FRACTIONAL HYPERGEOMETRIC ZETA FUNCTIONS

By

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I dedicate this thesis to my father Legesse Geleta and my mother Asegedech Geleta

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Abstract

Since Riemann, there have been a great number of generalizations of the Riemann zeta function. Of particular interest is the hypergeometric zeta functions a generalization due to Hassen and Nguyen that satisfies many of the same properties as the classical Riemann zeta function. In this dissertation we show that the second order Hypergeometric Zeta Function has zero free region in the right half of the complex plane and extend the hypergeometric zeta Functions to fractional hypergeometric zeta functions. We rewrite the series representation of the second order hypergeometric zeta function as a "power series" with Dirichlet series as its coefficients. We use this series representation to show that the second order hypergeometric zeta function has zero free region to the right half of the complex plane. We generalize the hypergeometric zeta function via the integral representation of the classical Riemann zeta function to fractional hypergeometric zeta functions. It is shown that fractional hypergeomeric zeta functions satisfy many of the properties satisfied by hypergeometric zeta functions.

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Introduction

The zeta function was introduced first by Euler [4], [7]. Since then the zeta function has captured the imaginations of great mathematicians. In particular, in 1859 the German Mathematician Bernhard Riemann showed that the variable defining the zeta function need not be a real number, and that the zeta function has a natural analytic continuation as a function of complex variable [4], [7]. Hence the function traditionally called the Riemann zeta function and defined in terms of a complex variable $s = \sigma + it$ for σ, t real numbers and $\sigma > 1$;

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The German Mathematician Bernhard Riemann published his now famous paper on his studies of prime numbers (1859). His paper introduced many new results in the fields of analytic number theory and complex analysis, but contained a number of statements which were unsupported by rigorous proofs (see [4], [3]). Proofs for many of these conjectures were found over the fifty or so years following Riemann's death in 1866. The Riemann Hypothesis is now among the most famous unsolved problems in mathematics. It is one of the famous Hilbert problems number eight of twenty-three

(1900). It is also one of the seven "Clay Mathematics Institute" millennium prize problems (2000).

The Riemann Hypothesis is a statement about the location of the complex zeros of the zeta function. Euler derived an infinite product representation for the zeta function that connected it to prime numbers ([1], [4], [7]). A century later Riemann discovered a close relationship between the zeta function and the asymptotic distributions of prime numbers. Except for certain "trivial" zeros on the negative Real axis, the zeros of the zeta function are easily seen to lie in the so called "critical" strip $\{s : 0 \leq \sigma \leq 1\}$. Riemann conjectured that the zeros in the critical strip all lie on the line with $\sigma = \frac{1}{2}$, but no proof was provided. Riemann checked the first few zeros of the zeta function by hand; they satisfy his hypothesis.

Since Riemann made his conjecture much work has been done towards his hypothesis [3]. The zeta function and related hypothesis are classical fields of study which in recent years have enjoyed renewed interest. Conjectured relationships between the hypothesis and results in mathematical physics have motivated extensive numerical investigations using modern supercomputers. By now over 1.5 billion nontrivial zeros have been checked by Van de Lune et al they demonstrated the correctness of the first 1.5 billion nontrivial zeros. Andrew Odlyzko ([3]) has verified the correctness of Riemann Hypothesis for large set of zeros around the $10^{20}th$ zero. The modern studies of the Riemann Hypothesis use different computational algorithms like the

Riemann Siegel formula, prior to it they used Euler-Maclaurine summation formula. These provide very strong experimental evidence but not a proof.

Following Riemann, many scholars tried to generalize the Riemann zeta function as: Multiple zeta function ([22]), Hurwitz zeta function ([1], [15], [8]), Hypergeometric zeta functions ([9]). In this paper we are only interested to review the hypergeometric zeta functions. Abdul Hassen and Heiu D. Nguyen ([9]) investigated and introduced a generalization of the Riemann zeta function called Hypergeometric zeta functions. They developed the analytic continuation of these families of functions to the entire complex plane except for finite number of poles and established many properties analogous to those satisfied by the classical zeta function ([9]). They also demonstrated their series representations with coefficients "$\mu_N(n, s)$" which depends on both n and s. But at present time no body knows whether this series representation is a generalized Dirichlet series. Thus in this thesis we study, first the coefficients "$\mu_N(n, s)$" and investigate the properties of the coefficients "$\mu_N(n, s)$" and rewrite the second order hypergeometric zeta function as a "power series" with Dirichlet series as a coefficient. We also use this series representation to show that the second order hypergeometric zeta function has zero free region to the right half of the complex plane. Second we investigate Fractional Hypergeometric zeta Functions, which is a generalization of the Hypergeometric zeta Functions via the integral representation of the Riemann zeta function. As the name indicates, in stead of considering only positive integers, we extend the definition to any positive real numbers. Analogous to its counterpart the fractional hypergeometric zeta functions is found to satisfy the same properties satisfied by hypergeometric zeta functions.

The outline of the dissertation is as follows: It has four chapters. The first chapter reviews on the classical zeta function, the second chapter reviews on a generalization of the integral representation of the Riemann zeta function known as Hypergeometric zeta function. These two chapters form a back ground for this thesis.

In the third chapter we will have some results concerning Series Representation of the second order Hypergeometric zeta Functions. We found that the second order hypergeometric zeta function can be rewritten as a "power series" with Dirichlet series as a coefficient. We also use this series representation to show that the second order Hypergeometric zeta Function has zero free region in the right half plane. The fourth chapter deals with what we call Fractional Hypergeometric zeta Functions, which is a generalization of the Hypergeometric zeta Functions via the integral representation of the Riemann zeta function. As the name indicates, in stead of considering only positive integers, we extend the definition to any positive real numbers. Analogous to its counterpart the fractional hypergeometric zeta functions is found to satisfy all properties satisfied by hypergeometric zeta functions.

Chapter 1

The Riemann Zeta Function

In this chapter we give an overview of the Riemann Zeta function. Here we mainly review on series representation, integral representation, analytic continuations and its functional equations. We also discuss on the zeros of the zeta function.

1.1 Introduction

Euler derived an infinite product representation for the zeta function that connected it to prime numbers. A century latter Riemann discovered a close relationship between the zeta function and the asymptotic distributions of prime numbers. Except for certain "trivial" zeros on the negative real axis, the zeros of the zeta function are easily seen to lie in the "critical" strip $\{s = \sigma + it : 0 \leqq \sigma \leqq 1\}$. Riemann conjectured that the zeros in the critical strip all lie on the line $\sigma = \frac{1}{2}$. This is the most famous unsolved problem in complex analysis if not in all mathematics [1], [4], [6], [7], [18]. In the next section it is shown that the zeta function has no zeros on the boundary lines of the critical strip, and this will lead to the prime number theorem. In the theory of the Riemann zeta function, it is traditional to denote the complex variable by $s = \sigma + it$, where σ and t are real numbers. With this notation we are going to discuss on the series of the form $\sum_{n=1}^{\infty} \frac{1}{n^s}$. We know that the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely for $\sigma > 1$ and converges uniformly for $\sigma \geq \sigma_0 > 1$.

Definition 1.1.1. For $\sigma > 1$ the Riemann Zeta function is defined by the following equation:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Proposition 1.1.1. *For $\sigma > 1$ the Riemann zeta function is analytic.*

Proof. : The function n^{-s} is an analytic function for each $n = 1, 2, 3, \cdots$. This implies that, $S_k = \sum_{n=1}^{k} \frac{1}{n^s}$ is an analytic function for each $n = 1, 2, 3, \cdots$. Now let $[\sigma_0, \rho]$ be a subset of $(1, \infty)$, then $|\frac{1}{n^{\sigma+it}}| = |\frac{1}{n^{\sigma}}| \leq \frac{1}{n^{\sigma_0}}$ for each $\sigma \in [\sigma_0, \rho]$. But we know that $\sum_{n=1}^{\infty} \frac{1}{n^{\sigma_0}}$ converges absolutely. Hence by Weierstrass M-Test, $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges uniformly for $\sigma \in [\sigma_0, \rho]$. Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges uniformly to $\zeta(s)$ for $\sigma \geq \sigma_0 > 1$ and this completes the proof. $\square$

1.2 Integral Representation of the Riemann Zeta Function

The integral representation of the Riemann zeta function is very important to extend the domain of definition of the zeta function to the whole complex plane. We have seen that $\zeta(s)$ converges absolutely for $\sigma > 1$. However, the most interesting properties of the zeta function is observed in the region where $\sigma \leq 1$ [1], [18]. Therefore, we need to find a way to extend it to this region to study those properties observed. For this purpose we use the gamma function. Recall that the gamma function is defined by the following equation for $\sigma > 0$:

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx$$

We use integration by parts to show that $\Gamma(s+1) = s\Gamma(s)$. In particular if $s = n$ is a nonnegative integer, one can show inductively that $\Gamma(n+1) = n!$. Actually the gamma function can be continued analytically to the whole complex plane except for isolated poles at non-positive integers. At $-k$ the gamma function has residue $\frac{(-1)^k}{k!}$ for $k \geq 0$ [1],[4], [7], [18].

Lemma 1.2.1. *For $\sigma > 1$ the zeta function can be represented by the following integral:*

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx.$$

Proof. Let $x = nu$, then $dx = ndu$. Thus as x goes to 0 then u goes to 0 and as x goes to ∞ then u goes to ∞. This implies that,

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx = \int_0^\infty (nu)^{s-1}e^{-nu}ndu = \int_0^\infty u^{s-1}e^{-(nu)}n^s du.$$

This implies that,

$$n^{-s}\Gamma(s) = \int_0^\infty u^{s-1}(e^{-u})^n du.$$

Which in turn implies that,

$$\sum_{n=1}^\infty n^{-s}\Gamma(s) = \sum_{n=1}^\infty \int_0^\infty u^{s-1}(e^{-u})^n du.$$

Thus it follows that,

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1}dx.$$

Here interchanging summation and integration yields the result for this see [4]. □

1.3 The Euler Product Representation of the Riemann Zeta Function

We now present the infinite product of the zeta function which connects the zeta function to prime numbers. This representation is used to show that the zeta function is not zero on the boundary of the critical strip.

Proposition 1.3.1. *If* $\sigma > 1$, *then the zeta function is given by,*

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}$$

where the product is over all prime numbers p.

Proof. $\sum_p \frac{1}{p^s}$ converges absolutely for $\sigma > 1$ and converges uniformly for $\sigma \geq \sigma_0 > 1$. Hence it follows from the theory of infinite product that

$$\prod_p (1 - p^{-s})$$

converges for $\sigma > 1$. Now consider the following geometric series:

$$\frac{1}{1 - p^{-s}} = 1 + p^{-s} + p^{-2s} + p^{-3s} + ...$$

for $\sigma > 1$. If we multiply together the m series corresponding to primes $p_1, p_2, ..., p_m$, we obtain,

$$(1 - p_1^{-s})^{-1}(1 - p_2^{-s})^{-1}...(1 - p_m^{-s})^{-1} = \sum_{k_1, k_2, ... k_m = 0}^{\infty} (p_1 p_2 ... p_m)^{-s}.$$

Since every integer $n \geq 1$ has a unique representation as a product of powers of distinct primes, a summand n^{-s} appears at most once in this sum. Thus the sum is a sub-sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$. Now as we incorporate more primes into the product, we eventually capture all terms n^{-s} and in the limit we have the required result. $\square$

As a result of the Euler product representation of $\zeta(s)$ it is easy to show that

$$\sum_p \frac{1}{p} = \infty \ .$$

This in turn implies that there are infinite number of prime numbers: otherwise the harmonic series becomes convergent. As a result of this product formula it can also be shown that $\zeta(s) \neq 0$ for $\sigma > 1$ (see [1], [4] and [7]).

1.4 Analytic Continuation of the Riemann Zeta Function

The development of complex analysis was a central preoccupation of Riemann's and so it comes as no surprise that from the beginning Riemann considered the zeta function as an analytic function. He first showed that the zeta function had an analytic continuation to the whole complex plane as a meromorphic function which has only one singularity, a simple pole of residue 1 at $s = 1$. He considered the integral,

$$\int_C \frac{(-x)^{s-1}}{e^x - 1} dx$$

where the contour C starts at infinity on the positive real axis, encircles the origin once in counterclockwise, excluding the points $2\pi in$ for an integer n. Thus Riemann deduced that for each $s \neq 1$

$$\zeta(s) = 2(2\pi)^{s-1}\zeta(1-s)\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)$$

as shown in [18]. Actually there are different methods for analytic continuation of the zeta function (see [1], [4], [6], [7] $\cdots$). We now use one of the methods in this paper and use the integral representation of the zeta function to extend the domain of definition of the zeta function to the whole complex plane except for a simple pole at $s = 1$ with residue 1 and drive the functional equation. To do this we consider the Laurent series expansion of $(e^s - 1)^{-1}$ which has a simple pole at $s = 0$ with residue 1 . Thus,

$$(e^s - 1)^{-1} = s^{-1} + a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \cdots .$$

It is also known that

$$e^s - 1 = s + \frac{s^2}{2!} + \frac{s^3}{3!} + \cdots .$$

Hence equating the two equations we get, $a_0 = \frac{-1}{2}$, $a_1 = \frac{1}{3}, \cdots$. From this it follows that $(e^s - 1)^{-1} - \frac{1}{s}$ goes to $\frac{-1}{2}$ as s goes to 0. Hence $(e^s - 1)^{-1} - \frac{1}{s}$ is bounded in a neighborhood of $s = 0$. Therefore, $\int_0^1 \left(\frac{1}{e^x - 1} - \frac{1}{x}\right) x^{s-1} dx$ converges uniformly on compact subsets of the right half-plane $\{s : \sigma > 0\}$ and represents an analytic function there. Now we can rewrite $\zeta(s)\Gamma(s)$ as follows:

$$\begin{aligned} \zeta(s)\Gamma(s) &= \int_0^1 \left(\frac{1}{e^x - 1} - \frac{1}{x}\right) x^{s-1} dx + \int_0^1 x^{s-2} dx + \int_1^\infty \frac{x^{s-1}}{e^x - 1} dx. \\ &= \int_0^1 \left(\frac{1}{e^x - 1} - \frac{1}{x}\right) x^{s-1} dt + \frac{1}{s-1} + \int_1^\infty \frac{x^{s-1}}{e^x - 1} dx. \end{aligned}$$

Thus one can define $\zeta(s)$ for $\{s : \sigma > 0\}$ by,

$$\zeta(s) = \frac{1}{\Gamma(s)} \left(\int_0^1 \left(\frac{1}{e^x - 1} - \frac{1}{x}\right) x^{s-1} dx + \frac{1}{s-1} + \int_1^\infty \frac{x^{s-1}}{e^x - 1} dx \right).$$

In this manner $\zeta(s)$ is meromorphic in the right half-plane with a simple pole at $s = 1$.

