DUALITY IN MULTIOBJECTIVE PROGRAMMING

ADDIS ABABA UNIVERSITY
COLLEGE OF COMPUTATIONAL AND NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS

"In partial fulfillment of the requirements of the degree of master of science in mathematics"

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Stream: Optimization
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June 30, 2014

ADDIS ABABA UNIVERSITY
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The undersigned hereby certify that they have read and recommend to the department of mathematics for acceptance of this project entitled "**DUALITY IN MULTIOBJECTIVE PROGRAMMING**" by **Tsega kidane** in partial fulfillment of the requirements for the degree of Master of Science in mathematics.

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Department: Mathematics

Degree: M.Sc.

Convocation: **June**

Year: 2014

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Abstract

In this project we see about the duality in multiobjective programming. We investigate a general convex multiobjective optimization problem with cone inequality constraints.
The main idea of constructing the multiobjective dual problem is to establish first a dual problem to the scalarized primal. In addition to these the project also explains some points about multiobjective optimization problem and conjugate duality for scalar optimization problems.

Keywords: Lagrange dual function, Conjugate function, Conjugate map, Dual cone, Multiobjective Optimization.

Acknowledgment

First and foremost, I would like to express my sincere gratitude to my advisor **Dr. Semu Mitiku** for his generous advice throughout the preparation of this project.

Next, my thanks goes to all staff members of the department of mathematics and to all **my friends** for their unmentioned cooperation on my project.

Finally, I would also like to express my appreciation to **my families** for their continuous encouragement.

Notations

- $\diamond$ $\mathbb{R}$ the set of real numbers
- $\diamond$ $\bar{\mathbb{R}}$ the extended set of real numbers
- $\diamond$ $\mathbb{R}^m_+$ the non-negative orthant of $\mathbb{R}^m$
- $\diamond$ K^* the dual cone of the cone K
- $\diamond$ $\underset{K}{\geqq}$ the partial ordering induced by the cone K
- $\diamond$ $\underset{K^*}{\geqq}$ the partial ordering induced by the dual cone K^*
- $\diamond$ $\underset{\mathbb{R}^m_+}{\geqq}$ the partial ordering induced by the non-negative orthant $\mathbb{R}^m_+$
- $\diamond$ f^* the conjugate of the function f
- $\diamond$ X_G the indicator function of the set G
- $\diamond$ $v-\min$ the notation for a multiobjective optimization problem in the sense of minimum
- $\diamond$ D_L Lagrange dual problem
- $\diamond$ D_F Fenchel dual problem
- $\diamond$ D_{FL} Fenchel-Lagrange dual problem
- $\diamond$ $dom(f)$ the domain of the function f
- $\diamond$ $inf(P)$ the optimal objective value of the scalar minimum optimization problem (P)
- $\diamond$ F^* conjugate map of the set-valued map
- $\diamond$ ∂f the subdifferential of the function f
- $\diamond$ $\partial F(x;y)$ subdifferential of the point-to-set maps F at (x,y).

Introduction

Multiobjective optimization (or Multiobjective programming), also known as Multicriteria or Multiattribute optimization, is the process of simultaneously optimizing two or more conflicting objectives subject to certain constraints. Its particularity is to try to optimize several objectives simultaneously from the same problem. It differs from the multidisciplinary optimization by the fact that the objectives to optimize focus here on only one problem.

Multiobjective optimization is a modern and fruitful research field with many practical applications, concerning especially engineering, economy and finance but also location and transports,even medicine. The rich literature on vector optimization mentions several types of solutions that can be attached to a multiobjective optimization problem. Let us enumerate here a few: efficient, properly efficient, strongly efficient, weakly efficient, strictly efficient, approximately efficient, critical efficient, ideal efficient, superefficient and epsilon-efficient solutions. In this project we use efficient and properly efficient solutions. Duality is an important tool in multiobjective optimization. Using the scalarized problem and its dual, it is possible to construct a multiobjective dual problem to the primal vector problem and some duality assertions are usually verified.
This project consists of three chapters and each chapter has sections and subsections.

Chapter 1 we presents the basic concepts and definition of the Lagrange dual function, conjugate function and multiobjective optimization problem. In addition to this we describe the convex, nonconvex, linear and nonlinear multiobjective optimization problem.

Chapter 2 describes conjugate duality in scalar optimization. To this problem we associate three conjugate dual problems, two of them proving to be the well known Lagrange and Fenchel dual problem this approach has the property that the so called weak duality always holds, namely, the optimal objective value of the primal problem is greater than or equal to the optimal objective value of the dual problem.

Chapter 3 we draw our attention to the duality for multiobjective optimization. This chapter contains two parts the first is general convex multiobjective problem with cone inequality constraints. In this case the ordering case in the objective space is the non negative orthant.
The general convex multiobjective problem with cone inequality constraints has the following formulation:

$$(P) \qquad\qquad v - \min_{x \in \mathcal{A}} f(x)$$

$$\mathcal{A} = \{x \in \mathbb{R}^n : g(x) = (g_1(x), \dots, g_k(x))^T \leqq_K 0\}$$

Where $f(x) = (f_1(x), \dots, f_m(x))^T$, $f_i : \mathbb{R}^n \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ $i = 1, \dots, m$ are proper functions, $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \dots, k$ and $K \subseteq \mathbb{R}^k$ is assumed to be a convex closed cone with $int(K) \neq \emptyset$ defining a partial ordering according to the definition that $x_2 \leqq_K x_1$ if and only if $x_1 - x_2 \in K$. To (P) is associated the following scalarized optimization problem In order to study the duality for the multiobjective problem (P) we study first the duality for the scalarized problem

$$(P^\lambda) \qquad \inf_{x \in \mathcal{A}} \sum_{i=1}^{m} \lambda_i f_i(x),$$

Where $\lambda = (\lambda_1, \dots, \lambda_m)^T \in int(\mathbb{R}^m_+)$. A scalar dual to it is constructed and the optimality conditions are derived. The structure of the scalar dual suggests the form of the multiobjective dual (D) to (P).

To study the duality for the scalarized problem (P^λ) the conjugacy approach is used. To the problem

$$(P) \qquad \inf_{x \in G} f(x)$$

$$G = \{x \in X : g(x) \leq 0\},$$

Where $X \subseteq \mathbb{R}^n$ is nonempty set, three different dual problems are constructed, namely, the well-known Lagrange and Fenchel duals (denoted by (D_L) and (D_F) respectively) and a combination of the above two, called the Fenchel-Lagrange dual (denoted by (D_{FL})).
The second one is conjugate duality in this part we develop a conjugate duality in vector optimization. Conjugate duality was fully developed in scalar optimization. The multiobjective optimization by introducing some new concepts such as conjugate maps and subgradients for vector valued, point-to-set maps. Their results are based on the efficiency. Besed on this definition of supremum, some useful concepts such as conjugate maps and subgradients are introduced for vector valued, point-to-set maps. These concepts enable us to develop the conjugate duality in vector optimization.

Chapter 1

Basic Concepts and Definitions

1.1 Lagrange Dual Function

We consider an equality and inequality constrained optimization problem:

$$\begin{aligned} &\textit{minimize} \;\; f(x) \\ &\textit{subject to} \;\; g_j(x) \leq 0, \quad j = 1, 2, \dots, r \\ &\qquad\qquad\quad h_i(x) = 0, \quad i = 1, 2, \dots, m \end{aligned}$$

making no assumption of f, g *and* h.
We denote by f^* the optimal value of the decision function under the constraints, i.e.,$f^* = f(x^*)$ if the minimum is reached at a global minimum x^*.
The **Lagrangian**of this problem is the function $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}$ defined by:

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{r} \mu_j g_j(x)$$

We define the **Lagrange dual function** $g : \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}$ as:

$$\begin{aligned} g(\lambda, \mu) &= \inf_{x \in \mathbb{R}^n} L(x, \lambda, \mu) \\ &= \inf_{x \in \mathbb{R}^n} (f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{r} \mu_j g_j(x)) \end{aligned}$$

Properties of the dual function
When L is unbounded from below in x, the dual function $g(\lambda, \mu)$ takes on the value $-\infty$.
It has two important properties:

1. g is concave in (λ, μ) even if the original problem is not convex.

   Proof. For each x, the function $(\lambda, \mu) \mapsto L(x, \lambda, \mu)$ is linear, and therefore both convex and concave in (λ, μ). The pointwise minimum of concave functions is concave, therefore g is concave. □

2. The dual function yields lower bounds on the optimal value f^* of the original problem when μ is nonnegative

$$g(\lambda, \mu) \leq f^*, \forall \lambda \in \mathbb{R}^m \ , \forall \mu \in \mathbb{R}^r \ , \mu \geq 0$$

Proof. Let $\bar{x}$ be any feasible point i.e, $h(\bar{x}) = 0$ and $g(\bar{x}) \leq 0$. Then we have, for any λ and $\mu \geq 0$

$$\sum_{i=1}^{m} \lambda_i h_i(\bar{x}) + \sum_{i=1}^{r} \mu_i g_i(\bar{x}) \leq 0,$$

$$L(\bar{x}, \lambda, \mu) = f(\bar{x}) + \sum_{i=1}^{m} \lambda_i h_i(\bar{x}) + \sum_{i=1}^{r} \mu_i g_i(\bar{x}) \leq f(\bar{x})$$

$$This \ \ imples \ \ g(\lambda, \mu) = \inf_x L(x, \lambda, \mu) \leq L(\bar{x}, \lambda, \mu) \leq f(\bar{x})$$

. □

Remark. We used the fact that the pointwise maximum (resp. minimum) of convex (resp. concave) functions is itself convex (concave).

Proof. Suppose that for each $y \in A$ the function $f(x, y)$ is convex in x, and let the function:

$$g(x) = \sup_{y \in A} f(x, y)$$

Then the domain of g is convex as an intersection of convex domains, and for any $\theta \in [0, 1]$ and x_1, x_2 in the domain of g

$$\begin{aligned} g(\theta x_1 + (1-\theta)x_2) &= \sup_{y \in A} f(\theta x_1 + (1-\theta)x_2, y) \\ &\leq \sup_{y \in A} (\theta f(x_1, y) + (1-\theta) f(x_2, y)) \\ &\leq \sup_{y \in A} (\theta f(x_1, y) + \sup_{y \in A} ((1-\theta) f(x_2, y)) \\ &= \theta g(x_1) + (1-\theta) g(x_2) \end{aligned}$$

□

Example 1.1. *[6]* Least-squares solution of linear equations:

$$\begin{aligned} minimize \ \ x^T x \\ subject \ \ to \ \ Ax = b \end{aligned}$$

Where $A \in \mathbb{R}^{p \times n}$. There are p equality constraints, the Lagrangian with domain $\mathbb{R}^n \times \mathbb{R}^r$ is:

$$L(x, \lambda) = x^T x + \lambda^T (Ax - b)$$

To minimize L over x for λ fixed, we set the gradient equal to zero

$$\bigtriangledown_x L(x, \lambda) = 2x + A^T \lambda = 0$$

$This\ \ imples\ x = \frac{-1}{2}A^T\lambda$ Plug it in L to obtain the dual function:

$$\begin{aligned} g(\lambda) &= L(x, \lambda) \\ &= L(\frac{-1}{2}A^T\lambda, \lambda) \\ &= \frac{-1}{4}\lambda^T AA^T\lambda - b^T\lambda \end{aligned}$$

g is a concave function of λ and the following lower bound holds

$$f^* \geq \frac{-1}{4}\lambda^T AA^T\lambda - b^T\lambda, \ \ \forall\lambda \in \mathbb{R}^p$$

Example 1.2. *[6]* Standard form LP:

$$\begin{aligned} minimize\ \ & c^T x \\ subject\ \ to\ \ & Ax = b \\ & x \geq 0 \end{aligned}$$

This reduced to

$$\begin{aligned} minimize\ \ & c^T x \\ subject\ \ to\ \ & Ax - b = 0 \\ & -x \leq 0 \end{aligned}$$

