On a Family of Riordan Arrays
and
Associated Integer Hankel Transforms

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SCHOOL OF GRADUATE STUDIES
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SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
AT
ADDIS ABABA UNIVERSITY
ADDIS ABABA, ETHIOPIA
JUNE 2013

ADDIS ABABA UNIVERSITY
DEPARTMENT OF
MATHEMATICS

The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a project entitled "**On a Family of Riordan Arrays and Associated Integer Hankel Transforms**" by Biruk Getachew in partial fulfillment of the requirements for the degree of **Master of Science**.

Dated: June 2013

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Date: June 2013

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Title: **On a Family of Riordan Arrays and Integer Hankel Transforms**

Department: **Mathematics**

Degree: **M.Sc.** Convocation: **June** Year: **2013**

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Summary

Hankel transforms are among the well-known transforms. In this paper integer Hankel transforms, Riordan Matrices and generating functions are defined. We also examine a set of special Riordan arrays, their inverses and associated integer Hankel transforms. We also prove the invariance of Hankel transform under the Binomial transform.

Acknowledgements

First of all I wish to express my most sincere gratitude to Dr. Seyoum Getu, my advisor and who has been helpful and patient.

To Freabeba Solomon, my sister, thank you for your advice and continuous support in every part of my life.

I want to thank the following: Yinka (for his advice and support); Tezamed and Birhanu (for their friendship and support); Abate (for printing all materials I needed and for his best commitment).

Lastly, thanks to my family and friends for all their support, especially Tesfaye and Emaye (for their advice); Hayelom and H/Michael (for their friendship and advice) and Hermela (for her best friendship), whose kindness I can only hope to return.

Introduction

The Riordan group, named after the Combinatorialist John Riordan. Riordan was an American mathematician who worked at Bell Laboratories (Bell Labs) for most of his working life. He had a strong influence on the development of Combinatorics. In 1989,The Riordan group, named in his honour, was first introduced by Shapiro, Getu, Woan and Woodson in a seminar paper.[4]

The Hankel transform of an integer sequence has attracted an increasing amount of attention recently. Although Hankel determinants are well-known for a long time, Layman first introduced the term “Hankel transform” in 2001. This is a transform action on the set of integer sequences.

In this note we explore the properties of a simply defined family of Riordan arrays. The inverse of these arrays are closely related to well-known Catalan-defined matrices. This motivates us to study the Hankel transforms of the images of some well-known families of sequences under the inverse matrices. We also give some examples of this phenomenon.

Chapter One

Generating Functions

1.1 Ordinary Generating Functions (OGF)

A sequence is a mapping from the set $\mathbb{N}$ of natural numbers into some other set of numbers. If $a : \mathbb{N} \longrightarrow \mathbb{R}$, the sequence is called a sequence of real numbers. Usually, the image of a $k \in \mathbb{N}$ is denoted by a_k instead of the traditional $a(k)$, and the whole sequence is abbreviated as $(a_0, a_1, a_{2,} \ldots) = (a_k)_{k \in \mathbb{N}}$. Because of this notation, an element $k \in \mathbb{N}$ is called an index.

Suppose we are given a sequence $a_0, a_1, \ldots$ The ordinary generating function associated with the sequence $a_0, a_1, \ldots$ is the function A(x) whose value at x is the power series $\sum_{n=0}^{\infty} a_n x^n$. In other words "ordinary generating function of " is a map(function) from sequences to power series that packages the entire series of numbers $a_0, a_1, \ldots$ in to a single function A(x) .

Definition 1.1 Ordinary generating function (OGF). Suppose we are given a sequence $a_0, a_1, \ldots$ The ordinary generating function (also called OGF) associated with this sequence is the function whose value at x is $\sum_{i=0}^{\infty} a_i x^i$. The sequence $a_0, a_1, \ldots$ is called the coefficients of the generating function.

The ordinary generating function also called a "power series" because it is the sum of a series whose terms involve powers of x.
Let $(a_n)_{n \geq 0}$ be a sequence of numbers. The ordinary generating function for this sequence is the power series

$$A(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots = \sum_{n=0}^{\infty} a_n x^n$$

Definition 1.2 Given a generating function A(x) we use $[x^n]$ A(x) to denote a_n , the coefficient of x^n.

Example 1.1 $(\frac{1}{0!}, \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \ldots)$ has as its generating function

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots=\sum_{n=0}^{\infty}\frac{x^n}{n!}=e^x.$$

Example 1.2 $1+x+x^2+\ldots=\sum_{n=0}^{\infty}x^n$ is the generating function for the sequence (1, 1, 1, 1, …).

Theorem 1.1: The Binomial Theorem

$$(x+y)^n=\binom{n}{0}x^n+\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y^2+\ldots+\binom{n}{n}y^n=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.$$

Example 1.3 Binomial coefficients. Set $a_{k,n}=\binom{n}{k}$. Remember that $a_{k,n}=0$ for k > n. From the Binomial Theorem,

$(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k$. Thus $\sum a_{k,n}x^k=(1+x)^n$ and so

$$A(x,y)=\sum_{n\geq0}\sum_{k\geq0}a_{k,n}x^k y^n=\sum_{n\geq0}(1+x)^n y^n=\sum_{n=0}^{\infty}(1+x)y)^n$$

From the formula $\sum_{k\geq0}a\,z^k=\frac{a}{(1-z)}$ for summing a geometric series,

$$A(x,y)=\frac{1}{1-(1+x)y}=\frac{1}{1-y-xy}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(*)$$

Now let's look at $[x^k]$ A(x,y). From (*) and the formula for a geometric series,

$$A(x,y)=\frac{1}{1-y-xy}=\frac{\frac{1}{(1-y)}}{1-\frac{xy}{(1-y)}}=\sum_{k\geq0}\frac{1}{(1-y)}(\frac{xy}{(1-y)})^k=\sum_{k\geq0}\frac{1}{(1-y)}(\frac{y}{(1-y)})^k x^k.$$

Thus $[x^k]\,A(x,y)=\frac{1}{(1-y)}(\frac{y}{(1-y)})^k$. In other words, we have the generating function

$$\sum_{k=0}^{\infty}\binom{n}{k}y^n=\frac{y^k}{(1-y)^{k+1}}.$$

Theorem 1.2: Convolution Formula. Let A(x), B(x), and C(x) be generating functions, then C(x) = A(x)B(x) if and only if

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \text{ for all } n \geq 0. \ldots\ldots\ldots\ldots\ldots\ldots(*)$$

We call (*) a convolution.

Proof: We first prove that C(x) = A(x)B(x) gives the claimed summation. Since we are not concerning ourselves with convergence, we can multiply generating functions like polynomials:

$$\begin{aligned} A(x)B(x) = \left(\sum_{k=0}^{\infty} a_k\, x^k \right)\left(\sum_{j=0}^{\infty} b_j\, x^j \right) &= \sum_{k,j=0}^{\infty} a_k\, b_j\, x^{k+j} \\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n} a_k\, b_{n-k} \right)x^n \end{aligned}$$

Where the last equality follows by letting $k+j = n$; that is $j = n-k$. The sum on k stops at n because $j \geq 0$ is equivalent to $n-k \geq 0$, which is equivalent to $k \leq n$. This proves that C(x) = A(x)B(x) implies (*).