Now suppose $0 < \sigma < 1$, then

$$\frac{1}{s-1} = -\int_1^\infty x^{s-2}dx.$$

Inserting this in the above expression of $\zeta(s)$ we have

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) x^{s-1}dx$$

for $0 < \sigma < 1$. Again considering the Laurent series expansion of $\frac{1}{e^s-1}$, we see that $\left(\frac{1}{e^s-1} - \frac{1}{s} + \frac{1}{2}\right)\frac{1}{s}$ goes to $a_1 = \frac{1}{3}$ as s goes to 0. Hence it follows that

$$\left| \frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} \right| \leq Cx$$

for some constant C and for each $x \in (0, 1]$. Similarly, $x(\frac{1}{x} - \frac{1}{e^x-1})$ goes to 1 as x goes to ∞ . Thus there is a constant K such that

$$\left| \frac{1}{x} - \frac{1}{e^x - 1} \right| \leq \frac{K}{x}$$

for $x \geq 1$. Hence $\int_0^1 \left(\frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2}\right) x^{s-1}dx$ converges uniformly on compact subset of $\{s : \sigma > -1\}$ and $\int_1^\infty \left(\frac{1}{e^x-1} - \frac{1}{x}\right) x^{s-1}dx$ converges uniformly on compact subset of $\{s : \sigma < 1\}$. Now using the last two integrals and the previous discussions $\zeta(s)$ can be represented by the following equations for $0 < \sigma < 1$:

$$\zeta(s) = \frac{1}{\Gamma(s)} \left(\int_0^1 \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} \right) x^{s-1}dx - \frac{1}{2s} + \int_1^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) x^{s-1}dx \right).$$

Here both integrals converge in the strip $\{-1 < \sigma < 1\}$ and this expression can be used to define $\zeta(s)$ in the strip. Thus this definition of $\zeta(s)$ makes it analytic in the strip. Therefore, $\zeta(s)$ is defined for $\sigma > -1$ with a simple pole at $s = 1$. If $-1 < \sigma < 1$ then

$$\int_1^\infty x^{s-1} dx = -\frac{1}{s}.$$

Making use of this and the representation of $\zeta(s)$ in the strip $\{s: \ -1 < \sigma < 1\}$ we have,

$$\zeta(s)\Gamma(s) = \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} \right) x^{s-1} dx,$$

for $-1 < \sigma < 0$. But we know that,

$$\frac{1}{e^x - 1} + \frac{1}{2} = \frac{1}{2} \left(\frac{e^x + 1}{e^x - 1} \right) = \frac{i}{2} \cot \left(\frac{ix}{2} \right).$$

A straightforward computation with the well known relation (see [1], [4], [7], [2],)

$$\pi \cot(\pi s) = \frac{1}{s} + \sum_{n=1}^\infty \frac{2s}{s^2 - n^2}.$$

This gives as,

$$\cot \left(\frac{ix}{2} \right) = \frac{2}{ix} - 4ix \sum_{n=1}^\infty \frac{1}{x^2 + 4n^2\pi^2}.$$

This implies that,

$$\left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} \right) \frac{1}{x} = 2 \sum_{n=1}^\infty \frac{1}{x^2 + 4n^2\pi^2}.$$

This again implies that,

$$\zeta(s)\Gamma(s) = \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2}\right) x^{s-1} dx = 2\int_0^\infty \sum_{n=1}^\infty \frac{x^s}{x^2 + 4n^2\pi^2} dx.$$

$$= 2\sum_{n=1}^\infty \int_0^\infty \frac{x^s}{x^2 + 4n^2\pi^2} dx = 2\sum_{n=1}^\infty (2\pi n)^{s-1} \int_0^\infty \frac{x^s}{x^2+1} dx.$$

$$= 2(2\pi)^{s-1}\zeta(1-s) \int_0^\infty \frac{x^s}{x^2+1} dx.$$

But using integration by substitution, we have

$$\int_0^\infty \frac{x^s}{x^2+1} dx = \frac{1}{2}\pi \sec\left(\frac{\pi s}{2}\right).$$

Now using the relation

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

and

$$\sin(\pi s) = 2\sin\left(\frac{\pi s}{2}\right)\cos\left(\frac{\pi s}{2}\right),$$

we get,

$$\zeta(s) = 2(2\pi)^{s-1}\zeta(1-s)\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right).$$

This last equation is called Riemann's Functional Equation. We notice that the right hand side of the functional equation is analytic in the left hand plane $\sigma < 0$. Thus we use the functional equation to extend the definition of $\zeta(s)$ to $\sigma < 0$. Now one can summarize what was done as follows: The Riemann zeta function can be defined to be meromorphic in the whole complex plane with only a simple pole at $s = 1$ and residue 1 . For $s \neq 1$ the Riemann zeta function satisfies Riemann's Functional Equation. We also know that the Riemann zeta function is analytic and $\zeta(s) \neq 0$ for $\sigma > 1$. In particular the Riemann zeta function is analytic at integers $2, 3, 4...$ and $\Gamma(1 - s)$ has poles at integers $1, 2, 3, ...$ So for the Riemann zeta function to be analytic, the simple poles of $\Gamma(1 - s)$ at integers $2, 3, 4...$ should be canceled by simple zeros of $\zeta(1-s)\sin(\frac{\pi s}{2})$. This implies that $\zeta(1-s)\sin\left(\frac{\pi s}{2}\right)$ has simple zeros at $s = 2, 3, 4, ...,$ but $\sin(\frac{\pi s}{2})$ has simple zeros at even integers $(s = 2, 4, 6, ...)$ and hence $\zeta(1 - s) = 0$ at $s = 3, 5, 7, \cdots$. This implies that $\zeta(-2n) = 0$ for each $n = 1, 2, 3, ...$ This implies that $\zeta(s) = 0$ for $s = -2, -4, -6, \cdots$. Similar reasoning gives that $\zeta(s)$ has no other zeros outside the closed strip $\{s : 0 \leq \sigma \leq 1\}$. We now give the following definitions and state the most celebrated open questions in all mathematics called the Riemann Hypothesis.

Definition 1.4.1. The points $s = -2, -4, -6, ...$ are called the "trivial" zeros of the zeta function $\zeta(s)$ and the strip $\{s : 0 \leq \sigma \leq 1\}$ is called the critical strip.

1.5 The Riemann Hypothesis

If $s = \sigma + it$ is a zero of the zeta function in the critical strip then $\sigma = \frac{1}{2}$. There are infinite number of zeros on the line $\sigma = \frac{1}{2}$ called the critical line. But no one has been able to show that $\zeta(s)$ has any zeros off the critical line and no one has been able to show that all zeros must lie on the critical line.

Chapter 2

Hypergeometric Zeta Functions

Following Riemann many scholars have generalized the Riemann zeta function for example, Multiple zeta function, Hurwitz zeta function, Hypergeometric zeta functions and so on see [22],[1], [15], [10]. In this chapter we give an overview of the Hypergeometric Zeta functions. Here we mainly review on integral representation, series representation and analytic continuations. We also discuss on some open problems concerning these families of functions.

2.1 Introduction

This chapter investigates a new family of special functions referred to as hypergeometric zeta functions, introduced by Abdul Hassen and Hieu D. Nguyen see [10]. The hypergeometric zeta functions are derived from the integral representation of the classical Riemann zeta function. It is also found to exhibit many properties analogous to the classical zeta function. For example the relation between the zeta function and

Bernoulli numbers is known to be $\zeta(2n) = (-1)^{n-1}2^{2n-1}\pi^{2n}\frac{B_{2n}}{(2n)!}$, for $n = 1, 2, 3, \cdots$ see [10]. A classical property of $\zeta(s)$ is its evaluation at negative integers. Euler demonstrated that its values are expressible in terms of Bernoulli numbers:

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}.$$

Here the Bernoulli numbers B_n are generated by,

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

see [18],[10]. In the case of hypergeometric zeta functions, it is found that they can be similarly expressed in terms of generalized Bernoulli numbers. For example, when $N = 2$, it is shown that $\zeta_2(-n) = (-1)^{n+1}\frac{2B_{2,n+1}}{n(n+1)}$. The coefficient $B_{2,n}$ above are likewise generated by,

$$\frac{x^2}{2(e^x - 1 - x)} = \sum_{n=0}^{\infty} \frac{B_{2,n}}{n!} x^n$$

see [10]. Now following these we define the hypergeometric zeta functions formally and establish domain of convergence.

Definition 2.1.1. Let N be a natural number and $T_N(x) = \sum_{n=0}^{N} \frac{x^n}{n!}$ be the Taylor (Maclaurin) polynomial of the exponential function e^x . We define the N^{th} order hypergeometric zeta functions (or just N^{th} order zeta function) to be,

$$\zeta_N(s) = \frac{1}{\Gamma(s + N - 1)} \int_0^{\infty} \frac{x^{s+N-2}}{e^x - T_{N-1}(x)} dx.$$

Observe that $\zeta_1(s) = \zeta(s)$, i.e.; when $N = 1$, we get the classical zeta function.

Lemma 2.1.1. *$\zeta_N(s)$ converges absolutely for $\sigma > 1$.*

Proof. Let $K > 0$ such that $e^x \geq e^{\frac{x}{2}} + T_{N-1}(x)$ for all $x \geq K$. Which is equivalent to saying, $e^x - T_{N-1}(x) \geq e^{\frac{x}{2}}$. Thus for $\sigma > 1$, we have

$$\zeta_N(s) = \frac{1}{\Gamma(s+N-1)}\left[\int_0^K \frac{x^{s+N-2}}{e^x - T_{N-1}(x)}dx + \int_K^\infty \frac{x^{s+N-2}}{e^x - T_{N-1}(x)}dx\right].$$

Thus it follows that,

$$|\zeta_N(s)| \leq \left|\frac{1}{\Gamma(s+N-1)}\right| \left[\int_0^K \left|\frac{x^{s+N-2}}{e^x - T_{N-1}(x)}\right| dx + \int_K^\infty \left|\frac{x^{s+N-2}}{e^x - T_{N-1}(x)}\right|\right].$$

This implies that,

$$|\zeta_N(s)| \leq \left|\frac{1}{\Gamma(s+N-1)}\right| \left[\int_0^K N! x^{\sigma-2} dx + \int_K^\infty x^{\sigma+N-2} e^{\frac{x}{2}} dx\right].$$

Now using integration by substitution, this yields

$$|\zeta_N(s)| \leq \left|\frac{1}{\Gamma(s+N-1)}\right| \left[\frac{N! K^{\sigma-1}}{\sigma-1} + 2^{\sigma+N-1}\int_{2K}^\infty u^{\sigma+N-2} e^{-u} du\right].$$

It follows that,

$$|\zeta_N(s)| \leq \left|\frac{1}{\Gamma(s+N-1)}\right| \left[\frac{N! K^{\sigma-1}}{\sigma-1} + 2^{\sigma+N-1}\Gamma(\sigma+N-1)\right] < \infty.$$

□

Thus we have shown that the hypergeometric zeta functions converge absolutely for $\sigma > 1$ and converges uniformly for $\sigma \geq \sigma_0 > 1$. Therefore, the hypergeometric zeta functions are analytic on $\{s : \sigma > 1\}$, where $s = \sigma + it$ for σ and t real numbers.

2.2 Series Representation of Hypergeometric Zeta Functions

We have seen that the hypergeometric zeta functions are generalization of the integral representation of the classical zeta function. In this section we try to see series representation of the hypergeometric zeta functions. The following lemmas provide hypergeometric zeta functions with a series representation, which reduces formally to the harmonic series at $s = 1$.

