



Order Systems, Ideals And Right Fixed Maps Of Subtraction Algebras .

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September 9, 2016

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The undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled **order systems, ideals and right fixed maps of subtraction algebras** By Abayneh Tilahun in partial fulfillment of the requirements for the degree of master of Science.

August, 2016

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August, 2016

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Acknowledgements

My deepest thanks goes to my advisor Dr.Zelalem Teshome who give valuable comments and suggestions for final accomplishment of the project .

The second deepest thanks goes to my seminar advisors Dr. Berhanu Bekele and Dr . Yebeltal Yitayew who gave me support for the successful completion of my project.

The third deepest thanks goes to Project 18 house of construction development workers and my family.

Abstract

This project deals with introduction of subtraction algebra, weak subtraction algebra and their the properties.order systems and ideals of a weak subtraction algebra and a subtraction algebra will be introduced.the concept of fixed map in weak subtraction algebra will be discussed. Conditions for an ideal to be irreducible will be provided. More over, relations between ideals and order systems are given.

Notations

\cap, \cup The symbol of intersection and unions .

\leq The symbol of partial order.

Chapter 1

Preliminary

In this part of the project we will define some important concepts that are useful in our subsequent discussions. Let us start by defining a meet semi lattice, semi lattice and a lattice and then a Boolean algebra. In the last part of this chapter we will discuss about subtraction algebra and weak subtraction algebra .

1.1 Meet Semilattice, Semilattice and Lattice

Let us start this section by defining what is meant by meet semilattice.

Definition 1.1.1. *Let \wedge be a binary operation on a non-empty set X . Then the algebraic structure (X, \wedge) is called meet semilattice if all $x, y, z \in X$, the following conditions are satisfied:*

- i. $x \wedge x = x$, (Idempotent law)*
- ii. $x \wedge y = y \wedge x$, (Commutative law)*
- iii. $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, (Associative law)*

A semilattice is a non-empty set X together with two binary operations on which it is a meet semilattice with each of these two operations.

Definition 1.1.2. *Let \vee and \wedge be two binary operations on a non-empty set X . Then the algebraic structure (X, \vee, \wedge) is called semilattice, if for all $x, y, z \in X$, it satisfies the following properties:*

- i. $x \vee x = x$ and $x \wedge x = x$. (Idempotent law)*
- ii. $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$. (Commutative law)*
- iii. $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$. (Associative law)*

The following is the most common example for a semilattice.

Example 1.1.1. Given a set X , then the power set $P(X)$ together with the two binary operations: union " \cup " and intersection " \cap " of elements of $P(X)$ is semilattice.

In this case we define \vee by \cup and \wedge by \cap .

If a semilattice satisfies one more property, called the Absorption law, then it is called a Lattice.

Definition 1.1.3. Let \wedge and \vee be two binary operations on a non empty set X . Then the algebraic structure (X, \wedge, \vee) is called lattice, if for all $x, y, z \in X$ it satisfies the following properties :

- (L_1) . $x \wedge x = x, x \vee x = x$. - - - - (Idempotent)
- (L_2) . $x \wedge y = y \wedge x, x \vee y = y \vee x$. - - - - - (Commutative)
- (L_3) . $x \wedge (y \wedge z) = (x \wedge y) \wedge z, x \vee (y \vee z) = (x \vee y) \vee z$. - - - - (Associative)
- (L_4) . $x \wedge (x \vee y) = x \vee (x \wedge y) = x$. - - - - - (Absorption)

1.2 Boolean Algebra

In this section we will give the definition of Boolean Algebra and give an example.

Definition 1.2.1. Let " \vee " and " \wedge " be two binary operations on a non-empty set X . Then the algebraic structure $(X, \vee, \wedge, ', 0, 1)$ is called a Boolean algebra, if for arbitrary element $x, y, z \in X$. The following conditions hold true:

1. It is Lattice on a set X with two binary operations. i.e, (X, \vee, \wedge) .
2. Distributive laws: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
3. Involution law: $x'' = x$.
4. Compliment laws: $y' \vee y = 1$ and $y' \wedge y = 0$.
5. Identity laws: There is a zero element 0 in X such that $y \vee 0 = y$, and there is a unit element 1 in X such that $y \wedge 1 = y$.
6. Both $y \vee 1 = 1$ and $y \wedge 0 = 0$ are satisfied
7. De' Morgan's laws: $(x \vee y)' = x' \wedge y'$ and $(x \wedge y)' = x' \vee y'$.

The two operations " \wedge " and " \vee " are called meet and join, respectively.

Example 1.2.1. Let X be a non-empty set. Then the algebraic structure

$$(P(X), \cup, \cap, ', 0, 1),$$

where $P(X)$ is the power set of X , \cup and \cap are the set operations \cup and \cap respectively, the unary operation $'$ is the set complement and the elements 0 for \emptyset and 1 for X , is a Boolean algebra.

1.3 Order Relations

Definition 1.3.1. A. An order relation " \leq " on a non-empty set X is called partial ordering, if the following conditions are true:

- a. Reflexive: If $x \leq x$, for all $x \in X$
- b. Antisymmetry: If $x \leq y$ and $y \leq x$, then $x = y$ for all $x, y \in X$.
- c. Transitive: If $x \leq y$ and $y \leq z$, then $x \leq z$, for all $x, y, z \in X$.

A set X together with a partial order " \leq " on X is denoted by $\langle X, \leq \rangle$ and we say it is a Partially Ordered Set (Poset).

Example 1.3.1. Let " \leq " be an order relation on \mathbb{N} , the set of natural numbers, defined by

$$a \leq b$$

if and only if a divides b .

Then " \leq " is a partial ordering on \mathbb{N} and (\mathbb{N}, \leq) is a poset.

B. A partially order set is called a totally order set if for any two elements $x, y \in X$, either $x \leq y$ or $y \leq x$.

A set X with a total ordering is called totally ordered set or a chain.

The most common examples of a totally ordered set are the different sets of numbers with the usual ordering, as we can see in the following example.

Example 1.3.2. The set of natural numbers, the set of integers, the set of rational numbers and real numbers with the usual ordering " \leq " are all chains.

1.4 Subtraction Algebra

The most important objects of discussions in Mathematics are algebraic structures. We are interested on one algebraic structure for our discussion in this project, which is subtraction algebra. A subtraction algebra is a set X together with the single binary operation, which is called subtraction that is denoted by " $-$ ". In this chapter we will define a subtraction algebra, study some properties of subtraction algebra and discuss some concepts related with subtraction algebra.

Definition 1.4.1. *A non-empty set X together with a binary operation " $-$ " is said to be a subtraction algebra, if it satisfies the following properties:*

$$S_1. x - (y - x) = x, \text{ for all } x, y \in X$$

$$S_2. x - (x - y) = y - (y - x), \text{ for all } x, y \in X$$

$$S_3. (x - y) - z = (x - z) - y, \text{ for all } x, y, z \in X$$

Let us see some examples of algebraic structures that are subtraction algebras and then algebraic structures that is not subtraction algebras.

Example 1.4.1. *Let $X = \{0, a, b\}$. Define " $-$ " on X by the following table.*

-	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Then $(X, -)$ is a subtraction algebra .

Verification :- Let us check all the properties of a subtraction algebra for each element in X .

$$(i) \text{ Let } x = 0, y = a. \text{ Then } x - (y - x) = 0 - (a - 0) = 0 - a = 0 \in X.$$

$$\text{Hence } 0 - (a - 0) = 0.$$

$$(ii) \text{ Let } x = 0, y = b. \text{ Then } x - (y - x) = 0 - (b - 0) = 0 - b = 0 \in X.$$

$$\text{Hence } 0 - (b - 0) = 0.$$

$$(iii) \text{ Let } x = a, y = 0. \text{ Then } x - (y - x) = a - (0 - a) = a - 0 = a \in X.$$

$$\text{Hence } a - (0 - a) = a.$$

(iv) Let $x = a, y = b$. Then $x - (y - x) = a - (b - a) = a - b = a \in X$.
Hence $a - (b - a) = a$.