Now suppose we are given (*). Multiply by x^n, sum over n ≥ 0, let $j = n-k$ and reverse the steps in the previous paragraph to obtain

$$C(x) = \sum_{n\geq 0} c_n\, x^n = \sum_{k,j\geq 0} a_k b_j x^{k+j} = A(x)B(x)\,. \qquad \square$$

1.2 Exponential Generating Functions (EGF)

The exponential generating function for the sequence of number (a_n) is defined to be the power series

$$f(x) = a_0 + a_1\, x + a_2\, \frac{x^2}{2!} + a_3\, \frac{x^3}{3!} + \ldots + a_n\, \frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty} a_n\, \frac{x^n}{n!}$$

Example 1.4 The exponential generating function for the sequence

0! , 1! , 2! , 3! … n! … is

$$1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} n!\frac{x^n}{n!} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}\,.$$

Example 1.5 The EGF for permutations:

Recall: $p_r^n = \binom{n}{r}.r!$

Then,

$$\sum_{r=0}^{n} p_r^n \frac{x^r}{r!} = \sum_{r=0}^{n} \binom{n}{r}.r!\frac{x^r}{r!} = \sum_{r=0}^{n} \binom{n}{r}.x^r = (1+x)^n.$$

Therefore, the EGF for permutations is $\sum_{r=0}^{n} \binom{n}{r}.x^r = (1+x)^n$.

Note: From the definition of the exponential generating function, we have $n![x^n]f(x) = [\frac{x^n}{n!}]f(x)$. Here, we use the operator $[x^n]$ to extract the n^{th} coefficient of the power series $f(x)$.

Theorem 1.3: Let $f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ and $g(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}$ be the exponential generating functions of the sequences (a_n) and (b_n) respectively.

Then,

$$f(x).g(x) = D(x) = d_0 + d_1 x + d_2 x^2 + \ldots\ldots$$

where $d_n = \binom{n}{0} a_0 b_n + \binom{n-1}{1} a_1 b_{n-1} + \ldots + \binom{n}{n} a_n b_0 = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$

Therefore, $D(x) = \sum_{n=0}^{\infty} d_n \frac{x^n}{n!} = \sum_{n=0}^{\infty}(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k})\frac{x^n}{n!}$.

Proof: $f(x)g(x) = (\sum_{n=0}^{\infty} a_n \frac{x^n}{n!})(\sum_{n=0}^{\infty} b_n \frac{x^n}{n!})$

$$= \sum_n \sum_{k,s} \frac{a_k b_s}{k!s!} x^{k+s} \quad \text{where, } n = k+s$$

$$= \sum_n \sum_k \frac{n! a_k b_{n-k}}{k!(n-k)!} \frac{x^n}{n!} = \sum_n \sum_k \binom{n}{k} a_k b_{n-k} \frac{x^n}{n!}.$$

□

Chapter Two

Riordan Arrays

2.1 Riordan Matrix

Definition 2.1 Let $g(x) = 1 + g_1 x + g_2 x^2 + g_3 x^3 + \ldots$and
$f(x) = x + f_2 x^2 + f_3 x^3 + \ldots$ The matrix corresponding to the pair g, f is denoted by (g, f), and is often called the ***Riordan array*** defined by g and f. When $g_0 = 0$, the array is called a monic Riordan Array. When $f_1 \neq 0$, the array is called a proper Riordan Array.
The matrix M is an infinite matrix given as follows.

$$M = (g(x), f(x)) = \begin{bmatrix} \cdot & \cdot & \cdot & & \cdot & & & & \\ \cdot & \cdot & \cdot & & \cdot & & & & \\ \cdot & \cdot & \cdot & & \cdot & & & & \\ g & gf & gf^2 & \ldots & gf^k & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & & & \\ \cdot & \cdot & \cdot & & \cdot & & & & \\ \cdot & \cdot & \cdot & & \cdot & & & & \end{bmatrix}$$

It is called a ***Riordan Matrix.***

Example 2.1 Pascal‟s Triangle

$$\begin{bmatrix} 1 & \ldots \\ 1 & 1 & \ldots \\ 1 & 2 & 1 & \ldots \\ 1 & 3 & 3 & 1 & \ldots \\ 1 & 4 & 6 & 4 & 1 & \ldots \\ 1 & 5 & 10 & 10 & 5 & 1 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

This is a Riordan matrix with $g(x) = \frac{1}{1-x}$ and $f(x) = \frac{x}{1-x}$

2.2 Riordan Group

Riordan arrays give us an intuitive method of solving combinatorial problems, helping to build an understanding of many number patterns. They provide an effective method of proving combinatorial identities and solving numerical puzzles rather than using computer based approaches.[4] Riordan arrays, named after the Combinatorist, John Riordan, were first used in 1990‟s by Shapiro as a method of exploring combinatorial patterns in numbers of Pascal‟s triangle. Shapiro saw the natural extension of Pascal‟s triangle due to its shape, to a lower triangular matrix, making use of matrix representation of transformations on sequences, then using this to explore patterns in the numbers of Pascal‟s triangle. This has become a classical example of a Riordan array. It was while exploring these extensions of Pascal‟s triangle that it was realized that Riordan arrays have a group structure.

Definition 2.2 The Riordan group **R**.
The Riordan group **R**, is a set of infinite lower – triangular integer matrices with 1‟s on the main diagonal.

Thus, $\mathbf{R} = \{(g(x), f(x)) \mid (g(x), f(x))$ is a Riordan array and

$$f(x) = f_0 + f_1\, x + f_2\, x^2 + f_3\, x^3 + \ldots, \text{ where } f_0 = 0,\ f_1 \neq 0.\}$$

Notation: The Riordan matrix multiplication in **R** (the group law) is given by

$$(g(x), f(x)) \,.\, (h(x), l(x)) = (g(x) \,.\, h(f(x), l(f(x))) \,.$$

Example 2.2 Let $g(x) = h(x) = \frac{1}{1-x}$ and $f(x) = l(x) = \frac{x}{1-x}$. Then,

$$\left(\frac{1}{1-x}, \frac{x}{1-x}\right).\left(\frac{1}{1-x}, \frac{x}{1-x}\right) = \left(\frac{1}{1-x} \,.\, \frac{1}{1-\frac{x}{1-x}}, \frac{\frac{x}{1-x}}{1-\frac{x}{1-x}}\right) = \left(\frac{1}{1-2x}, \frac{x}{1-2x}\right).$$

Definition 2.3 A lower-triangular infinite matrix, L, is a Riordan Array, if the generating function of the k^{th} column is

$$g(x)f(x)^k \qquad \text{for } k = 0, 1, 2, 3, \ldots$$

where

$$g(x) = 1 + g_1 x + g_2 x^2 + g_3 x^3 + \ldots,$$

$$f(x) = \quad x + f_2 x^2 + f_3 x^3 + \ldots.$$

The identity is $I = (1,x) = \begin{bmatrix} 1 & . & . & . & . & . & . & 0 \\ 0 & 1 & . & . & . & . & . & . \\ 0 & 0 & 1 & . & . & . & . & . \\ 0 & 0 & 0 & 1 & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & 1 \end{bmatrix}$ and the inverse of $(g(x), f(x))$ is

$$(g(x), f(x))^{-1} = \left(\frac{1}{g(\bar{f}(x))}, \bar{f}(x)\right)$$

Where, $\bar{f}(x)$ is the compositional inverse of $f(x)$.