Lemma 2.2.1. *For $\sigma > 1$, we have*

$$\zeta_N(s) = \sum_{n=1}^{\infty} f_n(N, s),$$

where

$$f_n(N, s) = \frac{1}{\Gamma(s+N-1)} \int_0^{\infty} x^{s+N-2} T_{N-1}^{n-1}(x) e^{-nx} dx.$$

Proof. Since $|T_{N-1}(x)e^{-x}| < 1$ for each $x > 0$, we have

$$\frac{x^{s+N-2}}{e^x - T_{N-1}(x)} = x^{s+N-2} e^{-x} \sum_{n=0}^{\infty} T_{N-1}^{n}(x) e^{-nx}.$$

This implies that

$$\frac{x^{s+N-2}}{e^x - T_{N-1}(x)} = x^{s+N-2} \sum_{n=1}^{\infty} T_{N-1}^{n-1}(x) e^{-nx}.$$

Thus,

$$\zeta_N(s) = \frac{1}{\Gamma(s+N-1)} \int_0^{\infty} \frac{x^{s+N-2}}{e^x - T_{N-1}(x)} dx.$$

Which implies that,

$$\zeta_N(s) = \frac{1}{\Gamma(s+N-1)} \int_0^\infty x^{s+N-2} \sum_{n=1}^{\infty} T_{N-1}^{n-1}(x) e^{-nx}.$$

The lemma now follows by reversing the order of integration and summation. □

Lemma 2.2.2. *For the function defined by* $f_n(N,s)$ *in lemma 2.2.1 we have* $f_n(N,1) = \frac{1}{n}$ *and hence, formally*

$$\sum_{n=1}^{\infty} f_n(N,1) = \sum_{n=1}^{\infty} \frac{1}{n}$$

which is the harmonic series.

Proof. Observe that, $x^{N-1} = (N-1)![T_{N-1}(x) - T_{N-2}(x)]$. Thus we have,

$$\begin{aligned} f_n(N,1) &= \frac{1}{(N-1)!} \int_0^\infty x^{N-1} T_{N-1}^{n-1}(x) e^{-nx} dx. \\ &= \int_0^\infty [T_{N-1}(x) - T_{N-2}(x)] x^{N-1} T_{N-1}^{n-1}(x) e^{-nx} dx. \\ &= \int_0^\infty T_{N-1}^{n}(x) e^{-nx} dx - \int_0^\infty T_{N-2}(x) T_{N-1}^{n-1}(x) e^{-nx} dx. \end{aligned}$$

Thus using integration by parts for the first integral we have the lemma. □

From lemma 2.2.2 it follows that $\zeta_N(1)$ formally generates the harmonic series for all positive integers N. This reveals the motivation for normalizing the gamma factor in $\zeta_N(s)$.

Lemma 2.2.3. *For* $\sigma > 1$ *we have*

$$\zeta_N(s) = \sum_{n=1}^{\infty} \frac{\mu_N(n,s)}{n^{s+N-1}},$$

where

$$\mu_N(n,s) = \sum_{k=0}^{(N-1)(n-1)} \frac{a_k(N,n)\Gamma(s+N+k-1)}{n^k\Gamma(s+N-1)},$$

here the $a_k(N,n)$ *is generated by*

$$(T_{N-1}(x))^{n-1} = (1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^{N-1}}{(N-1)!})^{n-1} = \sum_{k=0}^{(N-1)(n-1)} a_k(N,n)x^k.$$

The proof follows simply by using the definition of the hypergeometric zeta functions and using integration by substitution.

Lemma 2.2.4.

$$\mu_N(n,1) = n^{N-1}.$$

Proof. From lemma 2.2.1 and lemma 2.2.3 we have

$$f_n(N,s) = \frac{\mu_N(n,s)}{n^{s+N-1}}.$$

But then from lemma 2.2.2 $f_n(N,1) = \frac{1}{n}$, and hence the lemma follows. □

Theorem 2.2.5. *For* $N > 1$ *and* $\sigma > 1$ *we have* $\zeta_N(\sigma) > \zeta(\sigma)$.

Proof. $\mu_N(n,\sigma)$ is a strictly increasing function, since it is a polynomial with positive coefficients and positive domain. Thus $\mu_N(n,\sigma) > \mu_N(n,1)$. Hence, the lemma follows for $\sigma > 1$, by using the definition for $\zeta_N(\sigma)$ and $\zeta(\sigma)$. □

Remark 2.2.1. Observe that the coefficients $\mu_N(n,s)$ in the series representation of the hypergeometric zeta functions depend on both n and s. In this sense it is a generalized Dirichlet series. Of course, one would like to find an expression of $\mu_N(n,s)$ that allows him/her to write $\zeta_N(s)$ as an ordinary Dirichlet series. At the present moment, no body knows even for $N = 2$ if its series representation will lead to any such result. The other important property of the classical zeta function is the zero free region to the right half of the complex plane. Can we say something concerning zero free regions for $\zeta_N(s)$? But unlike the situation with classical zeta function there is no product formula for $\zeta_N(s)$ to take advantage of here in establishing a zero free region to the right half of the complex plane ([10]). The more difficult problem is finding functional equation or product representation of these functions remains open ([10]).

In the absence of a functional equation or product representation for $\zeta_N(s)$ it is suggested that one needs a good understanding of the properties of the "coefficients" $\mu_N(n,s)$ in order to investigate the existence of zeros to the right of the complex plane (see [10]).

2.3 Analytic Continuation of Hypergeometric Zeta Functions

We have seen that the hypergeometric zeta functions are analytic to the right half of the complex plane. Now we shall discuss on how to extend the domain of definition of these functions to the left of $\sigma = 1$. We can follow different approaches. But for the time being let us discuss on one of the methods which involves rewriting the integral representing of the hypergeometric zeta functions in stages to extend the domain of $\zeta_N(s)$ strip by strip. We have seen that for $\sigma > 1$ the hypergeometric zeta functions are given by,

$$\zeta_N(s) = \frac{1}{\Gamma(s+N-1)} \int_0^\infty \frac{x^{s+N-2}}{e^x - T_{N-1}(x)} dx.$$

This can be rewritten as:

$$\zeta_N(s) = \frac{1}{\Gamma(s+N-1)} \left[\int_0^1 \frac{x^{s+N-2}}{e^x - T_{N-1}(x)} dx + \int_1^\infty \frac{x^{s+N-2}}{e^x - T_{N-1}(x)} dx \right].$$

Now making use of the Laurent series expansion of $\frac{1}{e^w - T_{N-1}(w)}$ and the generalized Bernoulli numbers we have the following (see [10], [12]):

$$\begin{aligned}
\zeta_N(s) &= \frac{1}{\Gamma(s+N-1)}\left[\int_0^1 \frac{x^{s+N-2}}{e^x - T_{N-1}(x)}dx + \int_1^\infty \frac{x^{s+N-2}}{e^x - T_{N-1}(x)}dx\right]. \\
&= \frac{1}{\Gamma(s+N-1)}\left[\int_0^1 \left[\frac{1}{e^x - T_{N-1}(x)} - \frac{N!}{x^N}\right] x^{s+N-2}dx\right. \\
&\qquad \left. + \int_0^1 N! x^{s-2}dx + \int_1^\infty \frac{x^{s+N-2}}{e^x - T_{N-1}(x)}dx\right]. \\
&= \frac{1}{\Gamma(s+N-1)}\left[\int_0^1 \left[\frac{1}{e^x - T_{N-1}(x)} - \frac{N!}{x^N}\right] x^{s+N-2}dx\right. \\
&\qquad \left. + \frac{N!}{s-1} + \int_1^\infty \frac{x^{s+N-2}}{e^x - T_{N-1}(x)}dx\right]. \\
&= \frac{1}{\Gamma(s+N-1)}\left[\int_0^1 \left[\frac{1}{e^x - T_{N-1}(x)} - \frac{N!}{x^N}\right] x^{s+N-2}dx\right. \\
&\qquad \left. - \int_1^\infty N! x^{s-2}dx + \int_1^\infty \frac{x^{s+N-2}}{e^x - T_{N-1}(x)}dx\right]. \\
&= \frac{1}{\Gamma(s+N-1)}\int_0^\infty \left[\frac{1}{e^x - T_{N-1}(x)} - \frac{N!}{x^N}\right] x^{s+N-2}dx.
\end{aligned}$$

This series converges for $0 < \sigma < 1$. Observe that $s = 1$ is a simple pole for the hypergeometric zeta functions in the first strip. Now we use this representation of $\zeta_N(s)$ as a new definition for $0 < \sigma < 1$ and write as,

$$\begin{aligned}
\zeta_N(s) &= \frac{1}{\Gamma(s+N-1)}\left[\int_0^1 \left[\frac{1}{e^x - T_{N-1}(x)} - \frac{N!}{x^N}\right] x^{s+N-2}dx \right.\\
&\qquad \left. + \int_1^\infty \left[\frac{1}{e^x - T_{N-1}(x)} - \frac{N!}{x^N}\right] x^{s+N-2}dx\right].\\
&= \frac{1}{\Gamma(s+N-1)}\left[\int_0^1 \left[\frac{1}{e^x - T_{N-1}(x)} - \frac{N!}{x^N} + \frac{N!x^{1-N}}{N+1}\right] x^{s+N-2}dx \right.\\
&\qquad \left. - \int_0^1 \frac{N!x^{s-1}}{N+1}dx + \int_1^\infty \left[\frac{1}{e^x - T_{N-1}(x)} - \frac{N!}{x^N}\right] x^{s+N-2}dx\right].\\
&= \frac{1}{\Gamma(s+N-1)}\left[\int_0^1 \left[\frac{1}{e^x - T_{N-1}(x)} - \frac{N!}{x^N} + \frac{N!x^{1-N}}{N+1}\right] x^{s+N-2}dx \right.\\
&\qquad \left. - \frac{N!}{s(N+1)} + \int_1^\infty \left[\frac{1}{e^x - T_{N-1}(x)} - \frac{N!}{x^N}\right] x^{s+N-2}dx\right].
\end{aligned}$$

But for $-1 < \sigma < 0$ we have,

$$-\frac{N!}{s(N+1)} = \frac{N!}{N+1}\int_1^\infty x^{s-1}dx.$$

Hence we have the following representation of $\zeta_N(s)$ for $-1 < \sigma < 0$:

$$\zeta_N(s) = \frac{1}{\Gamma(s+N-1)}\int_0^\infty \left[\frac{1}{e^x - T_{N-1}(x)} - \frac{N!}{x^N} + \frac{N!x^{1-N}}{N+1}\right] x^{s+N-2}dx.$$

Observe that $s = 0$ is produced as a simple pole in the second strip for the hypergeometric zeta functions. Now this process can be repeated to extend $\zeta_N(s)$ analytically to the whole complex plane. But it may appear that $\zeta_N(s)$ has an infinite number of poles, since each application produces a pole in each strip, however after N repetitions the poles of $\Gamma(s+N-1)$ begin to make their appearance there by canceling the poles of $\zeta_N(s)$. Hence $\zeta_N(s)$ has at most a finite number of poles. Actually the poles are finite and, they are at integers $s = 1, 0, -1, -2, ..., 2-N$.

We can also consider the approach which uses contour integral and follow Riemann by using contour integral to develop the analytic continuation of the hypergeometric zeta functions. This method not only allows the statement about finite pole of the hypergeometric zeta functions but also make explicit the role of its residues in determining the values of $\zeta_N(s)$ at negative integers. To this end one can consider the integral,

$$I_N(s) = \frac{1}{2\pi i}\int_C \frac{(-w)^{s+N-1}}{e^w - T_{N-1}}\frac{dw}{w}$$

where the contour C starts at infinity on the positive real axis, encircles the origin once in counterclockwise, excluding the points $2\pi in$ for an integer n. In doing this we arrive at the results analogous to the classical Riemann zeta function (see [10]).

2.4 Some Open Problems Related to Hypergeometric Zeta Functions

In the previous sections we have seen that the hypergeometric zeta functions as a generalization of the integral representation of the classical zeta function. We have also seen series representation of the hypergeometric zeta functions. But the following problems remain open:

1. Do the hypergeometric zeta functions admit a Dirichlet series?

2. If so do these series admit functional equation?

3. Do they admit product formula?

4. What are the zeros of the hypergeometric zeta functions?

5. Do they have zero free region in the right half plane?

Chapter 3

Some Results Involving Series Representation of $\zeta_2(s)$

In this chapter we will prove some results involving series representation of the second order hypergeometric zeta function. It will be shown that this function is expressible as a power series whose coefficients are Dirichlet series. Using this Dirichlet series representation we also show that the second order hypergeometric zeta function has zero free region in a right half of the complex plane.

3.1 Introduction

As mentioned in chapter one, the classical Riemann zeta function has been the subject of an enormous amount of mathematical research since its introduction. The analysis of the zeta function has had a profound effect on number theory and this has in turn inspired more work on the zeta function. The different representation of the classical zeta function can be summarized as follows:

1. Series Representation:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\sigma > 1$ and is analytic there. It is a typical example of a Dirichlet series.

2. Euler product representation:

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1}$$

where the product is over all prime numbers p. This is very important to identify its zero free regions in the right half of the complex plane, $\sigma > 1$.

3. Integral Representation:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx.$$

This representation is very important in continuing the zeta function analytically to the whole complex plane except for a simple pole at $s = 1$. It also reveals Riemann's most celebrated functional equation.

4. Functional Equation:

$$\zeta(s) = 2(2\pi)^{s-1} \zeta(1-s)\Gamma(1-s) \sin\left(\frac{\pi s}{2}\right).$$

It is important to locate zeros to the left half of the complex plane $\{s : \sigma < 1\}$.

