_kt^k \in \mathcal{F}$, Then,

1. **Sum(+)**:
   $f(t) + g(t) = \sum_{k=0} \infty(f_k + g_k)t^k$.

   From this definition, it immediately follows that $\mathcal{F}$ is a commutative group with respect to "+".
   i.e $f(t) + g(t) = \sum_{k=0} \infty(f_k + g_k)t^k = \sum_{k=0} \infty(g_k + f_k)t^k = g(t) + f(t)$.

   The **identity element** with respect to the operation "+" is the formal power series $0 = 0 + 0t + 0t^2 + 0t^3 + ... = \sum_{k=0}^{\infty} 0t^k$.

   The **inverse** of formal power series $f(t)$ with respect to the operation "+" is the formal power series $-f(t) = \sum_{k=0}^{\infty}(-f_k)t^k$.

2. **Cauchy Product(.)**
$f(t)g(t) = (\sum_{k=0}^{\infty} f_k t^k)(\sum_{k=0}^{\infty} g_k t^k) = \sum_{n=0}^{\infty}(\sum_{k=0}^{n} f_k g_{n-k})t^n$.

We can write down explicitly the first term of the cauchy product as follow:

$f(t)g(t) = f_0g_0 + (f_0g_1 + f_1g_0)t + (f_0g_2 + f_1g_1 + f_2g_0)t^2 + (f_0g_3 + f_1g_2 + f_2g_1 + f_3g_0)t^3 + ...$

This shows that the product is commutative with respect to the operation "**.**"

The **identity element** with respect to the operation "." is the formal power series
$1 = 1 + 0t + 0t^2 + 0t^3 + ... = 1 + \sum_{k=1}^{\infty} 0t^k$
i.e $1.f(t) = (1 + \sum_{k=1}^{\infty} 0t^k)(\sum_{k=0}^{\infty} f_k t^k) = \sum_{k=0}^{\infty} f_k t^k = f(t)$, $\forall f(t) \in \mathcal{F}$

The order of identity 1 is zero(0).

Let $f(t)$ is an invertible element in $\mathcal{F}$,then there exist $f(t)^{-1} \in \mathcal{F}$ which is the inverse of $f(t)$,we should have $f(t)f(t)^{-1} = 1$ and therefore $ord(f(t)) = 0$
i.e if $f(t) \in \mathcal{F}_0$ (i.e $f(t) = f_0 + f_1t + f_2t^2 + ...$) with $f_0 \neq 0$,then $f(t)$ is invertible.

This shows that the power series is invertible if its order is zero.Because of that $\mathcal{F}_0$ is called the set of invertible formal power series.
Note that every $f(t) \in \mathcal{F}_0$ is invertible.

3. **Composition(o):**
Let $f(t) \in \mathcal{F}$ and $g(t) \notin \mathcal{F}_0$,then we define the "composition" of $f(t)$ by $g(t)$ as a formal power series:
$f(g(t)) = \sum_{k=0}^{\infty} f_k g(t)^k$

This definition justifies the fact that $g(t)$ cannot belongs to $\mathcal{F}_0$.

The formal power series , in $\mathcal{F}_1$ have a particular relevance for composition.

First of all,we wish to observe that the formal power series $t \in \mathcal{F}_1$ acts as an identity for composition.
As a second fact, we wish to observe that a formal power series, $f(t)$ has an inverse with respect to composition if and only if $f(t) \in \mathcal{F}_1$.
Note that $g(t)$ is the inverse of $f(t)$ if and only if $f(g(t)) = t$ and $g(f(t)) = t$.

From this we observe that $f(t) \notin \mathcal{F}_0$ and $g(t) \notin \mathcal{F}_0$.

1.4 Generating Functions

1.4.1 Definitions and Examples of generating functions

Generating functions are powerful tools for solving a number of problems in combinatorics.

Definition 1.4.1. *Given a sequence* $F = (f_0, f_1, f_2, ..., f_n, ...)$, *the generating function of* F, *denoted by* $\mathcal{G}(f_k)$ *is the formal power series* $\sum_{k=0}^{\infty} f_k t^k = \mathcal{G}(f_k) = f(t)$

In this case, $\mathcal{G}$ *is called the generating function* ***operator.***

Definition 1.4.2. ***The Ordinary Generating Function*** $(O.g.f)$ *of a sequence* $\{f_n\}_{n\geq 0}$ *is the formal power series:* $f(t) = \sum_{n=0}^{\infty} f_n t^n$

Example 1.4.1. *For any positive integer* m, *the generating function for the binomial coefficients* $\binom{m}{0}, \binom{m}{1}, \binom{m}{2}, ..., \binom{m}{m}, 0, 0, 0, ...$ *is the function* $f(t) = (1+t)^m = \sum_{n=0}^{\infty} \binom{m}{n} t^n$

Note:The Binomial expansion of the expression$(x+y)^n$ is:
$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + ... + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n$

Using the summation notation: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$,where each $\binom{n}{k}$ is a specific positive integers known as **a binomial coefficient**.

Example 1.4.2. *The function* $f(t) = 1 + t + t^2 + t^3 + ... = \sum_{k=0}^{\infty} t^k = \frac{1}{1-t}$ *is the generating function of the constant infinite sequence* $1, 1, 1, ...$*since* $\sum_{k=0}^{\infty} t^k = \frac{1}{1-t}$

Definition 1.4.3. ***The exponential Generating Function*** $(e.g.f)$ *of a sequence* $\{f_n\}n \geq 0$ *is the formal power series:* $\sum_{k=0}^{\infty} f_k \frac{t^k}{k!}$.

Example 1.4.3. *The e.g.f of the factorial numbers* $f_n = \frac{(2n)!}{n!}$ *is given by* $f(t) = \frac{1}{\sqrt{1-4t}}$.

1.5 Coefficient Extraction

Let $f(t) \in \mathcal{F}$, the notation $[t^n]f(t)$ indicates the **extraction** of the **coefficient** of $[t^n]$ from $f(t)$, and therefore we have:
$[t^n]f(t) = f_n$, $n \in \mathbb{N}$

In this case, $[t^n]$ is called an **operator** and exactly the coefficient operator.

The following are some of main properties of the "coefficient" operator.

Let $f(t)$ be the generating function of a sequence $\{f_k\}_{k\in\mathbb{N}}$, and $g(t)$ be the generating function of a sequence $\{g_k\}_{k\in\mathbb{N}}$ then,

1. **Linearity:**
   $[t^n](\alpha f(t) + \beta g(t)) = \alpha[t^n]f(t) + \beta[t^n]g(t)$,where $\alpha, \beta \in \mathbb{R}$ or $\alpha, \beta \in \mathbb{C}$.

2. **Shifting**: For any $k \in \mathbb{N}$,

i. **Right-shifting:**
   $[t^n]t^k f(t) = [t^{n-k}]f(t)$.

ii. **Left-shifting:**
   $[t^n]\frac{f(t)}{t^k} = [t^{n+k}]f(t)$.

3. **Differentiation**:
   $[t^n]f'(t)$ can be represented by $[t^n]Df(t)$

   Let $f(t) = \sum_{k=0}^{\infty} f_k t^k$.Then,

   $Df(t) = \sum_{k=1}^{\infty} k f_k t^{k-1} = \sum_{k=0}^{\infty}(k+1) f_{k+1} t^k$

   $\Rightarrow [t^n]f'(t) = (n+1)[t^{n+1}]f(t)$
   $\Rightarrow [t^{n-1}]f'(t) = n[t^n]f(t)$.

4. **Convolution or Cauchy product**
   $[t^n]f(t)g(t) = \sum_{k=0}^{n} f_k g_{n-k} = \sum_{k=0}^{n}[t^k]f(t)[t^{n-k}]g(t)$

5. **Composition**
   $[t^n]f(g(t)) = [t^n]\sum_{k=0}^{\infty} f_k g(t)^k = \sum_{k=0}^{\infty}[t^k]f(t)[y^n]g(y)^k$,$g(t) \notin \mathcal{F}_0$.

1.5.1 Properties of generating functions

The following are some of the main properties of generating function by considering $f(t)$ and $g(t)$ as formal power series. Let $f(t)$ be the generating function of a sequence $\{f_k\}_{k\in\mathbb{N}}$, $g(t)$ be the generating function of a sequence $\{g_k\}_{k\in\mathbb{N}}$ then,

1. **Linearity:**
   $\mathcal{G}(\alpha f_k + \beta g_k) = \alpha\mathcal{G}(f_k) + \beta\mathcal{G}(g_k)$,where α and β are constant.

2. **Shifting:**

i. **Right-shifting**
   $\mathcal{G}(f_k) = f_0 + f_1 t + f_2 t^2 + f_3 t^3 + ...$

   $\therefore t\mathcal{G}(f_k) = f_0 t + f_1 t^2 + f_2 t^3 + ...$ (shifting to the right).

ii. **Left-shifting**

$\mathcal{G}(f_k) = f_0 + f_1t + f_2t^2 + f_3t^3 + ...$

$\Rightarrow \mathcal{G}(f_{k+1}) = \frac{\mathcal{G}(f_k)-f_0}{t}$

Now,$\frac{\mathcal{G}(f_k)-f_0}{t} = \frac{1}{t}[f_1t + f_2t^2 + f_3t^3 + ...] = f_1 + f_2t + f_3t^2 + ...$ (shifting to the left).

3. **Differentiation:**

$f(t) = f_0 + f_1t + f_2t^2 + f_3t^3 + ... + f_kt^k + ...$

$\Rightarrow f'(t) = f_1 + 2f_2t + 3f_3t^2 + ... + kf_kt^{k-1} + ...$

$\Rightarrow [t^k]f'(t) = (k+1)[t^{k+1}]f(t)$

$\Rightarrow [t^k]f'(t) = (k+1)f_{k+1}$

$\Rightarrow [t^{k-1}]f'(t) = kf_k = k[t^k]f(t)$

$\therefore tD\mathcal{G}(f_k) = \mathcal{G}(kf_k)$.