Example 2.3 Pascal's Triangle, $\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$ and $\bar{f}(x) = \frac{x}{1+x}$.

$$\text{Then, } \left(\frac{1}{1-x}, \frac{x}{1-x}\right)^{-1} = \left(\frac{1}{1-\frac{x}{1+x}}, \frac{x}{1+x}\right) = \left(\frac{1}{1+x}, \frac{x}{1+x}\right).$$

Note: The row sums of the matrix (g, f) have generating function

$$(g, f)\frac{1}{1-x} = \frac{g(x)}{1-f(x)}$$

while the diagonal sums of (g, f) (sums of left-to right diagonals in the North East direction) have generating function $\frac{g(x)}{(1-xf(x))}$. This coincides with the row sums of the "generalized" Riordan array (g, xf):

$$(g, xf)\frac{1}{1-x} = \frac{g(x)}{1-xf(x)}.$$

Note: The row sums of the array $[g, f]$ have exponential generating function given by $g(x)e^{f(x)}$ since the sequence 1, 1, 1, 1, … has exponential generating function e^{x}.

Example 2.4 The Fibonacci numbers F_{n+1} are the diagonal sums of the binomial matrix B given by $(\frac{1}{1-x}, \frac{x}{1-x})$.

Theorem 2.1 Let D = $(g(x), f(x))$ be a Riordan array and $h(x) = \sum_{k\geq 0} h_k x^k$ be a generating function of the sequence $\{h_k\}_{k\geq 0}$. D(x) = $\sum_k d_{n,k} x^n$.
Then,

$$\sum_{k\geq 0} d_{n,k}\, h_k = [x^n]\, g(x).h(f(x))$$

Proof:

$$\sum_{k\geq 0} d_{n,k}\, h_k = \sum_{k\geq 0} [x^n] g(x)(f(x))^k\, h_k$$

$$= [x^n] g(x) \sum_{k\geq 0} h_k (f(x))^k$$

$$= [x^n] g(x).h(f(x)) \qquad \square$$

Example 2.5 (Row sum) Let D = $(\frac{1}{1-x}, \frac{x}{1-x})$ be a Riordan array and $d_{n,k}$ be the entries. If $h(x) = \frac{1}{1-x}$ then,

$$\sum_{k\geq 0} d_{n,k}\, h_k = [x^n]\, (\frac{1}{1-x}, \frac{1}{1-\frac{x}{1-x}})$$

$$= [x^n]\,(\tfrac{1}{1-x}, \tfrac{1-x}{1-2x})$$

$$= [x^n]\,(\tfrac{1}{1-2x})$$

$$= 2^n.$$

Theorem 2.2: Fundamental Theorem of Riordan Arrays (FTRA).

If M is the matrix (g, f), and $a = (a_0, a_1, a_2, \ldots)^T$ is an integer sequence with ordinary generating function A(x), then the sequence Ma has ordinary generating function g(x)A(f(x)). This result is often called „„The Fundamental Theorem of Riordan arrays‟‟

$$\mathrm{M}a = (\,g(x), f(x)\,)\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots & & \vdots & \\ g & gf & gf^2 & \cdots & gf^k & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{bmatrix}$$

Where $\mathrm{A}(x) = \sum_n a_n x^n$ and $\mathrm{B}(x) = \sum_n b_n x^n$

Proof: We look at the Riordan Array (g(x), f(x)) column by column, and multiply it by the column vector on the LHS (left hand side)

$$\begin{bmatrix} | & & & & \\ | & | & & & \\ g & gf & gf^2 & & \\ | & | & | & | & \\ | & | & | & | & | \\ & \cdots & & & \ddots \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{bmatrix}$$

This yield

$$a_0 g + a_1 g.f + a_2 g.f^2 + \ldots + a_k g.f^k + \ldots = B(x)$$

$$g\,(a_0 + a_1 f + a_2 f^2 + \ldots + a_k f^k + \ldots.) = B(x)$$

$$g(x)A\,(f(x)) = B(x),$$

and we have our result. □

Example 2.6 From Pascal Triangle. Let $A(x) = \frac{1}{1-x}$, $g(x) = \frac{1}{1-x}$, and $f(x) = \frac{x}{1-x}$.

$$\text{Then, } B(x) = \left(\frac{1}{1-x}, \frac{x}{1-x}\right).\frac{1}{1-x} = \frac{1}{1-x}\cdot\frac{1}{1-\frac{x}{1-x}} = \frac{1}{1-2x}$$

Therefore, $B(x) = \frac{1}{1-2x}$ and $[x^n]\,B(x) = 2^n$.

2.3 Riordan subgroups of R (The set of Riordan arrays)

Some of the important subgroups of **R** are the Appel, Associated, Bell subgroups.

(1) The Appel subgroup is $\{\, (g(x), x) \,\}$

Example 2.7 $\left(\frac{1}{1-2x}, x\right)$

$$\left(\frac{1}{1-2x}, x\right) = \begin{bmatrix} 1 \\ 1 & 1 \\ 2 & 1 & 1 \\ 4 & 2 & 1 & 1 \\ 8 & 4 & 2 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 & 1 \\ & & . & . & . & & \ddots \end{bmatrix}$$

(2) The associated subgroup is $\{\, (1, f(x)) \,\}$

Then we clearly have $(1, f(x))^{-1} = (1, \bar{f}(x))$

Example 2.8 We consider the Riordan array $\left(1, \frac{1-\sqrt{1-4x}}{2x}\right)$. This is the matrix

$$(1,\frac{1-\sqrt{1-4x}}{2x}) = \begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 1 & 1 & & & \\ 0 & 2 & 2 & 1 & & \\ 0 & 5 & 5 & 3 & 1 & \\ 0 & 14 & 14 & 9 & 4 & 1 \\ & . & . & . & & \ddots \end{bmatrix}$$

We have $(1,\frac{1-\sqrt{1-4x}}{2x})^{-1}$ = (1, x (1-x)) which has general term

$$= [x^n]1.\,(x\,(1-x))^k$$

$$= [x^n]\,x^k(1-x)^k$$

$$= [x^{n-k}]\,(1-x)^k$$

$$= [x^{n-k}]\sum_{i=0}^{k}\binom{k}{i}(-1)^i\,x^i$$

$$= \binom{k}{n-k}(-1)^{n-k} = (-1)^{n-k}\binom{k}{n-k}.$$

(3) The Bell subgroup is { (g(x), x g(x)) }

We let $f(x) = x\,g(x)$ and hence (g(x), x g(x)) = (g(x), f(x)). Then

$$(g(x), x\,g(x))^{-1} = (g(x),\ f(x))^{-1}$$

$$= (\frac{1}{g(\bar{f})}, \bar{f})$$

$$= (\frac{\bar{f}}{x}, \bar{f})$$

In addition, we have

$$(g(x), x\,g(x))\,(h(x), x\,h(x)) = (g(x)h(x\,g(x)), x\,g(x)h(x\,g(x))$$

$$= (\tilde{g}(x), x\tilde{g}(x))$$

Where $\tilde{g}(x) = g(x)\, h(x\, g(x)$. Thus the subset of Riordan arrays of the form $(g(x), x\, g(x))$ constitutes a sub-group of **R**.