In chapter two, we have seen that, following Riemann, Hassen and Nguyen introduced and investigated a generalization of the Riemann zeta function by replacing the denominator $e^x - 1$ in the integral representation with arbitrary Taylor difference $e^x - T_{N-1}(x)$, where N is a positive integer and $T_{N-1}(x)$ is the Taylor polynomial of e^x at the origin of degree $N-1$. This defines a family of what Hassen and Nguyen called hypergeometric zeta functions denoted by $\zeta_N(s)$ and defined by:

$$\zeta_N(s) = \frac{1}{\Gamma(s+N-1)} \int_0^\infty \frac{x^{s+N-2}}{e^x - T_{N-1}(x)} dx.$$

Observe that $\zeta_1(s) = \zeta(s)$, i.e.; when $N = 1$, we get the classical zeta function. In ([10]) Hassen and Nguyen established many of the properties of the classical zeta function hold for the hypergeometric zeta functions. But finding product formula and functional equation remain an open problem. They also demonstrated series representation of the hypergeometric zeta functions with coefficient $\mu_N(n, s)$ as follows: For $\sigma > 1$

$$\zeta_N(s) = \sum_{n=1}^{\infty} \frac{\mu_N(n,s)}{n^{s+N-1}},$$

where

$$\mu_N(n,s) = \sum_{k=0}^{(N-1)(n-1)} \frac{a_k(N,n)\Gamma(s+N+k-1)}{n^k \Gamma(s+N-1)}.$$

Here the $a_k(N,n)$ is generated by

$$(T_{N-1}(x))^{n-1} = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{N-1}}{(N-1)!}\right)^{n-1} = \sum_{k=0}^{(N-1)(n-1)} a_k(N,n)x^k.$$

But whether or not this series representation of $\zeta_N(s)$ is a generalization of the Dirichlet series; whether it has zero free region in the right half of the complex plane remain open problems (see [10]). In this chapter we will show the existence of zero free region for the second order hypergeometric zeta function in a right half of the complex plane.

Hassen and Nguyen developed the analytic continuation of the hypergeometric zeta functions $\zeta_N(s)$ to the entire complex plane, except for N simple poles at $s = 1, 0, -1, ..., 2-N$, and established many properties of $\zeta_N(s)$ analogous to those satisfied by the classical zeta function. But finding product formula, functional equation and whether the series representation demonstrated by Hassen and Nguyen is a generalized Dirichlet series or not are all remain open problems. As Hassen and Nguyen discussed in ([10]) in the absence of a product formula and a functional equation it is better to understand the properties of the coefficient $\mu_N(n, s)$ in order to investigate the existence of zeros to the right of the complex plane for $N > 1$ in general and for $N = 2$ in particular. The expression defining the coefficient $\mu_N(n, s)$ is implicit. One of our objective here is to make the expression defining the coefficient $\mu_N(n, s)$ explicit and write the hypergeometric zeta functions as a series whose coefficient is explicit. In this section we use this series representation to show that the second order hypergeometric zeta function has zero free region in a right half of the complex plane.

The coefficients $\mu_N(n,s)$ in the series representation of $\zeta_N(s)$ depends on both n and s. In this sense it is a generalized Dirichlet series as suggested in [10] . Of course, one would like to find an expression of $\mu_N(n,s)$ that allows him/her to write $\zeta_N(s)$ as an ordinary Dirichlet series. At the present moment, no body knows even for $N = 2$ if its series representation will lead to any such result ([10]) . As we have seen the Euler product formula and the functional equation are important tools to determine zero free regions in the complex plane for the classical zeta function. But unlike the situation with classical zeta function there is no product formula for $\zeta_N(s)$ to take advantage of here in establishing a zero free region to the right half of the complex plane. Here it is the series representation which reveals the existence of zero free region in the right half of the complex plane.

3.2 Preliminaries

In ([10], [11]) Hassen and Nguyen established many of the properties of the classical zeta function for the hypergeometric zeta functions. However, the hypergeometric zeta functions do not appear to have a product formula. The zero of the $e^x - T_{N-1}(x)$ can be approximated but cannot be found precisely. This makes it difficult to expect a functional equation. On a right half of the complex plane, the hypergeometric zeta functions can be represented in the form:

$$\zeta_N(s) = \sum_{n=1}^{\infty} \frac{\mu_N(n,s)}{n^{s+N-1}}, \tag{3.2.1}$$

for $\sigma > 1$, where

$$\mu_N(n,s) = \sum_{k=0}^{(N-1)(n-1)} \frac{a_k(N,n)\Gamma(s+N+k-1)}{n^k\Gamma(s+N-1)} \tag{3.2.2}$$

and $a_k(N,n)$ is generated by

$$(T_{N-1}(x))^{n-1} = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{N-1}}{(N-1)!}\right)^{n-1} = \sum_{k=0}^{(N-1)(n-1)} a_k(N,n)x^k.$$

Following Riemann, it is possible, as Hassen and Nguyen demonstrated, to extend the hypergeometric zeta functions to a left half of the complex plane. In fact, it has been shown that the hypergeometric zeta functions $\zeta_N(s)$ can be extended analytically to the entire complex plane, except for N simple poles at $s = 1, 0, -1, ..., 2-N$.

The coefficients $\mu_N(n,s)$ in the series representation of $\zeta_N(s)$ depend on both n and s. It would be desirable to find an expression of $\mu_N(n,s)$ that allows us to write $\zeta_N(s)$ as a linear combination of ordinary Dirichlet series. It is the objective of this chapter to investigate the properties of the coefficients $\mu_2(n,s)$ and write $\zeta_2(s)$ as a "power" series in s with Dirichlet series as coefficients. To this end, we will find an explicit form of the coefficients of these polynomials and rewrite $\zeta_2(s)$.

We note that the expression $\frac{\Gamma(s+N+k-1)}{\Gamma(s+N-1)}$ which appears in the definition of the coefficient of the hypergeometric zeta function of order N as given in(3.2.2) is just $(s+N-1)_k$:

$$\frac{\Gamma(s+N+k-1)}{\Gamma(s+N-1)} = (s+N-1)_k$$

where $(s+a)_k$ is Pochhammer symbol:

$$(s+a)_k = (s+a)(s+a+1)\cdots(s+a+k-1),$$

with initial value $(s+a)_0 = 1$. Moreover, the following recursive relation holds:

$$(s+a)_{k+1} = (s+a+k)(s+a)_k.$$

To express the polynomials $\mu_2(n,s)$ explicitly, we need to define a sequence of numbers recursively. For $m = 1, 2, \cdots$, and $k = m+1, m+2, m+3, \cdots$, we define

$$A_k^m = kA_{k-1}^m + A_{k-1}^{m-1}, \tag{3.2.3}$$

with initial values

$$A_k^0 = 0, \quad A_{k-1}^k = 1.$$

Note then that

$$A_k^1 = k! \quad \text{and} \quad A_k^k = 1+2+3+\cdots+k = \frac{k(k+1)}{2}. \tag{3.2.4}$$

We note that the recurrence relation above is the same as that of the unsigned Stirling numbers which are denoted by $\begin{bmatrix} k \\ m \end{bmatrix}$ and is given by

$$\begin{bmatrix} k \\ m \end{bmatrix} = k \begin{bmatrix} k-1 \\ m \end{bmatrix} + \begin{bmatrix} k-1 \\ m-1 \end{bmatrix}.$$

Lemma 3.2.1. $(s+1)_k$ *is a polynomial of degree* k *given as follows:*

$$(s+1)_k = s^k + A_k^k s^{k-1} + A_k^{k-1} s^{k-2} + A_k^{k-2} s^{k-3} + \cdots + A_k^2 s + A_k^1.$$

Proof. We shall use induction on k.

For $k=0$ we have $(s+1)_0 = 1 = A_0^1 s^0$, and assume the assertion is true for k:

$$(s+1)_k = s^k + A_k^k s^{k-1} + A_k^{k-1} s^{k-2} + A_k^{k-2} s^{k-3} + \cdots + A_k^2 s + A_k^1.$$

We first observe that the leading coefficient in $(s+1)_{k+1}$ is 1.

From the Pochhammer symbol since we have,

$$(s+1)_{k+1} = (s+1)_k (s+k+1)$$

the coefficient of s^{k-j} in $(s+1)_{k+1}$ is the sum of the coefficient of s^{k-j-1} in $(s+1)_k$ and $k+1$ times the coefficient of s^{k-j} in $(s+1)_k$. From the induction hypothesis the coefficient of s^{k-j-1} in $(s+1)_k$ is A_k^{k-j} and the coefficient of s^{k-j} in $(s+1)_k$ is A_k^{k+1-j}.

Therefore, the coefficient of s^{k-j} in $(s+1)_{k+1}$ is $(k+1)(A_k^{k+1-j}) + A_k^{k-j} = A_{k+1}^{k+1-j}$, as desired. $\square$

Returning to the series representation of the second order hypergeometric zeta functions given in (3.2.1), we note that for $N = 2$ we have,

$$\zeta_2(s) = \sum_{n=1}^{\infty} \frac{\mu_2(n, s)}{n^{s+1}},$$

where

$$\mu_2(n, s) = \sum_{k=0}^{n-1} \frac{a_k(2, n)\Gamma(s + k + 1)}{n^k \Gamma(s + 1)}$$

and the $a_k(2, n)$ is generated by,

$$(T_1(x))^{n-1} = (1 + x)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} x^k.$$

Thus,

$$\mu_2(n, s) = \sum_{k=0}^{n-1} \frac{\binom{n-1}{k}\Gamma(s + k + 1)}{n^k \Gamma(s + 1)} = \sum_{k=0}^{n-1} \frac{\binom{n-1}{k}(s + 1)_k}{n^k}. \tag{3.2.5}$$

Now we want to write these coefficients of the hypergeometric zeta functions as a polynomial whose coefficient is explicit as the following lemma shows:

Lemma 3.2.2. $\mu_2(n, s)$ *can be written as a polynomial of degree "$n - 1$" with its explicit coefficients as given below:*

$$\mu_2(n, s) = \sum_{m=1}^{n-1} b_{nm} s^{m-1}$$

where

$$b_{nm} = \sum_{j=m-1}^{n-1} n^{-j} \binom{n-1}{j} A_j^m.$$

Proof. To see this we begin from the very definition of $\mu_2(n,s)$,

$$\mu_2(n,s) = \sum_{k=0}^{n-1} n^{-k}\binom{n-1}{k}(s+1)_k.$$

By Lemma 1.1, the coefficients of s^{m-1} in $(s+1)_k$ is

$$\sum_{j=m-1}^{n-1} A_j^m$$

for each $m = 1, 2, \cdots$ and hence the coefficient of s^{m-1} in

$$\mu_2(n,s) = \sum_{k=0}^{n-1} n^{-k}\binom{n-1}{k}(s+1)_k$$

becomes

$$\sum_{j=m-1}^{n-1} n^{-j}\binom{n-1}{j}A_j^m$$

which is equal to b_{nm}. □

As an example we can have the first few $\mu_2(n,s)$ as follows:

$$\mu_2(1,s) = 1 = b_{11}.$$

$$\mu_2(2,s) = 1 + 2^{-1} + 2^{-1}s = b_{21} + b_{22}s.$$

$$\begin{aligned}\mu_2(3,s) &= 1 + \binom{2}{1}3^{-1} + \binom{2}{2}2!3^{-2} + [\binom{2}{1}3^{-1} + \binom{2}{2}3^{-2}A_2^2]s + \binom{2}{2}3^{-2}s^2 \\ &= b_{31} + b_{32}s + b_{33}s^2.\end{aligned}$$

$$\begin{aligned}\mu_2(4,s) &= 1+\binom{3}{1}4^{-1}+\binom{3}{2}2!4^{-2}+\binom{3}{3}3!4^{-3}+\left[\binom{3}{1}4^{-1}+\binom{3}{2}4^{-2}A_2^2+\binom{3}{3}4^{-3}A_3^2\right]s\\ &\quad+\left[\binom{3}{2}4^{-2}+\binom{3}{3}4^{-3}A_3^3\right]s^2+\binom{3}{3}4^{-3}s^3\\ &= b_{41}+b_{42}s+b_{43}s^2+b_{44}s^3.\end{aligned}$$

Observe that

$$\mu_2(4,s)=\sum_{m=1}^{3} b_{4m}s^{m-1},$$

where

$$b_{4m}=\sum_{j=m-1}^{3} 4^{-j}\binom{3}{j}A_j^m.$$

3.3 Series Representation

We are now in a position to state and prove our main result:

Theorem 3.3.1. *The hypergeometric zeta function of order* 2 *can be rewritten as follows*

$$\zeta_2(s)=\sum_{m=1}^{\infty} D_m(s)s^{m-1},$$

where the coefficient $D_m(s)$ *is a Dirichlet series for each* $m=1,2,3,\cdots$.