4. **Cauchy product**

$f(t)g(t) = \sum_{n=0}^{\infty}(\sum_{k=0}^{n} f_kg_{n-k})t^n$

$\Rightarrow [t^n]f(t)g(t) = \sum_{k=0}^{\infty}[t^k]f(t)[t^{n-k}]g(t)$

$\therefore \mathcal{G}(f_k)\mathcal{G}(g_k) = \mathcal{G}(\sum_{k=o}^{n} f_kg_{n-k})$.

5. **Composition**

$f(t)og(t) = f(g(t)) = \sum_{k=0}^{\infty} f_kg(t)^k$,$g(t) \notin \mathcal{F}_0$

$\Rightarrow [t^n]f(t)og(t)) = [t^n]\sum_{k=0}^{\infty} f_kg(t)^k = \sum_{k=0}^{\infty}[t^k]f(t)[y^n]g(y)^k$

$\therefore \mathcal{G}(f_k)o\mathcal{G}(g_k) = \sum_{k=0}^{\infty} f_k(\mathcal{G}(g_k))^k$.

6. **Diagonalization:**

For $w(t) = t\phi(t)$ is given,

$\mathcal{G}([t^n]F(t)\phi(t)^n) = \frac{F(w)}{1-t\phi'(w)}$,$w = t\phi(w), F \in \mathcal{F}$.

1.6 Matrix representation corresponding to formal power series

Let $f(t) \in \mathcal{F}_0$,and let $D = (d_{n,k})_{n,k\in\mathbb{N}}$ be an infinite lower triangular array, where $d_{n,k} = [t^{n-k}]f(t)$, $\forall n, k \in \mathbb{N}$,where n stands for row and k stands for column numbers:
Column 0 contains the coefficients $f_0, f_1, f_2, ...,$
Column 1 contains the same coefficients shifted down by one position and $d_{0,k} = 0$, $\forall k > 0$,
Column 2 contains the same coefficients shifted down by two position and $d_{1,k} = 0$, $\forall k > 1$ and Proceeding in the same way , column k contains the coefficients of $f(t)$ shifted down by k position and $d_{n,k} = 0$, $\forall k > n$, so that the first k positions are 0 and the array D denoted by $(f(t), 1) = (d_{n,k})_{n,k\in\mathbb{N}}$:

$$D = (f(t), 1) = (d_{n,k})_{n,k\in\mathbb{N}} = \begin{bmatrix} f_0 & 0 & 0 & 0 & 0 & . & . & . \\ f_1 & f_0 & 0 & 0 & 0 & . & . & . \\ f_2 & f_1 & f_0 & 0 & 0 & . & . & . \\ f_3 & f_2 & f_1 & f_0 & 0 & . & . & . \\ f_4 & f_3 & f_2 & f_1 & f_0 & . & . & . \\ . & . & . & . & . & . & & \\ . & . & . & . & . & & . & \\ . & . & . & . & . & & & . \end{bmatrix}$$

From this infinite, lower triangular array(matrix), we observe that:
$d_{n,k} = f_{n-k} = [t^{n-k}]f(t)$ is the $(k+1)^{th}$ column of the array.

1.7 Lagrange Inversion Formula(LIF)

The Lagrange Inversion Formula can be applied in a variety of ways such as:

•Determining the generating function of many combinatorial sequences.

•In the extraction of combinatorial sums.

•In the process of carrying out the inversion of combinatorial identities.

If $w = w(t)$ is a formal power series satisfying the functional equation $w(t) = t\phi(w(t))$ with $\phi(t) \in \mathcal{F}_0$, then the Lagrange Inversion Formula states that:
$[t^n]w(t)^k = \frac{k}{n}[t^{n-k}]\phi(t)^n$.

Let $f(t) \in \mathcal{F}_1$, Then, by setting $f(t) = \frac{t}{\phi(t)}$ it follows that $f(w(t)) = \frac{w(t)}{\phi(w(t))} = t$.

Therefore,w(t) can be considered the compositional inverse of $f(t)$. We therefore know that $w(t)$ is the uniquely determined and the LIF give us:
$[t^n]w(t) = \frac{1}{n}[t^{n-1}](\frac{t}{f(t)})^n = \frac{1}{n}[t^{n-1}]\phi(t)^n$. Let $F(t) \in \mathcal{F}$ and let us consider the composition $F(w(t))$, where $w = w(t)$, the solution to the functional equation $w = t\phi(w)$, with $\phi(w) \in \mathcal{F}_0$, Then for the coefficient of t^n in $F(w(t))$ we have:

$$
\begin{aligned}
[t^n]F(w(t)) &= [t^n]\sum_{k=0}^{\infty} F_k w(t)^k \\
&= \sum_{k=0}^{\infty} F_k [t^n] w(t)^k \\
&= \sum_{k=0}^{\infty} F_k \frac{k}{n} [t^{n-k}]\phi(t)^n \\
&= \frac{1}{n}[t^{n-1}](\sum_{k=0}^{\infty} kF_k t^{k-1})\phi(t)^n \\
&= \frac{1}{n}[t^{n-1}]F'(t)\phi(t)^n
\end{aligned}
$$

$$
\therefore [t^n]F(w(t)) = \frac{1}{n}[t^{n-1}]F'(t)\phi(t)^n.
$$

1.8 Catalan Numbers

The Catalan numbers are a sequence of positive integers that appear in many counting problems in combinatorics. They count certain types of combinatorial objects. They satisfy a fundamental recurrence relation, and have a closed-form formula in terms of binomial coefficients.
The first few Catalan numbers are 1,1,2,5,14,42,132...
The catalan numbers satisfy the recurrence relation:
$C_0 = 1$ and $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$
There are various ways of obtaining a simple formula for C_n, one of which is by using the generating function.
Let's define $C(t)$ to be the generating function of catalan numbers
i.e $C(t) = \sum_{n\geq 0} C_n t^n = 1 + t + 2t^2 + 5t^3 + 14t^4 + ...$
To derive a closed form for C_n from the recurrence relation,start by multiplying both sides of the catalan recurrence relation by t^n and summing up infinitely starting from zero as follows:
$\sum_{n=0}^{\infty} C_{n+1}t^n = \sum_{n=0}^{\infty}(\sum_{k=0}^{n} C_k C_{n-k})t^n$

$\frac{1}{t}\sum_{n=0}^{\infty} C_{n+1}t^{n+1} = C(t)^2$ (from Cauchy product rule)

$\frac{1}{t}\sum_{n=1}^{\infty} C_n t^n = C(t)^2$

$\frac{1}{t}(\sum_{n=0}^{\infty} C_n t^n - C_0) = C(t)^2$ but $C_0 = 1$

$\frac{1}{t}(\sum_{n=0}^{\infty} C_n t^n - 1) = C(t)^2$

$\frac{1}{t}(C(t) - 1) = C(t)^2$

$C(t) - 1 = tC(t)^2$

$tC(t)^2 - C(t) + 1 = 0$

The generating function $C(t) = \sum_{n=0}^{\infty} C_n t^n$ is the solution of the quadratic equation; $tC(t)^2 - C(t) + 1 = 0$ with $C(0) = 1$.
Now,from the quadratic equation $tC(t)^2 - C(t) + 1 = 0$ we have:

$C(t) = \frac{1 +- \sqrt{1-4t}}{2t}$

In fact, we must choose the minus sign here, otherwise the coefficients of the powers of t in the generating function of $C(t)$ become negative, whereas we want $C(t)$ to be the generating function of the Catalan numbers, all of which are positive.
Therefore,

$$C(t) = \frac{1 - \sqrt{1-4t}}{2t} \tag{1.1}$$

Indeed, if we expand the square root $\sqrt{1-4t} = (1-4t)^{1/2}$ as a formal power series using the binomial formula,then we have:

$$\sqrt{1-4t} = (1-4t)^{1/2} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} (-4t)^k = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} (-1)^k 4^k t^k \tag{1.2}$$

But

$$\begin{aligned}
\binom{\frac{1}{2}}{k} &= \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)...(\frac{1}{2}-k+1)}{k!} \\
&= \frac{(\frac{1}{2})(\frac{-3}{2})(\frac{-5}{2})...(\frac{3-2k}{2})}{k!} \\
&= \frac{(-1)^{k-1}(1)(3)(5)...(2k-3)}{2^k k!} \\
&= \frac{(-1)^{k-1}(1)(3)(5)...(2k-3)}{2^k k!} * \frac{(2k-1)(2)(4)(6)...(2k)}{(2k-1)(2)(4)(6)...(2k)} \\
&= \frac{(-1)^{k-1}(2k)!}{(2k-1)2^k 2^k k! k!} \\
&= \frac{(-1)^{k-1}(2k)!}{(2k-1)4^k k! k!}
\end{aligned}$$

Now, by substituting into equation (1.2) we have:

$$\sqrt{1-4t} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} (-1)^k 4^k t^k = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}(2k)!}{(2k-1)4^k k! k!} (-1)^k 4^k t^k = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}(2k)!}{(2k-1)k! k!} (-1)^k t^k \tag{1.3}$$

By inserting equation (1.3) into equation (1.1) and since $C(0) = 1$,we obtain:

$$\begin{aligned}
C(t) = \frac{1-\sqrt{1-4t}}{2t} &= \frac{1}{2}t^{-1}(1 - \sum_{k=0}^{\infty} \frac{(-1)^{k-1}(2k)!}{(2k-1)k!k!}(-1)^k t^k) \\
&= \frac{1}{2}t^{-1}(1 - \sum_{k=0}^{\infty} \frac{(-1)^{2k}(-1)(2k)!}{(2k-1)k!k!}(t^k)) \\
&= \frac{1}{2}t^{-1}(1 + \sum_{k=0}^{\infty} \frac{(2k)!}{(2k-1)k!k!}t^k) \\
&= \frac{1}{2}t^{-1} + \frac{1}{2}t^{-1} \sum_{k=0}^{\infty} \frac{(2k)!}{(2k-1)k!k!}t^k \\
&= \frac{1}{2}t^{-1} \sum_{k=0}^{\infty} \frac{(2k)!}{(2k-1)k!k!}t^k
\end{aligned}$$

Now, C_n is the n^{th} coefficient of $C(t) = \frac{1-\sqrt{1-4t}}{2t}$ which implies that:

$$\begin{aligned}
C_n = [t^n]C(t) = [t^n]\frac{1-\sqrt{1-4t}}{2t} &= [t^n]\frac{1}{2}t^{-1} \sum_{k=0}^{\infty} \frac{(2k)!}{(2k-1)k!k!}t^k \\
&= [t^{n+1}] \sum_{k=0}^{\infty} \frac{(2k)!}{(2k-1)2k!k!}t^k \\
&= \frac{(2(n+1))!}{2(n+1)!(n+1)!(2(n+1)-1)} \\
&= \frac{(2n+2)(2n+1)(2n)!}{2(n+1)n!(n+1)n!2n+1} \\
&= \frac{2(n+1)(2n+1)(2n)!}{2(n+1)n!(n+1)n!(2n+1)} \\
&= \frac{1}{n+1}\frac{(2n)!}{n!n!} \\
&= \frac{1}{n+1}\binom{2n}{n}
\end{aligned}$$

Hence,we conclude that $C_n = \frac{1}{n+1}\binom{2n}{n}$ for $n \geq 0$ is catalan numbers.

