On the theory of BRK-Algebras

By

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A thesis submitted to the Department of Mathematics

in partial fulfilment to the requirement

for the degree of

Doctor of Philosophy

Department of Mathematics

Addis Ababa University

December, 2018

Abstract

The notion of BRK-algebras were introduced by Ravi Kumar Bandaru in 2012 as a generalization of BCK/BCI/BCH/Q-algebras. He obtained some properties of BRK-algebras by introducing the classical notions such as subalgebras and ideals. In this thesis we further investigate the properties of BRK-algebras by considering homomorphisms, right maps, left maps and multipliers. We construct Quotient BRK-algebra using a special type of ideal called translation ideal. Moreover we introduce two new sub classes of BRK-algebras, namely anti-symmetric BRK-algebras and weak positive implicative BRK-algebras that have special properties. Finally we have proved that finite direct product of BRK-algebras is again a BRK-algebra and obtained certain properties.

Dedication

This work is dedicated to my Father, **Aklilu Abebe** and my Mother, **Meaza Zewdie**.

Acknowledgement

This work would not have been accomplished without the help and encouragement of a number of people. These few lines are intended to thank some of them.

First and foremost, I would like to record my gratitude to my principal supervisor Prof. K. Venkateswarlu for his inspiring guidance and constant encouragement through out my research work. Without him, this work would not have been completed. I also extend my thanks to my co-advisor Dr. Zelalem Teshome for his encouragement and support.

My sincere thanks to all staff members of the department of Mathematics, Addis Ababa University, in particular Dr. Tadesse Abdi, Dr.Berhanu Bekele, Dr. Addisalem Abathun, Dr. Yibeltal Yitayew, Dr. Mengistu Goa , Dr. Tesfa Biset, Dr. Tilahun Abebaw and Dr. Nega Arega for their constant encouragement.

My thanks goes to Dr. Terefe Getachew, Dr. Tadesse Regassa, Dr. Leta Sera and Dawit Gudeta for their help during my stay in Visakhapatnam, India.

I wish to express my heartfelt gratitude to the Dr. G.Madhu Kumar , Secretary and Correspondent for Dr. Lankapalli Bullayya College, Visakhapatnam, India for providing me the necessary facilities to carry out my research during my stay in India.

There are no words to express my thanks to to My wife Elizabeth Demissie , My sons Yeabsira and Amen, My aunt Ayelech Zewdie, brothers and sisters for their dedication and love.

Finally, I would like to thank Addis Ababa University and Department of Mathematics for giving me the opportunity to participate in the doctoral program and for the financial support for my research visit.

Girum Aklilu Abebe

Addis Ababa , Ethiopia.

Publications

- *Weak Postivie Implicative BRK- Algebras*, Annals of Pure and Applied Mathematics, Vol.13, No.2, 2017,235-240 http://dx.doi.org/10.12988/ija.2015.5635
- *Quotient BRK-Algebras* , Journal of Hyperstructures 7 (2) (2018), 94-103. ISSN: 2322-1666 print/2251-8436 online

Contents

Chapter 1

Introduction

The notion of BCK/BCI-algebras were first introduced by Y. Imai and K. Iseki [9, 10] in 1966 as a generalization of BCK/BCI prepositional Calculi. It is known that the class of BCK-algebras is a proper subclass of BCI-algebras. Since then many authors have introduced several new algebras as a generalization of BCK/BCI-algebras. and studied the properties of these generalized algebras.

In 1983, Hu and Li [7] introduced BCH-algebras as a generalization of BCI-algebras. In 2001, Neggers et al.[20] introduced a new class of algebras namely Q-algebras by generalizing BCH-algebras. As another generalization of BCH-algebras, Y. B. Jun, E. H. Roh and H. S. Kim [11] introduce the notion of BH-algebra . It is known that BH-algebra and Q-algebra are independent notions.

In 2002, Neggers and Kim [19] introduced a new class of algebras called B-algebras and obtained several results. In 2007, Walendziak [23] introduced BF-algebras as a generalization of B-algebras. C. B. Kim and H. S. Kim [14] introduced BG-algebras as another generalization of B-algebras. It is known that BG-algebra and BF algebra are different notions and every BG-algebra is a BH-algebra.

In 2012, Ravi Kumar Bandaru [2] introduced BRK-algebras, which is a generalization of BCK/BCI/BCH/Q-algebras. He established that BRK-algebras and B/BG/BH/BF-algebras are independent . Further he obtained many properties of BRK-algebras by introducing the classical notions such as subalgebra, ideal, G-part and p-radical and obtained several results(see section 2.3 and 2.4).

Thus, in this thesis we further investigate the properties of BRK-algebras by considering homomorphisms, quotient algebras and certain special maps. Also we introduce a new subclass of BRK-algebras, namely anti-symmetric BRK-algebras and weak positive imlicative BRK-algebras.

The rest of the thesis is divided into four chapters. The second chapter is meant for preliminaries. In this chapter we collect certain important definitions, examples and results concerning BCK/BCI-algebras which we would like to use in the sequel. We also recall basic definitions and results on BRK-algebras from([2])

In chapter 3, we introduce the notion of homomorphisms in BRK-algebras (see definition 3.2.7) and obtain certain results. We also introduce a special type of ideal called translation ideal (see definition 3.2.1) and furnish some examples. Further we (see example 3.2.5 and 3.2.4) show by examples that there are ideals that are not translation. Also we prove that the quotient algebra induced by a congruence relation on a BRK-algebra also form a BRK-algebra (see theorem 3.3.3). Also we obtain a congruence relation with a translation idea. In fact an arbitrary ideal may not give rise a congruence relation in general (see theorem 3.3.10. This we justify by giving an example 3.3.12). We conclude the chapter by introducing a new subclass of BRK-algebra called anti-symmetric BRK-algebra (see definition 3.4.1) and this subclass enjoys the property of the first isomorphism (homomorphism) theorem (see theorem 3.4.8).

In chapter 4, we introduce another new subclass of BRK-algebra called weak positive implicative BRK-algebra (see definition 4.1.1) and furnish some examples. Further we obtain certain basic properties as a consequence of the definition . Further we introduce right maps and left maps on BRK algebra and investigate the properties of BRK-algebra through these maps. We prove that the set of left maps on weak positive implicative BRK-algebra forms a weak positive implicative BRK-algebra (see theorem 4.2.8). We also introduce the notion of multipliers on weak positive implicative BRK-algebra and obtain basic results. Further we prove that the set of all right multipliers form a weak positive implicative BRK-algebra. Finally we conclude this chapter by obtaining a relationship between right(left) maps with left(right) multipliers.

Finally in chapter 5, we have introduce the notion of direct product of BRK-algebras and prove that finite direct product of BRK-algebras is again a BRK-algebra (see theorem 5.1.1). Many properties have been obtained as a consequence of the definition. In the last , we introduce two canonical mappings (see definition 5.2.2 and 5.2.5) and obtain their properties.

Chapter 2

Preliminaries

In this chapter, we collect certain important definitions, examples and results from the resulting literature. In the first part we recall main results on BCK/BCI-algebras which are related to our work whereas the second part is meant for BRK-algebras based on ([2])

2.1 Definitions and some results on BCI-Algebra

Definition 2.1.1. *An algebra* $(X, *, 0)$ *of type* $(2, 0)$ *is called a BCI-algebra if it satisfies the following conditions for any* $x, y, z \in X$,

BCI1: $((x * y) * (x * z)) * (z * y) = 0$

BCI2: $(x * (x * y)) * y = 0$

BCI3: $x * x = 0$

BCI4: $x * y = 0$ *and* $y * x = 0$ *implies* $x = y$.

Definition 2.1.2. *A BCI-algebra X is called BCK-algebra if it sat-*

isfies the condition,

BCK1: $0 * x = 0 \ \forall x \in X$.

Example 2.1.3. *Let S be a set and P(S) be power set of S. Then* $(P(S), -, \emptyset)$ *where* $-$ *is the usual set difference is a BCK-algebra.*

Example 2.1.4. *Let* $(G, +, 0)$ *be an abelian group. If we define* $*$ *on* G *by* $x * y = x - y$ *then* $(G, *, 0)$ *is a BCK-algebra.*

Example 2.1.5. *Let* $X = \{0, 1, 2\}$. *Define* $*$ *on* X *by the following table*

*	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Clearly $(X, *, 0)$ *is a BCI algebra, but not BCK-algebra since* $0*2 = 2 \neq 0$.

Remark 2.1.6. *A BCI algebra which is not BCK is called proper BCI algebra.*

Theorem 2.1.7. *In any BCI algebra* $(X, *, 0)$, $x * 0 = x \ \forall x \in X$.

Corollary 2.1.8. *Let* $(X, *, 0)$ *be a BCI-algebra. If* $x * 0 = 0$ *for some* $x \in X$, *then* $x = 0$.

Lemma 2.1.9. *Let* X *be a BCI algebra. Then for any* $x, y, z \in X$

$$x * y = 0 \Rightarrow (z * y) * (z * x) = 0.$$

Theorem 2.1.10. *Let* $(X, *, 0)$ *be a BCI algebra. Define a binary relation* $\leqslant$ *on* X *by* $x \leq y \Leftrightarrow x * y = 0$. *Then (X,*$\leqslant$*) is a poset.*

Definition 2.1.11. *The ordering* $\leqslant$ *in Theorem 2.1.10 is called*

BCI-ordering on X.

Remark 2.1.12. *In terms of the BCI ordering the axioms of BCI-algebra are*

BCI 1: $((x * y) * (x * z)) \leq (z * y)$

BCI 2: $(x * (x * y)) \leq y$

BCI 3: $x \leq x$

BCI 4: $x \leq y$ *and* $y \leq x$ *implies* $x = y$

Remark 2.1.13. *From corollary 2.1.8, it can be easily observed that 0 is a minimal element of the BCI-algebra* X*. Moreover if* X *is BCK, then 0 will be the least element.*

Theorem 2.1.14. *In any BCI-algebra* $(X, *, 0)$*,* $(x*y)*z = (x*z)*y$ $\forall x, y, z \in X$*.*

The identity in the above theorem is called **head fixed commutative law**. The following corollary is an immediate consequence of head fixed commutative law.

Corollary 2.1.15. *Let* $(X, \leqslant)$ *be a BCI-algebra. Then*

1. *If* $x * y \leq z$ *then* $x * z \leq y$*.*

2. *BCI1 is equivalent to* $(x * y) * (z * y) \leq x * z$*.*

Lemma 2.1.16. *Let* X *be a BCI-algebra. Then*

1. *If* $x \leq y$ *then* $z * y \leq z * x$*.*

2. *If* $x \leq y$ *then* $x * z \leq y * z$*.*

Lemma 2.1.17. *In any BCK-algebra* X*,* $(x*y)*x = 0$ *ie* $x*y \leq x$*.*

Theorem 2.1.18. *Let* X *be a BCI-algebra. Then* X *is BCK-algebra* *if and only if* $x * y \leq x \ \forall x, y \in X$.

Theorem 2.1.19. *In any BCI-algebra* X *the following properties are true for all* $x, y \in X$.

1. $x * (x * (x * y)) = x * y$ *(called the absorbent of element).*

2. $0 * (x * y) = (0 * x) * (0 * y)$ *(called left distributive of zero).*

Definition 2.1.20. *Let* $(X, *, 0)$ *be a BCI-algebra. A subset* Y *of* X *is called a subalgebra of* X *if the constant* 0 *of* X *is in* Y *and* $(Y, *, 0)$ *itself is a BCI-algebra.*

Theorem 2.1.21. *A non empty subset* Y *of a BCI-algebra* X *is a subalgebra of* X *if and only if* $x, y \in Y \Rightarrow x * y \in Y$.

Remark 2.1.22. X *and* $\{0\}$ *are obviously subalgebras of* X *which are called trivial subalgebras of* X. Y *is proper sub algebra of* X *if* $Y \subsetneqq X$.

Theorem 2.1.23. *Let* $(X, *, 0)$ *be a BCI algebra.*
Let $B = \{x \in X : 0 \leq x\}$ *and* P *be the set of all minimal elements of* X . *Then* B *and* P *are subalgebras of* X.

Definition 2.1.24. *A subset* I *of BCI-algebra* X *is called an ideal of* X *if*

1. $0 \in I$

2. $x \in I$ *and* $y * x \in I$ *imply* $y \in I$ *for all* $x, y \in X$.

Remark 2.1.25. *Clearly* $\{0\}$ *and X are ideals of X. I is called a proper ideal of* X *if* $I \neq X$.

In general an ideal of a BCI algebra may not be a subalgebra and vice verse as illustrated in the following examples

Example 2.1.26. *Let* $X = (Z, -, 0)$ *be the BCI algebra of set of integers under subtraction. Then the set* I *of all non negative integer forms an ideal which is not a subalgebra. Indeed clearly* $0 \in I$ *and if* $x, y - x \in I$ *then* $0 \leq x$ *and* $0 \leq y - x$ *which imply* $0 \leq y$ *and hence* $y \in I$*. Thus* I *is an ideal of* X*. On the other hand since* I *is not closed under subtraction,* I *is not a subalgebra of* X*.*

Example 2.1.27. *Let* X *be the BCI algebra in example 2.1.5 i.e* $X = \{0, 1, 2\}$ *and* $*$ *defined by the following table*

*	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Then $P = \{0, 2\}$ *(which is the set of all minimal elements of* X*) is a subalgebra of* X *which is not an ideal of* X *as* $2 \in P$ *and* $1*2 = 2 \in P$ *but* $1 \notin P$*.*

Definition 2.1.28. *An ideal* I *of a BCI algebra* X *is called closed ideal if it is closed under* $*$ *of* X *(i.e* I *a subalgebra of* X*).*

Remark 2.1.29. *The set* B *in Theorem 2.1.23 is closed ideal of* X *but not* P *(see example 2.1.27)*

Theorem 2.1.30. *An ideal* I *of a BCI algebra* X *is closed If and only if* $0 * x \in I$ *for any* $x \in I$*.*

2.2 Quotient BCI-Algebras

We begin this section by recalling the definition of a congruence relation.

