SINGLE-MODE THREE-LEVEL LASER COUPLED TO THERMAL RESERVOIR

By

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Abstract

In this thesis, we have studied the quantum properties of the light produced by degenerate three-level laser in which three-level atoms available in a cavity coupled to a thermal reservoir are pumped from the lower to the upper level at the rate r_a. Employing the master equation, we have obtained the quantum Langevin equations for the cavity mode and atomic operators. With the aid of the solutions of these equations, we have calculated the mean and variance of the photon number, the quadrature variance, the quadrature squeezing and power spectrum for the cavity light.

We have found that the variance of the photon number is greater than the mean photon number, indicating that the light produced by degenerate three-level laser has super-Poissonian photon statistics. On the other hand, we have realized that the presence of thermal reservoir decreases the quadrature squeezing. The maximum quadrature squeezing of the light generated by degenerate three-level laser is found to be 50% for $\bar{n} = 0$ and 36.6% for $\bar{n} = 0.2$ below a vacuum state level.

Chapter 1

Introduction

A three-level laser is a quantum optical system in which light is produced by three-level atoms inside a cavity usually coupled to a vacuum reservoir [1-4].
The statistical and squeezing properties of the light generated by three-level laser have been investigated by several authors [5-21]. It is found that degenerate three-level laser can produce squeezed light under certain conditions [6-8].

In this thesis, we consider the case in which N degenerate three-level atoms in cascade configuration and available in a cavity coupled to a thermal reservoir via a single port-mirror are pumped from the lower level to the upper level at a rate r_a. We study the case in which the cavity mode interacts with thermal reservoir via a single port-mirror and the three-level atoms interact with the cavity mode as well as a vacuum reservoir.
We denote the upper, middle and lower levels of the three-level atom by $|a_j\rangle$, $|b_j\rangle$ and $|c_j\rangle$ respectively. A three-level atom may then decay spontaneously from level $|a_j\rangle$ or level $|b_j\rangle$ to level $|c_j\rangle$ at the rate γ or make a transition from level $|a_j\rangle$ to level $|b_j\rangle$ and from level $|b_j\rangle$ to level $|c_j\rangle$ by emitting two photons of the same frequency ω
as shown in figure 1.1.

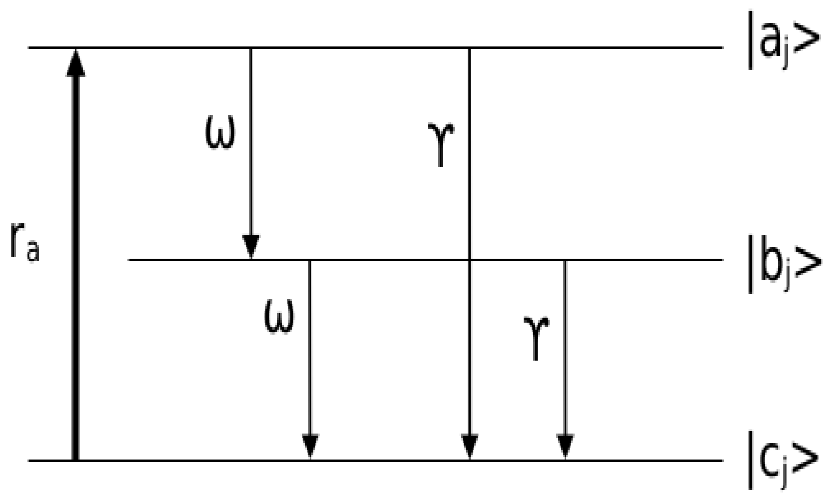

Figure 1.1: Degenerate three-level atom.

Applying the master equation for the system under consideration, we evaluate the equation of evolution for the expectation values of the cavity mode and atomic operators, and the correlation properties of the cavity mode and atomic noise operators. Thus taking the solutions of these equations into account, we determine the mean photon number, variance of the photon number, quadrature variance, quadrature squeezing and power spectrum for the cavity mode.

Chapter 2

Operator Dynamics

In this chapter we are going to study the dynamics of degenerate three-level laser. To this end, employing the master equation, we evaluate the equations of evolution for the expectation values of the cavity mode and atomic operators. Moreover, we determine the correlation properties of cavity mode and atomic noise operators.

The Hamiltonian describing the interaction of degenerate three-level atom with a cavity mode is expressible as [1]

$$\hat{H} = ig[\hat{a}^{\dagger}\left(|b_j\rangle\langle a_j| + |c_j\rangle\langle b_j|\right) - \left(|a_j\rangle\langle b_j| + |b_j\rangle\langle c_j|\right)\hat{a}], \tag{2.1}$$

where $|b_j\rangle\langle a_j|$ and $|c_j\rangle\langle b_j|$ are atomic operators, $\hat{a}$ is the annihilation operator for the cavity mode, g is the coupling constant between the atom and cavity mode.

The master equation that describes the interaction of degenerate three-level atom with cavity mode and vacuum reservoir, and interaction of cavity mode with the thermal reservoir is expressible as [1]

$$\begin{aligned}\frac{d\hat{\rho}}{dt} = &-i[\hat{H},\hat{\rho}] + \frac{\kappa}{2}(\bar{n}+1)(2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{a}) + \frac{\kappa}{2}\bar{n}(2\hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{a}\hat{a}^{\dagger}\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^{\dagger}) \\ &+ \frac{\gamma}{2}(2|c_j\rangle\langle a_j|\hat{\rho}|a_j\rangle\langle c_j| - |a_j\rangle\langle a_j|\hat{\rho} - \hat{\rho}|a_j\rangle\langle a_j| \\ &+ 2|c_j\rangle\langle b_j|\hat{\rho}|b_j\rangle\langle c_j| - |b_j\rangle\langle b_j|\hat{\rho} - \hat{\rho}|b_j\rangle\langle b_j|),\end{aligned} \tag{2.2}$$

where κ is the cavity damping constant, $\bar{n}$ is mean photon number of the thermal reservoir and γ is spontaneous emission decay constant.

Now using Eq. (2.1) in Eq. (2.2), the master equation for the system under consideration is expressible as

$$\frac{d\hat{\rho}}{dt} = g(\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{\rho} + \hat{a}^\dagger|c_j\rangle\langle b_j|\hat{\rho} - |a_j\rangle\langle b_j|\hat{a}\hat{\rho} - |b_j\rangle\langle c_j|\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger|b_j\rangle\langle a_j| - \hat{\rho}\hat{a}^\dagger|c_j\rangle\langle b_j|$$
$$+ \hat{\rho}|a_j\rangle\langle b_j|\hat{a} + \hat{\rho}|b_j\rangle\langle c_j|\hat{a}) + \frac{\kappa}{2}(\bar{n}+1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a})$$
$$+ \frac{\kappa}{2}\bar{n}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) + \frac{\gamma}{2}(2|c_j\rangle\langle a_j|\hat{\rho}|a_j\rangle\langle c_j| - |a_j\rangle\langle a_j|\hat{\rho} - \hat{\rho}|a_j\rangle\langle a_j|$$
$$+ 2|c_j\rangle\langle b_j|\hat{\rho}|b_j\rangle\langle c_j| - |b_j\rangle\langle b_j|\hat{\rho} - \hat{\rho}|b_j\rangle\langle b_j|). \tag{2.3}$$

2.1 Equation of evolution for the expectation values of the cavity mode and atomic operators

In this section we seek to evaluate the time evolution for the expectation values of the cavity mode and atomic operators. To this end, applying the relation

$$\frac{d}{dt}\langle\hat{A}\rangle \;=\; Tr\left(\frac{d\hat{\rho}}{dt}\hat{A}\right) \tag{2.4}$$

along with the master equation described by Eq. (2.3), we have

$$\frac{d}{dt}\langle\hat{a}\rangle \;=\; Tr\left(\frac{d\hat{\rho}}{dt}\hat{a}\right)$$
$$= gTr(\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{\rho}\hat{a} + \hat{a}^\dagger|c_j\rangle\langle b_j|\hat{\rho}\hat{a} - |a_j\rangle\langle b_j|\hat{a}\hat{\rho}\hat{a} - |b_j\rangle\langle c_j|\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{a}$$
$$- \hat{\rho}\hat{a}^\dagger|c_j\rangle\langle b_j|\hat{a} + \hat{\rho}|a_j\rangle\langle b_j|\hat{a}^2 + \hat{\rho}|b_j\rangle\langle c_j|\hat{a}^2) + \frac{\kappa}{2}(\bar{n}+1)Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2)$$
$$+ \frac{\kappa}{2}\bar{n}Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}^2 - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}) + \frac{\gamma}{2}Tr(2|c_j\rangle\langle a_j|\hat{\rho}|a_j\rangle\langle c_j|\hat{a} - |a_j\rangle\langle a_j|\hat{\rho}\hat{a} - \hat{\rho}|a_j\rangle\langle a_j|\hat{a}$$
$$+ 2|c_j\rangle\langle b_j|\hat{\rho}|b_j\rangle\langle c_j|\hat{a} - |b_j\rangle\langle b_j|\hat{\rho}\hat{a} - \hat{\rho}|b_j\rangle\langle b_j|\hat{a}). \tag{2.5}$$

Now applying the cyclic property of trace operation to Eq. (2.5), one obtains

$$\frac{d}{dt}\langle\hat{a}\rangle = g[Tr(\hat{\rho}\hat{a}\hat{a}^\dagger|b_j\rangle\langle a_j|) + Tr(\hat{\rho}\hat{a}\hat{a}^\dagger|c_j\rangle\langle b_j|) - Tr(\hat{\rho}\hat{a}|a_j\rangle\langle b_j|\hat{a}) - Tr(\hat{\rho}\hat{a}|b_j\rangle\langle c_j|\hat{a})$$
$$- Tr(\hat{\rho}\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{a}) - Tr(\hat{\rho}\hat{a}^\dagger|c_j\rangle\langle b_j|\hat{a}) + Tr(\hat{\rho}|a_j\rangle\langle b_j|\hat{a}^2) + Tr(\hat{\rho}|b_j\rangle\langle c_j|\hat{a}^2)]$$
$$+ \frac{\kappa}{2}(\bar{n}+1)[2Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^2) - Tr(\hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}) - Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^2)]$$
$$+ \frac{\kappa}{2}\bar{n}[2Tr(\hat{\rho}\hat{a}^2\hat{a}^\dagger) - Tr(\hat{\rho}\hat{a}^2\hat{a}^\dagger) - Tr(\hat{\rho}\hat{a}\hat{a}^\dagger\hat{a})]$$
$$+ \frac{\gamma}{2}[2Tr(\hat{\rho}|a_j\rangle\langle c_j|\hat{a}|c_j\rangle\langle a_j|) - Tr(\hat{\rho}\hat{a}|a_j\rangle\langle a_j|) - Tr(\hat{\rho}|a_j\rangle\langle a_j|\hat{a})$$
$$+ 2Tr(\hat{\rho}|b_j\rangle\langle c_j|\hat{a}|c_j\rangle\langle b_j|) - Tr(\hat{\rho}\hat{a}|b_j\rangle\langle b_j|) - Tr(\hat{\rho}|b_j\rangle\langle b_j|\hat{a})]. \tag{2.6}$$

Assuming that the cavity mode and atomic operators commute, and employing the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, we can rewrite Eq. (2.6) as

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}\rangle = g[&\langle(\hat{a}^\dagger\hat{a}+1)|b_j\rangle\langle a_j|\rangle + \langle(\hat{a}^\dagger\hat{a}+1)|c_j\rangle\langle b_j|\rangle - \langle|a_j\rangle\langle b_j|\hat{a}^2\rangle \\ &- \langle|b_j\rangle\langle c_j|\hat{a}^2\rangle - \langle\hat{a}^\dagger\hat{a}|b_j\rangle\langle a_j|\rangle - \langle\hat{a}^\dagger\hat{a}|c_j\rangle\langle b_j|\rangle + \langle|a_j\rangle\langle b_j|\hat{a}^2\rangle \\ &+ \langle|b_j\rangle\langle c_j|\hat{a}^2\rangle] + \frac{\kappa}{2}(\bar{n}+1)(\langle\hat{a}^\dagger\hat{a}^2\rangle - \langle\hat{a}^\dagger\hat{a}^2\rangle - \langle\hat{a}\rangle) + \frac{\kappa}{2}\bar{n}(\langle\hat{a}^2\hat{a}^\dagger\rangle \\ &- \langle\hat{a}^2\hat{a}^\dagger\rangle + \langle\hat{a}\rangle) + \frac{\gamma}{2}(2\langle|a_j\rangle\langle c_j|c_j\rangle\langle a_j|\hat{a}\rangle - \langle|a_j\rangle\langle a_j|\hat{a}\rangle \\ &- \langle|a_j\rangle\langle a_j|\hat{a}\rangle + 2\langle|b_j\rangle\langle c_j|c_j\rangle\langle b_j|\hat{a}\rangle - \langle|b_j\rangle\langle b_j|\hat{a}\rangle - \langle|b_j\rangle\langle b_j|\hat{a}\rangle) \\ = &\, g(\langle|b_j\rangle\langle a_j|\rangle + \langle|c_j\rangle\langle b_j|\rangle) - \frac{\kappa}{2}(\bar{n}+1)\langle\hat{a}\rangle + \frac{\kappa}{2}\bar{n}\langle\hat{a}\rangle \\ &+ \frac{\gamma}{2}(2\langle|a_j\rangle\langle a_j|\hat{a}\rangle - 2\langle|a_j\rangle\langle a_j|\hat{a}\rangle + 2\langle|b_j\rangle\langle b_j|\hat{a}\rangle - 2\langle|b_j\rangle\langle b_j|\hat{a}\rangle),\end{aligned} \tag{2.7}$$

or

$$\frac{d}{dt}\langle\hat{a}\rangle = -\frac{\kappa}{2}\langle\hat{a}\rangle + g(\langle\hat{\sigma}_a^j\rangle + \langle\hat{\sigma}_b^j\rangle), \tag{2.8}$$

where

$$\hat{\sigma}_a^j = |b_j\rangle\langle a_j|, \tag{2.9}$$

$$\hat{\sigma}_b^j = |c_j\rangle\langle b_j|, \tag{2.10}$$

are atomic operators.

Moreover, employing Eq. (2.4) along with Eq. (2.3), we see that

$$\begin{aligned}\frac{d}{dt}\langle|b_j\rangle\langle a_j|\rangle &= Tr(\frac{d\hat{\rho}}{dt}|b_j\rangle\langle a_j|) \\ &= gTr(\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{\rho}|b_j\rangle\langle a_j| + \hat{a}^\dagger|c_j\rangle\langle b_j|\hat{\rho}|b_j\rangle\langle a_j| - |a_j\rangle\langle b_j|\hat{a}\hat{\rho}|b_j\rangle\langle a_j| \\ &\quad - |b_j\rangle\langle c_j|\hat{a}\hat{\rho}|b_j\rangle\langle a_j| - \hat{\rho}\hat{a}^\dagger|b_j\rangle\langle a_j|b_j\rangle\langle a_j| - \hat{\rho}\hat{a}^\dagger|c_j\rangle\langle b_j|b_j\rangle\langle a_j| + \hat{\rho}|a_j\rangle\langle b_j|\hat{a}|b_j\rangle\langle a_j| \\ &\quad + \hat{\rho}|b_j\rangle\langle c_j|\hat{a}|b_j\rangle\langle a_j|) + \frac{\kappa}{2}(\bar{n}+1)Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger|b_j\rangle\langle a_j| - \hat{a}^\dagger\hat{a}\hat{\rho}|b_j\rangle\langle a_j| - \hat{\rho}\hat{a}^\dagger\hat{a}|b_j\rangle\langle a_j|) \\ &\quad + \frac{\kappa}{2}\bar{n}Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}|b_j\rangle\langle a_j| - \hat{a}\hat{a}^\dagger\hat{\rho}|b_j\rangle\langle a_j| - \hat{\rho}\hat{a}\hat{a}^\dagger|b_j\rangle\langle a_j|) \\ &\quad + \frac{\gamma}{2}Tr(2|c_j\rangle\langle a_j|\hat{\rho}|a_j\rangle\langle c_j|b_j\rangle\langle a_j| - |a_j\rangle\langle a_j|\hat{\rho}|b_j\rangle\langle a_j| - \hat{\rho}|a_j\rangle\langle a_j|b_j\rangle\langle a_j| \\ &\quad + 2|c_j\rangle\langle b_j|\hat{\rho}|b_j\rangle\langle c_j|b_j\rangle\langle a_j| - |b_j\rangle\langle b_j|\hat{\rho}|b_j\rangle\langle a_j| - \hat{\rho}|b_j\rangle\langle b_j|b_j\rangle\langle a_j|)\end{aligned}$$

$$
\begin{aligned}
&= gTr(\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{\rho}|b_j\rangle\langle a_j| + \hat{a}^\dagger|c_j\rangle\langle b_j|\hat{\rho}|b_j\rangle\langle a_j| - |a_j\rangle\langle b_j|\hat{a}\hat{\rho}|b_j\rangle\langle a_j| \\
&\quad - |b_j\rangle\langle c_j|\hat{a}\hat{\rho}|b_j\rangle\langle a_j| - \hat{\rho}\hat{a}^\dagger|c_j\rangle\langle a_j| + \hat{\rho}|a_j\rangle\langle b_j|\hat{a}|b_j\rangle\langle a_j| + \hat{\rho}|b_j\rangle\langle c_j|\hat{a}|b_j\rangle\langle a_j|) \\
&\quad + \frac{\kappa}{2}(\bar{n}+1)Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger|b_j\rangle\langle a_j| - \hat{a}^\dagger\hat{a}\hat{\rho}|b_j\rangle\langle a_j| - \hat{\rho}\hat{a}^\dagger\hat{a}|b_j\rangle\langle a_j|) \\
&\quad + \frac{\kappa}{2}\bar{n}Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}|b_j\rangle\langle a_j| - \hat{a}\hat{a}^\dagger\hat{\rho}|b_j\rangle\langle a_j| - \hat{\rho}\hat{a}\hat{a}^\dagger|b_j\rangle\langle a_j|) \\
&\quad + \frac{\gamma}{2}Tr(-|a_j\rangle\langle a_j|\hat{\rho}|b_j\rangle\langle a_j| - |b_j\rangle\langle b_j|\hat{\rho}|b_j\rangle\langle a_j| - \hat{\rho}|b_j\rangle\langle a_j|).
\end{aligned}
\tag{2.11}
$$

So that applying the cyclic property of trace operation to Eq. (2.11), we have

$$
\begin{aligned}
\frac{d}{dt}\langle|b_j\rangle\langle a_j|\rangle &= g[Tr(\hat{\rho}|b_j\rangle\langle a_j|\hat{a}^\dagger|b_j\rangle\langle a_j|) + Tr(\hat{\rho}|b_j\rangle\langle a_j|\hat{a}^\dagger|c_j\rangle\langle b_j|) \\
&\quad - Tr(\hat{\rho}|b_j\rangle\langle a_j|a_j\rangle\langle b_j|\hat{a}) - Tr(\hat{\rho}|b_j\rangle\langle a_j|b_j\rangle\langle c_j|\hat{a}) - Tr(\hat{\rho}\hat{a}^\dagger|c_j\rangle\langle a_j|) \\
&\quad + Tr(\hat{\rho}|a_j\rangle\langle b_j|\hat{a}|b_j\rangle\langle a_j|) + Tr(\hat{\rho}|b_j\rangle\langle c_j|\hat{a}|b_j\rangle\langle a_j|)] \\
&\quad + \frac{\kappa}{2}(\bar{n}+1)[2Tr(\hat{\rho}\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{a}) - Tr(\hat{\rho}|b_j\rangle\langle a_j|\hat{a}^\dagger\hat{a}) - Tr(\hat{\rho}\hat{a}^\dagger\hat{a}|b_j\rangle\langle a_j|)] \\
&\quad + \frac{\kappa}{2}\bar{n}[2Tr(\hat{\rho}\hat{a}|b_j\rangle\langle a_j|\hat{a}^\dagger) - Tr(\hat{\rho}|b_j\rangle\langle a_j|\hat{a}\hat{a}^\dagger) - Tr(\hat{\rho}\hat{a}\hat{a}^\dagger|b_j\rangle\langle a_j|)] \\
&\quad + \frac{\gamma}{2}[Tr(-\hat{\rho}|b_j\rangle\langle a_j|a_j\rangle\langle a_j|) - Tr(\hat{\rho}|b_j\rangle\langle a_j|b_j\rangle\langle b_j|) - Tr(\hat{\rho}|b_j\rangle\langle a_j|)]
\end{aligned}
$$

$$
\begin{aligned}
&= g(-\langle|b_j\rangle\langle b_j|\hat{a}\rangle - \langle\hat{a}^\dagger|c_j\rangle\langle a_j|\rangle + \langle|a_j\rangle\langle a_j|\hat{a}\rangle) \\
&\quad + \frac{\kappa}{2}(\bar{n}+1)(2\langle\hat{a}^\dagger\hat{a}|b_j\rangle\langle a_j|\rangle - 2\langle\hat{a}^\dagger\hat{a}|b_j\rangle\langle a_j|\rangle) \\
&\quad + \frac{\kappa}{2}\bar{n}(2\langle(\hat{a}^\dagger\hat{a}+1)|b_j\rangle\langle a_j|\rangle - 2\langle(\hat{a}^\dagger\hat{a}+1)|b_j\rangle\langle a_j|\rangle) \\
&\quad - \frac{\gamma}{2}(\langle|b_j\rangle\langle a_j|\rangle + \langle|b_j\rangle\langle a_j|\rangle)
\end{aligned}
\tag{2.12}
$$

or

$$
\frac{d}{dt}\langle\hat{\sigma}_a^j\rangle = g(\langle\hat{\eta}_a^j\hat{a}\rangle - \langle\hat{\eta}_b^j\hat{a}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_c^j\rangle) - \gamma\langle\hat{\sigma}_a^j\rangle, \tag{2.13}
$$

where

$$
\hat{\eta}_a^j = |a_j\rangle\langle a_j|, \tag{2.14}
$$

$$
\hat{\eta}_b^j = |b_j\rangle\langle b_j|, \tag{2.15}
$$

$$
\hat{\sigma}_c^j = |c_j\rangle\langle a_j|, \tag{2.16}
$$

are atomic operators.