Proof. We can rewrite $\zeta_2(s)$ as follows:

$$\begin{aligned}
\zeta_2(s) &= \sum_{n=1}^{\infty} \frac{\mu_2(n,s)}{n^{s+1}} \\
&= \mu_2(1,s) + \frac{\mu_2(2,s)}{2^{s+1}} + \frac{\mu_2(3,s)}{3^{s+1}} + \frac{\mu_2(4,s)}{4^{s+1}} + \cdots \\
&= 1 + \frac{1+2^{-1}+2^{-1}s}{2^{s+1}} + \frac{1+\binom{2}{1}3^{-1}+\binom{2}{2}2!3^{-2}+[\binom{2}{1}3^{-1}+\binom{2}{2}3^{-2}A_2^2]s+\binom{2}{2}3^{-2}s^2}{3^{s+1}} + \cdots \\
&= \frac{b_{11}}{1^{s+1}} + \frac{b_{21}+b_{22}s}{2^{s+1}} + \frac{b_{31}+b_{32}s+b_{33}s^2}{3^{s+1}} + \cdots \\
&= \frac{b_{11}}{1^{s+1}} + \frac{b_{21}}{2^{s+1}} + \frac{b_{31}}{3^{s+1}} + \cdots \\
&\quad + \left[\frac{b_{22}}{2^{s+1}} + \frac{b_{32}}{3^{s+1}} + \frac{b_{42}}{4^{s+1}} + \cdots\right] s + \cdots + \left[\frac{b_{nj}}{n^{s+1}} + \frac{b_{(n+1)j}}{(n+1)^{s+1}} + \cdots\right] s^{j-1} + \cdots \\
&= \sum_{n=1}^{\infty} \frac{b_{n1}}{n^{s+1}} + s\sum_{n=2}^{\infty} \frac{b_{n2}}{n^{s+1}} + s^2\sum_{n=3}^{\infty} \frac{b_{n3}}{n^{s+1}} + \cdots + s^{j-1}\sum_{n=j}^{\infty} \frac{b_{nj}}{n^{s+1}} + \cdots \\
&= \sum_{n=1}^{\infty} \frac{n^{-1}b_{n1}}{n^{s}} + s\sum_{n=2}^{\infty} \frac{n^{-1}b_{n2}}{n^{s}} + s^2\sum_{n=3}^{\infty} \frac{n^{-1}b_{n3}}{n^{s}} + \cdots + s^{j-1}\sum_{n=j}^{\infty} \frac{n^{-1}b_{nj}}{n^{s}} + \cdots .
\end{aligned}$$

Now we put,

$$D_1(s) = \sum_{n=1}^{\infty} \frac{n^{-1}b_{n1}}{n^s},$$

$$D_2(s) = \sum_{n=2}^{\infty} \frac{n^{-1}b_{n2}}{n^s},$$

$$D_3(s) = \sum_{n=3}^{\infty} \frac{n^{-1} b_{n3}}{n^s}.$$

So in general let $D_m(s)$ be the coefficient of s^{m-1} for each $m = 1, 2, 3, \cdots$, then,

$$D_m(s) = \sum_{n=m}^{\infty} \frac{n^{-1} b_{nm}}{n^s}.$$

Therefore,

$$\zeta_2(s) = \sum_{m=1}^{\infty} D_m(s) s^{m-1}.$$

This completes the proof. $\square$

For notational convenience, let us define

$$a_{nm} \quad = \quad \sum_{k=m-1}^{n-1} \binom{n-1}{k} A_k^m n^{-(k+1)} = \frac{b_{nm}}{n} \tag{3.3.1}$$

for each $m = 1, 2, 3, \cdots$, so that the Dirichlet series $D_m(s)$ can be expressed as

$$D_m(s) \quad = \quad \sum_{n=m}^{\infty} \frac{\sum_{k=m-1}^{n-1} \binom{n-1}{k} A_k^m n^{-(k+1)}}{n^s} = \sum_{n=m}^{\infty} \frac{n^{-1} b_{nm}}{n^s} = \sum_{n=m}^{\infty} \frac{a_{nm}}{n^s}, \tag{3.3.2}$$

where the A_k^m are given by (3.2.3).

We observe that the first few Dirichlet series given in (3.3.2) are

$$D_1(s) = \sum_{n=1}^{\infty} \frac{a_{n1}}{n^s}.$$

$$D_2(s) = \sum_{n=2}^{\infty} \frac{a_{n2}}{n^s}.$$

$$D_3(s) = \sum_{n=3}^{\infty} \frac{a_{n3}}{n^s}.$$

It is also interesting to list some few coefficients a_{nm} and look at what they represent as a remark.

Remark 3.3.1. The first few values of a_{nm} are given by,

$$a_{11} = 1, \quad a_{21} = \tfrac{3}{4}, \quad a_{31} = \tfrac{17}{27}, \quad a_{41} = \tfrac{142}{256}$$

$$a_{22} = \tfrac{1}{4}, \quad a_{32} = \tfrac{1}{3}, \quad a_{42} = \tfrac{95}{256}, \quad a_{52} = \tfrac{1220}{3125}$$

$$a_{33} = \tfrac{1}{27}, \quad a_{43} = \tfrac{18}{256}, \quad a_{53} = \tfrac{305}{3125}$$

We note that

$$a_{n1} = \sum_{k=0}^{n-1} \binom{n-1}{k} A_k^1 n^{-(k+1)} = \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{k!}{n^{k+1}}, \tag{3.3.3}$$

where we have used (3.2.4) in the last equality. It is interesting to note that a_{n1} has a closed form given by

$$a_{n1} = \frac{e^n \Gamma[n, n]}{n^n},$$

where $\Gamma[a, z]$ is the upper incomplete gamma functions. This can proven by mathematical induction. The number a_{n1} is the probability of selecting a ball from an urn containing n different balls, with replacement until exactly one ball has been selected twice and that ball was also the first ball selected once. Further more the sequence $\{n^n a_{n1}\}$ begins as

$$1, 3, 17, 142, 1569, 21576, 355081, 6805296, 148869153, 3660215680, \ 99920609601, \cdots$$

and is listed as $A001865$ in the On-Line Encyclopedia of Integer Sequences (OEIS) [16]. This sequence represents the number of connected functions on n-labeled nodes as indicated in OEIS. We also mention here that the sequence $\{n^n a_{n2}\}$ appears as $A065456$ on OEIS([17]), and it is the number of functions on n-labeled nodes whose representation as a digraph has two components. However, we have not seen a list that corresponds to other sequences $\{a_{nm}\}$ for $m \geq 3$ we have here. We will explore this in a future work.

Finally, if we define $a_{nm} = 0$ for $n < m$, then the Dirichlet series given in (3.3.2) is an ordinary Dirichlet series:

$$D_m(s) = \sum_{n=1}^{\infty} \frac{a_{nm}}{n^s}.$$

We do not know if these Dirichlet series have functional equations. We also note that each of the coefficients has the following relations on the real line:

$$\cdots, D_3(\sigma) < D_2(\sigma) < D_1(\sigma) < \zeta(\sigma) < \zeta_2(\sigma).$$

Moreover,

$$D_2(\sigma) < \frac{1}{4}\zeta(\sigma).$$

$$D_3(\sigma) < \frac{1}{27}\zeta(\sigma).$$

$$D_4(\sigma) < \frac{1}{256}\zeta(\sigma).$$

3.4 Zero Free Region in a Right half-plane

Zero free regions of the hypergeometric zeta functions of order 2 and order 3 on the left half of the complex plane was established by Hassen and Nguyen in [13]. In this section we will establish a zero free region for $\zeta_2(s)$ in the right half of the complex plane. In the case of the classical Riemann Zeta function, the Euler product formula

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

can be used to conclude that it is zero free for $\sigma > 1$(see [5],[19], [20] and [21]). The hypergeometric zeta functions are not known to have such a product formula. Furthermore, due to lack of knowledge of the precise locations of the zeros of $e^z - 1 - z = 0$, we do not have a functional equation. However one can use the Cauchy theory to express $\zeta_2(s)$ in terms of a series that involves the roots of $e^z - 1 - z = 0$ and establish zero free region on the left half of the complex plane. (See [13] for details.)

It follows from Theorem 2.1 that the second order hypergeometric zeta function has no real zeros if $\sigma > 1$ where $s = \sigma + it$. The following result extends this domain to $\sigma > 0$:

Theorem 3.4.1. *$\zeta_2(s) \neq 0$ for $s = \sigma > 0$*

Proof. As remarked above, we need only to show that $\zeta_2(\sigma) \neq 0$ for $0 < \sigma < 1$. For this we note that when $N = 2$, its integral representation becomes

$$\zeta_2(s) = \frac{1}{\Gamma(s+1)} \int_0^\infty \frac{x^s}{e^x - 1 - x} dx \tag{3.4.1}$$

and this can be rewritten as

$$\Gamma(s+1)\zeta_2(s) = \int_0^1 \frac{x^s}{e^x - 1 - x}\, dx + \int_1^\infty \frac{x^s}{e^x - 1 - x}\, dx \tag{3.4.2}$$

$$= \int_0^1 \left(\frac{1}{e^x - 1 - x} - \frac{2}{x^2} \right) x^s\, dx + \frac{2}{s-1} + \int_1^\infty \frac{x^s}{e^x - 1 - x} \tag{3.4.3}$$

The last formula in (3.4.3) is analytic in the strip $0 < \sigma \leq 1$, except for the pole at $s = 1$, since both integrals on the right hand side are convergent on this domain. Moreover, for $0 < \sigma < 1$,

$$\frac{1}{s-1} = -\int_1^\infty \frac{x^s}{x^2} dx.$$

Therefore, we can rewrite

$$\zeta_2(s) = \frac{1}{\Gamma(s+1)} \int_0^\infty \left(\frac{1}{e^x - 1 - x} - \frac{2}{x^2} \right) x^s\, dx.$$

Since $e^x > 1 + x + \frac{x^2}{2}$ for all $x > 0$, we see that

$$\left(\frac{1}{e^x - 1 - x} - \frac{2}{x^2} \right) x^\sigma < 0$$

for $\sigma > 0$ and all $x > 0$.

So the result follows. □

Now we show the second order zeta function has zero free region to the right half of the complex plane by showing that the limit of $\zeta_2(s)$ as the real part tends to infinity converges to 1 uniformly with some restrictions on the imaginary part. This is the content of the following theorem:

Theorem 3.4.2. *If* $F(s) = \zeta_2(s) - D_1(s)$*, where* $D_1(s)$ *is given by* (3.3.2)*, then*

$$\lim_{\sigma\to\infty} |F(s)| = 0$$

uniformly in t*, where* $s = \sigma + it$ *satisfies the inequality* $|s| < C\sigma$*, for some constant* C*.*

Proof. With

$$a_{nm} = \sum_{k=m-1}^{n-1} \binom{n-1}{k} A_k^m n^{-(k+1)}$$

we have,

$$D_m(s) = \sum_{n=m}^{\infty} \frac{a_{nm}}{n^s}.$$

We now use triangle inequality to obtain

$$\begin{aligned}
|F(s)| &= |\zeta_2(s) - D_1(s)| \\
&= \left|D_2(s)s + D_3(s)s^2 + D_4(s)s^3 + \cdots\right| \\
&\le |D_2(s)s| + \left|D_3(s)s^2\right| + \left|D_4(s)s^3\right| + \cdots \\
&\le \left|\frac{a_{22}}{2^s} + \frac{a_{32}}{3^s} + \frac{a_{42}}{4^s} + \cdots\right| |s| + \left|\frac{a_{33}}{3^s} + \frac{a_{43}}{4^s} + \frac{a_{53}}{5^s} + \cdots\right| |s|^2 + \cdots \\
&\le \left(\frac{a_{22}}{2^\sigma} + \frac{a_{32}}{3^\sigma} + \frac{a_{42}}{4^\sigma} + \cdots\right) |s| + \left(\frac{a_{33}}{3^\sigma} + \frac{a_{43}}{4^\sigma} + \frac{a_{53}}{5^\sigma} + \cdots\right) |s|^2 + \cdots \\
&< \frac{C\sigma}{2^\sigma}\left(a_{22} + \frac{2^\sigma}{3^\sigma}a_{32} + \cdots\right) + \frac{(C\sigma)^2}{3^\sigma}\left(a_{33} + \frac{3^\sigma}{4^\sigma}a_{43} + \cdots\right) + \cdots .
\end{aligned}$$

From this we get that,

$$\lim_{\sigma\to\infty} |F(s)| \le 0.$$

Hence,

$$\lim_{\sigma\to\infty} F(s) = 0.$$

□

We also note that the constant C is larger than 1.

Corollary 3.4.3. *Within the restriction given in Theorem 3.2,*

$$\lim_{\sigma\to\infty} |\zeta_2(s)| = 1.$$

Proof. Since $\lim_{\sigma\to\infty} |F(s)| = 0$ and $\lim_{\sigma\to\infty} |D(s)| = 1$, we have

$$\begin{aligned} |D_1(s)| &= |D_1(s) - \zeta_2(s) + \zeta_2(s)| \\ &\leq |D_1(s) - \zeta_2(s)| + |\zeta_2(s)|. \end{aligned}$$

Hence

$$\begin{aligned} 1 = \lim_{\sigma\to\infty} |D_1(s)| &\leq \lim_{\sigma\to\infty} |D_1(s) - \zeta_2(s)| + \lim_{\sigma\to\infty} |\zeta_2(s)| \\ &\leq \lim_{\sigma\to\infty} |\zeta_2(s)|. \end{aligned}$$

On the other hand,

$$\begin{aligned} |\zeta_2(s)| &= |D_1(s) - D_1(s) + \zeta_2(s)| \\ &\leq |D_1(s) - \zeta_2(s)| + |D_1(s)|. \end{aligned}$$

Thus we have,

$$\lim_{\sigma\to\infty} |\zeta_2(s)| \leq \lim_{\sigma\to\infty} |D_1(s) - \zeta_2(s)| + \lim_{\sigma\to\infty} |D_1(s)|$$

$$\leq \lim_{\sigma\to\infty} |\zeta_2(s)| \leq 1.$$

Therefore, we have,

$$\lim_{\sigma\to\infty} |\zeta_2(s)| = 1.$$

□

We note that the condition $|s| < C\sigma$ in the the above theorem can be strengthened to $|s| < \sigma^{\gamma}$ for any $\gamma > 1$. This zero free region is shown roughly as in the figure. We conjecture, based on numerical evidence, that there is a $\sigma_0 > 1$ such that $\zeta_2(s) \neq 0$ for all s for which $\sigma > \sigma_0$. We will return to this conjecture in the future work. We also expect similar result for other hypergeometric zeta function of order N.