Example 2.9 Binomial matrix

The so-called binomial matrix B is the element $(\frac{1}{1-x}, \frac{x}{1-x})$ of the Riordan group. It has general element $\binom{n}{k}$, More generally, B^m is the element $(\frac{1}{1-mx}, \frac{x}{1-mx})$ of the Riordan group, with general term $\binom{n}{k} m^{n-k}$. The inverse B^{-m} of B^m is given by $(\frac{1}{1+mx}, \frac{x}{1+mx})$

$$(\frac{1}{1-x}, \frac{x}{1-x})^{-1} = (\frac{1}{1+x}, \frac{x}{1+x}) = \begin{bmatrix} 1 \\ -1 & 1 \\ 1 & -2 & 1 \\ -1 & 3 & -3 & 1 \\ 1 & -4 & 6 & -4 & 1 \\ -1 & 5 & -10 & 10 & -5 & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix}$$

Example 2.10 Let $g(x) = C(x) = \frac{1-\sqrt{1-4x}}{2x}$ be the generating function of the Catalan numbers. Then if $f(x) = xg(x)$ and we have $\bar{f}(x) = x(1-x)$. Hence

$$(C(x), xC(x))^{-1} = (1-x, x(1-x))$$

Thus the inverse of the matrix

$$\begin{bmatrix} 1 \\ 1 & 1 \\ 2 & 2 & 1 \\ 5 & 5 & 3 & 1 \\ 14 & 14 & 9 & 4 & 1 \\ 42 & 42 & 28 & 14 & 5 & 1 \\ & . & . & . & & \ddots \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ 0 & -2 & 1 & & & \\ 0 & 1 & -3 & 1 & & \\ 0 & 0 & 3 & -4 & 1 & \\ 0 & 0 & -1 & 6 & -5 & 1 \\ & & . & . & . & \ddots \end{bmatrix}$$

With general term

$$[x^n]\,(1\text{-}x)(x(1-x))^k = [x^{n-k}](1-x)^{k+1} = (-1)^{n-k}\binom{k+1}{n-k}.$$

(4) The Derivative subgroup is $\{ (g(x), f(x)) \mid f'(x) = g(x)\}$

Let D = $\{ (g(x), f(x)) \mid f'(x) = g(x)\}$, a subset of the Riordan group. We want to show D is a subgroup of **R**.

Proof:
Closure: For any $(g_1(x), f_1(x)), (g_2(x), f_2(x)) \in$ D, we have

$$f_1'(x) = g_1(x),\ f_2'(x) = g_2(x).$$

Because $(g_1(x), f_1(x))(g_2(x), f_2(x)) = (g_1(x)\, g_2(f_1(x)), f_2(f_1(x)))$, and

$[f_2(f_1(x))]' = f_2'(f_1(x))f_1'(x) = g_2(f_1(x))g_1(x)$, so

$$(g_1(x), f_1(x))(g_2(x), f_2(x)) \in \text{D}.$$

Associativity: Because D is a subset of a group.

Identity: Since $x' = 1$, I $(1, x) \in$ D.

Inverse: For any $(g(x), f(x)) \in$ D, the inverse is $\left(\frac{1}{g(\bar{f}(x))}, \bar{f}(x)\right)$.

But $f(\bar{f}(x)) = x$ yields $\frac{d}{dx}(f(\bar{f}(x))) = f'(\bar{f}(x))\bar{f}'(x) = 1$.
Thus,

$\bar{f}'(x) = \frac{1}{f'(\bar{f}(x))} = \frac{1}{g(\bar{f}(x))}$ and $(g(x), f(x))^{-1} \in$ D. Then we finish our proof. □

Chapter Three

Riordan Arrays and Integer Hankel Transforms

3.1 The Matrix M

Let the matrix **M** be the Riordan array, $\mathbf{M} = (\frac{1}{1-x^2}, \frac{x}{(1+x)^2})$. We shall also consider the related matrices, $\widetilde{\mathbf{M}} = (\frac{1}{1-x^2}, \frac{-x}{(1-x)^2})$ and $\mathbf{M}^+ = (\frac{1}{1-x^2}, \frac{x}{(1-x)^2})$.

The matrix **M** begins

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 1 & 0 & 0 & 0 & 0 & . & . & . \\ 1 & -2 & 1 & 0 & 0 & 0 & . & . & . \\ 0 & 4 & -4 & 1 & 0 & 0 & . & . & . \\ 1 & -6 & 11 & -6 & 1 & 0 & . & . & . \\ 0 & 9 & -24 & 22 & -8 & 1 & . & . & . \\ . & . & . & . & . & . & . & & \\ . & . & . & . & . & . & & . & \\ . & . & . & . & . & . & & & . \end{bmatrix}$$

While $\widetilde{\mathbf{M}}$ begins

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & -1 & 0 & 0 & 0 & 0 & . & . & . \\ 1 & -2 & 1 & 0 & 0 & 0 & . & . & . \\ 0 & -4 & 4 & -1 & 0 & 0 & . & . & . \\ 1 & -6 & 11 & -6 & 1 & 0 & . & . & . \\ 0 & -9 & 24 & -22 & 8 & -1 & . & . & . \\ . & . & . & . & . & . & . & & \\ . & . & . & . & . & . & & . & \\ . & . & . & . & . & . & & & . \end{bmatrix}$$

The entries of $\mathbf{M}^{+}$ are the absolute value of entries of both these matrices.
The general term $\mathbf{M}_{n,k}$ of $\mathbf{M}$ as follows,

$$\mathbf{M}_{n,k} = [x^n]\left(\frac{1}{1-x^2}\frac{x^k}{(1+x)^{2k}}\right)$$

$$= [x^{n-k}](1-x)^{-1}(1+x)^{-(2k+1)}$$

$$= [x^{n-k}]\sum_{j=0}^{\infty} x^j \sum_{i=0}^{\infty}\binom{-(2k+1)}{i}x^i$$

$$= [x^{n-k}]\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\binom{2k+i}{i}(-1)^i x^{i+j}$$

$$= \sum_{j=0}^{n}\binom{n+k-j}{n-k-j}(-1)^{n-k-j}$$

$$= \sum_{j=0}^{n-k}\binom{n+k-j}{2k}(-1)^{n-k-j}.$$

An alternative expression for $\mathbf{M}_{n,k}$ can be obtained by noticing that

$$\mathbf{M} = \left(\frac{1}{1-x^2}, \frac{x}{(1+x)^2}\right) = \left(\frac{1}{1-x}, x\right) \cdot \left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right)$$

The general term of $\left(\frac{1}{1-x}, x\right)$ is $[x^n]\frac{1}{1+x}x^k = [x^{n-k}]\frac{1}{1+x} = [k \le n]\, 1$, while that of $\left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right)$ is

$$[x^n]\frac{1}{1+x}\left(\frac{x}{(1+x)^2}\right)^k = [x^n]\frac{1}{1+x}\frac{x^k}{(1+x)^{2k}} = [x^{n-k}]\frac{1}{(1+x)^{2k+1}} = (-1)^{n-k}\binom{n+k}{2k}.$$