(v) Let $x = b, y = 0$. Then $x - (y - x) = b - (0 - b) = b - 0 = b \in X$.
Hence $b - (0 - b) = b$.

(vi) Let $x = b, y = a$. Then $x - (y - x) = b - (a - b) = b - a = b \in X$.
Hence $b - (a - b) = b$.

(vii) Let $x = b, y = b$. Then $x - (y - x) = b - (b - b) = b - 0 = b \in X$.
Hence $b - (b - b) = b$. Thus property S_1 holds true.
Similarly property S_2 and S_3 hold true.

Therefore, $(X, -)$ is subtraction algebra.

Example 1.4.2. Let $X = \{0, a, b, c\}$ in which " $-$ " is defined by the table.

-	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Then $(X, -)$ is a subtraction algebra

Example 1.4.3. Let X be a set and $P(X)$ be the power set of X . Then $(P(X), -)$, where " $-$ " is the usual subtraction (relative complement), is a subtraction algebra.

Proof. Let $A, B, C \in P(X)$. Then $A - B := A \cap B'$.

$$\begin{aligned}
 1. \quad A - (B - A) &= A \cap (B - A)' \\
 &= A \cap (B \cap A)'' \\
 &= A \cap (B' \cup A''), \text{ by De'Morgan's law} \\
 &= A \cap (B' \cup A), \text{ by Involution's law } A'' = A \\
 &= (A \cap B') \cup (A \cap A), \text{ by Distributive law} \\
 &= (A \cap B') \cup A \\
 &= A, \text{ since } A \cap B' \subseteq A.
 \end{aligned}$$

Thus $A - (B - A) = A$

$$\begin{aligned}
2. \quad A - (A - B) &= A \cap (A - B)' \\
&= A \cap (A \cap B)' \\
&= A \cap (A' \cup B''), \text{ by De'Morgan's law} \\
&= A \cap (A' \cup B), \text{ by Involution's law } B'' = B \\
&= (A \cap A') \cup (A \cap B), \text{ by Distributive law} \\
&= \emptyset \cup A \cap B \\
&= A \cap B = B \cap A, \text{ by commutative property} \\
&= (B \cap B') \cup (B \cap A) \\
&= B \cap (B' \cup A) \\
&= B \cap (B \cap A)' \\
&= B - (B \cap A') \\
&= B - (B - A)
\end{aligned}$$

Hence $A - (A - B) = B - (B - A)$

$$\begin{aligned}
3. \quad (A - B) - C &= (A - B) \cap C' \\
&= (A \cap B') \cap C' \\
&= A \cap (B' \cap C'), \text{ by Associative property} \\
&= A \cap (C' \cap B'), \text{ by Commutative property} \\
&= (A \cap C') \cap B', \text{ by Associative property} \\
&= (A \cap C') - B \\
&= (A - C) - B
\end{aligned}$$

Thus $(A - B) - C = (A - C) - B$.

Therefore, $(P(X), -)$ is a subtraction algebra □

Example 1.4.4. *The set of integers together with the usual subtraction, $(\mathbb{Z}, -)$, is not a subtraction algebra.*

Proof. Take $x = 1$ and $y = 2$ from \mathbb{Z} . Then

$$x - (y - x) = 1 - (2 - 1) = 1 - 1 = 0.$$

This implies S_1 in the definition of a subtraction algebra, which states that $x - (y - x) = x$, is not satisfied. Hence, as a result $(\mathbb{Z}, -)$ is not subtraction algebra. □

1.4.1 Some Properties of Subtraction Algebras

Now let us see some properties of subtraction algebra.

property 1.4.1. *If $(X, -)$ is a subtraction algebra, then the following conditions hold true.*

property(1). $(x - y) - y = x - y$.

property(2). $x - x = y - y$.

Proof. property(1) For $x, y \in X$:

$$\begin{aligned}(x - y) - y &= (x - y) - [y - (x - y)], \text{ since } y = [y - (x - y)] \\ &= (x - y), \text{ by } S_1\end{aligned}$$

Hence $(x - y) - y = x - y$.

property(2) For $x, y \in X$.

$$\begin{aligned}x - x &= [x - (y - x)] - [x - (y - x)], \text{ since } x - (y - x) = x \\ &= [x - [x - (y - x)]] - (y - x), \text{ by } S_3 \\ &= [(y - x) - [(y - x) - x]] - (y - x), \text{ by } S_2 \\ &= [(y - x) - (y - x)] - (y - x), \text{ by theorem 1(i)} \\ &= [(y - (y - x)) - x] - (y - x), \text{ by } S_3 \\ &= [(x - (x - y)) - x] - (y - x), \text{ by } S_2 \\ &= [(x - (x - y)) - (y - x)] - x, \text{ by } S_3 \\ &= [(x - (y - x)) - (x - y)] - x, \text{ by } S_3 \\ &= [x - (x - y)] - x, \text{ since } x - (y - x) = x \\ x - x &= [x - (x - y)] - x \\ x - x &= [y - (y - x)] - x.\end{aligned}$$

Similarly, $y - y = [y - (y - x)] - y$. Then

$$\begin{aligned}
 x - x &= [y - (y - x)] - x \\
 &= [(y - (x - y)) - [(y - (x - y)) - x]] - x, \text{ by } S_1 \\
 &= [(y - (x - y)) - [(y - x) - (x - y)]] - x, \text{ by } S_3 \\
 &= [[(y - (x - y)) - x] - [(y - x) - (x - y)], \text{ by } S_3 \\
 &= [(y - x) - (x - y)] - [(y - x) - (x - y)], \text{ by } S_3 \\
 &= [(y - x) - [(y - x) - (x - y)]] - (x - y), \text{ by } S_3 \\
 &= [(x - y) - [(x - y) - (y - x)]] - (x - y), \text{ by } S_2 \\
 &= [(x - y) - [(x - (y - x)) - y]] - (x - y), \text{ by } S_3 \\
 &= [(x - y) - (x - y)] - (x - y), \text{ since } x - (y - x) = x \\
 &= [(x - (x - y)) - y] - (x - y), \text{ by } S_3 \\
 &= [(y - (y - x)) - (x - y)] - y, \text{ by } S_2 \\
 &= [(y - (x - y)) - (y - x)] - y, \text{ by } S_3 \\
 &= [y - (y - x)] - y, \text{ by } S_1 \\
 &= y - y
 \end{aligned}$$

Therefore

$$x - x = y - y.$$

□

From this, we can see that there is a unique element in X that is denoted by $x - x = y - y =: 0$, which does not depend on the choice of $x, y \in X$.

Property(3)

$$(x - 0) = x \text{ and } (0 - x) = 0.$$

Proof.

we need to show $x - 0 = x$ and $0 - x = 0$

Let $x - x = 0$ by P_2 then $x - 0 = x - (x - x) = x$ by S_1

therefore, $x - 0 = x$

Let $x - 0 = x$

We want to show that $0 - x = 0$

$$0 - x = 0 - (x - 0) = 0 \text{ ----- by } S_1$$

$$0 - x = 0$$

□

Property(4)

$$(x - y) - x = 0.$$

Proof.

We want to show $(x - y) - x = 0$

$$(x - y) - x = (x - x) - y \text{ ----- by } S_3$$

$$= 0 - y = 0 \text{ ----- by } P_3$$

$$\therefore (x - y) - x = 0. \quad \square$$

Property(5) .

$$(x - (x - y)) \leq y$$

define \leq by the subtraction algebra X is $x \leq y = x - y = 0$

Proof.

We want to show that $(x - (x - y)) \leq y$?

$$(x - (x - y)) - y = 0 \text{ by using order relation}$$

$$= (x - (x - y)) - y$$

$$= (y - (y - x)) - y \text{ ----- by } S_2$$

$$= (Y - Y) - (x - y) \text{ ----- by } S_3$$

$$= 0 - (x - y) = 0 \text{ ----- by } P_3$$

$$\therefore (x - (x - y)) \leq y. \quad \square$$

Property (6) .

$$(x - y) - (y - x) = x - y.$$

Proof.

