THE QUANTUM ANALYSIS OF THE LIGHT PRODUCED BY COHERENTLY DRIVEN DEGENERATE THREE-LEVEL ATOM IN A CAVITY CONTAINING PARAMETRIC AMPLIFIER AND COUPLED TO A VACUUM RESERVOIR

By

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Abstract

In this thesis, we analyze the statistical and squeezing properties of the light produced by a degenerate three-level atom, whose top and bottom levels are coupled by coherent light, and available in a cavity containing degenerate parametric amplifier and coupled to a vacuum reservoir via a port-mirror. Employing the master equation for the system under consideration, we obtain the equations of evolution for the expectation value of atomic operators and the quantum Langevin equation for the cavity mode operator. Using the steady-state solutions of these equations and the large time approximation, we have determined the mean and variance of photon number, the power spectrum of cavity mode, quadrature variance and quadrature squeezing. We observe that the increase of the amplitude of the driving coherent light and the presence of the parametric amplifier enhance the mean and variance of the photon number. We have also established that the maximum quadrature squeezing is 61% for $\lambda = 0.03$ and 70% for $\lambda = 0.06$ below the vacuum-state level. Thus, we note that the presence of parametric amplifier has positive impact on the quadrature squeezing.

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Chapter 1

Introduction

A three-level laser is a source of squeezed light generated by three-level atoms in a cavity coupled to a vacuum reservoir[1, 2]. The statistical and squeezing properties of light produced by three-level atom have been studied by several authors[1-10]. These studies have shown that degenerate three-level atom can produce squeezed light[4, 15].

A three-level laser with a parametric amplifier in which three-level atoms, initially prepared in a coherent supperposition of the top and bottom levels or when these levels are coupled by a strong coherent light is a source of squeezed light[4, 8-11].

In a three-level atom the upper, middle, and lower levels are denoted by $|a\rangle$, $|b\rangle$, and $|c\rangle$ respectively. The transitions between levels $|a\rangle$ and $|b\rangle$ and between levels $|b\rangle$ and $|c\rangle$ are assumed to be dipole allowed, with direct transitions between levels $|a\rangle$ and $|c\rangle$ to be dipole forbidden[1]. If the two photons corresponding to these allowed transitions have the same frequency(ω), then the three-level atom is referred as a degenerate three-level atom, otherwise it is referred as a nondegenerate[8, 11, 12, 14].

In this thesis we seek to study the quantum properties of the light emitted by coherently driven degenerate three-level atom available in a cavity containing parametric amplifier and coupled to a vacuum reservoir via a port mirror. We first drive the equations of time evolution for the expectation values of cavity mode and atomic operators applying the master equation. Using the steady-state solutions of the resulting equations, we obtain the mean and variance of photon number, power spectrum, quadrature variance, and quadrature squeezing.

Chapter 2

Dynamics of Operators

In this chapter we consider degenerate three-level atom driven by a coherent light and in a cavity containing parametric amplifier and coupled to a vacuum reservoir via a port mirror. We consider the case where a pump mode drives the nonlinear crystal and a coherent light couples the top and bottom of levels of a three- level atom. Moreover, the light emerging from the nonlinear crystal does not couple the top and bottom levels of the atom.

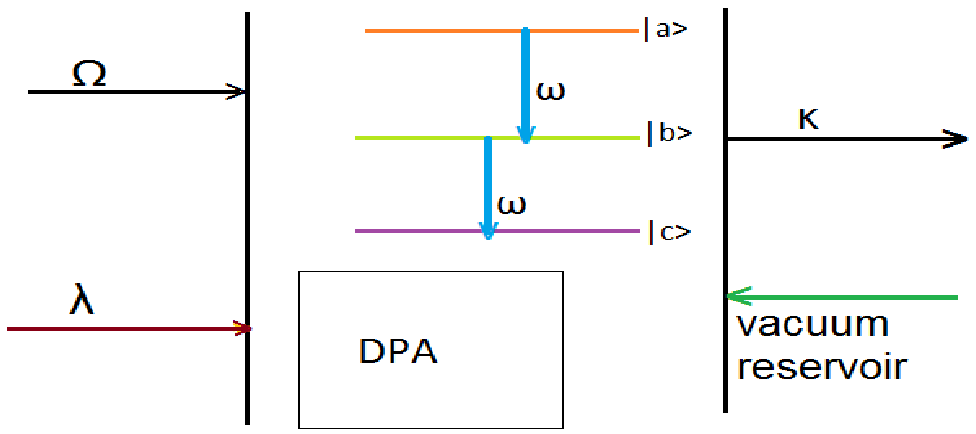

Figure 2.1: A degenerate three-level atom with a parametric amplifier.

The Hamiltonian that describes the coupling between the top and bottom levels of a three-level atom by the driving coherent light is expressible as

$$\hat{H}_1 = i\mu(\hat{\sigma}_c^\dagger \hat{b} - \hat{b}^\dagger \hat{\sigma}_c), \tag{2.1}$$

where $\hat{b}$ is annihilation operator for driving coherent light, $\hat{\sigma}_c = |c\rangle\langle a|$ is atomic operator, and μ is coupling constant between the driving coherent light and a degenerate three-level atom. Treating the driving coherent light classically, we can replace operator $\hat{b}$ by real and positive c-number ε_1, and hence the above Hamiltonian reduces to

$$\hat{H}_1 = \frac{i\Omega}{2}(\hat{\sigma}_c^\dagger - \hat{\sigma}_c), \tag{2.2}$$

where $\Omega = 2\varepsilon_1\mu$ is a real constant proportional to the amplitude of the driving coherent light. Moreover, the interaction of a three-level atom with the cavity mode is described by the Hamiltonian [1, 2]

$$\hat{H}_2 = ig[\hat{\sigma}_a^\dagger \hat{a} - \hat{a}^\dagger \hat{\sigma}_a + \hat{\sigma}_b^\dagger \hat{a} - \hat{a}^\dagger \hat{\sigma}_b], \tag{2.3}$$

where $\hat{\sigma}_a = |b\rangle\langle a|$, $\hat{\sigma}_b = |c\rangle\langle b|$, are atomic operators, g is the atom-cavity mode coupling constant, $\hat{a}$ is the annihilation operator for the cavity mode.

The process of parametric interaction is described by

$$\hat{H}_3 = i\frac{\beta}{2}(\hat{a}^2\hat{c}^\dagger - \hat{c}\hat{a}^{\dagger 2}), \tag{2.4}$$

in which β is the coupling constant, $\hat{c}$ is annihilation operator for the pump mode and $\hat{a}$ is annihilation operator for the signal mode. For strong pump mode, we can replace the operator $\hat{c}$ by real and positive c- number ε_2, and therefore re-express the Hamiltonian in Eq.(2.4) as

$$\hat{H}_3 = i\frac{\lambda}{2}(\hat{a}^2 - \hat{a}^{\dagger 2}), \tag{2.5}$$

where $\lambda = \beta\epsilon_2$ is a positive and real constant proportional to the amplitude of the pump mode.

Adding Eqs.(2.2), (2.3) and (2.5) the total Hamiltonian that describes parametric interaction, the interaction of degenerate three-level atom with the driving coherent light and cavity mode can be written as

$$\hat{H} = \frac{i\Omega}{2}(\hat{\sigma}_c^\dagger - \hat{\sigma}_c) + ig(\hat{\sigma}_a^\dagger \hat{a} - \hat{a}^\dagger \hat{\sigma}_a + \hat{\sigma}_b^\dagger \hat{a} - \hat{a}^\dagger \hat{\sigma}_b) + \frac{i\lambda}{2}(\hat{a}^2 - \hat{a}^{\dagger 2}). \tag{2.6}$$

We recall that the master equation for a cavity mode coupled to vacuum reservoir has the form [1, 2]

$$\frac{d}{dt}\hat{\rho} = -i[\hat{H}, \hat{\rho}] + \frac{\kappa}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}), \tag{2.7}$$

where κ is the cavity damping constant, $\hat{\rho}$ is the density operator for the cavity light, and $\hat{H}$ is given in Eq.(2.6).

2.1 The quantum Langevin equation

We seek here to determine the time evolution for the expectation values of the cavity mode and atomic operators.

Using the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr\Big(\frac{d\hat{\rho}}{dt}\hat{A}\Big), \tag{2.8}$$

along with the master equation described by Eq.(2.7), we have

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}\rangle &= Tr(\frac{d\hat{\rho}}{dt}\hat{a})\\ &= -iTr([\hat{H},\hat{\rho}]\hat{a}) + \frac{\kappa}{2}Tr(\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}),\end{aligned} \tag{2.9}$$

or

$$\frac{d}{dt}\langle\hat{a}\rangle = T_1 + T_2, \tag{2.10}$$

where

$$T_1 = -iTr([\hat{H},\hat{\rho}]\hat{a}), \tag{2.10a}$$

$$T_2 = \frac{\kappa}{2}Tr(\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}). \tag{2.10b}$$

Applying the cyclic property of trace operation, we see that

$$\begin{aligned}T_1 &= -Tr([\hat{H},\rho]\hat{a})\\ &= -iTr(\hat{H}\hat{\rho}\hat{a} - \hat{\rho}\hat{H}\hat{a})\\ &= -iTr(\hat{\rho}\hat{a}\hat{H} - \hat{\rho}\hat{H}\hat{a})\\ &= -i\langle[\hat{a},\hat{H}]\rangle.\end{aligned} \tag{2.11}$$

Substituting Eq.(2.6) into Eq.(2.11), we have

$$\begin{aligned}T_1 = -i\langle\{&\frac{i\Omega}{2}([\hat{a},\hat{\sigma}_c^\dagger] - [\hat{a},\hat{\sigma}_c]) + ig([\hat{a},\hat{\sigma}_a^\dagger\hat{a}] - [\hat{a},\hat{a}^\dagger\hat{\sigma}_a] + [\hat{a},\hat{\sigma}_b^\dagger\hat{a}] - [\hat{a},\hat{a}^\dagger\hat{\sigma}_b])\\ &+ i\frac{\lambda}{2}([\hat{a},\hat{a}^2] - [\hat{a},\hat{a}^{\dagger 2}])\}\rangle.\end{aligned} \tag{2.12}$$

Assuming that the cavity mode and atomic operators commute and taking in to account the commutation properties,

$$[\hat{A}, \hat{B} + \hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}] \tag{2.13a}$$

and

$$[\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}, \tag{2.13b}$$

we have

$$[\hat{a}, \hat{\sigma}_c^\dagger] = 0, \tag{2.14a}$$

$$[\hat{a}, \hat{\sigma}_c] = 0, \tag{2.14b}$$

$$[\hat{a}, \hat{\sigma}_a^\dagger \hat{a}] = \hat{\sigma}_a^\dagger [\hat{a}, \hat{a}] + [\hat{a}, \hat{\sigma}_a^\dagger]\hat{a} = 0, \tag{2.14c}$$

$$[\hat{a}, \hat{a}^\dagger \hat{\sigma}_a] = \hat{a}^\dagger [\hat{a}, \hat{\sigma}_a] + [\hat{a}, \hat{a}^\dagger]\hat{\sigma}_a = \hat{\sigma}_a, \tag{2.14d}$$

$$[\hat{a}, \hat{\sigma}_b^\dagger \hat{a}] = \hat{\sigma}_b^\dagger [\hat{a}, \hat{a}] + [\hat{a}, \hat{\sigma}_b^\dagger]\hat{a} = 0, \tag{2.14e}$$

$$[\hat{a}, \hat{a}^\dagger \hat{\sigma}_b] = \hat{a}^\dagger [\hat{a}, \hat{\sigma}_b] + [\hat{a}, \hat{a}^\dagger]\hat{\sigma}_b = \hat{\sigma}_b, \tag{2.14f}$$

$$[\hat{a}, \hat{a}^2] = 0, \tag{2.14g}$$

$$[\hat{a}, \hat{a}^{\dagger 2}] = \hat{a}^\dagger [\hat{a}, \hat{a}^\dagger] + [\hat{a}, \hat{a}^\dagger]\hat{a}^\dagger = 2\hat{a}^\dagger. \tag{2.14h}$$

Substituting Eqs.(2.14a) - (2.14h) into Eq.(2.11), we have

$$T_1 = -g(\langle\hat{\sigma}_a\rangle + \langle\hat{\sigma}_b\rangle) - \lambda\langle\hat{a}^\dagger\rangle. \tag{2.15}$$

Furthermore, applying the cyclic property of trace operation, we note that

$$\begin{aligned} T_2 &= \frac{\kappa}{2} Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2) \\ &= \frac{\kappa}{2} Tr(2\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2) \\ &= \frac{\kappa}{2} Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}) \\ &= \frac{\kappa}{2} Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}(1 + \hat{a}^\dagger\hat{a})\hat{a}) \\ &= \frac{\kappa}{2} Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2) \\ &= \frac{\kappa}{2} Tr(-\hat{\rho}\hat{a}) \\ &= -\frac{\kappa}{2}\langle\hat{a}\rangle. \end{aligned} \tag{2.16}$$

Substituting Eqs.(2.15) and (2.16) into Eq.(2.10), we obtain

$$\frac{d}{dt}\langle\hat{a}\rangle = -g(\langle\hat{\sigma}_a\rangle + \langle\hat{\sigma}_b\rangle) - \lambda\langle\hat{a}^\dagger\rangle - \frac{\kappa}{2}\langle\hat{a}\rangle. \tag{2.17}$$

In a similar manner, we can readily establish that

$$\frac{d}{dt}\langle\hat{a}^2\rangle = -g(\langle\hat{a}\hat{\sigma}_a\rangle + \langle\hat{a}\hat{\sigma}_b\rangle + \langle\hat{\sigma}_a\hat{a}\rangle + \langle\hat{\sigma}_b\hat{a}\rangle) - \lambda(\langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{a}^\dagger\hat{a}\rangle) - \kappa\langle\hat{a}^2\rangle, \tag{2.18}$$

$$\frac{d}{dt}\langle\hat{a}^{\dagger 2}\rangle = -g(\langle\hat{a}^\dagger\hat{\sigma}_a^\dagger\rangle + \langle\hat{a}^\dagger\hat{\sigma}_b^\dagger\rangle + \langle\hat{\sigma}_a^\dagger\hat{a}^\dagger\rangle + \langle\hat{\sigma}_b^\dagger\hat{a}^\dagger\rangle) - \lambda(\langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}\hat{a}^\dagger\rangle) - \kappa\langle\hat{a}^{\dagger 2}\rangle, \tag{2.19}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle = -\kappa\langle\hat{a}^\dagger\hat{a}\rangle - g(\langle\hat{\sigma}_a^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a\rangle + \langle\hat{\sigma}_b^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_b\rangle) - \lambda(\langle\hat{a}^2\rangle + \langle\hat{a}^{\dagger 2}\rangle), \tag{2.20}$$

$$\frac{d}{dt}\langle\hat{a}\hat{a}^\dagger\rangle = -\kappa\langle\hat{a}\hat{a}^\dagger\rangle - g(\langle\hat{a}\hat{\sigma}_a^\dagger\rangle + \langle\hat{\sigma}_a\hat{a}^\dagger\rangle + \langle\hat{a}\hat{\sigma}_b^\dagger\rangle + \langle\hat{\sigma}_b\hat{a}^\dagger\rangle) - \lambda(\langle\hat{a}^2\rangle + \langle\hat{a}^{\dagger 2}\rangle) + \kappa. \tag{2.21}$$