Where $A \in \mathbb{R}^{p\times n}$. There are p equality and n inequality constraints, the Lagrangian with domain $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^n$ is:

$$\begin{aligned} L(x, \lambda, \mu) &= c^T x + \lambda^T(Ax - b) - \mu^T x \\ &= -\lambda^T b + (c + A^T\lambda - \mu)^T x \end{aligned}$$

L is linear in x, and its minimum can only be 0 *or* $-\infty$:

$$g(\lambda, \mu) = \inf_{x\in\mathbb{R}^n} L(x, \lambda, \mu) = \begin{cases} -\lambda^T b, \ if\ A^T\lambda - \mu + c = 0 \\ -\infty, otherwise \end{cases}$$

g is linear on an affine subspace and therefore concave. The lower bound is non-trivial when λ and μ satisfy $\mu \geq 0$ and $A^T\lambda - \mu + c = 0$, giving the following bound:

$$f^* \geq -\lambda^T b, \ if\ A^T\lambda + c \geq 0$$

1.2 Conjugate Functions

Definition 1.1. (Linear functional) A vector space over F the mapping $T : V \to F$ satisfy the following condition

(LF1) $T(x+y) = Tx + Ty,\ \forall x, y \in V$

(LF2) $T(\alpha x) = \alpha Tx,\ \forall x \in V, \alpha \in F$

Definition 1.2. (Linear continuous functional) If a linear functional $f : V \to F$ has real or complex number values, then f is continuous for each $x \in V$ iff $\exists M$ such that

$$\frac{|f(x)|}{\|x\|} \leq M.$$

Definition 1.3. The topological vector space $L(X, \mathbb{R})$ is said to be the topological dual space of X being denoted by X^*. Further, we refer with dual to topological dual not to algebraical ones, unless other wise specified. Analogously to vector space by $\langle x^*, x\rangle$ we denote the value taken at $x \in X$ by the linear continuous functional $x^* \in X^*$.

Conjugate functions play an important role in the duality theory. Let U be a linear normed space (*e.g.* $\mathbb{R}^n$), U^* is topological dual space, while $\langle x^*, x\rangle$ denotes the value of the linear continuous functional $x^* \in U^*$ at the point $x \in U$.

Definition 1.4. *[7]* Let $f : U \to \bar{R}$ be a given function. Then the function $f^* : U^* \to \bar{R}$ given by

$$f^*(x^*) = \sup_{x \in U}\{\langle x^*, x\rangle - f(x)\}$$

is called the conjugate function of f.

Proposition 1.1. *[7]* (some elementary properties of conjugate functions)

i. (Young′s inequality)

$$f(x) + f^*(x^*) \geq \langle x^*, x\rangle,\ \forall x \in U,\ \forall x^* \in U^*$$

ii. $\inf_{x \in U} f(x) = -f^*(0)$

iii. $f \leq g \Rightarrow f^* \geq g^*$

iv. $(\lambda \cdot f)^*(x^*) = \lambda \cdot f^*(\frac{x^*}{\lambda})$ $,\forall \lambda \geq 0$.

Proof. **i)** follows immediately from the definition.

ii)

$$\begin{aligned} f^*(0) &= \sup_{x \in U}\{\langle 0, x\rangle - f(x)\} \\ &= -\inf_{x \in U} f(x) \end{aligned}$$

This implies

$$\inf_{x \in U} f(x) = -f^*(0)$$

iii) for any $x \in U$ and $x^* \in U^*$ we have

$$\langle x^*, x\rangle + (-g(x)) \leq \langle x^*, x\rangle + (-f(x))$$

and considering the supremum after $x \in U$ it holds

$$\sup_{x\in U}\{\langle x^*, x\rangle + (-g(x))\} \leq \sup_{x\in U}\{\langle x^*, x\rangle + (-f(x))\}$$

which actually means

$$g^*(x^*) \leq f^*(x^*)$$

This implies

$$g^* \leq f^*$$

iv)

$$\begin{aligned}(\lambda \cdot f)^*(x^*) &= \sup_{x\in U}\{\langle x^*, x\rangle - (\lambda \cdot f(x))\}\\ &= \sup_{x\in U}\{\lambda \cdot [\langle \frac{x^*}{\lambda}, x\rangle - f(x)]\}\\ &= \lambda \cdot \sup_{x\in U}\{\langle \frac{x^*}{\lambda}, x\rangle - f(x)\}\\ &= \lambda \cdot f^*(\frac{x^*}{\lambda})\end{aligned}$$

This implies $(\lambda \cdot f)^*(x^*) = \lambda \cdot f^*(\frac{x^*}{\lambda})$

□

Definition 1.5. *[7]* Let $f^* : U^* \to \bar{R}$ be a given function. Then $f^{**} : U \to \bar{R}$

$$f^{**}(x) = \sup_{x^*\in U^*}\{\langle x^*, x\rangle - f^*(x^*)\}$$

is called the biconjugate function to f.

1.3 Lagrange Dual Problem

For the (primal) problem:

$$\begin{aligned} \textit{minimize} \;\; & f(x) \\ \textit{subject to} \;\; & g(x) \leq 0, \\ & h(x) = 0 \end{aligned} \tag{1.1}$$

For each pair (λ, μ) with $\lambda \geq 0$, the Lagrange dual function gives us a lower bound on the optimal value f^* of the optimization problem (1.1). Thus we have a lower bound that depends on some parameters λ, μ. A natural question is: What is the best lower bound that can be obtained from the Lagrange dual function?

This leads to the optimization problem

$$\begin{aligned} &maximize \;\; g(\lambda,\mu) \\ &subject \;\; to \;\; \lambda \geq 0, \end{aligned} \tag{1.2}$$

This problem is called the Lagrange dual problem associated with the problem (1.1). In this context the original problem (1.1) is sometimes called the primal problem. The term dual feasible, to describe a pair (λ,μ) with $\lambda \geq 0$ and $g(\lambda,\mu) > -\infty$, now makes sense. It means, as the name implies, that (λ,μ) is feasible for the dual problem (1.2). We refer to (λ^*,μ^*) as dual optimal or optimal Lagrange multipliers if they are optimal for the problem (1.2).

Example 1.3. *[10]* Standard form LP:

$$\begin{aligned} minimize \;\; & c^T x \\ subject \;\; to \;\; & Ax = b \\ & x \geq 0 \end{aligned}$$

This reduced to

$$\begin{aligned} minimize \;\; & c^T x \\ subject \;\; to \;\; & Ax - b = 0 \\ & -x \leq 0 \end{aligned}$$

Where $A \in \mathbb{R}^{p \times n}$. There are p equality and n inequality constraints, the Lagrangian with domain $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^n$ is:

$$\begin{aligned} L(x,\lambda,\mu) &= c^T x + \mu^T (Ax - b) - \lambda^T x \\ &= -\mu^T b + (c + A^T \mu - \lambda)^T x \end{aligned}$$

L is linear in x, and its minimum can only be 0 *or* $-\infty$:

$$g(\lambda,\mu) = \inf_{x \in \mathbb{R}^n} L(x,\lambda,\mu) = \begin{cases} -\mu^T b, \; if \; A^T\mu - \lambda + c = 0 \\ -\infty, otherwise \end{cases}$$

The Lagrange dual problem of the standard form LP is to maximize this dual function g subject to $\lambda \geq 0$, i.e.,

$$\begin{aligned} &maximize \;\; g(\lambda,\mu) = \begin{cases} -\mu^T b, \; if \; A^T\mu - \lambda + c = 0 \\ -\infty, otherwise \end{cases} \\ &subject \;\; to \;\; \lambda \geq 0, \end{aligned}$$

Here g is finite only when $A^T\mu - \lambda + c = 0$. We can form an equivalent problem by making these equality constraints explicit:

$$\begin{aligned} maximize \;\; & -\mu^T b, \; if \\ subject \;\; to \;\; & A^T\mu - \lambda + c = 0 \\ & \lambda \geq 0, \end{aligned}$$

1.4 Weak and Strong Duality

1.4.1 Weak Duality

Let d^* be the optimal value of the Lagrange dual problem. Each $g(\lambda, \mu)$ is a lower bound for f^* and by definition d^* is the best lower bound that is obtained. The following weak duality inequality therefore always hold:

$$d^* \leq f^*$$

This inequality holds also when d^* *or* f^* are infinite. The difference $d^* - f^*$ is called the **optimal duality gap** of the original problem.

1.4.2 Strong Duality

We say that strong duality holds if the optimal duality gap is zero, i.e

$$d^* = f^*$$

Remark.

- If strong duality holds, then the best lower bound that can be obtained from the Lagrange dual function is tight
- Strong duality does not hold for general nonlinear problems
- It usually holds for convex problems
- Conditions that ensure strong duality for convex problems are called constraint qualification.

1.5 Multiobjective Optimization Problem

A multi-objective optimization problem is an optimization problem that involves multiple objective functions.
The multi-objective optimization problem is generally formulated as:

$$Minimize/Maximize \;\; f_m(x), \quad m = 1, 2, \ldots, M; \tag{1.3}$$

$$subject \;\; to \;\; g_j(x) \geq 0, \quad j = 1, 2, \ldots, J; \tag{1.4}$$

$$h_k(x) = 0, \quad k = 1, 2, \ldots, K; \tag{1.5}$$

$$x_i^L \leq x_i \leq x_i^u, \quad i = 1, 2, \ldots, n \tag{1.6}$$

A variable x is a vector of n decision variables: $x = (x_1, x_2, \ldots x_n)^T$. Equation(1.6) are called variables bound restricting each decision x_i to take a value with a lower x_i^L and an upper x_i^U bound. This bound constitute a decision variable space D or simply the

decision space. The term g_j and h_k are constrained functions.
Any $x \in \mathbb{R}^n$ that satisfies all constraints and variable bounds is known as a feasible solution. Otherwise it is called infeasible solution.
Multiobjective optimization is referred as Vector optimization because a vector objectives instead of a single objective is optimized.
If all objective functions and constraints are linear then the multiobjective optimization problem (MOOP) is called multiobjective linear program (MOLP). Thus the objective functions are

$$f_k(x) = c_k^T x, \quad k = 1, 2, \ldots, p$$

Where $c_k \in \mathbb{R}^n$. The constraints $g_j(x) \leq 0$ are similarly written in matrix form and as equality constraints $Ax = b$.
As usual in linear programming we restrict the variables to the nonnegative orthant of $\mathbb{R}^n : x \geq 0$ A multiple objective linear program (MOLP) is given by

$$Min\ Cx$$

$$subject\ to\ Ax = b$$

$$x \geq 0$$

with a $p \times n$ objective or criteria matrix C consisting of the rows $c_k^T, \quad k = 1, 2, \ldots, p$. The feasible set in decision space is defined by the $m \times n$ constraint matrix A and the right hand side vector $b \in \mathbb{R}^m$.
The feasible set in decision space is

$$X = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$$

The feasible set in objective space is

$$Y = CX = \{Cx : x \in X\}$$

Minimizing a vector-valued objective function needs some more explanation since there is no canonical ordering defined in $\mathbb{R}^p$ for $p \geq 2$. Throughout this report we use the Pareto concept of optimality for MOLPs which is based on the following two binary relations $<$ and $\leq$.Let $y^1, y^2 \in \mathbb{R}^p$. Then

$$y^1 \leq y^2 \Leftrightarrow, y_k^1 \leq y_k^2 \ \forall k = 1, 2, \ldots, p \ \&\ y^1 \neq y^2$$

$$y^1 < y^2 \Leftrightarrow y_k^1 < y_k^2 \ \forall k = 1, 2, \ldots, p$$

A point $y^2 \in \mathbb{R}^p$ is called dominated by $y^1 \in \mathbb{R}^p$ if $y^1 \leq y^2$.

Definition 1.6. *[1]* Let $\hat{x} \in X$ be a feasible solution of the MOLP and let $\hat{y} = C\hat{x}$.

1. $\hat{x}$ is called weakly efficient if there is no $x \in X$ such that $Cx < C\hat{x}; \hat{y} = C\hat{x}$ is called weakly nondominated.