We want to show that $(x - y) - (y - x) = x - y$?

$$(x - y) - (y - x) = (x - (y - x)) - y \text{ ----- by } S_3$$

$$= x - y \text{ ----- by } S_1$$

$$\therefore (x - y) - (y - x) = x - y. \quad \square$$

property(7).

$$x - (x - (x - y)) = x - y .$$

Proof.

We want to show that $x - (x - (x - y)) = x - y$?

$$x - (x - (x - y)) = (x - y) - ((x - y) - x) \text{ ---by } s_2$$

$$= (x - y) - ((x - x) - Y) \text{ ---by } s_3$$

$$= (x - y) - (0 - y) \text{ ---by } p_3$$

$$= (x - y) - 0 = x - y \text{ ---by } P_3$$

there fore $x - (x - (x - y)) = x - y$. □

Property(8) .

$$(x - y) - (z - y) \leq x - z.$$

Proof.

We want to show that $((x - y) - (z - y)) - (x - z) = 0$?

$$((x - y) - (z - y)) - (x - z)$$

$$= ((x - y) - (x - z)) - (z - y) \text{ by } S_3$$

$$= ((x - (x - z)) - y) - (z - y) \text{ by } S_3$$

$$= ((z - (z - x)) - y) - (z - y) \text{ by } S_2$$

$$= ((z - y) - (z - x)) - (z - y) \text{ by } S_1$$

$$= ((z - y) - (z - y)) - (z - x) \text{ by } S_3$$

$$= 0 - (z - x) \text{ by } P_2$$

$$= 0 \text{ since by } P_3$$

There fore $(x - y) - (z - y) \leq x - z$. □

Property(9) .

($x \leq y$) if and only if $x = y - w$ for some $w \in X$.

Proof. (\Rightarrow) given $x \leq y$

Let $x - y = 0$ and $y - x = w$

$$x = x - 0 = x - (x - y) \text{ ---by } S_2$$

$$y - (y - x)$$

$$y - w$$

$\therefore x = y - w$ (\Leftarrow) assume $x = y - w$ for some $w \in X$

We want to show that $x \leq y$

$x = y - w$ subtracted both sides y

$$x - y = (y - w) - y \text{ ----- by } S_3$$

$$x - y = (y - y) - w$$

$$= 0 - w = 0 \text{ ----- by } P_3$$

$$x - y = 0$$

$$\Rightarrow x \leq y$$

□

Property(10) . $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$.

Proof.

If $x \leq y$ then $x = y - w$ for some $w \in X$ hence we have subtracted both sides to z .

$$x - z = (y - w) - z$$

$$= (y - z) - w \leq (y - z)$$

$$\Rightarrow x - z \leq y - z$$

next If $x \leq y$ then $x - y = 0$ thus by P8

$$(z - y) - (z - x) = (z - (z - x)) - y$$

$$= (x - (x - z)) - y$$

$$= (x - y) - (x - z) = 0 - (x - z) = 0 \text{ ----- by P3}$$

$$\therefore z - y \leq z - x.$$

□

Property(11) . $x, y \leq z$ implies $x - y = x \wedge (z - y)$. define \wedge by $x \wedge y = x - (x - y)$

Proof.

If $x \leq z$ then $x - y \leq z - y$ but $x - y \leq x$ and thus $x - y \leq x \wedge (z - y)$

let $w = x \wedge (z - y)$ then $x - y \leq x$ and so $w = x \wedge w = x - (x - w)$ also $y \wedge (z - y) =$

$$(z - y) - ((z - y) - y)$$

$$= (z - y) - (z - y) = 0 \text{ hence } w - (w - y) = y \wedge w = x \wedge y \wedge (z - y) = 0$$

therefore $w - (x - y) = (w - 0) - (x - y)$

$$= (w - (w - y)) - (x - y)$$

$$= (w - y) - (x - y)$$

$$= (x - (x - w)) - (x - y)$$

$$= ((x - y) - (x - w)) - (x - y)$$

$$= ((x - y) - (x - y)) - (x - w)$$

$$= 0 - (x - w) = 0$$

and thus $x \wedge (z - y) = w \leq x - y$ consequently

$$x - y = x \wedge (z - y) .$$

□

Property(12) .

$$(x \wedge y) - (x \wedge z) \leq x \wedge (y - z).$$

Proof.

$$i \quad (x \wedge y) - (x \wedge z) = (x - (x - y)) - (x - (x - z)) = (y - (y - x)) - (z - (z - y)) \leq y - z$$

$$ii \quad (x \wedge y) - (x \wedge z) \leq x \wedge y = x - (x - y) = y - (y - x) \leq x$$

$$\text{thus } (x \wedge y) - (x \wedge z) \leq x \wedge (y - z) .$$

□

Property(13) .

$$(x - y) - z = (x - z) - (y - z).$$

Proof.

$$\begin{aligned} & (x - z) - (y - z) - (x - y) - z \\ &= (((x - z) - z) - (y - z)) - ((x - y) - z) \text{ ----- by } P_1 \\ &\leq ((x - z) - y) - ((x - y) - z) \text{ ----- by } P_8 \text{ and } P_{10} \\ &= ((x - y) - z) - ((x - y) - z) \text{ ----- by } S_3 \\ &= 0 \text{ ----- by } P_3 \end{aligned}$$

$$\text{and } ((x - z) - (y - z)) - ((x - y) - z) = 0$$

$$\Rightarrow (x - z) - (y - z) \leq (x - y) - z \text{ ----- (1)}$$

using S_3 , P_4 ,and P_8 we get

$$((x - y) - z) - ((x - z) - (y - z)) \text{ ----- by } S_3$$

$$\leq (y - z) - y = 0$$

$$\text{then } ((x - y) - z) - ((x - z) - (y - z)) = 0$$

$$= (x - y) - z \leq (x - z) - (y - z) \text{ ----- (2)}$$

$$\text{there for from (1) and (2) } (x - y) - z = (x - z) - (y - z) .$$

□

Theorem 1.4.1. Let $(X, -)$ be a subtraction algebra . Define the relation " \leq " on X by $a \leq b \Leftrightarrow a - b = 0$.

Claim:- " \leq *" is an order relation on* X *.*

Proof.

We want to show that ' \leq ' is reflexive, antisymmetric and transitive .

(i) reflexivity

Let $a \in X$. Then $a - a = 0$

$$\Rightarrow a \leq a$$

$\therefore \leq$ is reflexive

ii Antisymmetric

Let $a, b \in X$ such that $a \leq b$ and $b \leq a$

We want to show that $a = b$?

$$a = a - (b - a) \text{ by } S_1$$

$$= a - (a - b) \text{ since } b - a = 0 = a - b$$

$$= b - (b - a) \text{ by } S_2$$

$$= b - (a - b) = b \text{ by } S_1$$

$$\therefore a = b$$

iii Transitivity

Let $a, b, c \in X$ such that $a \leq b$ and $b \leq c$. Then, $a - b = 0$ and $b - c = 0$

We want to show that $a - c = 0 \Rightarrow a \leq c$

$$0 = a - b = (a - (b - a)) - b \text{ by } S_1$$

$$= (a - b) - (b - a) \text{ by } S_3$$

$$= (b - c) - (b - a) \text{ since } a - b = 0 \text{ and } b - c = 0$$

$$= (b - (b - a)) - c \text{ by } S_3$$

$$= (a - (a - b)) - c \text{ by } S_2$$

$$= (a - c) - (a - b) \text{ by } S_3$$

$$= (a - c) - 0 = a - c$$

thus $a - c = 0$ and hence $a \leq c$

$\therefore \leq$ is transitivity

Therefore, \leq is an order relation on X . □

Theorem 1.4.2. Let $(X, -, \leq)$ be a subtraction algebra. Then $0 \leq x$ for all $x \in X$.