We next proceed to obtain the time evolution of atomic operators using Eq. (2.8) along with (2.7). We thus see that

$$\begin{aligned}\frac{d}{dt}\langle\hat{\sigma}_a\rangle &= -iTr([\hat{H},\hat{\rho}]\hat{\sigma}_a) + \frac{\kappa}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{\sigma}_a - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{\sigma}_a - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{\sigma}_a)\\ &= R_1 + R_2,\end{aligned} \tag{2.22}$$

where

$$R_1 = -iTr([\hat{H},\hat{\rho}]\hat{\sigma}_a), \tag{2.22a}$$

$$R_2 = \frac{\kappa}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{\sigma}_a - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{\sigma}_a - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{\sigma}_a). \tag{2.22b}$$

These traces can be evaluated applying the cyclic property. We thus notice that

$$\begin{aligned}R_1 &= -iTr([\hat{H},\hat{\rho}]\hat{\sigma}_a)\\ &= -iTr(\hat{\rho}\hat{\sigma}_a\hat{H} - \hat{\rho}\hat{H}\hat{\sigma}_a)\\ &= -i\langle[\hat{\sigma}_a,\hat{H}]\rangle.\end{aligned} \tag{2.23}$$

Substituting Eq.(2.6) into Eq.(2.23), we have

$$\begin{aligned}R_1 &= -i\langle\frac{i\Omega}{2}[\hat{\sigma}_a,\hat{\sigma}_c^\dagger - \hat{\sigma}_c] + ig[\hat{\sigma}_a,\hat{\sigma}_a^\dagger\hat{a} - \hat{a}^\dagger\hat{\sigma}_a + \hat{\sigma}_b^\dagger\hat{a} - \hat{a}^\dagger\hat{\sigma}_b] + \frac{i\lambda}{2}[\hat{\sigma}_a,\hat{a}^2 - \hat{a}^{\dagger 2}]\rangle\\ &= -i\langle\frac{i\Omega}{2}([\hat{\sigma}_a,\hat{\sigma}_c^\dagger] - [\hat{\sigma}_a,\hat{\sigma}_c])\\ &+ ig([\hat{\sigma}_a,\hat{\sigma}_a^\dagger\hat{a}] - [\hat{\sigma}_a,\hat{a}^\dagger\hat{\sigma}_a] + [\hat{\sigma}_a,\hat{\sigma}_b^\dagger\hat{a}] - [\hat{\sigma}_a,\hat{a}^\dagger\hat{\sigma}_b])\\ &+ \frac{i\lambda}{2}([\hat{\sigma}_a,\hat{a}^2] - [\hat{\sigma}_a,\hat{a}^{\dagger 2}])\rangle.\end{aligned} \tag{2.24}$$

Assuming the cavity mode and atomic operators commute and taking into account the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, it can then be noticed that

$$[\hat{\sigma}_a, -\hat{\sigma}_c] = -(\hat{\sigma}_a\hat{\sigma}_c - \hat{\sigma}_c\hat{\sigma}_a) = -(|b\rangle\langle a|c\rangle\langle a| - |c\rangle\langle a|b\rangle\langle a|) = 0, \tag{2.24a}$$

$$[\hat{\sigma}_a, \hat{\sigma}_c^\dagger] = \hat{\sigma}_a\hat{\sigma}_c^\dagger - \hat{\sigma}_c^\dagger\hat{\sigma}_a = |b\rangle\langle a|a\rangle\langle c| - |a\rangle\langle c|b\rangle\langle a| = |b\rangle\langle c| = \hat{\sigma}_b^\dagger, \tag{2.24b}$$

$$\begin{aligned}[\hat{\sigma}_a, \hat{\sigma}_a^\dagger\hat{a}] &= \hat{\sigma}_a^\dagger[\hat{\sigma}_a, \hat{a}] + [\hat{\sigma}_a, \hat{\sigma}_a^\dagger]\hat{a} = (\hat{\sigma}_a\hat{\sigma}_a^\dagger - \hat{\sigma}_a^\dagger\hat{\sigma}_a)\hat{a} \\ &= (|b\rangle\langle a|a\rangle\langle b| - |a\rangle\langle b|b\rangle\langle a|)\hat{a} \\ &= (|b\rangle\langle b| - |a\rangle\langle a|)\hat{a} = (\hat{\eta}_b - \hat{\eta}_a)\hat{a},\end{aligned} \tag{2.24c}$$

$$[\hat{\sigma}_a, \hat{a}^\dagger\hat{\sigma}_a] = \hat{a}^\dagger[\hat{\sigma}_a, \hat{\sigma}_a] + [\hat{\sigma}_a, \hat{a}^\dagger]\hat{\sigma}_a = 0, \tag{2.24d}$$

$$\begin{aligned}[\hat{\sigma}_a, \hat{\sigma}_b^\dagger\hat{a}] &= \hat{\sigma}_b^\dagger[\hat{\sigma}_a, \hat{a}] + [\hat{\sigma}_a, \hat{\sigma}_b^\dagger]\hat{a} \\ &= [\hat{\sigma}_a, \hat{\sigma}_b^\dagger]\hat{a} = (\hat{\sigma}_a\hat{\sigma}_b^\dagger - \hat{\sigma}_b^\dagger\hat{\sigma}_a)\hat{a} \\ &= (|b\rangle\langle a|b\rangle c| - |b\rangle\langle c|b\rangle a|)\hat{a} = 0,\end{aligned} \tag{2.24e}$$

$$\begin{aligned}[\hat{\sigma}_a, \hat{a}^\dagger\hat{\sigma}_b] &= \hat{a}^\dagger[\hat{\sigma}_a, \hat{\sigma}_b] + [\hat{\sigma}_a, \hat{a}^\dagger]\hat{\sigma}_b \\ &= \hat{a}^\dagger[\hat{\sigma}_a, \hat{\sigma}_b] = \hat{a}^\dagger(\hat{\sigma}_a\hat{\sigma}_b - \hat{\sigma}_b\hat{\sigma}_a) \\ &= \hat{a}^\dagger(|b\rangle\langle a|c\rangle b| - |c\rangle\langle b|b\rangle a|) = -\hat{a}^\dagger(|c\rangle\rangle a|) \\ &= -\hat{a}^\dagger\hat{\sigma}_c,\end{aligned} \tag{2.24f}$$

$$[\hat{\sigma}_a, \hat{a}^2] = [\hat{\sigma}_a, \hat{a}^{\dagger 2}] = 0, \tag{2.24g}$$

where $\hat{\eta}_a = |a\rangle\langle a|$ and $\hat{\eta}_b = |b\rangle\langle b|$ are atomic operators.

Substituting Eqs.(2.24a) - (2.24g) into Eq.(2.24), we have

$$R_1 = \frac{\Omega}{2}\langle\hat{\sigma}_b^\dagger\rangle + g(\langle\hat{\eta}_b\hat{a}\rangle - \langle\hat{\eta}_a\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_c\rangle). \tag{2.25}$$

We also see that

$$\begin{aligned}R_2 &= \frac{\kappa}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{\sigma}_a - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{\sigma}_a - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{\sigma}_a) \\ &= \frac{\kappa}{2}Tr(2\hat{\rho}\hat{a}^\dagger\hat{\sigma}_a\hat{a} - \hat{\rho}\hat{\sigma}_a\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{\sigma}_a) \\ &= \frac{\kappa}{2}Tr(2\hat{\rho}\hat{a}^\dagger|b\rangle\langle a|\hat{a} - \hat{\rho}|b\rangle\langle a|\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}|b\rangle\langle a|) = 0.\end{aligned} \tag{2.26}$$

Substituting Eqs.(2.25) and (2.26) into (2.22), we obtain

$$\frac{d}{dt}\langle\hat{\sigma}_a\rangle = g(\langle\hat{\eta}_b\hat{a}\rangle - \langle\hat{\eta}_a\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_c\rangle) + \frac{\Omega}{2}\langle\hat{\sigma}_b^\dagger\rangle. \tag{2.27}$$

Following a procedure used in obtaining Eq.(2.27), one can establish that

$$\frac{d}{dt}\langle\hat{\sigma}_b\rangle = g(\langle\hat{\eta}_c\hat{a}\rangle - \langle\hat{\eta}_b\hat{a}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_c\rangle) - \frac{\Omega}{2}\langle\hat{\sigma}_a^\dagger\rangle, \tag{2.28}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c\rangle = g(\langle\hat{\sigma}_b\hat{a}\rangle - \langle\hat{\sigma}_a\hat{a}\rangle) + \frac{\Omega}{2}(\langle\hat{\eta}_c\rangle - \langle\hat{\eta}_a\rangle), \tag{2.29}$$

$$\frac{d}{dt}\langle\hat{\eta}_a\rangle = g(\langle\hat{\sigma}_a^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a\rangle) + \frac{\Omega}{2}(\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle), \tag{2.30}$$

$$\frac{d}{dt}\langle\hat{\eta}_b\rangle = g(\langle\hat{\sigma}_b^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_b\rangle - \langle\hat{\sigma}_a^\dagger\hat{a}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_a\rangle), \tag{2.31}$$

$$\frac{d}{dt}\langle\hat{\eta}_c\rangle = -g(\langle\hat{\sigma}_b^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_b\rangle) - \frac{\Omega}{2}(\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle). \tag{2.32}$$

The quantum Langevin equation for the cavity mode operator can be written, based on Eq.(2.17), as

$$\frac{d}{dt}\hat{a} = -\frac{\kappa}{2}\hat{a} - g(\hat{\sigma}_a + \hat{\sigma}_b) - \lambda\hat{a}^\dagger + \hat{g}_a(t), \tag{2.33}$$

where $\hat{g}_a(t)$ is a cavity mode noise operator whose correlation properties remain to be determined.

We observe that Eqs.(2.18)-(2.21) and Eqs.(2.27)-(2.32) are non linear coupled differential equations. Therefore, it is not possible to obtain exact time dependent solutions. Then to overcome this problem, we apply the large time approximation scheme to Eq.(2.33) and write the approximately valid relation

$$\hat{a}(t) = \frac{2}{\kappa}[-g(\hat{\sigma}_a(t) + \hat{\sigma}_b(t)) - \lambda\hat{a}^\dagger(t) + \hat{g}_a(t)], \tag{2.34}$$

from which follows

$$\hat{a}^\dagger(t) = \frac{2}{\kappa}[-g(\hat{\sigma}_a^\dagger(t) + \hat{\sigma}_b^\dagger(t)) - \lambda\hat{a}(t) + \hat{g}_a^\dagger(t)]. \tag{2.35}$$

Substituting Eq.(2.35) into (2.34), we have

$$\begin{aligned}\hat{a} = &-\frac{2g}{\kappa}(\frac{\kappa^2}{\kappa^2 - 4\lambda^2})(\hat{\sigma}_a + \hat{\sigma}_b) + \frac{4\lambda g}{\kappa^2}(\frac{\kappa^2}{\kappa^2 - 4\lambda^2})(\hat{\sigma}_a^\dagger + \hat{\sigma}_b^\dagger)\\ &-\frac{4\lambda}{\kappa^2}(\frac{\kappa^2}{\kappa^2 - 4\lambda^2})\hat{g}_a^\dagger(t) + \frac{2}{\kappa}(\frac{\kappa^2}{\kappa^2 - 4\lambda^2})\hat{g}_a(t),\end{aligned} \tag{2.36}$$

or

$$\hat{a} = -\frac{2g\Gamma}{\kappa}(\hat{\sigma}_a + \hat{\sigma}_b) + \frac{4\lambda g\Gamma}{\kappa^2}(\hat{\sigma}_a^\dagger + \hat{\sigma}_b^\dagger) - \frac{4\lambda\Gamma}{\kappa^2}\hat{g}_a^\dagger(t) + \frac{2\Gamma}{\kappa}\hat{g}_a(t), \tag{2.37}$$

where $\Gamma = \frac{\kappa^2}{\kappa^2 - 4\lambda^2}$.