Definition 2.2.1. *Let X be a BCI-algebra and let θ be an equivalence relation on X. Then θ is called a congruence relation on X if it has a compatibility property:*

$(x, y) \in \theta$ and $(u, v) \in \theta$ imply $(x * u, y * v) \in \theta$ for all $x, y, u, v \in X$.

Given a congruence relation θ on a BCI-algebra X, we denote θ_x for the equivalence class determined by x i.e. $\theta_x = \{y \in X : (y, x) \in \theta\}$ and X/θ for the quotient set $\{\theta_x : x \in X\}$.

Remark 2.2.2. *Let θ be a congruence relation on a BCI-algebra X. Then*

1. *$(x, y) \in \theta \Leftrightarrow \theta_x = \theta_y$*
2. *the binary operation $*$ on X/θ given by $\theta_x * \theta_y = \theta_{x*y}$ is well defined.*
3. *the equivalence class θ_0 is a closed ideal of X*
4. *we call the algebra $(X/\theta, *, \theta_0)$ the quotient algebra of X induced by θ*

Theorem 2.2.3. *Let θ be a congruence relation on a BCI-algebra X. Then the quotient algebra $(X/\theta, *, \theta_0)$ may not be a BCI-algebra.*

Next we define a congruence relation on X using ideal in such away that the induced quotient algebra becomes a BCI-algebra.

Lemma 2.2.4. *Let I be an ideal of a BCI algebra X. Define a relation* $\sim$ *on* X *by* $x \sim y \Leftrightarrow x * y, y * x \in I$. *Then* $\sim$ *is a congruence relation on X.*

We use the notation I_x for the equivalence class determined by x and X/I for the set of all equivalence classes of X.

Clearly $I_x = \{y \in X : x \sim y\}$ and $X/I = \{I_x : x \in X\}$.

Theorem 2.2.5. *Let X be a BCI-algebra and I be an ideal of X. Define* $*$ *on* X/I *by* $I_x * I_y = I_{x*y}$, *for all* $x, y \in X$. *Then* $(X/I, *, I_0)$ *is a BCI-algebra.*

Definition 2.2.6. $(X/I, *, I_0)$ *is called the quotient BCI algebra of X.*

Remark 2.2.7. *Let X be a BCI algebra. Then*

1. *in general for an ideal I of X,* I_0 *may not be equal to I (see example 2.2.8)*
2. *an ideal I of X is closed if and only if* $I_0 = I$.

Example 2.2.8. *Let* Q^* *be the set of all non zero rational numbers. Then we can very easily verify that set* Z^* *of all non negative integers forms an ideal of the BCI algebra* $(Q^*, \div, 1)$. *Now in the quotient algebra* Q^*/Z^* *the zero equivalence class* Z_1^* *of* $(Q^*, \div, 1)$ *is given by* $Z_1^* = \{-1, 1\}$ *which is a proper subset of* Z^*.

Definition 2.2.9. *Let X and Y be BCI-algebras and let* $f : X \longrightarrow Y$ *be a map. Then*

1. *f is called BCI-homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.*

2. *If f is an injective BCI-homomorphism, then it is called a monomorphism.*

3. *If f is a serjective BCI-homomorphism , then it is called an epimorphism.*

4. *If f is a bijective BCI-homomorphism , then it is called an isomorphism.*

Remark 2.2.10. *We say X is isomorphic to Y, symbolically $X \cong Y$, if there exists an isomorphism from X to Y and we say Y is a homomorphic image of X if there is an epimorphism from X to Y.*

Theorem 2.2.11. *Let X and Y be BCI algebras and $f : X \longrightarrow Y$ be a BCI homomorphism. Then*

1. *$f(0) = 0$.*

2. *f is isotone (preserves order) ie $x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$.*

Remark 2.2.12. *Let $f : X \longrightarrow Y$ be a BCI-homomorphism. We define kernel and image of f in the usual way by*

1. *$Ker(f) = \{x \in X : f(x) = 0\}$ and*

2. *$Im(f) = \{y \in Y : \exists x \in X \text{ such that} f(x) = y\}$.*

Theorem 2.2.13. *If $f : X \longrightarrow Y$ is a BCI homomorphism, then*

1. *Ker(f) is a closed ideal of X.*

2. *Im(f) is a subalgebra of Y.*

In general $Im(f)$ may not be an ideal of Y.

Example 2.2.14. *Let* $X = \{0, 1, 2\}$. *Define* $*$ *on* X *by*

*	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

Then $(X, *, 0)$ *is a BCI algebra. Let* $f : X \longrightarrow X$ *be a mapping defined by* $0 \longmapsto 0, 1 \longmapsto 0, 2 \longmapsto 2$. *Very easily we can show that* f *is a homomorpism on* X *and* $Im(f) = \{0, 2\}$. *Now since* $2, 1 * 2 = 0 \in Im(f)$ *and* $1 \notin Im(f)$, $Im(f)$ *is not an ideal of X.*

Nevertheless, if f is an epimorphism, we have the following result.

Theorem 2.2.15. *Let* $f : X \longrightarrow Y$ *be an epimorphism. Then if* I *is an ideal of X, then* $f(I)$ *is also an ideal of Y. Moreover if* I *is closed, then so is* $f(I)$.

Theorem 2.2.16. *Let* $f : X \longrightarrow Y$ *be a BCI homomorphism. If* $I = ker(f)$, *then* $X/I \cong Im(f)$.

2.3 Definitions and examples on BRK-Algebras

Before we give the definition of BRK-algebra, first let us recall the definitions and relationship of certain classes of algebras which are very useful for the study of BRK-algebra.

Definition 2.3.1. *Let* $X = (X, *, 0)$ *be an algebra of type* $(2, 0)$. *Then* X *is called*

1. *a BCH-algebra if it satisfies BCI3, BCI4 and*

$$BCH1: (x * y) * z = (x * z) * y$$

2. *a Q-algebra if it satisfies BCI3, BCH1 and Q1:* $x * 0 = x$.

3. *a B-algebra if it satisfies BCI3, Q1 and*

$$B1: (x * y) * z = x * (z * (0 * y))$$

4. *a BF-algebra if it satisfies BCI3, Q1 and BF1:* $0*(x*y) = y*x$

5. *a BM-algebra if it satisfies Q1 and BM1:* $(x*y)*(x*z) = z*y$

6. *a BH-algebra if it satisfies BCI1, BCI4 and Q1*

7. *a BG-algebra if it satisfies BCI1, Q1 and*

$$BG1: (x * y) * (0 * y) = x.$$

Definition 2.3.2.

1. *A Q-algebra X is called QS-algebra if it satisfies*

$$(x * y) * (x * z) = z * y \text{ for any } x, y, z \in X.$$

2. *A B-algebra X is called 0-commutative if it satisfies*

$$x * (0 * y) = y * (0 * x) \text{ for any } x, y \in X.$$

Remark 2.3.3. *It is known that*

1. *every BCI-algebra is a BCH-algebra but not conversely.*

2. *every BCH-algebra is a Q-algebra but not conversely.*

3. *every B-algebra is a BF-algebra but not conversely.*

4. *Q-algebra and B-algebra are different notions.*

Note: The relationship between these algebras is shown in the figure below.

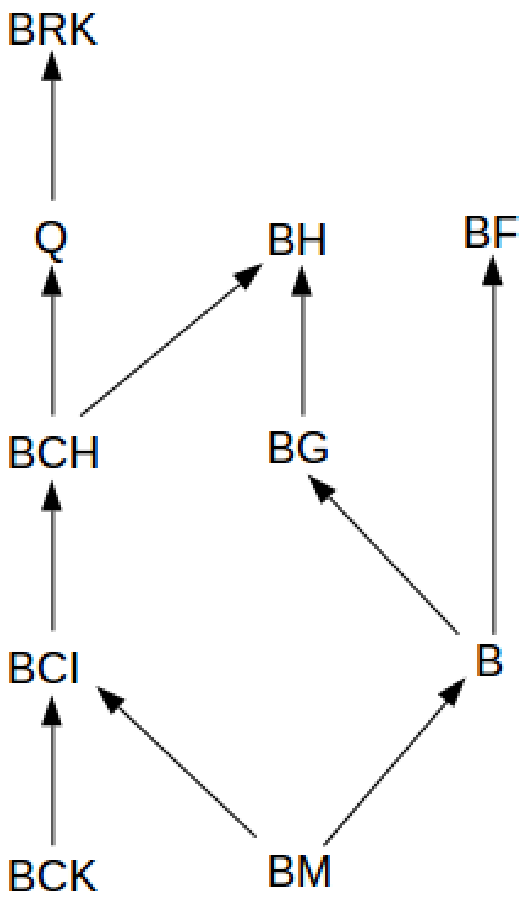


Figure 1.

Definition 2.3.4. *An algebra* $(X, *, 0)$ *of type* $(2, 0)$ *is called a BRK algebra if it satisfies Q1 and BRK 1:* $(x * y) * x = 0 * y$.

Example 2.3.5. *Let* $X = R \setminus \{-n\}$, $0 \neq n \in Z^+$ *where R is the set of real numbers and* Z^+ *is the set of positive integers. If we define a binary operation* $*$ *on X by*

$$x * y = \frac{n(x - y)}{n + y},$$

then $(X, *, 0)$ *is a BRK algebra.*

Example 2.3.6. *Let* $X = \{0, 1, 2\}$*. Define* $*$ *on* X *by the following table*

*	0	1	2
0	0	2	2
1	1	0	0
2	2	0	0

Then $(X, *, 0)$ *is a BRK algebra.*

Example 2.3.7. *Let* $X = \{0, 1, 2, 3\}$*. Define* $*$ *on* X *by the following table*

*	0	1	2	3
0	0	1	0	1
1	1	0	1	0
2	2	1	0	1
3	3	2	3	0

Then $(X, *, 0)$ *is a BRK-algebra which is not a BCK/BCI/BCH/Q-algebra (as* $(3 * 1) * 2 = 0 \neq 2 = (3 * 2) * 1$ *ie BCH1 dose not hold).*

We know that every QS-algebra is a BM-algebra and we can observe that every BM-algebra is a BRK-algebra but the converse is not true.

Example 2.3.8. *Let* $X = \{0, 1, 2, 3\}$*. Define* $*$ *on* X *by the following table*

*	0	1	2	3
0	0	2	2	0
1	1	0	0	2
2	2	0	0	2
3	3	1	1	0

Then $(X, *, 0)$ *is a BRK-algebra which is not a QS/BM-algebra.*

It is easy to see that B/BG/BF/BH- algebra and BRK-algebra are different notions . For example Example 2.3.6 is a BRK-algebra which is not a BH-algebra and Example 2.3.7 is a BRK-algebra which is not B/BG/BF-algebra.

Example 2.3.9. *Let* $X = \{0, 1, 2, 3, 4, 5\}$. *Define* $*$ *on* X *by the following table*

*	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X, *, 0)$ *is a B/BF/BG/BH-algebra which is not a BRK-algebra.*

Theorem 2.3.10. *If* $(X, *, 0)$ *is a BRK-algebra, then for any* $x, y \in X$ *the following properties are true.*

1. *BCI1:* $x * x = 0$

2. $x * y = 0 \Rightarrow 0 * x = 0 * y$

3. $0 * (x * y) = (0 * x) * (0 * y)$

Theorem 2.3.11. *Every BRK-algebra X satisfying* $x*(x*y) = x*y$ *for all* $x, y \in X$ *is a trivial algebra i. e.* $X = \{0\}$.

Theorem 2.3.12. *Every BRK-algebra X satisfying* $(x*y)*(x*z) = z * y$ *for all* $x, y, z \in X$ *is a BCI-algebra.*

2.4 Subalgebras and Ideals of BRK-algebra

Definition 2.4.1. *Let* X *be a BRK-algebra and let* I *be nonempty subset of* X*. Then*

1. I *is called a subalgebra of* X *if* $x * y \in I$ *for all* $x, y \in I$

2. I *is called an ideal of* X *if for any* $x, y \in X$*:*

   (i) $0 \in I$,

   (ii) $x * y \in I$ *and* $y \in I$ *imply* $x \in I$.

3. I *is called closed ideal of* X *if it is an ideal and* $0 * x \in I$ *for all* $x \in I$.

Example 2.4.2. *Let* X *be the BRK-algebra in Example 2.3.6. Put* $I = \{0, 2\}$*. Then* I *is a subalgebra, an ideal, and a closed ideal of* X.

Theorem 2.4.3. *Every closed ideal of a BRK-algebra is a subalgebra.*

Note: The converse of the preceding theorem is not true.