1.9 Fibonacci Numbers

The first few Fibonacci numbers are 0,1,1,2,3,5,8,13,...
For $F_0 = 0$,$F_1 = 1$ and for $n \geq 2$ we have:

$$F_n = F_{n-1} + F_{n-2} \tag{1.4}$$

We can find the generating function of sequence of Fibonacci numbers as follows:

Let $f(t) = \sum_{n=0}^{\infty} f_n t^n$ be generating function of the Fibonacci numbers.Then, by multiplying equation (1.4) by t^n and summing over all $n \geq 0$ we obtain the generating function as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} F_n t^n &= \sum_{n=0}^{\infty} F_{n-1} t^n + \sum_{n=0}^{\infty} F_{n-2} t^n \\
\sum_{n=0}^{\infty} F_{n+2} t^n &= \sum_{n=0}^{\infty} F_{n+1} t^n + \sum_{n=0}^{\infty} F_n t^n \\
\sum_{n=2}^{\infty} F_n t^{n-2} &= \sum_{n=1}^{\infty} F_n t^{n-1} + \sum_{n=0}^{\infty} F_n t^n
\end{aligned}$$

$$\begin{aligned}
\frac{1}{t^2}\Big(\sum_{n=0}^{\infty} F_n t^n - F_0 - tF_1\Big) &= \frac{1}{t}\Big(\sum_{n=0}^{\infty} F_n t^n - F_0\Big) + f(t) \\
\frac{f(t)}{t^2} - \frac{0}{t^2} - \frac{t}{t^2} &= \frac{1}{t} - 0 + f(t) \\
\frac{f(t)}{t^2} - \frac{1}{t} &= \frac{f(t)}{t} + f(t) \\
\frac{f(t)}{t^2} - \frac{f(t)}{t} - f(t) &= \frac{1}{t} \\
\Big(\frac{1}{t^2} - \frac{1}{t} - 1\Big)f(t) &= \frac{1}{t} \\
\Big(\frac{1-t-t^2}{t^2}\Big)f(t) &= \frac{1}{t} \\
f(t) &= \frac{t}{1-t-t^2}
\end{aligned}$$

$\therefore f(t) = \frac{t}{1-t-t^2}$ if the generating function of the Fibonacci numbers $0, 1, 1, 2, 3, 5, 8, 13, ...$

Chapter 2

Level Generating Trees And Proper Riordan Arrays

2.1 Definitions and Examples of Riordan Arrays

I.Definition Of Riordan Arrays

Definition 2.1.1. *A Riordan Array is an infinite lower triangular array(matrix)* $(d_{n,k})_{n,k\in\mathbb{N}}$ *including* 0 *defined by two formal power series* $(d(t), h(t))$,*where* $d(t) \in \mathcal{F}_0$, $h(t) \in \mathcal{F}_1$ *such that the generic element* $d_{n,k}$ *is the* n^{th} *coefficient in the series* $d(t)(h(t))^k$ *i.e*
$d_{n,k} = [t^n]d(t)(h(t))^k$, $n, k \in \mathbb{N}$ *implies that* $\sum_{n=0}^{\infty} d_{n,k}t^n = d(t)h(t)^k$.
In other words,an infinite lower, triangular array $D = (d(t), h(t)) = (d_{n,k})_{n,k\in\mathbb{N}}$ *(including* 0*) is said to be a Riordan Array if its* k^{th} *column is generated by* $d_k(t) = d(t)(h(t))^k$, *where* $k = 0, 1, 2, ...$ *for some formal power series* $d(t), h(t)$.
From this definition , we have $d_{n,k} = 0$, $\forall k > n$.
The bivariate generating function of a Riordan Array is given by:

$$\begin{aligned}
d(t, w) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} d_{n,k}w^k t^n \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} d_{n,k}w^k t^n \\
&= \sum_{k=0}^{\infty} d(t)h(t)^k w^k \\
&= d(t)\sum_{k=0}^{\infty}(wh(t))^k
\end{aligned}$$

$$d(t,w) = d(t)(\frac{1}{1-wht})$$
$$= \frac{d(t)}{1-wh(t)}$$

Definition 2.1.2. *The Riordan Array* $D = (d(t), h(t))$ *is said to be* ***proper Riordan Array****,if* $d(t), h(t) \in \mathcal{F}_0$. *In this report we will be mainly interest in proper Riordan arrays.*

II.Some Basic Examples Of Riordan Arrays

Example 2.1.1. *The most basic and important example of a Riordan Array which gives a good entry point into investigating the various theories, methods and techniques arising from the Riordan Array concept is the* ***Pascal triangle****. All Riordan arrays are generalization of the Pascal triangle that is equivalent to the binomial matrix* $\binom{n}{k}$, $\forall n \geq k$ *and* $\binom{n}{k} = 0$ *if* $k > n$ *which has all its leading diagonals elements equal to* 1 *that as shown as follow:*

n/k	0	1	2	3	4	.	.	.
0	1	0	0	0	0	.	.	.
1	1	1	0	0	0	.	.	.
2	1	2	1	0	0	.	.	.
3	1	3	3	1	0	.	.	.
4	1	4	6	4	1	.	.	.
.	.	.	.	.	.	.		
.	.	.	.	.	.		.	
.	.	.	.	.	.			.

$$= \begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ 1 & 3 & 3 & 1 & & & \\ 1 & 4 & 6 & 4 & 1 & & \\ . & . & . & . & . & . & \\ . & . & . & . & . & & . \\ . & . & . & . & . & & & . \end{bmatrix}$$

From this Pascal triangle we have: $d_0(t) = 1 + t + t^2 + t^3 + t^4 + ... = \frac{1}{1-t}$ *which is the generating function of column 0.*
This implies that:
$d_k(t) = d(t)(h(t))^k$
$d_0(t) = d(t)(h(t))^0 = d(t)$
$d_0(t) = \frac{1}{1-t} = d(t)$,*but* $d_0(t) = \frac{1}{1-t}$
$d(t) = \frac{1}{1-t}$.

The generating of column one is:

$$\begin{aligned} d_1(t) &= \sum_{k\geq 0} kt^k \\ &= 0 + t + 2t^2 + 3t^3 + ... \\ &= t + 2t^2 + 3t^3 + ... \\ &= t(1 + 2t + 3t^2 + 4t^3 + 5t^4 + ...) \\ &= tD(\frac{1}{1-t}) \\ &= \frac{t}{(1-t)^2}. \end{aligned}$$

now,since $d_1(t) = d(t)(h(t))^1$,*by substituting* $d_1(t) = \frac{t}{(1-t)^2}$ *and* $d(t) = \frac{1}{1-t}$ *in this equation we can find* $h(t)$ *as follow:*
$\frac{t}{(1-t)^2} = \frac{1}{1-t}h(t)$
$h(t) = \frac{t}{1-t}$.
In this way,we obtain the Riordan Array $D = (d(t), h(t)) = (\frac{1}{1-t}, \frac{t}{1-t})$ *that we call Pascal triangle. We can apply the column definition of a Riordan array and the properties of binomial coefficients to determine the general formula* $d_{n,k}$ *of Pascal triangle (*$d_{n,k}$ *of binomial matrix) as illustrated as follow:*

$$\begin{aligned} d_{n,k} &= [t^n]d(t)(h(t))^k \\ &= [t^n]\frac{1}{1-t}(\frac{t}{1-t})^k \\ &= [t^n]\frac{t^k}{(1-t)^{k+1}} \\ &= [t^{n-k}](1-t)^{-k-1} \\ &= [t^{n-k}]\sum_{j\geq 0}\binom{-k-1}{j}(-1)^j t^j \\ &= [t^{n-k}]\sum_{j\geq 0}\binom{j-(-k-1)-1}{j}(-1)^j(-1)^j t^j \\ &= [t^{n-k}]\sum_{j\geq 0}\binom{j+k}{j}t^j, but j = n-k \\ &= [t^{n-k}]\sum_{n-k\geq 0}\binom{n-k+k}{n-k}t^{n-k} \end{aligned}$$

$$d_{n,k} = [t^{n-k}] \sum_{n \geq k} \binom{n}{n-k} t^{n-k}$$
$$= \binom{n}{n-k} = \binom{n}{k}$$

is the entries of Pascal triangle.

This shows that a Riordan Array is generalization of the Pascal triangle.