Now comparing Eq.(2.17) and the expectation value of Eq.(2.33), we obtain

$$\langle \hat{g}_a(t)\rangle = 0, \tag{2.38}$$

and hence

$$\langle \hat{g}_a^\dagger(t)\rangle = 0. \tag{2.39}$$

Now substituting Eq.(2.37) and its adjoint into Eq.(2.18), we see that

$$\begin{aligned}
\frac{d}{dt}\langle \hat{a}^2\rangle = & -\kappa\langle \hat{a}^2\rangle - g\langle[-\frac{2g\Gamma}{\kappa}(\hat{\sigma}_a + \hat{\sigma}_b) + \frac{4\lambda g\Gamma}{\kappa^2}(\hat{\sigma}_a^\dagger + \hat{\sigma}_b^\dagger) - \frac{4\lambda\Gamma}{\kappa^2}\hat{g}_a^\dagger(t) + \frac{2\Gamma}{\kappa}\hat{g}_a(t)]\hat{\sigma}_a\rangle \\
& + \langle[-\frac{2g\Gamma}{\kappa}(\hat{\sigma}_a + \hat{\sigma}_b) + \frac{4\lambda g\Gamma}{\kappa^2}(\hat{\sigma}_a^\dagger + \hat{\sigma}_b^\dagger) - \frac{4\lambda\Gamma}{\kappa^2}\hat{g}_a^\dagger(t) + \frac{2\Gamma}{\kappa}\hat{g}_a(t)]\hat{\sigma}_b\rangle \\
& + \langle\hat{\sigma}_a[-\frac{2g\Gamma}{\kappa}(\hat{\sigma}_a + \hat{\sigma}_b) + \frac{4\lambda g\Gamma}{\kappa^2}(\hat{\sigma}_a^\dagger + \hat{\sigma}_b^\dagger) - \frac{4\lambda\Gamma}{\kappa^2}\hat{g}_a^\dagger(t) + \frac{2\Gamma}{\kappa}\hat{g}_a(t)]\rangle \\
& + \langle\hat{\sigma}_b[-\frac{2g\Gamma}{\kappa}(\hat{\sigma}_a + \hat{\sigma}_b) + \frac{4\lambda g\Gamma}{\kappa^2}(\hat{\sigma}_a^\dagger + \hat{\sigma}_b^\dagger) - \frac{4\lambda\Gamma}{\kappa^2}\hat{g}_a^\dagger(t) + \frac{2\Gamma}{\kappa}\hat{g}_a(t)]\rangle \\
& - \lambda(\langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{a}^\dagger\hat{a}\rangle) \\
= & -\kappa\langle\hat{a}^2\rangle + \frac{2g^2\Gamma}{\kappa}(\langle\hat{\sigma}_a\hat{\sigma}_a\rangle + \langle\hat{\sigma}_b\hat{\sigma}_a\rangle) - \frac{4\lambda g^2\Gamma}{\kappa^2}(\langle\hat{\sigma}_a^\dagger\hat{\sigma}_a\rangle + \langle\hat{\sigma}_b^\dagger\hat{\sigma}_a\rangle) + \frac{4\lambda g\Gamma}{\kappa^2}\langle\hat{g}_a^\dagger(t)\hat{\sigma}_a\rangle \\
& - \frac{2g\Gamma}{\kappa}\langle\hat{g}_a(t)\hat{\sigma}_a\rangle + \frac{2g^2\Gamma}{\kappa}(\langle\hat{\sigma}_a\hat{\sigma}_b\rangle + \langle\hat{\sigma}_b\hat{\sigma}_b\rangle) - \frac{4\lambda g^2\Gamma}{\kappa^2}(\langle\hat{\sigma}_a^\dagger\hat{\sigma}_b\rangle + \langle\hat{\sigma}_b^\dagger\hat{\sigma}_b\rangle) + \frac{4\lambda g\Gamma}{\kappa^2}\langle\hat{g}_a^\dagger(t)\hat{\sigma}_b\rangle \\
& - \frac{2g\Gamma}{\kappa}\langle\hat{g}_a(t)\hat{\sigma}_b\rangle + \frac{2g^2\Gamma}{\kappa}(\langle\hat{\sigma}_a\hat{\sigma}_a\rangle + \langle\hat{\sigma}_a\hat{\sigma}_b\rangle) - \frac{4\lambda g^2\Gamma}{\kappa^2}(\langle\hat{\sigma}_a\hat{\sigma}_a^\dagger\rangle + \langle\hat{\sigma}_a\hat{\sigma}_b^\dagger\rangle) + \frac{4\lambda g\Gamma}{\kappa^2}\langle\hat{\sigma}_a\hat{g}_a^\dagger(t)\rangle \\
& - \frac{2g\Gamma}{\kappa}\langle\hat{\sigma}_a\hat{g}_a(t)\rangle + \frac{2g^2\Gamma}{\kappa}(\langle\hat{\sigma}_b\hat{\sigma}_a\rangle + \langle\hat{\sigma}_b\hat{\sigma}_b\rangle) - \frac{4\lambda g^2\Gamma}{\kappa^2}(\langle\hat{\sigma}_b\hat{\sigma}_a^\dagger\rangle + \langle\hat{\sigma}_b\hat{\sigma}_b^\dagger\rangle) + \frac{4\lambda g\Gamma}{\kappa^2}\langle\hat{\sigma}_b\hat{g}_a^\dagger(t)\rangle \\
& - \frac{2g\Gamma}{\kappa}\langle\hat{\sigma}_b\hat{g}_a(t)\rangle - \lambda(\langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{a}^\dagger\hat{a}\rangle).
\end{aligned} \tag{2.41}$$

Assuming the cavity mode noise operator and atomic operator are not correlated and taking into account Eqs.(2.38) and (2.39), we can write

$$\langle\hat{\sigma}_a(t)\hat{g}_a(t)\rangle = \langle\hat{\sigma}_a(t)\rangle\langle\hat{g}_a(t)\rangle = 0, \tag{2.42}$$

$$\langle\hat{\sigma}_b(t)\hat{g}_a(t)\rangle = \langle\hat{\sigma}_b(t)\rangle\langle\hat{g}_a(t)\rangle = 0, \tag{2.43}$$

$$\langle\hat{\sigma}_a(t)\hat{g}_a^\dagger(t)\rangle = \langle\hat{\sigma}_a(t)\rangle\langle\hat{g}_a^\dagger(t)\rangle = 0, \tag{2.44}$$

$$\langle \hat{\sigma}_b(t)\hat{g}_a^\dagger(t)\rangle = \langle \hat{\sigma}_b(t)\rangle\langle \hat{g}_a^\dagger(t)\rangle = 0, \tag{2.45}$$

$$\langle \hat{g}_a(t)\hat{\sigma}_a(t)\rangle = \langle \hat{g}_a(t)\rangle\langle \hat{\sigma}_a(t)\rangle = 0, \tag{2.46}$$

$$\langle \hat{g}_a(t)\hat{\sigma}_b(t)\rangle = \langle \hat{g}_a(t)\rangle\langle \hat{\sigma}_b(t)\rangle = 0, \tag{2.47}$$

$$\langle \hat{g}_a^\dagger(t)\hat{\sigma}_a(t)\rangle = \langle \hat{g}_a^\dagger(t)\rangle\langle \hat{\sigma}_a(t)\rangle = 0, \tag{2.48}$$

$$\langle \hat{g}_a^\dagger(t)\hat{\sigma}_b(t)\rangle = \langle \hat{g}_a^\dagger(t)\rangle\langle \hat{\sigma}_b(t)\rangle = 0. \tag{2.49}$$

We also notice that

$$\hat{\sigma}_a\hat{\sigma}_a = |b\rangle\langle a|b\rangle\langle a| = 0, \tag{2.50}$$

$$\hat{\sigma}_b\hat{\sigma}_a = |c\rangle\langle b|b\rangle\langle a| = |c\rangle\langle a| = \hat{\sigma}_c, \tag{2.51}$$

$$\hat{\sigma}_a^\dagger\hat{\sigma}_a = |a\rangle\langle b|b\rangle\langle a| = |a\rangle\langle a| = \hat{\eta}_a, \tag{2.52}$$

$$\hat{\sigma}_b^\dagger\hat{\sigma}_a = |b\rangle\langle c|b\rangle\langle a| = 0, \tag{2.53}$$

$$\hat{\sigma}_a\hat{\sigma}_b = |b\rangle\langle a|c\rangle\langle b| = 0, \tag{2.54}$$

$$\hat{\sigma}_b\hat{\sigma}_b = |c\rangle\langle b|c\rangle\langle b| = 0, \tag{2.55}$$

$$\hat{\sigma}_a^\dagger\hat{\sigma}_b = |a\rangle\langle b|c\rangle\langle b| = 0, \tag{2.56}$$

$$\hat{\sigma}_b^\dagger\hat{\sigma}_b = |b\rangle\langle c|c\rangle\langle b| = |b\rangle\langle b| = \hat{\eta}_b, \tag{2.57}$$

$$\hat{\sigma}_a\hat{\sigma}_a^\dagger = |b\rangle\langle a|a\rangle\langle b| = |b\rangle\langle b| = \hat{\eta}_b, \tag{2.58}$$

$$\hat{\sigma}_a\hat{\sigma}_b^\dagger = |b\rangle\langle a|b\rangle\langle c| = 0, \tag{2.59}$$

$$\hat{\sigma}_b\hat{\sigma}_a^\dagger = |c\rangle\langle b|a\rangle\langle b| = 0, \tag{2.60}$$

$$\hat{\sigma}_b\hat{\sigma}_b^\dagger = |c\rangle\langle b|b\rangle\langle c| = |c\rangle\langle c| = \hat{\eta}_c. \tag{2.61}$$

On account of Eqs.(2.42)-(2.61), we can put Eq.(2.41) as

$$\frac{d}{dt}\langle\hat{a}^2\rangle = -\kappa\langle\hat{a}^2\rangle + \gamma_c\Gamma\langle\hat{\sigma}_c\rangle - \frac{\lambda\gamma_c\Gamma}{\kappa}(2\langle\hat{\eta}_b\rangle + \langle\hat{\eta}_a\rangle + \langle\hat{\eta}_c\rangle) - \lambda(\langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{a}^\dagger\hat{a}\rangle). \tag{2.62}$$

On the basis of the completeness relation

$$\hat{\eta}_a + \hat{\eta}_b + \hat{\eta}_c = \hat{I},$$

we observe that

$$\langle\hat{\eta}_a\rangle + \langle\hat{\eta}_b\rangle + \langle\hat{\eta}_c\rangle = 1. \tag{2.63}$$

Substituting Eq.(2.63) into Eq.(2.62), we get

$$\frac{d}{dt}\langle\hat{a}^2\rangle = -\kappa\langle\hat{a}^2\rangle + \gamma_c\Gamma\langle\hat{\sigma}_c\rangle - \frac{\lambda\gamma_c\Gamma}{\kappa}(1 + \langle\hat{\eta}_b\rangle) - \lambda(\langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{a}^\dagger\hat{a}\rangle), \tag{2.64}$$

where $\gamma_c = \frac{4g^2}{k}$ is the stimulated emission decay constant.
In a similar manner, the decoupled forms of Eqs.(2.19), (2.20), (2.21), (2.27), (2.28), (2.29), (2.30), (2.31) and (2.32) can be expressed as

$$\frac{d}{dt}\langle\hat{a}^{\dagger 2}\rangle = -\kappa\langle\hat{a}^{\dagger 2}\rangle + \gamma_c\Gamma\langle\hat{\sigma}_c^\dagger\rangle - \frac{\lambda\gamma_c\Gamma}{\kappa}(1 + \langle\hat{\eta}_b\rangle) - \lambda(\langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{a}^\dagger\hat{a}\rangle), \tag{2.65}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle = -\kappa\langle\hat{a}^\dagger\hat{a}\rangle - \frac{\lambda\gamma_c\Gamma}{\kappa}(\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle - \langle\hat{\eta}_b\rangle) + \gamma_c\Gamma\langle\hat{\eta}_a\rangle - \lambda(\langle\hat{a}^2\rangle + \langle\hat{a}^{\dagger 2}\rangle), \tag{2.66}$$

$$\frac{d}{dt}\langle\hat{a}\hat{a}^\dagger\rangle = -\kappa\langle\hat{a}\hat{a}^\dagger\rangle - \frac{\lambda\gamma_c\Gamma}{\kappa}(\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle) + \gamma_c\Gamma(\langle\hat{\eta}_b\rangle + \langle\hat{\eta}_c\rangle) - \lambda(\langle\hat{a}^2\rangle + \langle\hat{a}^{\dagger 2}\rangle) + \kappa, \tag{2.67}$$

$$\frac{d\langle\hat{\sigma}_a\rangle}{dt} = -\gamma_c\Gamma\langle\hat{\sigma}_a\rangle + \gamma_c\frac{\lambda}{\kappa}\Gamma(\langle\sigma_b^\dagger - \sigma_a^\dagger\rangle) + \frac{\Omega}{2}\langle\hat{\sigma}_b^\dagger\rangle, \tag{2.68}$$

$$\frac{d\langle\hat{\sigma}_b\rangle}{dt} = -\frac{\gamma_c\Gamma}{2}\langle\hat{\sigma}_b\rangle + \gamma_c\Gamma\langle\hat{\sigma}_a\rangle - \gamma_c\frac{\lambda}{\kappa}\Gamma\langle\sigma_b^\dagger\rangle - \frac{\Omega}{2}\langle\hat{\sigma}_a^\dagger\rangle, \tag{2.69}$$

$$\frac{d\langle\hat{\sigma}_c\rangle}{dt} = -\frac{\gamma_c\Gamma}{2}\langle\hat{\sigma}_c\rangle + \frac{\lambda\gamma_c\Gamma}{\kappa}[\langle\hat{\eta}_c\rangle - \langle\hat{\eta}_b\rangle] + \frac{\Omega}{2}[\langle\hat{\eta}_c\rangle - \langle\hat{\eta}_a\rangle], \tag{2.70}$$

$$\frac{d\langle\hat{\eta}_a\rangle}{dt} = -\gamma_c\Gamma\langle\hat{\eta}_a\rangle + (\frac{\lambda\gamma_c\Gamma}{\kappa} + \frac{\Omega}{2})(\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle), \tag{2.71}$$

$$\frac{d\langle\hat{\eta}_b\rangle}{dt} = -\gamma_c\Gamma\langle\hat{\eta}_b\rangle + \gamma_c\Gamma\langle\hat{\eta}_a\rangle - \frac{\lambda\gamma_c\Gamma}{\kappa}(\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle), \tag{2.72}$$

$$\frac{d\langle\hat{\eta}_c\rangle}{dt} = \gamma_c\Gamma\langle\hat{\eta}_b\rangle - \frac{\Omega}{2}(\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle). \tag{2.73}$$

Employing Eq.(2.63) into Eq.(2.73),we have

$$\frac{d\langle\hat{\eta}_c\rangle}{dt} = \gamma_c\Gamma - \gamma_c\Gamma(\langle\hat{\eta}_c\rangle + \langle\hat{\eta}_a\rangle) - \frac{\Omega}{2}(\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle). \tag{2.74}$$