2. $\hat{x}$ is called efficient if there is no $x \in X$ such that $Cx \leq C\hat{x}; \hat{y} = C\hat{x}$ is called nondominated.

3. $\hat{x}$ is called properly efficient if it is efficient and if there exists a real number $M > 0$ such that for all i and x with $C_i^T x < C_i^T \hat{x}$ there is an index j and $M > 0$ such that $C_j^T x > C_j^T \hat{x}$ and $\frac{C_i^T \hat{x} - C_i^T x}{C_j^T x - C_j^T \hat{x}} \leq M$.

1.5.1 Linear and Nonlinear MOOP

If all objective function and constraint function are linear, the resulting MOOP is called a multiobjective linear program (MOLP) like the linear programming problems MOLPs also have many theoretical properties. However, if any of the objective or constraint functions are nonlinear the resulting problem is called nonlinear multiobjective problem. Unfortunately, for nonlinear problems the solution techniques often do not have convergence proofs since most real world multiobjective optimization problems are nonlinear in nature we do not assume any particular structure of the objective and constraint function here.

1.5.2 Convex and Nonconvex MOOP

Before we discuss a convex multiobjective optimization problem let us first define a convex set and convex function.

Definition 1.7. A set S is convex if the line segment between any two points in S lies in S, i.e a set $S \subseteq \mathbb{R}^n$ is called a convex set if and only if $x^1, x^2 \in S$ and for all $\lambda \in [0, 1]$ it holds that $\lambda x^1 + (1 - \lambda)x^2 \in S$.

Definition 1.8. *[3]* A function $f : S \to \mathbb{R}$, where S is a non empty convex set is a convex function if

$$f(\lambda x^1 + (1 - \lambda)x^2) \leq \lambda f(x^1) + (1 - \lambda)f(x^2) \tag{1.7}$$

for all $x^1, x^2 \in S$ & $forall$ $\lambda \in [0, 1]$.

The above definition gives rise to the following properties of a convex function

1. The linear approximation of $f(x)$ at any point in the interval $[x, y]$ always underestimates the actual function value.

2. The Hessian matrix of $f(x)$ is positive semidefinite for all x.

3. For a convex function a local minimum is always a global minimum.

Equation (1.7) with a $' >'$ sign instead of $' \leq'$ sign is called a nonconvex function. To test if a function is convex within an interval the Hessian matrix $\nabla^2 f$ is calculated and checked for its positive definiteness at all points in the interval. One way to check the positive definiteness of a matrix is to compute the eigenvalues of the matrix and check to see if all eigenvalue are positive. To test if a function f is nonconvex in an interval the Hessian matrix $\nabla^2 f$ is checked for its positive definiteness. If it is not positive semidefinite the function f is nonconvex

Definition 1.9. *[3]* A multiobjective optimization problem is convex if all the objective functions are convex and the feasible region is convex (or all inequality constraints are convex and equality constraints are linear).

According to this definition an MOLP is a convex problem.

1.5.3 Methods to Solve Multiobjective Optimization Problems (MOOP)

Definition 1.10. *[5]* Let $X \subseteq \mathbb{R}^n$ be a non-empty set of feasible solutions and $f = (f_1, \dots, f_p) : \mathbb{R}^n \to \mathbb{R}^p$ be a function. Feasible solution $x^* \in X$ is called a Pareto optimal solution of the multi-objective optimization problem

$$\min\{f(x) : x \in X\}$$

if and only if there does not exist any $x \in X$ such that $f(x) \leq f(x^*)$.

Weighted sum method

The weighted sum method, as the name suggests, scalarizes a set of objectives in to a single objective by pre-multiplying each objective with a user supplied weight. This method is the simplest approach and is probably the most widely used classical approach. It is formulated as.[3]

$$minimize\ \ F(x) = \sum_{n=1}^{m} w_n f_n(x) \tag{1.8}$$

$$subject\ \ to\ \ g_j(x) \geq 0, \quad j = 1, 2, \dots, J;$$

$$h_k(x) = 0, \quad k = 1, 2, \dots, K;$$

$$x_i^L \leq x_i \leq x_i^u, \quad i = 1, 2, \dots, n$$

Where $w_n \in [0, 1]$ is the weight of the n^{th} objective function and $\sum_{n=1}^{m} w_n = 1$

Theorem 1.1. The solution of the problem represented by equation (1.8) is pareto optimal if the weights for all objectives are positive.[3]

Proof. Let x^* be a solution of the weighting problem with positive weighting coefficient. Let us suppose that it is not pareto optimal. This means that there is a solution x such that $f_i(x) \leq f_i(x^*)$ for all $i = 1, 2, \dots, k$ and $f_j(x) < f_j(x^*)$for at least one j. Since $w_i > 0$ for all $i = 1, 2, \dots, k$, we have

$$\sum_{i=1}^{k} w_i f_i(x) < \sum_{i=1}^{k} w_i f_i(x^*),$$

This contradicts the assumption that x^* is a solution of the weighting problem and, thus x^* must be pareto optimal. □

Theorem 1.2. *[3]* If x^* is a pareto-optimal solution of a convex multiobjective optimization problem, then there exist a nonzero positive weight w such that x^* is a solution to the problem given by equation (1.8).

Advantages of weighted sum method

This is probably the simplest way to solve a Multiobjective optimization problem. The concept is intuitive and easy to use, for the problems having a convex pareto-optimal, this method guarantees finding solution on the entire pareto-optimal set.[3]

Disadvantages of weighted sum method

Since most single objective optimization algorithms are designed to find a solution that satisfy the first optimality criterion, more test are needed to know whether the obtained solution is truly a minimum solution. This increases the computational complexity of the weighted sum approach. An property of interesting MOOPs is that, there may exists multiple minimum solution for a specific weighted vector. However, each of this minimum solution may represent a different solution in the pareto-optimal set. Some of these solutions can be weakly dominated to each other,thereby westing the search effort. Moreover, if the chosen single objective optimization algorithm can not find all minimum solutions for a weighted vector, some pareto-optimal solution can not be found.[3]

ϵ-constraint method

In order to alleviate the difficulties faced by weighted sum approach in solving problems with non convex objective functions, the ϵ-constraint method is used. This method keeps one of the objective and restricts the rest of the objectives within user specified values.
The modified problem is as follows:[3]

$$minimize \;\; f_\mu(x) \tag{1.9}$$

$$\begin{aligned} subject \;\; to \;\; f_m(x) \leq \epsilon_m \quad & m = 1, 2, \dots, M \; and \; m \neq \mu \\ g_j(x) \geq 0, \quad & j = 1, 2, \dots, J; \\ h_k(x) = 0, \quad & k = 1, 2, \dots, K; \\ x_i^L \leq x_i \leq x_i^u, \quad & i = 1, 2, \dots, n \end{aligned}$$

In the above formulation the parameter ϵ_m represents upper bound of the value of f_m and need not necessarily mean a small value close to zero.

Theorem 1.3. *[3]* The unique solution of the ϵ-constraint problem stated in equation(1.9) is pareto-optimal for any given upper bound vector $\epsilon = (\epsilon_1, \dots, \epsilon_{\mu-1}, \epsilon_{\mu+1}, \dots, \epsilon_m)^T$.

Proof. Let x^* be a unique solution of the ϵ-constraint problem. This mean that $f_l(x^*) < f_l(x)$ for all x in the decision space when $f_j(x) \leq \epsilon_j$ for every $j = 1, 2, \dots, K, \; j \neq l$. Let us assume that x^* is not pareto optimal. In this case, there is a vector x^0 in the decision space such that $f_i(x^0) \leq f_i(x^*)$ for all $i = 1, 2, \dots, K$ and the inequality is strict for at least one index j. If $j = l$, this means that $f_l(x^0) < f_l(x^*)$ and $f_i(x^0) \leq f_i(x^*) \leq \epsilon_j$ for all $i \neq l$. Here we have a contradiction with the fact x^* is a solution of the ϵ-constraint problem. On the other hand, if $j \neq l$, then $f_j(x^0) \leq f_j(x^*) \leq \epsilon_j$,$f_i(x^0) \leq \epsilon_j$ for all $i \neq l$ and $f_l(x^0) < f_l(x^*)$. This is a contradiction to x^* as a unique solution of the ϵ-constraint problem, and x^* has to be pareto optimal. □

Advantages

Different pareto-optimal solutions can be found by using different ϵ_m values. The same method can be used for the problems having convex or non convex objective spaces alike. In terms of the information needed from the user, this algorithm is similar to the weighted sum approach. In this approach a vector of ϵ values representing in some sense, the location of the pareto-optimal solution is needed. However, the advantage of this method is that it can be used for any arbitrary problem with either convex or non convex objective space.[3]

Disadvantages

The solution of the above problem largely depend on the choose of ϵ vector. It must be chosen so that it lies within the minimum or maximum value of the individual objective function. Moreover as the number of objective increases there exist more element in the ϵ vector, thereby requiring more information from the user.[3]

Weighted metric methods

Instead of using a weighted sum of the objectives, other means of combining multiple objectives in to a single objective can also be used. For this purpose weighted metrics such as l_p and l_∞ distance metrics are often used. For non-negative weights the weighted l_p distance measure of any solution x from the ideal solution z^* can be minimized as follows:[3]

$$minimize \;\; l_p(X) = \Big(\sum_{m=1}^{M} w_m \mid f_m(x) - z_m^* \mid^p \Big)^{\frac{1}{p}} \tag{1.10}$$

$$subject \;\; to \;\; g_j(x) \geq 0, \quad j = 1, 2, \ldots, J;$$

$$h_k(x) = 0, \quad k = 1, 2, \ldots, K;$$

$$x_i^L \leq x_i \leq x_i^u, \quad i = 1, 2, \ldots, n$$

The parameter p can take any value between 1 and ∞. when $p = 1$ is used, the resulting problem is equivalent to the weighted sum approach. When $p = 2$ is used, a weighted Euclidean distance of any point in the objective space from the ideal point is minimized. When a large p is used the above problem is reduced to a problem of minimizing the largest deviation $\mid f_m(x) - z_m^* \mid$. This problem has a spacial name − the weighted Tchebycheff problem:

$$minimize \;\; l_\infty(X) = \max_{m=1}^{M} w_m \mid f_m(x) - z_m^* \mid \tag{1.11}$$

$$subject \;\; to \;\; g_j(x) \geq 0, \quad j = 1, 2, \ldots, J;$$

$$h_k(x) = 0, \quad k = 1, 2, \ldots, K;$$

$$x_i^L \leq x_i \leq x_i^u, \quad i = 1, 2, \ldots, n$$

However, the resulting optimal solution obtained by the chosen l_p depend on the parameter p. When the weighted Tchebycheff metric is used, any pareto-optimal solution can be found.[3]

Theorem 1.4. Let x^* be a pareto-optimal solution Then there exist a positive weighting vector such that x^* is a solution of the weighted Tchebycheff problem shown in equation(1.11), where the reference point is the utopian objective vector Z^{**}.