Thus we obtain the alternative expression

$$\mathbf{M}_{n,k} = \sum_{j=0}^{n}(-1)^{j-k}\binom{k+j}{2k}.$$

This translates the combinatorial identity

$$\sum_{j=0}^{n}(-1)^{j-k}\binom{k+j}{2k} = \sum_{j=0}^{n-k}\binom{n+k-j}{2k}(-1)^{n-k-j}.$$

Note: $\mathbf{M} = (\frac{1}{1-x^2}, \frac{x}{(1+x)^2}) = (\frac{1}{(1-x)(1+x)}, \frac{x}{(1+x)^2})$ is the element $\mathbf{M}_1$ of the family of Riordan array $\mathbf{M}_r$ defined by

$$\mathbf{M}_r = (\frac{1}{(1-x)(1+rx)}, \frac{x}{(1+rx)^2})$$

The general term of $\mathbf{M}_r$ is given by

$$\sum_{j=0}^{n-k}\binom{n+k-j}{2k}(-r)^{n-k-j}.$$

The row sums of **M** are the periodic sequence

1, 1, 0, 1, 1, 0, 1, 1, 0 …

with generating function $(\frac{1}{1-x^2}, \frac{x}{(1+x)^2})\frac{1}{1-x} = \frac{\frac{1}{1-x^2}}{1-\frac{x}{(1+x)^2}} = \frac{1+x}{1-x^3}$. Where

$$\frac{1}{1-x^3} = \sum_{n=0}^{\infty}(x^3)^n = \sum_{n=0}^{\infty} x^{3n} = 1 + x^3 + x^6 + x^9 + \ldots$$

Let $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ be the generating function of the Catalan numbers C_n, we have

$$(\frac{1}{1-x^2}, \frac{x}{(1+x)^2}).\ \mathrm{C}(x) = \frac{1}{1-x^2}\,\frac{1-\sqrt{1-4\frac{x}{(1+x)^2}}}{2\frac{x}{(1+x)^2}} = \frac{1}{1-x},$$

Thus, the image of C_n under the matrix **M** is the all 1‟s sequence

$$1, 1, 1, 1, 1, 1, 1 \ldots$$

The image of C_{n+1} is also interesting. We get the sequence 1, 2, 2, 2 ... with generating function $\frac{1+x}{1-x}$. This can be generalized to the following result: The image of C_{n+k} under the matrix **M** has generating function $\frac{\sum_{j=0}^{k} a_{k,j}\, x^j}{1-x}$, where $a_{n,k}$ is the general term of the matrix (C (x), xC$(x)^2$). We have

$$a_{n,k} = \binom{2n}{n-k}\frac{2k+1}{n+k+1}.$$

Now taking $\binom{2n}{n}$ with generating function $\frac{1}{\sqrt{1-4x}}$, we obtain

$$\left(\frac{1}{1-x^2}, \frac{x}{(1+x)^2}\right)\frac{1}{\sqrt{1-4x}} = \frac{1}{1-x^2}\frac{1}{\sqrt{1-4\frac{x}{(1+x)^2}}} = \frac{1}{(1-x)^2}.$$

Thus the image of $\binom{2n}{n}$ under **M** is n+1 or the counting numbers 1, 2, 3, 4, … .

3.2 The Inverse Matrix $\mathbf{M}^{-1}$

Since **M** = $\left(\frac{1}{1-x}, x\right) . \left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right)$ we see that

$$\mathbf{M}^{-1} = \left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right)^{-1}(1-x, x) = (C(x), xC(x)^2)(1-x, x)$$

To find $\left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right)^{-1}$, first we have to determine the compositional inverse, $\bar{f}(x)$, of $f(x)$.

Approach to determine the inverse: Let y = $f(x) = \frac{x}{(1+x)^2}$.

$$x = \frac{y}{(1+y)^2} \Rightarrow x(1+y)^2 = y \Rightarrow x(1+2y+y^2) = y$$
$$\Rightarrow x + 2xy + xy^2 = y$$
$$\Rightarrow x = y(1 - 2x - xy)$$
$$\Rightarrow x = y - 2xy - xy^2$$

$$\Rightarrow \quad xy^2 + (2x - 1)y + x = 0$$

Solving for y we get $y = \bar{f}(x) = \mathrm{C}(x) - 1$, which is equal to $x\mathrm{C}(x)^2$, where $\mathrm{C}(x)$ is the generating function of Catalan numbers.

$$\left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right)^{-1} = \left(\frac{1}{\frac{1}{1+\mathrm{C}(x)-1}}, \mathrm{C}(x) - 1\right) = \left(\frac{1}{\frac{1}{C(x)}}, x\mathrm{C}(x)^2\right) = (\mathrm{C}(x), x\mathrm{C}(x)^2).$$

Thus, $\mathbf{M}^{-1} = (\mathrm{C}(x), x\mathrm{C}(x)^2)\,(1-x, x) = (\mathrm{C}(x)(1-x\mathrm{C}(x)^2), x\mathrm{C}(x)^2)$.

The general term of $(1-x, x)$ is

$$= [x^n](1-x)x^k$$

$$= [x^n]\sum_j \binom{1}{j}(-1)^j x^j x^k$$

$$= [x^n]\sum_j \binom{1}{j}(-1)^j x^{j+k}$$

$$= \binom{1}{n-k}(-1)^{n-k}.$$

And the general term of $(\mathrm{C}(x), x\mathrm{C}(x)^2)$ is $\binom{2n}{n-k}\frac{2k+1}{n+k+1}$.

Therefore, the general term of $\mathbf{M}^{-1}$ is given by

$$\mathbf{M}^{-1}_{n,k} = \sum_{j=0}^{n}(-1)^{j-k}\binom{1}{j-k}\binom{2n}{n-j}\frac{2j+1}{n+j+1}.$$

3.3 $\mathbf{M^{-1}}$ and Hankel Transforms

3.3.1 Hankel Matrix

Definition 3.1 The Hankel matrix H of the integer sequence $\{a_1, a_2, a_3, \ldots\}$ is the infinite matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ a_4 & a_5 & a_6 & a_7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & & \ddots \end{bmatrix},$$

With elements $h_{i,j} = a_{i+j-1}$. The Hankel matrix H_n of order *n* of A is the upper-left *n*×*n* submatrix of H, and h_n, the Hankel determinant of order *n* of A, is the determinant of the corresponding Hankel matrix of order *n*, $h_n = \det(H_n)$.

Example 3.1 The Hankel matrix of order 4 of the Fibonacci sequence
1, 1, 2, 3, 5, 8, 13… is

$$H_4 = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 5 \\ 2 & 3 & 5 & 8 \\ 3 & 5 & 8 & 13 \end{bmatrix}$$

Example 3.2 The Hankel matrix of order 4 of the sequence of Catalan numbers {1, 1, 2, 5, 14, 42, 132 …} is

$$H_4 = \begin{bmatrix} 1 & 1 & 2 & 5 \\ 1 & 2 & 5 & 14 \\ 2 & 5 & 14 & 42 \\ 5 & 14 & 42 & 132 \end{bmatrix}.$$

3.3.2 Hankel Transform

The Hankel transform of an integer sequence has attracted an increasing amount of attention recently. Although Hankel determinants are well-known for a long time, Layman[5] first introduced the term "Hankel transform" in 2001. This is a transform action on the set of integer sequences defined a follows.