Example 2.4.4. *Let X be the BRK-algebra in Example 2.3.7. Put $I = \{0, 1, 2\}$. but then I is a subalgebra which is not a closed ideal of X.*

Definition 2.4.5. *Let X be a BRK algebra. For any subset S of X, we define $G(S) = \{x \in S : 0 * x = x\}$. In particular, if $S = X$,then we say that $G(X)$ is called the G-part of X.*

Definition 2.4.6. *Let X be a BRK-algebra. The set $B(X) = \{x \in X : 0 * x = 0\}$ is called a p-radical of X.*

Remark 2.4.7. *If X is a BRK-algebra, then clearly $G(X) \cap B(X) = \{0\}$.*

Theorem 2.4.8. *For any BRK-algebra X, $B(X)$ is a subalgebra and an ideal of X.*

Theorem 2.4.9. *The G-part of every BRK-algebra is a subalgebra.*

Lemma 2.4.10. *In a BRK-algebra X if $a * b = a * c$ for some $a, b, c \in X$ then $0 * b = 0 * c$*

Theorem 2.4.11. *Let X be a BRK-algebra. Then left cancellation law holds in $G(X)$ ie. if for some $a, b, c \in G(X)$ $a * b = a * c$, then $b = c$*

Lemma 2.4.12. *In any BRK-algebra X if $x \in G(X)$ then $0 * x \in G(X)$.*

The converse of the above lemma need not be true, for example in Example 2.3.7 we can see that $0 * 1 = 2 \in G(X) = \{0, 2\}$ but $1 \notin G(X)$.

Theorem 2.4.13. *Let X be a BRK algebra. Then $y \in B(X)$ if and only if $(x * y) * x = 0$ for any $x \in X$.*

Definition 2.4.14. *Let $X = (X, *, 0)$ be a BRK-algebra. Then X is called*

1. *a p-semisimple if $B(X) = \{0\}$*
2. *a positive implicative if $((x * y) * y) * (0 * y) = x * y$ for any $x, y \in X$.*
3. *a medial BRK-algebra if $(x * y) * (z * u) = (x * z) * (y * u)$ for any $x, y, z, u \in X$.*
4. *an associative BRK-algebra if the binary operation $*$ is associative i.e $x * (y * z) = (x * y) * z$ for any $x, y, z \in X$.*

Theorem 2.4.15. *Let $(X, *, 0)$ be a BRK-algebra. If $G(X) = X$, then X is p-semisimple.*

Theorem 2.4.16. *Let $(X, *, 0)$ be medial BRK-algebra. Then for any $x, y, z \in X$ the following hold:*

1. *$x * (y * z) = (x * y) * (0 * z)$,*
2. *$(x * y) * z = (x * z) * y$*

Corollary 2.4.17. *Every medial BRK-algebra is a Q-algebra*

Theorem 2.4.18. *A BRK-algebra X is medial if and only if it satisfies:*

(i) $x * y = 0 * (y * x)$ for all $x, y \in X$

(ii) $(x * y) * z = (x * z) * y$for all $x, y, z \in X$.

Remark 2.4.19. *If* $(X, *, 0)$ *is associative BRK-algebra then* $B(X) = \{0\}$ *and* $G(X) = X$

Theorem 2.4.20. *Every associative BRK-algebra* $(X, *, 0)$ *is a group under the binary operation* $*$

Chapter 3

Quotient BRK-Algebras

3.1 Introduction

In this chapter we introduce the concept of quotient BRK-algebra and establish homomorphism theorem for a special subclass of BRK-algebra which we call anti-symmetric BRK-algebra.

3.2 Translation Ideals, Homomorphisms

We begin this section by introducing a special type of ideals in BRK-algebra.

Definition 3.2.1. *An ideal I of a BRK algebra X is called translation ideal if it satisfies the condition:*

$$x * y, y * x \in I \Rightarrow (x * z) * (y * z), (z * x) * (z * y) \in I \text{ for all } x, y, z \in X.$$

Example 3.2.2. *Let* $X = (Z, -, 0)$ *be the BRK-algebra of set of integers under the usual subtraction of integers. Then the set* I *of all non negative integers form a translation ideal of* X*. Indeed clearly* I *is an ideal of* X *(see Example 2.1.26).*
Now if $x - y, y - x \in I$*, then* $x - y$ *and* $y - x = -(x - y)$ *are non negative integers and hence* $x = y$*. But then for any* $z \in X$*,*

$$(x - z) - (y - z) = (x - z) - (x - z) = 0 \in I \text{ and}$$
$$(z - x) - (z - y) = (z - x) - (z - x) = 0 \in I.$$

Thus I *is translation.*

Remark 3.2.3. *Every ideal need not be translation.*

The following two examples clarify the above remark.

Example 3.2.4. *Let* $X = \{0, 1, 2, 3\}$ *be a set with the following Cayley table:*

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	0	1	0

Then $(X, *, 0)$ *is a BRK-algebra. Clearly* $I = \{0\}$ *is an ideal but not translation since* $3 * 1, 1 * 3 \in I$ *but* $(3 * 2) * (1 * 2) = 1 \notin I$*.*

Example 3.2.5. *Let* $X = \{0, 1, 2, 3\}$ *be a set with the following Cayley table:*

*	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	3	0	2
3	3	3	0	0

Then $(X, *, 0)$ *is a BRK-algebra and* $I = \{0, 1\}$ *is an ideal of* X *which is not translation since* $0*1, 1*0 \in I$ *but* $(2*0)*(2*1) = 2 \notin I$.

Remark 3.2.6. *In BCI-algebra, every ideal is a translation ideal. To see this let* I *be a an ideal of a BCI-algebra* X *and suppose* $x*y, y*x \in I$. *But then for any* $z \in X$, *we have* $(x*z)*(y*z) \leq x*y$ *and* $(z*x)*(z*y) \leq y*x$ *which implies* $(x*z)*(y*z), (z*x)*(z*y) \in I$ *as* $x * y, y * x \in I$. *Hence* I *is translation.*

Definition 3.2.7. *Let* X *and* Y *be BRK-algebras.*
A mapping $f : X \longrightarrow Y$ *is called a BRK-homomorphism from* X *into* Y *if* $f(x * y) = f(x) * f(y)$ *for all* $x, y \in X$.

Remark 3.2.8. *Let* f *be a BRK-homomorphism from* X *into* Y. *Then* f *is called:*

1. *a monomorphism if it is injective.*
2. *an epimorphism if it is surjective.*
3. *an isomorphism if it is bijective.*
4. X *and* Y *are said to be isomorphic, written* $X \cong Y$, *if there exists an isomorphism form* X *onto* Y

Definition 3.2.9. *Let* $f : X \longrightarrow Y$ *be a BRK-homomorphism.*

Then

1. *the kernel of f, denoted by $Ker(f)$ is given by:*

$$Ker(f) = \{x \in X : f(x) = 0\}.$$

2. *the image of f,denoted by $Im(f)$ is given by:*

$$Im(f) = \{f(x) : x \in X\}.$$

The following Lemma is a simple consequence of definition of a homomorphism.

Lemma 3.2.10. *Let $f : X \longrightarrow Y$ be a BRK-homomorphism. Then*

1. $f(0) = 0$

2. $x * y = 0$ *implies* $f(x) * f(y) = 0$.

Proof.

(1) $f(0) = f(0 * 0) = f(0) * f(0) = 0$.

(2) If $x*y = 0$, then $f(x*y) = f(0)$ which implies $f(x)*f(y) = 0$. □

Theorem 3.2.11. *Let $f : X \longrightarrow Y$ be a BRK-homomorphism.*

i. *If S is a subalgebra of X, then $f(S)$ a subalgebra of Y.*

ii. *If K is a subalgebra of Y, then $f^{-1}(K)$ is a subalgebra of X containing $ker f$.*

iii. *If I is an ideal(respectively a closed ideal) of X and f is injective, then $f(I)$ is an ideal (respectively a closed ideal) of $f(X)$.*

iv. *If J is an ideal(respectively a closed ideal) of Y, then $f^{-1}(J)$ is an ideal(respectively a closed ideal) of X.*

v. *If I is a translation ideal of X and f is injective, then $f(I)$ is a translation ideal of $f(X)$.*

vi. *If J is a translation ideal of Y, then $f^{-1}(J)$ is a translation ideal of X.*

Proof. i. Suppose S is a subalgebra of X and $y_1, y_2 \in f(S)$. But then there exists $x_1, x_2 \in S$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Now $y_1 * y_2 = f(x_1) * f(x_2) = f(x_1 * x_2)$ which implies $y_1 * y_2 \in f(S)$ as $x_1 * x_2 \in S$. Thus $f(S)$ is a subalgebra of Y.

ii. Let K be a subalgebra of Y and $x_1 * x_2 \in f^{-1}(K)$. Then $f(x_1), f(x_2) \in K$. Now $f(x_1 * x_2) = f(x_1) * f(x_2) \in K$. Thus $x_1 * x_2 \in f^{-1}(K)$ and so $f^{-1}(K)$ is a subalgebra of X.

iii. Suppose I is an ideal of X and f is injective. Then clearly $0 \in f(I)$ (as $0 = f(0)$ and $0 \in I$). Let $x, y \in f(X)$. Then there exist $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. Now if $x * y, y \in f(I)$, then there exists $c, d \in I$ such that $f(c) = x * y$ and $f(d) = y$. But then $f(a * d) = f(a) * f(d) = x * y \in f(I)$ and so $a * d \in I$ as f is injective. Now since I is an ideal and $d \in I$, $a \in I$ and hence $x = f(a) \in f(I)$. Therefore $f(I)$ is an ideal.

iv. Let J be an ideal of Y. Then clearly $0 \in f^{-1}(J)$. If $x * y, y \in f^{-1}(J)$, then $f(x * y) = f(x) * f(y), f(y) \in J$ which implies $f(x) \in J$. Thus $x \in f^{-1}(J)$ and so $f^{-1}(J)$ is an ideal of X. If $x \in Ker(f)$, then $f(x) = 0 \in J$ so $x \in f^{-1}(J)$ Therefore

$Ker(f) \subseteq f^{-1}(J)$.

v. Let I be a translation ideal of X and f is injective. By (iii), $f(I)$ is an ideal of $f(X)$. Let $x, y, z \in f(X)$ then there exist a, b, c such that $f(a) = x, f(b) = y$ and $f(c) = z$. If $x*y, y*x \in f(I)$, then $f(a*b) = f(a)*f(b) = x*y, f(b*a) = f(b)*f(a) = y*x \in f(I)$ and so $a*b, b*a \in I$ as f is injective. Since I is translation ideal, $(a*c)*(b*c), (c*a)*(c*b) \in I$.Now we have

$$\begin{aligned}(x*z)*(y*z) &= (f(a)*f(c))*(f(b)*f(c)) \\ &= f(a*c)*f(b*c) \\ &= f((a*c)*(b*c)) \in f(I).\end{aligned}$$

Similarly $(z*x)*(z*y) \in f(I)$ and hence $f(I)$ is a translation ideal.

vi. Let J be a translation ideal of Y. By (iv) $f^{-1}(J)$ is an ideal of X. If $x*y, y*x \in f^{-1}(J)$, then $f(x*y) = f(x)*f(y), f(y*x) = f(y)*f(x) \in J$.But then since J is a translation ideal, for any $z \in X$ we have

$$\begin{aligned}&(f(x)*f(z))*(f(y)*f(z)) \in J \\ &\Rightarrow f((x*z)*(y*z)) \in J \\ &\Rightarrow (x*z)*(y*z) \in f^{-1}(J).\end{aligned}$$

Similarly $(z*x)*(z*y) \in f^{-1}(J)$.

Therefore $f^{-1}(J)$ is a translation ideal.

□

Corollary 3.2.12. *If $f : X \longrightarrow Y$ is homomorphism of BRK alge-*

bras, then $kerf$ *is a closed ideal of* X *and* Imf *is a subalgebra of* Y*.*

Remark 3.2.13. *For any BRK-homomorphism* f*,*

1. $kerf = \{0\}$ *may not imply* f *is injective.*

   For instance, consider the BRK-algebra $X = (X, *, 0)$ *where* $X = \{0, 1, 2\}$ *and* $*$ *is given by the Cayley table:*

*	0	1	2
0	0	2	2
1	1	0	0
2	2	0	0

   Let $f : X \longrightarrow X$ *be defined by* $f(0) = 0$ *and* $f(1) = f(2) = 2$*.It can be easily see that* f *is a homomorphism with* $kerf = \{0\}$ *but* f *is not injective.*

2. $kerf$ *may not be a translation ideal of* X*.*

   To see this, consider the BRK-algebra X *in Example 3.2.4. Clearly the identity map* $id : X \longrightarrow X$ *is a homomorphism, but* $kerf = \{0\}$ *is not a translation ideal.*

3.3 Congruence in BRK-Algebra

We define a congruence relation on BRK-algebra as in the usual way.

Definition 3.3.1. *An equivalence relation* θ *on a BRK-algebra* X *is called a congruence relation if it has a compatibility property:*

$(x, y) \in \theta$ and $(u, v) \in \theta$ imply $(x * u, y * v) \in \theta$ for all $x, y, u, v \in X$.

Given a congruence relation θ on a BRK-algebra X, we use the notation θ_x for the equivalence class determined by x i.e. $\theta_x = \{y \in X : (y, x) \in \theta\}$ and X/θ for the quotient set $\{\theta_x : x \in X\}$.

Remark 3.3.2. *Let θ be a congruence relation on a BRK-algebra X. Then*

1. *$(x, y) \in \theta \Leftrightarrow \theta_x = \theta_y$*
2. *the binary operation $*$ on X/θ given by $\theta_x * \theta_y = \theta_{x*y}$ is well defined.*
3. *the equivalence class θ_0 is closed ideal of X*
4. *we call the algebra $(X/\theta, *, \theta_0)$ the quotient algebra of X induced by θ*

Theorem 3.3.3. *Let X be a BRK-algebra. Define $*$ on the quotient algebra X/θ by $\theta_x * \theta_y = \theta_{x*y}$.Then $(X/\theta, *, \theta_0)$ is a BRK-algebra which is called a quotient BRK-algebra induced by the congruence θ.*

Proof. Since θ is a congruence relation $*$ is well defined. For any $\theta_x, \theta_y \in X/\theta$ we have

(1) $\theta_x * \theta_0 = \theta_{x*0} = \theta_0$, and

(2) $(\theta_x * \theta_y) * \theta_x = \theta_{x*y} * \theta_x = \theta_{(x*y)*x} = \theta_{0*y} = \theta_0 * \theta_y$.