Proof. Let x^* in decision space be pareto optimal. Let us assume that there does not exist a weighting vector $w > 0$ such that x^* is a solution of the weighted Tchebycheff problem. We know that $f_i(x) > z_i^*$ for all $i = 1, 2, \ldots, k$ and for all x in the decision space. Now we choose $w_i = \frac{\beta}{f_i(x) - z_i^*}$ for all $i = 1, 2, \ldots, k$ where $\beta > 0$ is same normalizing factor. If x^* is not a solution of the weighted Tchebycheff problem, there exists another point x^0 in decition space that is a solution of the weighted Tchebycheff problem, mean that

$$\begin{aligned} \max_i [w_i(f_i(x^0) - z_i^*)] &< \max_i [w_i(f_i(x^*) - z_i^{**})] \\ &= \max_i [\frac{\beta}{f_i(x) - z_i^*}(f_i(x^*) - z_i^*)] \\ &= \beta \end{aligned}$$

Thus $w_i(f_i(x^0) - z_i^*) < \beta$ for all $i = 1, 2, \ldots, k$ this mean that

$$\frac{\beta}{f_i(x^0) - z_i^*}(f_i(x^*) - z_i^*) < \beta$$

And after simplifying the expression we have $f_i(x^0) < f_i(x^*)$ for all $i = 1, 2, \ldots, k$. Here we have a contradiction with the pareto optimality of x^*. □

Advantages

The weighted Tchebycheff metric guarantee finding each and every pareto-optimal solution when z^* is a utopian objective vector. Although in the above discussion only l_p metrics are suggested, other distance metrics are also used.[3]

Disadvantages

Since different objectives may take values of different orders of magnitude, it is advisable to normalize the objective function. This requires a knowledge of minimum and maximum function values of each objective. moreover, this method also requires the ideal solution z^* therefore all M objectives need to be independently optimized before optimizing the l_p metrics.[3]

Benson's method

This procedure is similar to the weighted metric approach, except that the reference solution is taken as a feasible non-pareto optimal solution. A solution z^0 is randomly chosen from the feasible region. Thereafter the non-negative difference $(z_m^0 - f_m(x))$ of each objective is calculated and their sum is maximized.

$$maximize \sum_{m=1}^{M} \max(0, (z_m^0 - f_m(x))) \tag{1.12}$$

$$subject\ \ to\ \ f_m(x) \leq z_m^0\ \ m = 1, 2, \ldots, M$$
$$g_j(x) \geq 0, \quad j = 1, 2, \ldots, J;$$
$$h_k(x) = 0, \quad k = 1, 2, \ldots, K;$$
$$x_i^L \leq x_i \leq x_i^u, \quad i = 1, 2, \ldots, n$$

For any solution x, the above objective function has a value equal to half of the perimeter of a hyperbole having z^0 and $f(x)$ as the diagonal points. The only requirement is that the solution x must weakly dominate the solution at z^0. The maximization of the above objective is similar to finding a hypercube with the maximum perimeter. Since the pareto-optimal region lies at the extreme of the feasible search space, the optimum solution of the above optimization problem is a member of the pareto optimal.

Advantages

To avoid scaling problems individual difference can be normalized before the summation. To obtain different pareto optimal solution the difference can be weighted before summation. Thereafter by changing the weight vector, different pareto optimal solution can be obtained. In such a scenario, the use of the nadir point z^{nad}, as chosen point may be found as suitable. If z^0 is chosen appropriately this method can be used to find solutions in the non-convex pareto optimal region.[3]

Disadvantages

The optimization problem formulated above has an additional number of constraints needs to restrict the search in the region dominating the chosen solution z^0. Moreover, the objective function is non-differentiable, thereby chousing difficulties for gradient based methods to solve the above problem.[3]

Value function method

In the value function(or utility function) method the user provide a mathematical value function $U : \mathbb{R}^m \rightarrow \mathbb{R}$ relating all M objectives. The value function must be valid over the entire feasible search space. The task is then to maximize the value function as follow:[3]

$$maximize\ \ U(f(x)) \tag{1.13}$$

$$subject\ \ to\ \ g_j(x) \geq 0, \quad j = 1, 2, \ldots, J;$$
$$h_k(x) = 0, \quad k = 1, 2, \ldots, K;$$
$$x_i^L \leq x_i \leq x_i^u, \quad i = 1, 2, \ldots, n$$

Here, $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$. As seen from the above problem the value functions provides interactions among different objectives. Among the two solution i and j if $U(f(i)) > U(f(j))$ solution i is then preferred to solution j.

Advantages

This idea is simple and ideal, if adequate value function information is available. The value function methods are mainly used in practice to multi-attribute decision analysis problem with a decreta set of feasible solutions, although the principle can also be used in continuous search space.[3]

Disadvantages

As evident from the above discussion the obtained solution entirely depends on the chosen value function. It also requires users to came up with a value function, which is globally applicable over the entire search space. Thus there is a danger of using an over simplified function.[3]

Chapter 2

Conjugate Duality in Scalar Optimization Problem

We consider three different conjugate dual problems the well-known Lagrange and Fenchel dual problems (denoted by (D_L) and (D_F), respectively) and a combination of the above two, which we shall call the Fenchel-Lagrange dual problem (denoted by (D_{FL})).

2.1 Problem Formulation

Before we discuss a problem formulation let us first define a cone, convex cone, dual cone and closed convex cone.

Definition 2.1. Let C be any non empty subset of a vector space V. The set C is said to be a cone if $x \in C$ implies $\lambda x \in C, \forall \lambda \geq 0$.

Definition 2.2. A subset C of a vector space V is a convex cone if $\alpha x + \beta y$ belongs to C for any positive scalars α, β and $x, y \in C$.

Definition 2.3. Let K be a cone. The set $K^* = \{y \mid x^T y \geq 0, \forall x \in K\}$ is called the dual cone of K. As the name suggests K^* is a cone and is always convex even when the original cone K is not.

Definition 2.4. Let $K \subseteq V$ be a nonempty closed set. K is called a closed convex cone if the following two properties hold.

(i) For all $x \in K$ and all nonnegative real numbers λ, we have $\lambda x \in K$.

(ii) For all $x, x' \in K$, we have $x + x' \in K$.

Let $X \subseteq \mathbb{R}^n$ be a nonempty set and $K \subseteq \mathbb{R}^k$ a nonempty closed convex cone with $int(K) \neq \emptyset$. The set $K^* := \{k^* \in \mathbb{R}^k : k^{*T} k \geq 0,\ \forall k \in K\}$ is the dual cone of K. Consider the partial order " $\leq$ " induced by K in $\mathbb{R}^k$, namely for
$y, z \in \mathbb{R}^k$ we have that $y \leq z$,iff $z - y \in K$. Let $f : \mathbb{R}^n \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ and $g = (g_1, \ldots, g^k)^T : \mathbb{R}^n \to \mathbb{R}^k$. The optimization problem we investigate in this section is

$$(P) \qquad \inf_{x \in G} f(x)$$

Where

$$G = \{x \in X : g(x) \leq 0\},\ X \subseteq \mathbb{R}^n$$

In the following we suppose that the feasible set G is nonempty. Assume further that $dom(f) = X$, where $dom(f) := \{x \in \mathbb{R}^n : f(x) < +\infty\}$.
The problem (P) is said to be an optimal and its optimal objective value is denoted $inf(P)$.

Definition 2.5. *[2]* An element $\bar{x} \in G$ is said to be an optimal solution for (P) if $f(\bar{x}) = inf(P)$

The aim of this section is to construct different dual problems to (P). Now let us first consider the general optimization problem without constraints

$$(PG) \qquad \inf_{x \in \mathbb{R}^n} F(x)$$

With F a mapping from $\mathbb{R}^n$ in to $\bar{\mathbb{R}}$.

Remark. By the assumption we made for f, we have

$$f^*(p^*) = \sup_{x \in \mathbb{R}^n} \{p^{*T}x - f(x)\} = \sup_{x \in X} \{p^{*T}x - f(x)\}.$$

The approach is based on the construction of a so called perturbation function $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \bar{\mathbb{R}}$ with the property that $\Phi(x, 0) = F(x)$ for each $x \in \mathbb{R}^n$.
Here, $\mathbb{R}^m$ is the space of the perturbation variables. For each $p \in \mathbb{R}^m$ we obtain a new optimization problem

$$(PG_p) \qquad \inf_{x \in \mathbb{R}^n} \Phi(x, p)$$

For $p \in \mathbb{R}^m$ the problem (PG_p) is called the perturbed problem of (PG).
By Definition 1.4 the conjugate of Φ is the function $\Phi^* : \mathbb{R}^n \times \mathbb{R}^m \to \bar{\mathbb{R}}$

$$\begin{aligned} \Phi^*(x^*, p^*) &= \sup_{x \in \mathbb{R}^n, p \in \mathbb{R}^m} \{(x^*, p^*)^T(x, p) - \Phi(x, p)\} \\ &= \sup_{x \in \mathbb{R}^n, p \in \mathbb{R}^m} \{x^{*T}x + p^{*T}p - \Phi(x, p)\} \end{aligned} \tag{2.1}$$

Now at $x^* = 0$ we can define the following optimization problem

$$(DG) \qquad \sup_{p^* \in \mathbb{R}^m} \{p^{*T}p - \Phi^*(0, p^*)\}$$

The problem (DG) is called the dual problem to (PG) and its optimal objective value is denoted by $sup(DG)$.

Theorem 2.1. *[2]* The relation

$$-\infty \leq sup(DG) \leq inf(PG) \leq +\infty \tag{2.2}$$

always holds.

Proof. Let $p^* \in \mathbb{R}^m$ from (2.1) we obtain

$$\begin{aligned}\Phi^*(0, p^*) &= \sup_{x\in\mathbb{R}^n, p\in\mathbb{R}^m} \{0^T x + p^{*T} p - \Phi(x, p)\} \\ &= \sup_{x\in\mathbb{R}^n, p\in\mathbb{R}^m} \{p^{*T} p - \Phi(x, p)\} \\ &\geq \sup_{x\in\mathbb{R}^n} \{p^{*T} 0 - \Phi(x, 0)\} \\ &= \sup_{x\in\mathbb{R}^n} \{-\Phi(x, 0)\}\end{aligned}$$

This means that, for each $p^* \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$ it holds

$$-\Phi^*(0, p^*) \leq \Phi(x, 0) = F(x)$$

which implies that $sup(DG) \leq inf(PG)$. □

Our next aim is to show how we can apply this approach to the constrained optimization problem (P). Therefore, let $F : \mathbb{R}^n \to \bar{\mathbb{R}}$ be the function given by

$$F(x) = \begin{cases} f(x), if\ x \in G \\ +\infty,\ otherwise \end{cases}$$

The primal problem (P) is then equivalent to

$$(PG) \qquad\qquad \inf_{x\in\mathbb{R}^n} F(x)$$

and, since the perturbation function $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \bar{\mathbb{R}}$ satisfies $\Phi(x, 0) = F(x)$ for each $x \in \mathbb{R}^n$, we obtain that

$$\Phi(x, 0) = f(x),\ \forall x \in G \tag{2.3}$$

and

$$\Phi(x, 0) = +\infty,\ \forall x \in \mathbb{R}^n \setminus G. \tag{2.4}$$

2.2 The Lagrange Dual Problem

Let the function $\Phi_L : \mathbb{R}^n \times \mathbb{R}^k \to \bar{\mathbb{R}}$ be defined by

$$\Phi_L(x, q) = \begin{cases} f(x),\ if\ x \in X\ , g(x) \underset{K}{\leqq} q \\ +\infty,\ otherwise \end{cases}$$

With the perturbation variable $q \in \mathbb{R}^k$. It is obvious that the relation (2.3) and (2.4) are fulfilled. For the conjugate of Φ_L we have

$$\begin{aligned}\Phi_L^*(x^*, q^*) &= \sup_{x\in\mathbb{R}^n, q\in\mathbb{R}^k} \{x^{*T} x + q^{*T} q - \Phi_L(x, q)\} \\ &= \sup_{x\in X, q\in\mathbb{R}^k, g(x)\underset{K}{\leqq} q} \{x^{*T} x + q^{*T} q - f(x)\}\end{aligned}$$

In order to calculate this expression we introduce the variable s instead of q by $s = q - g(x) \in K$. This implies

$$\begin{aligned}
\Phi_L^*(x^*, q^*) &= \sup_{x \in X, s \in K} \{x^{*T}x + q^{*T}[s + g(x)] - f(x)\} \\
&= \sup_{x \in X}\{x^{*T}x + q^{*T}g(x) - f(x)\} + \sup_{s \in K} q^{*T}s \\
&= \begin{cases} \sup_{x \in X}\{x^{*T}x + q^{*T}g(x) - f(x)\}\ , & \textbf{if } q^* \in -K^* \\ +\infty, & otherwise. \end{cases}
\end{aligned}$$

As we have seen, the dual of (P)obtained by the perturbation function Φ_L is

$$(D_L) \qquad \sup_{q^* \in \mathbb{R}^k} \{-\Phi_L^*(0, q^*)\}$$

and, since

$$\sup_{q^* \in -K^*} \{-\sup_{x \in X}[q^{*T}g(x) - f(x)]\} = \sup_{q^* \in -K^*} \{\inf_{x \in X}[-q^{*T}g(x) + f(x)]\}$$

the dual has the following from

$$(D_L) \qquad \sup_{q^* \underset{K^*}{\geqq} 0} \inf_{x \in X}[f(x) + q^{*T}g(x)] \tag{2.5}$$

The problem D_L is actually the well known lagrange dual problem. Its optimal objective value is denoted by $sup(D_L)$ and Theorem 2.1 implies

$$sup(D_L) \leq inf(P) \tag{2.6}$$

Example 2.1. *[2]* Let $K = \mathbb{R}_+, X = [0, +\infty) \subseteq \mathbb{R}$, $f : \mathbb{R} \to \bar{\mathbb{R}},\ g : \mathbb{R} \to \mathbb{R}$, be defined by

$$f(x) = \begin{cases} -x^2, & if\ x \in X \\ +\infty, & otherwise \end{cases}$$

and

$$g(x) = x^2 - 1.$$

Now the optimal objective value of the Lagrange dual is

$$\begin{aligned}
sup(D_L) &= \sup_{q^* \geq 0} \inf_{x \geq 0}[f(x) + q^*g(x)] \\
&= \sup_{q^* \geq 0} \inf_{x \geq 0}[-x^2 + q^*(x^2 - 1)] \\
&= \sup_{q^* \geq 0} \inf_{x \geq 0}[(q^* - 1)x^2 - q^*] \\
&= \sup_{q^* \geq 1}(-q^*) \\
&= -1
\end{aligned}$$

This implies that the optimal objective value of D_L is -1.
We are now interested to obtain dual problem for (P), different from the classical lagrange problem.