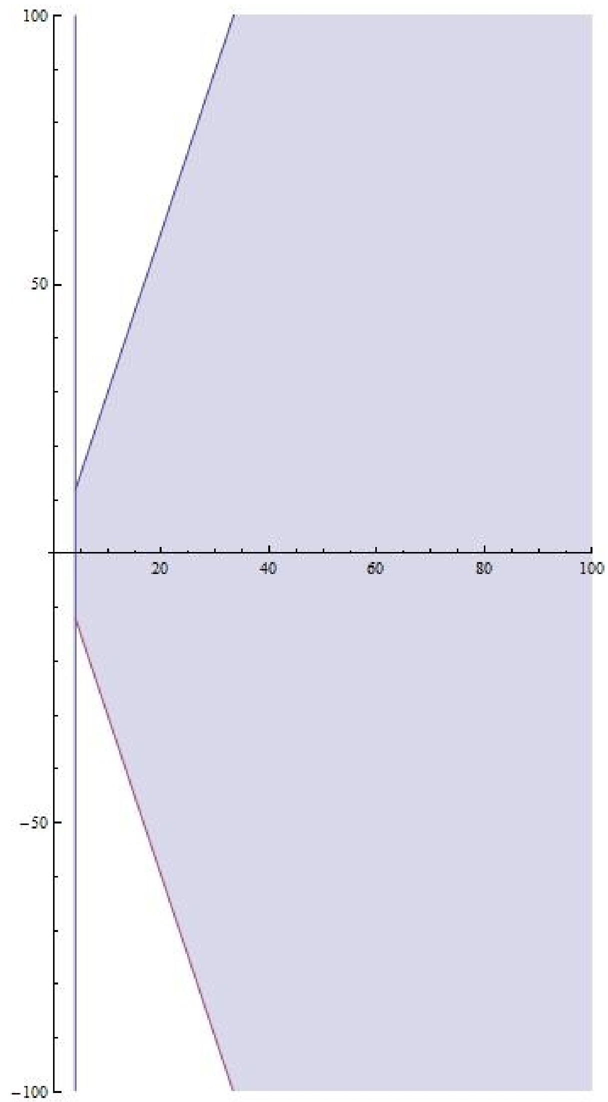

Figure 3.1: A Zero Free Region in the Right Half Plane

Chapter 4

Fractional Hypergeometric Zeta Functions

In [10] Abdul Hassen and Heiu D. Nguyen investigated a generalization of the Riemann zeta function based on its integral representation. They also investigated a continuous version of hypergeometric zeta functions by generalizing to all real positive values of N in [11]. But in [11] they focused only on $N = 1/2$ which they called it the error zeta function. In this chapter we tried to investigate a continuous version of the hypergeometric zeta functions for any real positive number "a", demonstrate the analytic continuation using strip-by-strip method for any real positive number "a" and finally we use the contour integral for half odd integers, that is for $a = \frac{2N+1}{2}$, where N is a nonnegative integer.

4.1 Introduction

In chapter 2 we have seen that, Hassen and Nguyen introduced and investigated a generalization of the Riemann zeta function by replacing $e^x - 1$ the denominator in the integral representation with arbitrary Taylor difference $e^x - T_{N-1}(x)$, where N is a positive integer and $T_{N-1}(x)$ is the Taylor polynomial of e^x at the origin having degree $N-1$. This defines a family of what Hassen and Nguyen called hypergeometric zeta function denoted by $\zeta_N(s)$ defined by:

$$\zeta_N(s) = \frac{1}{\Gamma(s+N-1)} \int_0^\infty \frac{x^{s+N-2}}{e^x - T_{N-1}(x)} dx.$$

Observe that $\zeta_1(s) = \zeta(s)$, i.e.; when $N = 1$, we get the classical zeta function. In the same paper, ([9]) they developed the analytic continuation of $\zeta_N(s)$ to the entire complex plane, except for N simple poles at $s = 1, 0, -1, \cdots, 2-N$; and established many of the properties of the classical zeta function were established for the hypergeometric zeta functions including generalized Bernoulli numbers. The same authors also investigated a continuous version of the hypergeometric zeta functions called the error zeta function in ([11]) by generalizing the definition involving a positive integer N to all real positive N but focused only on $N = \frac{1}{2}$. Regarding the continuous version they developed the analytic continuation to the entire complex plane except for infinite number of poles and discovered that, the error zeta function shares many of the same properties found in hypergeometric zeta functions of integer order.

In this chapter we introduce what we call fractional hypergeometric zeta function of order "a" where a is a positive real number. This can be done allowing the natural number N to be any positive real number a. The authors of ([11]) and ([9])defined $\zeta_N(s)$ for $\Re(s) > 1$ as,

$$\zeta_N(s) = \frac{\Gamma(N+1)}{\Gamma(s+N-1)} \int_0^\infty \frac{x^{s-2}}{{}_1F_1(1, N+1, x)} dx$$

where ${}_1F_1(b, c, x)$ is the confluent hypergeometric function. We now take this as our definition of $\zeta_N(s)$ which is formally defined for all real positive values of N. Now in our case we replace N by a and have a look at ${}_1F_1(1, a+1, x)$.

$$\begin{aligned} {}_1F_1(1, a+1, x) &= e^x x^{-a}(\Gamma(a+1) - a\Gamma(a, x)) \\ &= ae^x x^{-a}(\Gamma(a) - \Gamma(a, x)) \\ &= ae^x x^{-a}\gamma(a, x), \end{aligned} \tag{4.1.1}$$

where

$$\gamma(a, x) = \int_0^x t^{a-1}e^{-t}dt$$

is the lower incomplete gamma function and

$$\Gamma(a, x) = \int_x^\infty t^{a-1}e^{-t}dt$$

is the upper incomplete gamma function. Therefore, the fractional hypergeometric zeta function of order a is given as,

$$\zeta_a(s) = \frac{\Gamma(a+1)}{\Gamma(s+a-1)} \int_0^\infty \frac{x^{s+a-2}e^{-x}}{a\gamma(a, x)} dx.$$

It is discovered that this fractional hypergeometric zeta function of order "a" shares many of the same properties found in hypergeometric zeta functions of integer order. Observe that, if a is a positive integer we have the hypergeometric zeta function. This chapter is organized as follows. In section 2, we formally define fractional hypergeometric zeta functions, establish its convergence in a right half of the complex plane, and develop its series representation. In section 3, we reveal its analytic continuation to all complex plane except at infinite number of poles. We show this in two approaches: in the first approach we require strip-by-strip continuation to the left half of the complex plane for any positive real number a, and in the second approach we use contour integral for half odd integers a and express $\zeta_a(s)$ as a contour integral.

4.2 Preliminaries

In this section we formally define the fractional hypergeometric zeta functions, establish its domain of convergence, and demonstrate its series representation.

Definition 4.2.1. The fractional hypergeometric zeta function is defined for all positive real number a as,

$$\zeta_a(s) = \frac{\Gamma(a+1)}{\Gamma(s+a-1)} \int_0^\infty \frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)} dx.$$

Observe that when $a = N$, a natural number we get the classical hypergeometric zeta function.

Lemma 4.2.1. *Let r be a fixed real number. Then for any real number x in $[1,\infty)$ there is a constant C such that $x^r < Ce^{\frac{x}{2}}$.*

Proof. The function $x^r e^{\frac{-x}{2}}$ is continuous on $[1,\infty)$ and converges to zero as x tends to infinity for any real number r. Hence there is a constant C such that $x^r e^{\frac{-x}{2}} < C$. Thus it follows that $x^r < Ce^{\frac{x}{2}}$.

□

Lemma 4.2.2. *Let a be a fixed positive real number. Then $ae^x\gamma(a,x) \geq x^a$ for any positive real numbers x.*

Proof. The proof follows from the relation $ae^x x^{-a}\gamma(a,x) =_1 F_1(1,a+1,x)$ and the fact that ${}_1F_1(1,a+1,x) > 1$ for positive real number x. □

Lemma 4.2.3. *Let a be a fixed positive real number. Then for any real number x in $[1,\infty)$ there is a constant $\delta > 0$ such that $\gamma(a,x) \geq \delta$.*

Proof. Since $\lim_{x\to\infty}\gamma(a,x) = \Gamma(a)$, there is a number M such that $\gamma(a,x) > \frac{\Gamma(a)}{2}$. Since $\gamma(a,x)$ is continuous on $[1,M]$ it has a minimum value on $[1,M]$ say m. Then the lemma follows if we let δ to be the minimum of m and $\frac{\Gamma(a)}{2}$. □

Theorem 4.2.4. *Let a be a fixed positive real number. Then $\zeta_a(s)$ converges absolutely for $\sigma > 1$, where $s = \sigma + it$ and both σ and t are real numbers.*

Proof. Since $ae^x\gamma(a,x) \geq x^a$ on $[0,\infty)$, it holds on $(0,1]$. Let s be a complex number with $\sigma \geq \sigma_0 > 1$, then on $(0, 1]$, we have

$$\left| \frac{x^{s+a-2}}{ae^x\gamma(a,x)} \right| \leq \left| \frac{x^{s+a-2}}{x^a} \right| \leq x^{\sigma_0 - 2}.$$

Since $\sigma_0 > 1$, we have

$$\int_0^1 x^{\sigma_0 - 2} dx = \frac{1}{\sigma_0 - 1}.$$

Hence,

$$\left| \int_0^1 \frac{x^{s+a-2}}{ae^x\gamma(a,x)} dx \right| \leq \int_0^1 \left| \frac{x^{s+a-2}}{ae^x\gamma(a,x)} dx \right| \leq \frac{1}{\sigma_0 - 1}.$$

On the other hand, let s be a complex number with $\sigma > 1$ and x be a real number in $[1, \infty)$, then there is a constant C such that $x^{\sigma+a-2} < Ce^{\frac{x}{2}}$. Thus it follows that

$$\left| \frac{x^{s+a-2}}{ae^x\gamma(a,x)} \right| \leq \frac{Ce^{\frac{x}{2}}}{ae^x\gamma(a,x)} \leq \frac{Ce^{\frac{-x}{2}}}{a\delta}.$$

This holds true since from the previous lemma we have,$\gamma(a, x) \geq \delta$ and,

$$\int_1^\infty \frac{Ce^{\frac{-x}{2}}}{a\delta} dx = \frac{2Ce^{\frac{-1}{2}}}{a\delta}.$$

$$\left| \int_1^\infty \frac{x^{s+a-2}}{ae^x\gamma(a,x)} dx \right| \leq \int_1^\infty \left| \frac{x^{s+a-2}}{ae^x\gamma(a,x)} dx \right| \leq \frac{2Ce^{\frac{-1}{2}}}{a\delta}.$$

Therefore the theorem follows from the following inequality.

$$\zeta_a(s) = \frac{\Gamma(a+1)}{\Gamma(s+a-1)} \left[\int_0^1 \frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)} dx + \int_1^\infty \frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)} dx \right].$$

thus it follows that,

$$|\zeta_a(s)| \leq \left|\frac{\Gamma(a+1)}{\Gamma(s+a-1)}\right| \left[\int_0^1 \left|\frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)}\right| dx + \int_1^\infty \left|\frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)}\right| dx\right].$$

Here the right hand side of the integrals are both finite. □

Thus we have shown that the fractional hypergeometric zeta functions converges absolutely for $\sigma > 1$ and converges uniformly for $\sigma \geq \sigma_0 > 1$. Therefore, the fractional hypergeometric zeta function is analytic on $\{s : \sigma > 1\}$ where $s = \sigma + it$ for σ and t real numbers. We now develop series representation using its expansion as follows:

Lemma 4.2.5. *For* $\Re(s) = \sigma > 1$, *we have,*

$$\zeta_a(s) = \sum_{n=1}^{\infty} f_n(a,s)$$

where

$$f_n(a,s) = \int_0^\infty \frac{x^{s+a-2}e^{-x}}{\Gamma(s+a-1)} \left(\frac{\Gamma(a,x)}{\Gamma(a)}\right)^n dx.$$

Proof. From integral representation of $\zeta_a(s)$, we have,

$$\begin{aligned} \zeta_a(s) &= \frac{\Gamma(a+1)}{\Gamma(s+a-1)} \int_0^\infty \frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)} dx \\ &= \frac{\Gamma(a)}{\Gamma(s+a-1)} \int_0^\infty \frac{x^{s+a-2}e^{-x}}{\gamma(a,x)} dx. \end{aligned} \tag{4.2.1}$$

Since,$\gamma(a,x) > 0$, $\Gamma(a,x) > 0$ for $x > 0$ and$\gamma(a,x) + \Gamma(a,x) = \Gamma(a)$, we have,

$$\gamma(a,x) = \Gamma(a)\left(1 - \frac{\Gamma(a,x)}{\Gamma(a)}\right).$$

Since, $0 < \frac{\Gamma(a,x)}{\Gamma(a)} < 1$, using geometric series and reversing the order of summation and integration we have the required result. □

Lemma 4.2.6. *For each $n = 0, 1, 2, \cdots$, we have,*

$$f_n(a, 1) = \frac{1}{n+1}.$$

Proof. Substituting 1 for s in the definition of $f_n(a, s)$, we have,

$$f_n(a, 1) = \int_0^\infty \frac{x^{a-1}e^{-x}}{\Gamma(a)} \left(\frac{\Gamma(a, x)}{\Gamma(a)} \right)^n dx.$$

Using integration by substitution, $u = \frac{\Gamma(a,x)}{\Gamma(a)}$ and integrating we have the required result. □

The lemma reveals that

$$\zeta_a(1) = \sum_{n=1}^{\infty} \frac{1}{n}$$

formally generates the harmonic series.

4.3 Analytic Continuation

In this section we develop the analytic continuation of fractional hypergeometric zeta function to the entire complex plane. We shall discuss two different approaches. The first involves rewriting the integral representation defining the fractional hypergeometric zeta functions in stages to extend the domain strip-by-strip. The second uses contour integration to perform the analytic continuation in one stroke. But in the second method we specify our a and focus on specific value of a because of the complexity of the contour integral. As we will see each method has its advantages.