Thus $(X/\theta, *, \theta_0)$ is a BRK-algebra. □

Remark 3.3.4. *Unlike BCI-algebra, the quotient algebra determined by any congruence relation on BRK-algebra is also a BRK-algebra.*

Theorem 3.3.5. *Let $X/\theta = (X/\theta, *, \theta_0)$ be the quotient BRK-algebra induced by a congruence θ. Then θ_0 is a closed ideal of X.*

Proof. Since $(0,0) \in \theta$, $0 \in \theta_0$. Suppose $x, y * x \in \theta_0$, then $(x,0) \in \theta$ and $(y * x, 0) \in \theta$. Now from $(y,y) \in \theta$ and $(x,0) \in \theta$ we have $(y * x, y) \in \theta$. Also from $(y * x, y) \in \theta$ and $(y * x, 0) \in \theta$ we get $(0, y) \in \theta$ and hence by symmetry we have $(y, 0) \in \theta$ i.e. $y \in \theta_0$. Therefor θ_0 is an ideal of X. Next, if $x, y \in \theta_0$ then $(x,0) \in \theta$ and $(y,0) \in \theta$ and hence $(x * y, 0) \in \theta$ i.e. $x * y \in \theta_0$. Thus it is closed. □

Next, we will construct a congruence relation on BRK-algebra X by using a homomorphism and a translation ideal.

Congruence relation determined by a Homomorphism

Theorem 3.3.6. *Let $f : X \longrightarrow Y$ be a homomorphism of BRK-algebras. Define a relation $\backsim$ on X by $x \backsim y$ if and only if $f(x) = f(y)$ for all $x, y \in X$. Then $\backsim$ is a congruence relation on X which is called the congurence relation determined by the homomorphism f.*

Proof. Clearly $\backsim$ is an equivalence relation on X. Next suppose

$x \backsim y$ and $u \backsim v$. Then $f(x) = f(y)$ and $f(u) = f(v)$. Now since $f(x * u) = f(x) * f(u) = f(y) * f(v) = f(y * v)$, $x * u \backsim y * v$. Thus $\backsim$ is a congruence relation on X. □

Note: We denote the equivalence class of x determined by $\backsim$ by $[x]_f$ and the set of all equivalence classes by X/f i.e. $[x]_f = \{y \in X : x \backsim y\}$ and $X/f = \{[x]_f : x \in X\}$.

Corollary 3.3.7. *Let $f : X \longrightarrow Y$ be a homomorphism of BRK-algebras. Define $*$ on X/f by $[x]_f * [y]_f = [x*y]_f$. Then $(X/f, *, [0]_f)$ is a BRK algebra. It is called the quotient BRK-algebra determined by the homomorphism f.*

Proof. Since $\backsim$ is a congruence relation on X, $(X/f, *, [0]_f)$ is a BRK-algebra by Theorem 3.3.3. □

Remark 3.3.8. *Clearly $[0]_f = Kerf$.*

Theorem 3.3.9. *Let $f : X \longrightarrow Y$ be homomorphism of BRK-algebras. Then the image of f is isomorphic to the quotient BRK-algebra determined by f, i.e. $X/f \cong Imf$.*

Proof. Define a mapping $\theta : X/f \longrightarrow Imf$ by $\theta([x]_f) = f(x)$. Then

1) θ is well defined. Indeed suppose $[x]_f = [y]_f$. Then
   $[x]_f = [y]_f \Rightarrow x \in [y]_f \Rightarrow f(x) = f(y)$. Thus $\theta([x]_f) = \theta([y]_f)$.

2) θ is homomorphism. Indeed for any $[x]_f, [y]_f \in X/f$ we have
   $\theta([x]_f * [y]_f) = \theta([x * y]_f) = f(x * y) = f(x) * f(y) = \theta([x]_f) * \theta([y]_f)$

3) Clearly θ is bijective.

Hence $X/f \cong Imf$. □

Congruence relation determined by a translation ideal

Theorem 3.3.10. *Let I be a translation ideal of a BRK-algebra X. Define a relation $\backsim$ on X by $x \backsim y \Leftrightarrow x * y, y * x \in I$. Then $\backsim$ is a congruence relation on X which is called the congruence relation of X induced by a translation ideal I.*

Proof. For any $x \in X$, since $x * x = 0 \in I$, we have $x \backsim x$ hence $\backsim$ is reflexive. From the definition of $\backsim$, it is clear that $\backsim$ is symmetric. If $x \backsim y$ and $y \backsim z$ then $x*y, y*x, y*z, z*y \in I$. Now $x*y, y*x \in I$ implies $(x*z)*(y*z) \in I$. Since I is an ideal and $y*z \in I$, $x*z \in I$. Similarly $z*x \in I$. Thus $\backsim$ is transitive. Hence $\backsim$ is an equivalence relation. Next suppose $x \backsim y$ and $u \backsim v$. But then since I is a translation ideal we have $x * y, y * x \in I \Rightarrow (x * u) * (y * v) \in I \Rightarrow ((x*u)*(y*v))*((y*u)*(y*v)) \in I$ and hence $(x*u)*(y*v) \in I$ (as I is an ideal and $(y * u) * (y * v) \in I$). Similarly we can show that $(y * v) * (x * u) \in I$. Thus $\backsim$ is a congruence relation. □

Remark 3.3.11. *In theorem 3.3.10 the relation $\backsim$ may not be a congruence relation if we consider an arbitrary ideal instead of translation ideal. This we justify here under.*

Example 3.3.12. *Consider the BRK-algebra X and its ideal I in Example 3.2.5. Define $\sim$ on X by $x \sim y \Leftrightarrow x * y, y * x \in I$. Then*

$2 \sim 2$ *and* $0 \sim 1$ *but* $2 * 0 = 2 \nsim 3 = 2 * 1$ *as* $2 * 3 = 2 \notin I$ *hence not a congruence relation.*

Note: For any translation ideal I of a BRK-algebra X we use the notation I_x for the equivalence class determined by x and X/I for the set of all equivalence classes of X for the congruence relation of X induced by I. Clearly $I_x = \{y \in X : x \sim y\}$ and $X/I = \{I_x : x \in X\}$.

Corollary 3.3.13. *Let X be a BRK-algebra and I be a translation ideal of X. Define* $*$ *on* X/I *by* $I_x * I_y = I_{x*y}$*, for all* $x, y \in X$*. Then* $(X/I, *, I_0)$ *is a BRK-algebra.*

Definition 3.3.14. $(X/I, *, I_0)$ *is called the quotient BRK-algebra of X determined by a translation ideal I*

Remark 3.3.15. *Let X be a BRK-algebra. Then*

1. *in general for a translation ideal I of X,* I_0 *may not be equal to I and*
2. *a translation ideal I of X is closed if and only if* $I_0 = I$*.*

3.4 Anti-symmetric BRK-Algebras

In this section we introduce a new subclass of BRK-algebra called anti-symmetric BRK-algebra and we will establish the homomorpism theorem for this subclass.

Definition 3.4.1. *A BRK-algebra X is called an anti-symmetric*

BRK-algebra if it satisfies BCI4: $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$ *for any* $x, y \in X$*.*

The following examples illustrate such algebra exist.

Example 3.4.2. *Let* $X = \{0, 1, 2\}$ *be a set with cayley table:*

*	0	1	2
0	0	2	2
1	1	0	0
2	2	0	0

Then $(X, *, 0)$ *is a BRK algebra which is not anti-symmetric BRK-algebra as* $2 * 1 = 1 * 2 = 0$ *but* $1 \neq 2$*.*

Example 3.4.3. *Let* $X = \{0, 1, 2, 3\}$ *be a set with cayley table:*

*	0	1	2	3
0	0	1	0	1
1	1	0	1	0
2	2	1	0	1
3	3	2	3	0

Then $(X, *, 0)$ *is an anti-symmetric BRK-algebra which is not BCH as* $(3 * 1) * 1 = 0 \neq 2 = (3 * 2) * 1$*.*

Example 3.4.4. *Let* $X = \{0, 1, 2, 3\}$ *be a set with cayley table:*

$*$	0	1	2	3
0	0	3	0	2
1	1	0	0	0
2	2	2	0	3
3	3	3	1	0

Then $(X, *, 0)$ *is a BH-algebra which is not an anti-symmetric BRK-algebra as* $(2 * 3) * 2 = 1 \neq 2 = 0 * 3$.

Remark 3.4.5. *Every anti-symmetric BRK-algebra is a BH algebra but not conversely.*

The following figure shows the relation ship of the algebras.

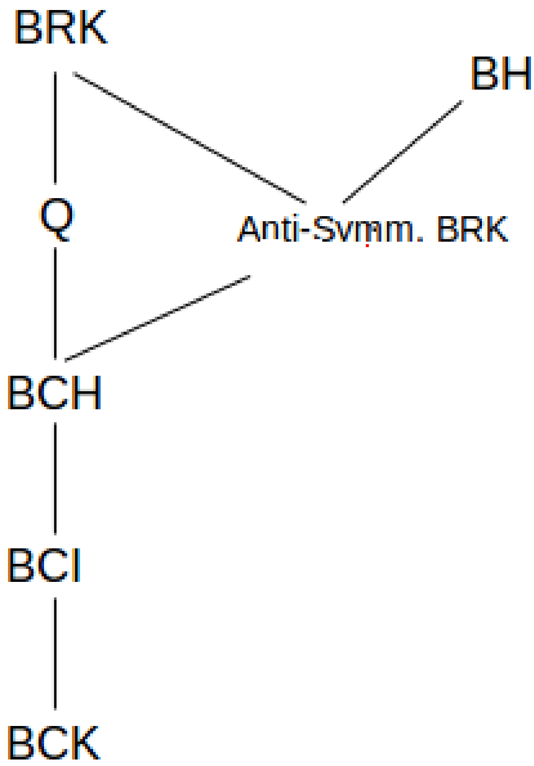


Figure 2.

Theorem 3.4.6. *Let X and Y be anti-symmetric BRK-algebra and*

$f : X \longrightarrow Y$ be BRK-homomorhism. Then f is injective if and only if $kerf = \{0\}$.

Proof. Obviously if f is injective then clearly $Kerf = \{0\}$. On the other hand, suppose that $x, y \in X$ and $f(x) = f(y)$. Then $f(x * y) = f(x) * f(y) = f(x) * f(x) = 0$. Hence $x * y \in Kerf$ and so $x * y = 0$. Similarly, we have $y * x = 0$. Thus $x = y$ and hence f is injective. □

Theorem 3.4.7. *Let X and Y be anti-symmetric BRK-algebra and $f : X \longrightarrow Y$ be a BRK-homomorhism. Then $Kerf$ is a translation ideal of X.*

Proof. Since $Kerf$ is an ideal of X, it is enough to show that it is translation. If $x * y, y * x \in kerf$, then $f(x) * f(y) = f(y) * f(x) = 0 \Rightarrow f(x) = f(y)$. For any $z \in X$,

$$\begin{aligned} f((x * z) * (y * z)) &= (f(x) * f(z)) * (f(y) * f(z)) \\ &= (f(x) * f(z)) * ((f(x) * f(z)) \\ &= 0 \end{aligned}$$

Hence $(x * z) * (y * z) \in kerf$. Similarly, $(z * x) * (z * y) \in kerf$. Thus $Kerf$ is a translation ideal. □

Theorem 3.4.8. *Let X and Y be anti-symmetric BRK-algebras and $f : X \longrightarrow Y$ be a BRK homomorphism. If $I = Kerf$, then $X/I \cong Imf$.*

Proof. As I is a translation ideal of X, X/I is a BRK algebra. Define a mapping $\alpha : X/I \longrightarrow Imf$ by $\alpha(I_x) = f(x)$. Then

(1) α is well defined. If $I_x = I_y$ for some $I_x, I_y \in X/I$, then we have

$$\begin{aligned} I_x = I_y &\Rightarrow x * y, y * x \in I \\ &\Rightarrow f(x * y) = f(y * x) = 0 \\ &\Rightarrow f(x) * f(y) = f(y) * f(x) = 0 \\ &\Rightarrow f(x) = f(y) \\ &\Rightarrow \alpha(I_x) = \alpha(I_y). \end{aligned}$$

Thus α is well defined.

(2) α is a homomorpism. For any $I_x, I_y \in X/I$ we have

$\alpha(I_x * I_y) = \alpha(I_{x*y}) = f(x * y) = f(x) * f(y) = \alpha(I_x) * \alpha(I_y)$.

Therefore α is a homomorphism.

(3) α is injecive. Suppose $\alpha(I_x) = \alpha(I_y)$ for some $I_x, I_y \in X/I$. Then

$$\begin{aligned} \alpha(I_x) = \alpha(I_y) &\Rightarrow f(x) = f(y) \\ &\Rightarrow f(x) * f(y) = f(y) * f(x) = 0 \\ &\Rightarrow f(x * y) = f(y * x) = 0 \\ &\Rightarrow x * y, y * x \in I \\ &\Rightarrow I_x = I_y. \end{aligned}$$

Hence α is injective.