Proof.

since $0 - x = 0$ by p_3 ,

we have $0 \leq x$ □

From the above discussion , we see that a subtraction algebra $(X, -)$ can be considered as an order set ordered by the order relation $' \leq'$ defined above .

Now , let $(X, -)$ be a subtraction algebra with the order relation $' \leq'$. For $a \in X$, define the set $[0, a] = \{ x \in X : 0 \leq x \leq a \} \subseteq X$. On $[0, a]$, define $\wedge, \vee, '$ by

$$x \wedge y = x - (x - y)$$

$$x' = a - x$$

$$x \vee y = (x' \wedge y')' = a - ((a - x) \wedge (a - y))$$

Theorem 1.4.3. *We have ,*

(i) . $([0, a], \wedge)$ is a meet semi lattice.

(ii) . $([0, a], \wedge, \vee)$ is a boolean algebra.

Proof. :- (i) . $([0, a], \wedge)$ is a meet semi lattice.

let $x, y \in [0, a]$

We want to show that $([0, a], \wedge)$ Is a meet semi lattice?

(i) Idempotent law

We want to show that $x \wedge x = x$?

$$x \wedge x = x - (x - x) = x - 0 = x$$

ii Commutative law

We want to show $x \wedge y = y \wedge x$

$$x \wedge y = x - (x - y) = y - (y - x) = y \wedge x$$

$$\therefore x \wedge y = y \wedge x$$

iii Associative law

$x, y, z \in [0, a]$

$$x \wedge (y \wedge z) = x - (x - (y \wedge z)) \text{ by } S_2$$

$$= (y \wedge z) - ((y \wedge z) - x) \text{ - - - - by } S_2$$

$$= (y - (y - z)) - ((y - (y - z)) - x) \text{ - - - by the definition of the above .}$$

$$= (y - (y - z)) - (y - x) - (y - z) \text{ - - - - - by } S_3$$

$$= (y - (y - x)) - (y - z)$$

$$(y \wedge x) - (y - z) \text{ --- (1)}$$

$$(x \wedge y) \wedge z = (x \wedge y) - ((x \wedge y) - z) \text{ ---- by the definition of the above.}$$

$$= (x - (x - y)) - ((x - (x - y)) - z) \text{ ---- by the definition of the above.}$$

$$= (y - (y - z)) - ((y - (y - x)) - z) \text{ ---- by } S_2$$

$$= (y - (y - z)) - (y - x) \text{ ----by } S_3$$

$$= y - (y - z) - (y - x) \text{ ---- by } S_3$$

$$= y - (y - x) - (y - z)$$

$$= (y \wedge x) - (y - z) \text{ (2)}$$

From (1) and (2) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

$\therefore \wedge$ is associative

Therefore , $[0, a]$ with \wedge is a meet semi lattice .

□

Proof. :- Let $x, y \in [0, a]$ in which 0 ,1 be identity elements with the binary operation \vee, \wedge ($[0, a], \vee, \wedge, 0, 1$) is boolean algebra.

1 Idempotent law (i_1) $a \vee a = a$. and

$$(ii_1) a \wedge a = a.$$

$$\text{We want to show that (i1)} a \vee a = (a' \wedge a')' = a - ((a - a) \wedge (a - a))$$

$$= a - (0 \wedge 0)$$

$$= a - 0$$

$$= a$$

(i_2) is proof on (i) $\therefore \vee, \wedge$ is idempotent

2 Commutative law

We want to show that (i_2) $a \vee x = x \vee a$ and (ii_2) $a \wedge x = x \wedge a$

$$a \vee x = (a' \wedge x')' = a - ((a - a) \wedge (a - x)) = a - (0 \wedge (a - x))$$

$$= a - (a - (0 - (0 - x))) = a - 0 = a$$

$$x \vee a = (x' \wedge a')' = a - (x' \wedge a') = a - ((a - x) \wedge (a - a))$$

$$= a - ((a - x) - (a - x) - a)$$

$$= a - ((0 - x) - (a - x))$$

$$= a - 0 = a.$$

$$\therefore x \wedge y = y \wedge x$$

3. Associative law

We want to show that $i_3 (a \wedge x) \wedge y = a \wedge (y \wedge x)$

$$ii_3. (a \vee x) \vee z = a \vee (x \vee z)$$

i_3 proof on (i)

$$\begin{aligned} ii_3 \quad a \vee (x \vee y) &= a \vee (x' \wedge y')' = (y' \wedge (a - ((a - x) \wedge (a - y))))' \\ &= a - ((a' \wedge ((a - x) \wedge (a - y)))) \\ &= a - ((a - a) \wedge (a - ((a - x) \wedge (a - y)))) \\ &= a - (0 \wedge (a - ((a - x) \wedge (a - y)))) \\ &= a - 0 = a \end{aligned}$$

4 Absorption law .

We want to show that $(i_4) a \vee (a \wedge x) = a$ and $(ii_4) a \wedge (a \vee x) = a$

$$\begin{aligned} i_4 \quad a \vee (a \wedge x) &= a \vee (a - (a - x)) = (a' \wedge (a - (a - x)))' \\ &= a - ((a - a) \wedge (a - (a - x))) \\ &= a - (0 \wedge (a - x)) \\ &= a - (0 - (0 - (a - x))) = a - 0 = a. \\ \therefore a \vee (a \wedge x) &= a. \end{aligned}$$

$$ii_4 \quad a \wedge (a \vee x) = a \wedge x = a - (a - a) = a - 0 = x \therefore a \wedge (a \vee x) = a.$$

5 Distributive law

We want to show that $(i_5) (a \wedge (x \vee y)) = (a \wedge x) \vee (a \wedge y)$, $(ii_5) a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$

$$\begin{aligned} i_5 \quad a \wedge (x \vee y) &= a \wedge (x' \wedge y')' = a \wedge (a - ((a - x) \wedge (a - y))) \\ &= a - (a - (a - ((a - x) \wedge (a - y)))) \\ &= a - ((a - x) \wedge (a - y)) \end{aligned}$$

$$\begin{aligned}
&= (x' \wedge y')' = x \vee y \\
&(a \wedge x) \vee (a \wedge y) = ((a - (a - x))' \wedge (a - (a - y))')' \\
&= a - (a - (a - (a - x)) \wedge (a - (a - (a - y)))) \\
&= a - ((a - x) \wedge (a - y)) = (x' \wedge y')' = x \vee y \\
&\therefore a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y).
\end{aligned}$$

$$ii_5 \quad a \vee (x \wedge y) = a' \wedge (x \wedge y)' = a - ((a - a) \wedge (a - (x \wedge)))$$

$$\begin{aligned}
&= a - (0 - (0 - (a - (x \wedge y)))) \\
&= x - 0 = x \\
&(a \vee x) \wedge (a \vee y) = a \wedge a = a - (a -) = a - 0 = a.
\end{aligned}$$

6 Involution

We want to show that $a'' = a$
 $a'' = (a - a)' = 0' = a - 0 = a.$

7 Compliments

We want to show that

$$(i_7) \quad x' \vee x = a \text{ and}$$

$$(ii_7) \quad x' \wedge x = 0$$

$$i_7 \quad x' \vee x = (a - x) \vee x = ((a - x)' \wedge b')'$$

$$\begin{aligned}
&= (a - (a - x) \wedge (a - x))' \\
&= a - (a - (a - x) \wedge (a - x)) \\
&= a - ((a - x) - ((a - x) - (a - (a - x)))) \\
&= a - ((a - x) - (a - x)) \\
&= a - 0 = a
\end{aligned}$$

$$\begin{aligned}
ii_7 \quad x' \wedge x &= (a - x) \wedge x = x - (x - (a - x)) \\
&= x - x = 0
\end{aligned}$$