2.3 Fenchel Dual Problems

Let us consider the perturbation function $\Phi_F : \mathbb{R}^n \times \mathbb{R}^n \to \bar{\mathbb{R}}$ given by

$$\Phi_F(x,p) = \begin{cases} f(x+p), & if\ x \in G \\ +\infty, & otherwise \end{cases}$$

With the perturbation variable $p \in \mathbb{R}^n$. The relation (2.3) and (2.4) are also fulfilled and it holds

$$\begin{aligned}\Phi_F^*(x^*,p^*) &= \sup_{x\in\mathbb{R}^n, p\in\mathbb{R}^n} \{x^{*T}x + p^{*T}p - \Phi_F(x,p)\} \\ &= \sup_{x\in X, p\in\mathbb{R}^n, g(x) \underset{K}{\leqq} 0} \{x^{*T}x + p^{*T}p - f(x+p)\}\end{aligned}$$

Introducing a new variable $r = x + p \in \mathbb{R}^n$, we have

$$\begin{aligned}\Phi_F^*(x^*,p^*) &= \sup_{x\in X, r\in\mathbb{R}^n, g(x) \underset{K}{\leqq} 0} \{x^{*T}x + p^{*T}(r-x) - f(r)\} \\ &= \sup_{r\in\mathbb{R}^n}\{p^{*T}r - f(r)\} + \sup_{x\in X, g(x) \underset{K}{\leqq} 0} \{(x^* - p^*)^T x\} \\ &= f^*(p^*) - \inf_{x\in X, g(x) \underset{K}{\leqq} 0} \{(p^* - x^*)^T x\} \\ &= f^*(p^*) - \inf_{x\in G}\{(p^* - x^*)^T x\}\end{aligned}$$

Now the dual of (P) will be:

$$(D_F) \qquad \sup_{p^*\in\mathbb{R}^n} \{-\Phi_F^*(0,p^*)\}$$

and can be written in the form

$$(D_F) \qquad \sup_{p^*\in\mathbb{R}^n} \{-f^*(p^*) + \inf_{x\in X, g(x) \underset{K}{\leqq} 0} p^{*T}x\}$$

Denoting by

$$X_G(x) = \begin{cases} 0, & if\ \ x \in G \\ +\infty, & otherwise \end{cases}$$

The indicator function of the set G we have that $X_G^*(-p^*) = -\inf_{x\in G} p^{*T}x$. The dual D_F becomes then

$$(D_F) \qquad \sup_{p^*\in\mathbb{R}^n} \{-f^*(p^*) - X_G^*(-p^*)\} \tag{2.7}$$

Let us call (D_F) the Fenchel dual problem and denote its optimal objective value by $sup(D_F)$. The weak duality

$$sup(D_F) \leq inf(P) \tag{2.8}$$

Example 2.2. *[2]* Let $K = \mathbb{R}_+, X = [0, +\infty) \subseteq \mathbb{R}$, $f : \mathbb{R} \to \bar{\mathbb{R}},\ g : \mathbb{R} \to \mathbb{R}$, be defined by

$$f(x) = \begin{cases} x, & if\ x \in X \\ +\infty, & otherwise \end{cases}$$

and

$$g(x) = 1 - x^2.$$

Now the optimal objective value of the Fenchel dual problem is

$$\begin{aligned} sup(D_F) &= \sup_{p^* \in \mathbb{R}} \{-\sup_{x \geq 0} [p^* x - x] + \inf_{x \geq 0, 1 - x^2 \leq 0} p^* x\} \\ &= \sup_{p^* \in \mathbb{R}} \{\sup_{x \geq 0} (1 - p^*) x + \inf_{x \geq 1} p^* x\} \\ &= \sup_{0 \leq p^* \leq 1} (p^*) \\ &= 1 \end{aligned}$$

This implies that the optimal objective value of D_F is 1.

2.4 Fenchel-Lagrange Dual Problems

Another dual problem different from (D_L) *and* (D_F) can be obtained by considering the perturbation function as a combination of the functions Φ_L and Φ_F. Let this be defined by $\Phi_{FL} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \to \bar{\mathbb{R}}$

$$\Phi_{FL}(x, p, q) = \begin{cases} f(x + p), \ if\ x \in X,\ g(x) \leqq_K q \\ +\infty, \ otherwise \end{cases}$$

With the perturbation variables $p \in \mathbb{R}$ and $q \in \mathbb{R}$, Φ_{FL} satisfies the relation (2.3) and (2.4) and its conjugate is

$$\begin{aligned} \Phi^*_{FL}(x^*, p^*, q^*) &= \sup_{x \in \mathbb{R}^n, p \in \mathbb{R}^n, q \in \mathbb{R}^k} \{(x^*, p^*, q^*)^T (x, p, q) - \Phi_{FL}(x, p, q)\} \\ &= \sup_{x \in \mathbb{R}^n, p \in \mathbb{R}^n, q \in \mathbb{R}^k} \{x^{*T} x + p^{*T} p + q^{*T} q - \Phi_{FL}(x, p, q)\} \\ &= \sup_{x \in X, g(x) \leqq_K q, p \in \mathbb{R}^n, q \in \mathbb{R}^k} \{x^{*T} x + p^{*T} p + q^{*T} q - f(x + p)\} \end{aligned}$$

Like in the previous subsection we introduce new variables $r = x + p \in \mathbb{R}^n$ and $s = q - g(x) \in K$. Then we have

$$\begin{aligned} \Phi^*_{FL}(x^*, p^*, q^*) &= \sup_{r \in \mathbb{R}^n, s \in K, x \in X} \{x^{*T} x + p^{*T} (r - x) + q^{*T} [s + g(x)] - f(r)\} \\ &= \sup_{r \in \mathbb{R}^n} \{p^{*T} r - f(r)\} + \sup_{x \in X} \{(x^* - p^*)^T x + q^{*T} g(x)\} + \sup_{s \in K} \{q^{*T} s\} \end{aligned}$$

Computing the first supremum we get

$$\sup_{r\in\mathbb{R}}\{p^{*T}r - f(r)\} = f^*(p^*)$$

While for the last it holds

$$\sup_{s\in K} q^{*T}s = \begin{cases} 0, & \textbf{if } q^* \in -K^*, \\ +\infty, & otherwise \end{cases}$$

In this case the dual problem

$$(D_{FL}) \qquad \sup_{p^*\in\mathbb{R}^n, q^*\in\mathbb{R}^k}\{-\Phi^*_{FL}(0, p^*, q^*)\}$$

becomes

$$(D_{FL}) \qquad \sup_{p^*\in\mathbb{R}^n, q^*\in -K}\{-f^*(p^*) - \sup_{x\in X}[-p^{*T}x + q^{*T}g(x)]\}$$

or, equivalently

$$(D_{FL}) \qquad \sup_{p^*\in\mathbb{R}^n, q^* \underset{K^*}{\geqq} 0}\{-f^*(p^*) + \inf_{x\in X}[-p^{*T}x + q^{*T}g(x)]\} \tag{2.9}$$

We will call (D_{FL}) the Fenchel-Lagrange dual problem and denote its optimal objective value by $sup(D_{FL})$. By Theorem 2.1 the weak duality

$$sup(D_{FL}) \leq inf(P) \tag{2.10}$$

also holds.

Example 2.3. *[2]* Let $K = \mathbb{R}_+, X = [0, +\infty) \subseteq \mathbb{R}$, $f : \mathbb{R} \to \bar{\mathbb{R}},\ g : \mathbb{R} \to \mathbb{R}$, be defined by

$$f(x) = \begin{cases} -x^2, & if\ x \in X \\ +\infty, & otherwise \end{cases}$$

and

$$g(x) = x^2 - 1.$$

Now the optimal objective value of the Fenchel-Lagrange dual problem is

$$\begin{aligned} sup(D_{FL}) &= \sup_{p^*\in\mathbb{R}, q^*\geq 0}\{-\sup_{x\geq 0}[p^*x + x^2] + \inf_{x\geq 0}[p^*x + q^*(x^2-1)]\} \\ &= \sup_{p^*\in\mathbb{R}, q^*\geq 0}\{-\infty + \inf_{x\geq 0}[p^*x + q^*(x^2-1)]\} \\ &= -\infty \end{aligned}$$

Chapter 3

Duality in Multiobjective Programming

The third chapter of this work deals with duality in multiobjective optimization. It contains a general convex multiobjective problem with cone inequality constraints. In this cases, the basic idea is to establish a dual problem to a scalarized problem associated to the multiobjective primal. The scalar dual is formulated in terms of conjugate functions and its structure gives an idea about how to construct a multiobjective dual in a natural way. And the concept of conjugate duality.

3.1 General Convex Multiobjective Problem with Cone Inequality Constraints

3.1.1 Problem Formulation

Before we discuss a problem formulation let us first discuss some points about affine set, a ball, interior points and relative interior.

Definition 3.1. *[10]* A set $C \subseteq \mathbb{R}^n$ is affine if the line through any two distinct points in C lies in C, i.e., if for any $x^1, x^2 \in C$ and $\Theta \in \mathbb{R}$, we have $\Theta x^1 + (1-\Theta)x^2 \in C$.

In other words, C contains the linear combination of any two points in C, provided the coefficients in the linear combination sum to one. This idea can be generalized to more than two points. We refer to a point of the form $\Theta_1 x_1 + \ldots + \Theta_k x_k$, where $\Theta_1 + \ldots + \Theta_k = 1$, as an affine combination of the points $x_1, \ldots, x_k$.

The set of all affine combinations of points in some set $C \subseteq \mathbb{R}^n$ is called the affine hull of C and denoted affC:
affC= $\{\Theta_1 x_1 + \ldots + \Theta_k x_k \mid x_1, \ldots, x_k \in C, \Theta_1 + \ldots + \Theta_k = 1\}$
The affine hull is the smallest affine set that contains C.

Definition 3.2. (Ball) A ball in $\mathbb{R}^n$ is a set of the form

$$B(x,r) = \{y \in \mathbb{R}^n | \|y - x\| < r\}$$

where $x \in \mathbb{R}^n$ is a center and $r > 0$ is radius of the ball.