Example 2.1.2. *Other well known examples of Riordan arrays that have received much attention are the Catalan arrays.*
The catalan array is represented as $C = (d(t), h(t)) = (c(t), tc(t))$ *but we have seen the generating function of the catalan numbers which is* $c(t) = \frac{1-\sqrt{1-4t}}{2t}$ *and from this we can determine* $h(t)$ *as* $h(t) = tc(t) = t(\frac{1-\sqrt{1-4t}}{2t}) = \frac{1-\sqrt{1-4t}}{2}$.
Therefore,$d_{n,k} = C_{n,k} = [t^n]d(t)h(t)^k = [t^n]\frac{1-\sqrt{1-4t}}{2t}(\frac{1-\sqrt{1-4t}}{2})^k = [t^{n+1}](\frac{1-\sqrt{1-4t}}{2})^{k+1} = \frac{k+1}{n+1}\binom{2n-k}{n-k}$ *that forming the Riordan matrix as follow:*

C=

n/k	0	1	2	3	4	
0	1					
1	1	1				
2	2	2	1			
3	5	5	3	1		
4	14	14	9	4	1	
5	42	42	28	14	5	1

$$= \begin{pmatrix} 1 \\ 1 & 1 \\ 2 & 2 & 1 \\ 5 & 5 & 3 & 1 \\ 14 & 14 & 9 & 4 & 1 \\ 42 & 42 & 28 & 14 & 5 & 1 \end{pmatrix}$$

2.1.1 Computation of Combinatorial Sums Using Riordan Arrays

Theorem 2.1.1. *If* (f_k), $k \in \mathbb{N}$ *is any sequence and* $f(t) = \sum_k^{\infty} f_k t^k$ *is its generating function, then for every Riordan array* $D = (d(t), h(t))$, *we have:*

$$\sum_{k=0}^{n} d_{n,k} f_k = [t^n] d(t) f(th(t)) \tag{2.1}$$

This theorem allows us to translate a combinatorial sum into the extraction of a coefficient from a generating function, thus showing the close connection between Riordan arrays and the method of coefficients as exposed.

Proof. The proof consists in a straight-forward computation:

$$\sum_{k=0}^{n} d_{n,k} f_k = \sum_{k=0}^{\infty} d_{n,k} f_k$$

$$\begin{aligned}\sum_{k=0}^{n} d_{n,k} f_k &= \sum_{k=0}^{\infty} [t^n] d(t)(th(t))^k f_k \\ &= [t^n] d(t) \sum_{k=0} f_k (th(t))^k \\ &= [t^n] d(t) f(th(t)).\end{aligned}$$

□

Partial sum theorem:

$$\sum_{k=0}^{n} f_k = [t^n] \frac{f(t)}{1-t}. \tag{2.2}$$

Euler transformation:

$$\sum_{k=0}^{n} \binom{n}{k} f_k = [t^n] \frac{1}{1-t} f(\frac{t}{1-t}). \tag{2.3}$$

Proof. We know that $\binom{n}{k} = \binom{n}{n-k} = \binom{n-k-n-1}{n-k}(-1)^{n-k} = \binom{-k-1}{n-k}(-1)^{n-k}$. Therefore,

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} f_k = \sum_{k=0}^{\infty} \binom{n}{k} f_k &= \sum_{k=0}^{\infty} \binom{-k-1}{n-k} (-1)^{n-k} f_k \\ &= \sum_{k=0}^{\infty} [t^{n-k}](1-t)^{-k-1} f_k \\ &= \sum_{k=0}^{\infty} [t^n] t^k (1-t)^{-k} (1-t)^{-1} f_k \\ &= \frac{1}{1-t} \sum_{k=0}^{\infty} (\frac{t}{1-t})^k f_k \\ &= \frac{1}{1-t} f(\frac{t}{1-t}) since \sum_{k=0}^{\infty} (\frac{t}{1-t})^k f_k = f(\frac{t}{1-t})\end{aligned}$$

Hence, $\sum_{k=0}^{n} \binom{n}{k} f_k = \frac{1}{1-t} f(\frac{t}{1-t})$ □

Remark:Let $\mathcal{G}$ be the generating function operator. Then,
$\mathcal{G}(1) = \frac{1}{1-t}$.
$\mathcal{G}((-1)^k) = \frac{1}{1+t}$.
$\mathcal{G}(k) = \frac{t}{(1-t)^2}$.
For any Riordan Array $D = (d(t), h(t))$ we have:

1. **Row Sum:**
   $\sum_{k=0}^{n} d_{n,k} f_k = [t^n] d(t) f(th(t))$ (with $f_k = 1, \forall k$, $\mathcal{G}(f_k) = \mathcal{G}(1) = f(t) = \frac{1}{1-t}$)

$\Rightarrow \sum_{k=0}^{n} d_{n,k}.1 = [t^n]\frac{d(t)}{1-th(t)}$
$\Rightarrow \sum_{k=0}^{n} d_{n,k} = [t^n]\frac{d(t)}{1-th(t)}$.

2. **Alternative Sum:**
$\sum_{k=0}^{n} d_{n,k} f_k = [t^n]d(t)f(th(t))$ (with $f_k = (-1)^k$, $\forall k$, $\mathcal{G}((-1)^k) = f(t) = \frac{1}{1+t}$)
$\Rightarrow \sum_{k=0}^{n} d_{n,k}(-1)^k = [t^n]\frac{d(t)}{1+th(t)}$.

3. **Weighted Sum:**
$\sum_{k=0}^{n} d_{n,k} f_k = [t^n]d(t)f(th(t))$ (with $f_k = k$,$\forall k$,$\mathcal{G}(f_k) = \mathcal{G}(k) = f(t) = \frac{t}{(1-t)^2}$)
$\Rightarrow \sum_{k=0}^{n} kd_{n,k} = [t^n]\frac{d(t)th(t)}{(1-th(t))^2}$.

4. **Diagonal Sum of the Riordan Array:**
Consider the Riordan Array $\hat{D} = (d(t), th(t))$, where $\hat{d}_{n,k} = [t^n]d(t)f(t.th(t))$ and a Riordan Array $D = (d(t), h(t))$, where $d_{n,k} = [t^n]d(t)f(th(t))$
The Riordan Array $\hat{D} = (d(t), th(t))$ is a Riordan Array whose rows are the diagonals of a Riordan Array $D = (d(t), h(t))$
Now,

$$\begin{aligned}\hat{d}_{n,k} = [t^n]d(t)f(t.th(t)) &= [t^n]d(t)(t.th(t))^k \\ &= [t^n]d(t)t^k(th(t))^k \\ &= [t^n]t^k d(t)(th(t))^k \\ &= [t^{n-k}]d(t)(th(t))^k \\ &= d_{n-k,k}\end{aligned}$$
$$\Rightarrow \hat{d}_{n,k} = d_{n-k,k}$$

Therefore,$\sum_{k=0}^{n} d_{n-k,k} f_k = \sum_{k=0}^{n} \hat{d}_{n,k} f_k = [t^n]d(t)f(t^2h(t))$ (with $f_k = 1$,$\forall k$,$f(t) = \frac{1}{1-t}$)
$\Rightarrow \sum_{k=0}^{n} d_{n-k,k} = \sum_{k=0}^{n} \hat{d}_{n,k} = [t^n]\frac{d(t)}{(1-t^2h(t))}$

2.1.2 Some Examples Related To Binomial Coefficients

Recall that a Riordan array is infinite,lower triangular matrix $D = \{d_{n,k}\}_{n,k\in\mathbb{N}}$, where $d_{n,k} = [t^n]d(t)(h(t))^k$ and $d(t) \in \mathcal{F}_0, h(t) \in \mathcal{F}_1$.
Consider the binomial coefficients of the form: $d_{n,k} = \binom{n+ak}{m+bk}$ where a, b are two parameters(constants) and k is a nonnegative variables.
To find $d(t)$ and $h(t)$ we have two cases:
Case-1: When n is fixed or $b > 0$:

$$\begin{aligned}d_{n,k} = \binom{n+ak}{m+bk} &= [t^{m+bk}](1+t)^{n+ak} \\ &= [t^m]t^{-bk}(1+t)^n(1+t)^{ak} \\ &= [t^m](1+t)^n(t^{-b}(1+t)^a)^k.\end{aligned}$$

Therefore,$D = \{d_{n,k}\}_{n,k\in\mathbb{N}}$, where $d_{n,k} = [t^m](1+t)^n(t^{-b}(1+t)^a)^k$ is a Riordan Array with the two column generating functions:
$d(t) = (1+t)^n$ and $h(t) = t^{-b}(1+t)^a$
Case-2: When m is fixed or $b > a$;

$$\begin{aligned} d_{n,k} = \binom{n+ak}{m+bk} &= \binom{n+ak}{n+ak-(m+bk)} \\ &= \binom{n+ak}{n-m+(ak-bk)} \\ &= \binom{n-m+ak-bk-(n+ak)-1}{(n-m)+(a-b)k}(-1)^{n-m+ak-bk} \\ &= \binom{-m-bk-1}{(n-m)+(a-b)k}(-1)^{n-m+ak-bk} \\ &= [t^{n-m+ak-bk}](1-t)^{-m-bk-1} \\ &= [t^{n-m+ak-bk}]\frac{1}{(1-t)^{m+bk+1}} \\ &= [t^n]t^m t^{(a-b)k}\frac{1}{(1-t)^{m+bk+1}} \\ &= [t^n]\frac{t^m}{(1-t)^{m+1}}(\frac{t^{b-a}}{(1-t)^b})^k. \end{aligned}$$

Hence,$D = \{d_{n,k}\}_{n,k\in\mathbb{N}}$, where $d_{n,k} = [t^n]\frac{t^m}{(1-t)^{m+1}}(\frac{t^{b-a}}{(1-t)^b})^k$ is a Riordan array with the two column generating functions:
$d(t) = \frac{t^m}{(1-t)^{m+1}}$ and $h(t) = \frac{t^{b-a}}{(1-t)^b}$.