Furthermore, with the aid of Eq. (2.4) along Eq. (2.3), we find

$$
\begin{aligned}
\frac{d}{dt}\langle|c_j\rangle\langle b_j|\rangle &= Tr(\frac{d\hat{\rho}}{dt}|c_j\rangle\langle b_j|) \\
&= gTr(\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{\rho}|c_j\rangle\langle b_j| + \hat{a}^\dagger|c_j\rangle\langle b_j|\hat{\rho}|c_j\rangle\langle b_j| - |a_j\rangle\langle b_j|\hat{a}\hat{\rho}|c_j\rangle\langle b_j| \\
&\quad - |b_j\rangle\langle c_j|\hat{a}\hat{\rho}|c_j\rangle\langle b_j| - \hat{\rho}\hat{a}^\dagger|b_j\rangle\langle a_j|c_j\rangle b_j| - \hat{\rho}\hat{a}^\dagger|c_j\rangle\langle b_j|c_j\rangle\langle b_j| \\
&\quad + \hat{\rho}|a_j\rangle\langle b_j|\hat{a}|c_j\rangle\langle b_j| + \hat{\rho}|b_j\rangle\langle c_j|\hat{a}|c_j\rangle\langle b_j|) + \frac{\kappa}{2}(\bar{n}+1)Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger|c_j\rangle\langle b_j| \\
&\quad - \hat{a}^\dagger\hat{a}\hat{\rho}|c_j\rangle\langle b_j| - \hat{\rho}\hat{a}^\dagger\hat{a}|c_j\rangle\langle b_j|) + \frac{\kappa}{2}\bar{n}Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}|c_j\rangle\langle b_j| - \hat{a}\hat{a}^\dagger\hat{\rho}|c_j\rangle\langle b_j| \\
&\quad - \hat{\rho}\hat{a}\hat{a}^\dagger|c_j\rangle\langle b_j|) + \frac{\gamma}{2}Tr(2|c_j\rangle\langle a_j|\hat{\rho}|a_j\rangle\langle c_j|c_j\rangle\langle b_j| - |a_j\rangle\langle a_j|\hat{\rho}|c_j\rangle\langle b_j| \\
&\quad - \hat{\rho}|a_j\rangle\langle a_j|c_j\rangle\langle b_j| + 2|c_j\rangle\langle b_j|\hat{\rho}|b_j\rangle\langle c_j|c_j\rangle\langle b_j| - |b_j\rangle\langle b_j|\hat{\rho}|c_j\rangle\langle b_j| \\
&\quad - \hat{\rho}|b_j\rangle\langle b_j|c_j\rangle\langle b_j|) \\
&= gTr(\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{\rho}|c_j\rangle\langle b_j| + \hat{a}^\dagger|c_j\rangle\langle b_j|\hat{\rho}|c_j\rangle\langle b_j| - |a_j\rangle\langle b_j|\hat{a}\hat{\rho}|c_j\rangle\langle b_j| \\
&\quad - |b_j\rangle\langle c_j|\hat{a}\hat{\rho}|c_j\rangle\langle b_j| + \hat{\rho}|a_j\rangle\langle b_j|\hat{a}|c_j\rangle\langle b_j| + \hat{\rho}|b_j\rangle\langle c_j|\hat{a}|c_j\rangle\langle b_j|) \\
&\quad + \frac{\kappa}{2}(\bar{n}+1)Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger|c_j\rangle\langle b_j| - \hat{a}^\dagger\hat{a}\hat{\rho}|c_j\rangle\langle b_j| - \hat{\rho}\hat{a}^\dagger\hat{a}|c_j\rangle\langle b_j|) \\
&\quad + \frac{\kappa}{2}\bar{n}Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}|c_j\rangle\langle b_j| - \hat{a}\hat{a}^\dagger\hat{\rho}|c_j\rangle\langle b_j| - \hat{\rho}\hat{a}\hat{a}^\dagger|c_j\rangle\langle b_j| \\
&\quad + \frac{\gamma}{2}Tr(2|c_j\rangle\langle a_j|\hat{\rho}|a_j\rangle\langle b_j| - |a_j\rangle\langle a_j|\hat{\rho}|c_j\rangle\langle b_j| + 2|c_j\rangle\langle b_j|\hat{\rho}|b_j\rangle\langle b_j| \\
&\quad - |b_j\rangle\langle b_j|\hat{\rho}|c_j\rangle\langle b_j|).
\end{aligned}
\tag{2.17}
$$

Then employing the cyclic property of trace operation to Eq. (2.17), it follows that

$$
\begin{aligned}
\frac{d}{dt}\langle|c_j\rangle\langle b_j|\rangle &= g[Tr(\hat{\rho}|c_j\rangle\langle b_j|\hat{a}^\dagger|b_j\rangle\langle a_j|) + Tr(\hat{\rho}|c_j\rangle\langle b_j|\hat{a}^\dagger|c_j\rangle\langle b_j|) \\
&\quad - Tr(\hat{\rho}|c_j\rangle\langle b_j|a_j\rangle\langle b_j|\hat{a}) - Tr(\hat{\rho}|c_j\rangle\langle b_j|b_j\rangle\langle c_j|\hat{a}) \\
&\quad + Tr(\hat{\rho}|a_j\rangle\langle b_j|\hat{a}|c_j\rangle\langle b_j|) + Tr(\hat{\rho}|b_j\rangle\langle c_j|\hat{a}|c_j\rangle\langle b_j|)] \\
&\quad + \frac{\kappa}{2}(\bar{n}+1)[2Tr(\hat{\rho}\hat{a}^\dagger|c_j\rangle\langle b_j|\hat{a}) - Tr(\hat{\rho}|c_j\rangle\langle b_j|\hat{a}^\dagger\hat{a}) - Tr(\hat{\rho}\hat{a}^\dagger\hat{a}|c_j\rangle\langle b_j|)] \\
&\quad + \frac{\kappa}{2}\bar{n}[2Tr(\hat{\rho}\hat{a}|c_j\rangle\langle b_j|\hat{a}^\dagger) - Tr(\hat{\rho}|c_j\rangle\langle b_j|\hat{a}\hat{a}^\dagger) - Tr(\hat{\rho}\hat{a}\hat{a}^\dagger|c_j\rangle\langle b_j|)] \\
&\quad + \frac{\gamma}{2}[-Tr(\hat{\rho}|c_j\rangle\langle b_j|a_j\rangle\langle a_j|) - Tr(\hat{\rho}|c_j\rangle\langle b_j|b_j\rangle\langle b_j|) \\
&\quad - Tr(\hat{\rho}|c_j\rangle\langle b_j|c_j\rangle\langle c_j|) - Tr(\hat{\rho}|c_j\rangle\langle b_j|)]
\end{aligned}
$$

$$
\begin{aligned}
&= g(\langle \hat{a}^\dagger|c_j\rangle\langle a_j|\rangle - \langle|c_j\rangle\langle c_j|\hat{a}\rangle + \langle|b_j\rangle\langle b_j|\hat{a}\rangle) \\
&\quad + \frac{\kappa}{2}(\bar{n}+1)(2\langle \hat{a}^\dagger|\hat{a}|c_j\rangle\langle b_j\rangle - \langle \hat{a}^\dagger\hat{a}|c_j\rangle\langle b_j|\rangle - \langle \hat{a}^\dagger\hat{a}|c_j\rangle\langle b_j|\rangle) \\
&\quad + \frac{\kappa}{2}\bar{n}(2\langle(\hat{a}^\dagger\hat{a}+1)|c_j\rangle\langle b_j|\rangle - \langle(\hat{a}^\dagger\hat{a}+1)|c_j\rangle\langle b_j|\rangle \\
&\quad - \langle(\hat{a}^\dagger\hat{a}+1)|c_j\rangle\langle b_j|\rangle) - \frac{\gamma}{2}(\langle|c_j\rangle\langle b_j|\rangle + \langle|c_j\rangle\langle b_j|\rangle)
\end{aligned}
\tag{2.18}
$$

or

$$\frac{d}{dt}\langle\hat{\sigma}_b^j\rangle \;= g(\langle\hat{a}^\dagger\hat{\sigma}_c^j\rangle + \langle\hat{\eta}_b^j\hat{a}\rangle - \langle\hat{\eta}_c^j\hat{a}\rangle) - \gamma\langle\hat{\sigma}_b^j\rangle, \tag{2.19}$$

where

$$\hat{\eta}_c^j \;= |c_j\rangle\langle c_j|, \tag{2.20}$$

is atomic operator.

In a similar manner, it can be established that

$$\frac{d}{dt}\langle\hat{\sigma}_c^j\rangle = g(\langle\hat{\sigma}_a^j\hat{a}\rangle - \langle\hat{\sigma}_b^j\hat{a}\rangle) - \frac{\gamma}{2}\langle\hat{\sigma}_c^j\rangle, \tag{2.21}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^j\rangle = -g(\langle\hat{a}^\dagger\hat{\sigma}_a^j\rangle + \langle\hat{\sigma}_a^{\dagger j}\hat{a}\rangle) - \gamma\langle\hat{\eta}_a^j\rangle, \tag{2.22}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^j\rangle = g(\langle\hat{a}^\dagger\hat{\sigma}_a^j\rangle + \langle\hat{\sigma}_a^{\dagger j}\hat{a}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_b^j\rangle - \langle\hat{\sigma}_b^{\dagger j}\hat{a}\rangle) - \gamma\langle\hat{\eta}_b^j\rangle, \tag{2.23}$$

$$\frac{d}{dt}\langle\hat{\eta}_c^j\rangle = g(\langle\hat{a}^\dagger\hat{\sigma}_b^j\rangle + \langle\hat{\sigma}_b^{\dagger j}\hat{a}\rangle) + \gamma(\langle\hat{\eta}_a^j\rangle + \langle\hat{\eta}_b^j\rangle). \tag{2.24}$$

Based on Eq. (2.8), we can write

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + g(\hat{\sigma}_a^j + \hat{\sigma}_b^j) + \hat{g}_a(t), \tag{2.25}$$

where $\hat{g}_a(t)$ is cavity mode noise operator when the cavity mode interacts with a single three-level atom and whose correlation properties remain to be determined.

Taking the expectation value of Eq. (2.25) and comparing the resulting expression with Eq. (2.8), it immediately follows that

$$\langle\hat{g}_a(t)\rangle = 0. \tag{2.26}$$

We observe that Eqs. (2.13), (2.19), (2.21), (2.22), (2.23) and (2.24) are coupled non-linear differential equations and it is thus difficult to obtain their exact solutions.

We need to overcome this problem by applying the large-time approximation scheme [9]. We therefore use this approximation scheme to Eq. (2.25) and write

$$\hat{a}(t) \ = \frac{2}{\kappa}\Big[g(\hat{\sigma}_a^j(t) + \hat{\sigma}_b^j(t)) + \hat{g}_a(t)\Big]. \tag{2.27}$$

Now substituting Eq. (2.27) and its adjoint into Eqs. (2.13), (2.19), (2.21), (2.22), (2.23) and (2.24) respectively, we find

$$\frac{d}{dt}\langle\hat{\sigma}_a^j\rangle = \frac{2g}{\kappa}\Big[\langle\hat{\eta}_a^j(t)\hat{g}_a(t)\rangle - \langle\hat{\eta}_b^j(t)\hat{g}_a(t)\rangle - 2g\langle\hat{\sigma}_a^j(t)\rangle - \langle\hat{g}_a^\dagger(t)\hat{\sigma}_c^j(t)\rangle\Big] - \gamma\langle\hat{\sigma}_a^j(t)\rangle, \tag{2.28}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{\sigma}_b^j\rangle = \frac{2g}{\kappa}\Big[&2g\langle\hat{\sigma}_a^j(t)\rangle + \langle\hat{g}_a^\dagger(t)\hat{\sigma}_c^j(t)\rangle + \langle\hat{\eta}_b^j(t)\hat{g}_a(t)\rangle - g\langle\hat{\sigma}_b^j(t)\rangle - \langle\hat{\eta}_c^j(t)\hat{g}_a(t)\rangle\Big]\\ &- \gamma\langle\hat{\sigma}_b^j(t)\rangle,\end{aligned} \tag{2.29}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^j\rangle = \frac{2g}{\kappa}\Big[\langle\hat{\sigma}_a^j(t)\hat{g}_a(t)\rangle - g\langle\hat{\sigma}_c^j(t)\rangle - \langle\hat{\sigma}_b^j(t)\hat{g}_a(t)\rangle\Big] - \frac{\gamma}{2}\langle\hat{\sigma}_c^j(t)\rangle, \tag{2.30}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^j\rangle = -\frac{2g}{\kappa}\Big[2g\langle\hat{\eta}_a^j(t)\rangle + \langle\hat{g}_a^\dagger(t)\hat{\sigma}_a^j(t)\rangle + \langle\hat{\sigma}_a^{\dagger j}(t)\hat{g}_a(t)\rangle\Big] - \gamma\langle\hat{\eta}_a^j(t)\rangle, \tag{2.31}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{\eta}_b^j\rangle = \frac{2g}{\kappa}\Big[&2g\langle\hat{\eta}_a^j(t)\rangle + \langle\hat{g}_a^\dagger(t)\hat{\sigma}_a^j(t)\rangle + \langle\hat{\sigma}_a^{\dagger j}(t)\hat{g}_a(t)\rangle - 2g\langle\hat{\eta}_b^j(t)\rangle\\ &- \langle\hat{g}_a^\dagger(t)\hat{\sigma}_b^j(t)\rangle - \langle\hat{\sigma}_b^{\dagger j}(t)\hat{g}_a(t)\rangle\Big] - \gamma\langle\hat{\eta}_b^j(t)\rangle,\end{aligned} \tag{2.32}$$

$$\frac{d}{dt}\langle\hat{\eta}_c^j\rangle = \frac{2g}{\kappa}\Big[2g\langle\hat{\eta}_b^j(t)\rangle + \langle\hat{g}_a^\dagger(t)\hat{\sigma}_b^j(t)\rangle + \langle\hat{\sigma}_b^{\dagger j}(t)\hat{g}_a(t)\rangle\Big] + \gamma(\langle\hat{\eta}_a^j(t)\rangle + \langle\hat{\eta}_b^j(t)\rangle). \tag{2.33}$$

We next proceed to determine the expectation values of the products of cavity mode noise operator and atomic operator appearing in Eqs. (2.28)-(2.33).

Now employing Eq. (2.4) along with Eq. (2.3), we have

$$\frac{d}{dt}\langle|a_j\rangle\langle a_j|\hat{a}\rangle = Tr(\frac{d\hat{\rho}}{dt}|a_j\rangle\langle a_j|\hat{a})$$

$$
\begin{aligned}
&= gTr(\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{\rho}|a_j\rangle\langle a_j|\hat{a} + \hat{a}^\dagger|c_j\rangle\langle b_j|\hat{\rho}|a_j\rangle\langle a_j|\hat{a} - |a_j\rangle\langle b_j|\hat{a}\hat{\rho}|a_j\rangle\langle a_j|\hat{a} \\
&\quad - |b_j\rangle\langle c_j|\hat{a}\hat{\rho}|a_j\rangle\langle a_j|\hat{a} - \hat{\rho}\hat{a}^\dagger|b_j\rangle\langle a_j|a_j\rangle\langle a_j|\hat{a} - \hat{\rho}\hat{a}^\dagger|c_j\rangle\langle b_j|a_j\rangle\langle a_j|\hat{a} \\
&\quad + \hat{\rho}|a_j\rangle\langle b_j|\hat{a}|a_j\rangle\langle a_j|\hat{a} + \hat{\rho}|b_j\rangle\langle c_j|\hat{a}|a_j\rangle\langle a_j|\hat{a}) \\
&\quad + \frac{\kappa}{2}(\bar{n}+1)Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger|a_j\rangle\langle a_j|\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}|a_j\rangle\langle a_j|\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}|a_j\rangle\langle a_j|\hat{a}) \\
&\quad + \frac{\kappa}{2}\bar{n}Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}|a_j\rangle\langle a_j|\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho}|a_j\rangle\langle a_j|\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger|a_j\rangle\langle a_j|\hat{a}) \\
&\quad + \frac{\gamma}{2}Tr(2|c_j\rangle\langle a_j|\hat{\rho}|a_j\rangle\langle c_j|a_j\rangle\langle a_j|\hat{a} - |a_j\rangle\langle a_j|\hat{\rho}|a_j\rangle\langle a_j|\hat{a} - \hat{\rho}|a_j\rangle\langle a_j|a_j\rangle\langle a_j|\hat{a} \\
&\quad + 2|c_j\rangle\langle b_j|\hat{\rho}|b_j\rangle\langle c_j|a_j\rangle\langle a_j|\hat{a} - |b_j\rangle\langle b_j|\hat{\rho}|a_j\rangle\langle a_j|\hat{a} - \hat{\rho}|b_j\rangle\langle b_j|a_j\rangle\langle a_j|\hat{a} \\
&= gTr(\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{\rho}|a_j\rangle\langle a_j|\hat{a} + \hat{a}^\dagger|c_j\rangle\langle b_j|\hat{\rho}|a_jb\rangle\langle a_j|\hat{a} - |a_j\rangle\langle b_j|\hat{a}\hat{\rho}|a_j\rangle\langle a_j|\hat{a} \\
&\quad - |b_j\rangle\langle c_j|\hat{a}\hat{\rho}|a_j\rangle\langle a_j|\hat{a} - \hat{\rho}\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{a} + \hat{\rho}|a_j\rangle\langle b_j|\hat{a}|a_j\rangle\langle a_j|\hat{a} + \hat{\rho}|b_j\rangle\langle c_j|\hat{a}|a_j\rangle\langle a_j|\hat{a}) \\
&\quad + \frac{\kappa}{2}(\bar{n}+1)Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger|a_j\rangle\langle a_j|\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}|a_j\rangle\langle a_j|\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}|a_j\rangle\langle a_j|\hat{a}) \\
&\quad + \frac{\kappa}{2}\bar{n}Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}|a_j\rangle\langle a_j|\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho}|a_j\rangle\langle a_j|\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger|a_j\rangle\langle a_j|\hat{a}) \\
&\quad + \frac{\gamma}{2}Tr(-|a_j\rangle\langle a_j|\hat{\rho}|a_j\rangle\langle a_j|\hat{a} - \hat{\rho}|a_j\rangle\langle a_j|\hat{a} - |b_j\rangle\langle b_j|\hat{\rho}|a_j\rangle\langle a_j|\hat{a}).
\end{aligned} \tag{2.34}
$$