Definition 3.2 The Hankel transform of a given sequence $a = \{a_n\}_{n \in \mathbb{N}_0}$ is the sequence of Hankel determinants h = H(a) = $\{h_n\}_{n \in \mathbb{N}_0}$ where

$$h_n = \det\left[a_{i+j-2}\right]_{i,j=1}^{n}, \quad \text{i.e}$$

$$a = \{a_n\}_{n \in \mathbb{N}_0} \Rightarrow^{H} h = \{h_n\}_{n \in \mathbb{N}_0} : h_n = \det\begin{bmatrix} a_0 & a_1 & . & . & . & a_n \\ a_1 & a_2 & . & . & . & a_{n+1} \\ . & & . & & & \\ . & & & . & & \\ . & & & & . & \\ a_n & a_{n+1} & . & . & . & a_{2n} \end{bmatrix}$$

Example 3.3 The Hankel transform of the central binomial coefficients $\left\{\binom{2n}{n}\right\}_{n \in \mathbb{N}_0}$ is the sequence $\{2^n\}_{n \in \mathbb{N}_0}$.

That is $[\,1\,] = 1$, $\begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} = 2$, $\begin{bmatrix} 1 & 2 & 6 \\ 2 & 6 & 20 \\ 6 & 20 & 70 \end{bmatrix} = 4 \ldots$

Example 3.4 The Hankel transform of the sequence of Catalan numbers ($C_n = \frac{1}{n+1}\binom{2n}{n}$) is the sequence of all 1‟s. Thus each of the determinants

$$[\,1\,],\ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},\ \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 5 & 14 \end{bmatrix}, \ldots\ldots$$

has value 1.

Theorem 3.1: Consider an ordinary generating function $g(x) = \frac{1}{1-f(x)}$ of a sequence (a_n), with $f(0) = 0$ and suppose that

$$h(x) = \frac{f(\text{x})}{x} - f'(0)$$

satisfies

$$h(\text{x}) = x(\lambda + \mu h(x) + vh^2(x)).$$

Then the Hankel transform (h_n) of (a_n)[3] is given by

$$h_n = \lambda^{n(n+1)/2} v^{n(n-1)/2}.$$

Thus we let $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ be the o.g.f of the Catalan numbers, and let

$g(x) = C(x) = \frac{1-\sqrt{1-4x}}{2x}$, and $f(x) = 1 - \frac{1}{g(x)} = \frac{1-\sqrt{1-4x}}{2}$.

We note that $f'(0) = 1$ and so we can define

$$h(\text{x}) = \frac{f(\text{x})}{x} - f'(0) = \frac{1-2x-\sqrt{1-4x}}{2x}.$$

We now seek λ, μ and v such that

$$l(x) = \lambda + \mu h(x) + vh^2(x)$$

then we have

$$l(x) = \frac{h(\text{x})}{x}.$$

In this case, we find that

$$\lambda = 1, \mu = 2, \ v = 1.$$

Thus the Hankel transform of C_n is given by

$$\lambda^{n(n+1)/2} v^{n(n-1)/2} = 1.$$

□

Definition 3.3 Binomial transform.
A transformation that is widely used in the study of integer sequences is the so-called Binomial transform, which associates to the sequence with general term a_n and the sequence with general term b_n where

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$$

If we consider the sequence to be the vector $(a_0, a_1, \ldots)$ then we obtain the binomial transform of the sequence by multiplying this (infinite) vector with the lower-triangle matrix B whose (i, j)-th element is equal to $\binom{i}{j}$:

$$B = \begin{bmatrix} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 1 & 2 & 1 & . & . & . & . & . & . & . & . & . & . \\ 1 & 3 & 3 & 1 & . & . & . & . & . & . & . & . & . & . \\ 1 & 4 & 6 & 4 & 1 & . & . & . & . & . & . & . & . \\ 1 & 5 & 10 & 10 & 5 & 1 & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}$$

This transformation is invertible, with

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k.$$

Example 3.5 The Binomial transform of 0^n is 1^n.

$$\sum_{k=0}^{n} \binom{n}{k} 0^k = 1^n = 1.$$

Note: If A(x) is the ordinary generating function of the sequence a_n, the generating function of the transformed sequence b_n, is $\frac{1}{(1-x)}$ A($\frac{x}{(1-x)}$).

Proof: $\binom{n}{k} = \binom{n}{n-k} = \binom{-n+n-k-1}{n-k}(-1)^{n-k} = \binom{-k-1}{n-k}(-1)^{n-k}$ and this is the coefficient of x^{n-k} in $(1-x)^{-k-1}$.

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k = \sum_{k=0}^{n} \binom{-k-1}{n-k} (-1)^{n-k} a_k = \sum_{k=0}^{\infty} [x^{n-k}] (1-x)^{-k-1} [y^k] \mathrm{A}(y)$$

$$= [x^n] \frac{1}{1-x} \sum_{k=0}^{\infty} [y^k] \mathrm{A}(y) \left(\frac{x}{1-x}\right)^k$$

$$= [x^n] \frac{1}{1-x} \mathrm{A}\left(\frac{x}{1-x}\right).$$

3.3.3 Invariance of the Hankel transform under the binomial transform

Further computational investigation reveals numerous instances in which one member of a pair of sequences with the same Hankel transform is the Binomial transform of the other.

Layman proves that H(B(A)) = H(A) for any sequence A. He does this by showing that the Hankel matrix of order *n* of B(A) can be obtained by multiplying the Hankel matrix of order *n* of A by certain upper and lower triangular matrices, each of which have determinant 1.

We present another proof of this result. Our proof technique suggests generalizations of the binomial transform. We require the following Lemma.

Lemma 3.1: Given a sequence A = $\{a_0, a_1, a_2, \ldots\}$, create a triangle of numbers T using the following rule:

1. The left diagonal of the triangle consists of the elements of A.

2. Any number off the left diagonal is the sum of the number to its left and the number diagonally above it to the left.

Then the sequence on the right diagonal is the binomial transform of A.

For example, the binomial transform of the derangement numbers is the factorial numbers. The figure 3.1 illustrates how the factorial numbers can be generated from the derangement numbers using the triangle described in Lemma 3.1.

$$\begin{array}{c}
1 \\
0 \quad 1 \\
1 \quad 1 \quad 2 \\
2 \quad 3 \quad 4 \quad 6 \\
9 \quad 11 \quad 14 \quad 18 \quad 24 \\
44 \quad 53 \quad 64 \quad 78 \quad 96 \quad 120 \\
265 \quad 309 \quad 362 \quad 426 \quad 504 \quad 600 \quad 720
\end{array}$$

Figure 1: derangement triangle.

Proof: Let t_n be the n^{th} element on the right diagonal of the triangle. By construction of the triangle, we can see that the number of times element a_i contributes to the value of t_n is the number of paths from a_i to t_n. To move from a_i to t_n requires *n* path segments, *i* of which move directly to the right. Thus there are $\binom{n}{i}$ ways to choose which of the *n* ordered segments the rightward-moving segments are, and the down segments are completely determined by this choice. Therefore the contribution of a_i to t_n is $\binom{n}{i} a_i$, and the value of t_n, in terms of the elements on the left diagonal, is $\sum_{i=0}^{n} \binom{n}{i} a_i$. But this is the definition of the binomial transform, making the right diagonal of the triangle the binomial transform of A.