Example 2.1.3. *Find $d(t)$ and $h(t)$ from combinatorial coefficient $\binom{n+k}{m+2k}$.*

Solution: Since $b = 2 \geq a = 1$,m is fixed and we use the formula $d_{n,k} = [t^n]\frac{t^m}{(1-t)^{m+1}}(\frac{t^{b-a}}{(1-t)^b})^k$ such that $d(t) = \frac{t^m}{(1-t)^{m+1}}$ and $h(t) = \frac{t^{b-a}}{(1-t)^b} = \frac{t^{2-1}}{(1-t)^2} = \frac{t}{(1-t)^2}$.

proposition 2.1.1. *If f(t) is a generating function of $\{f_k\}_{n=0}^{\infty}$,then for $b < 0$:*
$\sum_{k=0}^{n}\binom{n+ak}{m+bk}f_k = [t^m](1+t)^n f(t^{-b}(1+t)^a)$, *where $d(t) = (1+t)^n$ and $h(t) = t^{-b}(1+t)^a$.*

proposition 2.1.2. *If f(t) is a generating function of $\{f_k\}_{n=0}^{\infty}$, then for $b > a$:*
$\sum_{k=0}^{n}\binom{n+ak}{m+bk}f_k = [t^n\frac{t^m}{(1-t)^{m+1}}f(\frac{t^{b-a}}{(1-t)^b})$ *where $d(t) = \frac{t^m}{(1-t)^{m+1}}$ and $h(t) = \frac{t^{b-a}}{(1-t)^b}$.*

2.1.3 Algebraic Structure Of Riordan Arrays

Let $d(t) \in \mathcal{F}_0$, $h(t) \in \mathcal{F}_1$, $h(0) = 0$ and $h'(0) \neq 0$. Then, $D = (d(t), h(t))$ is proper Riordan Array.
Consider the set of all Proper Riordan Arrays:
The main operation required to determine that a set of Riordan arrays forms group is the usual row by column matrix multiplication(*).
The multiplication rule for two arbitrary Riordan arrays,
$D = (d(t), h(t))$ and $\hat{D} = (a(t), b(t))$ is also Riordan Array(R) defined as:

$$\begin{aligned} D * \hat{D} = (d(t), h(t)) * (a(t), b(t)) &= (\sum_{j \geq 0} d_{n,j} \hat{d}_{j,k}) \\ &= (\sum_{j \geq 0} [t^n] d(t) h(t)^j [y^j] a(y) b(y)^k) \\ &= ([t^n] d(t) \sum_{j \geq 0} [y^j] a(y) b(y)^k h(t)^j) \\ &= ([t^n] d(t) a(h(t)) (b(h(t)))^k) \\ &= (d(t) a(h(t)), b(h(t))) \end{aligned}$$

Therefore,$D * \hat{D} = (d(t) * (aoh), boh) = R$ (product of two Riordan Arrays).
The key criteria for (R,*) to be a group under the multiplication operation is stated on the following conditions listed below:

1. The existence of $(1, t)$ as the identity element.

2. The existence of inverse(i.e every element is invertible). Let's consider a proper Riordan Array $D = (d(t), h(t))$ and to determine its inverse,
   Suppose $D' = (f(t), g(t))$ be the inverse of D, Then,

$$\begin{aligned} D * D' &= (1, t) \\ &= (d(t), h(t)) * (f(t), g(t)) = (1, t) \\ &= (d(t) f(h(t)), g(h(t))) = (1, t) \end{aligned}$$

   By equating with the same coordinates we have:

$$d(t) f(h(t)) = 1 \tag{2.4}$$

$$g(h(t)) = t. \tag{2.5}$$

   We note that for a formal power series $h(t) = \sum_{n=0}^{\infty} h_n t^n$ with $h(0) = 0$,
   we define the compositional inverse of h to be the formal power series $\bar{h}(t)$ such that $h(\bar{h}(t)) = \bar{h}(h(t)) = t$.

Now,from equation (2.4) we have:

$$\begin{aligned} d(t)f(h(t)) &= 1 \\ f(h(t)) &= \frac{1}{d(t)} \\ f(h(\bar{h}(t))) &= \frac{1}{d(\bar{h}(t)} \\ f(t) = \frac{1}{d(\bar{h}(t)} &= (d(\bar{h}(t))^{-1}. \end{aligned}$$

From equation (2.5) we have:

$$\begin{aligned} g(h(t)) &= t \\ g(h(\bar{h}(t))) &= \bar{h}(t) \\ g(t) &= \bar{h}(t). \end{aligned}$$

Therefore, $D' = (f(t), g(t)) = (\frac{1}{d(\bar{h}(t)}, \bar{h}(t))$ is the inverse of $D = (d(t), h(t))$.
In the same way we can check $D' * D = (1, t)$ and since inverse is unique, the D inverse is also $D' = (f(t), g(t)) = (\frac{1}{d(\bar{h}(t)}, \bar{h}(t))$

3. The set is closed under the multiplication matrices.

4. It is associative since multiplication of matrices is associative.

Example 2.1.4. *Consider the Pascal triangle given by $P = (\frac{1}{1-t}, \frac{t}{1-t})$ to show the product of it by itself is also Riordan array and to determine the inverse.*
Product:

$$\begin{aligned} P * P &= (\frac{1}{1-t}, \frac{t}{1-t}) * (\frac{1}{1-t}, \frac{t}{1-t}) \\ &= (\frac{1}{1-t} * \frac{1}{1 - \frac{t}{1-t}}, \frac{\frac{t}{1-t}}{1 - \frac{t}{1-t}}) \\ &= (\frac{1}{1-t} * \frac{1-t}{1-2t}, \frac{t}{1-t}\frac{1-t}{1-2t}) \\ &= (\frac{1}{1-2t}, \frac{t}{1-2t}). \end{aligned}$$

which is also Riordan Array.
Inverse:
To find the inverse, let's consider $\bar{h}(t)$ as the compositional inverse of $h(t)$ such that $h(\bar{h}(t)) = t$
Now we want to determine $\bar{h}(t)$ and $\frac{1}{d(\bar{h}(t)}$ as follow;

we have $h(t) = \frac{t}{1-t}$

$\Rightarrow h(\bar{h}(t)) = \frac{\bar{h}(t)}{1-\bar{h}(t)}$

$\Rightarrow t = \frac{\bar{h}(t)}{1-\bar{h}(t)}$

$\Rightarrow \bar{h}(t) = t - \bar{h}(t)t$

$\Rightarrow \bar{h}(t) + \bar{h}(t)t = t$

$\Rightarrow \bar{h}(t) = \frac{t}{1+t}$ *is the compositional inverse of* $h(t)$.

And also we have $d(t) = \frac{1}{1-t}$:

$\Rightarrow \frac{1}{d(\bar{h}(t))} = \frac{1}{1-\bar{h}(t)} = \frac{1}{1-\frac{t}{1+t}} = \frac{1}{1+t}$

Therefore,$P^{-1} = (\frac{1}{1-t}, \frac{t}{1-t})^{-1} = (\frac{1}{1+t}, \frac{t}{1+t})$

The existence of the inverse element of Riordan Arrays that makes them suitable to perform operations involving combinatorial sum inversion.

2.1.4 Subgroups Of The Riordan Group

Some important sub-groups of the Riordan group arising from the Riordan Arrays of $(d(t), h(t))$ are listed below:
The **Appell subgroup** that has the form $(d(t), t)$, where $d(t) \in \mathcal{F}_0$.
The **Lagrange subgroup** that has the form $(1, h(t))$,where $h(t) \in \mathcal{F}_1$ which is also known as the **associated subgroup**.

The **Bell subgroup** that has the form $(d(t), td(t))$,where $d(t) \in \mathcal{F}_0$ which is also referred as the **Renewal subgroup**.

Any Proper Riordan Array $(d(t), h(t))$ can be given as a product of an Appell subgroup and Lagrange Subgroup such that:
$(d(t), t) * (1, h(t)) = (d(t) * 1, h(t)) = (d(t), h(t))$ by using the product of two Riordan Array rule.

2.1.5 Sequence Characterization Of Riordan Arrays

The fundamental theorem of Riordan Arrays togethers with the theorems on the A and Z sequences which characterizes the formation of a Riordan matrix.
The A and Z sequences characterizations of a Riordan arrays $(d(t), h(t))$ involves determining both $h(t)$ and $d(t)$ respectively that define any such arrays.The theorem arising from the sequence characterizations of Riordan Arrays present an alternative definition of a Riordan Array in terms of the recursive formation of its elements. These theorems are summarized below.

The Fundamental Theorem Of Riordan Arrays

Theorem 2.1.2. *Suppose* $D = (d(t), h(t))$ *is a Riordan Array.Let* $A(t) = \sum_{k=0}^{\infty} a_k t^k$ *be generating function of the sequence* $\{a_k\}k \geq 0$ *and* $B(t) = \sum_{k=0}^{\infty} b_k t^k$ *be the generating function of the sequence* (b_k),$k \geq 0$ *with* A *and* B *representing column vectors such that* $A = (a_0, a_1, a_2, ...)^T$ *and* $B = (b_1, b_2, b_3, ...)^T$. *Then,* $D * A = (d, h) * A = B$ *if and only if* $B(t) = d(t)A(h(t))$

The A and Z sequences

An alternative definition, is in terms of the so-called A-sequence and Z-sequence, with generating functions A(t) and Z(t) satisfying the relations:

$h(t) = tA(h(t))$ and $d(t) = \frac{d_0}{1-tZ(h(t))}$ with $d_0 = d(0)$

I.The A-Sequence

The A-sequence characterization, the column elements after the first column.

Theorem 2.1.3. *An infinite lower triangular array* $D = \{d_{n,k}\}_{n,k\in\mathbb{N}}$ *is a proper Riordan Array if and only if there exists a sequence* $A = \{a_i\}_{i\in\mathbb{N}}$ *including zero with* $a_0 \neq 0$ *such that every element* $d_{n+1,k+1}$ *(not lying in column* 0 *or row* 0*) can be expressed as a linear combination with coefficients in* A *of the elements in the preceding row, starting from the preceding column on, i.e.:*

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + ... = \sum_{i=0}^{n-k} a_i d_{n,k+i} \tag{2.6}$$

Proof. Let us suppose that D is the Riordan Array $(d(t), h(t))$ and let us consider the Riordan Array $(\frac{d(t)h(t)}{t}, h(t))$, we define the Riordan Array $(A(t), B(t))$ by the relation:

$$(A(t), B(t)) = (d(t), h(t)^{-1} * (\frac{d(t)h(t)}{t}, h(t)) \text{ or } (d(t), h(t)) * (A(t), B(t)) = (\frac{d(t)h(t)}{t}, h(t))$$

By performing the product of two Riordan Arrays, we find: $d(t)A(h(t)) = \frac{d(t)h(t)}{t}$ and $B(h(t)) = h(t)$.