The steady state solutions of Eqs.(2.63)-(2.71) and Eq.(2.74), are expressible as

$$\langle\hat{a}^2\rangle = \frac{\gamma_c\Gamma}{\kappa}\langle\hat{\sigma}_c\rangle - \frac{\lambda\gamma_c\Gamma}{\kappa^2}(1 + \langle\hat{\eta}_b\rangle) - \frac{\lambda}{\kappa}(\langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}\hat{a}^\dagger\rangle), \tag{2.75}$$

$$\langle\hat{a}^{\dagger 2}\rangle = \frac{\gamma_c\Gamma}{\kappa}\langle\hat{\sigma}_c^\dagger\rangle - \frac{\lambda\gamma_c\Gamma}{\kappa^2}(1 + \langle\hat{\eta}_b\rangle) - \frac{\lambda}{\kappa}(\langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}\hat{a}^\dagger\rangle), \tag{2.76}$$

$$\langle\hat{a}^\dagger\hat{a}\rangle = -\frac{\lambda\gamma_c\Gamma}{\kappa^2}(\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle - \langle\hat{\eta}_b\rangle) + \frac{\gamma_c\Gamma}{\kappa}\langle\hat{\eta}_a\rangle - \frac{\lambda}{\kappa}(\langle\hat{a}^2\rangle + \langle\hat{a}^{\dagger 2}\rangle), \tag{2.77}$$

$$\langle\hat{a}\hat{a}^\dagger\rangle = -\frac{\lambda\gamma_c\Gamma}{\kappa^2}(\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle) + \frac{\gamma_c\Gamma}{\kappa}(\langle\hat{\eta}_b\rangle + \langle\hat{\eta}_c\rangle) - \frac{\lambda}{\kappa}(\langle\hat{a}^2\rangle + \langle\hat{a}^{\dagger 2}\rangle) + 1, \tag{2.78}$$

$$\langle\hat{\sigma}_a\rangle = -\frac{\lambda}{\kappa}\langle\hat{\sigma}_a^\dagger\rangle + (\frac{2\lambda\gamma_c\Gamma + \Omega\kappa}{2\kappa\gamma_c\Gamma})\langle\hat{\sigma}_b^\dagger\rangle, \tag{2.79}$$

$$\langle\hat{\sigma}_b\rangle = 2\langle\hat{\sigma}_a\rangle - \frac{2\lambda}{\kappa}\langle\hat{\sigma}_b^\dagger\rangle - \frac{\Omega}{\gamma_c\Gamma}\langle\hat{\sigma}_a^\dagger\rangle, \tag{2.80}$$

$$\langle\hat{\sigma}_c\rangle = \frac{2\lambda}{\kappa}[\langle\hat{\eta}_c\rangle - \langle\hat{\eta}_b\rangle] + \frac{\Omega}{\gamma_c\Gamma}[\langle\hat{\eta}_c\rangle - \langle\hat{\eta}_a\rangle], \tag{2.81}$$

$$\langle\hat{\eta}_a\rangle = (\frac{2\lambda\gamma_c\Gamma + \Omega\kappa}{2\kappa\gamma_c\Gamma})(\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle), \tag{2.82}$$

$$\langle\hat{\eta}_b\rangle = \langle\hat{\eta}_a\rangle - \frac{\lambda}{\kappa}(\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle), \tag{2.83}$$

$$\langle\hat{\eta}_c\rangle = 1 - \langle\hat{\eta}_a\rangle - \frac{\Omega}{2\gamma_c\Gamma}(\langle\hat{\sigma}_c^\dagger\rangle + \langle\hat{\sigma}_c\rangle). \tag{2.84}$$

Substituting Eq.(2.79) into Eq.(2.80), we have

$$\langle\hat{\sigma}_b\rangle = -(\frac{2\lambda\gamma_c\Gamma + \Omega\kappa}{\kappa\gamma_c\Gamma})\langle\hat{\sigma}_a^\dagger\rangle + (\frac{\Omega\kappa}{\kappa\gamma_c\Gamma})\langle\hat{\sigma}_b^\dagger\rangle. \quad (2.85)$$

Taking conjugate of Eqs.(2.79) and (2.85), we find that

$$\langle\hat{\sigma}_a^\dagger\rangle = -\frac{\lambda}{\kappa}\langle\hat{\sigma}_a\rangle + (\frac{2\lambda\gamma_c\Gamma + \Omega\kappa}{2\kappa\gamma_c\Gamma})\langle\hat{\sigma}_b\rangle, \quad (2.86)$$

$$\langle\hat{\sigma}_b^\dagger\rangle = -(\frac{2\lambda\gamma_c\Gamma + \Omega\kappa}{\kappa\gamma_c\Gamma})\langle\hat{\sigma}_a\rangle + (\frac{\Omega\kappa}{\kappa\gamma_c\Gamma})\langle\hat{\sigma}_b\rangle. \quad (2.87)$$

Subtracting Eq.(2.86) from Eq.(2.79), we obtain

$$\langle\hat{\sigma}_a\rangle - \langle\hat{\sigma}_a^\dagger\rangle = (\frac{2\lambda\gamma_c\Gamma + \Omega\kappa}{(\kappa+\lambda)2\gamma_c\Gamma)}(\langle\hat{\sigma}_b\rangle - \langle\hat{\sigma}_b^\dagger\rangle), \quad (2.88)$$

and also subtracting Eq.(2.87) from Eq.(2.85), we have

$$\langle\hat{\sigma}_b\rangle - \langle\hat{\sigma}_b^\dagger\rangle = (\frac{2\lambda\gamma_c\Gamma + \Omega\kappa}{\kappa\lambda\gamma_c\Gamma + \Omega\kappa})(\langle\hat{\sigma}_a\rangle - \langle\hat{\sigma}_a^\dagger\rangle). \quad (2.89)$$

Adding Eq.(2.79) and Eq.(2.86), we obtain

$$\langle\hat{\sigma}_a\rangle + \langle\hat{\sigma}_a^\dagger\rangle = (\frac{2\lambda\gamma_c\Gamma + \Omega\kappa}{(\kappa+\lambda)2\gamma_c\Gamma)}(\langle\hat{\sigma}_b\rangle + \langle\hat{\sigma}_b^\dagger\rangle), \quad (2.90)$$

and also adding Eq.(2.85) and Eq.(2.87), we obtain

$$\langle\hat{\sigma}_b\rangle + \langle\hat{\sigma}_b^\dagger\rangle = (\frac{2\lambda\gamma_c\Gamma + \Omega\kappa}{\kappa\lambda\gamma_c\Gamma + \Omega\kappa})(\langle\hat{\sigma}_a\rangle + \langle\hat{\sigma}_a^\dagger\rangle). \quad (2.91)$$

Substituting Eq.(2.89) into Eq.(2.88), we have

$$\langle\hat{\sigma}_a\rangle - \langle\hat{\sigma}_a^\dagger\rangle = 0, \quad (2.92)$$

and substituting Eq.(2.91) into Eq.(2.90),we obtain

$$\langle\hat{\sigma}_a\rangle + \langle\hat{\sigma}_a^\dagger\rangle = 0. \quad (2.93)$$

Using Eq.(92) into Eq.(2.89), we get

$$\langle\hat{\sigma}_b\rangle - \langle\hat{\sigma}_b^\dagger\rangle = 0, \quad (2.94)$$

and also combining Eq.(2.93) with Eq.(2.91), we have

$$\langle\hat{\sigma}_b\rangle + \langle\hat{\sigma}_b^\dagger\rangle = 0. \quad (2.95)$$

In view of Eqs.(2.92)-(2.95), we find that

$$\langle\hat{\sigma}_a\rangle = 0, \tag{2.96}$$

$$\langle\hat{\sigma}_b\rangle = 0. \tag{2.97}$$

We notice from Eq.(2.81) that $\langle\hat{\sigma}_c\rangle = \langle\hat{\sigma}_c^\dagger\rangle$ and hence Eq.(2.82) results in

$$\langle\hat{\sigma}_c\rangle = (\frac{\kappa\gamma_c\Gamma}{2\lambda\gamma_c\Gamma + \Omega\kappa})\langle\hat{\eta}_a\rangle. \tag{2.98}$$

Substituting Eq.(2.98) into Eq.(2.83) and Eq.(2.84), we get

$$\langle\hat{\eta}_b\rangle = (\frac{\Omega\kappa}{2\lambda\gamma_c\Gamma + \Omega\kappa})\langle\hat{\eta}_a\rangle, \tag{2.99}$$

$$\langle\hat{\eta}_c\rangle = 1 - (\frac{2\Omega\kappa + 2\lambda\gamma_c\Gamma}{2\lambda\gamma_c\Gamma + \Omega\kappa})\langle\hat{\eta}_a\rangle. \tag{2.100}$$

Moreover, with the use of Eqs.(2.98) and (2.99) in Eqs.(2.75)-(2.77), we obtain

$$\langle\hat{a}^2\rangle = -\frac{\lambda\gamma_c\Gamma}{\kappa^2} - (\frac{\Omega\lambda\gamma_c\Gamma - \kappa\gamma_c^2\Gamma^2}{\Omega\kappa^2 + 2\kappa\lambda\gamma_c\Gamma})\langle\hat{\eta}_a\rangle - \frac{\lambda}{\kappa}(\langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}\hat{a}^\dagger\rangle), \tag{2.101}$$

$$\langle\hat{a}^{\dagger 2}\rangle = -\frac{\lambda\gamma_c\Gamma}{\kappa^2} - (\frac{\Omega\lambda\gamma_c\Gamma - \kappa\gamma_c^2\Gamma^2}{\Omega\kappa^2 + 2\kappa\lambda\gamma_c\Gamma})\langle\hat{\eta}_a\rangle - \frac{\lambda}{\kappa}(\langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}\hat{a}^\dagger\rangle), \tag{2.102}$$

$$\langle\hat{a}^\dagger\hat{a}\rangle = (\frac{\Omega\lambda\gamma_c\Gamma + \Omega\kappa\gamma_c\Gamma}{\Omega\kappa^2 + 2\kappa\lambda\gamma_c\Gamma})\langle\hat{\eta}_a\rangle - \frac{2\lambda}{\kappa}\langle\hat{a}^2\rangle, \tag{2.103}$$

and also substituting Eq.(2.98) and Eq.(2.100) into Eq.(2.78), we obtain

$$\langle\hat{a}\hat{a}^\dagger\rangle = \frac{\gamma_c\Gamma}{\kappa} - (\frac{\Omega\kappa\gamma_c\Gamma + 4\lambda\gamma_c^2\Gamma^2}{\Omega\kappa^2 + 2\kappa\lambda\gamma_c\Gamma})\langle\hat{\eta}_a\rangle - \frac{2\lambda}{\kappa}\langle\hat{a}^2\rangle + 1. \tag{2.104}$$

Now inserting Eq.(2.83) into Eq.(2.63), we have

$$\langle\hat{\eta}_c\rangle = 1 - 2\langle\hat{\eta}_a\rangle + \frac{2\lambda}{\kappa}\langle\hat{\sigma}_c\rangle. \tag{2.105}$$

Upon substituting Eqs.(2.83) and (2.105) into Eq.(2.81), we find

$$\langle\hat{\sigma}_c\rangle = \frac{2\lambda\gamma_c\Gamma + \Omega\kappa}{\kappa\gamma_c\Gamma} - (\frac{6\lambda\gamma_c\Gamma + 3\Omega\kappa}{\kappa\gamma_c\Gamma})\langle\hat{\eta}_a\rangle + (\frac{8\lambda^2\gamma_c\Gamma + 2\Omega\kappa\lambda}{\kappa^2\gamma_c\Gamma})\langle\hat{\sigma}_c\rangle, \tag{2.106}$$

which with the aid of Eq.(2.82) becomes

$$\langle\hat{\sigma}_c\rangle = \frac{2\kappa\lambda\gamma_c^2\Gamma^2 + \Omega\kappa^2\gamma_c\Gamma}{(\Omega\kappa + 2\lambda\gamma_c\Gamma)(3\Omega\kappa + 2\lambda\gamma_c\Gamma) + \kappa\gamma_c\Gamma(\kappa\gamma_c\Gamma + 2\Omega\lambda)}. \tag{2.107}$$

Putting Eq.(2.107) into Eq.(2.82), we have

$$\langle\hat{\eta}_a\rangle = \frac{4\lambda^2\gamma_c^2\Gamma^2 + 4\Omega\kappa\lambda\gamma_c\Gamma + \Omega^2\kappa^2}{(\Omega\kappa + 2\lambda\gamma_c\Gamma)(3\Omega\kappa + 2\lambda\gamma_c\Gamma) + \kappa\gamma_c\Gamma(\kappa\gamma_c\Gamma + 2\Omega\lambda)}. \tag{2.108}$$

Substituting Eqs.(2.107) and (2.108) into Eq.(2.105), we have

$$\langle\hat{\eta}_c\rangle = \frac{\kappa^2\gamma_c^2\Gamma^2 + 4\Omega\kappa\lambda\gamma_c\Gamma + \Omega^2\kappa^2}{(\Omega\kappa + 2\lambda\gamma_c\Gamma)(3\Omega\kappa + 2\lambda\gamma_c\Gamma) + \kappa\gamma_c\Gamma(\kappa\gamma_c\Gamma + 2\Omega\lambda)}. \tag{2.109}$$

In addition, using Eqs.(2.107) and (2.108) and into Eq.(2.83), we obtain

$$\langle\hat{\eta}_b\rangle = \frac{2\Omega\kappa\lambda\gamma_c\Gamma + \Omega^2\kappa^2}{(\Omega\kappa + 2\lambda\gamma_c\Gamma)(3\Omega\kappa + 2\lambda\gamma_c\Gamma) + \kappa\gamma_c\Gamma(\kappa\gamma_c\Gamma + 2\Omega\lambda)}. \tag{2.110}$$

Now adding Eqs.(2.103) and (2.104), we have

$$\langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}\hat{a}^\dagger\rangle = \frac{\gamma_c\Gamma}{\kappa} + (\frac{\Omega\lambda\gamma_c\Gamma - 4\lambda\gamma_c^2\Gamma^2}{\Omega\kappa^2 + 2\kappa\lambda\gamma_c\Gamma})\langle\hat{\eta}_a\rangle - \frac{4\lambda}{\kappa}\langle\hat{a}^2\rangle + 1. \tag{2.111}$$

Substituting Eq.(2.111) into Eq.(2.101), we obtain

$$\begin{aligned}\langle\hat{a}^2\rangle = &-\frac{\lambda\gamma_c\Gamma}{\kappa^2} - (\frac{\Omega\lambda\gamma_c\Gamma - \kappa\gamma_c^2\Gamma^2}{\Omega\kappa^2 + 2\kappa\lambda\gamma_c\Gamma})\langle\hat{\eta}_a\rangle \\ &- \frac{\lambda}{\kappa}[\frac{\gamma_c\Gamma}{\kappa} + (\frac{\Omega\lambda\gamma_c\Gamma - 4\lambda\gamma_c^2\Gamma^2}{\Omega\kappa^2 + 2\kappa\lambda\gamma_c\Gamma})\langle\hat{\eta}_a\rangle - \frac{4\lambda}{\kappa}\langle\hat{a}^2\rangle + 1],\end{aligned} \tag{2.112}$$

from which follows

$$\langle\hat{a}^2\rangle = \frac{-2\lambda\gamma_c\Gamma - \kappa\lambda}{\kappa^2 - 4\lambda^2} + [\frac{\gamma_c\Gamma(\gamma_c\Gamma(\kappa^2 + 4\lambda^2) - \Omega\lambda(\kappa + \lambda))}{(\kappa^2 - 4\lambda^2)(\Omega\kappa + 2\lambda\gamma_c\Gamma)}]\langle\hat{\eta}_a\rangle. \tag{2.113}$$

Substituting Eq.(2.113) into Eqs.(2.103) and (2.104), we obtain

$$\begin{aligned}\langle\hat{a}^\dagger\hat{a}\rangle = &\frac{4\lambda^2\gamma_c\Gamma + 2\kappa\lambda^2}{\kappa(\kappa^2 - 4\lambda^2)} \\ &+ [\frac{\Omega\gamma_c\Gamma(\kappa + \lambda)(\kappa^2 - 4\lambda^2) - 2\lambda\gamma_c\Gamma(\gamma_c\Gamma(\kappa^2 + 4\lambda^2) - \Omega\lambda(\kappa + \lambda))}{\kappa(\kappa^2 - 4\lambda^2)(\Omega\kappa + 2\lambda\gamma_c\Gamma)}]\langle\hat{\eta}_a\rangle,\end{aligned} \tag{2.114}$$

$$\begin{aligned}\langle\hat{a}\hat{a}^\dagger\rangle = &\frac{\kappa\gamma_c\Gamma + \kappa^2 - 2\lambda^2}{\kappa^2 - 4\lambda^2} \\ &- [\frac{\gamma_c\Gamma(\kappa^2 - 4\lambda^2)(\Omega\kappa + 4\lambda\gamma_c\Gamma) + 2\lambda\gamma_c\Gamma(\gamma_c\Gamma(\kappa^2 + 4\lambda^2) - \Omega\lambda(\kappa + \lambda))}{\kappa(\kappa^2 - 4\lambda^2)(\Omega\kappa + 2\lambda\gamma_c\Gamma)}]\langle\hat{\eta}_a\rangle.\end{aligned} \tag{2.115}$$

2.2 Correlation properties of cavity mode noise operator

In this section we seek to determine the correlation properties of the cavity mode noise operator.