8 Identities.

We want to show that

$$(i_8)x \vee 0 = x \text{ and}$$

$$(ii_8) x \wedge a = x$$

$$\begin{aligned} (i_8)x \vee 0 &= (x' \wedge 0')' = a - ((a - x) \wedge (a - 0)) \\ &= a - ((a - x) - ((a - x) - a)) \\ &= a - (a - x) \\ &= x'' = x \\ (ii_8)y \wedge a &= x - (x - a) \\ &= a - (a - x) \\ &= x'' = x \end{aligned}$$

9 $x \wedge 0 = 0$ and $x \vee 1 = 1$

$$x - (x - 0) = x - x = 0 \text{ and}$$

$$x \vee 1 = 1 \vee x = 1.$$

10 Demorgan's law

We want to show that

$$i (a \vee x)' = a' \wedge x' \text{ and}$$

$$ii (a \wedge x)' = a \vee x'$$

$$i (a \vee x)' = a' = a - a = 0$$

$$a' \wedge x'' \wedge = (a - a) \wedge (a - x) = 0 - (0 - (a - x)) = 0$$

$$=(a \vee x) = a' \wedge x'$$

$$ii (a \wedge x)' = a - (a - (a - x)) = a - x = x'$$

$$a' \vee x' = a - a \vee (a - x) = (0' \wedge (a - x)')'$$

$$= ((a - 0) \wedge a - (a - x)')'$$

$$= (a - (a - (a - (a - x))))'$$

$$= a - (a - (a - x))$$

$$= 0 - x = x'$$

$$\therefore (a \wedge x)' = a' \vee x' = x'$$

□

1.5 weak subtraction algebra

Definition 1.5.1. By a weak subtraction algebra (WS-algebra), we mean a triplet $(W, -, 0)$ where W is a nonempty set, $'-'$ is a binary operation on W and $0 \in W$ is a nullary operation, called zero element, such that

$$(w_1) (\forall x, y, z \in W)((x - y) - z = (x - z) - y)$$

$$(w_2) (\forall x \in W)(x - 0 = x, x - x = 0)$$

$$(w_3) (\forall x, y, z \in W)((x - y) - z = (x - z) - (y - z))$$

Example 1.5.1. Let $W = \{0, a, b, c\}$ be a set with the following table.

-	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	0	0	0
c	c	0	0	0

$(W, -)$ is a WS-algebra

Remark 1.5.1. Every subtraction algebra is a WS-algebra. The converse may not be true .

Example 1.5.2. Let $X = \{0, a, b, c, d\}$ be a set with the following table

-	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	b
b	b	b	0	0	b
c	c	c	c	0	c
d	d	d	d	d	0

Then $(X, -)$ is a WS-algebra which is not subtraction algebra
because $b - (c - b) = b - c = 0 \neq b$

Definition 1.5.2. Let $(X, -)$ be a subtraction algebra if there exist $x_0 \in X : x \leq x_0$
for all x in X we call X a bounded subtraction algebra .

Remark 1.5.2. If there exists such an $x_0 \in X$, then it is unique

Proof. :- Suppose there exist $x_0, y_0 \in X$ such that $x \leq x_0 \forall x \in X$ and $x \leq y_0 \forall x \in X$.

Then , $x_0 \leq y_0$ since $y_0 \in X$ and $y_0 \leq x_0$ since $x_0 \in X$
Therefore $x_0 = y_0$ because \leq is antisymmetric .

□

Notation we denote the unique element satisfying the above condition by 1 .

Remark 1.5.3. If the subtraction algebra $(X, -)$ is bounded then $X = [0, 1]$

Chapter 2

ORDER SYSTEMS AND IDEALS OF WEAK SUBTRACTION ALGEBRAS

In what follows , let X denote a WS- algebra unless otherwise specified .

2.1 Ideals

Definition 2.1.1. *A nonempty subset A of X is an ideal of X if it satisfies:*

$$(b_1) 0 \in A$$

$$(b_2) (\forall x \in X)(\forall y \in A)(x - y \in A \Rightarrow x \in A)$$

Notation 1. *The set of all ideals in X will be denoted by $Id(X)$.*

Lemma 2.1.1. *An ideal A of a subtraction algebra X has the following property:*

$$(\forall x \in X)(\forall y \in A)(x \leq y \Rightarrow x \in A)$$

Theorem 2.1.1. *Let A be a nonempty subset of X . Then the set*

$K = \{x \in X : (\dots((x - a_1) - a_2) - \dots) - a_n = 0 \text{ for some } a_1, a_2, \dots, a_n \in A\}$ *is a minimal ideal of X containing A .*

Proof. :- K is non empty since $0 \in K$

Let $x \in X$ $y \in K$ such that $x - y \in K$

$\Rightarrow (\dots((y - a_1) - a_2) - \dots) - a_n = 0$, for some $a_1, a_2, \dots, a_n \in A$ and

$(\dots(((x - y) - b_1) - b_2) - \dots) - b_m = 0$ for some $b_1, b_2, \dots, b_m \in A$

$\Rightarrow ((\dots((x - b_1) - b_2) - \dots) - b_m) - y$ by S_3

$\Rightarrow (\dots((x - b_1) - b_2) - \dots) - b_m \leq y$

$= (\dots(((\dots((x - b_1) - b_2) - \dots) - b_m) - a_1) - \dots) - a_n \leq (\dots((y - a_1) - a_2) - \dots) - a_n = 0$

and so $(\dots(((\dots((x - b_1) - b_2) - \dots) - b_m) - a_1) - \dots) - a_n = 0$

It follow that $x \in K$ so that K is an ideal of X .

Let G be an ideal of X such that $A \subseteq G$

We need to show that $k \subseteq G$.

Let $x \in K$. Then , there exist $a_1, a_2, \dots, a_n \in A$ such that,

$(\dots((x - a_1) - a_2) - \dots) - a_n = 0 \in G$

since $a_n \in G$ by the second condition in the definition of an ideal we have ,

$(\dots((x - a_1) - a_2) - \dots) - a_{n-1} \in G$

By a similar argument ,we have :

$(\dots((x - a_1) - a_2) - \dots) - a_{n-2} \in G$.

$\Rightarrow (\dots((x - a_1) - a_2) - \dots) - a_{n-3} \in G$.

.

.

.

$\Rightarrow x - a_1 \in G$

$\Rightarrow x \in G$

Therefore, K is a minimal ideal of X containing A . □

Notation 2. *The ideal K described in the above theorem is called the ideal generated by A and denoted by $\langle A \rangle$.*

2.2 Irreducible Ideal

Definition 2.2.1. An ideal A of X is said to be irreducible if for any ideals C and D of X , $A = C \cap D$ implies $A = C$ or $A = D$.

Theorem 2.2.1. If $A \in \text{Id}(X)$ satisfies the following assertion .

$$(\forall x, y \in X \setminus A) (\exists z \in X \setminus A) (z - x \in A, z - y \in A) \text{ --- (1)}$$

then A is an irreducible ideal of X .

Proof. :- Assume that $A \in \text{Id}(X)$ satisfies (1) and let $C, D \in \text{Id}(X)$ such that $A = C \cap D$,

We want to show that either $A = C$ or $A = D$.

Suppose $A \neq C$ and $A \neq D$. Then $\exists x \in C \setminus A \subseteq X \setminus A$ and $y \in D \setminus A \subseteq X \setminus A$.

It follows from (1), that $\exists z \in X \setminus A$ such that $z - x \in A$ and $z - y \in A$.

since $x \in C$ and $z - x \in A = C \cap D \subseteq C$ we have $z \in C$ because of C is an ideal of X .

Moreover, $y \in D$ and $z - y \in D$ implies $z \in D$. Hence $z \in C \cap D = A$

which is a contradiction. Then, $A = C$ or $A = D$. Hence A is an irreducible ideal of X .

□

Corollary 2.2.1. Let $A \in \text{Id}(X)$. Assume that for any $x, y \in X \setminus A$ there exist $z \in X \setminus A$ such that $z \leq x$ and $z \leq y$ then A is an irreducible ideal of X .