4.3.1 Method I-Strip-by-Strip

From the definition of fractional hypergeometric zeta function of order a for $\Re(s) = \sigma > 1$, we have,

$$\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s) = \int_0^\infty \frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)}dx.$$

So using series representation of $\frac{1}{a\gamma(a,x)}$, we can add and subtract each of the terms in the series to the integrand without violating convergence. This can be shown in the following theorem:

Theorem 4.3.1. *For* $0 < \Re(s) = \sigma < 1$,

$$\zeta_a(s) = \frac{\Gamma(a+1)}{\Gamma(s+a-1)}\left[\Gamma(s-1) + \int_0^\infty \left(\frac{1}{a\gamma(a,x)} - \frac{1}{x^a}\right)x^{s+a-2}e^{-x}dx\right].$$

Proof. For $\Re(s) = \sigma > 1$, we have,

$$\begin{aligned}\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s) &= \int_0^\infty \frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)}dx \\ &= \int_0^1 \left[\frac{1}{a\gamma(a,x)} - \frac{1}{x^a}\right]x^{s+a-2}e^{-x}dx + \int_0^1 x^{s-2}e^{-x}dx + \int_1^\infty \frac{x^{s+a-2}e^{-x}}{a\gamma(a,x)}dx.\end{aligned}$$

But,

$$\begin{aligned}\int_0^1 x^{s-2}e^{-x}dx &= \Gamma(s-1) - \Gamma(s-1,1) \\ &= \Gamma(s-1) - \int_1^\infty \frac{x^{s+a-2}e^{-x}}{x^a}dx.\end{aligned} \tag{4.3.1}$$

So we have,

$$\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s) = \Gamma(s-1) + \int_0^\infty \left(\frac{1}{a\gamma(a,x)} - \frac{1}{x^a}\right) x^{s+a-2}e^{-x}dx.$$

Hence the theorem follows. □

From this theorem observe that $\zeta_a(s)$ has a zero at $s = 1-a$ and a pole at $s = 1$. Moreover, this process of analytic continuation can be repeated to extend the domain of $\zeta_a(s)$ to $\Re(s) > -1$. To this end we write

$$\begin{aligned}\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s) &= \Gamma(s-1) + \int_0^1 \left[\frac{1}{a\gamma(a,x)} - \frac{1}{x^a} - \frac{a}{(1+a)x^{a-1}}\right] x^{s+a-2}e^{-x}dx \\ &+ \frac{a}{1+a}\int_0^1 x^{s-1}e^{-x}dx + \int_1^\infty \left[\frac{1}{a\gamma(a,x)} - \frac{1}{x^a}\right] x^{s+a-2}e^{-x}dx.\end{aligned}$$

But then,

$$\frac{a}{1+a}\int_0^1 x^{s-1}e^{-x}dx = \frac{a}{1+a}\Gamma(s) - \frac{a}{1+a}\int_1^\infty x^{s-1}e^{-x}dx.$$

Therefore,

$$\frac{\Gamma(s+a-1)}{\Gamma(a+1)}\zeta_a(s) = \Gamma(s-1) + \frac{a}{1+a}\Gamma(s) + \int_0^\infty \left[\frac{1}{a\gamma(a,x)} - \frac{1}{x^a} - \frac{a}{(1+a)x^{a-1}}\right] x^{s+a-2}e^{-x}dx.$$

From this, observe that $\zeta_a(s)$ has a zero at $s = -a$ and a pole at $s = 0$. Continuing in this fashion, we observe that:$\zeta_a(s)$ has a zeros at $s = 1 - a, -a, -(1 + a), -(2 + a), \cdots$, and poles at $s = 1, 0, -1, -2, -3, \cdots$. These zeros are called the trivial zeros of $\zeta_a(s)$. For example, if $a = \frac{3}{2}$, the trivial zeros of $\zeta_{\frac{3}{2}}(s)$ are at $s = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \cdots$.

The main advantage of analytic continuation using the method of strip-by-strip is that it reveals the behavior of $\zeta_a(s)$ near the distinguished pole $s = 1$. The following theorem is valid for all positive real values of a and generalizes the corresponding result on hypergeometric zeta functions of positive integer order. This is the content of the next theorem.

Theorem 4.3.2.

$$\lim_{s \to 1} \left[\zeta_a(s) - \frac{a}{s-1} \right] = \log(\Gamma(a+1)) - a\frac{\Gamma'(a)}{\Gamma(a)}.$$

Proof.

$$\begin{aligned} \lim_{s \to 1} \left[\zeta_a(s) - \frac{a}{s-1} \right] &= \frac{\Gamma(a+1)}{\Gamma(a)} \int_0^\infty \left[\frac{1}{{}_1F_1(1, a+1; x)} - e^{-x} \right] \frac{dx}{x} \\ &= \frac{\Gamma(a+1)}{\Gamma(a)} \left[\gamma + \frac{\log(\Gamma(a+1))}{a} \right] \\ &= a\gamma + \log(\Gamma(a+1)) \end{aligned} \tag{4.3.2}$$

where γ is Euler's constant. But on the other hand we have,

$$\begin{aligned}\lim_{s\to 1}\left[\zeta_a(s)-\frac{a}{s-1}\right] &= \lim_{s\to 1}\left[\zeta_a(s)-\frac{\Gamma(a+1)\Gamma(s-1)}{\Gamma(s+a-1)}\right]-\lim_{s\to 1}\left[\frac{a}{s-1}-\frac{\Gamma(a+1)\Gamma(s-1)}{\Gamma(s+a-1)}\right]\\ &= a\gamma+\log(\Gamma(a+1))-\left[a\gamma+a\frac{\Gamma^{'}(a)}{\Gamma(a)}\right]\\ &= \log(\Gamma(a+1))-a\frac{\Gamma^{'}(a)}{\Gamma(a)}\end{aligned} \tag{4.3.0}$$

as desired. □

Observe that this result is analogous to the classic result for $\zeta(s)$

$$\lim_{s\to 1}\left[\zeta(s)-\frac{1}{s-1}\right]=\frac{\Gamma^{'}(1)}{\Gamma(1)}=\gamma\approx 0.577.$$

4.3.2 Method II-Contour Integral

We now take a different approach and follow Riemann by using contour integration to develop the analytic continuation of the fractional hypergeometric zeta function of order a. This will not only allow us to make precise our earlier statements about $\zeta_a(s)$ having an infinite number of poles but also make explicit the role of the zeros of incomplete gamma function in determining the values of $\zeta_a(s)$ at negative integers. To this end we consider the contour integral for half odd integers a. In this section we consider the case $a=\frac{2N+1}{2}$, and so that,

$$\zeta_{\frac{2N+1}{2}}(s)=\frac{\Gamma(\frac{2N+1}{2}+1)}{\Gamma(s+\frac{2N+1}{2}-1)}\int_0^\infty\left[\frac{1}{\frac{2N+1}{2}\gamma(\frac{2N+1}{2},x^2)}\right]x^{2(s+\frac{2N+1}{2}-2)}e^{-x^2}(2x)dx.$$

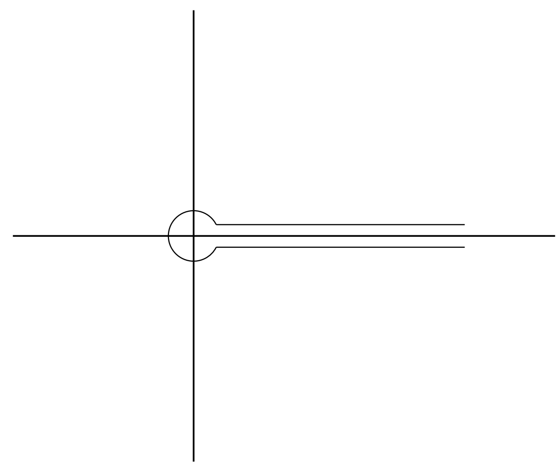

Figure 4.1: Contour C

Simplifying this we get,

$$\zeta_{\frac{2N+1}{2}}(s) = \frac{(2N+1)\Gamma\left(\frac{2N+1}{2}\right)}{\Gamma(s+\frac{2N-1}{2})} \int_0^\infty \frac{2x^{2(s+N-1)+1}e^{-x^2}}{(2N+1)\gamma(\frac{2N+1}{2}, x^2)} \frac{dx}{x}.$$

We now consider the contour integral,

$$I(s) = \frac{1}{2\pi i} \int_C \frac{2(-w)^{2(s+N-1)+1}e^{-w^2}}{(2N+1)\gamma(\frac{2N+1}{2}, w^2)} \frac{dw}{w}$$

where the contour C is given as follows:

$$C = C_- + C_\delta + C_+$$

C_- is taken along the real axis from ∞ to $\delta > 0$. C_δ is taken counterclockwise around a circle of radius $\delta > 0$. C_+ is taken along the real axis from $\delta > 0$ to ∞. Moreover, we let $-w$ have argument $-\pi$ backwards along ∞ to $\delta > 0$, and argument π when going from $\delta > 0$ to ∞. Also we choose the radius $\delta > 0$ to be sufficiently small so that there are no roots of $\gamma(\frac{2N+1}{2}, w^2)$ inside the circle of radius $\delta > 0$ besides the trivial zeros at $w_0 = 0$. This follows from the fact that $w_0 = 0$ is an isolated zero. It is then clear from this assumption that $I(s)$ must converge for all complex s and therefore defines an entire function. Now we begin by evaluating $I(s)$ at integer values. To this end we decompose the integral as follows:

$$I(s) = \frac{1}{2\pi i}\int_{C_-} \frac{2(-x)^{2(s+N-1)+1}e^{-x^2}}{(2N+1)\gamma(\frac{2N+1}{2}, x^2)}\frac{dx}{x}$$

$$+\frac{1}{2\pi i}\int_{C_\delta} \frac{2(-w)^{2(s+N-1)+1}e^{-w^2}}{(2N+1)\gamma(\frac{2N+1}{2}, w^2)}\frac{dw}{w}$$

$$+\frac{1}{2\pi i}\int_{C_+} \frac{2(-x)^{2(s+N-1)+1}e^{-x^2}}{(2N+1)\gamma(\frac{2N+1}{2}, x^2)}\frac{dx}{x}.$$

Now if $2(s+N-1)+1$ is an integer the two integrals along C_- and C_+ cancel each other and we have only

$$I(s) = \frac{1}{2\pi i}\int_{C_\delta} \frac{2(-w)^{2(s+N-1)+1}e^{-w^2}}{(2N+1)\gamma(\frac{2N+1}{2}, w^2)}\frac{dw}{w}$$

But the function defined by,

$$f(w) = \frac{2w^{2N+1}e^{-w^2}}{(2N+1)\gamma(\frac{2N+1}{2}, w^2)}$$

in the integrand inside C_δ has a removable singularity at the origin, hence by Cauchy's theorem we have $I(s) = 0$ for integer values $2s-3 \geq 0$; that is for $s = \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, \cdots$.

For integers $2s-3 \leq -1$; that is for $s = 1, \frac{1}{2}, 0, \frac{-1}{2}, -1, \frac{-3}{2}, \cdots$, we consider power series expansion at the origin for $f(w)$,

$$f(w) = \frac{2w^{2N+1}e^{-w^2}}{(2N+1)\gamma(\frac{2N+1}{2}, w^2)} = \sum_{m=0}^{\infty} B_{\frac{2N+1}{2},m} \frac{w^m}{m!}.$$

From Cauchy integral formula we have,

$$B_{\frac{2N+1}{2},m} = \frac{m!}{2\pi i} \int_{C_\delta} \frac{f(w)}{w^{m+1}} dw.$$

So we have,

$$\begin{aligned} I(s) &= (-1)^{2(s+N-1)+1} \frac{1}{2\pi i} \int_{C_\delta} \left(\frac{2(-w)^{2N+1}e^{-w^2}}{(2N+1)\gamma(\frac{2N+1}{2}, w^2)} \right) \frac{dw}{w^{2(1-s)+1}} \\ &= (-1)^{2(s+N-1)+1} \frac{1}{2\pi i} \int_{C_\delta} \frac{f(w)}{w^{2(1-s)+1}} dw \\ &= (-1)^{2(s+N-1)+1} \frac{B_{\frac{2N+1}{2},2(1-s)}}{(2(1-s))!}. \end{aligned} \tag{4.3.0}$$

But at $s = \frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}, \cdots$, it can be easily shown that

$$\frac{B_{\frac{2N+1}{2},2(1-s)}}{(2(1-s))!} = 0.$$

Thus $I(s) = 0$ for $s = \frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}, \cdots ,$.

Therefore, the zeros of $I(s)$ can be summarized as follows:

$s = \frac{n}{2}$, n is an integer and $n \geq 3$, or $s = \frac{2n+1}{2}$, n is an integer and $n \leq 0$.