Moreover, using the cyclic property of trace operation to Eq. (2.34), we see that

$$
\begin{aligned}
\frac{d}{dt}\langle|a_j\rangle\langle a_j|\hat{a}\rangle &= g[Tr(\hat{\rho}|a_j\rangle\langle a_j|\hat{a}\hat{a}^\dagger|b_j\rangle\langle a_j|) + Tr(\hat{\rho}|a_j\rangle\langle a_j|\hat{a}\hat{a}^\dagger|c_j\rangle\langle b_j|) \\
&\quad - Tr(\hat{\rho}|a_j\rangle\langle a_j|\hat{a}|a_j\rangle\langle b_j|\hat{a}) - Tr(\hat{\rho}|a_j\rangle\langle a_j|\hat{a}|b_j\rangle\langle c_j|\hat{a}) \\
&\quad - Tr(\hat{\rho}\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{a}) + Tr(\hat{\rho}|a_j\rangle\langle b_j|\hat{a}|a_j\rangle\langle a_j|\hat{a}) \\
&\quad + Tr(\hat{\rho}|b_j\rangle\langle c_j|\hat{a}|a_j\rangle\langle a_j|\hat{a})] + \frac{\kappa}{2}(\bar{n}+1)[Tr(2\hat{\rho}\hat{a}^\dagger|a_j\rangle\langle a_j|\hat{a}\hat{a}) \\
&\quad - Tr(\hat{\rho}|a_j\rangle\langle a_j|\hat{a}\hat{a}^\dagger\hat{a}) - Tr(\hat{\rho}\hat{a}^\dagger\hat{a}|a_j\rangle\langle a_j|\hat{a})] + \frac{\kappa}{2}\bar{n}[Tr(2\hat{\rho}\hat{a}|a_j\rangle\langle a_j|\hat{a}\hat{a}^\dagger) \\
&\quad - Tr(\hat{\rho}|a_j\rangle\langle a_j|\hat{a}\hat{a}\hat{a}^\dagger) - Tr(\hat{\rho}\hat{a}\hat{a}^\dagger|a_j\rangle\langle a_j|\hat{a})] \\
&\quad + \frac{\gamma}{2}[-Tr(\hat{\rho}|a_j\rangle\langle a_j|\hat{a}|a_j\rangle\langle a_j|) - Tr(\hat{\rho}|a_j\rangle\langle a_j|\hat{a}) \\
&\quad - Tr(\hat{\rho}|a_j\rangle\langle a_j|\hat{a}|b_j\rangle\langle b_j|)], \\
&= -g(\langle\hat{a}^\dagger\hat{a}|b_j\rangle\langle a_j|\rangle + \langle|a_j\rangle\langle b_j|\hat{a}^2\rangle) + \frac{\kappa}{2}(\bar{n}+1)(\langle\hat{a}^\dagger\hat{a}^2|a_j\rangle\langle a_j|\rangle - \langle\hat{a}\hat{a}^\dagger\hat{a}|a_j\rangle\langle a_j|\rangle) \\
&\quad + \frac{\kappa}{2}\bar{n}(\langle\hat{a}^2\hat{a}^\dagger|a_j\rangle\langle a_j|\rangle - \langle\hat{a}\hat{a}^\dagger\hat{a}|a_j\rangle\langle a_j|\rangle) - \gamma\langle|a_j\rangle\langle a_j|\hat{a}\rangle,
\end{aligned}
$$

or

$$\frac{d}{dt}\langle\hat{\eta}_a^j\hat{a}\rangle = -g(\langle\hat{a}^\dagger\hat{a}\hat{\sigma}_a^j\rangle + \langle\hat{\sigma}_a^{\dagger j}\hat{a}^2\rangle) - \frac{\kappa}{2}\langle\hat{\eta}_a^j\hat{a}\rangle - \gamma\langle\hat{\eta}_a^j\hat{a}\rangle. \tag{2.35}$$

Furthermore, taking Eqs. (2.9) and (2.14) into consideration, we can rewrite Eq. (2.22) including a noise operator as

$$\frac{d}{dt}|a_j\rangle\langle a_j| = -g(\hat{a}^\dagger|b_j\rangle\langle a_j| + |a_j\rangle\langle|b_j|\hat{a}) - \gamma|a_j\rangle\langle a_j| + \hat{G}_{aj}(t), \tag{2.36}$$

where $\hat{G}_{aj}(t)$ is the atomic noise operator.

Now writting the expectation value of Eq. (2.36) as

$$\frac{d}{dt}\langle|a_j\rangle\langle a_j|\rangle = -g(\langle\hat{a}^\dagger|b_j\rangle\langle a_j|\rangle + \langle|a_j\rangle\langle|b_j|\hat{a}\rangle) - \gamma\langle|a_j\rangle\langle a_j|\rangle + \langle\hat{G}_{aj}(t)\rangle \tag{2.37}$$

and comparing with Eq. (2.22), we find

$$\langle\hat{G}_{aj}(t)\rangle = 0. \tag{2.38}$$

Using the mathematical relation

$$\frac{d}{dt}\Big\langle|a_j\rangle\langle a_j|\hat{a}\Big\rangle = \Big\langle\frac{d|a_j\rangle\langle a_j|}{dt}\hat{a}\Big\rangle + \Big\langle|a_j\rangle\langle a_j|\frac{d\hat{a}}{dt}\Big\rangle, \tag{2.39}$$

along with Eqs. (2.25) and (2.36), we have

$$\begin{aligned}\frac{d}{dt}\Big\langle|a_j\rangle\langle a_j|\hat{a}\Big\rangle &= \Big\langle -g(\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{a} + |a_j\rangle\langle b_j|\hat{a}^2) - \gamma|a_j\rangle\langle a_j|\hat{a} + \hat{G}_{aj}(t)\hat{a}\Big\rangle \\ &+ \Big\langle|a_j\rangle\langle a_j|\Big[g(|b_j\rangle\langle a_j| + |c_j\rangle\langle b_j|) - \frac{\kappa}{2}\hat{a} + \hat{g}_a(t)\Big]\Big\rangle \\ &= -g(\langle\hat{a}^\dagger\hat{a}|b_j\rangle\langle a_j| + |a_j\rangle\langle b_j|\hat{a}^2) - \frac{\kappa}{2}\langle|a_j\rangle\langle a_j|\hat{a}\rangle \\ &- \gamma\langle|a_j\rangle\langle a_j|\hat{a}\rangle + \langle|a_j\rangle\langle a_j|\hat{g}_a(t)\rangle + \langle\hat{G}_{aj}(t)\hat{a}(t)\rangle\end{aligned}$$

or

$$\frac{d}{dt}\langle\hat{\eta}_a^j\hat{a}\rangle = -g(\langle\hat{a}^\dagger\hat{a}\hat{\sigma}_a^j\rangle + \langle\hat{\sigma}_a^{\dagger j}\hat{a}^2\rangle) - \frac{\kappa}{2}\langle\hat{\eta}_a^j\hat{a}\rangle - \gamma\langle\hat{\eta}_a^j\hat{a}\rangle + \langle\hat{\eta}_a^j(t)\hat{g}_a(t)\rangle + \langle\hat{G}_{aj}(t)\hat{a}(t)\rangle. \tag{2.40}$$

Upon comparing Eqs. (2.35) and (2.40), we see that

$$\langle\hat{\eta}_a^j(t)\hat{g}_a(t)\rangle + \langle\hat{G}_{aj}(t)\hat{a}(t)\rangle = 0. \tag{2.41}$$

The formal solution of Eq. (2.36) can be written as

$$\hat{\eta}_a^j(t) = \hat{\eta}_a^j(0)e^{-\gamma t} + \int_0^t e^{-\gamma(t-t')}\Big[-g\hat{a}^\dagger(t')\hat{\sigma}_a^j(t') - g\hat{\sigma}_a^{\dagger j}(t')\hat{a}(t') + \hat{G}_{aj}(t')\Big]dt'. \tag{2.42}$$

Now multiplying Eq. (2.42) from the right by $\hat{g}_a(t)$ and taking the expectation value of the resulting expression, we obtain

$$\begin{aligned}\langle\hat{\eta}_a^j(t)\hat{g}_a(t)\rangle = \langle\hat{\eta}_a^j(0)\hat{g}_a(t)\rangle e^{-\gamma t} + \int_0^t e^{-\gamma(t-t')}\Big[&-g\langle\hat{a}^\dagger(t')\hat{\sigma}_a^j(t')\hat{g}_a(t)\rangle \\ &- g\langle\hat{\sigma}_a^{\dagger j}(t')\hat{a}(t')\hat{g}_a(t)\rangle + \langle\hat{G}_{aj}(t')\hat{g}_a(t)\rangle\Big]dt'.\end{aligned} \tag{2.43}$$

Taking Eq. (2.26) into account along with the assertion that a noise operator at a certain time t should not affect the atomic and cavity mode operators at an earlier time, we note that

$$\langle\hat{\eta}_a^j(0)\hat{g}_a(t)\rangle = \langle\hat{\eta}_a^j(0)\rangle\langle\hat{g}_a(t)\rangle = 0, \tag{2.44}$$

$$\langle\hat{a}^\dagger(t')\hat{\sigma}_a^j(t')\hat{g}_a(t)\rangle = \langle\hat{a}^\dagger(t')\hat{\sigma}_a^j(t')\rangle\langle\hat{g}_a(t)\rangle = 0, \tag{2.45}$$

$$\langle\hat{\sigma}_a^{\dagger j}(t')\hat{a}(t')\hat{g}_a(t)\rangle = \langle\hat{\sigma}_a^{\dagger j}(t')\hat{a}(t')\rangle\langle\hat{g}_a(t)\rangle = 0. \tag{2.46}$$

Eq. (2.43) thus becomes

$$\langle\hat{\eta}_a^j(t)\hat{g}_a(t)\rangle = \int_0^t e^{-\gamma(t-t')}\langle\hat{G}_{aj}(t')\hat{g}_a(t)\rangle dt'. \tag{2.47}$$

Moreover, a formal solution of Eq. (2.25) can be written as

$$\hat{a}(t) = \hat{a}(0)e^{\frac{-\kappa}{2}(t)} + g\int_0^t e^{\frac{-\kappa}{2}(t-t')}\Big[\hat{\sigma}_a^j(t') + \hat{\sigma}_b^j(t')\Big]dt' + \int_0^t e^{\frac{-\kappa}{2}(t-t')}\hat{g}_a(t')dt'. \tag{2.48}$$

Then multiplying Eq. (2.48) on the left by $\hat{G}_{aj}(t)$ and taking the expectation value of the resulting expression, we get

$$\begin{aligned}\langle\hat{G}_{aj}(t)\hat{a}(t)\rangle = \langle\hat{G}_{aj}(t)\hat{a}(0)\rangle e^{\frac{-\kappa}{2}(t)} + \int_0^t e^{\frac{-\kappa}{2}(t-t')}\Big[&g\langle\hat{G}_{aj}(t)\hat{\sigma}_a^j(t')\rangle \\ &+ g\langle\hat{G}_{aj}(t)\hat{\sigma}_b^j(t')\rangle + \langle\hat{G}_{aj}(t)\hat{g}_a(t')\rangle\Big]dt',\end{aligned} \tag{2.49}$$

Also since the noise operator at a certain time t has not affect atomic and cavity mode operators at an earlier time and in view of Eq. (2.38), we see that

$$\langle\hat{G}_{aj}(t)\hat{a}(0)\rangle = \langle\hat{G}_{aj}(t)\rangle\langle\hat{a}(0)\rangle = 0, \tag{2.50}$$

$$\langle \hat{G}_{aj}(t)\hat{\sigma}_a^j(t')\rangle = \langle \hat{G}_{aj}(t)\rangle\langle \hat{\sigma}_a^j(t')\rangle = 0, \tag{2.51}$$

$$\langle \hat{G}_{aj}(t)\hat{\sigma}_b^j(t')\rangle = \langle \hat{G}_{aj}(t)\rangle\langle \hat{\sigma}_b^j(t')\rangle = 0. \tag{2.52}$$

Then Eq. (2.49) reduces to

$$\langle \hat{G}_{aj}(t)\hat{a}(t)\rangle \ = \int_0^t e^{\frac{-\kappa}{2}(t-t')}\langle \hat{G}_{aj}(t)\hat{g}_a(t')\rangle dt'. \tag{2.53}$$

Now introducing Eqs. (2.47) and (2.53) into Eq. (2.41), we obtain

$$\int_0^t e^{-\gamma(t-t')}\langle \hat{G}_{aj}(t')\hat{g}_a(t)\rangle dt' + \int_0^t e^{\frac{-\kappa}{2}(t-t')}\langle \hat{G}_{aj}(t)\hat{g}_a(t')\rangle dt' = 0, \tag{2.54}$$

and assuming that $\langle \hat{G}_{aj}(t')\hat{g}_a(t)\rangle = \langle \hat{G}_{aj}(t)\hat{g}_a(t')\rangle$, we see that

$$\left[\int_0^t e^{-\gamma(t-t')} + \int_0^t e^{\frac{-\kappa}{2}(t-t')}\right] dt'\langle \hat{G}_{aj}(t)\hat{g}_a(t')\rangle = 0, \tag{2.55}$$

It then follows that

$$\langle \hat{G}_{aj}(t)\hat{g}_a(t')\rangle = 0. \tag{2.56}$$

So that using Eq. (2.56) in Eq. (2.53), one obtains

$$\langle \hat{G}_{aj}(t)\hat{a}(t)\rangle = 0, \tag{2.57}$$

with the aid of which Eq. (2.41) becomes

$$\langle \hat{\eta}_a^j(t)\hat{g}_a(t)\rangle = 0. \tag{2.58}$$

Following the same procedure, we can readily check that

$$\langle \hat{\eta}_b^j(t)\hat{g}_a(t)\rangle = 0, \tag{2.59}$$

$$\langle \hat{\eta}_c^j(t)\hat{g}_a(t)\rangle = 0, \tag{2.60}$$

$$\langle \hat{\sigma}_a^j(t)\hat{g}_a(t)\rangle = 0, \tag{2.61}$$

$$\langle \hat{\sigma}_b^j(t)\hat{g}_a(t)\rangle = 0, \tag{2.62}$$

$$\langle \hat{\sigma}_a^{\dagger j}(t)\hat{g}_a(t)\rangle = 0, \tag{2.63}$$

$$\langle\hat{\sigma}_b^{\dagger j}(t)\hat{g}_a(t)\rangle = 0, \tag{2.64}$$

$$\langle\hat{g}_a^{\dagger}(t)\hat{\sigma}_a^{j}(t)\rangle = 0, \tag{2.65}$$

$$\langle\hat{g}_a^{\dagger}(t)\hat{\sigma}_b^{j}(t)\rangle = 0, \tag{2.66}$$

$$\langle\hat{g}_a^{\dagger}(t)\hat{\sigma}_c^{j}(t)\rangle = 0. \tag{2.67}$$

Therefore, employing Eqs. (2.58)-(2.67), we can rewrite Eqs. (2.28)-(2.33) as follows

$$\frac{d}{dt}\langle\hat{\sigma}_a^j\rangle = -(\gamma_c+\gamma)\langle\hat{\sigma}_a^j\rangle, \tag{2.68}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b^j\rangle = \gamma_c\langle\hat{\sigma}_a^j\rangle - \frac{1}{2}(\gamma_c+2\gamma)\langle\hat{\sigma}_b^j\rangle, \tag{2.69}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^j\rangle = -\frac{1}{2}(\gamma_c+\gamma)\langle\hat{\sigma}_c^j\rangle, \tag{2.70}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^j\rangle = -(\gamma_c+\gamma)\langle\hat{\eta}_a^j\rangle, \tag{2.71}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^j\rangle = \gamma_c\langle\hat{\eta}_a^j\rangle - (\gamma_c+\gamma)\langle\hat{\eta}_b^j\rangle, \tag{2.72}$$

$$\frac{d}{dt}\langle\hat{\eta}_c^j\rangle = (\gamma_c+\gamma)\langle\hat{\eta}_b^j\rangle + \gamma\langle\hat{\eta}_a^j\rangle, \tag{2.73}$$

where

$$\gamma_c = \frac{4g^2}{\kappa} \tag{2.74}$$

is the stimulated emission decay constant.

The degenerate three-level atoms available in the cavity are pumped from the lower level to the upper level at a rate of r_a. The pumping process affects the rate of change of number of atoms in the upper level and in the lower level. Thus, $\langle\hat{\eta}_a^j\rangle$ increases at the rate of $r_a\langle\hat{\eta}_c^j\rangle$ and $\langle\hat{\eta}_c^j\rangle$ decreases at the same rate of $r_a\langle\hat{\eta}_c^j\rangle$ [12].

Incorporating the effect of the pumping process, we can rewrite Eqs. (2.71) and (2.73) as

$$\frac{d}{dt}\langle\hat{\eta}_a^j\rangle = -(\gamma_c+\gamma)\langle\hat{\eta}_a^j\rangle + r_a\langle\hat{\eta}_c^j\rangle, \tag{2.75}$$

$$\frac{d}{dt}\langle\hat{\eta}_c^j\rangle = (\gamma_c+\gamma)\langle\hat{\eta}_b^j\rangle + \gamma\langle\hat{\eta}_a^j\rangle - r_a\langle\hat{\eta}_c^j\rangle. \tag{2.76}$$

Now summing over N three-level atoms, we can rewrite Eqs. (2.68), (2.69), (2.70), (2.72), (2.75) and (2.76) in the form of

$$\frac{d}{dt}\langle\hat{m}_a\rangle = -(\gamma_c+\gamma)\langle\hat{m}_a\rangle, \tag{2.77}$$

$$\frac{d}{dt}\langle\hat{m}_b\rangle = \gamma_c\langle\hat{m}_a\rangle - \frac{1}{2}(\gamma_c+2\gamma)\langle\hat{m}_b\rangle, \tag{2.78}$$

$$\frac{d}{dt}\langle\hat{m}_c\rangle = -\frac{1}{2}(\gamma_c+\gamma)\langle\hat{m}_c\rangle, \tag{2.79}$$

$$\frac{d}{dt}\langle\hat{N}_a\rangle = -(\gamma_c+\gamma)\langle\hat{N}_a\rangle + r_a\langle\hat{N}_c\rangle, \tag{2.80}$$

$$\frac{d}{dt}\langle\hat{N}_b\rangle = \gamma_c\langle\hat{N}_a\rangle - (\gamma_c+\gamma)\langle\hat{N}_b\rangle, \tag{2.81}$$

$$\frac{d}{dt}\langle\hat{N}_c\rangle = (\gamma_c+\gamma)\langle\hat{N}_b\rangle + \gamma\langle\hat{N}_a\rangle - r_a\langle\hat{N}_c\rangle, \tag{2.82}$$

in which

$$\hat{m}_a = \sum_{j=1}^{N}\hat{\sigma}_a^j = \sum_{j=1}^{N}|b_j\rangle\langle a_j| = N|b\rangle\langle a|, \tag{2.83}$$

$$\hat{m}_b = \sum_{j=1}^{N}\hat{\sigma}_b^j = \sum_{j=1}^{N}|c_j\rangle\langle b_j| = N|c\rangle\langle b|, \tag{2.84}$$

$$\hat{m}_c = \sum_{j=1}^{N}\hat{\sigma}_c^j = \sum_{j=1}^{N}|c_j\rangle\langle a_j| = N|c\rangle\langle a|, \tag{2.85}$$

$$\hat{N}_a = \sum_{j=1}^{N}\hat{\eta}_a^j = \sum_{j=1}^{N}|a_j\rangle\langle a_j| = N|a\rangle\langle a|, \tag{2.86}$$

$$\hat{N}_b = \sum_{j=1}^{N}\hat{\eta}_b^j = \sum_{j=1}^{N}|b_j\rangle\langle b_j| = N|b\rangle\langle b|, \tag{2.87}$$

$$\hat{N}_c = \sum_{j=1}^{N}\hat{\eta}_c^j = \sum_{j=1}^{N}|c_j\rangle\langle c_j| = N|c\rangle\langle c|, \tag{2.88}$$

where $\hat{N}_a$, $\hat{N}_b$ and $\hat{N}_c$ are atomic operators representing number of atoms in the upper, middle and lower levels respectively.