The latter identity implies $B(t) = t$. Therefore we have $(d(t), h(t)) * (A(t), t) = (\frac{d(t)h(t)}{t}, h(t))$.

The element $f_{n,k}$ of the left hand member is $\sum_{i=0}^{\infty} d_{n,i} a_{i-k} = \sum_{i=0}^{\infty} d_{n,k+i} a_i$, if as usual we interpret a_{i-k} as 0 when $i < k$.

The same element in the right hand member is:

$$[t^n]\frac{d(t)h(t)}{t} h(t)^k = [t^{n+1}]d(t)h(t)^{k+1} = d_{n+1,k+1}$$

By equating these two quantities, we have the identity (2.6).
For the converse, let us observe that (2.6) uniquely defines the array D when the elements of column 0 $(d_{0,0}, d_{1,0}, d_{2,0}, ...)$ are given. Let $d(t)$ be the generating function of this column, $A(t)$ the generating function of the sequence A and define $h(t)$ as the solution of the functional equation $h(t) = tA(h(t))$, which is uniquely determined because of our hypothesis $a_0 \neq 0$. We can therefore consider the proper Riordan Array $\hat{D} = (d(t), h(t))$, by the first part of the theorem, $\hat{D}$ satisfies relation (2.6), for every $n, k \in \mathbb{N}$ and therefore, by our previous observation, it must coincide with D.This completes the proof. □

corollary 2.1.1. *If $D = (d(t), h(t))$ be a Riordan Array, then the generating function of A-sequence such that $A(t) = \sum_{i=0}^{\infty} a_i t^i$ satisfies the equation:*
$h(t) = tA(h(t)) \Rightarrow A(t) = \frac{t}{\bar{h}(t)}$, where $\bar{h}$ is the compositional inverse of h.

Example 2.1.5. *Let us consider Pascal triangle and Catalan triangle:*
For Pascal triangle;

n/k	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

The A-sequence of Pascal triangle is $\{1, 1, 0, 0, 0, ...\} \Rightarrow A(t) = 1 + t$

The A-sequence is unique and only depend on $h(t)$ that satisfy the functional equation:
$h(t) = tA(h(t))$
Now by using the A-sequence we can determine $h(t)$ of Pascal triangle as follow:

$h(t) = tA(h(t))$

$h(t) = t(1 + h(t))$

$h(t) = t + th(t)$

$h(t) - th(t) = t$

$(1 - t)h(t) = t$

$h(t) = \frac{t}{1-t}$.

For Pascal triangle the relation of Theorem (2.1.3) reduces to the well-known recurrence relation for binomial coefficients: $d_{n+1,k+1} = \binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$.

For Catalan triangle;

n/k	0	1	2	3	4	5
0	1					
1	1	1				
2	2	2	1			
3	5	5	3	1		
4	14	14	9	4	1	
5	42	42	28	14	5	1

The A-sequence for the catalan triangle is $A = \{1, 1, 1, ...\} \Rightarrow A(t) = \frac{1}{1-t}$

And also by using the A-sequence of the catalan triangle we can determine $h(t)$ of the catalan triangle as follow:

$h(t) = tA(h(t))$

$h(t) = t\frac{1}{1-h(t)}$

$h(t)(1 - h(t)) = t$

$h(t) - (h(t))^2 - t = 0$

$(h(t))^2 - h(t) + t = 0$ from this quadratic equation we have two solutions of $h(t)$ such as $h(t) = \frac{1-\sqrt{1-4t}}{2}$ 0r $h(t) = \frac{1+\sqrt{1-4t}}{2}$ but we used only $h(t) = \frac{1-\sqrt{1-4t}}{2}$ because of the hypothesis $h(0) = 0$

Therefore,$h(t) = \frac{1-\sqrt{1-4t}}{2}$ is the $h(t)$ of catalan triangle.

corollary 2.1.2. *Let $D = (d(t), h(t))$ be a proper Riordan Array,and let $A = \{a_j\}_{j\in\mathbb{N}}$ be its A-sequence. Then,if $A(t)$ is the generating function of the sequence A, we have the functional equation:*

$$h(t) = A(th(t))$$

corollary 2.1.3. *Let $D = (d(t), h(t))$ be a proper Riordan Array,and let $A = \{a_j\}_{j\in\mathbb{N}}$ be its A-sequence. Then, if $d(t) = 1$ we have:*

$$d_{n,k} = \frac{k}{n}[t^{n-k}]A(t)^n$$

and if $d(t) = h(t)$ we have:

$$d_{n,k} = \frac{k+1}{n+1}[t^{n-k}]A(t)^{n+1}.$$

The A-sequence does not characterize completely Riordan array $(d(t), h(t))$ because $d(t)$ is independent of $A(t)$. But we have the following:

II.The Z-Sequence

The Z-sequence characterizes the elements of the first column of Proper Riordan Arrays as follow:

Theorem 2.1.4. *Let* $\{d_{n,k}\}_{n,k\in\mathbb{N}} = (d(t), h(t))$ *be any infinite lower triangular array with* $d_{n,n} \neq 0$*,* $\forall n \in \mathbb{N}$ *(in particular,let it be a proper Riordan Array);then a unique sequence* $Z = (z_0, z_1, z_3, ...)$ *exists such that every element in column* 0 *excluding the element in the first row can be expressed as a linear combination of all the elements in the preceding row with the coefficients identified as the elements of Z-sequence,i'e,:*

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + ... = \sum_{i=0}^{n} z_i d_{n,i}, \forall n \in \mathbb{N} \tag{2.7}$$

Proof. Let $z_0 = \frac{d_{1,0}}{d_{0,0}}$.Now we can uniquely determine the value of z_1 by expressing $d_{2,0}$ in terms of the elements in row 1, i.e,:

$d_{2,0} = z_0 d_{1,0} + z_1 d_{1,1} \Rightarrow z_1 = \frac{d_{0,0}d_{2,0} - d_{1,0}^2}{d_{0,0}d_{1,1}}$.

In the same way,we can determine z_2 by expressing $d_{3,0}$ in terms of the elements in row 2,and by substituting the values just obtained for z_0 and z_1.

By proceeding in this way, we determine the Z-sequence in a unique way. □

Note: The Z-sequence characterizes column 0,while the A-sequence characterizes all the other columns.

We can conclude that the triple $(d_0, Z(t), A(t))$,with $d_0 = d(0)$ characterizes any proper Riordan Array:

Theorem 2.1.5. *Let* $D = (d(t), h(t))$ *be a proper Riordan Array and let* $Z(t)$ *be the generating function of the array's Z-sequence.Then,*

$$d(t) = \frac{d_0}{1 - t(Z(th(t)))} \tag{2.8}$$

Proof. By the preceding theorem(i.e.theorem (2.1.4) above), the Z-sequence exists and it is unique. Therefore, equation (2.7) is valid for every $n \in \mathbb{N}$,and we can go on to the generating functions.

Since $d(t)(th(t))^k$ is the generating function for column k,we have:

$$\begin{aligned}\frac{d(t) - d_0}{t} &= z_0 d(t) + z_1 d(t)th(t) + z_2 d(t)t^2 h(t)^2 + ... \\ &= d(t)(z_0 + z_1 th(t) + z_2 t^2 h(t)^2 + ...) \\ &= d(t)Z(th(t))\end{aligned}$$

By solving this equation in $d(t)$,we obtain $d(t) = \frac{d_0}{1-t(Z(th(t)))} \Rightarrow Z(th(t)) = \frac{d_0-d(t)}{td(t)}$where $d_0 = d_{0,0}$ □

Theorem 2.1.6. *If $d_0 = h_0 \neq 0$,then $A(y) = d_0 + yZ(y)$, where $y = th(t)$ if and only if $d(t) = h(t)$.*

Proof. ($\Rightarrow$) Let us assume that $A(y) = d_0 + yZ(y)$ or $Z(y) = \frac{A(y)-d_0}{y}$ by theorem 2.1.5 where $y = th(t)$
We have:

$$d(t) = \frac{d_0}{1 - tZ(th(t))} = \frac{d_0}{1 - (\frac{tA(th(t))-d_0t}{th(t)})}$$
$$= \frac{d_0th(t)}{d_0t} = h(t)$$

because $A(th(t)) = h(t)$
($\Leftarrow$) By the formula for $Z(th(t))$, we obtain from the hypothesis $d(t) = h(t)$:

$$d_0 + yZ(y) = d_0 + y(\frac{1}{t} - \frac{d_0}{th(t)})$$
$$= d_0 + (\frac{th(t)}{t} - \frac{d_0th(t)}{th(t)}$$
$$= h(t)$$
$$= A(th(t))$$
$$= A(y)$$

Riordan Arrays having $d(t) = h(t)$ is called "renewal Arrays" □

2.2 Level Generating Trees

2.2.1 Definitions and Examples Of Generating Trees

2.2.2 Generating Trees

In this project we extend the correspondence between generating trees and proper Riordan Arrays to the whole group of monic integer proper Riordan Arrays. The primary diffeculty in doing so is the fact the coefficients in proper Riordan arrays can be negative, and this requires a particular interpretation of the labels in a generating tree. The idea is rather simple, but effective. Since, as we are going to show, the construction of the generating tree related to a given monic integer proper Riordan Array requires the concept of the A-sequence and Z-sequence, the results obtained in the previous section will be important in applying the theory.