(4) α is surjective. Let y be any element in Imf. But then there exists $x \in X$ such that $f(x) = y$. Now $I_x \in X/I$ and $\alpha(I_x) = f(x) = y$ and hence α is Surjective.

Hence α is an isomorphism and $X/I \cong Imf$. □

Theorem 3.4.9. *Let X, Y and Z be anti-symmetric BRK-algebras* *and let $h : X \longrightarrow Y$ be an epimorphism and $g : X \longrightarrow Z$ be a homomorphism. If $Kerh \subseteq Kerg$, then there is a unique homomorphism $f : Y \longrightarrow Z$ such that $f \circ h = g$.*

Proof. Since h is onto, for any $y \in Y$ there exists $x \in X$ such that $y = h(x)$. Now define a mapping $f : Y \longrightarrow Z$ by $f(y) = g(x)$. Then:

(1) f is well defined and $f \circ h = g$. Indeed suppose $y_1 = y_2$ but then there exist $x_1, x_2 \in X$ such that $y_1 = h(x_1) = h(x_2) = y_2$. Now $h(x_1 * x_2) = h(x_1) * h(x_2) = 0$ which implies $x_1 * x_2 \in Kerh$. Since $Kerh \subseteq Kerg$ $x_1 * x_2 \in Kerg$. So $0 = g(x_1 * x_2) = g(x_1) * g(x_2)$. Similarly we can show that $g(x_2 * g(x_1) = 0$. Thus $g(x_1) = g(x_2)$ which means $f(y_1) = f(y_2)$ i.e. f is well defined. Clearly $g(x) = f(h(x))$ for any $x \in X$.

(2) f is homomorphism. Indeed for any $y_1, y_2 \in Y$ there exists $x_1, x_2 \in X$ such that $y_1 = h(x_1), y_2 = h(x_2)$. Now we we have $f(y_1 * y_2) = f(h(x_1 * h(x_2) = f(h(x_1 * x_2)) = g(x_1 * x_2) = g(x_1) * g(x_2) = f(y_1) * f(y_2)$. Hence f is a homomorphism.

(3) f is unique. Indeed suppose $\overline{f} : Y \longrightarrow Z$ such that $\overline{f} \circ h = g$. For any $y \in Y$ there exists $x \in X$ such that $h(x) = y$. Now $\overline{f}(y) = \overline{f}(h(x)) = g(x) = f(y)$ and hence $\overline{f} = f$.

□

Theorem 3.4.10. *Let X, Y and Z be anti-symmetric BRK-algebras and let $g : X \longrightarrow Z$ be a homomorphism and $h : Y \longrightarrow Z$ be a monomorphism. If $Img \subseteq Imh$, then there is a unique homomorphism $f : X \longrightarrow Y$ such that $h \circ f = g$.*

Proof. For any $x \in X$, $g(x) \in Img \subseteq Imh$. Since h is a monomorhism there is unique $y \in Y$ such that $h(y) = g(x)$. Now define $f : X \longrightarrow Y$ by $f(x) = y$. Then clearly f is well defined and $h \circ f = g$. To show f is a homomorphism, for any $x_1, x_2 \in X$ we have $h(f(x_1 * x_2)) = g(x_1 * x_2) = g(x_1) * g(x_2) = h(f(x_1)) * h(f(x_2)) = h(f(x_1) * f(x_2))$. Since h is monomorphism $f(x_1 * x_2) = f(x_1) * f(x_2)$, hence f is homomrphism. For the uniqueness suppose $\overline{f} : X \longrightarrow Y$ such that $h \circ \overline{f} = g$. Now for any $x \in X$, we have $h(\overline{f}(x)) = g(x) = h(f(x))$. But then since h is monomorphism $\overline{f}(x) = f(x)$. So it is unique. □

If I is a translation ideal of an anti-symmetric BRK-algebra X, then a map $p : X \longrightarrow X/I$ defined by $p(x) = I_x$ is a homomorphism and called the canonical mapping. Note that $Kerp = I$

Lemma 3.4.11. *Let X and Y be anti-symmetric BRK-algebras and let $f : X \longrightarrow Y$ be a homomorphism. If I is a translation ideal of X such that $I \subseteq Kerf$, then a map $\overline{f} : X/I \longrightarrow Y$ defined by $\overline{f}(I_x) = f(x)$ for all $x \in X$ is a homomorphism .*

Proof. We show that $\overline{f}$ is well-defined. Now if $I_x = I_y$, then $x * y \in I \subseteq Kerf$. So we have $f(x) * f(y) = f(x * y) = 0$. Similarly we can show that $f(y) * f(x) = 0$. Hence $f(x) = f(y)$. Therefor $\overline{f}$ is well-defined. Clearly $\overline{f}$ is a homomorphism. □

Theorem 3.4.12. *Let X and Y be anti-symmetric BRK-algebras and let $f : X \longrightarrow Y$ be a homomorphism I be a translation ideal of X.Then the following are equivalent:*

(i) there is a unique homomorphism $\overline{f} : X/I \longrightarrow Y$ such that $\overline{f} \circ p = f$ where p is the canonical mapping.

(ii) $I \subseteq Kerf$.

Furthermore, $\overline{f}$ is monomorphism if and only if $I = Kerf$

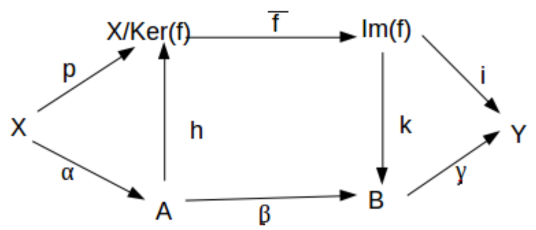
Proof. $(i) \Rightarrow (ii)$. If $a \in I$, then $\overline{f}(p(a)) = f(0) = 0$ for all $a \in I$, since $\overline{f} \circ p = f$ and $Kerp = I$. Hence $a \in Kerf$.
$(ii) \Rightarrow (i)$. By lemma 3.4.11, we have a homomorphism $\overline{f} : X/I \longrightarrow Y$ defined by $\overline{f}(I_x) = f(x)$ for all $x \in X$. Since $(\overline{f} \circ p)(x) = \overline{f}(I_x) = f(x)$ for all $x \in X$, we have $\overline{f} \circ p = f$. The uniqueness of $\overline{f}$ follows from the fact that p is surjective.
Moreover, $\overline{f}$ is a monomorphism if and only if $f(x) = 0$ implies $I_x = I_0 = I$, i.e., if and only if $Kerf \subseteq I$. Thus $Kerf = I$ □

Theorem 3.4.13. *Let X and Y be anti-symmetric BRK algebras. If a homomorphism $f : X \longrightarrow Y$ can be expressed as a composite homomrphisms of anti-symmetric BRK-algebras*

$$X \underset{\rightarrow}{\alpha} A \underset{\rightarrow}{\beta} B \underset{\rightarrow}{\gamma} Y,$$

where α is epimorphism, β is an isomorphism, and γ is monomorphism, then $A \cong X/Kerf$ and $B \cong Imf$.

Proof. Consider the following diagram



where $i \circ \overline{f} \circ p$ is canonical decomposition of f. Now Since $f = \gamma \circ \beta \circ \alpha$ and γ, β monomorphism, we have $f(x) = 0$ if and only if $\alpha(x) = 0$. Hence $ker(\alpha) = ker(f) = ker(p)$. By theorem 3.4.9, there exists unique homomorhpism $h : A \longrightarrow X/ker(f)$ such that $h \circ \alpha = p$. Now since $ker(\alpha) = ker(p)$, h is injective and h is surjective as p is surjective. Thus h is an isomorphism.

On the other hand since $im(\gamma) = Im(f)$, by Theorem 3.4.10 we have a unique homomomorphism $k : im(f) \longrightarrow B$ such that $\gamma \circ k = i$. Clearly k is surjective and since i is injective so is k. Thus k is an isomorphism.

□

Chapter 4

Special Maps in BRK-algebras

In this chapter we discuss about some special maps on BRK-algebras namely right map, left map, right multiplier and left multiplier. We begin this chapter by introducing a special subclass of BRK-algebra called a weak positive implicative BRK-algebra.

4.1 Weak Positive Implicative BRK-algebra

Definition 4.1.1. *A BRK-algebra $X = (X, *, 0)$ is said to be weak positive implicative if it satisfies the condition:*

$$(x * y) * z = (x * z) * (y * z) \text{ for all } x, y, z \in X$$

Example 4.1.2. *Let $X = \{0, 1, 2, 3\}$ be a set with the following cayley table:*

*	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	3	0

Then $(X, *, 0)$ *is a weak positive implicative BRK-algebra.*

The following example shows the existence of weak positive implicative BRK-algebra which is not BCH-algebra.

Example 4.1.3. *Let* $\mathbb{Z}$ *be the set of integers. Define* $*$ *on* $\mathbb{Z}$ *by*

$$x * y = \begin{cases} x, & \text{if } y = 0 \\ 0, & \text{if } y \neq 0 \end{cases}$$

Then $(\mathbb{Z}, *, 0)$ *is a positive implicative BRK-algebra which is not BCH-algebra (as* $2 * 3 = 3 * 2 = 0$ *but* $2 \neq 3$ *hence BCI4 not hold).*

Lemma 4.1.4. *In any weak positive implicative BRK-algebra* X*, the following hold for all* $x, y \in X$*.*

1. $0 * x = 0$
2. $(x * y) * x = 0$
3. $x * y = (x * y) * y$
4. $(x * (x * y)) * y = 0$

Proof. Let $x, y \in X$. Then

1. $0 * x = (x * x) * x$

$$= (x * x) * (x * x)$$

$= 0 * 0 = 0.$

Therefore $0 * x = 0$ for all $x \in X$

2. $(x * y) * x = 0 * y = 0.$

   Thus $(x * y) * x = 0$ for all $x, y \in X$.

3. $x * y = (x * y) * 0$

   $= (x * y) * (y * y)$

   $= (x * y) * y.$

   Thus $x * y = (x * y) * y$ for all $x, y \in X$.

4. $(x * (x * y)) * y = (x * y) * ((x * y) * y)$

   $= (x * y) * (x * y) = 0.$

   Hence $(x * (x * y)) * y = 0$ for all $x, y \in X$.

□

Remark 4.1.5. *If X is a weak positive implicative BRK-algebra, then $G(X) = \{0\}$ and $B(X) = X$.*

Theorem 4.1.6. *Every weak positive implicative BRK-algebra is positive implicative.*

Proof. Let $X = (X, *, 0)$ be a weak positive implicative BRK-algebra. For any $x, y \in X$, we have

$$((x * y) * y) * (0 * y) = ((x * y) * y) * 0$$
$$= (x * y) * y = x * y.$$

Thus X is positive implicative BRK-algebra. □

Remark 4.1.7. *The converse of the above theorem is not true.*

The following example justifies the above remark.

Example 4.1.8. *Let* $X = \{0, 1, 2\}$ *be a set with Cayley table:*

$*$	0	1	2
0	0	2	2
1	1	0	0
2	2	0	0

Then $(X, *, 0)$ *is a positive implicative BRK-algebra which is not a weak positive implicative (as* $(1 * 1) * 1 = 2 \neq 0 = (1 * 1) * (1 * 1)$*).*

Theorem 4.1.9. *Let* $X = (X, *, 0)$ *be a weak positive implicative BRK-algebra. If* X *is associative, then* X *is the trivial algebra i. e.* $X = \{0\}$.

Proof. Let X be associative weak positive implicative BRK-algebra. Then for any $x \in X$ we have:

$$x = x * 0 = x * (x * x) = (x * x) * x = 0 * x = 0.$$

Thus $X = \{0\}$ □

4.2 Left and Right maps

In this section we investigate the properties of R-maps and L-maps and show that the set of all left maps over weak positive implicative BRK-algebra becomes a weak positive implicative BRK-algebra.

Definition 4.2.1. *Let* $X = (X, *, 0)$ *be a BRK-algebra and* $a \in X$ *be a fixed element. Then:*

1. *the map* $R_a : X \longrightarrow X$ *given by* $R_a(x) = x * a$ *is called right map of* X *and*

2. *the map* $L_a : X \longrightarrow X$ *given by* $L_a(x) = a * x$ *is called left map of* X.

Definition 4.2.2. *A right map* R_a *is called idempotent if* $R_a \circ R_a = R_a$ *where* $\circ$ *is the usual composition of maps.*

Lemma 4.2.3. *Let* X *be a BRK-algebra and* $a \in X$. *Then* R_a *is idempotent if and only if* $(x * a) * a = x * a$ *for all* $x \in X$.

Proof. R_a is idempotent

$$\begin{aligned} &\Leftrightarrow R_a \circ R_a = R_a \\ &\Leftrightarrow R_a(R_a(x)) = R_a(x), \quad \forall x \in X \\ &\Leftrightarrow R_a(x * a) = x * a, \quad \forall x \in X \\ &\Leftrightarrow (x * a) * a = x * a, \quad \forall x \in X. \end{aligned}$$

Thus R_a is idempotent if and only if $(x * a) * a = x * a$. □

Theorem 4.2.4. *If a BRK-algebra* $X = (X, *, 0)$ *is weak positive implicative, then every right map on* X *is idempotent.*

Proof. Let $a \in X$. Then for all $x \in X$ we have:

$$\begin{aligned} R_a(x) = x * a &= (x * a) * a \\ &= R_a(R_a(x)) \\ &= (R_a \circ R_a)(x). \end{aligned}$$

Hence $R_a \circ R_a = R_a$. Therefore R_a is idempotent. □

Theorem 4.2.5. *A BRK-algebra* $X = (X, *, 0)$ *is a weak positive implicative if and only if every right map is a homomorphism.*

Proof. Suppose X is a weak positive implicative BRK-algebra. Then for each $a \in X$,

$$\begin{aligned} R_a(x * y) &= (x * y) * a \\ &= (x * a) * (y * a) \\ &= R_a(x) * R_a(y). \end{aligned}$$

Thus R_a is a homomorphism.