4.3.3 Relations between $I(s)$ and $\zeta_{\frac{2N+1}{2}}(s)$

We now express $\zeta_{\frac{2N+1}{2}}(s)$ in terms of $I(s)$. For $\Re(s) > 1$, the integral over C_δ goes to zero as δ goes to zero. Hence the integral over C_+ and C_- yields that,

$$\begin{aligned} I(s) &= \frac{1}{2\pi i}\left[e^{2s+2N-1)\pi i} - e^{-(2s+2N-1)\pi i}\right]\int_{C_-}\left(\frac{2(x)^{2(s+N-1)}e^{-x^2}}{(2N+1)\gamma(\frac{2N+1}{2},x^2)}\right)dx \\ &= \frac{\sin((2s+2N-1)\pi)}{\pi}\frac{\Gamma(s+\frac{2N-1}{2})}{(2N+1)\Gamma(\frac{2N+1}{2})}\zeta_{\frac{2N+1}{2}}(s). \end{aligned} \tag{4.3.1}$$

Thus we have,

$$\zeta_{\frac{2N+1}{2}}(s) = \frac{(2N+1)\Gamma\left(\frac{2N+1}{2}\right)}{\Gamma\left(s+\frac{2N-1}{2}\right)}\frac{\pi}{\sin(2(s+N)-1)\pi)}I(s).$$

Now using trigonometric identity, the double angle formula,

$$\sin(2(s+N)-1)\pi = 2\cos(s+N-\frac{1}{2})\pi\sin(s+N-\frac{1}{2})\pi,$$

$$\zeta_{\frac{2N+1}{2}}(s) = \frac{2N+1}{2}\frac{\Gamma\left(\frac{2N+1}{2}\right)}{\Gamma\left(s+\frac{2N-1}{2}\right)}\frac{\pi}{2\cos(s+N-\frac{1}{2})\pi\sin(s+N-\frac{1}{2})\pi}I(s).$$

By using the functional equation of the gamma function,

$$\frac{\pi}{\sin((s+\frac{1}{2})\pi)\Gamma(s+\frac{1}{2})} = \Gamma(\frac{1}{2}-s).$$

Therefore, for $\Re(s) > 1$, we have the relation,

$$\zeta_{\frac{2N+1}{2}}(s) = \frac{2N+1}{2}\Gamma\left(\frac{2N+1}{2}\right)\frac{\Gamma\left(\frac{1}{2}-(s+N-1)\right)}{\cos(s+N-\frac{1}{2})}I(s).$$

From this relation we observe that the zeros of $I(s)$ at $\frac{n}{2}$ for $n \geq 3$ are simple, since we know by definition that $\zeta_{\frac{2N+1}{2}}(n) > 0$ for $n > 1$. We have also the following results as a consequence of the above relations:

Theorem 4.3.3. *$\zeta_{\frac{2N+1}{2}}(s)$ is analytic on the entire complex plane except for simple poles at $s = n + 1 - N$ where $n = N, N-1, N-2, N-3, \cdots$, with residue,*

$$Res(\zeta_{\frac{2N+1}{2}}(s), s = n) = \frac{2N+1}{2} \frac{\Gamma\left(\frac{2N+1}{2}\right)\Gamma\left(\frac{1}{2}-n\right)}{\pi \sin\left(n+\frac{1}{2}\right)} \frac{B_{\frac{2N+1}{2},(2(N-n))}}{(2N-2n)!}.$$

Furthermore,$\zeta_{\frac{2N+1}{2}}(s) = 0$ at $s = \frac{-(2n+1)}{2}$ for $n = 0, 1, 2, 3, \cdots$.

Proof. Since $\Gamma(\frac{1}{2} - (s + N - 1)$ has only simple poles at $s = \frac{2(n-N)+1}{2}$ for $n = 1, 2, 3, \cdots$, which are canceled by the zeros of $I(s)$ at $s = \frac{n}{2}$ for integer $n \geq 3$, and $\cos((s + N - \frac{1}{2})\pi)$ has simple zeros at the integers, it follows that $\zeta_{\frac{2N+1}{2}}(s)$ is analytic on the entire complex plane except for simple poles at $s = n + N - 1$ where $n = N, N-1, N-2, N-3, \cdots$. The residue at these simple poles is given by

$$\begin{aligned} & Res(\zeta_{\frac{2N+1}{2}}(s), s = n + N - 1) = \lim_{s \to n+N-1} (s - (n + N - 1))\zeta_{\frac{2N+1}{2}}(s) \\ = \; & \frac{2N+1}{2}\Gamma\left(\frac{2N+1}{2}\right)\Gamma\left(\frac{1}{2}-n\right) I(n+1-N) \lim_{s \to n+1-N} \frac{s-n-1+N}{\cos((s+N-\frac{1}{2})\pi)} \\ = \; & \frac{2N+1}{2}\Gamma\left(\frac{2N+1}{2}\right)\Gamma\left(\frac{1}{2}-n\right) I(n+1-N) \frac{-1}{\pi \sin(n+\frac{1}{2})\pi} \\ = \; & \frac{2N+1}{2}\Gamma\left(\frac{2N+1}{2}\right)\Gamma\left(\frac{1}{2}-n\right) \frac{B_{\frac{2N+1}{2},2(N-1)}}{(2N-2n)!\pi \sin(n+\frac{1}{2})\pi}. \end{aligned} \tag{4.3.2}$$

Since, $I(s) = 0$ for $s = \frac{-(2n+1)}{2}$ for $n = 0, 1, 2, 3, \cdots$, we have that $\zeta_{\frac{2N+1}{2}}(s) = 0$ at $s = \frac{-(2n+1)}{2}$ for $n = 0, 1, 2, 3, \cdots$. □

4.4 Pre-functional Equation

In this section we discuss on a pre-functional equation satisfied by $\zeta_{\frac{2N+1}{2}}(s)$. For this we need some properties of the lower incomplete gamma function. From the definition of the lower incomplete gamma function:

$$\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt,$$

it follows that $\gamma(a,x)$ tends to $\Gamma(a)$ as x tends to infinity for real parameter x. This also holds true if we replace x by a complex parameter z and let the modulus of z tends to infinity for the positive real part of z. This can be shown as follows:

Theorem 4.4.1. *Let z be a complex number such that its real part is positive, then*

$$\lim_{|z|\to\infty} |\gamma(a,z) - \gamma(a,|z|)| = 0.$$

Proof. Let $|argz| < \delta < \frac{\pi}{2}$ with fixed δ and $\gamma(a,z)$ be in the principal branch of this sector and u be any complex number from the sector. Then

$$|\gamma(a,u) - \gamma(a,|u|)| = \left| \int_u^{|u|} z^{a-1}e^{-z}dz \right|.$$

Now if we integrate along the arc with radius $R = |u|$ around zero connecting u and $|u|$, then we have

$$|\gamma(a,u) - \gamma(a,|u|)| \leq \int_u^{|u|} \left| z^{a-1}e^{-z}dz \right|.$$

Since $|argz| < \delta$ and hence $\cos\delta \leq \cos|argu|$ the integrand has a maximum value in this sector. Thus considering the length of the arc connecting u and $|u|$, we have

$$|\gamma(a,u) - \gamma(a,|u|)| \leq R^a \delta e^{-R\cos\delta}.$$

Thus letting R tends to infinity we have the required result. □

Corollary 4.4.2. *Let z be a complex number such that its real part is positive, then*

$$\lim_{|z|\to\infty} \gamma(a,z) = \Gamma(a).$$

Proof.

$$|\Gamma(a) - \gamma(a,z)| = |\Gamma(a) - \gamma(a,|z|) + \gamma(a,|z|) - \gamma(a,z)|\,.$$

Thus we have,

$$|\Gamma(a) - \gamma(a,z)| \leq |\Gamma(a) - \gamma(a,|z|)| + |\gamma(a,|z|) - \gamma(a,z)|\,.$$

Hence using the theorem and letting the modulus of z tends to infinity we have the required result. □

From this theorem and its corollary it follows that $\gamma(a,z)$ has only a finite number of simple zeros to the right half of the complex plane. Moreover these zeros are symmetric with respect to the Real-axis. Thus all these zeros can be arranged in a sequence of increasing modulus. Let for each $n = 1, 2, 3, \cdots,\ z_1^n, z_2^n, \cdots, z_{k_n}^n$ be the non-zero roots in the first quadrant of the lower incomplete gamma function having the same distance r_n from the origin with different arguments $\theta_1^n, \theta_2^n, \cdots, \theta_{k_n}^n$, respectively. Since the lower incomplete gamma function is symmetric about the real axis, the conjugates of these roots are also its roots in the fourth quadrant.

Let $C_{R,\delta}$ be the annulus-shaped contour consisting of two concentric circles centered at the origin. The outer wedge C_R having radius R, the outer circle centered at the origin having radius $\frac{\delta}{R}$, and the inner circle C_δ having radius $\frac{\delta}{2R}$. Here $\delta > 0$ is chosen so that no other zeros of the lower incomplete gamma function included besides the root zero inside the circle. This is possible since zero is an isolated zero of the lower incomplete gamma function. R is chosen so that all the zeros to the right half of the complex plane are included in the wedge. This is possible , since there are only finite number of zeros of the lower incomplete gamma function occurred in the right half of the complex plane. The outer circle and the wedge traversed clockwise, the inner circle counterclockwise and the radial segment along the positive real axis is traversed in both directions. We define

$$I_{C_{R,\delta}}(s) = \frac{1}{2\pi i}\int_{C_{R,\delta}} \left(\frac{2(-w)^{2(s+N-1)}e^{-w^2}}{(2N+1)\gamma(\frac{2N+1}{2}, w^2)} \right) \frac{dw}{w}.$$

The claim is that $I_{C_{R,\delta}}(s)$ converges to $I(s)$ as R tends to infinity, for $\Re(s) < 1 - N$. Where

$$I(s) = \frac{1}{2\pi i}\int_{C} \frac{2(-w)^{2(s+N-1)}e^{-w^2}}{(2N+1)\gamma(\frac{3}{2}, w^2)})\frac{dw}{w}$$

and the contour C is given as follows:

$$C = C_- + C_\delta + C_+$$

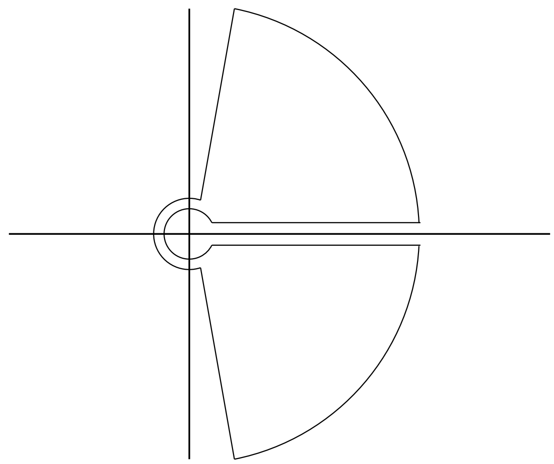

Figure 4.2: Contour $C_{R,\delta}$

C_- is taken along the real axis from ∞ to $\delta > 0$. C_δ is taken counterclockwise around a circle of radius $\delta > 0$. C_+ is taken along the real axis from $\delta > 0$ to ∞. Moreover, we let $-w$ have argument $-\pi$ backwards along ∞ to $\delta > 0$, and argument π when going from $\delta > 0$ to ∞. as defined in the previous section.

Since out side the wedge with radius R, $\gamma(\frac{2N+1}{2}, w^2) \neq 0$ we have $|\gamma(\frac{2N+1}{2}, w^2)| > M > 0$.

Thus if $|w| = R$, we have

$$\left| \frac{w^{2(s+N-1)} e^{-w^2}}{\gamma(\frac{2N+1}{2}, w^2)} \right| \leq \left| \frac{R^{2(s+N-1)}}{M} \right| = \frac{R^{2\Re(s+N-1)}}{M} \to 0$$

as R tends to infinity, since $\Re(s) < 1 - N$.

Therefore,

$$I(s) = \lim_{R \to \infty} I_{C_{R,\delta}}(s).$$

On the other hand, we have by residue theory that,

$$I_{C_{R,\delta}}(s) = \sum_{j=1}^{k_n} Res \left[\frac{2(-z)^{2(s+N-1)}e^{-z^2}}{(2N+1)\gamma(\frac{2N+1}{2}, z^2)}, z = z_j^n, z = \overline{z_j^n} \right]$$

But $Res \left[\frac{2(-z)^{2(s+N-1)}e^{-z^2}}{(2N+1)\gamma(\frac{2N+1}{2}, z^2)}, z = w \right] = \frac{w^{2s-2}}{2N+1}$.

Hence,

$$I_{C_{R,\delta}}(s) = \frac{2}{2N+1} \sum_{j=1}^{n} k_j r_j^{2s-2} \sum_{i=1}^{k_j} \cos(2(s-1)\theta_i^j).$$

Now taking limit as n tends to infinity we have the required result. Thus,

$$I(s) = \frac{2}{2N+1} \sum_{j=1}^{\infty} k_j r_j^{2s-2} \sum_{i=1}^{k_j} \cos(2(s-1)\theta_i^j),$$

since

$$\zeta_{\frac{2N+1}{2}}(s) = \frac{2N+1}{2} \Gamma\left(\frac{2N+1}{2}\right) \frac{\Gamma\left(\frac{1}{2} - (s+N-1)\right)}{\cos(s+N-\frac{1}{2})} I(s).$$

Then we have,

$$\zeta_{\frac{2N+1}{2}}(s) = \frac{2N+1}{2} \Gamma\left(\frac{2N+1}{2}\right) \frac{\Gamma\left(\frac{1}{2} - (s+N-1)\right)}{\cos(s+N-\frac{1}{2})} \frac{2}{2N+1} \sum_{j=1}^{\infty} k_j r_j^{2s-2} \sum_{i=1}^{k_j} \cos(2(s-1)\theta_i^j)$$

In particular, if $k_j = 1$ for all $j = 1, 2, 3, \cdots$ we put $\theta_i^j = \theta_j$ so that the above relation becomes,

$$\zeta_{\frac{2N+1}{2}}(s) = \frac{2N+1}{2} \Gamma\left(\frac{2N+1}{2}\right) \frac{\Gamma\left(\frac{1}{2} - (s+N-1)\right)}{\cos(s+N-\frac{1}{2})} \frac{2}{2N+1} \sum_{j=1}^{\infty} r_j^{2s-2} \cos(2(s-1)\theta_j).$$

We will explore some other properties of the fractional hypergeometric zeta functions in the future work. One trivial problem with non-trivial answer is whether or not the pre-functional equation can be made functional equation.

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