□

The binomial transform has the following combinatorial interpretation: If a_n represents the number of arrangements of *n* labeled objects with some property P, then b_n represents the number of ways of dividing n labeled objects into two groups such that the first group has property P. In terms of the derangement and factorial numbers, then, as D_n is the number of permutations of *n* ordered objects in which no object remains in its original position, *n*! is the number of ways that one can divide *n* labeled objects into two groups, order the objects in the first group, and then permute the first group objects so that none remains in its original position.

The number in row i, position j, in the triangle is the number of permutations of i ordered objects such that every object after j does not remain in its original position.

We now give the proof of Layman‟s result.

Theorem 3.2: (Layman) The Hankel transform is invariant under the binomial transform.

Proof: We define a procedure for transforming the Hankel matrix of order n of a sequence A to the Hankel matrix of order n of B(A) using only matrix row and column addition. The procedure is as follows:

1. Given a sequence A = $\{a_0, a_1, a_2, \ldots\}$, create the triangle of numbers described in Lemma 3.1, where $T_{i,j}$ is the $(i,j)^{th}$ entry in the triangle.

2. Let T_n be the following matrix consisting of number from the left diagonal of T.

$$\begin{bmatrix} T_{0,0} & T_{1,0} & T_{2,0} & \cdots & T_{n,0} \\ T_{1,0} & T_{2,0} & T_{3,0} & & T_{n+1,0} \\ \vdots & \vdots & \vdots & & \vdots \\ T_{n,0} & T_{n+1,0} & T_{n+2,0} & \cdots & T_{2n,o} \end{bmatrix}$$

Since $a_i = T_{i,0}$, T_n is the Hankel matrix of order n of A.

3. Then apply the following transformations to T_n, where rows and columns of the matrix are indexed beginning with 0.
   (a) Let i range from 1 to n. During stage i, for each row $j \geq i$, add row j-1 to row j and replace row j with the result.

   (b) Then let i again range from 1 to n. During stage i for each column $j \geq i$, add column j-1 to column j and replace column j with the result.

Claim1: After stage i in 3(a), row m of the matrix is of the following form:

$$[T_{m,m}, T_{m+1,m} \ldots\ldots T_{n+m,m}], \text{ if m} \le i$$

$$[T_{m,i}, T_{m+1,i} \ldots\ldots T_{n+m,i}], \qquad \text{if m} > i.$$

The claim is clearly true initially, when $i = 0$. Now, assume the claim is true for all values of i from 0 to k-1. Then in stage $i = k$, the only rows that change are rows k, k+1, ... , n. row m, for $m \ge k$, is sum of rows m and m-1 from the previous iteration:

$$[T_{m,k-1}+T_{m-1,k-1} \quad T_{m+1,k-1} + T_{m,k-1} \ldots T_{n+m,k-1} + T_{n+m-1,k-1}].$$

But, by the definition of T, $T_{i,j} + T_{i-1,j} = T_{i,j+1}$.

Thus this row is equal to

$$[T_{m,k} \;\; T_{m+1,k} \ldots T_{n+m,k}],$$

Proving the claim.

After the transformations in 3(a) are applied, then, we have the matrix

$$\begin{bmatrix} T_{0,0} & T_{1,0} & T_{2,0} & \cdots & T_{n,0} \\ T_{1,1} & T_{2,1} & T_{3,0} & & T_{n+1,1} \\ . & . & . & & . \\ . & . & . & & . \\ T_{n,n} & T_{n+1,n} & T_{n+2,n} & \cdots & T_{2n,n} \end{bmatrix}.$$

Claim2: **A**fter stage i in 3(b), column m of the matrix is of the following form:

$$[T_{m,m} \;\; T_{m+1,m+1} \ldots T_{n+m,n+m}]^{\mathrm{T}}, \text{ if } m \le i;$$

$$[T_{m,m} \;\; T_{m+1,m+1} \ldots T_{n+m,n+m}]^{\mathrm{T}}, \text{ if } m > i.$$

The proof is almost the same as that for claim 1. The claim is clearly true for $i = 0$. Assume the claim is true for all values of i from 0 to k-1. In stage $i = k$, the only columns that change are columns k, k+1, ... , n. column m, for $m \geq k$, is sum of columns m and m-1 from the previous iteration:

$$[T_{m,k-1} + T_{m-1,k-1} \quad T_{m+1,k} + T_{m,k} \; \cdots \; T_{n+m,n+k-1} + T_{n+m-1,n+k-1}]^{\mathrm{T}}$$

Again, by the definition of T, $T_{i,j} + T_{i-1,j} = T_{i,j+1}$.

Thus this column is equal to $[T_{m,k} \; T_{m+1,k+1} \; \cdots . \, T_{n+m,n+k}]^{\mathrm{T}}$,
which prove the claim.

After applying the transformation in 3(b), then, we have the matrix

$$\begin{bmatrix} T_{0,0} & T_{1,1} & T_{2,2} & \cdots & T_{n,n} \\ T_{1,1} & T_{2,2} & T_{3,3} & & T_{n+1,n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ T_{n,n} & T_{n+1,n+1} & T_{n+2,n+2} & \cdots & T_{2n,2n} \end{bmatrix}$$

But this is the Hankel matrix of order n of B(A), as B(A) is the right diagonal of triangle T. Since the only matrix manipulation we used were adding a row to another row and adding a column to another column, and the determinant of a matrix is invariant under these operations, the determinant of the Hankel matrix of order n of A is equal to the determinants of the Hankel matrix of order n of B(A).

□

3.4 The Matrix $\widetilde{\mathbf{M}} = (\frac{1}{1-x^2}, \frac{-x}{(1-x)^2})$

The general term $\widetilde{\mathbf{M}}_{n,k}$ of the matrix $\widetilde{\mathbf{M}}$ is given by

$$\widetilde{\mathbf{M}}_{n,k} = [x^n]\,\frac{\mathbf{1}}{\mathbf{1}-\boldsymbol{x}^2}\,\left(\frac{-\boldsymbol{x}}{(\mathbf{1}-\boldsymbol{x})^2}\right)^k$$

$$= [x^n]\left(\frac{1}{1-x^2}\frac{-x^k}{(1-x)^{2k}}\right)$$

$$= [x^{n-k}]\left(\frac{(-1)^k}{(1+x)}\frac{1}{(1-x)^{2k+1}}\right)$$

$$= [x^{n-k}](-1)^k \sum_{j=0}^{\infty}(-1)^j x^j \sum_{i=0}^{\infty}\binom{i+2k+1-1}{2k+1-1}x^i$$

$$= [x^{n-k}]\sum_{j=0}^{\infty}(-1)^{j+k}x^j \sum_{i=0}^{\infty}\binom{i+2k}{2k}x^i$$

$$= [x^{n-k}]\sum_{j=0}^{\infty}(-1)^{j+k}\sum_{i=0}^{\infty}\binom{2k+i}{i}x^{i+j}$$

$$= \sum_{j=0}^{n-k}\binom{n+k-j}{2k}(-1)^{j+k}.$$

The row sums of this matrix are the periodic sequence 1, -1, 0, 1, -1, 0 … with generating function

$$\frac{\frac{1}{1-x^2}}{1-\frac{-x}{(1-x)^2}} = \frac{1}{1-x^2}\,\frac{(1-x)^2}{(1-x)^2+x} = \frac{1-x}{(1+x)(1-x+x^2)} = \frac{1-x}{1+x^3}\,.$$