Conversely, suppose that every right map is a homomorphism. Now for any $x, y, z \in X$ we have

$$\begin{aligned} (x * y) * z &= R_z(x * y) \\ &= R_z(x) * R_z(y) \\ &= (x * z) * (y * z). \end{aligned}$$

Hence X is weak positive implicative □

Theorem 4.2.6. *In any BRK-algebra $X = (X, *, 0)$, if L_a is a homomorphism, then $a = 0$.*

Proof. Suppose L_a is a homomorphism for some $a \in X$. Then

$$\begin{aligned} a = a * 0 &= L_a(0) \\ &= L_a(0 * 0) \\ &= L_a(0) * L_a(0) \\ &= (a * 0) * (a * 0) = a * a = 0. \end{aligned}$$

Thus $a = 0$ □

Now let us denote the set of all left maps on a BRK-algebra X by $\mathbb{L}(X)$ and on $\mathbb{L}(X)$ we define a binary operation $\circledast$ by $(L_a \circledast L_b)(x) := L_a(x) * L_b(x)$ for any $L_a, L_b \in \mathbb{L}(X)$. We have the following Lemma.

Lemma 4.2.7. *Let $X = (X, *, 0)$ be weak positive implicatice BRK-algebra. For any $L_a, L_b, L_c \in \mathbb{L}(X)$, we have*

i. $L_a \circledast L_b = L_{a*b}$ *i.e.* $L_a \circledast L_b \in \mathbb{L}(X)$.

ii. $(L_a \circledast L_b) \circledast L_c = (L_a \circledast L_c) \circledast (L_a \circledast L_c)$.

Proof. Let $L_a, L_b, L_c \in \mathbb{L}(X)$. Then:

i. For any $x \in X$ we have

$$\begin{aligned}(L_a \circledast L_b)(x) &= L_a(x) \circledast L_b(x)\\ &= (a * x) * (b * x)\\ &= (a * b) * x\\ &= L_{a*b}(x).\end{aligned}$$

Therefore $L_a \circledast L_b = L_{a*b}$.

ii. By (i.) we have

$$\begin{aligned}(L_a \circledast L_b) \circledast L_c &= L_{a*b} \circledast L_c\\ &= L_{(a*b)*c}\\ &= L_{(a*c)*(b*c)}\\ &= L_{a*c} \circledast L_{b*c}\\ &= (L_a \circledast L_c) \circledast (L_a \circledast L_c).\end{aligned}$$

Thus $(L_a \circledast L_b) \circledast L_c = (L_a \circledast L_c) \circledast (L_a \circledast L_c)$.

□

Theorem 4.2.8. *If $X = (X, *, 0)$ is a weak positive implicative BRK-algebra, then $\mathbb{L}(X) = (\mathbb{L}(X), \circledast, L_0)$ is a weak positive implicative BRK-algebra.*

Proof. It is enough to show that $\mathbb{L}(X) = (\mathbb{L}(X), \circledast, L_0)$ is a BRK-algebra. Now for any $L_a, L_b \in \mathbb{L}(X)$ we have

1. $L_a \circledast L_0 = L_{a*0} = L_a$, and
2. $(L_a \circledast L_b) \circledast L_a = L_{(a*b)*a} = L_{0*b} = L_0 \circledast L_b$.

Therefore $\mathbb{L}(X)$ is a BRK-algebra and hence by the above lemma it is weak positive implicative. □

Theorem 4.2.9. *Let $X = (X, *, 0)$ be a weak positive implicative BRK-algebra. Then the map $f : X \longrightarrow \mathbb{L}(X)$ given by $f(x) = L_x$ is an epimorphism and $X/f \cong \mathbb{L}(X)$ where X/f is the quotient BRK-algebra determined by the homomorphism f.*

Proof. Let X be a weak positive implicative BRK-algebra and let $f : X \longrightarrow \mathbb{L}(X)$ be a map given by $f(x) = L_x$. But then for any $x, y \in X$, $f(x * y) = L_{x*y} = L_x \circledast L_y$ which implies f is a homomorphism. Clearly f is onto and hence f is an epimorphism. Therefore, by Theorem 3.3.9 we have $X/f \cong \mathbb{L}(X)$.

□

4.3 Multipliers

Here we introduce the notion of multipliers in a weak positive implicative BRK-algebra and discuss about their properties.Through out this section by X we shall mean a weak positive implicative BRK-algebra unless stated otherwise.

Definition 4.3.1. . *Let $f : X \longrightarrow X$ be a map. Then f is called:*

1. *a left multiplier on X if $f(x * y) = x * f(y)$ for all $x, y \in X$.*
2. *a right multiplier on X if $f(x * y) = f(x) * y$ for all $x, y \in X$.*

Example 4.3.2.

1. *The identity map $I : X \longrightarrow X$ given by $I(x) = x$ is both right multiplier and left multiplier.*
2. *The zero map $\overline{0} : X \longrightarrow X$ given by $\overline{0}(x) = 0$ is right multiplier but not left multiplier. Indeed for any $x, y \in X$, $0 = \overline{0}(x * y) = 0 * y = \overline{0}(x) * y$, $x * \overline{0}(y) = x * 0 = x \neq 0 = \overline{0}(x * y)$*

Lemma 4.3.3. . *Let $f : X \longrightarrow X$ be a right multiplier, then*

1. *$f(0) = 0$*
2. *$f(x) \leq x$ for all $x \in X$*
3. *$x \leq y \Rightarrow f(x) \leq y$ for all $x, y \in X$*

Proof. (1) $f(0) = f(0 * f(0)) = f(0) * f(0) = 0$

(2) Let $x \in X$. Then $0 = f(0) = f(x * x) = f(x) * x$. Hence $f(x) \leq x$.

(3) Let $x, y \in X$ and $x \leq y$. Then $0 = f(0) = f(x * y) = f(x) * y$. Thus $f(x) \leq y$.

□

Lemma 4.3.4. *Let $f : X \longrightarrow X$ be a left multiplier, then*

1. *$f(0) = 0$*
2. *$x \leq f(x)$ for all $x \in X$*

3. $x \leq y \Rightarrow x \leq f(y)$ *for all* $x, y \in X$

Proof. (1) $f(0) = f(0 * 0) = 0 * f(0) = 0$

(2) Let $x \in X$. Then $0 = f(0) = f(x * x) = x * f(x)$. so $x \leq f(x)$.

(3) Let $x, y \in X$ and $x \leq y$. Then $0 = f(0) = f(x * y) = x * f(y)$. Thus $x \leq f(y)$.

□

Lemma 4.3.5. *Let f and g be right(left) multipliers on X. Then their composition $f \circ g$ is a right(left) multiplier on X.*

Proof. Let f and g be right multipliers on X. Then for any $x, y \in X$ $(f \circ g)(x * y) = f(g(x * y)) = f(g(x) * y) = f(g(x)) * y = (f \circ g)(x) * y$. Hence $f \circ g$ is a right multiplier. Similarly we can show for left multipliers □

Remark 4.3.6. *Let f be a right(left) multiplier on X. For any non negative integer n we define f^n inductively by $f^0 = I$ and $f^n = f^{n-1} \circ f$ for $n \geq 1$.*

Theorem 4.3.7. *If f is a right(left) multiplier, then $f^n(x) \leq f^{n-1}(x)$ ($f^{n-1}(x) \leq f^n(x)$ for all $n \geq 1$and for all $x \in X$.*

Proof. Let f be a right multiplier on X. Then
$0 = f(0) = f(f^{n-1}(x) * f^{n-1}(x)) = f(f^{n-1}(x)) * f^{n-1}(x) = f^n(x) * f^{n-1}(x)$. This implies $f^n(x) \leq f^{n-1}(x)$.
Similarly we can show for left multiplier. □

Theorem 4.3.8. *Let $RM(X)$ be the set of all right multipliers on X. Define a binary operation $\circledast$ on $RM(X)$ by $(f \circledast g)(x) = f(x) * g(x)$ then $(RM(X), \circledast, \overline{0})$ is a weak positive multiplicative BRK-algebra.*

Proof. Since the zero map $\overline{0}$ is a right multiplier, $RM(X) \neq \emptyset$.To Show $\circledast$ is well defined suppose $f_1, f_2, g_1, g_2 \in RM(X)$ with $f_1 = f_2$ and $g_1 = g_2$. But then for any $x \in X$ we have $(f_1 \circledast g_1)(x) = f_1(x)*g_1(x) = f_2(x)*g_2(x) = (f_2 \circledast g_2)(x)$ and hence $f_1 \circledast g_1 = f_2 \circledast g_2$. Thus it is well defined. Now for any for any $f, g, h \in RM(X)$

(1) $$\begin{aligned}(f \circledast g)(x * y) &= f(x * y) * g(x * y)\\ &= (f(x) * y) * (g(x) * y)\\ &= (f(x) * g(x)) * y\\ &= (f \circledast g)(x) * y.\end{aligned}$$

Thus $f \circledast g \in RM(X)$.

(2) $$\begin{aligned}(f \circledast \overline{0})(x) &= f(x) * \overline{0}(x)\\ &= f(x) * 0 = f(x).\end{aligned}$$

So $f \circledast \overline{0} = f$

(3) $$\begin{aligned}((f \circledast g) \circledast f)(x) &= (f(x) * g(x)) * f(x)\\ &= 0 * g(x) = (\overline{0} * g)(x).\end{aligned}$$

Thus $(f \circledast g) \circledast f = \overline{0} * g$

(4) $$\begin{aligned}((f \circledast g) \circledast h)(x) &= (f(x) * g(x)) * h(x)\\ &= (f(x) * h(x)) * (g(x) * h(x))\\ &= ((f \circledast h)(x) * (g \circledast h)(x)\end{aligned}$$

$$= ((f \circledast h) \circledast (g \circledast h))(x).$$

Hence $(f \circledast g) \circledast h = (f \circledast h) \circledast (g \circledast h)$.

Therefore $(RM(X), \circledast, \overline{0})$ is a weak positive implicative BRK-algebra.

□

Theorem 4.3.9. *Let f be a right multiplier on X. Then*

1. *$Ker(f) = \{x \in X : f(x) = 0\}$ is a subalgebra of X.*
2. *$Im(f) = \{y \in X : \exists x \in X$ such that $f(x) = y\}$ is a sub algebra of X.*
3. *$Fix(f) = \{x \in X : f(x) = x\}$ is a subalgebra of X*

Proof. Let f be a right multiplier on X. Then:

1. Since $0 \in Ker(f)$, $Ker(f) \neq \emptyset$. Suppose $x, y \in Ker(f)$. Then $f(x) = f(y) = 0$. Now $f(x * y) = f(x) * y = 0 * y = 0$. Thus $x * y \in Ker(f)$ so it is a subalgebra.
2. Clearly $Im(f) \neq \emptyset$ (as $0 \in Im(f)$). Let $x, y \in Im(f)$. Then there exist $x', y' \in X$ such that $f(x') = x$ and $f(y') = y$. Now since $x' * y \in X$ and $f(x' * y) = f(x') * y = x * y$, $x * y \in Im(f)$. Hence $Im(f)$ is a subalgebra of X
3. Clearly $Fix(f) \neq \emptyset$ (as $0 \in Fix(f)$). Let $x, y \in Fix(f)$. But then $f(x) = x$ and $f(y) = y$. Now since $f(x * y) = f(x) * y = x * y$, $x * y \in Fix(f)$. Thus $Fix(f)$is a subalgebra of X

□

For left multipliers we have the following theorem.

Theorem 4.3.10. *Let f be a left multiplier on X. Then*

1. $Im(f) = \{y \in X : \exists x \in X$ *such that* $f(x) = y\}$ *is a sub algebra of* X.
2. $Fix(f) = \{x \in X : f(x) = x\}$ *is a subalgebra of* X.

Proof. Let f be a left multiplier on X. Then:

1. Clearly $Im(f) \neq \emptyset$ (as $0 \in Im(f)$). Let $x, y \in Im(f)$. Then there exist $x', y' \in X$ such that $f(x') = x$ and $f(y') = y$. Now since $x * y' \in X$ and $f(x * y') = x * f(y') = x * y$, $x * y \in Im(f)$. Hence $Im(f)$ is a subalgebra of X
2. Clearly $Fix(f) \neq \emptyset$ (as $0 \in Fix(f)$). Let $x, y \in Fix(f)$. But then $f(x) = x$ and $f(y) = y$. Now since $f(x * y) = x * f(y) = x * y$, $x * y \in Fix(f)$. Thus $Fix(f)$ is a subalgebra of X.

□

The following two theorems discuss about the relationship between multipliers with right maps and left maps.