Proof. $z \leq x \Rightarrow z - x = 0$ and $z \leq y \Rightarrow z - y = 0$

$\Rightarrow z - x, z - y \in A \therefore A$ is irreducible.

□

2.3 Order Systems

Definition 2.3.1. Let X be a poset. A non empty subset I of X is called an Order system of X if it satisfies .

$$(b_3) x \in X, y \in I : x \leq y \Rightarrow x \in I.$$

$$(b_4) x, y \in I \exists z \in I : x \leq z, y \leq z.$$

Remark 2.3.1. The set of all order systems of a poset X will be denoted by $O_s(X)$.

If X is a poset with the bottom element \perp then every order system of X contains the bottom element .

Proof.

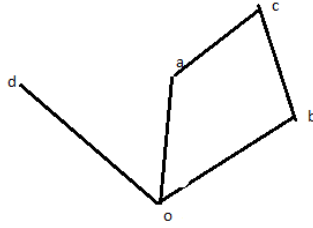
Let I be an order system of X .

since I is non empty, there exist $y \in X$.

Then \perp is bottom element implies $\perp \leq y$.

By (b_3) , $\perp \in I$.

□



Example 2.3.1. Let $X = \{0, a, b, c, d\}$ be a poset with the above Hasse diagram.

then $(I_1) = \{0, a\} \in Os(X)$

$(I_2) = \{0, a, b, c\} \in Os(X)$

$(I_3) = \{0, b, c\}$ note element of $Os(X)$ since a is not in I_3

$(I_4) = \{0, a, d\}$ is not in $Os(X)$ since a, d is not in the system

Theorem 2.3.1. For every WS-algebra X , we have $Os(X) \subseteq Id(X)$.

Proof.

Let X be a WS-algebra, and let $I \in Os(X)$.

since 0 is a bottom element of the WS-algebra X , by remark(3.2.1) $0 \in I$ (because for any $x \in X, 0 - x = 0 \Rightarrow 0 \leq x$)

Now let $x \in X, y \in I$ such that $x - y \in I$.

Since I is an order system and y and $x - y \in I \exists z \in I$ such that, $x - y \leq z$ and $y \leq z$.

$$\begin{aligned} \text{Thus, } x - z &= (x - z) - 0 = (x - z) - (y - z) \\ &= (x - y) - z \\ &= 0 \end{aligned}$$

$\therefore x \leq z$.

By (b_3) , since $z \in I$, we have $x \in I$.

Hence , I is an ideal of X .

□

Example 2.3.2. *the following example show that an ideal is not an order system .*

Let $X = \{0, a, b, c, d\}$ be a set with the following table.

-	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	b
c	c	b	a	0	c
d	d	d	d	d	0

Then $(X, -)$ is a subtraction algebra and hence a WS-algebra . It is easy to verify that .

$I_1 = \{0, a, d\}$ is an ideal of X but not an order system because $a, d \in I_1$ but there is no $z \in I_1$ such that $a \leq z$ and $d \leq z$.

Example 2.3.3. *Let $X = \{0, a, b, c, d\}$ be a set with the following cayley table}.*

-	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	0
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

Then $(X, -)$ is a WS-algebra which is not subtraction algebra it is easy see that $(I_1) = \{0, a, c\}$ in $Id(X)$ but not in $Os(X)$.

2.4 Complicated Subtraction Algebra

Definition 2.4.1. A subtraction algebra X is said to be complicated if for any $a, b \in X$ the set

$S(a, b) = \{x \in X : x - a \leq b\}$ has a greatest element. The greatest element of $S(a, b)$ is denoted by $a + b$.

Notation 3. The set $S(a, b)$ is non empty since $0 \in S(a, b)$

Example 2.4.1. Let $X = \{0, a, b, c\}$ be a subtraction algebra with the following table.

-	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

then $S(0, 0) = \{0\}$, $S(0, a) = \{0, a\} = S(a, a)$, $S(0, b) = \{0, b\} = S(b, b)$

$S(0, c) = X = S(a, b) = S(a, c) = S(b, c) = S(c, c)$

In fact, $(X, -)$ is a complicated subtraction algebra, with .

$$0 + 0 = 0.$$

$$0 + a = a.$$

$$0 + b = b.$$

$$0 + c = c.$$

$$a + a = a .$$

$$b + b = c.$$

$$a + c = c.$$

$$b + b = b.$$

$$b + c = c.$$

$$c + c = c.$$

Lemma 2.4.1. *If X is a complicated subtraction algebra then, for any $a, b \in X$, $a \leq a + b$ and $b \leq a + b$*

Proof. :- since X a complicated subtraction algebra, the set

$S(a, b) = \{x \in X : x - a \leq b\}$ has a greatest element denoted by $a + b$.

Since $a - a = 0 \leq b$, we have $a \in S(a, b)$. Similarly, $(b - a) - b = (b - b) - a = 0 - a = 0$ implies $b \in S(a, b)$.

Therefore, $a \leq a + b$, $b \leq a + b$.

□

Theorem 2.4.1. *In a complicated subtraction algebra X , every ideal is an order system.*

Proof. :- let Q be an ideal of complicated subtraction algebra X .

(i) Let $x \in X$ and $y \in Q$ such that $x \leq y$.

Then, $x \leq y$ implies $x - y = 0 \in Q$

$\Rightarrow x - y \in Q$

$\Rightarrow x \in Q$ - - - because Q is an ideal.

(ii) Let $x, y \in Q$. consider $S(x, y)$.

Then, $x + y \in S(x, y)$

Now, let $z = x + y$. We need to show that $x + y$ is in Q .

Since $x + y \in S(x, y)$, we have

$(x + y) - x \leq y$

Since $y \in Q$, we have $(x + y) - x \in Q$

$\Rightarrow x + y \in Q$ - - - because $x \in Q$

Therefore, $\exists z = x + y \in Q$ such that

$x \leq z$ and $y \leq z$.

Hence, Q is an order system.

□

Corollary 2.4.1. *Let Q be a non empty subset of a complicated subtraction algebra X . Then, Q is an ideal of X if and only if Q is an order system of X .*

Theorem 2.4.2. *Let $Q \in \text{Os}(X)$. If Q is irreducible as an ideal of X , then
(2) $(\forall a, b \in X \setminus Q)(\exists c \in X \setminus Q)(c \leq a, c \leq b)$.*

Proof. Assume that

$$(\exists a, b \in X \setminus Q)(\forall c \in X)(c \leq a, c \leq b \Rightarrow c \in Q).$$

Let $Q(a)$ and $Q(b)$ be the ideal of X generated by $Q \cup \{a\}$ and $Q \cup \{b\}$ respectively .
Then $Q \subseteq Q(a) \cap Q(b)$. Then $x \in Q(a)$ and $x \in Q(b)$. thus

$$(\dots(((x - a) - c_1) - c_2) - \dots) - c_m = 0$$

and

$$(\dots(((x - b) - d_1) - d_2) - \dots) - d_n = 0$$

for some $c_1, c_2, \dots, c_m, d_1, d_2, \dots, d_n \in Q$ is an ideal of X ,it follows from (b_1) and (b_2)
that $x - a \in Q$ and $x - b \in Q$ so from (b_4) that there exists $z \in Q$ such that $x - a \leq z$
and $x - b \leq z$. Hence

$(x - z) - a = (x - a) - z = 0$ and $(x - z) - b = (x - b) - z = 0$, and so $x - z \in Q$
by(2). but $Q \in \text{Id}(x)$ and $z \in Q$ and thus $x \in Q$ by (b_2) . thus $Q(a) \cap Q(b) \subseteq Q$, and
consequently $Q = Q(a) \cap Q(b)$ which is a contradiction.

□

Chapter 3

RIGHT FIXED MAPS OF WEAK SUBTRACTION ALGEBRAS

Definition 3.0.2. A right fixed map α of X defined to be self map $\alpha : X \rightarrow X$ satisfying $\alpha(x - y) = \alpha(x) - y$ for $\forall x, y \in X$

Example 3.0.2. Let $X = \{0, a, b\}$ be a set with the following cayley table.