Applying the mathematical relation,

$$\frac{d}{dt}\langle\hat{a}^2\rangle = \langle\hat{a}\frac{d\hat{a}}{dt}\rangle + \langle\frac{d\hat{a}}{dt}\hat{a}\rangle, \tag{2.122}$$

along with Eq.(2.33), we see that

$$\frac{d}{dt}\langle\hat{a}^2\rangle = -\kappa\langle\hat{a}\rangle - g(\langle\hat{a}\hat{\sigma}_a\rangle + \langle\hat{a}\hat{\sigma}_b\rangle + \langle\hat{\sigma}_a\hat{a}\rangle + \langle\hat{\sigma}_b\hat{a}\rangle) + \lambda(\langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}\hat{a}^\dagger\rangle) + \langle\hat{a}\hat{g}_a(t)\rangle + \langle\hat{g}_a(t)\hat{a}\rangle. \tag{2.123}$$

Comparison of Eqs.(2.123) and (2.18) yields

$$\langle\hat{a}(t)\hat{g}_a(t)\rangle + \langle\hat{g}_a(t)\hat{a}(t)\rangle = 0. \tag{2.124}$$

Multiplying the solution of Eq.(2.33) from the right by $\hat{g}_a(t)$ and taking the expectation value of the resulting expression, we obtain

$$\begin{aligned}\langle\hat{a}(t)\hat{g}_a(t)\rangle = \langle\hat{a}(0)\hat{g}_a(t)\rangle e^{-\frac{\kappa}{2}t} - g\int_0^t e^{-\frac{\kappa}{2}(t-t')}[\langle\hat{\sigma}_a(t')\hat{g}_a(t)\rangle + \langle\hat{\sigma}_b(t')\hat{g}_a(t)\rangle]dt'\\ + \lambda\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{a}^\dagger(t')\hat{g}_a(t)\rangle dt'\\ + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a(t')\hat{g}_a(t)\rangle dt'.\end{aligned} \tag{2.125}$$

Taking into account the notion that a noise operator at time t has no effect on cavity mode and atomic operators at earlier times and with the use of Eq.(2.38), we can write

$$\langle\hat{a}(0)\hat{g}_a(t)\rangle = \langle\hat{a}(0)\rangle\langle\hat{g}_a(t)\rangle = 0, \tag{2.125a}$$

$$\langle\hat{\sigma}_a(t')\hat{g}_a(t)\rangle = \langle\hat{\sigma}_a(t')\rangle\langle\hat{g}_a(t)\rangle = 0, \tag{2.125b}$$

$$\langle\hat{\sigma}_b(t')\hat{g}_a(t)\rangle = \langle\hat{\sigma}_b(t')\rangle\langle\hat{g}_a(t)\rangle = 0, \tag{2.125c}$$

$$\langle\hat{a}^\dagger(t')\hat{g}_a(t)\rangle = \langle\hat{a}^\dagger(t')\rangle\langle\hat{g}_a(t)\rangle = 0, \tag{2.125d}$$

so that Eq.(2.125) can be written as

$$\langle\hat{a}(t)\hat{g}_a(t)\rangle = \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a(t')\hat{g}_a(t)\rangle dt'. \tag{2.126}$$

In a similar pattern, we observe that

$$\langle\hat{g}_a(t)\hat{a}(t)\rangle = \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a(t)\hat{g}_a(t')\rangle dt'. \tag{2.127}$$

Employing Eqs.(2.126) and (2.127) into Eq.(2.124), we find

$$\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a(t')\hat{g}_a(t)\rangle dt' + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a(t)\hat{g}_a(t')\rangle dt' = 0, \tag{2.128}$$

and assuming $\langle\hat{g}_a(t')\hat{g}_a(t)\rangle = \langle\hat{g}_a(t)\hat{g}_a(t')\rangle$, we can write Eq.(2.128) as

$$\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a(t)\hat{g}_a(t')\rangle dt' = 0, \tag{2.129}$$

from which follows

$$\langle\hat{g}_a(t)\hat{g}_a(t')\rangle = 0. \tag{2.130}$$

Using the mathematical relation,

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle = \langle\frac{d\hat{a}^\dagger}{dt}\hat{a}\rangle + \langle\hat{a}^\dagger\frac{d\hat{a}}{dt}\rangle, \tag{2.131}$$

along with Eqs.(2.33) and its adjoint, we obtain

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle = & -\kappa\langle\hat{a}^\dagger\hat{a}\rangle - g(\langle\hat{\sigma}_a^\dagger\hat{a}\rangle + \langle\hat{\sigma}_b^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a\rangle + \langle\hat{a}^\dagger\hat{\sigma}_b\rangle) \\ & + \lambda(\langle\hat{a}^2\rangle + \langle\hat{a}^{\dagger 2}\rangle) + \langle\hat{g}_a^\dagger(t)\hat{a}\rangle + \langle\hat{a}^\dagger\hat{g}_a(t)\rangle.\end{aligned} \tag{2.132}$$

Comparison of Eqs.(2.132) and (2.20) yields

$$\langle\hat{g}_a^\dagger(t)\hat{a}(t)\rangle + \langle\hat{a}^\dagger(t)\hat{g}_a(t)\rangle = 0. \tag{2.133}$$

Multiplying the solution of Eq.(2.38) from right by $\hat{g}_a(t)$ and taking the expectation value of the resulting expression, we get

$$\begin{aligned}\langle\hat{a}^\dagger(t)\hat{g}_a(t)\rangle = & \langle\hat{a}^\dagger(0)\hat{g}_a(t)\rangle e^{-\frac{\kappa}{2}t} - g\int_0^t e^{-\frac{\kappa}{2}(t-t')}[\langle\hat{\sigma}_a^\dagger(t')\hat{g}_a(t)\rangle + \langle\hat{\sigma}_b^\dagger(t')\hat{g}_a(t)\rangle]dt' \\ & + \lambda\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{a}(t')\hat{g}_a(t)\rangle dt' \\ & + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a^\dagger(t')\hat{g}_a(t)\rangle dt'.\end{aligned} \tag{2.134}$$

Taking into account the notion that a noise operator at a time t has no effect on cavity mode and atomic operators at earlier times along with Eq.(2.38), we can write

$$\langle\hat{a}^\dagger(0)\hat{g}_a(t)\rangle = \langle\hat{a}^\dagger(0)\rangle\langle\hat{g}_a(t)\rangle = 0, \tag{2.134a}$$

$$\langle\hat{\sigma}_a^\dagger(t')\hat{g}_a(t)\rangle = \langle\hat{\sigma}_a^\dagger(t')\rangle\langle\hat{g}_a(t)\rangle = 0, \tag{2.134b}$$

$$\langle\hat{\sigma}_b^\dagger(t')\hat{g}_a(t)\rangle = \langle\hat{\sigma}_b^\dagger(t')\rangle\langle\hat{g}_a(t)\rangle = 0, \tag{2.134c}$$

$$\langle\hat{a}(t')\hat{g}_a(t)\rangle = \langle\hat{a}(t')\rangle\langle\hat{g}_a(t)\rangle = 0, \tag{2.134d}$$

so that Eq.(2.134) reduces to

$$\langle\hat{a}^\dagger(t)\hat{g}_a(t)\rangle = \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a^\dagger(t')\hat{g}_a(t)\rangle dt'. \tag{2.135}$$

Multiplying the solution of Eq.(2.33) from the left by $\hat{g}_a^\dagger(t)$ and taking the expectation value of the resulting expression, we get

$$\begin{aligned}\langle\hat{g}_a^\dagger(t)\hat{a}(t)\rangle = \langle\hat{g}_a^\dagger(t)\hat{a}(0)\rangle e^{-\frac{\kappa}{2}t} - g\int_0^t e^{-\frac{\kappa}{2}(t-t')}[\langle\hat{g}_a^\dagger(t)\hat{\sigma}_a(t')\rangle + \langle\hat{g}_a^\dagger(t)\hat{\sigma}_b(t')\rangle]dt'\\ + \lambda\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a^\dagger(t)\hat{a}^\dagger(t')\rangle dt'\\ + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a^\dagger(t)\hat{g}_a(t')\rangle dt'.\end{aligned} \tag{2.136}$$

Taking into account the notion that a noise operator at time t should not affect cavity mode and atomic operators at an earlier times along with Eq.(2.39), we can write

$$\langle\hat{g}_a^\dagger(t)\hat{a}(0)\rangle = \langle\hat{g}_a^\dagger(t)\rangle\langle\hat{a}(0)\rangle = 0, \tag{2.136a}$$

$$\langle\hat{g}_a^\dagger(t)\hat{\sigma}_a(t')\rangle = \langle\hat{g}_a^\dagger(t)\rangle\langle\hat{\sigma}_a(t')\rangle = 0, \tag{2.136b}$$

$$\langle\hat{g}_a^\dagger(t)\hat{\sigma}_b(t')\rangle = \langle\hat{g}_a^\dagger(t)\rangle\langle\hat{\sigma}_b(t')\rangle = 0, \tag{2.136c}$$

$$\langle\hat{g}_a^\dagger(t)\hat{a}^\dagger(t')\rangle = \langle\hat{g}_a^\dagger(t)\rangle\langle\hat{a}^\dagger(t')\rangle = 0, \tag{2.136d}$$

Thus Eq.(2.136) becomes

$$\langle\hat{g}_a^\dagger(t)\hat{a}(t)\rangle = \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a^\dagger(t)\hat{g}_a(t')\rangle dt'. \tag{2.137}$$

Putting Eqs.(2.135) and (2.137) into (2.133) gives

$$\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a^\dagger(t')\hat{g}_a(t)\rangle dt' + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a^\dagger(t)\hat{g}_a(t')\rangle dt' = 0, \tag{2.138}$$

so that assuming $\langle\hat{g}_a^\dagger(t')\hat{g}_a(t)\rangle = \langle\hat{g}_a^\dagger(t)\hat{g}_a(t')\rangle$, we obtain

$$2\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\hat{g}_a^\dagger(t')\hat{g}_a(t)\rangle dt' = 0, \tag{2.139}$$

or

$$\langle \hat{g}_a^\dagger(t')\hat{g}_a(t)\rangle = 0. \tag{2.140}$$

Following a similar procedure, we readily get

$$\langle \hat{g}_a(t)\hat{g}_a^\dagger(t')\rangle = \kappa\delta(t-t'). \tag{2.141}$$

Chapter 3

Photon Statistics

In this chapter, we wish to study the statistical properties of the cavity mode by analyzing the mean and variance of the cavity photon number and the power spectrum of the cavity light emitted by coherently driven degenerate three-level atom in a cavity containing degenerate parametric amplifier and coupled to a vacuum reservoir via a port mirror.

3.1 The mean photon number

Substituting Eq.(2.108) into Eq.(2.114), the steady state mean photon number of the cavity light is expressible as

$$\begin{aligned}\bar{n} = \langle \hat{a}^{\dagger}\hat{a}\rangle = &\frac{4\lambda^2\gamma_c\Gamma + 2\kappa\lambda^2}{\kappa(\kappa^2 - 4\lambda^2)}\\ &+ \frac{\Omega\gamma_c\Gamma(\kappa+\lambda)(\kappa^2-4\lambda^2)[4\lambda\gamma_c\Gamma(\lambda\gamma_c\Gamma+\Omega\kappa)+\Omega^2+\kappa^2]}{\kappa(\kappa^2-4\lambda^2)(\Omega\kappa+2\lambda\gamma_c\Gamma)[(\Omega\kappa+2\lambda\gamma_c\Gamma)(3\Omega\kappa+2\lambda\gamma_c\Gamma)+\kappa\gamma_c\Gamma(\kappa\gamma_c\Gamma+2\Omega\lambda)]}\\ &- \frac{2\lambda\gamma_c\Gamma[\gamma_c\Gamma(\kappa^2+4\lambda^2)-\Omega\lambda(\kappa+\lambda)][4\lambda\gamma_c\Gamma(\lambda\gamma_c\Gamma+\Omega\kappa)+\Omega^2+\kappa^2]}{\kappa(\kappa^2-4\lambda^2)(\Omega\kappa+2\lambda\gamma_c\Gamma)[(\Omega\kappa+2\lambda\gamma_c\Gamma)(3\Omega\kappa+2\lambda\gamma_c\Gamma)+\kappa\gamma_c\Gamma(\kappa\gamma_c\Gamma+2\Omega\lambda)]}.\end{aligned} \tag{3.1}$$

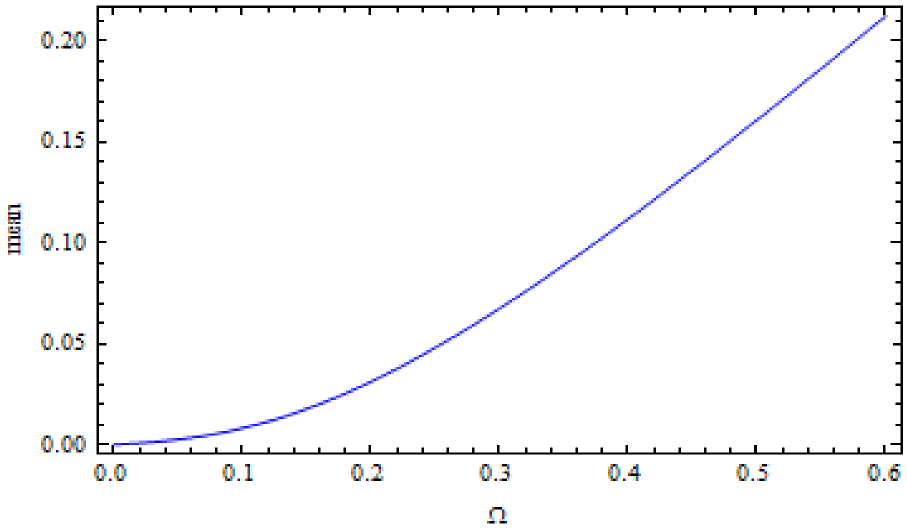

Figure 3.1: Plot of the mean of photon number[Eq.(3.1)] versus Ω for $\kappa = 0.8$, $\gamma_c = 0.5$, $\lambda = 0.1$.