Theorem 4.3.11. *Let X be a BRK-algebra. X associative if and only if every left map is a right multiplier on X.*

Proof. Suppose X is associative. Then for any $a, x, y \in X$, $L_a(x * y) = a * (x * y) = (a * x) * y = L_a(x) * y$. Thus L_a is right multiplier. Conversely suppose every left map is a right multiplier. For any $x, y, z \in X$, $x * (y * z) = L_x(y * z) = L_x(y) * z = (x * y) * z$. Hence associative. □

Theorem 4.3.12. *Let X be a BRK-algebra. X associative if and only if every right map is a left multiplier on X.*

Proof. Suppose X is associative. Then for any $a, x, y \in X$, $R_a(x * y) = (x * y) * a = x * (y * a) = x * R_a(y)$. Thus R_a is left multiplier. Conversely suppose every right map is a left multiplier. For any $x, y, z \in X$, $(x * y) * z = R_z(x * y) = x * R_z(y) = x * (y * z)$. Hence associative. □

Corollary 4.3.13. *If every right(left)map on X is a left(right)multiplier on X, then X is trivial algebra i. e. $X = \{0\}$.*

Proof. It follows from Theorem 4.3.12(4.3.11) and Theorem 4.1.9

□

Chapter 5

Direct product of BRK-algebras

In this chapter we will define the notion of direct product of BRK-algebras and study some basic properties of the direct product. We begin with the definition and elementary properties of direct product of BRK-algebras.

5.1 Definition and basic properties

Let $\{X_i = (X_i, *, 0_i) : i = 1, 2, \cdots n\}$ be a finite family of BRK-algebras. Define the direct product of BRK-algebras $X_1, X_2, \cdots, X_n$ to be the structure $\prod_{i=1}^{n} X_i = (\prod_{i=1}^{n} X_i, \circledast, (0_1, \cdots, 0_n)$ where $\prod_{i=1}^{n} X_i = X_1 \times \cdots \times X_n = \{(x_1, \cdots x_n) : x_i \in X_i \forall i = 1, 2, \cdots n\}$ and $\circledast$ is given by $(x_1, \cdots, x_n) \circledast (y_1, \cdots, y_n) = (x_1 * y_1, \cdots, x_n * y_n)$. The following theorem shows that the direct product of finite BRK-algebras is also a BRK-algebra

Theorem 5.1.1. *If* $\{X_i = (X_i, *, 0_i) : i = 1, 2, \cdots n\}$ *be a finite* *family of BRK-algebras, then* $\prod_{i=1}^{n} X_i = (\prod_{i=1}^{n} X_i, \circledast, (0_1, \cdots, 0_n)$ *is a BRK-algebra.*

Proof. Let $(x_1, \cdots, x_n), (y_1, \cdots, y_n) \in \prod_{i=1}^{n} X_i$. Then:

1. $(x_1, \cdots, x_n) \circledast (0_1, \cdots, 0_n) = (x_1 * 0_1, \cdots, x_n * 0_n) = (x_1, \cdots, x_n)$

2. $((x_1, \cdots, x_n) \circledast (y_1, \cdots, y_n)) \circledast (x_1, \cdots, x_n) = ((x_1 * y_1) * x_1, \cdots, (x_n * y_n) * x_n)$

$$= (0_1 * y_1, \cdots, 0_n * y_n)$$
$$= (0_1, \cdots, 0_n) \circledast (y_1, \cdots, y_n)$$

Therefore $\prod_{i=1}^{n} X_i$ is a BRK-algebra.

□

Theorem 5.1.2. *Let* $\{X_i = (X_i, *, 0_i) : i = 1, 2, \cdots n\}$ *be a finite family of BRK-algebras and* $\prod_{i=1}^{n} X_i$ *be their direct product. Then*

1. $\prod_{i=1}^{n} X_i$ *is associative if and only if each* X_i *is associative.*
2. $\prod_{i=1}^{n} X_i$ *is positive implicative if and only if each* X_i *is positive implicative.*
3. $\prod_{i=1}^{n} X_i$ *is anti-symmetric if and only if each* X_i *is anti-symmetric.*
4. $\prod_{i=1}^{n} X_i$ *is weak positive implicative if and only if each* X_i *is weak positive implicative.*

Proof. It follows from respective definitions. □

Theorem 5.1.3. *Let $\{X_i = (X_i, *, 0_i) : i = 1, 2, \cdots n\}$ be a finite family of BRK-algebras and $\prod_{i=1}^{n} X_i$ be their direct product. Then*

1. *If S_i is a subalgebra of X_i for each $i = 1, 2, \cdots, n$, then $\prod_{i=1}^{n} S_i$ is a subalgebra of $\prod_{i=1}^{n} X_i$.*

2. *If I_i is an ideal of X_i for each $i = 1, 2, \cdots, n$, then $\prod_{i=1}^{n} I_i$ is an ideal of $\prod_{i=1}^{n} X_i$. Moreover If each I_i is translation, then $\prod_{i=1}^{n} I_i$ is also translation.*

Proof. 1. Suppose S_i is a subalgebra of X_i for any all $i = 1, 2, \cdots, n$. Since each S_i is non empty, $\prod_{i=1}^{n} S_i \neq \emptyset$. Let $(x_1, \cdots, x_n), (y_1, \cdots, y_n) \in \prod_{i=1}^{n} S_i$.Then: $(x_1, \cdots, x_n) \circledast (y_1, \cdots, y_n) = (x_1 * y_1, \cdots, x_n * y_n) \in \prod_{i=1}^{n} S_i$ as $x_i * y_i \in S_i$.. Thus $\prod_{i=1}^{n} S_i$ is a subalgebra of $\prod_{i=1}^{n} X_i$

2. Suppose I_i is an ideal of X_i. Clearly $(0_1, \cdots, 0_n) \in \prod_{i=1}^{n} I_i$. Next if $(x_1, \cdots, x_n), (y_1, \cdots, y_n) \circledast (x_1, \cdots, x_n) \in \prod_{i=1}^{n} I_i$, then $x_i, y_i * x_i \in I_i \forall i = 1, 2, \cdots, n$. But the since I_i is an ideal of X_i, $y_i \in I_i$ which implies $(y_1, \cdots, y_n) \in \prod_{i=1}^{n} I_i$. Thus $\prod_{i=1}^{n} I_i$ is an ideal of $\prod_{i=1}^{n} X_i$. For the second part suppose each I_i is translation and $(x_1, \cdots, x_n) \circledast (y_1, \cdots, y_n), (y_1, \cdots, y_n) \circledast (x_1, \cdots, x_n) \in \prod_{i=1}^{n} I_i$. Then for any $(z_1, \cdots, z_n) \in \prod_{i=1}^{n} X_i$,

$$
\begin{aligned}
&((x_1, \cdots, x_n) \circledast (z_1, \cdots, z_n)) \circledast ((y_1, \cdots, y_n) \circledast (z_1, \cdots, z_n)) \\
&\quad = (x_1 * z_1, \cdots, x_n * z_n) \circledast (y_1 * z_1, \cdots, y_n * z_n)
\end{aligned}
$$

$$= ((x_1 * z_1) * (y_1 * z_1), \cdots, (x_n * z_n) * (y_n * z_n)).$$

But then since I_i is a translation ideal, $(x_i * z_i) * (y_i * z_i \in I_i$ for each i which implies $((x_1 * z_1) * (y_1 * z_1), \cdots, (x_n * z_n) * (y_n * z_n)) \in \prod_{i=1}^{n} I_i$. Thus $\prod_{i=1}^{n} I_i$ is translation.

□

Theorem 5.1.4. *Let $\{\theta_i : X_i \longrightarrow Y_i : i = 1, 2, \cdots n\}$ be a family of BRK-homomorphisms. Define $\theta : \prod_{i=1}^{n} X_i \longrightarrow \prod_{i=1}^{n} Y_i$ by $\theta(x_1, \cdots, x_n) = (\theta_1(x_1), \cdots, \theta_n(x_n)$. Then θ is a homomorphism and $Ker\theta = \prod_{i=1}^{n} Ker\theta_i$ and $\theta(\prod_{i=1}^{n} X_i) = \prod_{i=1}^{n} \theta_i(X_i)$.*

Proof. Let $\{\theta_i : X_i \longrightarrow Y_i : i = 1, 2, \cdots n\}$ be a family of BRK-homomorphisms and let $\theta : \prod_{i=1}^{n} X_i \longrightarrow \prod_{i=1}^{n} Y_i$ be a mapping given by $\theta(x_1, \cdots, x_n) = (\theta_1(x_1), \cdots, \theta_n(x_n)$. For any $(x_1, \cdots x_n)), (y_1, \cdots, y_n) \in \prod_{i=1}^{n} X_i$, we have

$$\begin{aligned}
\theta((x_1, \cdots, x_n) \circledast (y_1, \cdots, y_n)) &= \theta((x_1 * y_1, \cdots, x_n * y_n)) \\
&= (\theta_1(x_1 * y_1), \cdots, \theta_n(x_n * y_n)) \\
&= (\theta_1(x_1) * \theta_1(y_1), \cdots, \theta_n(x_n) * \theta_n(y_n)) \\
&= (\theta_1(x_1), \cdots, \theta_n(x_n)) \circledast (\theta_1(y_1), \cdots, \theta_n(y_n)) \\
&= \theta((x_1, \cdots, x_n)) \circledast \theta((y_1, \cdots, y_n))
\end{aligned}$$

Thus θ is a homomorphism.

$$\begin{aligned}
ker\theta &= \{(x_1, \cdots, x_n) \in \prod_{i=1}^{n} X_i : \theta((x_1, \cdots, x_n)) = (0_1, \cdots, 0_n)\} \\
&= \{(x_1, \cdots, x_n) \in \prod_{i=1}^{n} X_i : (\theta_1(x_1), \cdots, \theta_n(x_n)) = (0_1, \cdots, 0_n)\}
\end{aligned}$$

$$= \{(x_1, \cdots, x_n) \in \prod_{i=1}^{n} X_i : \theta_1(x_1) = 0_1, \cdots, \theta_n(x_n) = 0_n\}$$
$$= \prod_{i=1}^{n} Ker\theta_i$$
□

Remark 5.1.5. *θ is a monomorphism (epimorphism) if and only if each θ_i is a monomorphism (epimorphism)*

Theorem 5.1.6. *Let $\{X_i = (X_i, *, 0_i) : i = 1, 2, \cdots n\}$ be a finite family of BRK-algebras and let I_i be a translation ideal of X_i. Then $\prod_{i=1}^{n} X_i / \prod_{i=1}^{n} I_i \cong \prod_{i=1}^{n} X_i/I_i$.*

Proof. Let $\{X_i = (X_i, *, 0_i) : i = 1, 2, \cdots n\}$ be a finite family of BRK-algebras and let I_i be a translation ideal of X_i. Then by Theorem 5.1.3, $\prod_{i=1}^{n} I_i$ is a translation ideal of $\prod_{i=1}^{n} X_i$. Now for simplicity put $I = \prod_{i=1}^{n} I_i$ and $X = \prod_{i=1}^{n} X_i$. Define $f : X/I \longrightarrow \prod_{i=1}^{n} X_i/I_i$ by $f((x_1, \cdots, x_n)/I) = (x_1/I_1, \cdots, x_n/I_n)$ for all $(x_1, \cdots, x_n)/I \in X/I$.

1. f is well defined. Indeed suppose $(x_1, \cdots, x_n)/I = (y_1, \cdots, y_n)/I$. Then $(x_1, \cdots, x_n) \circledast (y_1, \cdots, y_n), (y_1, \cdots, y_n) \circledast (x_1, \cdots, x_n) \in I$. Hence $(x_1 * y_1, \cdots, x_n * y_n), (y_1 * x_1, \cdots, y_n * x_n) \in I$. Thus for each i $x_i * y_i, y_i * x_i \in I_i$ it follows that $x_i/I_i = y_i/I_i$.

   Now $f((x_1, \cdots, x_n)/I) = (x_1/I_1, \cdots, x_n/I_n)$
   $$= (y_1/I_1, \cdots, y_n/I_n)$$
   $$= f((y_1, \cdots, y_n)/I).$$
   Thus f is well defined.

2. f is a homomorphism. For any $(x_1, \cdots, x_n)/I, (y_1, \cdots, y_n)/I) \in$

X/I, we have $f(((x_1, \cdots, x_n)/I) * ((y_1, \cdots, y_n)/I))) = f((x_1, \cdots, x_n) \circledast (y_1, \cdots, y_n)/I$

3. f is bijective. Clearly f is surjective. For injectivity suppose $f(x_1, \cdots, x_n)/I) = f(y_1, \cdots, y_n)/I)$. Then

$$\begin{aligned}(x_1/I_1, \cdots, x_n/I_n) &= f(x_1, \cdots, x_n)/I) \\ &= f(y_1, \cdots, y_n)/I) \\ &= (y_1/I_1, \cdots, y_n/I_n)\end{aligned}$$

Thus injective.

□

5.2 Canonical Mappings of the Direct Product

This section presents about two canonical mappings of the direct product of BRK-algebras, namely canonical projections and canonical injections. We begin with the following theorem.

Theorem 5.2.1. *Let* $\{X_i = (X_i, *, 0_i) : i = 1, 2, \cdots n\}$ *be a family of BRK-algebras. Then* $f_k : \prod_{i=1}^{n} X_i \longrightarrow X_k$ *given by* $f_k(x_1, \cdots, x_k, \cdots, x_n) = x_k$ *is an epimorphism for each* $k = 1, 2, \cdots, n$.