-	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

then $(X, -)$ is a subtraction algebra, and hence a WS-algebra. It can be easily verified that the self map α of X defined by $\alpha(0) = 0$, $\alpha(a) = 0$ and $\alpha(b) = b$ is a right fixed map. because,

$$\alpha(0 - 0) = \alpha(0) = 0 = 0 - 0 = \alpha(0) - 0.$$

$$\alpha(0 - a) = \alpha(0) = 0 = 0 - a = \alpha(0) - a.$$

$$\alpha(0 - b) = \alpha(0) = 0 = 0 - b = \alpha(0) - b.$$

$$\alpha(a - 0) = \alpha(a) = 0 = 0 - 0 = \alpha(a) - 0.$$

$$\alpha(a - a) = \alpha(0) = 0 - a = \alpha(a) - a.$$

$$\alpha(a - b) = \alpha(a) = 0 = 0 - b = \alpha(a) - b.$$

$$\alpha(b - 0) = \alpha(b) = b = 0 - b = \alpha(b) - 0.$$

$$\alpha(b - a) = \alpha(b) = b = b - a = \alpha(b) - a.$$

$$\alpha(b - b) = \alpha(0) = 0 = b - b = \alpha(b) - b.$$

Example 3.0.3. Consider the subtraction algebra and hence a weak subtraction algebra $X : \{0, a, b, c\}$ with the following cayley table.

-	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

Let $\beta : X \rightarrow X$ be defined by $\beta(0) = 0$, $\beta(a) = 0$

$\beta(b) = c$ and $\beta(c) = c$ then β is not right fixed map since $\beta(b - c) = \beta(b) = c$ and $\beta(b) - c = c - c = 0$ and $c \neq 0$.

Example 3.0.4. Let $X = \{0, a, b, c, d\}$ be a set with the following table.

-	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

then $(X, -)$ is a WS-algebra. which is not a subtraction algebra, because using S_2 , $a - (b - a) = a - b = 0 \neq a$ therefor S_2 is not satisfied .

Let β be a self map of X defined by $\beta(0) = \beta(a) = \beta(b) = 0$, $\beta(c) = c$ and $\beta(d) = d$ then

β is a right fixed map.

Example 3.0.5. Let $X = \{0, a, b, c, d\}$ be a set with the following Cayley table.

-	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	b
c	c	c	c	0	c
d	d	d	d	d	0

then $(X, -)$ is a WS-algebra, which is not a subtraction algebra because S_2 is failed, since $a - (b - a) = a - b = 0 \neq a$.

Let β be a self map of X defined by

$$\beta(x) = \begin{cases} 0, & \text{if } x \in \{0, a\} \\ x, & \text{\{otherwise\}} \end{cases} \quad \text{then } \alpha \text{ is a right fixed map of } X$$

Proposition 3.0.1. If α is a right fixed map of X then

(i) $\alpha(0) = 0$

(ii) $\forall x \in X : \alpha(0 - x) = 0$

(iii) $\forall x \in X : \alpha(x) \leq x$

(iv) $\forall x, y \in X : x \leq y \Leftrightarrow \alpha(x) \leq y$

Proof.

(i) We have

$$\alpha(0) = \alpha(0 - \alpha(0)) = \alpha(0) - \alpha(0) = 0$$

Therefore, $\alpha(0) = 0$.

(ii) For every $x \in X$, we have

$\alpha(0 - x) = \alpha(0) = 0$ by P3 and (i) above .

(iii) For any $x \in X$, we get

$$0 = \alpha(0) = \alpha(x - x) = \alpha(x) - x$$

$$\Rightarrow \alpha(x) - x = 0$$

$$\Rightarrow \alpha(x) \leq x$$

$$\text{so } \alpha(x) \leq x$$

(iv) Let $x, y \in X$ such that $x \leq y$.

$$\text{then , } x - y = 0$$

$$\Rightarrow \alpha(x - y) = \alpha(0) = 0$$

$$\Rightarrow \alpha(x) - y = 0$$

$$\Rightarrow \alpha(x) \leq y.$$

□

Definition 3.0.3. For a right fixed map α of X , the kernel of α denoted by $\ker(\alpha)$ is defined to be the set .

$$\ker(\alpha) = \{x \in X / \alpha(x) = 0\}.$$

Clearly $\ker(\alpha) \neq \emptyset$ since $0 \in \ker(\alpha)$.

Theorem 3.0.3. Let α be a right fixed map of X then α is one to one if and only if $\ker(\alpha) = \{0\}$.

Proof.

(\Rightarrow) Assume that α is one to one and let $x \in \ker(\alpha)$ then $\alpha(x) = 0 = \alpha(0)$ and thus $x = 0$.

$$\text{i.e } \ker(\alpha) = \{0\}$$

(\Leftarrow) Suppose that $\ker(\alpha) = \{0\}$

Let $x, y \in X$ such that , $\alpha(x) = \alpha(y)$.

We want to show that α is one to one ,i.e , $x = y$

since $\alpha(y) \leq y$, it follows from (P_{10}) that

$$\alpha(x - y) = \alpha(x) - y \leq \alpha(x) - \alpha(y) = 0 .$$

so that $\alpha(x - y) = 0$ then $x - y \in \ker(\alpha)$ and so $x - y = 0$. Thus $x \leq y$ similarly $y - x = 0$ and hence $y \leq x$.

$$\Rightarrow x = y$$

Therefore, α is one to one □

Theorem 3.0.4. *Let α be a right fixed map of X . Then α is one - to - one if and only if α is the identity map.*

Proof.

\Rightarrow Suppose that α is one to one. For every x in X we have

$$\alpha(x - \alpha(x)) = \alpha(x) - \alpha(x) = 0 .$$

$x - \alpha(x) \in \ker(\alpha)$ and hence $x - \alpha(x) = 0$.

$$\Rightarrow x \leq \alpha(x) \text{---(1)}$$

But, from proposition (2)(iii), $\alpha(x) \leq x$.

$\therefore \alpha(x) = x$ and thus α is identity .

\Leftarrow suppose α is identity .

Then , $\ker(\alpha) = \{0\}$

$\Rightarrow \alpha$ 1-1. □

Theorem 3.0.5. *Let α be a right fixed map of X ,if α is idempotent .*

i.e $\alpha(\alpha(x)) = \alpha(x) \forall x \in X$ then,

(i) $\forall x \in X : \alpha(x) = x \Leftrightarrow x \in \text{Im}(\alpha)$

(ii) $\ker(\alpha) \cap \text{Im}(\alpha) = \{0\}$, where $\text{Im}(\alpha)$ is the image of α .

Proof.

(i) If $x \in \text{Im}(\alpha)$, then there exist for some $y \in X$ such that $\alpha(y) = x$.Then $\alpha(x) = \alpha(\alpha(y)) = \alpha(y) = x$. If $\alpha(x) = x$, then $x \in \text{Im}(\alpha)$.

(ii) If $x \in \ker(\alpha) \cap \text{Im}(\alpha)$, then $\alpha(x) = 0$ and $\alpha(y) = x$ for some $y \in X$ it follows that

$$0 = \alpha(x) = \alpha(\alpha(y)) = \alpha(y) = x$$

so that $\ker(\alpha) \cap \text{Im}(\alpha) = \{0\}$. □

Theorem 3.0.6. *Let α be a right fixed map of a subtraction algebra X . Then*

(i) . $\forall x \in X, \text{there exist } y \in \ker(\alpha), \text{ there exist } z \in \text{Im}(\alpha) : (z = x - y)$

(ii) . α is idempotent, i.e, every right fixed map of a subtraction algebra is idempotent .

Proof.