In addition, employing the completeness relation

$$\hat{\eta}_a^j + \hat{\eta}_b^j + \hat{\eta}_c^j = \hat{I} \tag{2.89}$$

and summing over N three-level atoms, we obtain

$$\hat{N}_a + \hat{N}_b + \hat{N}_c \ = N\hat{I}. \tag{2.90}$$

Now taking the expectation value both sides of Eq. (2.90), we arrive at

$$\langle\hat{N}_a\rangle + \langle\hat{N}_b\rangle + \langle\hat{N}_c\rangle \ = N. \tag{2.91}$$

The steady-state solution of Eqs. (2.80)-(2.82) can be written as

$$\langle\hat{N}_a\rangle = \frac{r_a}{(\gamma_c+\gamma)}\langle\hat{N}_c\rangle, \tag{2.92}$$

$$\langle\hat{N}_b\rangle = \frac{\gamma_c}{(\gamma_c+\gamma)}\langle\hat{N}_a\rangle, \tag{2.93}$$

$$\langle\hat{N}_c\rangle = \frac{(\gamma_c+\gamma)}{r_a}\langle\hat{N}_b\rangle + \frac{\gamma}{r_a}\langle\hat{N}_a\rangle. \tag{2.94}$$

We note from Eq. (2.91) that

$$\langle\hat{N}_c\rangle \ = \ N - (\langle\hat{N}_a\rangle + \langle\hat{N}_b\rangle). \tag{2.95}$$

Now combination of Eqs. (2.95) and (2.92) yields

$$\langle\hat{N}_a\rangle = \frac{r_a}{(\gamma_c+\gamma+r_a)}(N - \langle\hat{N}_b\rangle). \tag{2.96}$$

Moreover, substituting Eq. (2.93) into Eq. (2.96), we readily get

$$\langle\hat{N}_a\rangle = \frac{N r_a(\gamma_c+\gamma)}{(\gamma_c+\gamma)(\gamma_c+\gamma+r_a)+\gamma_c r_a}. \tag{2.97}$$

On the other hand, using Eq. (2.97) in Eq. (2.93), we find

$$\langle\hat{N}_b\rangle = \frac{N\gamma_c r_a}{(\gamma_c+\gamma)(\gamma_c+\gamma+r_a)+\gamma_c r_a}. \tag{2.98}$$

Finally, inserting Eqs. (2.97) and (2.98) into Eq. (2.94), we easily arrive at

$$\langle\hat{N}_c\rangle = \frac{N(\gamma_c+\gamma)^2}{(\gamma_c+\gamma)(\gamma_c+\gamma+r_a)+\gamma_c r_a}. \tag{2.99}$$

Moreover, defining

$$\hat{m} \ = \hat{m}_a + \hat{m}_b \tag{2.100}$$

and taking into account Eqs. (2.83)-(2.88), it can be readily established that

$$\hat{m}^{\dagger}\hat{m} = N(\hat{N}_a + \hat{N}_b), \tag{2.101}$$

$$\hat{m}\hat{m}^{\dagger} = N(\hat{N}_b + \hat{N}_c), \tag{2.102}$$

$$\hat{m}^2 = N\hat{m}_c. \tag{2.103}$$

In the presence of N three-level atoms in the cavity, we can rewrite Eq. (2.25) as [1,12]

$$\frac{d}{dt}\hat{a}(t) = -\frac{\kappa}{2}\hat{a}(t) + \lambda\hat{m}(t) + \beta\hat{g}_a(t), \tag{2.104}$$

in which λ and β are constants whose values remain to be fixed.

We now proceed to evaluate the values of λ and β.

Employing Eq. (2.27) along with its adjoint, we see that

$$\begin{aligned}[\hat{a}, \hat{a}^{\dagger}]_j &= \hat{a}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a} \\ &= \frac{4}{\kappa^2}\Big[g(\hat{\sigma}_a^j + \hat{\sigma}_b^j) + \hat{g}_a(t)\Big]\Big[g(\hat{\sigma}_a^{\dagger j} + \hat{\sigma}_b^{\dagger j}) + \hat{g}_a^{\dagger}(t)\Big] - \frac{4}{\kappa^2}\Big[g(\hat{\sigma}_a^{\dagger j} + \hat{\sigma}_b^{\dagger j}) + \hat{g}_a^{\dagger}(t)\Big]\Big[g(\hat{\sigma}_a^j + \hat{\sigma}_b^j) + \hat{g}_a(t)\Big] \\ &= \frac{4g^2}{\kappa^2}(\hat{\eta}_c^j - \hat{\eta}_a^j) + \frac{4g}{\kappa^2}[\hat{\sigma}_a^j(t), \hat{g}_a^{\dagger}(t)] + \frac{4g}{\kappa^2}[\hat{\sigma}_b^j(t), \hat{g}_a^{\dagger}(t)] + \frac{4g}{\kappa^2}[\hat{g}_a(t), \hat{\sigma}_b^{\dagger j}(t)] \\ &\quad + \frac{4}{\kappa^2}[\hat{g}_a(t), \hat{g}_a^{\dagger}(t)]\end{aligned} \tag{2.105}$$

and assuming that atomic operator commute with cavity mode noise operator, we can write Eq. (2.105) as

$$[\hat{a}, \hat{a}^{\dagger}]_j = \frac{4g^2}{\kappa^2}\Big(\hat{\eta}_c^j - \hat{\eta}_a^j\Big) + \frac{4}{\kappa^2}\Big[\hat{g}_a(t)\hat{g}_a^{\dagger}(t) - \hat{g}_a^{\dagger}(t)\hat{g}_a(t)\Big] \tag{2.106}$$

and on summing over N three-level atoms, one obtains

$$[\hat{a}, \hat{a}^{\dagger}] = \frac{4g^2}{\kappa^2}\Big(\hat{N}_c - \hat{N}_a\Big) + \frac{4N}{\kappa^2}\Big[\hat{g}_a(t)\hat{g}_a^{\dagger}(t) - \hat{g}_a^{\dagger}(t)\hat{g}_a(t)\Big], \tag{2.107}$$

where

$$[\hat{a}, \hat{a}^{\dagger}] = \sum_{j=1}^{N}[\hat{a}, \hat{a}^{\dagger}]_j. \tag{2.108}$$

On the other hand, application of the large-time approximation scheme to Eq. (2.104) yields

$$\hat{a}(t) \;= \frac{2}{\kappa}\Big[\lambda\hat{m}(t) + \beta\hat{g}_a(t)\Big]. \tag{2.109}$$

Moreover, employing Eq. (2.109) together with its adjoint, we have

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}$$

$$= \frac{4}{\kappa^2}\Big[\lambda\hat{m}(t)+\beta\hat{g}_a(t)\Big]\Big[\lambda\hat{m}^\dagger(t)+\beta\hat{g}_a^\dagger(t)\Big] - \frac{4}{\kappa^2}\Big[\lambda\hat{m}^\dagger(t)+\beta\hat{g}_a^\dagger(t)\Big]\Big[\lambda\hat{m}(t)+\beta\hat{g}_a(t)\Big]$$

$$= \frac{4\lambda^2}{\kappa^2}(\hat{m}\hat{m}^\dagger - \hat{m}^\dagger\hat{m}) + \frac{4\lambda\beta}{\kappa^2}[\hat{m}(t), \hat{g}_a^\dagger(t)] + \frac{4\beta\lambda}{\kappa^2}[\hat{g}_a(t), \hat{m}^\dagger(t)] + \frac{4\beta^2}{\kappa^2}[\hat{g}_a(t), \hat{g}_a^\dagger(t)]. \tag{2.110}$$

Again assuming that atomic operator commute with cavity mode noise operator, Eq. (2.110) becomes

$$[\hat{a}, \hat{a}^\dagger] = \frac{4\lambda^2}{\kappa^2}\Big(\hat{m}\hat{m}^\dagger - \hat{m}^\dagger\hat{m}\Big) + \frac{4\beta^2}{\kappa^2}\Big[\hat{g}_a(t)\hat{g}_a^\dagger(t) - \hat{g}_a^\dagger(t)\hat{g}_a(t)\Big]. \tag{2.111}$$

Now substituting Eqs. (2.101) and (2.102) into Eq. (2.111), we find

$$[\hat{a}, \hat{a}^\dagger] \;= \frac{4\lambda^2 N}{\kappa^2}\Big(\hat{N}_c - \hat{N}_a\Big) + \frac{4\beta^2}{\kappa^2}\Big[\hat{g}_a(t)\hat{g}_a^\dagger(t) - \hat{g}_a^\dagger(t)\hat{g}_a(t)\Big]. \tag{2.112}$$

On account of Eqs. (2.107) and (2.112), we observe that

$$\lambda \;= \pm\frac{g}{\sqrt{N}} \tag{2.113}$$

and

$$\beta = \pm\sqrt{N}. \tag{2.114}$$

So that inserting Eqs. (2.113) and (2.114) into Eq. (2.104), we readily have

$$\frac{d}{dt}\hat{a}(t) \;= -\frac{\kappa}{2}\hat{a}(t) + \frac{g}{\sqrt{N}}\hat{m}(t) + \sqrt{N}\hat{g}_a(t) \tag{2.115}$$

or

$$\frac{d}{dt}\hat{a}(t) \;= -\frac{\kappa}{2}\hat{a}(t) + \frac{g}{\sqrt{N}}\hat{m}(t) + \hat{G}(t), \tag{2.116}$$

where $\hat{G}(t) = \sqrt{N}\hat{g}_a(t)$ is cavity mode noise operator when the cavity mode is interacting with N three-level atoms. Its also a vanishing mean.

2.2 Correlation properties of noise operators

Under this section we want to determine the correlation properties of the cavity mode noise operator and atomic noise operators.

2.2.1 The correlation properties of the cavity mode noise operator

Employing the relation

$$\frac{d}{dt}\langle\hat{a}^{\dagger}\hat{a}\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{a}^{\dagger}\hat{a}) \tag{2.117}$$

together with the master equation described by Eq. (2.3), we have

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^{\dagger}\hat{a}\rangle = {} & gTr(\hat{a}^{\dagger}|b_j\rangle\langle a_j|\hat{\rho}\hat{a}^{\dagger}\hat{a} + \hat{a}^{\dagger}|c_j\rangle\langle b_j|\hat{\rho}\hat{a}^{\dagger}\hat{a} - |a_j\rangle\langle b_j|\hat{a}\hat{\rho}\hat{a}^{\dagger}\hat{a} - |b_jb\rangle\langle c_j|\hat{a}\hat{\rho}\hat{a}^{\dagger}\hat{a} \\ & - \hat{\rho}\hat{a}^{\dagger}|b_j\rangle\langle a_j|\hat{a}^{\dagger}\hat{a} - \hat{\rho}\hat{a}^{\dagger}|c_j\rangle\langle b_j|\hat{a}^{\dagger}\hat{a} + \hat{\rho}|a_j\rangle\langle b_j|\hat{a}\hat{a}^{\dagger}\hat{a} + \hat{\rho}|b_j\rangle\langle c_j|\hat{a}\hat{a}^{\dagger}\hat{a}) \\ & + \frac{\kappa}{2}(\bar{n}+1)Tr(2\hat{a}\hat{\rho}\hat{a}^{\dagger 2}\hat{a} - \hat{a}^{\dagger}\hat{a}\hat{\rho}\hat{a}^{\dagger}\hat{a} - \hat{\rho}\hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}\hat{a}) \\ & + \frac{\kappa}{2}\bar{n}Tr(2\hat{a}^{\dagger}\hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a} - \hat{a}\hat{a}^{\dagger}\hat{\rho}\hat{a}^{\dagger}\hat{a} - \hat{\rho}\hat{a}\hat{a}^{\dagger 2}\hat{a}) \\ & + \frac{\gamma}{2}Tr(2|c_j\rangle\langle a_j|\hat{\rho}|a_j\rangle\langle c_j|\hat{a}^{\dagger}\hat{a} - |a_j\rangle\langle a_j|\hat{\rho}\hat{a}^{\dagger}\hat{a} - \hat{\rho}|a_j\rangle\langle a_j|\hat{a}^{\dagger}\hat{a} \\ & + 2|c_j\rangle\langle b_j|\hat{\rho}|b_j\rangle\langle c_j|\hat{a}^{\dagger}\hat{a} - |b_j\rangle\langle b_j|\hat{\rho}\hat{a}^{\dagger}\hat{a} - \hat{\rho}|b_j\rangle\langle b_j|\hat{a}^{\dagger}\hat{a}).\end{aligned} \tag{2.118}$$

Applying the cyclic property of trace operation to Eq. (2.118), we see that

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^{\dagger}\hat{a}\rangle = {} & gTr(\hat{\rho}\hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}|b_j\rangle\langle a_j|) + Tr(\hat{\rho}\hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}|c_j\rangle\langle b_j|) - Tr(\hat{\rho}\hat{a}^{\dagger}\hat{a}^{2}|a_j\rangle\langle b_j|) - Tr(\hat{\rho}\hat{a}^{\dagger}\hat{a}^{2}|b_j\rangle\langle c_j|) \\ & - Tr(\hat{\rho}\hat{a}^{\dagger 2}\hat{a}|b_j\rangle\langle a_j|) - Tr(\hat{\rho}\hat{a}^{\dagger 2}\hat{a}|c_j\rangle\langle b_j|) + Tr(\hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a}|a_j\rangle\langle b_j|) + Tr(\hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a}|b_j\rangle\langle c_j|) \\ & + \frac{\kappa}{2}(\bar{n}+1)Tr(2\hat{\rho}\hat{a}^{\dagger 2}\hat{a}^{2}) - Tr(\hat{\rho}\hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}\hat{a}) - Tr(\hat{\rho}\hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}\hat{a}) \\ & + \frac{\kappa}{2}\bar{n}Tr(2\hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}) - Tr(\hat{\rho}\hat{a}^{\dagger}\hat{a}\hat{a}\hat{a}^{\dagger}) - Tr(\hat{\rho}\hat{a}\hat{a}^{\dagger 2}\hat{a}) \\ & + \frac{\gamma}{2}Tr(2\hat{\rho}|a_j\rangle\langle c_j|\hat{a}^{\dagger}\hat{a}|c_j\rangle\langle a_j|) - Tr(\hat{\rho}\hat{a}^{\dagger}\hat{a}|a_j\rangle\langle a_j|) - Tr(\hat{\rho}|a_j\rangle\langle a_j|\hat{a}^{\dagger}\hat{a}) \\ & + Tr(2\hat{\rho}|b_j\rangle\langle c_j|\hat{a}^{\dagger}\hat{a}|c_j\rangle\langle b_j|) - Tr(\hat{\rho}\hat{a}^{\dagger}\hat{a}|b_j\rangle\langle b_j|) - Tr(\hat{\rho}|b_j\rangle\langle b_j|\hat{a}^{\dagger}\hat{a}),\end{aligned}$$

$$= g(\langle \hat{a}^{\dagger 2}\hat{a}|b_j\rangle\langle a_j|\rangle + \langle \hat{a}^{\dagger}|b_j\rangle\langle a_j|\rangle + \langle \hat{a}^{\dagger 2}\hat{a}|c_j\rangle\langle b_j|\rangle + \langle \hat{a}^{\dagger}|c_j\rangle\langle b_j|\rangle$$
$$- \langle \hat{a}^{\dagger}\hat{a}^2|a_j\rangle\langle b_j|\rangle - \langle \hat{a}^{\dagger}\hat{a}^2|b_j\rangle\langle c_j|\rangle - \langle \hat{a}^{\dagger 2}\hat{a}|b_j\rangle\langle a_j|\rangle - \langle \hat{a}^{\dagger 2}\hat{a}|c_j\rangle\langle b_j|\rangle$$
$$+ \langle \hat{a}^{\dagger}\hat{a}^2|a_j\rangle\langle b_j|\rangle + \langle \hat{a}|a_j\rangle\langle b_j|\rangle + \langle \hat{a}^{\dagger}\hat{a}^2|b_j\rangle\langle c_j|\rangle + \langle \hat{a}|b_j\rangle\langle c_j|\rangle)$$
$$+ \frac{\kappa}{2}(\bar{n}+1)(2\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle - 2\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle - 2\langle \hat{a}^{\dagger}\hat{a}\rangle) + \frac{\kappa}{2}\bar{n}(2\langle \hat{a}\hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}\rangle - 2\langle \hat{a}\hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}\rangle + 2\langle \hat{a}\hat{a}^{\dagger}\rangle)$$
$$+ \frac{\gamma}{2}(2\langle \hat{a}^{\dagger}\hat{a}|a_j\rangle\langle a_j|\rangle - 2\langle \hat{a}^{\dagger}\hat{a}|a_j\rangle\langle a_j|\rangle + 2\langle \hat{a}^{\dagger}\hat{a}|b_j\rangle\langle b_j|\rangle - 2\langle \hat{a}^{\dagger}\hat{a}|b_j\rangle\langle b_j|\rangle),$$

or

$$\frac{d}{dt}\langle \hat{a}^{\dagger}\hat{a}\rangle = g(\langle \hat{a}^{\dagger}\hat{\sigma}_a^j\rangle + \langle \hat{a}^{\dagger}\hat{\sigma}_b^j\rangle + \langle \hat{\sigma}_a^{\dagger j}\hat{a}\rangle + \langle \hat{\sigma}_b^{\dagger j}\hat{a}\rangle) - \kappa\langle \hat{a}^{\dagger}\hat{a}\rangle + \kappa\bar{n}. \tag{2.119}$$

Now using the mathematical relation

$$\frac{d}{dt}\langle \hat{a}^{\dagger}\hat{a}\rangle = \langle \frac{d\hat{a}^{\dagger}}{dt}\hat{a}\rangle + \langle \hat{a}^{\dagger}\frac{d\hat{a}}{dt}\rangle \tag{2.120}$$

along with Eq. (2.25) and its adjoint, one obtains

$$\begin{aligned}\frac{d}{dt}\langle \hat{a}^{\dagger}\hat{a}\rangle = {} & g(\langle \hat{a}^{\dagger}\hat{\sigma}_a^j\rangle + \langle \hat{a}^{\dagger}\hat{\sigma}_b^j\rangle + \langle \hat{\sigma}_a^{\dagger j}\hat{a}\rangle + \langle \hat{\sigma}_b^{\dagger j}\hat{a}\rangle) - \kappa\langle \hat{a}^{\dagger}\hat{a}\rangle \\ & + \langle \hat{a}^{\dagger}(t)\hat{g}_a(t)\rangle + \langle \hat{g}_a^{\dagger}(t)\hat{a}(t)\rangle.\end{aligned} \tag{2.121}$$

On account of Eqs. (2.119) and (2.121), we see that

$$\langle \hat{a}^{\dagger}(t)\hat{g}_a(t)\rangle + \langle \hat{g}_a^{\dagger}(t)\hat{a}(t)\rangle = \kappa\bar{n}. \tag{2.122}$$

With the aid of Eq. (2.48) and its adjoint, Eq. (2.122) becomes

$$\begin{aligned}&\langle \hat{a}^{\dagger}(0)\hat{g}_a(t)\rangle e^{-\frac{\kappa t}{2}} + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\Big[g\langle \hat{\sigma}_a^{\dagger j}(t')\hat{g}_a(t)\rangle + g\langle \hat{\sigma}_b^{\dagger j}(t')\hat{g}_a(t)\rangle + \langle \hat{g}_a^{\dagger}(t')\hat{g}_a(t)\rangle\Big]dt' \\ &+ \langle \hat{g}_a^{\dagger}(t)\hat{a}(0)\rangle e^{-\frac{\kappa t}{2}} + g\int_0^t e^{-\frac{\kappa}{2}(t-t')}\Big[\langle \hat{g}_a^{\dagger}(t)\hat{\sigma}_a^j(t')\rangle + +\langle \hat{g}_a^{\dagger}(t)\hat{\sigma}_b^j(t')\rangle\Big]dt' \\ &+ \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle \hat{g}_a^{\dagger}(t)\hat{g}_a(t')\rangle dt' = \kappa\bar{n}.\end{aligned} \tag{2.123}$$

Since cavity mode noise operator at an certain time t could not affect cavity mode and atomic operators at an earlier time, we note that

$$\langle \hat{a}^{\dagger}(0)\hat{g}_a(t)\rangle = \langle \hat{a}^{\dagger}(0)\rangle\langle \hat{g}_a(t)\rangle = 0, \tag{2.124}$$

$$\langle\hat{\sigma}_a^{\dagger j}(t')\hat{g}_a(t)\rangle = \langle\hat{\sigma}_a^{\dagger j}(t')\rangle\langle\hat{g}_a(t)\rangle = 0, \tag{2.125}$$

$$\langle\hat{\sigma}_b^{\dagger j}(t')\hat{g}_a(t)\rangle = \langle\hat{\sigma}_b^{\dagger j}(t')\rangle\langle\hat{g}_a(t)\rangle = 0, \tag{2.126}$$

$$\langle\hat{g}_a^{\dagger}(t)\hat{a}(0)\rangle = \langle\hat{g}_a^{\dagger}(t)\rangle\langle\hat{a}(0)\rangle = 0, \tag{2.127}$$

$$\langle\hat{g}_a^{\dagger}(t)\hat{\sigma}_a^{j}(t')\rangle = \langle\hat{g}_a^{\dagger}(t)\rangle\langle\hat{\sigma}_a^{j}(t')\rangle = 0, \tag{2.128}$$

$$\langle\hat{g}_a^{\dagger}(t)\hat{\sigma}_b^{j}(t')\rangle = \langle\hat{g}_a^{\dagger}(t)\rangle\langle\hat{\sigma}_b^{j}(t')\rangle = 0. \tag{2.129}$$

In view of Eqs. (2.124)-(2.129), we can put Eq. (2.123) as

$$\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a^{\dagger}(t')\hat{g}_a(t)\rangle dt' + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a^{\dagger}(t)\hat{g}_a(t')\rangle dt' = \kappa\bar{n}. \tag{2.130}$$