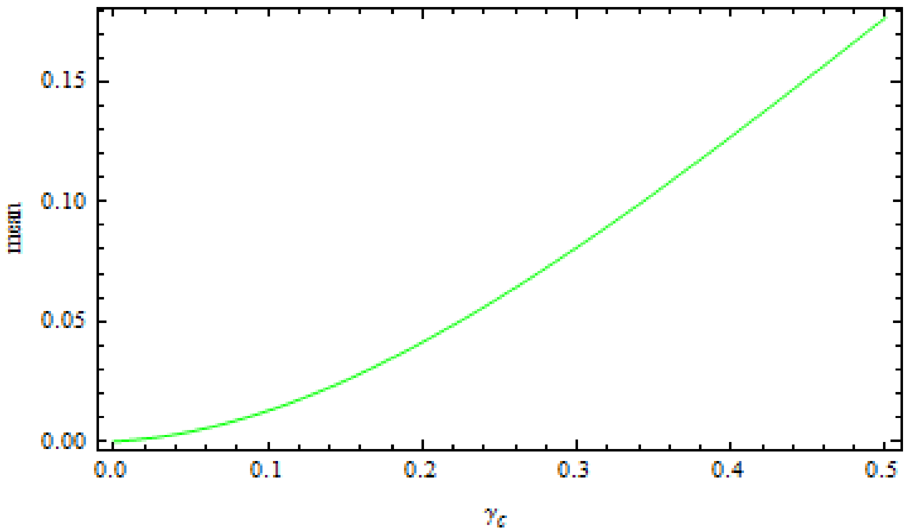

Figure 3.2: Plot of the mean of photon number[Eq.(3.1)] versus γ_c for $\kappa = 0.8$, $\Omega = 0.6$, $\lambda = 0.1$.

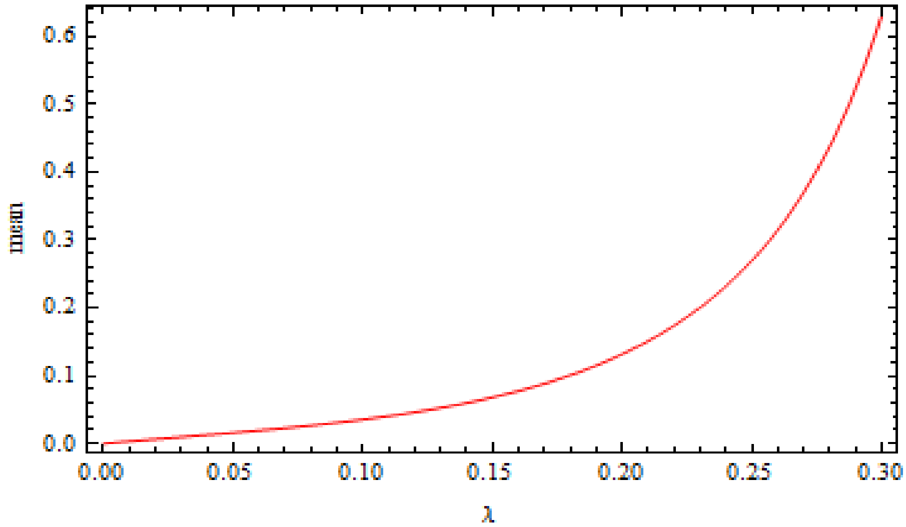

Figure 3.3: Plot of the mean of photon number[Eq.(3.1)] versus λ for $\kappa = 0.8$, $\Omega = 0.6$, $\gamma_c = 0.5$.

The plots in Figures 3.1, 3.2 and 3.3 show that, the amplitude of the driving coherent light(Ω), stimulated emission decay constant(γ_c), and the parametric interaction parameter(λ) have the effect of increasing the mean photon number.

We next wish to examine some special cases. First we consider the case in which the parametric amplifier is absent($\lambda = 0$) in Eq.(3.1). The steady state cavity mean photon number then reduces to

$$\bar{n} = \frac{\gamma_c}{\kappa}\left(\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2}\right). \tag{3.2}$$

This represents the steady-state mean photon number of cavity mode produced by coherently driven degenerate three-level atom available in a cavity coupled to vacuum reservoir.

Furthermore, we consider the case in which the driving coherent light is absent ($\Omega = 0$). The steady-state mean photon number described by Eq.(3.1) goes over into

$$\bar{n} = \frac{2\lambda^2}{\kappa^2 - 4\lambda^2}. \tag{3.3}$$

This represent the steady-state mean photon number for degenerate parametric amplifier.

3.2 Variance of the photon number

The variance of the photon number for the cavity mode is expressible as

$$\begin{aligned}(\Delta n)^2 &= \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 \\ &= \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2. \end{aligned} \tag{3.4}$$

Taking the expectation value of Eq.(2.37) and with the use of Eqs.(2.96), (2.97) and (2.38), we see that

$$\langle \hat{a} \rangle = 0. \tag{3.5}$$

In view of Eqs.(2.33) and (3.5) we observe that $\hat{a}$ is a Gaussian variable with zero mean. Therefore using this fact, we can write[1]

$$\langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle = \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}^\dagger \hat{a}^\dagger \rangle \langle \hat{a} \hat{a} \rangle + \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a} \hat{a}^\dagger \rangle. \tag{3.6}$$

Substituting Eq.(3.6) into Eq.(3.4), we get

$$(\Delta n)^2 = \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a} \hat{a}^\dagger \rangle + \langle \hat{a}^{\dagger 2} \rangle \langle \hat{a}^2 \rangle. \tag{3.7}$$

On the basis of Eqs.(3.1), (2.113) and (2.115), variance of photon number, we obtain

$$\begin{aligned}(\Delta n)^2 = \bar{n}[&\frac{\kappa\gamma_c\Gamma + \kappa^2 - 2\lambda^2}{\kappa^2 - 4\lambda^2} \\ &- (\frac{\gamma_c\Gamma(\kappa^2 - 4\lambda^2)(\Omega\kappa + 4\lambda\gamma_c\Gamma) + 2\lambda\gamma_c\Gamma(\gamma_c\Gamma(\kappa^2 + 4\lambda^2) - \Omega\lambda(\kappa + \lambda))}{\kappa(\kappa^2 - 4\lambda^2)(\Omega\kappa + 2\lambda\gamma_c\Gamma)})\langle \hat{\eta}_a \rangle] \\ &+ [\frac{-2\lambda\gamma_c\Gamma - \kappa\lambda}{\kappa^2 - 4\lambda^2} + (\frac{\gamma_c\Gamma(\gamma_c\Gamma(\kappa^2 + 4\lambda^2) - \Omega\lambda(\kappa + \lambda))}{(\kappa^2 - 4\lambda^2)(\Omega\kappa + 2\lambda\gamma_c\Gamma)})\langle \hat{\eta}_a \rangle]^2, \end{aligned} \tag{3.8}$$

where $\langle \hat{\eta}_a \rangle$ is given by Eq.(2.108).

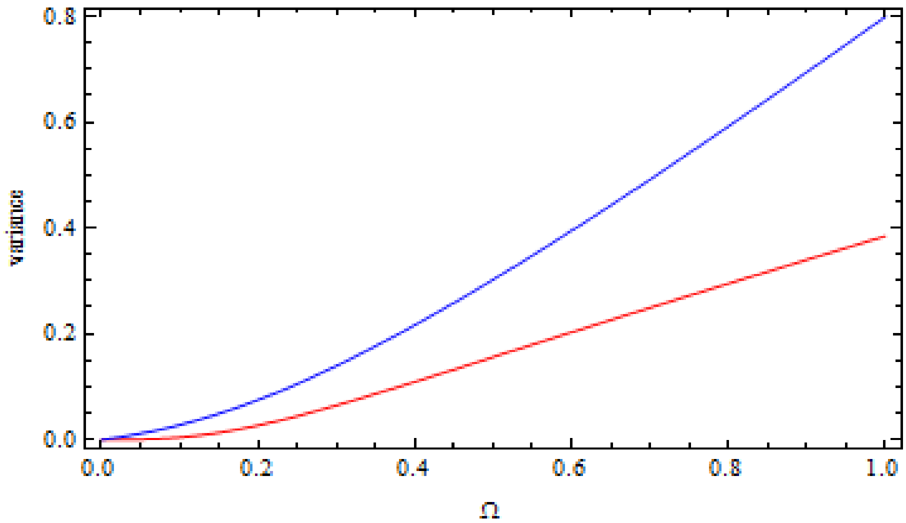

Figure 3.4: Plots of the variance$(\Delta n)^2$ of photon number[Eq.(3.8)] at steady state versus Ω for $\kappa = 0.8$, $\gamma_c = 0.5$, $\lambda = 0$(red) and $\lambda = 0.2$, (blue).

From Figure 3.4 we see that, like the mean of photon number, the variance of photon number enhances as the amplitude of the driving coherent light(Ω) and the parametric interaction(λ) increase.

In the absence of the parametric amplifier ($\lambda = 0$), the variance of the photon number reduces to

$$(\Delta n)^2 = (\frac{\gamma_c}{\kappa})^2(\frac{2\Omega^2}{\gamma_c^2 + 3\Omega^2})(\frac{2\Omega^2 + \gamma_c^2}{\gamma_c^2 + 3\Omega^2}) + (\frac{\Omega}{\kappa})^2(\frac{\gamma_c^2}{\gamma_c^2 + 3\Omega^2})^2. \tag{3.10}$$

On account of Eq.(3.2), we easily establish that

$$(\Delta n)^2 = \bar{n}^2(1 + \frac{3\gamma_c^2}{4\Omega^2}), \tag{3.11}$$

where $\bar{n}$ is given by Eq.(3.2).

3.3 The power spectrum

In this section we want to obtain mean photon number of the cavity light in a given frequency interval, employing the power spectrum of a single mode cavity light. The power spectrum of a single mode light with central frequency ω_o is defined as[3]

$$P(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\langle\hat{a}^{\dagger}(t)\hat{a}(t+\tau)\rangle_{ss}e^{i(\omega-\omega_o)\tau}d\tau. \tag{3.12}$$

Upon integrating both sides of Eq.(3.12) over ω, we obtain

$$\int_{-\infty}^{\infty}P(\omega)d\omega = \int_{-\infty}^{\infty}\langle\hat{a}^{\dagger}(t)\hat{a}(t+\tau)\rangle_{ss}e^{-i\omega_o\tau}d\tau x\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega\tau}d\omega, \tag{3.13}$$

using the fact that

$$\delta(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega\tau}d\omega, \tag{3.14}$$

we have

$$\int_{-\infty}^{\infty}P(\omega)d\omega = \int_{-\infty}^{\infty}\langle\hat{a}^{\dagger}(t)\hat{a}(t+\tau)\rangle_{ss}e^{-i\omega_o\tau}\delta(\tau)d\tau. \tag{3.15}$$

In view of the relation

$$\int_{-\infty}^{\infty}f(x)\delta(x)dx = f(x)|_{x=0}. \tag{3.16}$$

Eq.(3.15) reduces

$$\int_{-\infty}^{\infty}P(\omega)d\omega = \bar{n}, \tag{3.17}$$

with $\bar{n} = \langle\hat{a}^{\dagger}\hat{a}\rangle$ being the steady-state mean photon number. From this relation, we observe that $P(\omega)d\omega$ is steady-state mean photon number in the frequency interval between ω and $\omega + d\omega$[6].

We realize that the spectrum of the mean photon number in the interval between $\omega' = -\beta$ and $\omega' = \beta$ is expressible as

$$\bar{n}_{\pm\beta} = \int_{-\beta}^{\beta}P(\omega')d\omega', \tag{3.18}$$

in which $\omega' = \omega - \omega_o$.

For convenience, Eq.(3.12) can be rewritten as

$$P(\omega) = \frac{1}{2\pi}\int_{-\infty}^{0} \langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss} e^{i(\omega-\omega_o)\tau} d\tau + \frac{1}{2\pi}\int_{0}^{\infty} \langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss} e^{i(\omega-\omega_o)\tau} d\tau, \qquad (3.19)$$

by replacing τ by $-\tau$ and then t by $t+\tau$ in the first integral, one finds

$$P(\omega) = \frac{1}{2\pi}\int_{0}^{\infty} \langle \hat{a}^\dagger(t+\tau)\hat{a}(t)\rangle_{ss} e^{-i(\omega-\omega_o)\tau} d\tau + \frac{1}{2\pi}\int_{0}^{\infty} \langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss} e^{i(\omega-\omega_o)\tau} d\tau. \qquad (3.20)$$

From Eq.(3.20), we observe that one integral is the complex conjugate of other. Thus the power spectrum can be rewritten as

$$P(\omega) = \frac{1}{\pi} Re \int_{0}^{\infty} \langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss} e^{i(\omega-\omega_o)\tau} d\tau, \qquad (3.21)$$

in which 'Re' denotes the real part.