The image of C_n under $\widetilde{\mathbf{M}}$ is $(-1)^n$. Similarly the image of $\binom{2n}{n}$ is $(-1)^n(n+1)$. The inverse matrix $(\widetilde{\mathbf{M}})^{-1} = (2\mathrm{C}(x) - \mathrm{C}(x)^2, 1 - \mathrm{C}(x)) = (1+\mathrm{C}(x), 1 - \mathrm{C}(x))$ has general term

$$\sum_{j=0}^{n}(-1)^j\binom{1}{j-k}\binom{2n}{n-j}\frac{2j+1}{n+j+1}.$$

3.4 The Positive Matrix $\mathbf{M}^+ = (\frac{1}{1-x^2}, \frac{x}{(1-x)^2})$

The matrix $\mathbf{M}^+$ begins

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 4 & 4 & 1 & 0 & 0 & \cdots \\ 1 & 6 & 11 & 6 & 1 & 0 & \cdots \\ 0 & 9 & 24 & 22 & 8 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

It has general term

$$\mathbf{M}^+_{n,k} = [x^n]\,\frac{\mathbf{1}}{\mathbf{1-x^2}}\,\left(\frac{\boldsymbol{x}}{(\mathbf{1}-\boldsymbol{x})^2}\right)^k$$

$$= [x^n]\left(\frac{1}{1-x^2}\frac{x^k}{(1-x)^{2k}}\right)$$

$$= [x^{n-k}]\left(\frac{1}{(1+x)}\frac{1}{(1-x)^{2k+1}}\right)$$

$$= [x^{n-k}]\sum_{j=0}^{\infty}(-1)^j x^j \sum_{i=0}^{\infty}\binom{i+2k+1-1}{2k+1-1}x^i$$

$$= [x^{n-k}]\sum_{j=0}^{\infty}(-1)^j \sum_{i=0}^{\infty}\binom{i+2k}{2k}x^{i+j}$$

$$= \sum_{j=0}^{n}(-1)^j\binom{n+k-j}{2k}$$

$$= \sum_{j=0}^{n-k}\binom{n+k-j}{2k}(-1)^j.$$

Yet another expression for $\mathbf{M}^{+}_{n,k}$ can be obtained by observing that we have the factorization

$$\mathbf{M}^{+} = \left(\frac{1}{1-x^2}, \frac{x}{(1-x)^2}\right) = \left(\frac{1}{1-x^2}, x\right)\left(1, \frac{x}{(1-x)^2}\right)$$

To get the general term of $\left(\frac{1}{1-x^2}, x\right)$ we can use the rule for the "coefficient of" functions:

$$[x^{\mathrm{n}}]\left(\frac{1}{(1+\mathrm{r}x)(1+\mathrm{s}x)}\right. = \frac{\mathrm{r}^{\mathrm{n}+1} - \mathrm{s}^{\mathrm{n}+1}}{\mathrm{r}-\mathrm{s}}(-1)^{\mathrm{n}}.$$

Thus, we have

$$= [x^n]\frac{1}{1-x^2}x^k = [x^{n-k}]\frac{1}{1-x^2} = [x^{n-k}]\frac{1}{(1+x)(1-x)} = \frac{1+(-1)^{n-k}}{2}.$$

The general term of $\left(1, \frac{x}{(1-x)^2}\right)$ is given by

$$= [x^n]1.\left(\frac{x}{(1-x)^2}\right)^k$$

$$= [x^n]\frac{x^k}{(1-x)^{2k}}$$

$$= [x^{n-k}]\frac{1}{(1-x)^{2k}} = \binom{n+k-1}{n-k}.$$

This leads to the expression

$$\mathbf{M}^{+}_{n,k} = \sum_{j=0}^{n}\binom{j+k-j}{j-k}\frac{1+(-1)^{n-j}}{2}.$$

The row sums of this matrix have generating function,

$$\frac{\frac{1}{1-x^2}}{1-\frac{x}{(1-x)^2}} = \frac{1}{1-x^2}\frac{1-2x+x^2}{1-3x+x^2}$$

$$= \frac{1}{(1+x)(1-x)}\frac{(1-x)^2}{1-3x+x^2}$$

$$= \frac{1-x}{(1+x)(1-3x+x^2)}.$$

Conclusions

- In this note we have seen some properties of a family of Riordan arrays. The inverse of these arrays are closely related to well-known Catalan defined matrices. We also studied the Hankel transforms of the images of some well-known families of sequences under the inverse matrices.

- Also we have seen the invariance of the Hankel transform in which one member of a pair of sequences with the same Hankel transform is the Binomial transform of the other.

- Among questions raised by this investigation in to properties of the Hankel transform we mention one which needs further study. Are there other transforms, T, of an integer sequence S, in addition to the Binomial transforms and Invert transforms [1], with the property that the Hankel transform of S is the same as the Hankel transform of the T transform of S ?

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Theorem 3.1 Let A be an integer sequence and B its Binomial transform. Then A and B have the same Hankel transform, therefore

$$\mathrm{H}(B(A)) = \mathrm{H}(A) \qquad \text{for any sequence } A.$$

Proof: Let $A = \{a_1, a_2, a_3, \dots\}$ and $\mathrm{B} = \{b_1, b_2, b_3, \dots\}$, and define H^* to be the matrix $\mathrm{H}^* = \mathrm{RHC}$, where the elements of R, H, and C are given by

$$r_{i,j} = \begin{cases} 0, & if\ i < j, \\ \binom{i-1}{j-1}, & if\ i \geq j \end{cases}, \qquad h_{k,m} = a_{k+m-1}, \quad \text{and} \quad c_{i,j} = \begin{cases} 0, & if\ i > j, \\ \binom{j-1}{i-1}, & if\ i \leq j \end{cases},$$

and $\binom{i}{j}$ denotes the usual binomial coefficient. Then the elements of H^* are

$$h^*_{i,j} = \sum_{k=1}^{i} \sum_{m=1}^{j} \binom{i-1}{k-1} a_{k+m-1} \binom{j-1}{m-1},$$

which, by making slight change of variables, gives

$$h^*_{i,j} = \sum_{k=0}^{i-1} \sum_{m=0}^{j-1} \binom{i-1}{k} \binom{j-1}{m} a_{k+m-1}.$$

By Vandermonde‟s convolution formula, this reduces to

$$h^*_{i,j} = \sum_{s=1}^{i+j-1} \binom{i+j-2}{s-1} a_s,$$

which, by the definition of the Binomial transform, is b_{i+j-1}, thus showing that H^* is the Hankel matrix of sequence B. Thus the terms of the Hankel transforms of the sequences A and B are det (H_n) and det $(R_n H_n C_n)$, respectively, where R_n, H_n, and C_n are the upper-left submatrices of order n of H, R, and C, respectively. But R_n and C_n are both triangular with all 1‟s on the main diagonal, thus det (R_n) and det (C_n) are both 1, and therefore det (H_n) = det $(R_n H_n C_n)$, completing the proof.

□