So that on assuming $\langle\hat{g}_a^{\dagger}(t')\hat{g}_a(t)\rangle = \langle\hat{g}_a^{\dagger}(t)\hat{g}_a(t')\rangle$, we obtain

$$\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a^{\dagger}(t)\hat{g}_a(t')\rangle dt' \ = \frac{\kappa}{2}\bar{n}. \tag{2.131}$$

In view of the relation

$$\int_0^t e^{-a(t-t')}\langle\hat{f}(t)\hat{g}(t')\rangle dt' = D, \tag{2.132}$$

we note that

$$\langle\hat{f}(t)\hat{g}(t')\rangle = 2D\delta(t-t'), \tag{2.133}$$

where a and D are some constant values. Based on Eqs. (2.132) and (2.133), we can put Eq. (2.131) as

$$\langle\hat{g}_a^{\dagger}(t)\hat{g}_a(t')\rangle = \kappa\bar{n}\delta(t-t'). \tag{2.134}$$

Now for N three−level atoms in the cavity, Eq. (2.134) goes over to

$$\langle\hat{G}^{\dagger}(t)\hat{G}(t')\rangle = \kappa N\bar{n}\delta(t-t'). \tag{2.135}$$

In another case, using the relation

$$\frac{d}{dt}\langle\hat{a}\hat{a}^{\dagger}\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{a}\hat{a}^{\dagger}) \tag{2.136}$$

along with Eq. (2.3), we get

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}\hat{a}^\dagger\rangle &= gTr(\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{\rho}\hat{a}\hat{a}^\dagger + \hat{a}^\dagger|c_j\rangle\langle b_j|\hat{\rho}\hat{a}\hat{a}^\dagger - |a_j\rangle\langle b_j|\hat{a}\hat{\rho}\hat{a}\hat{a}^\dagger - |b_j\rangle\langle c_j|\hat{a}\hat{\rho}\hat{a}\hat{a}^\dagger \\ &\quad - \hat{\rho}\hat{a}^\dagger|b_j\rangle\langle a_j|\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger|c_j\rangle\langle b_j|\hat{a}\hat{a}^\dagger + \hat{\rho}|a_j\rangle\langle b_j|\hat{a}^2\hat{a}^\dagger + \hat{\rho}|b_j\rangle\langle c_j|\hat{a}^2\hat{a}^\dagger) \\ &\quad + \frac{\kappa}{2}(\bar{n}+1)Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger) + \frac{\kappa}{2}\bar{n}Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}^2\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}\hat{a}^\dagger \\ &\quad - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger) + \frac{\gamma}{2}Tr(2|c_j\rangle\langle a_j|\hat{\rho}|a_j\rangle\langle c_j|\hat{a}\hat{a}^\dagger - |a_j\rangle\langle a_j|\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}|a_j\rangle\langle a_j|\hat{a}\hat{a}^\dagger \\ &\quad + 2|c_j\rangle\langle b_j|\hat{\rho}|b_j\rangle\langle c_j|\hat{a}\hat{a}^\dagger - |b_j\rangle\langle b_j|\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}|b_j\rangle\langle b_j|\hat{a}\hat{a}^\dagger.\end{aligned} \tag{2.137}$$

Applying the cyclic properties of trace operation to Eq. (2.137) leads to

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}\hat{a}^\dagger\rangle &= gTr(\hat{\rho}\hat{a}\hat{a}^{\dagger 2}|b_j\rangle\langle a_j|) + Tr(\hat{\rho}\hat{a}\hat{a}^{\dagger 2}|c_j\rangle\langle b_j|) - Tr(\hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}|a_j\rangle\langle b_j|) \\ &\quad - Tr(\hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}|b_j\rangle\langle c_j|) - Tr(\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger|b_j\rangle\langle a_j|) - Tr(\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger|c_j\rangle\langle b_j|) \\ &\quad + Tr(\hat{\rho}\hat{a}^2\hat{a}^\dagger|a_j\rangle\langle b_j|) + Tr(\hat{\rho}\hat{a}^2\hat{a}^\dagger|b_j\rangle\langle c_j|) + \frac{\kappa}{2}(\bar{n}+1)Tr(2\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}) \\ &\quad - Tr(\hat{\rho}\hat{a}\hat{a}^{\dagger 2}\hat{a}) - Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger) + \frac{\kappa}{2}\bar{n}Tr(2\hat{\rho}\hat{a}^2\hat{a}^{\dagger 2}) - Tr(\hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger) \\ &\quad - Tr(\hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger) + \frac{\gamma}{2}Tr(2\hat{\rho}|a_j\rangle\langle c_j|\hat{a}\hat{a}^\dagger|c_j\rangle\langle a_j|) - Tr(\hat{\rho}\hat{a}\hat{a}^\dagger|a_j\rangle\langle a_j|) \\ &\quad - Tr(\hat{\rho}|a_j\rangle\langle a_j|\hat{a}\hat{a}^\dagger) + Tr(2\hat{\rho}|b_j\rangle\langle c_j|\hat{a}\hat{a}^\dagger|c_j\rangle\langle b_j|) - Tr(\hat{\rho}\hat{a}\hat{a}^\dagger|b_j\rangle\langle b_j|) \\ &\quad - Tr(\hat{\rho}|b_j\rangle\langle b_j|\hat{a}\hat{a}^\dagger), \\ &= g(\hat{a}\hat{a}^{\dagger 2}|b_j\rangle\langle a_j|\rangle + \langle\hat{a}\hat{a}^{\dagger 2}|c_j\rangle\langle b_j|\rangle - \langle\hat{a}^2\hat{a}^\dagger|a_j\rangle\langle b_j|\rangle \\ &\quad + \langle\hat{a}|a_j\rangle\langle b_j|\rangle - \langle\hat{a}^2\hat{a}^\dagger|b_j\rangle\langle c_j|\rangle + \langle\hat{a}|b_j\rangle\langle c_j|\rangle - \langle\hat{a}\hat{a}^{\dagger 2}|b_j\rangle\langle a_j|\rangle \\ &\quad + \langle\hat{a}^\dagger|b_j\rangle\langle a_j|\rangle - \langle\hat{a}\hat{a}^{\dagger 2}|c_j\rangle\langle b_j|\rangle + \langle\hat{a}^\dagger|cj\rangle\langle b_j|\rangle + \langle\hat{a}^2\hat{a}^\dagger|a_j\rangle\langle b_j|\rangle \\ &\quad + \langle\hat{a}^2\hat{a}^\dagger|b_j\rangle\langle c_j|\rangle) + \frac{\kappa}{2}(\bar{n}+1)(2\langle\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\rangle - \langle\hat{a}\hat{a}^{\dagger 2}\hat{a}\rangle - \langle\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger\rangle) \\ &\quad + \frac{\kappa}{2}\bar{n}(2\langle\hat{a}^2\hat{a}^{\dagger 2}\rangle - \langle\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\rangle - \langle\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\rangle) + \frac{\gamma}{2}(2\langle\hat{a}\hat{a}^\dagger|a_j\rangle\langle a_j|\rangle \\ &\quad - 2\langle\hat{a}\hat{a}^\dagger|a_j\rangle\langle a_j|\rangle + 2\langle\hat{a}\hat{a}^\dagger|b_j\rangle\langle b_j|\rangle - 2\langle\hat{a}\hat{a}^\dagger|b_j\rangle\langle b_j|\rangle),\end{aligned}$$

$$\frac{d}{dt}\langle\hat{a}\hat{a}^\dagger\rangle = g\bigg(\langle\hat{\sigma}_a^j\hat{a}^\dagger\rangle + \langle\hat{\sigma}_b^j\hat{a}^\dagger\rangle + \langle\hat{a}\hat{\sigma}_a^{\dagger j}\rangle + \langle\hat{a}\hat{\sigma}_b^{\dagger j}\rangle\bigg) - \kappa\langle\hat{a}\hat{a}^\dagger\rangle + \kappa(\bar{n}+1). \tag{2.138}$$

Now summing over N three-level atoms, Eq. (2.138) is expressible as

$$\frac{d}{dt}\langle\hat{a}\hat{a}^\dagger\rangle = \frac{g}{\sqrt{N}}\bigg(\langle\hat{m}_a\hat{a}^\dagger\rangle + \langle\hat{m}_b\hat{a}^\dagger\rangle + \langle\hat{a}\hat{m}_a^\dagger\rangle + \langle\hat{a}\hat{m}_b^\dagger\rangle\bigg) - \kappa\langle\hat{a}\hat{a}^\dagger\rangle + \kappa N(\bar{n}+1). \tag{2.139}$$

Now employing the mathematical relation

$$\frac{d}{dt}\langle\hat{a}\hat{a}^\dagger\rangle = \langle\frac{d\hat{a}}{dt}\hat{a}^\dagger\rangle + \langle\hat{a}\frac{d\hat{a}^\dagger}{dt}\rangle \tag{2.140}$$

together with Eq. (2.116) and its adjoint, we obtain

$$\frac{d}{dt}\langle\hat{a}\hat{a}^\dagger\rangle = \frac{g}{\sqrt{N}}(\langle\hat{m}\hat{a}^\dagger\rangle + \langle\hat{a}\hat{m}^\dagger\rangle) - \kappa\langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{a}(t)\hat{G}^\dagger(t)\rangle + \langle\hat{G}(t)\hat{a}^\dagger(t)\rangle. \tag{2.141}$$

Comparison of Eqs. (2.139) and (2.141) shows that

$$\langle\hat{a}(t)\hat{G}^\dagger(t)\rangle + \langle\hat{G}(t)\hat{a}^\dagger(t)\rangle = \kappa N(\bar{n}+1). \tag{2.142}$$

We can write the formal solution of Eq. (2.116) as

$$\hat{a}(t) = \hat{a}(0)e^{-\frac{\kappa}{2}t} + \frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(t-t')}\hat{m}(t')dt' + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\hat{G}(t')dt' \tag{2.143}$$

and using Eq. (2.143) and its adjoint in Eq. (2.142), we find

$$\begin{aligned}
&\langle\hat{a}(0)\hat{G}^\dagger(t)\rangle e^{-\frac{\kappa t}{2}} + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\left[\frac{g}{\sqrt{N}}\langle\hat{m}(t')\hat{G}^\dagger(t)\rangle + \langle\hat{G}(t')\hat{G}^\dagger(t)\rangle\right]dt' \\
&+ \langle\hat{G}(t)\hat{a}^\dagger(0)\rangle e^{-\frac{\kappa t}{2}} + \frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{G}(t)\hat{m}^\dagger(t')\rangle dt' \\
&+ \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{G}(t)\hat{G}^\dagger(t')\rangle dt' = \kappa N(\bar{n}+1).
\end{aligned} \tag{2.144}$$

Based on the fact that the noise operator at a certain time t does not affect the cavity mode and atomic operators at earlier times, then Eq. (2.144) reduces to

$$\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{G}(t')\hat{G}^\dagger(t)\rangle dt' + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{G}(t)\hat{G}^\dagger(t')\rangle dt' = \kappa N(\bar{n}+1) \tag{2.145}$$

and assuming that $\langle\hat{G}(t')\hat{G}^\dagger(t)\rangle = \langle\hat{G}(t)\hat{G}^\dagger(t')\rangle$, we assert that

$$\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{G}(t)\hat{G}^\dagger(t')\rangle dt' = \frac{\kappa}{2}N(\bar{n}+1). \tag{2.146}$$

So that based on the result of Eqs. (2.132) and (2.133), Eq. (2.146) becomes

$$\langle\hat{G}(t)\hat{G}^\dagger(t')\rangle = \kappa N(\bar{n}+1)\delta(t-t'). \tag{2.147}$$

Following the same procedure, we can verify that

$$\langle \hat{G}(t)\hat{G}(t')\rangle = 0, \tag{2.148}$$

$$\langle \hat{G}^{\dagger}(t)\hat{G}^{\dagger}(t')\rangle = 0. \tag{2.149}$$

Therefore Eqs. (2.135), (2.147), (2.148) and (2.149) are the correlation properties of cavity mode noise operators.

2.2.2 The correlation properties of atomic noise operators

Making use of the master equation described by Eq. (2.3) along with the relation

$$\frac{d}{dt}\langle \hat{m}_a^{\dagger}\hat{m}_a\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{m}_a^{\dagger}\hat{m}_a) \tag{2.150}$$

and noting that $\hat{m}_a^{\dagger}\hat{m}_a = N\hat{N}_a$, one obtains

$$\frac{d}{dt}\langle \hat{m}_a^{\dagger}\hat{m}_a\rangle = N\frac{d}{dt}\langle \hat{N}_a\rangle. \tag{2.151}$$

Upon combining Eqs. (2.80) and (2.151), we get

$$\frac{d}{dt}\langle \hat{m}_a^{\dagger}\hat{m}_a\rangle = -(\gamma_c+\gamma)N\langle \hat{N}_a\rangle + \ r_a N\langle \hat{N}_c\rangle. \tag{2.152}$$

On the basis of Eqs. (2.77) and (2.78), we can write

$$\frac{d}{dt}\hat{m}_a = -(\gamma_c+\gamma)\hat{m}_a + \hat{f}_a(t) \tag{2.153}$$

and

$$\frac{d}{dt}\hat{m}_b = -\frac{1}{2}(\gamma_c+2\gamma)\hat{m}_b + \gamma_c\hat{m}_a + \hat{f}_b(t), \tag{2.154}$$

where $\hat{f}_a(t)$ and $\hat{f}_b(t)$ are atomic noise operators with a vanishing mean.

The solution of Eqs. (2.153) and (2.154) can be written as

$$\hat{m}_a(t) \ = \hat{m}_a(0)e^{-(\gamma_c+\gamma)t} + \int_0^t e^{-(\gamma_c+\gamma)(t-t')}\hat{f}_a(t')dt', \tag{2.155}$$

and

$$\hat{m}_b(t) = \hat{m}_b(0)e^{-\frac{1}{2}(\gamma_c+2\gamma)t} + \int_0^t e^{-\frac{1}{2}(\gamma_c+2\gamma)(t-t')}[\gamma_c\hat{m}_a(t') + \hat{f}_b(t')]dt'. \tag{2.156}$$

Now employing the mathematical relation

$$\frac{d}{dt}\langle\hat{m}_a^\dagger\hat{m}_a\rangle = \langle\frac{d\hat{m}_a^\dagger}{dt}\hat{m}_a\rangle + \langle\hat{m}_a^\dagger\frac{d\hat{m}_a}{dt}\rangle \tag{2.157}$$

along with Eq. (2.153) and its adjoint, we obtain

$$\frac{d}{dt}\langle\hat{m}_a^\dagger\hat{m}_a\rangle = -2(\gamma_c+\gamma)N\langle\hat{N}_a\rangle + \langle\hat{m}_a^\dagger(t)\hat{f}_a(t)\rangle + \langle\hat{f}_a^\dagger(t)\hat{m}_a(t)\rangle, \tag{2.158}$$

Upon comparing Eqs. (2.152) and (2.158), we see that

$$\langle\hat{m}_a^\dagger(t)\hat{f}_a(t)\rangle + \langle\hat{f}_a^\dagger(t)\hat{m}_a(t)\rangle = N[(\gamma_c+\gamma)\langle\hat{N}_a\rangle + \ r_a\langle\hat{N}_c\rangle]. \tag{2.159}$$

Now introducing Eq. (2.155) and its adjoint into Eq. (2.159), we find

$$\begin{aligned}\langle\hat{m}_a^\dagger(0)\hat{f}_a(t)\rangle e^{-(\gamma_c+\gamma)t} + \int_0^t e^{-(\gamma_c+\gamma)(t-t')}\langle\hat{f}_a^\dagger(t')\hat{f}_a(t)\rangle dt' + \langle\hat{f}_a^\dagger(t)\hat{m}_a(0)\rangle e^{-(\gamma_c+\gamma)t}\\ + \int_0^t e^{-(\gamma_c+\gamma)(t-t')}\langle\hat{f}_a^\dagger(t)\hat{f}_a(t')\rangle dt' = N[(\gamma_c+\gamma)\langle\hat{N}_a\rangle + \ r_a\langle\hat{N}_c\rangle].\end{aligned} \tag{2.160}$$

Since atomic noise operator at some time t should not affect atomic operator at an earlier time, we note that

$$\langle\hat{m}_a^\dagger(0)\hat{f}_a(t)\rangle = \langle\hat{m}_a^\dagger(0)\rangle\langle\hat{f}_a(t)\rangle = 0, \quad \langle\hat{f}_a^\dagger(t)\hat{m}_a(0)\rangle = \langle\hat{f}_a^\dagger(t)\rangle\langle\hat{m}_a(0)\rangle = 0, \tag{2.161}$$

in view of which Eq. (2.160) reduces to

$$\begin{aligned}\int_0^t e^{-(\gamma_c+\gamma)(t-t')}\langle\hat{f}_a^\dagger(t')\hat{f}_a(t)\rangle dt' + \int_0^t e^{-(\gamma_c+\gamma)(t-t')}\langle\hat{f}_a^\dagger(t)\hat{f}_a(t')\rangle dt'\\ = N[(\gamma_c+\gamma)\langle\hat{N}_a\rangle + \ r_a\langle\hat{N}_c\rangle].\end{aligned} \tag{2.162}$$

On assuming $\langle\hat{f}_a^\dagger(t')\hat{f}_a(t)\rangle = \langle\hat{f}_a^\dagger(t)\hat{f}_a(t')\rangle$, we have

$$\int_0^t e^{-(\gamma_c+\gamma)(t-t')}\langle\hat{f}_a^\dagger(t)\hat{f}_a(t')\rangle dt' = \frac{N}{2}\Big[(\gamma_c+\gamma)\langle\hat{N}_a\rangle + \ r_a\langle\hat{N}_c\rangle\Big]. \tag{2.163}$$

Taking Eqs. (2.132) and (2.133) into account, we can put Eq. (2.163) as

$$\langle\hat{f}_a^\dagger(t)\hat{f}_a(t')\rangle = N\Big[(\gamma_c+\gamma)\langle\hat{N}_a\rangle + \ r_a\langle\hat{N}_c\rangle\Big]\delta(t-t'). \tag{2.164}$$

Then, using Eqs. (2.97) and (2.99) in Eq. (2.164), there follows that

$$\langle \hat{f}_a^\dagger(t)\hat{f}_a(t')\rangle = \frac{2N^2 r_a(\gamma_c+\gamma)^2}{(\gamma_c+\gamma)(\gamma_c+\gamma+r_a)+\gamma_c r_a}\delta(t-t'). \tag{2.165}$$

Furthermore, employing the master equation in

$$\frac{d}{dt}\langle \hat{m}_a\hat{m}_a^\dagger\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{m}_a\hat{m}_a^\dagger) \tag{2.166}$$

and using $\hat{m}_a\hat{m}_a^\dagger = N\hat{N}_b$, we obtain

$$\frac{d}{dt}\langle \hat{m}_a\hat{m}_a^\dagger\rangle = NTr(\frac{d\hat{\rho}}{dt}\hat{N}_b), \tag{2.167}$$

or

$$\frac{d}{dt}\langle \hat{m}_a\hat{m}_a^\dagger\rangle = N\frac{d}{dt}\langle \hat{N}_b\rangle. \tag{2.168}$$

Now substitution of Eq. (2.81) into Eq. (2.168) leads to

$$\frac{d}{dt}\langle \hat{m}_a\hat{m}_a^\dagger\rangle = \gamma_c N\langle \hat{N}_a\rangle - (\gamma_c+\gamma)N\langle \hat{N}_b\rangle. \tag{2.169}$$

On the other hand, using Eq. (2.153) along with its conjugate in the relation

$$\frac{d}{dt}\langle \hat{m}_a\hat{m}_a^\dagger\rangle = \langle\frac{d\hat{m}_a}{dt}\hat{m}_a^\dagger\rangle + \langle \hat{m}_a\frac{d\hat{m}_a^\dagger}{dt}\rangle, \tag{2.170}$$

we obtain

$$\frac{d}{dt}\langle \hat{m}_a\hat{m}_a^\dagger\rangle = -2(\gamma_c+\gamma)\langle \hat{m}_a\hat{m}_a^\dagger\rangle + \langle \hat{m}_a(t)\hat{f}_a^\dagger(t)\rangle + \langle \hat{f}_a(t)\hat{m}_a^\dagger(t)\rangle, \tag{2.171}$$

or

$$\frac{d}{dt}\langle \hat{m}_a\hat{m}_a^\dagger\rangle = -2(\gamma_c+\gamma)N\langle \hat{N}_b\rangle + \langle \hat{m}_a(t)\hat{f}_a^\dagger(t)\rangle + \langle \hat{f}_a(t)\hat{m}_a^\dagger(t)\rangle. \tag{2.172}$$