Definition 3.3. (Interior) The interior of a set $M \subseteq \mathbb{R}^n$ is the set of all points interior to M, i.e,

$$int(M) = \{x \in M | \exists \epsilon > 0 : B(x, \epsilon) \subseteq M\}.$$

Definition 3.4. *[10]* Let $M \subseteq \mathbb{R}^n$, we say that a point $x \in M$ is relative interior for M, if M contains the intersection of a small enough ball centered at x with Aff(M):
$\exists r > 0 \;\; B_r(x) \cap$ Aff(M)$\equiv \{y | y \in$ Aff(M) $, |y - x| \leq r\} \subset M$
The set of all relative interior point of M is called its relative interior [notation riM].

The primal multiobjective optimization problem with cone inequality constraints which we consider here is the following one

$$(P) \qquad\qquad v - \min_{x \in \mathcal{A}} f(x)$$

$$\mathcal{A} = \{x \in \mathbb{R}^n : g(x) \underset{K}{\leqq} 0\}$$

$$f(x) = (f_1(x), \ldots, f_m(x))^T$$

$$g(x) = (g_1(x), \ldots, g_k(x))^T$$

For $i = 1, \ldots, m$, $f_i : \mathbb{R}^n \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ are proper and convex functions with the property that $\bigcap_{i=1}^{m} ri(dom(f_i)) \neq \emptyset$, where $ri(dom(f_i))$ represents the relative interior of the set $dom(f_i) = \{x \in \mathbb{R}^n : f_i(x) < +\infty\}$. The function $g : \mathbb{R}^n \to \mathbb{R}^k$ is convex relative to the cone $K \subseteq \mathbb{R}^k$. K is a convex closed cone with $int(K) \neq \emptyset$ which defines a partial ordering on $\mathbb{R}^k$ according to the definition that $x_1 \underset{K}{\leqq} x_2$ if and only if $x_2 - x_1 \in K$.

The "$v - \min$" term means that we ask for Pareto-efficient solution of the problem (P). This kind of solution is obtained by using the dominance structure given by the non-negative orthant $\mathbb{R}^m_+ = \{x = (x_1, \ldots, x_m)^T \in \mathbb{R}^m : x_i \geq 0, i = 1, \ldots, m\}$ on $\mathbb{R}^m$.

Definition 3.5. *[2]* An element $\bar{x} \in \mathcal{A}$ is said to be efficient (or Pareto-efficient) with respect to (P) if from $f(\bar{x}) \underset{\mathbb{R}^m_+}{\geqq} f(x)$, for $x \in \mathcal{A}$, follows $f(\bar{x}) = f(x)$.

Definition 3.6. *[2]* An element $\bar{x} \in \mathcal{A}$ is said to be properly efficient with respect to (P) if it is efficient and if there exists a number $M > 0$ such that for each $i \in \{i = 1, \ldots, m\}$ and $x \in \mathcal{A}$ satisfying $f_i(x) < f_i(\bar{x})$ there exists at least one $j \in \{j = 1, \ldots, m\}$ such that $f_j(\bar{x}) < f_j(x)$ and

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leq M$$

Definition 3.7. *[2]* An element $\bar{x} \in \mathcal{A}$ is said to be properly efficient with respect to (P) if there exists $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in int(\mathbb{R}^m_+)(i.e \lambda_i > 0, i = 1, \ldots, m)$ such that $\sum_{i=1}^{m} \lambda_i f_i(\bar{x}) \leq \sum_{i=1}^{m} \lambda_i f_i(x), \; \forall x \in \mathcal{A}$.

3.1.2 Duality for The Scalarized Problem

In order to study the duality for the multiobjective problem (P) we study first the duality for the scalarized problem

$$(P^\lambda) \qquad \inf_{x\in\mathcal{A}} \sum_{i=1}^{m} \lambda_i f_i(x),$$

Where $\lambda = (\lambda_1, \ldots, \lambda_m)^T$ is a fixed vector in $int(\mathbb{R}^m_+)$.
For $\tilde{f} : \mathbb{R}^n \to \bar{\mathbb{R}},\ \tilde{f}(x) = \sum_{i=1}^m \lambda_i f_i(x)$, the problem (P^λ)can be written as

$$(P^\lambda) \qquad \inf_{x\in\mathcal{A}} \tilde{f}(x)$$

Where $\mathcal{A} = \{x \in \mathbb{R}^n : g(x) \underset{K}{\leqq} 0\}$. Then the Fenchel-Lagrange dual of problem (P^λ) is

$$(D^\lambda) \qquad \sup_{\tilde{p}\in\mathbb{R}^n, q \underset{K^*}{\geqq} 0} \Big\{ -\tilde{f}^*(\tilde{p}) + \inf_{x\in\mathbb{R}^n} [\tilde{p}^T x + q^T g(x)] \Big\}$$

Replacing $\tilde{f}$ by its formula, we get

$$(D^\lambda) \qquad \sup_{\tilde{p}\in\mathbb{R}^n, q \underset{K^*}{\geqq} 0} \left\{ -\left(\sum_{i=1}^m \lambda_i f_i\right)^* (\tilde{p}) + \inf_{x\in\mathbb{R}^n} [\tilde{p}^T x + q^T g(x)] \right\}$$

Because of $\bigcap_{i=1}^m ri(dom(f_i)) \neq \emptyset$ we have

$$\left(\sum_{i=1}^m \lambda_i f_i\right)^* (\tilde{p}) = \inf \left\{ \sum_{i=1}^m (\lambda_i f_i)^*(\tilde{p}_i) : \sum_{i=1}^m \tilde{p}_i = \tilde{p} \right\}$$

and the dual (D^λ) becomes

$$(D^\lambda) \qquad \sup_{\tilde{p}\in\mathbb{R}^n, q \underset{K^*}{\geqq} 0, \tilde{p}_i\in\mathbb{R}^n, \sum_{i=1}^m \tilde{p}_i = \tilde{p}} \left\{ -\sum_{i=1}^m (\lambda_i f_i)^*(\tilde{p}_i) + \inf_{x\in\mathbb{R}^n} [\tilde{p}^T x + q^T g(x)] \right\}$$

But $(\lambda_i f_i)^*(\tilde{p}_i) = \lambda_i f_i^*(\frac{\tilde{p}_i}{\lambda_i})$ for $i = 1, \ldots, m$. Therefore, we can make the substitutions $p_i := \frac{\tilde{p}_i}{\lambda_i},\ i = 1, \ldots, m$ so, $\tilde{p} = \sum_{i=1}^m \lambda_i p_i$ and omitting $\tilde{p}$ we obtain for the dual of (P^λ)

$$(D^\lambda) \qquad \sup_{p_i\in\mathbb{R}^n, i=1,\ldots,m, q \underset{K^*}{\geqq} 0} \left\{ -\sum_{i=1}^m \lambda_i f_i^*(p_i) + \inf_{x\in\mathbb{R}^n} \left[\left(\sum_{i=1}^m \lambda_i p_i\right)^T x + q^T g(x) \right] \right\},$$

The reason why we consider the dual in this form is because as one can see in the next subsection (D^λ) will suggest us the dual for the vector problem (P). we are able to present a strong duality theorem for (P^λ) and (D^λ). Therefore, we need the following constraint qualification(CQ).

$$(CQ) \quad \Big| \textit{there exists an element } x' \in \bigcap_{i=1}^m dom(f_i) \textit{ such that } g(x') = (g_1(x'), \ldots, g_m(x'))^T \in -int(K).$$

Now we can formulate the following strong duality theorem.

Theorem 3.1. *[2]* Let the optimal objective value of (P^λ) be finite and assume that there exist an element $x' \in \bigcap_{i=1}^{m} dom(f_i)$ *such that* $g(x') \in -int(K)$ (i.e the constraint qualification(CQ) is fulfilled). Then the dual problem (D^λ) has an optimal solution and strong duality holds

$$inf(P^\lambda) = max(D^\lambda)$$

Theorem 3.2. *[2]*

(a) Let the constraint qualification(CQ) be fulfilled and let $\bar{x}$ be a solution to (P^λ) there exists $(\bar{p}, \bar{q}),\ \bar{p} = (\bar{p}_1, \dots, \bar{p}_m) \in \mathbb{R}^n \times, \dots, \times \mathbb{R}^n,\ \bar{q} \underset{K^*}{\geqq} 0$, optimal solution to (D^λ) such that the following optimality conditions are satisfies

(i) $f_i^*(\bar{p}_i) + f_i(\bar{x}) = \bar{p}_i^T \bar{x}, \quad i = 1, \dots, m,$

(ii) $\bar{q}^T g(\bar{x}) = 0$

(iii) $\left(\sum_{i=1}^{m} \lambda_i \bar{p}_i\right)^T \bar{x} = \inf_{x \in \mathbb{R}^n} \left[\left(\sum_{i=1}^{m} \lambda_i \bar{p}_i\right)^T x + \bar{q}^T g(x)\right]$

(b) Let $\bar{x}$ be admissible to (P^λ) and $(\bar{p}, \bar{q})$ be admissible to (D^λ) satisfying $(i), (ii)$ *and* (iii). Then $\bar{x}$ is an optimal solution to (P^λ), $(\bar{p}, \bar{q})$ is an optimal solution to (D^λ) and the strong duality holds

$$\sum_{i=1}^{m} \lambda_i f_i(\bar{x}) = -\sum_{i=1}^{m} \lambda_i f_i^*(\bar{p}_i) + \inf_{x \in \mathbb{R}^n} \left[\left(\sum_{i=1}^{m} \lambda_i \bar{p}_i\right)^T x + \bar{q}^T g(x)\right].$$

Proof. Let $\bar{x}$ be an optimal solution to P^λ there exists $(\bar{p}, \bar{q}),\ \bar{p} = (\bar{p}_1, \dots, \bar{p}_m) \in \mathbb{R}^n \times, \dots, \times \mathbb{R}^n,\ \bar{q} \underset{K^*}{\geqq} 0$, is a solution to D^λ such that

$$\inf(P^\lambda) = f_i(\bar{x}) = -f_i^*(\bar{p}_i) + \inf_{x \in X} \left[(\sum_{i=1}^{m} \lambda_i \bar{p}_i)^T x + \bar{q}^T g(x)\right]$$

or, equivalently,

$$f_i(\bar{x}) + f_i^*(\bar{p}_i) - \bar{p}_i^T \bar{x} + (\sum_{i=1}^{m} \lambda_i \bar{p}_i)^T \bar{x} + \bar{q}^T g(\bar{x}) - \inf_{x \in X} \left[(\sum_{i=1}^{m} \lambda_i \bar{p}_i)^T x + \bar{q}^T g(x)\right] - \bar{q}^T g(\bar{x}) = 0 \quad (3.1)$$

On the other hand the following inequality hold

$$f_i(\bar{x}) + f_i^*(\bar{p}_i) - \bar{p}_i{}^T \bar{x} \geq 0,$$

$$(\sum_{i=1}^{m} \lambda_i \bar{p}_i)^T \bar{x} + \bar{q}^T g(\bar{x}) - \inf_{x \in X} \left[(\sum_{i=1}^{m} \lambda_i \bar{p}_i)^T x + \bar{q}^T g(x)\right] \geq 0,$$

$$-\bar{q}^T g(\bar{x}) \geq 0.$$

By (3.1) it follows that all these inequality have to be in fact fulfilled as equality. The conducts us to the optimality condition $(i), (ii)$ & (iii).