Definition 2.2.1. *A generating tree is a rooted, labelled and typically infinite tree such that the label of a node determines the labels of its children with the property that if v_1 and v_2 are any two nodes with the same label, then for each label l, v_1 and v_2 have exactly the same number of children with label l.*
In order to specify a generating tree it therefore suffices to specify:

1. *the label of the root*

2. *a set of rules explaining how to derive from the label of a parent the labels of all of its children*

Example 2.2.1. *Illustrate the upper part of the generating tree which corresponds to the following specification:*

$$\begin{cases} root : (2), \\ rule : (k) \rightarrow (k)(k+1), \end{cases} \tag{2.9}$$

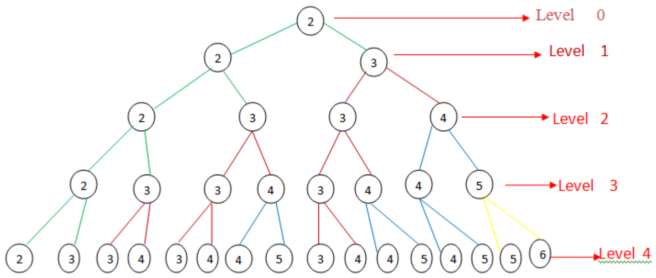

Figure 2.1: Generating tree of specification (2.9)

Example 2.2.2. *Illustrate the upper part of the generating tree which corresponds to the following specification:*

$$\begin{cases} root : (1), \\ rule : (k) \rightarrow (1)...(k)(k+1), \end{cases} \tag{2.10}$$

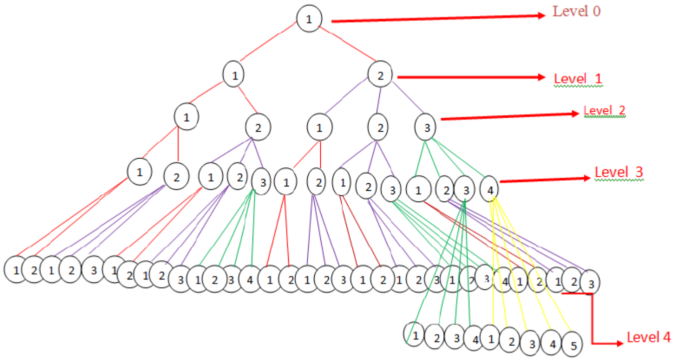

Figure 2.2: Generating tree of specification (2.10)

2.2.3 Marked Generating Trees

The generating trees have been extended to deal with marked labels: a label is any positive integer, generated according to the generating tree specification; a marked label is any positive integer, marked by a bar, for which appropriate rules are given in the specification:

Definition 2.2.2. *A marked generating tree is a rooted labeled tree (the labels can be marked or non-marked) with the property that if v_1 and v_2 are any two nodes with the same label ,then for each label l, v_1 and v_2 have exactly the same number of children with label l.*
To specify a generating tree it is therefore suffices to specify:

1. *the label of the root*

2. *a set of rules explaining how to derive from the label of a parent the labels of all of its children*

Example 2.2.3. *A simple example is given by the following generating tree specification:*

$$\begin{cases} root : (2), \\ rule : (k) \to (\bar{k})(k+1), \\ (\bar{k}) \to (k)(\overline{k+1}), \end{cases} \tag{2.11}$$

The first 4 levels of the corresponding generating tree are shown in Figure (2.3) below.

The idea is that marked labels kill or annihilate the non-marked labels with the same number.

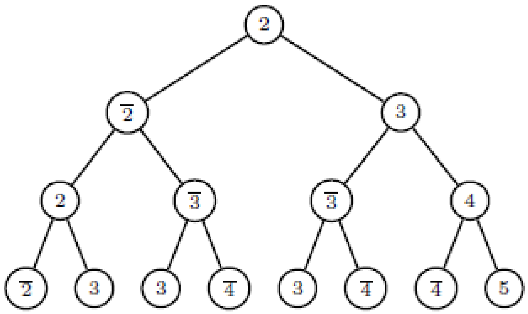

Figure 2.3: Marked Generating tree of specification (2.11)

2.3 Level Generating Trees and Proper Riordan Arrays

In the previous sections we have seen all the theoretical tools necessary to study the relation between generating trees and proper Riordan Arrays.

2.3.1 Relation Between Level Generating Trees and Proper Riordan Arrays

Before stating the main result , we introduce the following special product notations for generating tree specifications which plays an important role in the proof and in the application of some theorems.

$$
\begin{array}{rcl}
(x) & = & (\overline{\overline{x}}) \\
(x)^p & = & \underbrace{(x)\cdots(x)}_{p\ times},\ p \geq 0 \\
(x)^p & = & \underbrace{(\overline{x})\cdots(\overline{x})}_{-p\ times},\ p < 0 \\
\overline{(x)^p} & = & (\overline{x})^p,\ p > 0 \\
\overline{(x)^p} & = & (x)^{-p},\ p < 0 \\
\prod_{j=0}^{i}(k-j)^{\alpha_j} & = & (k)^{\alpha_0}(k-1)^{\alpha_1}\cdots(k-i)^{\alpha_i}
\end{array}
$$

We note that $(x)^0$ is the empty sequence.

2.3.2 Matrix Associated to a Generating Tree

We associate to a generating tree an infinite matrix $(d_{n,k})_{n,k\in\mathbb{N}}$ defined as follows:

Definition 2.3.1. *An infinite matrix* $(d_{n,k})_{n,k\in\mathbb{N}}$ *is said to be 'associated' to a generating tree with root*(c) *(AGT matrix for short) if* $d_{n,k}$ *is the number of nodes at level* n *with label* $k+c$*: By convention, the level of the root is* 0.

Example 2.3.1. *By referring to rule* (2.9) *and to Figure* (2.1) *above we have the following partial associated matrix:*

n/k	0	1	2	3	4
0	1				
1	1	1			
2	1	2	1		
3	1	3	3	1	
4	1	4	6	4	1

where we recognize the Pascal triangle.

Example 2.3.2. *By referring to specification* (2.10) *and to Figure* (2.2) *above we have the following partial associated matrix:*

n/k	0	1	2	3	4
0	1				
1	1	1			
2	2	2	1		
3	5	5	3	1	
4	14	14	9	4	1

where we recognize the Catalan triangle.
We observe that an AGT matrix is always infinite in the direction of n but can be finite in the direction of k because the node's labels of the generating tree can be all less than r for some $r \in N$

Definition 2.3.2. *An infinite matrix* $(d_{n,k})_{n,k\in\mathbb{N}}$ *is said to be associated to a marked generating tree with root* (c) *(AGT or ALGT matrix for short) if* $d_{n,k}$ *is the difference between the number of nodes at level* n *with label* $k+c$ *and the number of nodes at level* n *with label* $\overline{k+c}$.
This gives a negative count if marked labels are more numerous than non-marked ones.
By convention, the level of the root is 0.

Example 2.3.3. *By referring to specification* (2.11) *and to Figure* (2.3) *above we have the following partial associated matrix:*

n/k	0	1	2	3	4
0	1				
1	-1	1			
2	1	-2	1		
3	-1	3	-3	1	
4	1	-4	6	-4	1

where we recognize the inverse of Pascal triangle.

We observe that the row sums of an AGT matrix can be simply evaluated by formula $\sum_{k=0}^{n} d_{n,k} f_k = [t^n]d(t)f(h(t))$.

Some Basic Theorems

Theorem 2.3.1. *Let* $c \in \mathbb{N}$,$a_j \in \mathbb{N}$,$a_0 \neq 0$,$b_k \in \mathbb{Z}$,*and* $b_{c+j} + a_{j+1} \geq 0, \forall j \geq 0$ *and* $k \geq c$ *and let*

$$\begin{cases} root : (c), \\ rule : (k) \rightarrow (c)^{b_k} \prod_{j=0}^{k+1-c} (k+1-j)^{a_j}, \end{cases} \tag{2.12}$$

be a generating tree specification. Then, the AGT matrix associated to (2.12) *is a proper Riordan Array D defined by the triple* (d_0, A, Z) *such that* $d_0 = 1$,$A = (a_0, a_1, a_2, ...)$ *and* $Z = (b_c + a_1, b_{c+1} + a_2, b_{c+2} + a_3, ...)$
Vice versa,if D is a proper Riordan Array defined by the triple (d_0, A, Z) *with* $d_0 = 1$ *and* $a_j, z_j \in \mathbb{N}, \forall j \geq 0$, *then D is the AGT matrix associated to the generating tree specification (2.12) with* $b_{c+j} = z_j - a_j + 1, \forall j \geq 0$.