(i) since $\alpha(x) \leq x$ for $\forall x \in X$,it follows from (P9) that there exist $w \in X$ such that $\alpha(x) = x - w$.

so from (P7) that $x - (x - \alpha(x)) = x - (x - w) = x - w = \alpha(x)$

Noticing that $x - \alpha(x) \in \ker(\alpha)$ (because $\alpha(x - \alpha(x)) = \alpha(x) - \alpha(x) = 0$)and $\alpha(x) \in \text{Im}(\alpha)$, we have the result in (i) because ,

$\alpha(x) = x - (x - \alpha(x))$. $\alpha(x) \in \text{Im}(\alpha)$ and $(x - \alpha(x)) \in \ker(\alpha)$

(ii) From P_1 we get

$\alpha(\alpha(x)) = \alpha(x - w) = \alpha(x) - w = (x - w) - w = x - w = \alpha(x)$

$\therefore \alpha$ is idempotent . □

Notation 4. By $RF(X)$ we denote the set of all right fixed maps of X .

Let Θ be a binary operation on $RF(X)$ defined by $(\alpha \Theta \beta)(x) = \alpha(x) - \beta(x)$, for all $\alpha, \beta \in RF(X)$ and $x \in X$.

Lemma 3.0.2. If $(X, -)$ is a WS-algebra , then $(RF(X), \Theta)$ is a WS-algebra .

Proof. Let $\alpha, \beta, \gamma \in RF(X)$, and let $x \in X$. Then,

(i) $[(\alpha \Theta \beta) \Theta \gamma](x) = (\alpha \Theta \beta)(x) - \gamma(x)$

$$\begin{aligned} &= (\alpha(x) - \beta(x)) - \gamma(x) \\ &= (\alpha(x) - \gamma(x)) - \beta(x) \\ &= (\alpha \Theta \gamma)(x) - \beta(x) \\ &= [(\alpha \Theta \gamma) \Theta \beta](x) \end{aligned}$$

Henc, $(\alpha \Theta \beta) \Theta \gamma = (\alpha \Theta \gamma) \Theta \beta$ (ii) Let 0 be the zero map on X , i.e, $0(x) = 0$.

Then , $(\alpha \Theta 0)(x) = \alpha(x) - 0(x) = \alpha(x) - 0 = \alpha(x)$

$\therefore \alpha \Theta 0 = \alpha$ Moreover $(\alpha \Theta \alpha)(x) = \alpha(x) - \alpha(x) = 0 = 0(x)$

$\therefore \alpha\Theta\alpha = 0$.

$$\begin{aligned}
\text{(iii)} \quad [(\alpha\Theta\beta)\Theta\gamma](x) &= (\alpha(x) - \beta(x)) - \gamma(x) \\
&= (\alpha(x) - \gamma(x)) - (\beta(x) - \gamma(x)) \\
&= (\alpha\Theta\gamma)(x) - (\beta\Theta\gamma)(x) \\
&= [(\alpha\Theta\gamma)\Theta(\beta\Theta\gamma)](x)
\end{aligned}$$

$\therefore (\alpha\Theta\beta)\Theta\gamma = (\alpha\Theta\gamma)\Theta(\beta - \gamma)$

Hence , $(RF(x), \Theta)$ is a WS-algebra . □

Let $IRF(X)$ denote the set of all idempotent right fixed maps of X .

Theorem 3.0.7.

For every $\alpha, \beta \in IRF(X)$,if $\alpha \Theta \beta = 0$ in $RF(X)$ then $Im(\alpha) \subseteq Im(\beta)$

Proof.

Let $\alpha, \beta \in IRF(X)$ satisfy $\alpha\Theta\beta = 0$. If $y \in Im(\alpha)$,then $\alpha(y) = y$ by theorem (3.0.4) and hence

$$0 = (\alpha\Theta\beta)(y) = \alpha(y) - \beta(y) = y - \beta(y)$$

i.e $y \leq \beta(y)$ combining this with preposition (3.0.1)(iii) we get

$$y = \beta(y) \in Im(\beta)$$

then $Im(\alpha) \subseteq Im(\beta)$. □

Theorem 3.0.8. *Let $\alpha, \beta \in IRF(X)$. Then*

(i) $\alpha\Theta\beta \in RF(X)$

(ii) *If $\alpha(\beta(x)) = \beta(\alpha(x))$ for all $x \in X$,then $\alpha\Theta\beta \in IRF(X)$*

(iii) *If $Im(\alpha) \subseteq Im(\beta)$ and $\alpha(\beta(x)) = \beta(\alpha(x))$ for all $x \in X$ then $\alpha\Theta\beta = 0$ in $RF(X)$*

(iv) $Im(\alpha) \cap ker(\beta) \subseteq Im(\alpha\Theta\beta)$

Proof.

(i) For every $x, y \in X$, we want to show that $(\alpha \ominus \beta)(x - y) = (\alpha \Theta \beta)(x) - y$.

But $(\alpha \Theta \beta)(x - y) = \alpha(x - y) - \beta(x - y)$

$$\begin{aligned} &= (\alpha(x) - y) - (\beta(x) - y) \text{ --- by definition} \\ &= (\alpha(x) - \beta(x)) - y \text{ --- by } (P_{13}) \\ &= (\alpha \Theta \beta)(x) - y \end{aligned}$$

$\therefore \alpha \Theta \beta \in \text{RF}(X)$

(ii) Assume that $\alpha(\beta(x)) = \beta(\alpha(x)) \forall x \in X$. By (i), $(\alpha \Theta \beta) \in \text{RF}$. So it remains to show that $\alpha \Theta \beta$ is idempotent.

Let $x \in X$. Then $(\alpha \Theta \beta)((\alpha \Theta \beta)(x)) = (\alpha \Theta \beta)(\alpha(x) - \beta(x))$.

$$\begin{aligned} &= \alpha(\alpha(x) - \beta(x)) - \beta(\alpha(x) - \beta(x)) \\ &= (\alpha(\alpha(x) - \beta(x)) - (\beta(\alpha(x)) - \beta(x))) \\ &= (\alpha(\alpha(x) - \beta(x)) - (\beta(\alpha(x)) - \beta(x))) \\ &= (\alpha(x) - \beta(x)) - (\alpha(\beta(x)) - \beta(x)) \\ &= (\alpha(x) - \beta(x)) - (\alpha(\beta(x)) - \beta(x)) \\ &= (\alpha(x) - \beta(x)) - \alpha(0) \\ &= \alpha(x) - \beta(x) \\ &= (\alpha \Theta \beta)(x) \end{aligned}$$

that is $\alpha \Theta \beta$ is idempotent. Hence $\alpha \Theta \beta \in \text{IRF}(X)$

(iii) Suppose that $\text{Im}(\alpha) \subseteq \text{Im}(\beta)$ and $\alpha(\beta(x)) = \beta(\alpha(x))$ for $x \in X$

since $\alpha(x) \in \text{Im}(\alpha) \subseteq \text{Im}(\beta)$

For all $x \in X$ it follows from theorem (3.0.7) that $(\alpha\Theta\beta)(x)$

$$\begin{aligned} &= \alpha(x) - \beta(x) = \beta(\alpha(x)) - \beta(x) \\ &= \alpha(\beta(x)) - \beta(x) \\ &= \alpha(\beta(x) - \beta(x)) \\ &= \alpha(0) = 0 \end{aligned}$$

For all $x \in X$ there for $\alpha\Theta\beta = 0$.

(iv) . If $y \in \text{Im}(\alpha) \cap \text{ker}(\beta)$ then

$\beta(y) = 0$ and $\alpha(x) = y$ for some $x \in X$

It follows from (P3) that

$$\begin{aligned} y &= \alpha(x) = \alpha(\alpha(x) - 0) \\ &= \alpha(y) - \beta(y) \\ &= (\alpha\Theta\beta) \in \text{Im}(\alpha\Theta\beta) \end{aligned}$$

There fore $\text{Im}(\alpha) \cap \text{ker}(\beta) \subseteq \text{Im}(\alpha\Theta\beta)$. □

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