Now we proceed to calculate the two time correlation function that appears in Eq.(3.21) for the cavity light. Taking the complex conjugate of Eq.(2.33), we have

$$\frac{d}{dt}\langle \hat{a}^\dagger\rangle = -\frac{\kappa}{2}\langle \hat{a}^\dagger(t)\rangle - g(\langle \hat{\sigma}_a^\dagger(t)\rangle + \langle \hat{\sigma}_b^\dagger(t)\rangle) - \lambda\langle \hat{a}(t)\rangle. \qquad (3.22)$$

Adding Eq.(2.33) and Eq.(3.22), we obtain

$$\begin{aligned}\frac{d}{dt}(\langle \hat{a}\rangle + \langle \hat{a}^\dagger\rangle) &= -\frac{\kappa}{2}(\langle \hat{a}(t)\rangle + \langle \hat{a}^\dagger(t)\rangle) - g(\langle \hat{\sigma}_a(t)\rangle + \langle \hat{\sigma}_a^\dagger(t)\rangle + \langle \hat{\sigma}_b(t)\rangle + \langle \hat{\sigma}_b^\dagger(t)\rangle)\\ &- \lambda(\langle \hat{a}(t)\rangle + \langle \hat{a}^\dagger(t)\rangle)\\ &= -\frac{1}{2}(\kappa + 2\lambda)(\langle \hat{a}(t)\rangle + \langle \hat{a}^\dagger(t)\rangle)\\ &- g(\langle \hat{\sigma}_a(t)\rangle + \langle \hat{\sigma}_a^\dagger(t)\rangle + \langle \hat{\sigma}_b(t)\rangle + \langle \hat{\sigma}_b^\dagger(t)\rangle), \qquad (3.23)\end{aligned}$$

and also subtracting Eq.(3.22) to Eq.(2.33), we have

$$\begin{aligned}\frac{d}{dt}(\langle \hat{a}\rangle - \langle \hat{a}^\dagger\rangle) &= -\frac{\kappa}{2}(\langle \hat{a}(t)\rangle - \langle \hat{a}^\dagger(t)\rangle) - g(\langle \hat{\sigma}_a(t)\rangle - \langle \hat{\sigma}_a^\dagger(t)\rangle + \langle \hat{\sigma}_b(t)\rangle - \langle \hat{\sigma}_b^\dagger(t)\rangle)\\ &- \lambda(\langle \hat{a}(t)\rangle - \langle \hat{a}^\dagger(t)\rangle)\\ &= -\frac{1}{2}(\kappa - 2\lambda)(\langle \hat{a}(t)\rangle - \langle \hat{a}^\dagger(t)\rangle)\\ &- g(\langle \hat{\sigma}_a(t)\rangle - \langle \hat{\sigma}_a^\dagger(t)\rangle + \langle \hat{\sigma}_b(t)\rangle - \langle \hat{\sigma}_b^\dagger(t)\rangle). \qquad (3.24)\end{aligned}$$

Eqs.(3.23) and (3.24), can be reduces to

$$\frac{d}{dt}(\langle \hat{a}_+\rangle = -\frac{1}{2}\upsilon_+\langle \hat{a}_+(t)\rangle - g(\langle \hat{\xi}_{a+}(t)\rangle + \langle \hat{\xi}_{b+}(t)\rangle), \qquad (3.25)$$

$$\frac{d}{dt}(\langle\hat{a}_-\rangle = -\frac{1}{2}\upsilon_-\langle\hat{a}_+(t)\rangle - g(\langle\hat{\xi}_{a-}(t)\rangle + \langle\hat{\xi}_{b-}(t)\rangle), \tag{3.26}$$

where

$$\hat{a}_\pm = \hat{a} \pm \hat{a}^\dagger, \tag{3.27}$$

$$\upsilon_\pm = \kappa \pm 2\lambda, \tag{3.28}$$

$$\hat{\xi}_{a\pm} = \hat{\sigma}_a \pm \hat{\sigma}_a^\dagger, \tag{3.29}$$

$$\hat{\xi}_{b\pm} = \hat{\sigma}_b \pm \hat{\sigma}_b^\dagger. \tag{3.30}$$

In view of Eqs.(2.92)-(2.95), we can express Eqs.(3.25) and (3.26)as

$$\frac{d}{dt}\langle\hat{a}_+\rangle = -\frac{1}{2}\upsilon_+\langle\hat{a}_+(t)\rangle, \tag{3.31}$$

$$\frac{d}{dt}\langle\hat{a}_-\rangle = -\frac{1}{2}\upsilon_-\langle\hat{a}_+(t)\rangle. \tag{3.32}$$

The solutions of Eqs.(3.31) and (3.32) are expressible as

$$\langle\hat{a}_+(t+\tau)\rangle = \langle\hat{a}_+(t)\rangle e^{-(\upsilon_+)\frac{\tau}{2}}, \tag{3.33}$$

$$\langle\hat{a}_-(t+\tau)\rangle = \langle\hat{a}_-(t)\rangle e^{-(\upsilon_-)\frac{\tau}{2}}. \tag{3.34}$$

Substituting Eq.(3.27) into Eqs.(3.33) and (3.34), we have

$$\langle\hat{a}(t+\tau)\rangle + \langle\hat{a}^\dagger(t+\tau)\rangle = (\langle\hat{a}(t)\rangle + \langle\hat{a}^\dagger(t)\rangle)e^{-(\upsilon_+)\frac{\tau}{2}}, \tag{3.35}$$

$$\langle\hat{a}(t+\tau)\rangle - \langle\hat{a}^\dagger(t+\tau)\rangle = (\langle\hat{a}(t)\rangle - \langle\hat{a}^\dagger(t)\rangle)e^{-(\upsilon_-)\frac{\tau}{2}}. \tag{3.36}$$

Now adding Eqs.(3.35) and (3.36), we find that

$$\langle\hat{a}(t+\tau)\rangle = \frac{1}{2}\langle\hat{a}(t)\rangle(e^{-(\upsilon_-)\frac{\tau}{2}} + e^{-(\upsilon_+)\frac{\tau}{2}}) + \frac{1}{2}\langle\hat{a}^\dagger(t)\rangle(e^{-(\upsilon_-)\frac{\tau}{2}} - e^{-(\upsilon_+)\frac{\tau}{2}}). \tag{3.37}$$

Applying the quantum regration theorem to Eq.(3.37), we obtain

$$\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \frac{1}{2}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle(e^{-(\upsilon_-)\frac{\tau}{2}} + e^{-(\upsilon_+)\frac{\tau}{2}}) + \frac{1}{2}\langle\hat{a}^{\dagger 2}(t)\rangle(e^{-(\upsilon_-)\frac{\tau}{2}} - e^{-(\upsilon_+)\frac{\tau}{2}}). \tag{3.38}$$

Substituting Eq.(3.38) into Eq.(3.21), we have

$$P(\omega) = \frac{1}{2\pi}\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle[Re\int_{0}^{\infty}e^{-[\frac{\upsilon_{+}}{2}-i(\omega-\omega_{o})]\tau}d\tau + Re\int_{0}^{\infty}e^{-[\frac{\upsilon_{-}}{2}-i(\omega-\omega_{o})]\tau}d\tau]$$
$$+\frac{1}{2\pi}\langle\hat{a}^{\dagger 2}(\langle\hat{a}^{\dagger 2}\rangle(t))\rangle[Re\int_{0}^{\infty}e^{-[\frac{\upsilon_{+}}{2}-i(\omega-\omega_{o})]\tau}d\tau + Re\int_{0}^{\infty}e^{-[\frac{\upsilon_{-}}{2}-i(\omega-\omega_{o})]\tau}d\tau]. \quad (3.39)$$

Carrying out the integration, and taking the real parts lead to

$$P(\omega) = \langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle[\frac{\frac{\upsilon_{+}}{4\pi}}{(\frac{\upsilon_{+}}{2})^{2}+(\omega-\omega_{o})^{2}} + \frac{\frac{\upsilon_{-}}{4\pi}}{(\frac{\upsilon_{-}}{2})^{2}+(\omega-\omega_{o})^{2}}]$$
$$+\langle\hat{a}^{\dagger 2}(t)\rangle[\frac{\frac{\upsilon_{+}}{4\pi}}{(\frac{\upsilon_{+}}{2})^{2}+(\omega-\omega_{o})^{2}} - \frac{\frac{\upsilon_{-}}{4\pi}}{(\frac{\upsilon_{-}}{2})^{2}+(\omega-\omega_{o})^{2}}]. \quad (3.40)$$

Therefore, on account of Eqs.(3.18) and (3.40), we see that

$$\bar{n}_{\pm\beta} = \langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle[\int_{-\beta}^{\beta}\frac{\frac{\upsilon_{+}}{4\pi}}{(\frac{\upsilon_{+}}{2})^{2}+(\omega-\omega_{o})^{2}}d\omega' + \int_{-\beta}^{\beta}\frac{\frac{\upsilon_{-}}{4\pi}}{(\frac{\upsilon_{-}}{2})^{2}+(\omega-\omega_{o})^{2}}d\omega']$$
$$+\langle\hat{a}^{\dagger 2}(t)\rangle[\int_{-\beta}^{\beta}\frac{\frac{\upsilon_{+}}{4\pi}}{(\frac{\upsilon_{+}}{2})^{2}+(\omega-\omega_{o})^{2}}d\omega' - \int_{-\beta}^{\beta}\frac{\frac{\upsilon_{-}}{4\pi}}{(\frac{\upsilon_{-}}{2})^{2}+(\omega-\omega_{o})^{2}}d\omega']. \quad (3.41)$$

Employing the relation

$$\int_{-\beta}^{\beta}\frac{dx}{x^{2}+a^{2}} = \frac{2}{a}\arctan(\frac{\beta}{a}), \quad (3.42)$$

and carrying out the integration, we see that

$$\bar{n}_{\pm\beta} = \frac{1}{\pi}\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle[\arctan(\frac{2\beta}{\upsilon_{+}}) + \arctan(\frac{2\beta}{\upsilon_{-}})]$$
$$+\frac{1}{\pi}\langle\hat{a}^{\dagger 2}(t)\rangle[\arctan(\frac{2\beta}{\upsilon_{+}}) + \arctan(\frac{2\beta}{\upsilon_{-}})]. \quad (3.43)$$

On the basis of (3.1), we can express Eq.(3.43)as

$$\bar{n}_{\pm\beta} = \frac{1}{\pi}\bar{n}[\arctan(\frac{2\beta}{\upsilon_{+}}) + \arctan(\frac{2\beta}{\upsilon_{-}})]$$
$$+\frac{\langle\hat{a}^{\dagger 2}(t)\rangle}{\pi}[\arctan(\frac{2\beta}{\upsilon_{+}}) - \arctan(\frac{2\beta}{\upsilon_{-}})]. \quad (3.44)$$

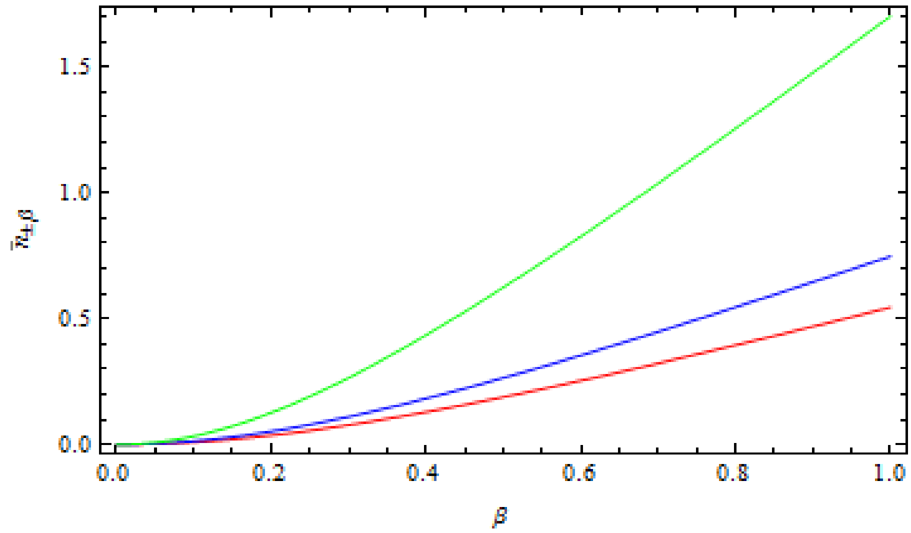

Figure 3.5: Plots of $n_{\pm\beta}$ [Eq.(3.44)] versus β for $\Omega = 0.4$, $\kappa = 0.8$, $\gamma_c = 0.5$, $\lambda = 0$(red), $\lambda = 0.2$(blue) and $\lambda = 0.3$(green).

From Figure 3.5 we observe that the local mean photon number ($\bar{n}_{\pm\beta}$) increases as the frequency interval and parametric interaction parameter(λ) increase.

Chapter 4

Quadrature Squeezing

In this chapter we seek to study the squeezing properties of a light generated by degenerate three-level atom in a cavity containing parametric amplifier coupled to a vacuum reservoir via a port mirror. To this end, we calculate the quadrature variance and the quadrature squeezing of the light. The squeezing properties of single mode cavity light are described by plus and minus quadrature operators defined by

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a}, \tag{4.1}$$

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}). \tag{4.2}$$

The variance of plus quadrature is defined as [3]

$$(\Delta a_+)^2 = \langle \hat{a}_+^2 \rangle - \langle \hat{a}_+ \rangle^2. \tag{4.3}$$

Substituting Eq.(4.1) into Eq.(4.3), we have

$$\begin{aligned}(\Delta a_+)^2 = {} & \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}\hat{a}^\dagger \rangle + \langle \hat{a}^{\dagger 2} \rangle + \langle \hat{a}^2 \rangle \\ & - (\langle \hat{a}^\dagger \rangle^2 + \langle \hat{a} \rangle^2 + \langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle + \langle \hat{a} \rangle \langle \hat{a}^\dagger \rangle).\end{aligned} \tag{4.4}$$

On account of Eq.(3.7) and its conjugate, Eq.(4.4) reduces to

$$(\Delta a_+)^2 = \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}\hat{a}^\dagger \rangle + \langle \hat{a}^{\dagger 2} \rangle + \langle \hat{a}^2 \rangle. \tag{4.5}$$

Similarly, the variance of minus quadrature can be expressed as

$$(\Delta a_-)^2 = \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}\hat{a}^\dagger \rangle - \langle \hat{a}^{\dagger 2} \rangle - \langle \hat{a}^2 \rangle. \tag{4.6}$$

Based on Eqs.(4.5) and (4.6), we note that

$$(\Delta a_\pm)^2 = \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}\hat{a}^\dagger \rangle \pm \langle \hat{a}^{\dagger 2} \rangle \pm \langle \hat{a}^2 \rangle. \tag{4.7}$$