(b) All calculations done within part (a) may be carried out in the inverse direction starting from $(i), (ii)$ & (iii). Then $\bar{x}$ solves (P^λ), $(\bar{p}, \bar{q})$ solves (D^λ) and the strong duality holds. □

3.1.3 The Multiobjective Dual Problem

Now we are able to formulate a multiobjective dual to (P). The dual (D) will be a vector maximum problem and for it Pareto-efficient solution in the sense of maximum are considered. After we introduce the multiobjective dual (D)we prove the weak and strong duality theorems.
The dual multiobjective optimization problem (D) is

$$(D) \qquad v - \max_{(p,q,\lambda,t)\in\mathcal{B}} h(p,q,\lambda,t)$$

with

$$h(p,q,\lambda,t) = \begin{pmatrix} h_1(p,q,\lambda,t) \\ \vdots \\ h_m(p,q,\lambda,t) \end{pmatrix}$$

$$h_j(p,q,\lambda,t) = -f_j^*(p_j) - (q_j^T g)^*\left(-\frac{1}{m\lambda_j}\sum_{i=1}^m \lambda_i p_i \right) + t_j,\ j = 1,\dots,m$$

the dual variables

$$p = (p_1,\dots,p_m) \in \mathbb{R}^n\times,\dots,\times\mathbb{R}^n,\ q = (q_1,\dots,q_m) \in \mathbb{R}^k\times,\dots,\times\mathbb{R}^k$$

$$\lambda = (\lambda_1,\dots,\lambda_m) \in \mathbb{R}^m,\ t = (t_1,\dots,t_m) \in \mathbb{R}^m$$

and the set of constraints

$$\mathcal{B} = \left\{ (p,q,\lambda,t) : \lambda \in int(\mathbb{R}^m_+),\ \sum_{i=1}^m \lambda_i p_i \underset{K^*}{\geqq} 0, \sum_{i=1}^m \lambda_i t_i = 0 \right\}. \tag{3.2}$$

Definition 3.8. *[2]* An element $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}$ is said to be efficient (or pareto efficient) with respect to (D) if from $h(p,q,\lambda,t) \underset{\mathbb{R}^m_+}{\geqq} h(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ for $(p,q,\lambda,t) \in \mathcal{B}$ follows $h(p,q,\lambda,t) = h(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$.

The following theorem states the weak duality assertion for the vector problem (P) and (D).

Theorem 3.3. *[2]* There is no $x \in A$ and no $(p,q,\lambda,t) \in \mathcal{B}$ fulfilling $h(p,q,\lambda,t) \underset{\mathbb{R}^m_+}{\geqq} f(x)$ and $h(p,q,\lambda,t) \neq f(x)$.

Proof. We assume that there exist $x \in A$ and $(p, q, \lambda, t) \in \mathcal{B}$ such that $f_i(x) \le h_i(p, q\lambda, t), \forall i \in \{1, 2, \ldots, m\}$ and $f_j(x) < h_j(p, q\lambda, t)$ for at least one $j \in \{1, 2, \ldots, m\}$ this implies

$$\sum_{i=1}^{m} \lambda_i f_i(x) < \sum_{i=1}^{m} \lambda_i h_i(p, q\lambda, t) \tag{3.3}$$

on the other hand we have

$$\sum_{i=1}^{m} \lambda_i h_i(p, q, \lambda, t) = -\sum_{i=1}^{m} \lambda_i f_i^*(p_i) - \sum_{i=1}^{m} \lambda_i (q_i^T g)^* \left(-\frac{1}{m\lambda_i} \sum_{i=1}^{m} \lambda_i p_i \right) + \sum_{i=1}^{m} \lambda_i t_i$$

For f_i and $q_i^T g, i = 1, \ldots, m$, we can apply the inequality of Young

$$-f_i^*(p_i) \le f_i(x) - p_i^T x,$$

$$-(q_i^T g)^* \left(-\frac{1}{m\lambda_i} \sum_{i=1}^{m} \lambda_i p_i \right) \le q_i^T g(x) + \left(\frac{1}{m\lambda_i} \sum_{i=1}^{m} \lambda_i p_i \right)^T x$$

and so we obtain

$$\begin{aligned}
\sum_{i=1}^{m} \lambda_i h_i(p, q, \lambda, t) &\le \sum_{i=1}^{m} \lambda_i f_i(x) - \Big(\sum_{i=1}^{m} \lambda_i p_i \Big)^T x + \sum_{i=1}^{m} \lambda_i \left[q_i^T g(x) + \left(\frac{1}{m\lambda_i} \sum_{i=1}^{m} \lambda_i p_i \right)^T x \right] \\
&= \sum_{i=1}^{m} \lambda_i f_i(x) + \Big(\sum_{i=1}^{m} \lambda_i q_i \Big)^T g(x) \\
&\le \sum_{i=1}^{m} \lambda_i f_i(x).
\end{aligned}$$

The resulting inequality $\sum_{i=1}^{m} \lambda_i h_i(p, q, \lambda, t) \le \sum_{i=1}^{m} \lambda_i f_i(x)$ contradicts relation(3.3) □

The following theorem expresses the strong duality between the two multiobjective problems (P) and (D).

Theorem 3.4. *[2] Assume the existence of an element $x' \in \cap_{i=1}^{m} dom(f_i)$ fulfilling $g(x') \in -int(K)$ let $\bar{x}$ be a properly efficient element to (P). Then there exists an efficient solution $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}$ to the dual (D) and the strong duality $f(\bar{x}) = h(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ holds.*

Proof. Assume $\bar{x}$ to be properly efficient to (P). From Definition 3.7 there follows the existence of a corresponding vector $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_m)^T \in int(\mathbb{R}_+^m)$ such that $\bar{x}$ solves the scalar problem

$$(P^{\bar{\lambda}}) \qquad \inf_{x \in A} \sum_{i=1}^{m} \bar{\lambda}_i f_i(x)$$

The constraint qualification (CQ) being fulfilled by Theorem 3.2 there exists $(\bar{p}, \bar{q})$ an optimal solution to the dual $D^{\bar{\lambda}}$ such that the optimality condition $(i), (ii)$ *and* (iii) are satisfied. By means of $\bar{x}$ and $(\bar{p}, \bar{q})$ we construct now an efficient solution $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ to (D). Therefor let

$\bar{\lambda} = (\bar{\lambda_1}, \dots, \bar{\lambda_m})^T$ be the vector given by the proper efficiency of $\bar{x}$ and $\bar{p} = (\bar{p_1}, \dots, \bar{p_m}) := (\tilde{p}_1, \dots, \tilde{p}_m) = \tilde{p}$. It remains us to define $\tilde{q} = (\tilde{q_1}, \dots, \tilde{q_m})$ *and* $\bar{t} = (\bar{t_1}, \dots, \bar{t_m})^T$
let for $i = 1, \dots, m$, be

$$\begin{aligned} \bar{q_i} &:= \frac{1}{m\bar{\lambda}_i}\tilde{q} \in \mathbb{R}^k \\ \bar{t_i} &:= \bar{p_i}^T\bar{x} + (\bar{q_i}^T g)^*\Big(- \frac{1}{m\bar{\lambda_i}} \sum_{i=1}^{m} \bar{\lambda}_i \bar{p_i}\Big) \in \mathbb{R} \end{aligned} \tag{3.4}$$

for $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ it holds $\bar{\lambda} \in int(\mathbb{R}^m_+)$, $\sum_{i=1}^m \bar{\lambda}_i \bar{p_i} = \tilde{q} \underset{K^*}{\geqq} 0$ and

$$\begin{aligned} \sum_{i=1}^{m} \bar{\lambda}_i \bar{t_i} &= \Big(\sum_{i=1}^{m} \bar{\lambda}_i \bar{p_i}\Big)^T \bar{x} \sum_{i=1}^{m} \bar{\lambda}_i \Big(\frac{1}{m\bar{\lambda_i}} \bar{q_i}^T g\Big)^* \Big(- \frac{1}{m\bar{\lambda_i}} \sum_{i=1}^{m} \bar{\lambda}_i \bar{p_i}\Big) \\ &= \Big(\sum_{i=1}^{m} \bar{\lambda}_i \bar{p_i}\Big)^T \bar{x} + \sum_{i=1}^{m} \bar{\lambda}_i \frac{1}{m\bar{\lambda_i}} (\bar{q_i}^T g)^* \Big(- \sum_{i=1}^{m} \bar{\lambda}_i \bar{p_i}\Big) \\ &= \Big(\sum_{i=1}^{m} \bar{\lambda}_i \bar{p_i}\Big)^T \bar{x} + (\bar{q_i}^T g)^* \Big(- \sum_{i=1}^{m} \bar{\lambda}_i \bar{p_i}\Big) \\ &= 0 \end{aligned}$$

In conclusion the element $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ is feasible to (D)
It remains to show that the value of the objective function are equal, namely that $f(\bar{x}) = h(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$. Therefor, we prove that $f_i(\bar{x}) = h_i(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ holds for each $i = 1, \dots, m$, for this we use the relation (i) in the Theorem 3.2 and the equation (3.4) then it holds

$$\begin{aligned} h_i(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) &= -f_i^*(\bar{p_i}) - (\bar{q_i}^T g)^* \Big(- \frac{1}{m\bar{\lambda_i}} \sum_{i=1}^{m} \bar{\lambda}_i \bar{p_i}\Big) + \bar{t_i} \\ &= -f_i^*(\bar{p_i}) - (\bar{q_i}^T g)^* \Big(- \frac{1}{m\bar{\lambda_i}} \sum_{i=1}^{m} \bar{\lambda}_i \bar{p_i}\Big) + \bar{p_i}^T \bar{x} + (\bar{q_i}^T g)^* \Big(- \frac{1}{m\bar{\lambda_i}} \sum_{i=1}^{m} \bar{\lambda}_i \bar{p_i}\Big) \\ &= -f_i^*(\bar{p_i}) + \bar{p_i}^T \bar{x} \\ &= f_i(\bar{x}). \end{aligned}$$

□

3.2 Conjugate Duality

In this section we develop a conjugate duality in vector optimization. Conjugate duality was fully developed in scalar optimization. The multiobjective optimization by introducing some new concepts such as conjugate maps and subgradients for vector valued, point-to-set maps. Their results are based on the efficiency (Pareto maximality). Besed on this definition of supremum, some useful concepts such as conjugate maps and subgradients are introduced for vector valued, point-to-set maps. These concepts enable us to develop the conjugate duality in vector optimization.

3.2.1 Conjugate Maps

First we discus the concepts of point-to-set maps

- **point-to-set maps**

A point-to-set map F from a set X into a set Y is a map that associates a subset of Y with each point of X. Equivalently, F can be viewed as a function from the set X into the power set 2^Y.
In a multiobjective optimization problem, it is rather difficult to obtain a unique optimal solution. Solving the problem often leads to a solution set. Thus, if the problem has a parameter, the solution set defines a point-to-set maps from the parameter space into the objective (or decision) space.

In this subsection, we introduce the concept of conjugate maps for vector valued functions and point-to-set maps. When f is an extended real-valued function on $\mathbb{R}^n$ (i.e., when $f : \mathbb{R}^n \to \bar{\mathbb{R}}$) its conjugate function$f^*$ is defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in \mathbb{R}^n\}, \;\; for\ x^* \in \mathbb{R}^n$$

In order to consider the vector-valued case we must define a paired space of $\mathbb{R}^n$ with respect to $\mathbb{R}^p$. A most natural paired space is the set of all $p \times n$ matrices which is denoted by $\mathbb{R}^{p\times n}$. However, its dimension $p \times n$ is often too large. Another idea is to take $(\mathbb{R}^n)^* = \mathbb{R}^n$ as paired space as in the scalar.

Definition 3.9. *[8]* Let F be a point-to-set maps from $\mathbb{R}^n$ in to $\mathbb{R}^p$). The point-to-set map $F^* : \mathbb{R}^{p\times n} \to \mathbb{R}^p$ defined by

$$F^*(T) = \sup \bigcup_{x\in\mathbb{R}^n} [Tx - F(x)], \; for\ T \in \mathbb{R}^{p\times n}$$

is called the conjugate map of F. Moreover, A point-to-set maps F^{**} defined by

$$F^{**}(x) = \sup \bigcup_{T\in\mathbb{R}^{p\times n}} [Tx - F^*(T)], \; for\ x \in \mathbb{R}^n$$

is called the biconjugate map of F.

When f is a vector-valued function from $\mathbb{R}^n$ to $\mathbb{R}^p \cup \{+\infty\}$, let $domf = \{x \in \mathbb{R}^n : f(x) \neq +\infty\}$ and define the conjugate map f^* of f by

$$f^*(T) = \sup\{Tx - f(x) : x \in domf\}$$

Here $+\infty$ is the imaginary point whose every component is $+\infty$. We identify the function f as the point-to-set map that is equal to {f(x)} for $x \in domf$ and is empty otherwise. The biconjugate map f^{**} can be defined as the conjugate map of f^*.