We call proper generating trees the generating trees corresponding to Theorem 2.3.1

Proof. Let us consider the *AGT* matrix $D = \{d_{n,k}\}_{n,k \in \mathbb{N}}$ associated to (2.12). Then, $d_{n,k}$ counts the number of nodes at level n with label $k+c$. We have obviously $d_{0,0} = 1$. Moreover, we observe that the maximum label's value at each level increases by one with respect to the previous level, hence $d_{n,j} = 0$ for $j > n$. Now, (2.12) tells us that a node at level $n+1$ with label $k+1+c$ can be determined, in a_j different ways, from the nodes at level n with label $m+c$, such that $m+c+1-j = k+c+1$; i.e., $m = k+j$,$j > 0$, hence
$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + ...$
Finally, a node at level $n+1$ with label c can be obtained in $b_m + a_{m+1-c}$ different ways from the nodes at level n with label m, $m > c$. Hence we have,
$d_{n+1,0} = (b_c + a_1)d_{n,0} + (b_{c+1} + a_2)d_{n,1} + ...$
Conversely, we only need to observe that the elements of an *AGT* matrix are all positive and this is assured for the elements of a Riordan Array having $d_0 = 1$ and $a_j, z_j \in \mathbb{N}, \forall j > 0$. □

Theorem 2.3.2. *Let* $c \in \mathbb{N}$,$a_j, b_k \in \mathbb{Z}, \forall j \geq 0$ *and* $k \geq c$,$a_0 = 1$,*and let*

$$\begin{cases} root : (c), \\ rule : (k) \rightarrow (c)^{b_k} \prod_{j=o}^{k+1-c} (k+1-j)^{a_j}, \\ (\bar{k}) \rightarrow \overline{(c)}^{b_k} \prod_{j=0}^{k+1-c} \overline{(k+1-j)}^{a_j}, \end{cases} \tag{2.13}$$

be a marked generating tree specification. Then, the AGT matrix associated to (2.13) *is a monic integer proper Riordan array defined by the triple* (d_0, A, Z), *such that* $d_0 = 1$, $A = (a_0, a_1, a_2, ...)$, $Z = (b_c + a_1, b_{c+1} + a_2, b_{c+2} + a_3, ...)$.
Vice versa, if D is a monic integer proper Riordan Array defined by the triple $(1, A, Z)$ *with*

$a_j, z_j \in \mathbb{Z}$, $\forall j \geq 0$ and $a_0 = 1$, then D is the AGT matrix associated to the generating tree specification (2.13) *with $b_{c+j} = z_j - a_{j+1}$,$\forall j \geq 0$. A generating tree corresponding to the specification*(2.13) *will be called a Proper generating tree.*
An important consequence of Theorem (2.3.2) *is that we can define the product of two proper generating trees and the inverse of proper generating tree:*

Definition 2.3.3. *Given two generating tree specification $S_1$and S_2 of type* (2.13) *and the corresponding AGT matrices M_1 and M_2,we define the generating tree specification product of S_1 and S_2 as the specification of S_3 having $M_3 = M_1 * M_2$ as AGT matrix.*

Definition 2.3.4. *Given generating tree specification S_1 of type* (2.13) *and the corresponding AGT matrix M_1, we define the generating tree specification inverse of S_1 as the specification S_2 having $M_2 = M_1^{-1}$ as AGT matrix.*

Definition 2.3.5. *The identity generating tree specification S_1 is the one having the identity matrix I as AGT matrix. The identity generating tree specification and the corresponding generating trees are shown as below:*

$$\begin{cases} root : (c), \\ rule : (k) \to (k+1) \end{cases} \tag{2.14}$$

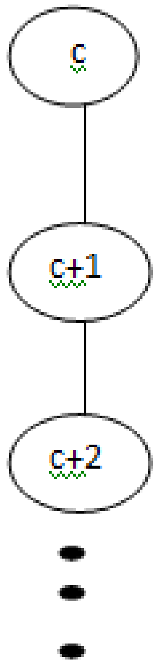

Figure 2.4: The identity generating tree

For example, specification (2.11) *is the inverse of the specification* (2.9)*, as can be easily verified by using formulas in section* 2*.In fact,for Pascal triangle we have* $d_0 = 1$*,*$A = \{1, 1, 0, 0, ...\}$ *and* $Z = \{1, 0, 0, ...\}$ *and for its inverse we have* $d_0^{-1} = 1$*,*$A^{-1} = \{1, -1, 0, 0, 0, ...\}$ *and* $Z^{-1} = \{-1, 0, 0, 0, ...\}$
In what follows, we examine some other generating tree specifications,very well known in this work, by finding for each of corresponding inverse as follows:

Example 2.3.4. *The following example is related to the Catalan numbers* $C_n = \{1, 1, 2, 5, 14, ...\} = \frac{1}{n+1}\binom{2n}{n}$*, for some* $n \in \mathbb{N}$ *including zero.*
To illustrate the Catalan generating tree, we have the specification:

$$\begin{cases} root : (2), \\ rule : (k) \to (2)...(k)(k+1) \end{cases} \tag{2.15}$$

To illustrate the inverse of the Catalan generating tree, we have the specification:

$$\begin{cases} root : (2), \\ rule : (k) \to (k+1)\prod_{j=1}^{k-1}\overline{(k+1-j)}^{C_{j-1}}, \\ (\bar{k}) \to \overline{(k+1)}\prod_{j=1}^{k-1}(k+1-j)^{C_{j-1}} \end{cases} \tag{2.16}$$

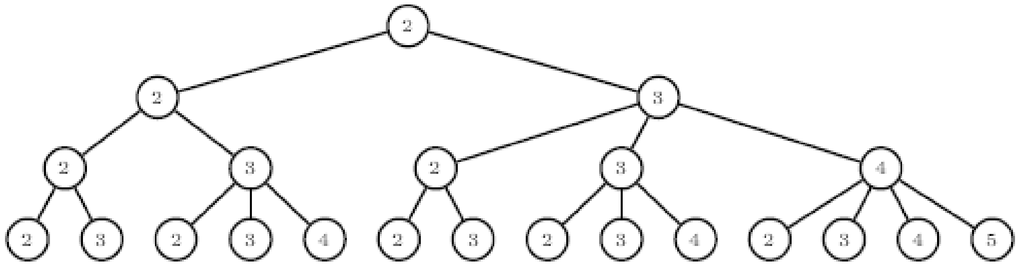

Figure 2.5: The Catalan generating tree of specification(2.15)

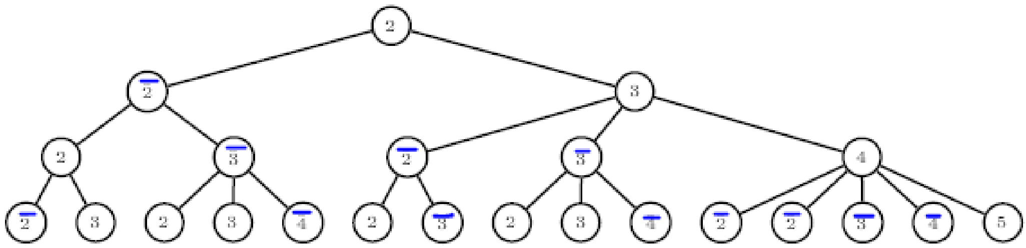

Figure 2.6: The inverse of Catalan generating tree of specification (2.16)

The AGT matrix of the Catalan generating tree of specification (2.15) *is:*

n/k	*0*	*1*	*2*	*3*	*4*
0	*1*				
1	*1*	*1*			
2	*2*	*2*	*1*		
3	*5*	*5*	*3*	*1*	
4	*14*	*14*	*9*	*4*	*1*

The AGT matrix of the inverse of Catalan generating tree of specification (2.16) is:

n/k	*0*	*1*	*2*	*3*	*4*
0	*1*				
1	*-1*	*1*			
2	*0*	*-2*	*1*		
3	*0*	*1*	*-3*	*1*	
4	*0*	*0*	*3*	*-4*	*1*

Some Examples

In this section we describe some applications of Theorem 2.4.1 in two directions:

1. we illustrate some well-known generating tree specifications and then find the associated proper Riordan arrays, this will allow us to obtain some very important combinatorial results, which are easily and directly derived in the Riordan array approach
2. Starting from some well-known proper Riordan arrays , we find the corresponding generating tree specifications.

Example 2.3.5. *Let us start by applying the Theorem 2.4.1 to generating tree specification:*

$$\begin{cases} root : (1), \\ rule : (k) \to (1)...(k)(k+1), \end{cases} \tag{2.17}$$

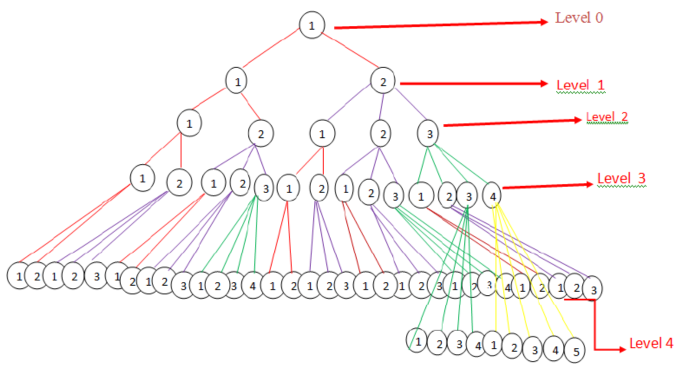

Figure 2.7: Generating tree of specification (2.17)

n/k	0	1	2	3	4
0	1				
1	1	1			
2	2	2	1		
3	5	5	3	1	
4	14	14	9	4	1

where recognize the catalan triangle

We have $c = 1$, $b_k = 0$, and $a_k = 1, \forall k \geq 0$.We therefore obtain the proper Riordan Array $(d_{n,k})_{n,k\in\mathbb{N}}$ defined by the triple (d_0, A, Z) with:
$d_0 = 1$,$A = (1, 1, 1, ...)$,$Z = (1, 1, 1, ...)$
We have $A(t) = Z(t) = \frac{1}{1-t}$ and from corollary 2.1.2 and Theorem 2.1.6 we have:

$$d(t) = h(t) = A(th(t)) = \frac{1}{1-th(t)}$$

by solving functional equation $h(t) = A(th(t))$ above we can find the generating function of the catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$:

$$d(t) = h(t) = 1 + th(t)^2 = \frac{1-\sqrt{1-4t}}{2t}$$

$c(t) = \frac{1-\sqrt{1-4t}}{2t}$ is the generating function of the catalan numbers.

The generic element of the Riordan array is given by corollary 2.1.3:

$$C_{n,k} = \frac{k+1}{n+1}[t^{n-k}](\frac{1}{1-t})^{n+1} = \frac{k+1}{n+1}\binom{-n-1}{n-k}(-1)^{n-k} = \frac{k+1}{n+1}\binom{2n-k}{n-k}$$

These values represent the number of nodes at level n having label $k+1$ in the rooted labelled tree describe in Figure
that obtained from specification(2.12).

Conclusion

In this project, the concept of level generating trees corresponding to Riordan arrays in particular, proper Riordan Array whose generic element $d_{n,k}, where n, k \in \mathbb{N}$ depends in a simple way from the elements of several previous rows are discussed.The main result is to know a theorem which states the conditions under which a matrix associated to a Level generating tree is a proper Riordan array, and vice versa. we have seen several examples that allow us to associate the new concept of level generating trees to a class of Riordan arrays.

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