Comparison of Eqs. (2.169) and (2.172), we see that

$$\langle \hat{m}_a(t)\hat{f}_a^\dagger(t)\rangle + \langle \hat{f}_a(t)\hat{m}_a^\dagger(t)\rangle = N[(\gamma_c+\gamma)\langle \hat{N}_b\rangle + \gamma_c\langle \hat{N}_a\rangle]. \tag{2.173}$$

Again introduction of Eq. (2.155) and its adjoint into Eq. (2.173) yields

$$\begin{aligned}\langle \hat{m}_a(0)\hat{f}_a^\dagger(t)\rangle e^{-(\gamma_c+\gamma)t} + \int_0^t e^{-(\gamma_c+\gamma)(t-t')}\langle \hat{f}_a(t')\hat{f}_a^\dagger(t)\rangle dt' + \langle \hat{f}_a(t)\hat{m}_a^\dagger(0)\rangle e^{-(\gamma_c+\gamma)t} \\ + \int_0^t e^{-(\gamma_c+\gamma)(t-t')}\langle \hat{f}_a(t)\hat{f}_a^\dagger(t')\rangle dt' = N\Big[(\gamma_c+\gamma)\langle \hat{N}_b\rangle + \gamma_c\langle \hat{N}_a\rangle\Big].\end{aligned} \tag{2.174}$$

So that in view of Eq. (2.161), we can write Eq. (2.174) as

$$\int_0^t e^{-(\gamma_c+\gamma)(t-t')}\langle \hat{f}_a(t')\hat{f}_a^\dagger(t)\rangle dt' + \int_0^t e^{-(\gamma_c+\gamma)(t-t')}\langle \hat{f}_a(t)\hat{f}_a^\dagger(t')\rangle dt$$
$$= N[(\gamma_c+\gamma)\langle \hat{N}_b\rangle + \gamma_c\langle \hat{N}_a\rangle]. \tag{2.175}$$

Assuming that $\langle \hat{f}_a(t')\hat{f}_a^\dagger(t)\rangle = \langle \hat{f}_a(t)\hat{f}_a^\dagger(t')\rangle$, we note that

$$\int_0^t e^{-(\gamma_c+\gamma)(t-t')}\langle \hat{f}_a(t)\hat{f}_a^\dagger(t')\rangle dt' = \frac{N}{2}\Big[(\gamma_c+\gamma)\langle \hat{N}_b\rangle + \gamma_c\langle \hat{N}_a\rangle\Big], \tag{2.176}$$

Now taking into account Eqs. (2.132) and (2.133), we can put Eq. (2.176) as

$$\langle \hat{f}_a(t)\hat{f}_a^\dagger(t')\rangle = N\Big[(\gamma_c+\gamma)\langle \hat{N}_b\rangle + \gamma_c\langle \hat{N}_a\rangle\Big]\delta(t-t'). \tag{2.177}$$

Upon substituting Eqs. (2.97) and (2.98) into Eq. (2.177), we find

$$\langle \hat{f}_a(t)\hat{f}_a^\dagger(t')\rangle = \frac{2N^2\gamma_c r_a(\gamma_c+\gamma)}{(\gamma_c+\gamma)(\gamma_c+\gamma+r_a)+\gamma_c r_a}\delta(t-t'). \tag{2.178}$$

Following the same procedure, we can verify that

$$\langle \hat{f}_a(t)\hat{f}_a(t')\rangle = 0, \tag{2.179}$$

$$\langle \hat{f}_a^\dagger(t)\hat{f}_a^\dagger(t')\rangle = 0, \tag{2.180}$$

$$\langle \hat{f}_b^\dagger(t)\hat{f}_b(t')\rangle = \frac{N^2\gamma_c r_a(\gamma_c+2\gamma)}{(\gamma_c+\gamma)(\gamma_c+\gamma+r_a)+\gamma_c r_a}\delta(t-t'), \tag{2.181}$$

$$\langle \hat{f}_b(t)\hat{f}_b^\dagger(t')\rangle = \frac{N^2(\gamma_c+\gamma)}{(\gamma_c+\gamma)(\gamma_c+\gamma+r_a)+\gamma_c r_a}\Big[(\gamma_c^2+\gamma(3\gamma_c+2\gamma)\Big]\delta(t-t'), \tag{2.182}$$

$$\langle \hat{f}_b(t)\hat{f}_b(t')\rangle = 0, \tag{2.183}$$

$$\langle \hat{f}_b^\dagger(t)\hat{f}_b^\dagger(t')\rangle = 0, \tag{2.184}$$

$$\langle \hat{f}_b(t)\hat{f}_a(t')\rangle = \frac{N}{4}\Big(2\gamma_c+3\gamma\Big)\langle \hat{m}_c\rangle\delta(t-t'), \tag{2.185}$$

$$\langle \hat{f}_a^\dagger(t)\hat{f}_b^\dagger(t')\rangle = \frac{N}{4}\Big(2\gamma_c+3\gamma\Big)\langle \hat{m}_c^\dagger\rangle\delta(t-t'), \tag{2.186}$$

$$\langle \hat{f}_a^\dagger(t)\hat{f}_b(t')\rangle = 0, \tag{2.187}$$

$$\langle \hat{f}_b^\dagger(t)\hat{f}_a(t')\rangle = 0, \tag{2.188}$$

$$\langle \hat{f}_a(t)\hat{f}_b^\dagger(t')\rangle = 0, \tag{2.189}$$

$$\langle \hat{f}_b(t)\hat{f}_a^\dagger(t')\rangle = 0. \tag{2.190}$$

Chapter 3

Photon Statistics

In this chapter we are going to study the statistical properties of a light generated by degenerate three-level laser in which degenerate three-level atoms available in cavity are pumped from lower level to upper level at a rate r_a. To this end, we evaluate the mean and variance of the photon number, and the power spectrum for the cavity mode .

3.1 The mean photon number

Employing the solution

$$\hat{a}(t) = \hat{a}(0)e^{-\frac{\kappa}{2}t} + \frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(t-t')}\hat{m}(t')dt' + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\hat{G}(t')dt' \tag{3.1}$$

of Eq. (2.116) and its adjoint

$$\hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{-\frac{\kappa}{2}t} + \frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(t-t'')}\hat{m}^\dagger(t'')dt'' + \int_0^t e^{-\frac{\kappa}{2}(t-t'')}\hat{G}^\dagger(t'')dt'', \tag{3.2}$$

the mean photon number of the cavity mode is

$$\begin{aligned}
\bar{n}_c &= \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle \\
&= \langle\hat{a}^\dagger(0)\hat{a}(0)\rangle e^{-\kappa t} + \int_0^t e^{-\frac{\kappa}{2}(2t-t')}\left[\frac{g}{\sqrt{N}}\langle\hat{a}^\dagger(0)\hat{m}(t')\rangle + \langle\hat{a}^\dagger(0)\hat{G}(t')\rangle\right]dt' \\
&\quad + \frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(2t-t'')}\langle\hat{m}^\dagger(t'')\hat{a}(0)\rangle dt'' + \frac{g^2}{N}e^{-\kappa t}\int_0^t\int_0^t e^{\frac{\kappa}{2}(t''+t')}\langle\hat{m}^\dagger(t'')\hat{m}(t')\rangle dt''dt' \\
&\quad + \frac{g}{\sqrt{N}}e^{-\kappa t}\int_0^t\int_0^t e^{\frac{\kappa}{2}(t''+t')}\langle\hat{m}^\dagger(t'')\hat{G}(t')\rangle dt''dt' + \int_0^t e^{-\frac{\kappa}{2}(2t-t'')}\langle\hat{G}^\dagger(t'')\hat{a}(0)\rangle dt'' \\
&\quad + e^{-\kappa t}\int_0^t\int_0^t e^{\frac{\kappa}{2}(t''+t')}\left[\frac{g}{\sqrt{N}}\langle\hat{G}^\dagger(t'')\hat{m}(t')\rangle + \langle\hat{G}^\dagger(t'')\hat{G}(t')\rangle\right]dt''dt'.
\end{aligned} \tag{3.3}$$

Based on the fact that the cavity mode noise operator at a certain time does not affect a cavity mode operator at earlier times, we see that

$$\langle\hat{a}^\dagger(0)\hat{G}(t')\rangle = \langle\hat{a}^\dagger(0)\rangle\langle\hat{G}(t')\rangle = 0, \quad \langle\hat{G}^\dagger(t'')\hat{a}(0)\rangle = \langle\hat{G}^\dagger(t'')\rangle\langle\hat{a}(0)\rangle = 0. \tag{3.4}$$

On the other hand, summing Eqs. (2.125), (2.126), (2.128) and (2.129) over N three-level atoms, we can write as

$$\langle\hat{m}^\dagger(t)\hat{G}(t)\rangle = \langle\hat{m}_a^\dagger(t)\hat{G}(t)\rangle + \langle\hat{m}_b^\dagger(t)\hat{G}(t)\rangle = 0, \tag{3.5}$$

$$\langle\hat{G}^\dagger(t)\hat{m}(t)\rangle = \langle\hat{G}^\dagger(t)\hat{m}_a(t)\rangle + \langle\hat{G}^\dagger(t)\hat{m}_b(t)\rangle = 0. \tag{3.6}$$

Based on Eqs. (3.4)-(3.6), we can reduce Eq. (3.3) as

$$\begin{aligned}\bar{n}_c = {} & \langle\hat{a}^\dagger(0)\hat{a}(0)\rangle e^{-\kappa t} + \frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(2t-t')}\langle\hat{a}^\dagger(0)\hat{m}(t')\rangle dt' \\ & + \frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(2t-t'')}\langle\hat{m}^\dagger(t'')\hat{a}(0)\rangle dt'' \\ & + e^{-\kappa t}\int_0^t\int_0^t e^{\frac{\kappa}{2}(t''+t')}\left[\frac{g^2}{N}\langle\hat{m}^\dagger(t'')\hat{m}(t')\rangle + \langle\hat{G}^\dagger(t'')\hat{G}(t')\rangle\right]dt''dt'.\end{aligned} \tag{3.7}$$

Now for a cavity mode initially assumed to be in a vacuum state, we can write

$$\langle\hat{a}(0)\rangle = Tr(\hat{\rho}(0)\hat{a}) = Tr(|0\rangle\langle 0|\hat{a}) = \langle 0|\hat{a}|0\rangle = 0 \tag{3.8}$$

and

$$\begin{aligned}\langle\hat{a}^\dagger(0)\hat{a}(0)\rangle &= Tr(\hat{\rho}(0)\hat{a}^\dagger\hat{a}) \\ &= Tr(|0\rangle\langle 0|\hat{a}^\dagger\hat{a}) \\ &= \langle 0|\hat{a}^\dagger\hat{a}|0\rangle = 0,\end{aligned} \tag{3.9}$$

where $\hat{\rho}(0) = |0\rangle\langle 0|$ is the initial density operator for the cavity mode initially in a vacuum state.

Assuming that cavity mode and atomic operators are not initially correrated,

we note that

$$\langle\hat{a}^\dagger(0)\hat{m}(t')\rangle = \langle\hat{a}^\dagger(0)\rangle\langle\hat{m}(t')\rangle = 0, \quad \langle\hat{m}^\dagger(t'')\hat{a}(0)\rangle = \langle\hat{m}^\dagger(t'')\rangle\langle\hat{a}(0)\rangle = 0. \tag{3.10}$$

Therefore, using Eqs. (3.9) and (3.10) in Eq. (3.7), we find

$$\bar{n}_c = e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t''+t')} \left[\frac{g^2}{N} \langle \hat{m}^\dagger(t'')\hat{m}(t') \rangle + \langle \hat{G}^\dagger(t'')\hat{G}(t') \rangle \right] dt'' dt' \tag{3.11}$$

or

$$\bar{n}_c = \langle \hat{a}'^\dagger(t)\hat{a}'(t) \rangle + e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t''+t')} \langle \hat{G}^\dagger(t'')\hat{G}(t') \rangle dt'' dt', \tag{3.12}$$

where

$$\hat{a}'(t) = \frac{g}{\sqrt{N}} e^{-\frac{\kappa}{2}t} \int_0^t e^{\frac{\kappa}{2}t'} \hat{m}(t') dt', \tag{3.13}$$

$$\hat{a}'^\dagger(t) = \frac{g}{\sqrt{N}} e^{-\frac{\kappa}{2}t} \int_0^t e^{\frac{\kappa}{2}t''} \hat{m}^\dagger(t'') dt''. \tag{3.14}$$

Now differentiating Eq. (3.13) with respect to time t, one obtains

$$\frac{d}{dt}\hat{a}'(t) = -\frac{\kappa}{2}\frac{g}{\sqrt{N}} e^{-\frac{\kappa}{2}t} \int_0^t e^{\frac{\kappa}{2}t'} \hat{m}(t') dt' + \frac{g}{\sqrt{N}} e^{-\frac{\kappa}{2}t} \frac{d}{dt} \int_0^t e^{\frac{\kappa}{2}t'} \hat{m}(t') dt'. \tag{3.15}$$

Applying the relation

$$\frac{d}{dx} \int_a^x f(x,x') dx' = f(x,x) - f(x,a) + \int_a^x \frac{d}{dx} f(x,x') dx', \tag{3.16}$$

the time derivative of the integral in Eq. (3.15) becomes

$$\begin{aligned} \frac{d}{dt} \int_0^t e^{\frac{\kappa}{2}t'} \hat{m}(t') dt' &= e^{\frac{\kappa}{2}t} \hat{m}(t) - \hat{m}(0) + \int_0^t \frac{d}{dt} e^{\frac{\kappa}{2}t'} \hat{m}(t') dt' \\ &= e^{\frac{\kappa}{2}t} \hat{m}(t) - \hat{m}(0). \end{aligned} \tag{3.17}$$

Now using Eqs. (3.13) and (3.17) in Eq. (3.15), we note that

$$\frac{d}{dt}\hat{a}'(t) = -\frac{\kappa}{2}\hat{a}'(t) + \frac{g}{\sqrt{N}}\hat{m}(t) - \frac{g}{\sqrt{N}}\hat{m}(0) e^{-\frac{\kappa}{2}t}. \tag{3.18}$$

Applying the large-time approximation scheme to Eq. (3.18), we can write as

$$\hat{a}'(t) = \frac{2g}{\kappa\sqrt{N}}\hat{m}(t). \tag{3.19}$$

Then taking the expectation value of the product of Eq. (3.19) and its conjugate, we find

$$\langle \hat{a}'^\dagger(t)\hat{a}'(t) \rangle = \frac{4g^2}{\kappa^2 N} \langle \hat{m}^\dagger(t)\hat{m}(t) \rangle,$$

or

$$\langle \hat{a}'^{\dagger}(t)\hat{a}'(t)\rangle = \frac{\gamma_c}{\kappa N}\langle \hat{m}^{\dagger}(t)\hat{m}(t)\rangle. \tag{3.20}$$

Employing Eq. (2.101) along with Eqs. (2.97) and (2.98), we obtain

$$\langle \hat{m}^{\dagger}\hat{m}\rangle = \frac{N^2 r_a}{(\gamma_c+\gamma)(\gamma_c+\gamma+r_a)+\gamma_c r_a}[2\gamma_c+\gamma], \tag{3.21}$$

in view of which Eq. (3.20) becomes

$$\langle \hat{a}'^{\dagger}(t)\hat{a}'(t)\rangle = \frac{N\gamma_c r_a}{\kappa((\gamma_c+\gamma)(\gamma_c+\gamma+r_a)+\gamma_c r_a)}[2\gamma_c+\gamma]. \tag{3.22}$$

Finally, taking into account Eqs. (2.135) and (3.22) and carrying out the integration, the steady-state mean photon number is expressible as

$$\bar{n}_c = \frac{N\gamma_c r_a}{\kappa((\gamma_c+\gamma)(\gamma_c+\gamma+r_a)+\gamma_c r_a)}[2\gamma_c+\gamma] + N\bar{n} \tag{3.23}$$

where $\bar{n}_c$ is mean photon number of a cavity mode and $\bar{n}$ is the mean photon of thermal reservoir.

Therefore, we see from Eq. (3.23) that, the mean photon number of a cavity mode is the sum of the mean photon number of laser light and the mean photon number due to thermal reservoir.

3.2 The variance of the photon number

Under this section we want to calculate the variance of the photon number for the cavity light, which is expressible as

$$(\Delta n)^2 = \langle \hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}\hat{a}\rangle - \langle \hat{a}^{\dagger}\hat{a}\rangle^2$$

$$= \langle \hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}\hat{a}\rangle - \bar{n}_c^2. \tag{3.24}$$

Now with the atoms considered to be initially in the lower level, the expectation value of Eqs. (2.155) and (2.156) leads to

$$\langle \hat{m}_a(t)\rangle = 0 \tag{3.25}$$

and

$$\langle \hat{m}_b(t) \rangle = 0. \tag{3.26}$$

On account of Eqs. (3.25) and (3.26), the expectation value of Eq.(2.100) turns out to

$$\langle \hat{m}(t) \rangle = 0. \tag{3.27}$$

On the other hand, in view of Eq. (3.27) and the assumption that the cavity light is initially in a vacuum state, the expectation value of Eq. (3.1) goes over into

$$\langle \hat{a}(t) \rangle = 0. \tag{3.28}$$

Now in view of Eqs. (2.116) and (3.28), we note that $\hat{a}$ is a Gaussian variable with a vanishing mean.

Therefore, we can write

$$\begin{aligned}\langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle &= \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}^\dagger \hat{a}^\dagger \rangle \langle \hat{a} \hat{a} \rangle + \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a} \hat{a}^\dagger \rangle \\ &= \bar{n}_c^2 + \langle \hat{a}^{\dagger 2} \rangle \langle \hat{a}^2 \rangle + \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a} \hat{a}^\dagger \rangle. \end{aligned} \tag{3.29}$$

Now using Eq. (3.29) in Eq. (3.24), we have

$$(\Delta n)^2 = \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a} \hat{a}^\dagger \rangle + \langle \hat{a}^{\dagger 2} \rangle \langle \hat{a}^2 \rangle. \tag{3.30}$$

Again with the aid of Eqs. (3.1) and (3.2), one obtains

$$\begin{aligned}\langle \hat{a}(t) \hat{a}^\dagger(t) \rangle = \langle \hat{a}(0) \hat{a}^\dagger(0) \rangle e^{-\kappa t} &+ \int_0^t e^{-\frac{\kappa}{2}(2t-t'')} \left[\frac{g}{\sqrt{N}} \langle \hat{a}(0) \hat{m}^\dagger(t'') \rangle + \langle \hat{a}(0) \hat{G}^\dagger(t'') \rangle \right] dt'' \\ &+ e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t'+t'')} \left[\frac{g^2}{N} \langle \hat{m}(t') \hat{m}^\dagger(t'') \rangle + \frac{g}{\sqrt{N}} \langle \hat{m}(t') \hat{G}^\dagger(t'') \rangle \right] dt' dt'' \\ &+ \frac{g}{\sqrt{N}} \int_0^t e^{-\frac{\kappa}{2}(2t-t')} \langle \hat{m}(t') \hat{a}^\dagger(0) \rangle dt' + \int_0^t e^{-\frac{\kappa}{2}(2t-t')} \langle \hat{G}(t') \hat{a}^\dagger(0) \rangle dt'' \\ &+ e^{-\kappa t} \int_0^t \int_0^t e^{\frac{\kappa}{2}(t'+t'')} \left[\frac{g}{\sqrt{N}} \langle \hat{G}(t') \hat{m}^\dagger(t'') \rangle + \langle \hat{G}(t') \hat{G}^\dagger(t'') \rangle \right] dt' dt''. \end{aligned} \tag{3.31}$$

Then in view of Eqs. (3.4), (3.5), (3.6) and (3.10), we can put Eq. (3.31) as

$$\langle\hat{a}(t)\hat{a}^{\dagger}(t)\rangle = \langle\hat{a}(0)\hat{a}^{\dagger}(0)\rangle e^{-\kappa t} + \frac{g^2}{N}e^{-\kappa t}\int_0^t\int_0^t e^{\frac{\kappa}{2}(t'+t'')}\langle\hat{m}(t')\hat{m}^{\dagger}(t'')\rangle dt'dt'' + e^{-\kappa t}\int_0^t\int_0^t e^{\frac{\kappa}{2}(t'+t'')}\langle\hat{G}(t')\hat{G}^{\dagger}(t'')\rangle dt'dt''. \tag{3.32}$$

Again on assuming a cavity mode to be initially in a vacuum state, we can write

$$\begin{aligned}\langle\hat{a}(0)\hat{a}^{\dagger}(0)\rangle &= Tr(\hat{\rho}(0)\hat{a}\hat{a}^{\dagger}) \\ &= Tr(|0\rangle\langle 0|\hat{a}\hat{a}^{\dagger}) \\ &= \langle 0|\hat{a}\hat{a}^{\dagger}|0\rangle = 1,\end{aligned} \tag{3.33}$$

in view of which Eq. (3.32) becomes

$$\langle\hat{a}(t)\hat{a}^{\dagger}(t)\rangle = e^{-\kappa t} + e^{-\kappa t}\int_0^t\int_0^t e^{\frac{\kappa}{2}(t'+t'')}\left[\frac{g^2}{N}\langle\hat{m}(t')\hat{m}^{\dagger}(t'')\rangle + \langle\hat{G}(t')\hat{G}^{\dagger}(t'')\rangle\right]dt'dt'' \tag{3.34}$$

or

$$\langle\hat{a}(t)\hat{a}^{\dagger}(t)\rangle = e^{-\kappa t} + \langle\hat{a}'(t)\hat{a}'^{\dagger}(t)\rangle + e^{-\kappa t}\int_0^t e^{\frac{\kappa}{2}(t'+t'')}\langle\hat{G}(t')\hat{G}^{\dagger}(t'')\rangle dt'dt'', \tag{3.35}$$

where $\hat{a}'(t)$ is given by Eq. (3.13).