Proposition 3.1. *[8]* Let F be a point-to-set map from $\mathbb{R}^n$ into $\mathbb{R}^p$ and $x \in \mathbb{R}^n$. If we define another point-to-set map G by $G(x) = F(x + \bar{x})$ for all $x \in X$, then

(i) $G^*(T) = F^*(T) - T\bar{x},\ forall\ T \in \mathbb{R}^{p\times n}$

(ii) $G^{**}(x) = F^{**}(x + \bar{x}),\ forall\ x \in \mathbb{R}^n.$

Proof. **(i)**

$$\begin{aligned} G^*(T) &= \sup \bigcup_x [Tx - G(x)] \\ &= \sup \bigcup_x [Tx - F(x + \bar{x})] \\ &= \sup \bigcup_{x'} \{[Tx' - F(x')] - T\bar{x}\} \\ &= \sup \bigcup_{x'} [Tx' - F(x')] - T\bar{x} \\ &= F^*(T) - T\bar{x} \end{aligned}$$

(ii)

$$\begin{aligned} G^{**}(x) &= \sup \bigcup_T [Tx - G^*(T)] \\ &= \sup \bigcup_T [Tx - F^*(T) + T\bar{x}] \\ &= F^{**}(x + \bar{x}) \end{aligned}$$

□

Proposition 3.2. *[8]* Let F be a point-to-set map from $\mathbb{R}^n$ into $\mathbb{R}^p$ and $\bar{y} \in \mathbb{R}^p$. Then

(i) $(F + \bar{y})^*(T) = F^*(T) - \bar{y}$

(ii) $(F + \bar{y})^{**}(x) = F^{**}(x) + \bar{y}.$

Proof. **(i)**

$$\begin{aligned} (F + \bar{y})^*(T) &= \sup \bigcup_x [Tx - F(x) - \bar{y}] \\ &= \sup \bigcup_x [Tx - F(x)] - \bar{y} \\ &= F^*(T) - \bar{y} \end{aligned}$$

(ii)

$$\begin{aligned} (F + \bar{y})^{**}(x) &= \sup \bigcup_T [Tx - F(T) + \bar{y}] \\ &= \sup \bigcup_T [Tx - F(T)] + \bar{y} \\ &= F^{**}(x) + \bar{y} \end{aligned}$$

□

Lemma 3.1. *[8]* Let $InfF$ be another point-to-set-maps from $\mathbb{R}^n$ to $\mathbb{R}^p$ defined by $(InfF)(x) = InfF(x)$ for all $x \in \mathbb{R}^n$. Then

$$F^*(T) = (InfF)^*(T) \ \ and \ \ F^{**}(x) = (InfF)^{**}(x)$$

Proof.

$$\begin{aligned}(InfF)^*(T) &= \sup \bigcup_{x\in\mathbb{R}^n} [Tx - (InfF)(x)] \\ &= \sup \bigcup_{x\in\mathbb{R}^n} [Tx - InfF(x)] \\ &= \sup \bigcup_{x\in\mathbb{R}^n} \sup[Tx - F(x)] \\ &= \sup \bigcup_{x\in\mathbb{R}^n} [Tx - F(x)] \\ &= F^*(T)\end{aligned}$$

$F^{**}(x) = (InfF)^{**}(x)$ follows directly from the above relation. □

Proposition 3.3. *[9]* Let F be a point-to-set map from $\mathbb{R}^n$ into $\mathbb{R}^p$. If $y \in F(x)$ and $y' \in F^*(T)$, then

$$y + y' \nleq T\hat{x} \ \ (i.e., T\hat{x} - (y + y') \notin \mathbb{R}^p_+ \setminus \{0\}).$$

Proof. Since $y \in F(\hat{x})$, $T\hat{x} - y \in \bigcup_x [Tx - F(x)]$. Hence, if $y' \leq T\hat{x} - y$, it contradicts the assumption $y' \in F^*(T) = \sup \bigcup_x [Tx - F(x)]$. □

Lemma 3.2. *[9]* Let F_1 and F_2 be point-to-set maps from $\mathbb{R}^n$ into $\mathbb{R}^p$. Then

$$Max \bigcup_x [F_1(x) + F_2(x)] \subset Max \bigcup_x [F_1(x) + MaxF_2(x)]$$

If $MaxF_2(x)$ is externally stable $(i.e., F_2 \subset MaxF_2(x) - \mathbb{R}^p_+)$ for every $x \in \mathbb{R}^n$, then the converse inclusion also holds.

Proof. Let $\hat{y} \in Max \bigcup_x [F_1(x) + F_2(x)]$. Then there exists $\hat{x} \in \mathbb{R}^n$ such that $\hat{y} = y^1 + y^2$ for some $y^1 \in F_1(\hat{x})$ and $y^2 \in F_2(\hat{x})$. If we suppose that $y^2 \notin MaxF_2(\hat{x})$, there exists $\bar{y}^2 \in F_2(\hat{x})$ such that $y^2 \leq \bar{y}^2$.Then $\hat{y} = y^1 + y^2 \leq y^1 + \bar{y}^2$ which is contradiction.Therefore ,$y^2 \in MaxF_2(\hat{x})$.since $\bigcup_x [F_1(x) + F_2(x)] \supset \bigcup_x [F_1(x) + MaxF_2(x)]$, then

$$\hat{y} \in Max \bigcup_x [F_1(x) + MaxF_2(x)]$$

Next, suppose that $MaxF_2(x)$ is externally stable for every x, then

$$F_2(x) - \mathbb{R}^p_+ = MaxF_2(x) - \mathbb{R}^p_+ \ \ for\ every\ x$$

Thus

$$F_1(x) + F_2(x) - \mathbb{R}^p_+ = F_1(x) + MaxF_2(x) - \mathbb{R}^p_+ \ \ for\ every\ x$$

$$\bigcup_x [F_1(x) + F_2(x)] - \mathbb{R}^p_+ = \bigcup_x [F_1(x) + MaxF_2(x)] - \mathbb{R}^p_+$$

Taking the Max of both sides, we have

$$Max \bigcup_x [F_1(x) + F_2(x)] = Max \bigcup_x [F_1(x) + MaxF_2(x)]$$

□

3.2.2 Subgradients

First define the subgradient at a point

Definition 3.10. *[9]* Let f be a convex function from $\mathbb{R}^n$ to $\mathbb{R}$.A vector $x^* \in \mathbb{R}^n$ is said to be a subgradient of f at x if

$$f(x') \geq f(x) + \langle x^*, x' - x \rangle, \quad for \ all \ x' \in \mathbb{R}^n.$$

The set of all subgradients of f at x is called the subdifferential of f at x and is denoted by $\partial f(x)$.If $\partial f(x)$ is not empty, f is said to be subdifferential at x.

In this subsection we introduce the concepts of subgradients of vector-valued functions and point-to-set maps. A subgradient x^* of a scalar-valued function f at x is defined in Definition 3.10. The definition can be formally extended to a nonconvex function, though it essentially requires convexity (at least locally). We may extend the definition of subgradients to the vector-valued case, which is also an intuitively direct extension of Definition 3.10.

Definition 3.11. *[9]*

(i) Let f be a function from $\mathbb{R}^n$ to $\mathbb{R}^p \cup \{+\infty\}$.A $p \times n$ matrix T is said to be a subgradient of f at $\hat{x} \in domf$ if

$$f(x) \nleq f(\hat{x}) + T(x - \hat{x}) \ \ for \ all \ x \in \mathbb{R}^n$$

i.e., if

$$f(\hat{x}) - T\hat{x} \in Min\{f(x) - Tx \in \mathbb{R}^p : x \in \mathbb{R}^n\} = Min\{f(x) - Tx : x \in domf\}.$$

The set of all subgradients of f at $\hat{x}$ is called the subdifferential of f at $\hat{x}$ and is denoted by $\partial f(\hat{x})$. If $\partial f(\hat{x})$ is not empty, then f is said to be subdifferentiable at $\hat{x}$.

(ii) Let F be a point-to-set map from $\mathbb{R}^n$ into $\mathbb{R}^p$ and $\hat{y} \in F(\hat{x})$. $p \times n$ matrix is said to be a subgradient of F at $(\hat{x}; \hat{y})$ if

$$\hat{y} - T\hat{x} \in Max \bigcup_{x \in \mathbb{R}^n} [Tx - F(x)]$$

The set of all subgradients of F at $(\hat{x}; \hat{y})$ is called the subdifferential of F at $(\hat{x}; \hat{y})$ and is denoted by $\partial F(\hat{x}; \hat{y})$. When $\partial F(\hat{x}; \hat{y}) \neq \emptyset$ for every $\hat{y} \in F(\hat{x})$ F is said to be subdifferentiable at $\hat{x}$.

The following proposition provides a characterization of minimal solution by the subgradient 0, that is, the stationary condition in an extended sense.

Proposition 3.4. *[9]*

(i) Let f be a function from $\mathbb{R}^n$ to $\mathbb{R}^p \cup \{+\infty\}$. Then

$$f(\hat{x}) \in Min\{f(x) : x \in dom f\} \;\; if \; and \; only \; if \;\; 0 \in \partial f(\hat{x}).$$

(ii) Let F be a point-to-set map from $\mathbb{R}^n$ into $\mathbb{R}^p$. For $\hat{y} \in F(\hat{x})$,

$$\hat{y} \in Min \bigcup_x F(x) \;\; if \; and \; only \; if \;\; 0 \in \partial F(\hat{x}; \hat{y}).$$

Proof. Immediate from the definition of subgradients. □

The following propositions shows the relationships between conjugate or biconjugate maps and subgradients.

Proposition 3.5. *[9]*

(i) Let f be a function from $\mathbb{R}^n$ to $\mathbb{R}^p \cup \{+\infty\}$. Then

$$T \in \partial f(x) \;\; if \; and \; only \; if \;\; Tx - f(x) \in f^*(T).$$

(ii) Let F be a point-to-set map from $\mathbb{R}^n$ to $\mathbb{R}^p$. Then, for $y \in F(x)$

$$T \in \partial F(x; y) \;\; if \; and \; only \; if \;\; Tx - y \in F^*(T).$$

Proof. This proposition is obvious from the definition of conjugate maps and subgradients. □

Summary

The central point of this project is represented by the study of the duality in multiobjective optimization problem.
Having the following form:

$$(P) \qquad v - \min_{x \in \mathcal{A}} f(x)$$

$$\mathcal{A} = \{x \in \mathbb{R}^n : g(x) = (g_1(x), \dots, g_k(x))^T \underset{K}{\leqq} 0\}$$

where $f(x) = (f_1(x), \dots, f_m(x))^T$, $f_i : \mathbb{R}^n \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ $i = 1, \dots, m$ are proper functions, $g_j : \mathbb{R}^n \to \mathbb{R},\ j = 1, \dots, k$ and $K \subseteq \mathbb{R}^k$ is assumed to be a convex closed cone with $int(K) \neq \emptyset$ defining a partial ordering on $\mathbb{R}^k$. To (P) is associated the following scalarized optimization problem In order to study the duality for the multiobjective problem (P) we study first the duality for the scalarized problem

$$(P^\lambda) \qquad \inf_{x \in \mathcal{A}} \sum_{i=1}^{m} \lambda_i f_i(x),$$

where $\lambda = (\lambda_1, \dots, \lambda_m)^T \in int(\mathbb{R}^m_+)$. A scalar dual to it is constructed and the optimality conditions are derived. The structure of the scalar dual suggests the form of the multiobjective dual (D) to (P).

To study the duality for the scalarized problem (P^λ) the conjugacy approach is used. To the problem

$$(P) \qquad \inf_{x \in G} f(x)$$

$$G = \{x \in X : g(x) \leq 0\},$$

Where $X \subseteq \mathbb{R}^n$ is nonempty set, three different dual problems are constructed, namely, the well-known Lagrange and Fenchel duals (denoted by (D_L) and (D_F), respectively) and a combination of the above two, called the Fenchel-Lagrange dual (denoted by (D_{FL})).

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