Now employing Eqs.(2.113), (2.114) and (2.115) into Eq.(4.7), we obtain

$$(\Delta a_\pm)^2 = \frac{\gamma_c\Gamma(\kappa^2+4\lambda^2\mp 4\kappa\lambda)+\kappa(\kappa^2\mp\kappa\lambda)}{\kappa(\kappa^2-4\lambda^2)} + [\frac{\gamma_c\Gamma(\kappa^2-4\lambda^2)(\Omega\lambda-4\lambda\gamma_c\Gamma)-4\lambda\gamma_c\Gamma(\gamma_c\Gamma(\kappa^2+4\lambda^2)-\Omega\lambda(\kappa+\lambda))}{\kappa(\kappa^2-4\lambda^2)(\Omega\kappa+2\lambda\gamma_c\Gamma)}]\langle\hat{\eta}_a\rangle \pm [\frac{\gamma_c\Gamma(\gamma_c\Gamma(\kappa^2+4\lambda^2)-\Omega\lambda(\kappa+\lambda))}{\kappa(\kappa^2-4\lambda^2)(\Omega\kappa+2\lambda\gamma_c\Gamma)}]\langle\hat{\eta}_a\rangle, \tag{4.8}$$

where $\langle\hat{\eta}_a\rangle$ is given by Eq.(2.108). So that, the plus and minus quadrature variance can be written respectively as

$$(\Delta a_+)^2 = \frac{\gamma_c\Gamma(\kappa^2+4\lambda^2-4\kappa\lambda)+\kappa(\kappa^2-\kappa\lambda)}{\kappa(\kappa^2-4\lambda^2)} + [\frac{\gamma_c\Gamma(\kappa^2-4\lambda^2)(\Omega\lambda-4\lambda\gamma_c\Gamma)-4\lambda\gamma_c\Gamma(\gamma_c\Gamma(\kappa^2+4\lambda^2)-\Omega\lambda(\kappa+\lambda))}{\kappa(\kappa^2-4\lambda^2)(\Omega\kappa+2\lambda\gamma_c\Gamma)}]\langle\hat{\eta}_a\rangle + [\frac{\gamma_c\Gamma(\gamma_c\Gamma(\kappa^2+4\lambda^2)-\Omega\lambda(\kappa+\lambda))}{\kappa(\kappa^2-4\lambda^2)(\Omega\kappa+2\lambda\gamma_c\Gamma)}]\langle\hat{\eta}_a\rangle, \tag{4.9}$$

and

$$(\Delta a_-)^2 = \frac{\gamma_c\Gamma(\kappa^2+4\lambda^2+4\kappa\lambda)+\kappa(\kappa^2+\kappa\lambda)}{\kappa(\kappa^2-4\lambda^2)} + [\frac{\gamma_c\Gamma(\kappa^2-4\lambda^2)(\Omega\lambda-4\lambda\gamma_c\Gamma)-4\lambda\gamma_c\Gamma(\gamma_c\Gamma(\kappa^2+4\lambda^2)-\Omega\lambda(\kappa+\lambda))}{\kappa(\kappa^2-4\lambda^2)(\Omega\kappa+2\lambda\gamma_c\Gamma)}]\langle\hat{\eta}_a\rangle - [\frac{\gamma_c\Gamma(\gamma_c\Gamma(\kappa^2+4\lambda^2)-\Omega\lambda(\kappa+\lambda))}{\kappa(\kappa^2-4\lambda^2)(\Omega\kappa+2\lambda\gamma_c\Gamma)}]\langle\hat{\eta}_a\rangle. \tag{4.10}$$

The quadrature variance of the cavity light in a vacuum state can be obtained by setting $\Omega = 0$ and $\lambda = 0$ in Eq. (4.9), we thus have [3]

$$(\Delta a_+)^2_v = \frac{\gamma_c}{\kappa} + 1. \tag{4.11}$$

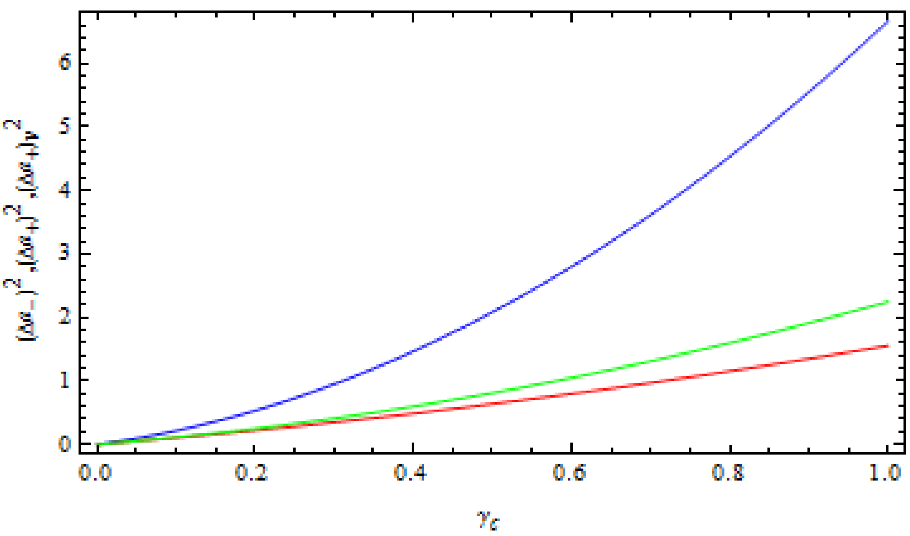

Figure 4.1: Plots of the minus quadrature variance[Eq.(4.9)](blue), plus quadrature variance[Eq.(4.10)](red) and plus quadrature variance in vacuum level[Eq.(4.11)](green) versus γ_c for $\kappa = 0.8$, $\Omega = 0.6$, $\lambda = 0.2$.

From Figure 4.1 we see that the cavity light is in a squeezed state and the squeezing occurs in the plus quadrature.

Now proceed to determine the quadrature squeezing(S) of the cavity light relative to the quadrature variance of the cavity vacuum light which is expressible as [3],

$$S = \frac{(\Delta a_+)_v^2 - (\Delta a_+)^2}{(\Delta a_+)_v^2}, \tag{4.12}$$

where $(\Delta a_+)_v^2$ is the quadrature variance of cavity light in a vacuum state.

Employing Eqs.(4.10) and (4.11) into Eq.(4.12), we obtain

$$\begin{aligned} S = {} & \frac{(\gamma_c + \kappa)(\kappa^2 - 4\lambda^2) - \gamma_c\Gamma(\kappa^2 + 4\lambda^2 - 4\kappa\lambda) - \kappa(\kappa^2 - \kappa\lambda)}{(\gamma_c + \kappa)(\kappa^2 - 4\lambda^2)} \\ & - [\frac{\gamma_c\Gamma(\kappa^2 - 4\lambda^2)(\Omega\lambda - 4\lambda\gamma_c\Gamma) - 4\lambda\gamma_c\Gamma(\gamma_c\Gamma(\kappa^2 + 4\lambda^2) - \Omega\lambda(\kappa + \lambda))}{(\gamma_c + \kappa)(\kappa^2 - 4\lambda^2)(\Omega\kappa + 2\lambda\gamma_c\Gamma)}]\langle\hat{\eta}_a\rangle \\ & - [\frac{\gamma_c\Gamma(\gamma_c\Gamma(\kappa^2 + 4\lambda^2) - \Omega\lambda(\kappa + \lambda))}{(\gamma_c + \kappa)(\kappa^2 - 4\lambda^2)(\Omega\kappa + 2\lambda\gamma_c\Gamma)}]\langle\hat{\eta}_a\rangle. \end{aligned} \tag{4.13}$$

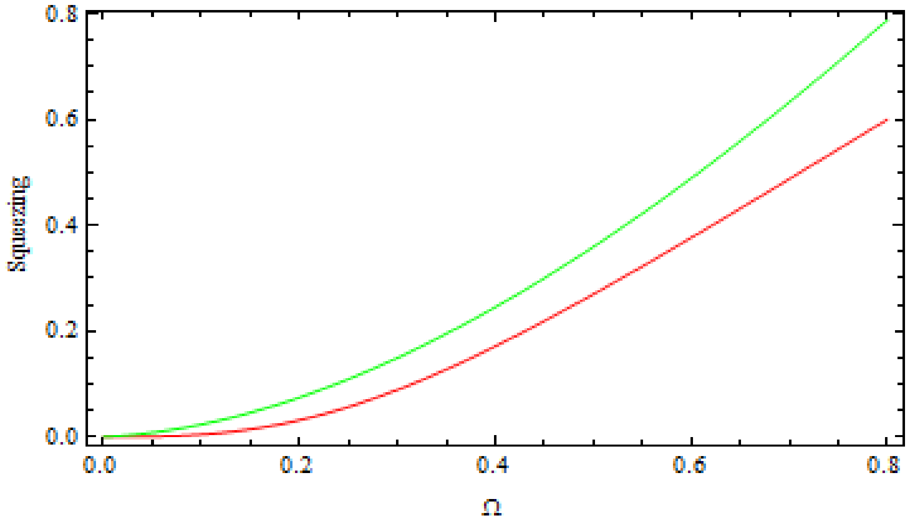

Figure 4.2: plots of quadrature squeezing(S) versus Ω [Eq.(4.13)] for $\kappa = 0.8$, $\gamma_c = 0.8$, $\lambda = 0.3$(green), $\lambda = 0$(red).

From Figure 4.2 we observe that squeezing increases with the increase of the amplitude of driving coherent light and the parametric interaction. We also see that the maximum squeezing attainable being 80% for $\lambda = 0.3$ and 60% for $\lambda = 0$, below the vacuum state.

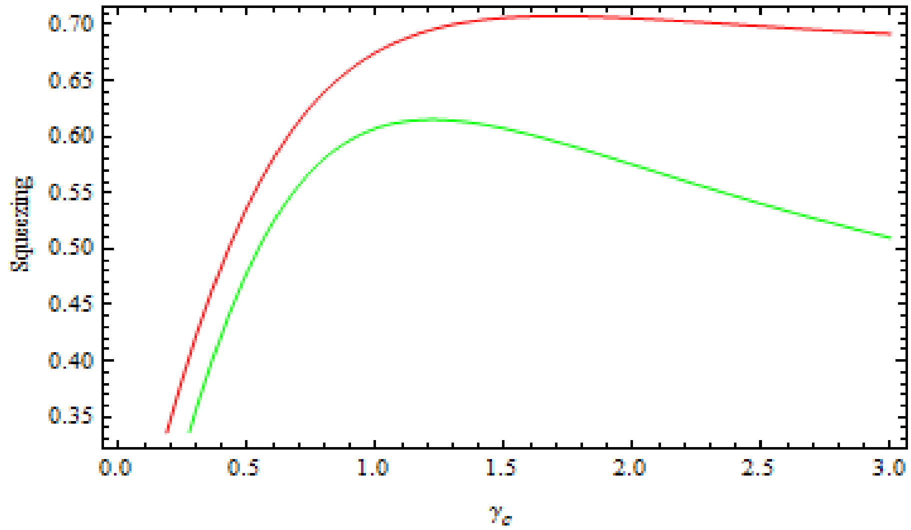

Figure 4.3: plots of quadrature squeezing(S) versus γ_c [Eq.(4.13)] for $\kappa = 0.8$, $\Omega = 0.6$, $\lambda = 0.03$(green), $\lambda = 0.06$(red).

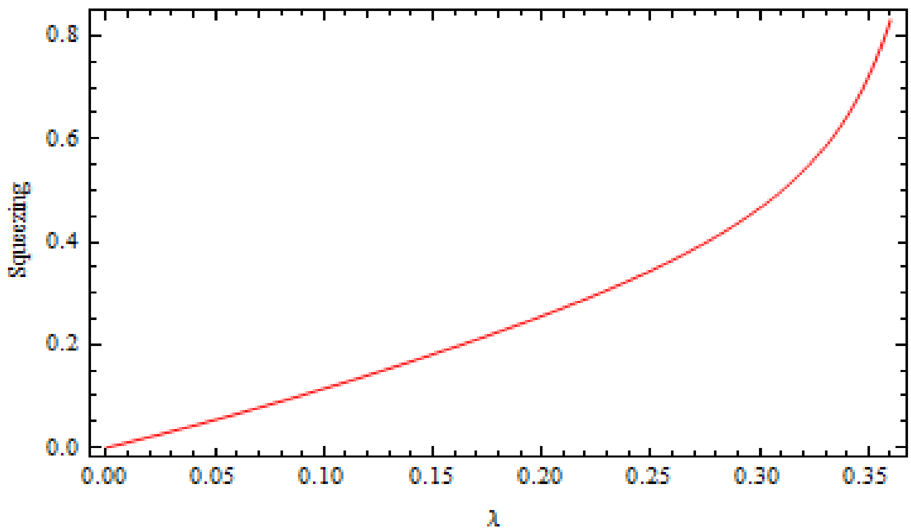

Figure 4.4: A plot of quadrature squeezing(S) versus λ [Eq.(4.13)] for $\kappa = 0.8$, $\gamma_c = 0.8$, $\Omega = 0.4$.

From Figure 4.3 and 4.4 we observe that squeezing increases with the increase of the amplitude of rate of stimulated emission and the parametric interaction. We also see that the maximum squeezing attainable being 70% for $\lambda = 0.06$ and 61% for $\lambda = 0.03$, below the vacuum state, which indicates that the presence of degenerate parametric amplifier enhances the squeezing of the cavity light.

Chapter 5

Conclusion

We have studied the statistical and squeezing properties of the light produced by degenerate three-level atom, whose top and bottom levels are coupled by coherent light, and available in a cavity containing parametric amplifier and coupled to vacuum reservoir via a single port-mirror. Employing the master equation for the system under consideration, we obtained the quantum Langevin equation for the cavity mode and atomic operators. Using the solution of these equations, we have calculated mean and variance of the photon number for the cavity mode. From the plots in Figures 3.1, 3.2, 3.3, and 3.4, we observed that the mean and variance of photon number of the cavity light increase with the presence of parametric amplifier. Moreover, we have also evaluated the quadrature variance and quadrature squeezing of the cavity light. We have realized that the light produced by the system is in squeezed state and the squeezing occurs in the plus quadrature. From the plot of squeezing versus the amplitude of driving coherent light, we observed that the maximum squeezing attainable being 80% for $\lambda = 0.3$ and 60% for $\lambda = 0$, below the vacuum state level. And also from the plot of squeezing versus rate of stimulated emission decay constant, we observed that the maximum squeezing attainable being 70% for $\lambda = 0.06$ and 61% for $\lambda = 0.03$, below the vacuum state level. Furthermore, we have seen that the rate of stimulated emission and driving coherent light enhances the degree of squeezing.

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DECLARATION

I, hereby declare that this thesis is my original work and has not been presented for a degree in any other university, and that all sources of materials have been duly acknowledged.

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