Now with the aid of Eq. (3.19) and its adjoint, we note that

$$\langle\hat{a}'(t)\hat{a}'^{\dagger}\rangle = \frac{\gamma_c}{\kappa N}\langle\hat{m}(t)\hat{m}^{\dagger}(t)\rangle. \tag{3.36}$$

Employing Eq.(2.102) along with Eqs. (2.98) and (2.99), we can put Eq. (3.36) in the form

$$\langle\hat{a}'(t)\hat{a}'^{\dagger}(t)\rangle = \frac{N\gamma_c}{\kappa((\gamma_c+\gamma)(\gamma_c+\gamma+r_a)+\gamma_c r_a)}\Big[\gamma_c r_a + (\gamma_c+\gamma)^2\Big]. \tag{3.37}$$

Finally, taking into account Eqs. (2.147) and (3.37) and carrying out the integration, then Eq. (3.35) can be written at steady-state as

$$\langle\hat{a}\hat{a}^{\dagger}\rangle = \frac{N\gamma_c}{\kappa((\gamma_c+\gamma)(\gamma_c+\gamma+r_a)+\gamma_c r_a)}\Big[\gamma_c r_a + (\gamma_c+\gamma)^2\Big] + N(\bar{n}+1). \tag{3.38}$$

We now can rewrite Eq. (3.2) as

$$\hat{a}^{\dagger}(t) = \hat{a}^{\dagger}(0)e^{-\frac{\kappa}{2}t} + \frac{g}{\sqrt{N}}\int_0^t e^{-\frac{\kappa}{2}(t-t')}\hat{m}^{\dagger}(t')dt' + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\hat{G}^{\dagger}(t')dt'. \tag{3.39}$$

Then with the help of Eqs. (3.2) and (3.39), we can determine the expectation value of their product and write

$$\begin{aligned}\langle\hat{a}^\dagger(t)\hat{a}^\dagger(t)\rangle = \langle\hat{a}^\dagger(0)\hat{a}^\dagger(0)\rangle e^{-\kappa t} + \int_0^t e^{-\frac{\kappa}{2}(2t-t')}\left[\frac{g}{\sqrt{N}}\langle\hat{a}^\dagger(0)\hat{m}^\dagger(t')\rangle + \langle\hat{a}^\dagger(0)\hat{G}^\dagger(t')\rangle\right]dt' \\ + \int_0^t e^{-\frac{\kappa}{2}(2t-t'')}\left[\frac{g}{\sqrt{N}}\langle\hat{m}^\dagger(t'')\hat{a}^\dagger(0)\rangle + \langle\hat{G}^\dagger(t'')\hat{a}^\dagger(0)\rangle\right]dt'' \\ + \int_0^t\int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\left[\frac{g^2}{N}\langle\hat{m}^\dagger(t'')\hat{m}^\dagger(t')\rangle + \frac{g}{\sqrt{N}}\langle\hat{m}^\dagger(t'')\hat{G}^\dagger(t')\rangle\right]dt''dt' \\ + \int_0^t\int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\left[\frac{g}{\sqrt{N}}\langle\hat{G}^\dagger(t'')\hat{m}^\dagger(t')\rangle + \langle\hat{G}^\dagger(t'')\hat{G}^\dagger(t')\rangle\right]dt''dt'.\end{aligned} \tag{3.40}$$

Now in view of Eqs. (3.4), (3.5), (3.6) and (3.10) and using Eq. (2.149) in Eq. (3.40), we find

$$\langle\hat{a}^\dagger(t)\hat{a}^\dagger(t)\rangle = \langle\hat{a}^\dagger(0)\hat{a}^\dagger(0)\rangle e^{-\kappa t} + \frac{g^2}{N}\int_0^t\int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle\hat{m}^\dagger(t'')\hat{m}^\dagger(t')\rangle dt''dt'. \tag{3.41}$$

We again assume that a cavity mode to be initially in a vacuum state and write

$$\begin{aligned}\langle\hat{a}^\dagger(0)\hat{a}^\dagger(0)\rangle &= Tr(\hat{\rho}(0)\hat{a}^\dagger\hat{a}^\dagger) \\ &= Tr(|0\rangle\langle 0|\hat{a}^\dagger\hat{a}^\dagger) \\ &= \langle 0|\hat{a}^\dagger\hat{a}^\dagger|0\rangle = 0.\end{aligned} \tag{3.42}$$

Using Eqs. (3.42) in Eq. (3.41), we obtain

$$\langle\hat{a}^\dagger(t)\hat{a}^\dagger(t)\rangle = \frac{g^2}{N}\int_0^t\int_0^t e^{-\frac{\kappa}{2}(2t-t'-t'')}\langle\hat{m}^\dagger(t'')\hat{m}^\dagger(t')\rangle dt''dt' \tag{3.43}$$

or

$$\langle\hat{a}^\dagger(t)\hat{a}^\dagger(t)\rangle = \langle\hat{a}'^\dagger(t)\hat{a}'^\dagger(t)\rangle, \tag{3.44}$$

where $\hat{a}'^\dagger(t)$ is given by Eq. (3.14).

Employing the adjoint of Eq. (3.19) along with the adjoint of Eq. (2.103), we find that

$$\langle\hat{a}'^\dagger(t)\hat{a}'^\dagger(t)\rangle = \frac{\gamma_c}{\kappa}\langle\hat{m}_{c'}^\dagger\rangle, \tag{3.45}$$

with the aid of which Eq. (3.44) reduces to

$$\langle \hat{a}^{\dagger 2}\rangle = \frac{\gamma_c}{\kappa}\langle \hat{m}_c^{\dagger}\rangle. \tag{3.46}$$

Following a similar procedure, we can verify that

$$\langle \hat{a}^{2}\rangle = \frac{\gamma_c}{\kappa}\langle \hat{m}_c\rangle. \tag{3.47}$$

Then substituting Eqs. (3.38), (3.46) and (3.47) into Eq. (3.30), one finds

$$\begin{aligned}(\Delta n)^2 = \bar{n}_c\bigg\{&\frac{N\gamma_c}{\kappa((\gamma_c+\gamma)(\gamma_c+\gamma+r_a)+\gamma_c r_a)}[\gamma_c r_a + (\gamma_c+\gamma)^2] + N(\bar{n}+1)\bigg\}\\ &+\frac{\gamma_c^2}{\kappa^2}\langle \hat{m}_c^{\dagger}\rangle\langle \hat{m}_c\rangle.\end{aligned} \tag{3.48}$$

We now proceed to determine the expectation value of the atomic operator $\hat{m}_c$. Then expressing the state vector of a three-level atom as [1,3,12]

$$|\psi_j\rangle = C_a|a_j\rangle + C_b|b_j\rangle + C_c|c_j\rangle, \tag{3.49}$$

where C_a, C_b and C_c are the probability amplitudes of finding the atom in states $|a_j\rangle$, $|b_j\rangle$ and $|c_j\rangle$, respectively. Now multiplying this equation from the left by $\langle a_j|$ and $\langle c_j|$ respectively, we readily find

$$C_a = \langle a_j|\psi_j\rangle \tag{3.50}$$

and

$$C_c = \langle c_j|\psi_j\rangle. \tag{3.51}$$

In view of Eqs. (3.50) and (3.51), we observe that

$$C_a C_a^* = \langle \hat{\eta}_a^j\rangle, \tag{3.52}$$

$$C_c C_c^* = \langle \hat{\eta}_c^j\rangle, \tag{3.53}$$

$$C_c C_a^* = \langle \hat{\sigma}_c^j\rangle. \tag{3.54}$$

In order to have a mathematically manageable analysis, we take $\hat{\sigma}_c^j$ to be real and write

$$C_c C_a^* = C_c^* C_a \tag{3.55}$$

and on subtracting C_cC_a from both sides of this equation, we have

$$C_c(C_a^* - C_a) + C_a(C_c - C_c^*) = 0. \tag{3.56}$$

Hence for all possible values of C_a and C_c, we see that

$$C_a^* = C_a, \tag{3.57}$$

$$C_c^* = C_c. \tag{3.58}$$

So that on account Eqs. (3.52) and (3.53) together with Eqs. (3.57) and (3.58), we can put Eq. (3.54) in the form of

$$\langle \hat{\sigma}_c^j \rangle = \sqrt{\langle \hat{\eta}_c^j \rangle \langle \hat{\eta}_a^j \rangle}. \tag{3.59}$$

Finally, on summing over N three-level atoms in the cavity and taking into account of Eqs. (2.85), (2.86) and (2.88), we can write Eq. (3.59) as

$$\langle \hat{m}_c \rangle = \sqrt{\langle \hat{N}_c \rangle \langle \hat{N}_a \rangle}. \tag{3.60}$$

Now using Eqs. (2.97) and (2.99) in Eq. (3.60), we obtain

$$\langle \hat{m}_c \rangle = \frac{N(\gamma_c + \gamma)}{(\gamma_c + \gamma)(\gamma_c + \gamma + r_a) + \gamma_c r_a} \sqrt{r_a(\gamma_c + \gamma)}. \tag{3.61}$$

Finally, introducing Eq. (3.61) and its conjugate into Eq. (3.48), we arrive at

$$\begin{aligned}(\Delta n)^2 = \bar{n}_c \bigg\{ &\frac{N\gamma_c}{\kappa((\gamma_c + \gamma)(\gamma_c + \gamma + r_a) + \gamma_c r_a)} [\gamma_c r_a + (\gamma_c + \gamma)^2] + N(\bar{n} + 1) \bigg\} \\ &+ \left[\frac{N\gamma_c(\gamma_c + \gamma)}{\kappa((\gamma_c + \gamma)(\gamma_c + \gamma + r_a) + \gamma_c r_a)} \right]^2 r_a(\gamma_c + \gamma)\end{aligned} \tag{3.62}$$

where $\bar{n}_c$ is given by Eq. (3.23).

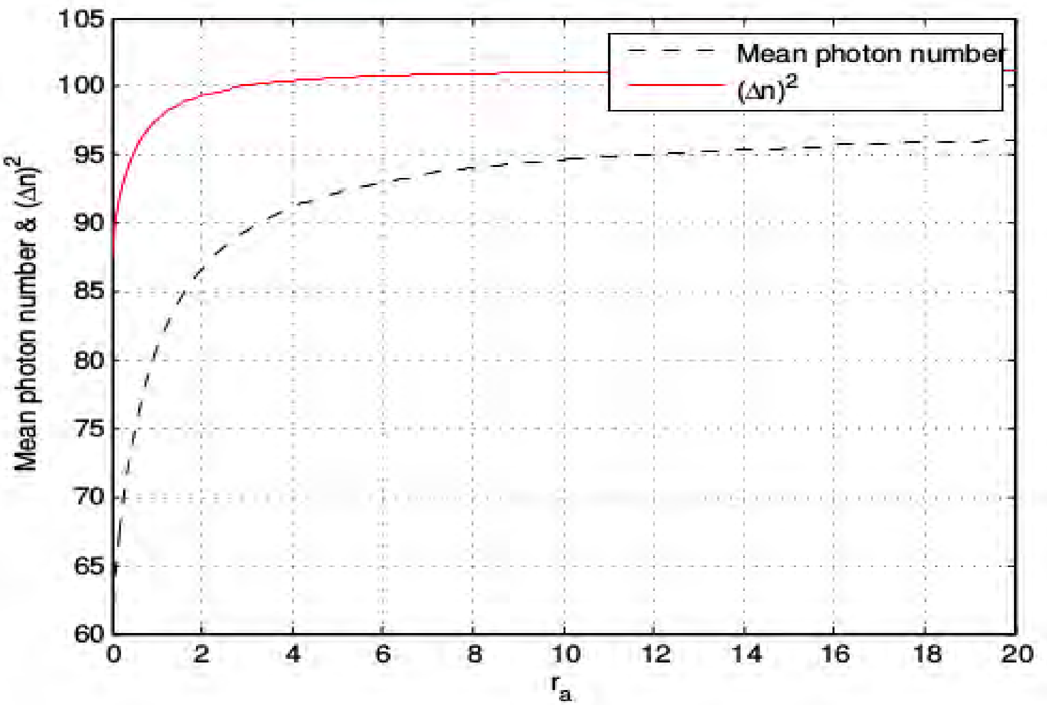

Figure 3.1: A plot of Eqs. (3.23) and (3.62) versus r_a, for $N = 30, \gamma_c = 1, \gamma = 0.4$, $\kappa = 0.8$ and $\bar{n} = 2$.

A plot in Figure 3.1 shows that the mean and variance of the photon number versus pump rate. From this plot, we observe that the variance of the photon number is greater than the mean photon number, then the light produced by degenerate three-level laser exhibits super-Poissonian photon statistics [2].

3.3 The power spectrum

In this section we want to evaluate the power spectrum of the cavity light. So that the power spectrum of a single-mode light with central frequency ω_0 is expressible as [15]

$$P(\omega) = \frac{1}{\pi} Re \int_0^\infty \langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss} e^{i(\omega-\omega_0)\tau} d\tau \tag{3.63}$$

where the subscript "ss" represents the steady-state.

Now upon integrating both sides of Eq. (3.63) over ω, we readily obtain

$$\int_{-\infty}^{\infty} P(\omega) d\omega = \bar{n}_c \tag{3.64}$$

in which $\bar{n}_c$ is the steady-state mean photon number of cavity mode. On the basis of this result, we assert that $P(\omega)d\omega$ is the steady-state mean photon number in the interval between ω and $\omega + d\omega$ [12,15].

We then proceed to evaluate the two-time correlation function that appears in Eq. (3.63) for the cavity light. To this end, we realize that Eq. (2.143) can be rewritten as

$$\hat{a}(t+\tau) = \hat{a}(t)e^{-\frac{\kappa}{2}\tau} + \frac{g}{\sqrt{N}}\int_0^{\tau} e^{-\frac{\kappa}{2}(\tau-\tau')}\hat{m}(t+\tau')d\tau' + \int_0^{\tau} e^{-\frac{\kappa}{2}(\tau-\tau')}\hat{G}(t+\tau')d\tau'. \quad (3.65)$$

Furthermore, adding Eqs. (2.153) and (2.154) together, we find

$$\frac{d}{dt}\hat{m} = -\gamma\hat{m} - \frac{\gamma_c}{2}\hat{m}_b + \hat{f}_a(t) + \hat{f}_b(t) \quad (3.66)$$

and setting $\hat{m}_b = \hat{m} - \hat{m}_a$, we obtain

$$\frac{d}{dt}\hat{m} = -\frac{1}{2}(\gamma_c + 2\gamma)\hat{m} + \frac{\gamma_c}{2}\hat{m}_a + \hat{f}_a(t) + \hat{f}_b(t), \quad (3.67)$$

or

$$\frac{d}{dt}\hat{m} \ = -\alpha\hat{m} + \frac{\gamma_c}{2}\hat{m}_a + \hat{f}_m(t), \quad (3.68)$$

where $\hat{f}_m(t) = \hat{f}_a(t) + \hat{f}_b(t)$ and $\alpha = \frac{1}{2}(\gamma_c + 2\gamma)$

Moreover, the solution of Eq. (3.68) can be written as

$$\hat{m}(t+\tau') = \hat{m}(t)e^{-\alpha\tau'} + \int_0^{\tau'} e^{-\alpha(\tau'-\tau'')}\left[\frac{\gamma_c}{2}\hat{m}_a(t+\tau'') + \hat{f}_m(t+\tau'')\right]d\tau'' \quad (3.69)$$

Application of the large-time approximation scheme to Eq. (2.153) yields

$$\hat{m}_a(t+\tau) = \frac{1}{(\gamma_c+\gamma)}\hat{f}_a(t+\tau). \quad (3.70)$$

Then using Eq. (3.70) in Eq. (3.69), we note that

$$\hat{m}(t+\tau') = \hat{m}(t)e^{-\alpha\tau'} + \int_0^{\tau'} e^{-\alpha(\tau'-\tau'')}\left[\frac{\gamma_c}{2(\gamma_c+\gamma)}\hat{f}_a(t+\tau'') + \hat{f}_m(t+\tau'')\right]d\tau''. \quad (3.71)$$

Upon introducing Eq. (3.71) into Eq. (3.65), we get

$$\begin{aligned}\hat{a}(t+\tau) = \hat{a}(t)e^{-\frac{\kappa}{2}\tau} &+ \frac{g}{\sqrt{N}}\hat{m}(t)e^{-\frac{\kappa}{2}\tau}\int_0^{\tau} e^{(\frac{\kappa}{2}-\alpha)\tau'}d\tau' + \frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}\tau}\int_0^{\tau}\Big\{e^{(\frac{\kappa}{2}-\alpha)\tau'} \\ &\times \int_0^{\tau'} e^{\alpha\tau''}\left[\frac{\gamma_c}{2(\gamma_c+\gamma)}\hat{f}_a(t+\tau'') + \hat{f}_m(t+\tau'')\right]d\tau''\Big\}d\tau' \\ &+ \int_0^{\tau} e^{-\frac{\kappa}{2}(\tau-\tau')}\hat{G}(t+\tau')d\tau'.\end{aligned} \quad (3.72)$$

So that integrating the first integral, we find

$$\hat{a}(t+\tau) = \hat{a}(t)e^{-\frac{\kappa}{2}\tau} + \frac{2g}{\sqrt{N}(\kappa-2\alpha)}\hat{m}(t)[e^{-\alpha\tau} - e^{-\frac{\kappa}{2}\tau}] + \frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}\tau}\int_0^{\tau}\Big\{e^{(\frac{\kappa}{2}-\alpha)\tau'}$$
$$\times\int_0^{\tau'} e^{\alpha\tau''}\left[\frac{\gamma_c}{2(\gamma_c+\gamma)}\hat{f}_a(t+\tau'') + \hat{f}_m(t+\tau'')\right]d\tau''\Big\}d\tau'$$
$$+\int_0^{\tau} e^{-\frac{\kappa}{2}(\tau-\tau')}\hat{G}(t+\tau')d\tau'. \tag{3.73}$$

Upon multiplying both sides of Eq. (3.73) by $\hat{a}^\dagger(t)$ from the left side and taking the expectation value of the resulting expression, we have

$$\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle e^{-\frac{\kappa}{2}\tau} + \frac{2g}{\sqrt{N}(\kappa-2\alpha)}\langle\hat{a}^\dagger(t)\hat{m}(t)\rangle[e^{-\alpha\tau} - e^{-\frac{\kappa}{2}\tau}]$$
$$+\frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}\tau}\Big\{\int_0^{\tau} e^{(\frac{\kappa}{2}-\alpha)\tau'}\int_0^{\tau'} e^{\alpha\tau''}\Big[\frac{\gamma_c}{2(\gamma_c+\gamma)}\langle\hat{a}^\dagger(t)\hat{f}_a(t+\tau'')\rangle$$
$$+\langle\hat{a}^\dagger(t)\hat{f}_m(t+\tau'')\rangle\Big]d\tau''\Big\}d\tau' + \int_0^{\tau} e^{-\frac{\kappa}{2}(\tau-\tau')}\langle\hat{a}^\dagger(t)\hat{G}(t+\tau')\rangle d\tau'. \tag{3.74}$$

Since the noise operators at some time has no effect on the cavity mode operator at an earlier time, we see that

$$\langle\hat{a}^\dagger(t)\hat{f}_a(t+\tau'')\rangle = \langle\hat{a}^\dagger(t)\rangle\langle\hat{f}_a(t+\tau'')\rangle = 0, \tag{3.75}$$

$$\langle\hat{a}^\dagger(t)\hat{f}_m(t+\tau'')\rangle = \langle\hat{a}^\dagger(t)\rangle\langle\hat{f}_m(t+\tau'')\rangle = 0, \tag{3.76}$$

$$\langle\hat{a}^\dagger(t)\hat{G}(t+\tau')\rangle = \langle\hat{a}^\dagger(t)\rangle\langle\hat{G}(t+\tau')\rangle = 0. \tag{3.77}$$

In view of these results, Eq. (3.74) reduces to

$$\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle e^{-\frac{\kappa}{2}\tau} + \frac{2g}{\sqrt{N}(\kappa-2\alpha)}\langle\hat{a}^\dagger(t)\hat{m}(t)\rangle\Big[e^{-\alpha\tau} - e^{-\frac{\kappa}{2}\tau}\Big]. \tag{3.78}$$

Applying once more the large-time approximation, one obtains from Eq. (2.116)

$$\hat{m}(t) = \frac{\kappa\sqrt{N}}{2g}\hat{a}(t) - \frac{\sqrt{N}}{g}\hat{G}(t), \tag{3.79}$$

with the aid of which Eq. (3.78) becomes

$$\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \frac{\kappa\bar{n}_c}{(\kappa-2\alpha)}e^{-\alpha\tau} - \frac{2\alpha\bar{n}_c}{(\kappa-2\alpha)}e^{-\frac{\kappa}{2}\tau}. \tag{3.80}$$

Upon substituting of Eq. (3.80) into Eq. (3.63), we have

$$P(\omega) = \frac{\kappa\bar{n}_c}{\pi(\kappa - 2\alpha)} Re \int_0^\infty e^{[i(\omega-\omega_0)-\alpha]\tau} d\tau - \frac{2\alpha\bar{n}_c}{\pi(\kappa - 2\alpha)} Re \int_0^\infty e^{[i(\omega-\omega_0)-\frac{\kappa}{2}]\tau} d\tau$$

$$= \frac{\kappa\bar{n}_c}{\pi(\kappa - 2\alpha)} Re \frac{1}{i(\omega-\omega_0)-\alpha} e^{[i(\omega-\omega_0)-\alpha]\tau}\Big|_0^\infty - \frac{2\alpha\bar{n}_c}{\pi(\kappa - 2\alpha)} Re \frac{1}{i(\omega-\omega_0)-\frac{\kappa}{2}} e^{[i(\omega\omega_0)-\frac{\kappa}{2}]\tau}\Big|_0^\infty$$

$$= \frac{\kappa\bar{n}_c}{(\kappa - 2\alpha)}\left[\frac{\alpha/\pi}{(\omega-\omega_0)^2+\alpha^2}\right] - \frac{2\alpha\bar{n}_c}{(\kappa - 2\alpha)}\left[\frac{\kappa/2\pi}{(\omega-\omega_0)^2+(\frac{\kappa}{2})^2}\right]. \tag{3.81}$$

We now realize in view of Eq. (3.64) that the mean photon number in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$ is expressible as

$$\bar{n}_{c\pm\lambda} = \int_{-\lambda}^{\lambda} P(\omega')d\omega' \tag{3.82}$$

in which $\omega' = \omega - \omega_0$. Upon introducing Eq. (3.81) into Eq. (3.82), and carrying out the integration applying the relation

$$\int_{-\lambda}^{\lambda} \frac{dy}{y^2+a^2} = \frac{2}{a}\arctan\left(\frac{\lambda}{a}\right), \tag{3.83}$$

we readily arrive at

$$\bar{n}_{c\pm\lambda} = \bar{n}_c z(\lambda) \tag{3.84}$$

where

$$z(\lambda) = \frac{2\kappa/\pi}{(\kappa - 2\alpha)}\arctan\left(\frac{\lambda}{\alpha}\right) - \frac{4\alpha/\pi}{(\kappa - 2\alpha)}\arctan\left(\frac{2\lambda}{\kappa}\right). \tag{3.85}$$

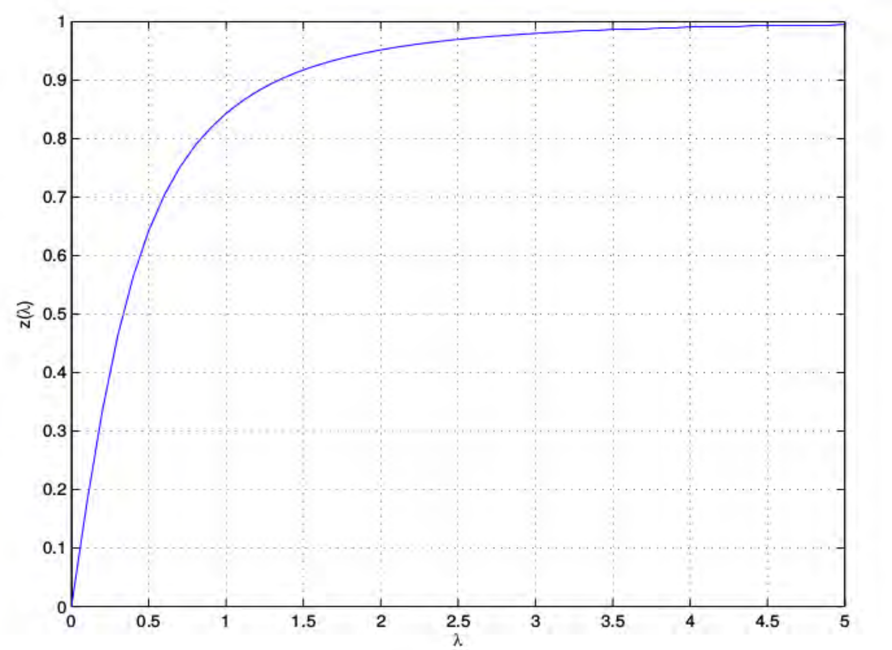

Figure 3.2: A plot of Eq. (3.85) versus λ, for $\kappa = 0.8$ and $\alpha = 3$.

A plot in Fig.(3.2) shows that $z(\lambda)$ versus λ, which is approach to 1 for a relatively small value of λ. This shows that the total mean photon number is confined in a relatively small frequency interval near the central frequency. From this plot, we can easily find $z(0.5) = 0.64$, $z(1) = 0.84$, $z(1.5) = 0.92$, $z(2) = 0.95$. Then combination of these results with Eq. (3.84) gives $\bar{n}_{c\pm0.5} = 0.64\bar{n}_c$, $\bar{n}_{c\pm1} = 0.84\bar{n}_c$, $\bar{n}_{c\pm1.5} = 0.92\bar{n}_c$, $\bar{n}_{c\pm2} = 0.95\bar{n}_c$.

Chapter 4

Quadrature Squeezing

In this chapter we are going to study the squeezing properties of a light produced by degenerate three-level laser in which degenerate three-level atoms are pumped from lower level to upper level r_a, by means of electron bombardment. To this end, we evaluate the quadrature variance and the quadrature squeezing of a light produced by degenerate three-level laser.

4.1 The quadrature variance

In this section we determine the quadrature variance of the plus and minus quadrature operators. The squeezing properties of the cavity light are described by the plus and minus quadrature operators defined by [1,12]

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a} \tag{4.1}$$

and

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}) \tag{4.2}$$

The variance of the plus quadrature operator is defined as

$$(\Delta a_+)^2 = \langle \hat{a}_+^2 \rangle - \langle \hat{a}_+ \rangle^2, \tag{4.3}$$

so that taking into account Eq. (4.1), we have

$$(\Delta a_+)^2 = \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}\hat{a}^\dagger \rangle + \langle \hat{a}^{\dagger 2} \rangle + \langle \hat{a}^2 \rangle - (\langle \hat{a}^\dagger \rangle^2 + \langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle + \langle \hat{a} \rangle \langle \hat{a}^\dagger \rangle + \langle \hat{a} \rangle^2). \tag{4.4}$$

Then based on Eq. (3.28) and its conjugate, Eq. (4.4) reduces to

$$(\Delta a_+)^2 = \langle \hat{a}^\dagger \hat{a}\rangle + \langle \hat{a}\hat{a}^\dagger\rangle + \langle \hat{a}^{\dagger 2}\rangle + \langle \hat{a}^2\rangle. \tag{4.5}$$

In a similar manner, the variance of the minus quadrature operator can be shown to be

$$(\Delta a_-)^2 = \langle \hat{a}^\dagger \hat{a}\rangle + \langle \hat{a}\hat{a}^\dagger\rangle - \langle \hat{a}^{\dagger 2}\rangle - \langle \hat{a}^2\rangle. \tag{4.6}$$

On the basis of Eqs. (4.5) and (4.6) the plus and minus quadrature variances can be expressed as

$$(\Delta a_\pm)^2 = \langle \hat{a}^\dagger \hat{a}\rangle + \langle \hat{a}\hat{a}^\dagger\rangle \pm \langle \hat{a}^{\dagger 2}\rangle \pm \langle \hat{a}^2\rangle. \tag{4.7}$$

Now employing Eq. (4.7) along with Eqs. (3.23), (3.38), (3.46) and (3.47), we find

$$\begin{aligned}(\Delta a_\pm)^2 = {} & \frac{N\gamma_c r_a(2\gamma_c+\gamma)}{\kappa((\gamma_c+\gamma)(\gamma_c+\gamma+r_a)+\gamma_c r_a)} + N\bar{n} + \frac{N\gamma_c[(\gamma_c+\gamma)^2+\gamma_c r_a]}{\kappa((\gamma_c+\gamma)(\gamma_c+\gamma+r_a)+\gamma_c r_a)} \\ & + N(\bar{n}+1) \pm \frac{\gamma_c}{\kappa}(\langle \hat{m}_c^\dagger\rangle + \langle \hat{m}_c\rangle).\end{aligned} \tag{4.8}$$

Then using Eq. (3.61) and its adjoint in Eq. (4.8), we readily arrive at

$$\begin{aligned}(\Delta a_\pm)^2 = {} & \frac{N\gamma_c}{\kappa((\gamma_c+\gamma)(\gamma_c+\gamma+r_a)+\gamma_c r_a)}\Big[r_a(3\gamma_c+\gamma)+(\gamma_c+\gamma)^2 \\ & \pm 2(\gamma_c+\gamma)\sqrt{r_a(\gamma_c+\gamma)}\Big] + N(2\bar{n}+1).\end{aligned} \tag{4.9}$$

On account of the result described by Eq. (4.9), the variance of the plus and minus quadrature operators can be written respectively as

$$\begin{aligned}(\Delta a_+)^2 = {} & \frac{N\gamma_c}{\kappa((\gamma_c+\gamma)(\gamma_c+\gamma+r_a)+\gamma_c r_a)}\Big[r_a(3\gamma_c+\gamma)+(\gamma_c+\gamma)^2 \\ & + 2(\gamma_c+\gamma)\sqrt{r_a(\gamma_c+\gamma)}\Big] + N(2\bar{n}+1)\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}(\Delta a_-)^2 = {} & \frac{N\gamma_c}{\kappa((\gamma_c+\gamma)(\gamma_c+\gamma+r_a)+\gamma_c r_a)}\Big[r_a(3\gamma_c+\gamma)+(\gamma_c+\gamma)^2 \\ & - 2(\gamma_c+\gamma)\sqrt{r_a(\gamma_c+\gamma)}\Big] + N(2\bar{n}+1).\end{aligned} \tag{4.11}$$

Now for $\gamma_c+\gamma >> r_a$, Eq. (4.9) becomes

$$(\Delta a_\pm)^2 = \frac{N\gamma_c}{\kappa} + N(2\bar{n}+1). \tag{4.12}$$

The product of the uncertainities in the two quadrature variances is thus

$$\Delta a_+ \Delta a_- = \frac{N\gamma_c}{\kappa} + N(2\bar{n} + 1). \tag{4.13}$$

Employing Eqs. (4.1) and (4.2), we easily find

$$[\hat{a}_+, \hat{a}_-] = 2i[\hat{a}, \hat{a}^\dagger], \tag{4.14}$$

or

$$\langle [\hat{a}_+, \hat{a}_-] \rangle = 2i(\langle \hat{a}\hat{a}^\dagger \rangle - \langle \hat{a}^\dagger \hat{a} \rangle). \tag{4.15}$$

Based on Eq. (4.14), the minimum uncertainity relation for the plus and minus quadrature operators can be written as

$$\begin{aligned} \Delta a_+ \Delta a_- &= \frac{1}{2} \Big| \langle [\hat{a}_+, \hat{a}_-] \rangle \Big| \\ &= \Big| \langle \hat{a}\hat{a}^\dagger \rangle - \langle \hat{a}^\dagger \hat{a} \rangle \Big|. \end{aligned} \tag{4.16}$$

Introduction of Eqs. (3.23) and (3.38) into Eq. (4.16) gives

$$\Delta a_+ \Delta a_- = \left| \frac{N\gamma_c}{\kappa((\gamma_c + \gamma)(\gamma_c + \gamma + r_a) + \gamma_c r_a)} [(\gamma_c + \gamma)^2 - r_a(\gamma_c + \gamma)] + N \right|. \tag{4.17}$$

For the condition of $\gamma_c + \gamma >> r_a$, Eq. (4.17) becomes

$$\Delta a_+ \Delta a_- = \frac{N\gamma_c}{\kappa} + N. \tag{4.18}$$

Now for $\bar{n} = 0$, Eq. (4.13) is equal to Eq. (4.18). Moreover, we note from Eq. (3.23) that the mean photon number of the cavity mode is zero for $\bar{n} = 0$ and $\gamma_c + \gamma >> r_a$. Thus the cavity mode is in a vacuum state under these conditions.

In the other case, for $\bar{n} = 0$, Eq. (4.12) can be expressed as

$$(\Delta a_\pm)^2 = \frac{N\gamma_c}{\kappa} + N, \tag{4.19}$$

which is quadrature variance of the cavity light in a vacuum state.

4.2 The quadrature squeezing

In this section we seek to study the squeezing properties of the light generated by degenerate three-level laser. We now seek to calculate the quadrature squeezing relative to the quadrature variance of the cavity light in a vacuum state. That is;

$$S = \frac{(\Delta a_{\pm})^2_{vac} - (\Delta a_{-})^2}{(\Delta a_{\pm})^2_{vac}} \tag{4.20}$$

where $(\Delta a_{\pm})^2_{vac}$ is quadrature variance of the cavity light in a vacuum state given by Eq. (4.19), S is quadrature squeezing and $(\Delta a_{-})^2$ is given by Eq. (4.11).

With the aid of Eqs. (4.11) and (4.19), we can put the quadrature squeezing as

$$S = \frac{\frac{N\gamma_c}{\kappa} - \frac{N\gamma_c}{\kappa((\gamma_c+\gamma)(\gamma_c+\gamma+r_a)+\gamma_c r_a)}\left[(\gamma_c+\gamma)^2 + r_a(3\gamma_c+\gamma) - 2(\gamma_c+\gamma)\sqrt{r_a(\gamma_c+\gamma)}\right] - 2N\bar{n}}{\frac{N\gamma_c}{\kappa} + N}$$

$$S = \frac{\gamma_c\left[\frac{2(\gamma_c+\gamma)\sqrt{r_a(\gamma_c+\gamma)} - \gamma_c r_a}{(\gamma_c+\gamma)(\gamma_c+\gamma+r_a)+\gamma_c r_a}\right] - 2\bar{n}\kappa}{\gamma_c+\kappa}, \tag{4.21}$$

or

$$S = \frac{2\sqrt{\gamma_c r_a(1+\eta)} - \frac{r_a}{1+\eta}}{(\gamma_c+\kappa)[1+\eta+\frac{r_a}{\gamma_c}(\frac{2+\eta}{1+\eta})]} - \frac{2\bar{n}\kappa}{\gamma_c+\kappa}, \tag{4.22}$$

where

$$\eta = \gamma/\gamma_c. \tag{4.23}$$

Therefore, we observe from Eq. (4.22) that, unlike the mean photon number, the quadrature squeezing does not depend on the number of atoms. This implies that the quadrature squeezing of the cavity light is independent of the number of cavity photons. Moreover, we note that the interaction of cavity mode with thermal reservoir decreases the quadrature squeezing.

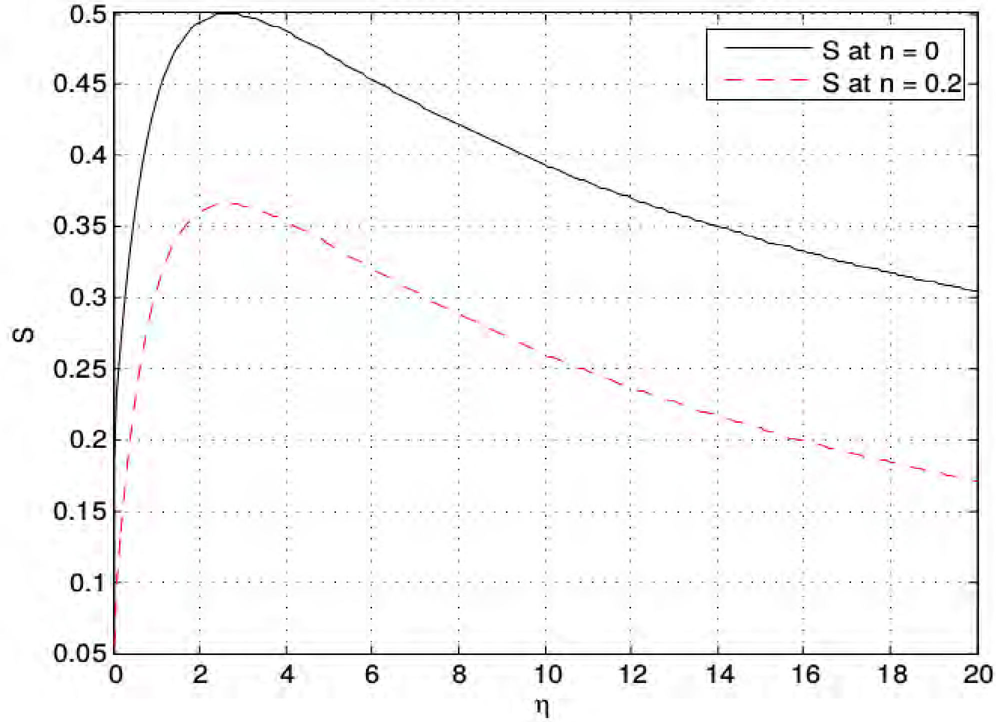

Figure 4.1: A plot of Eq. (4.22) versus η, for $\gamma_c = 1.6, r_a = 2, \kappa = 0.8$, and $\bar{n} = 0$ and 0.2.

The plot in Fig.(4.1) shows that, the maximum squeezing of the cavity light is 50% below a vacuum state level for the values $\bar{n} = 0$ and $\eta = 2.7$. On the other hand, we observe that the maximum squeezing of the cavity light is 36.6% below a vacuum state level for the values $\bar{n} = 0.2$ and $\eta = 2.7$.

Chapter 5

Conclusion

In this thesis, we have considered degenerate three-level laser in which degenerate three-level atoms available in the cavity are pumped from lower level to upper level at a rate of r_a. Employing the master equation for the system under consideration, we have obtained the equation of evolution for the expectation values of the cavity mode and atomic operators. Applying the large-time approximation scheme and the correlation properties of the cavity mode and atomic noise operators, we have determined the mean and variance of photon number for the cavity mode. We have found that the photon statistics of the cavity light is super-Poissonian.

We have realized that the light produced by degenerate three-level laser is in a vacuum state for $\gamma_c + \gamma >> r_a$ and $\bar{n} = 0$. Moreover, we have evaluated the quadrature variance and quadrature squeezing of a light produced by degenerate three-level laser. We have observed from Fig. (4.1) that, the maximum squeezing of the cavity light is 50% and 36.6% below a vacuum state level in the absence and presence of thermal reservoir, respectively.

On the other hand, we have seen that the interaction of cavity mode with thermal reservoir reduces the quadrature squeezing.

Furthermore, we have calculated the power spectrum of a light produced by degenerate three-level laser. From Fig. (3.2) the value of $z(\lambda)$ described by Eq.(3.85) approaches to 1 for a relatively small values of λ, indicating that the total mean photon number is confined in a relatively small frequency interval.

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DECLARATION

I hereby declare that this Thesis is my original work and has not been presented for a degree in any other university, and that all sources of material used for the thesis have been dully acknowledged.

Name: Belay Wedajo

Signature: ——————————

Place and time of submission: Addis Ababa University, June 2016

This thesis has been submitted for examination with my approval as University advisor.

Name: Dr. Deribe Hirpo

Signature: ——————————