Proof. For any $k = 1, 2, \cdots, n$ let $f_k : \prod_{i=1}^{n} X_i \longrightarrow X_k$ given by $f_k(x_1, \cdots, x_k, \cdots, x_n) = x_k$. Then for any $(x_1, \cdots, x_k, \cdots, x_n), (y_1, \cdots, y_k, \cdots, y_n$ $\prod_{i=1}^{n} X_i$, we have
$f_k((x_1, \cdots, x_k, \cdots, x_n) \circledast (y_1, \cdots, y_k, \cdots, y_n)) = f_k(x_1 * y_1, \cdots, x_k * y_k, \cdots, x_n * y_n) = x_k * y_k = f_k(x_1, \cdots, x_k, \cdots, x_n) * f_k(y_1, \cdots, y_k, \cdots, y_n)$

and hence f_k is a homomorphism. For serjectivity, for any $x_k \in X_k$ $(0_1, \cdots, x_k, \cdots, 0_k) \in \prod_{i=1}^{n} X_i$ such that $f_k(0_1, \cdots, x_k, \cdots, 0_k) = x_k$ therefor onto. □

Definition 5.2.2. *The map in the preceeding theorem is called the canonical projection of the direct product.*

Theorem 5.2.3. *Let $\{X_i = (X_i, *, 0_i) : i = 1, 2, \cdots n\}$ be a family of BRK-algebras. Then there exists a BRK-algebra X, together with a family of homomorphisms $\{f_i : X \longrightarrow X_i : \forall i = 1, 2, \cdots, n\}$ with the following property: for any BRK algebra Y and a family of BRK-homomorpihsms $\{g_i : Y \longrightarrow X_i : \forall i = 1, 2, \cdots, n\}$, there exists a unique homomorpism $g : Y \longrightarrow X$ such that $f_i \circ g = g_i \forall i = 1, 2, \cdots, n$.*

Proof. Put $X = \prod_{i=1}^{n} X_i$. Then X is a BRK-algebra. Let $\{f_i : X \longrightarrow X_i : \forall i = 1, 2, \cdots, n\}$ be the family of canonical projections. Suppose that Y is any BRK-algebra and $\{g_i : Y \longrightarrow X_i : \forall i = 1, 2, \cdots, n\}$ be a family of BRK-homomorphisms. Define $g : Y \longrightarrow X$ by $g(y) = (g_1(y), \cdots, g_n(y))$ for all $y \in Y$. Then

(1) for any $y, z \in Y$
$g(y * z) = (g_1(y * z), \cdots, g_n(y * z)) = (g_1(y) * g_1(z), \cdot, g_n(y) * g_n(z)) = (g_1(y), \cdots, g_n(y)) \circledast (g_1(z), \cdots, g_n(z)) = g(y) * g(z)$.
Hence G is a homomorphism.

(2) for any $y \in Y$ $(f_i \circ g)(y) = f_i(g(y)) = f_i((g_1(y), \cdots, g_i(y), \cdots, g_n(y))) = g_i(y)$. Thus $f_i \circ g = g_i$.

(3) to show g is unique, let $g^{'} : Y \longrightarrow X$ be another homomorphism such that $f_i \circ g^{'} = g_i$. Now let $y \in Y$. Then $(f_i \circ g)(y) = g_i(y) = (f_i \circ g^{'})(y)$. Suppose that $g^{'}(y) = (x_1, \cdots, x_i, \cdots, x_n)$ But then for each i,
$x_i = f_i((x_1, \cdots, x_i, \cdots, x_n)) = f_i(g^{'}(y)) = (f_i \circ g^{'})(y) = (f_i \circ g)(y) = f_i(g(y)) = f_i(g_1(y), \cdots, g_i(y), \cdots, g_n(y)) = g_i(y)$.
Hence $g(y) = (g_1(y), \cdots, g_i(y), \cdots, g_n(y)) = (x_1, \cdots, x_i, \cdots, x_n) = g^{'}(y)$. Therefore, g is unique.

□

Theorem 5.2.4. *Let $\{X_i = (X_i, *, 0_i) : i = 1, 2, \cdots n\}$ be a family of BRK-algebras. Then $g_k : X_k \longrightarrow \prod_{i=1}^{n} X_i$ given by $g_k(x_k) = (0_1, \cdots, 0_{k-1}, x_k, 0_{k+1}, \cdots, 0_n)$ is a monomorphism of BRK-algebras $\forall k = 1, 2, \cdots, n$.*

Proof. For each $k = 1, 2, \cdots, n$, let $x_k, y_k \in X_k$. Then

$$\begin{aligned} g_k(x_k * y_k) &= (0_1, \cdots, 0_{k-1}, x_k * y_k, 0_{k+1}, \cdots, 0_n) \\ &= (0_1, \cdots, 0_{k-1}, x_k, 0_{k+1}, \cdots, 0_n) \circledast (0_1, \cdots, 0_{k-1}, y_k, 0_{k+1}, \cdots, 0_n) \\ &= g_k(x_k) \circledast g_k(y_k). \end{aligned}$$

Thus g_k is a homomorphism. Clearly g_k is one to one and hence it is a monomorphism. □

Definition 5.2.5. *The maps in the preceeding theorem are called the canonical injections.*

Lemma 5.2.6. *Let $\{X_i = (X_i, *, 0_i) : i = 1, 2, \cdots n\}$ be a family of BRK-algebras. $\forall k = 1, 2, \cdots, n$, if g_k is the canonical injection,* *then* $g_k(A_k)$ *is a closed ideal of* $\prod_{i=1}^{n} A_i$.

Proof. For each $k = 1, 2, \cdots, n$, clearly $(0_1, \cdots, 0_k, \cdots, 0_n) \in g_k(X_k)$. Suppose for any $(y_1, \cdots, y_k, \cdots, y_n), (x_1, \cdots, x_k, \cdots, x_n) \in \prod_{i=1}^{n} X_i$, $(y_1, \cdots, y_k, \cdots, y_n) \circledast (x_1, \cdots, x_k, \cdots, x_n), (x_1, \cdots, x_k, \cdots, x_n) \in g_k(X_k)$. But then for $i \neq k$, $x_i = y_i = 0_i$. Now since $y_k \in X_k$, $(y_1, \cdots, y_k, \cdots, y_n) \in g_k(X_k)$ hence $g_k(X_k)$ is an ideal of $\prod_{i=1}^{n} X_i$.

□

Remark 5.2.7. *In general* $g_k(X_k)$ *may not be a translation ideal of* $\prod_{i=1}^{n} X_i$.

Example 5.2.8. *Let* $X = \{0, 1, 2, 3\}$ *be a set with the following Cayley table:*

*	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	3	0	2
3	3	3	0	0

Then $(X, *, 0)$ *is a BRK-algebra Let* $Y = \{0, 1, 2, 3\}$ *be a set with the following Cayley table:*

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	0	1	0

Then $(Y, *, 0)$ *is a BRK-algebra. Let* $g : X \longrightarrow X \times Y$ *be the canonical injection given by* $g(x) = (x, 0)$*. Then* $g(X)$ *is not a translation ideal of* $X \times Y$ *as* $(2,3) \circledast (0,1) = (2*0, 3*1) = (2,0), (0,1) \circledast (2,3) = (0*1, 1*3) = (0,0) \in g(X)$ *but* $(2,3) \circledast (0,2)) \circledast ((0,1) \circledast (0,2)) = ((2*0)*(0*0), (3*2)*(1*2)) = (2,1) \notin g(X)$

Theorem 5.2.9. *Let* $\{X_i = (X_i, *, 0_i) : i = 1, 2, \cdots n\}$ *be a family of anti-symmetric BRK-algebras. If* g_k *is the canonical injection for each* $k = 1, \cdots, n$*, then* $g_k(A_k)$ *is a translation ideal of* $\prod_{i=1}^{n} A_i$ *and* $\prod_{i=1}^{n} A_i / g_k(A_k) \cong \prod_{i \neq k} A_i$*.*

Proof. Let g_k be the canonical injection. By Lemma 5.2.6, $g_k(X_k)$ is an ideal of $\prod_{i=1}^{n} X_i$. Let $(w_1, \cdots, w_k, \cdots, w_n), (x_1, \cdots, x_k, \cdots, x_n), (y_1, \cdots, y_k, \cdots, y$ $\prod_{i=1}^{n} X_i$. Suppose $(w_1, \cdots, w_k, \cdots, w_n) \circledast (x_1, \cdots, x_k, \cdots, x_n), (x_1, \cdots, x_k, \cdots, x_n) \circledast$ $(w_1, \cdots, w_k, \cdots, w_n) \in g_k(X_k)$. But then for $i \neq k$ $w_i * x_i = x_i * w_i = 0$ and hence $w_i = x_i$. Now $((w_1, \cdots, w_k, \cdots, w_n) \circledast (y_1, \cdots, y_k, \cdots, y_n)) \circledast ((x_1, \cdots, x_k, \cdots, x_n) \circledast (y_1, \cdots, y_k, \cdots, y_n)) = ((w_1 * y_1) * (x_1 * y_1), \cdots, (w_k * y_k) * (x_k * y_k), \cdots, (w_n * y_n) * (x_n * y_n)) = ((w_1 * y_1) * (w_1 * y_1), \cdots, (w_k * y_k) * (x_k * y_k), \cdots, (w_n * y_n) * (w_n * y_n)) = (0_1, \cdots, (w_k * y_k) * (x_k * y_k), \cdots, 0_n)$

Thus since $(w_k * y_k) * (x_k * y_k \in X_k$, $((w_1, \cdots, w_k, \cdots, w_n) \circledast (y_1, \cdots, y_k, \cdots, y_n)) \circledast$ $((x_1, \cdots, x_k, \cdots, x_n) \circledast (y_1, \cdots, y_k, \cdots, y_n)) \in g_k(X_k)$. Similarly $((y_1, \cdots, y_k, \cdots, y_n) \circledast (w_1, \cdots, w_k, \cdots, w_n)) \circledast ((y_1, \cdots, y_k, \cdots, y_n) \circledast (x_1, \cdots, x_k, \cdots, x_n)) \in g_k(X_k)$. Hence $g_k(X_k)$ is a translation ideal. For the proof of the isomorphism let $X = \prod_{i=1}^{n} X_i$ and $I = g_k(X_k)$.

Define $f : X/I \longrightarrow \prod_{i \neq k} A_i$ by $f((x_1, \cdots, x_n)/I) = (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n)$ for all $(x_1, \cdots, x_n)/I \in X/I$. Then

(i) Suppose that $(x_1, \cdots, x_n)/I = (y_1, \cdots, y_n)/I$ But then $(x_1, \cdots, x_n) \circledast (y_1, \cdots, y_n) = (x_1 * y_1, \cdots, x_n * y_n), (y_1, \cdots, y_n) \circledast (x_1, \cdots, x_n) = (y_1 * x_1, \cdots, y_n * x_n) \in I$ so that $x_i * y_i = y_i * x_i = 0$ for all $i \neq k$. Which implies $x_i = y_i$for all $i \neq k$. Thus $f((x_1, \cdots, x_n)/I) = f((y_1, \cdots, y_n)/I)$ which shows that f is well defined.

(ii) To show it is a homomorphism for any $(x_1, \cdots, x_n)/I, (y_1, \cdots, y_n)/I \in X/I$, $f((x_1, \cdots, x_n)/I) * ((y_1, \cdots, y_n)/I) = f((x_1, \cdots, x_n) \circledast (y_1, \cdots, y_n))/I) = f(x_1 * y_1, \cdots, x_n * y_n)/I) = (x_1 * y_1, \cdots, x_{k-1} * y_{k-1}, x_{k+1} * y_{k+1}, \cdots, x_n * y_n) = (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) \circledast (y_1, \cdots, y_{k-1}, y_{k+1}, \cdots, y_n) = f((x_1, \cdots, x_n)/I \circledast f(y_1, \cdots, y_n)/I)$. Thus f is a homomorphism.

(iii) Clearly f is onto. To show it is one to one suppose $f((x_1, \cdots, x_n)/I) = f((y_1, \cdots, y_n)/I)$. Then $(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) = (y_1, \cdots, y_{k-1}, y_{k+1}, \cdots$ which implies $x_i = y_i$ for all $i \neq k$. Hence $(x_1, \cdots, x_n) \circledast (y_1, \cdots, y_n) = (x_1 * y_1, \cdots, x_n * y_n), (y_1, \cdots, y_n) \circledast (x_1, \cdots, x_n) = (y_1 * x_1, \cdots, y_n * x_n) \in I$, so $(x_1, \cdots, x_n)/I = (y_1, \cdots, y_n)/I$. Thus f is one to one.

□

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Annex-A: First Paper

`Weak Positive Implicative BRK-Algebras`

Annals of Pure and Applied Mathematics
Vol. 13, No. 2, 2017, 235-240
ISSN: 2279-087X (P), 2279-0888(online)
Published on 29 April 2017
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v13n2a10

Annals of
Pure and Applied Mathematics

Weak Positive Implicative BRK-Algebras

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Received 19 March 2017; accepted 6 April 2017

Abstract. In this paper we introduce the notion of weak positive implicative BRK-algebra and study its properties via left maps.

Keywords: Quotient BRK-Algebra, Weak positive implicative BRK-Algebra, R-map, L-map, Homomorphism.

AMS Mathematics Subject Classification (2010): 06F35, 08A35

1. Introduction

In 1996, two classes of abstract algebras, BCK-algebras and BCI-algebras,were introduced by Imai and Iseki [1, 2]. It is known that the class of BCK-algebras is a proper subclass of BCI-algebras. Since then many researchers introduced and studied different classes of new algebras as a generalization of BCK/BCI-algebras.

In 1983, Hu and Li introduced the notion of BCH-algebras [4] as a generalization of BCI-algebras and studied certain properties of these algebras. In this direction, Jun, Roh and Kim introduced a new class of algebra namely BH-algebras [6] as a generalization of BCH-algebras. Q-Algebras and QS-algebras [5] are further generalizations of BCH algebras. Recently, Ravi Kumar Bandaru introduced the notion of BRK-algebras [3] which is a generalization of BCI/BCI/BCH/Q/QS-algebras and studied various properties of BRK-Algebras. His study was confined to give various characterizations for these BCK/BCI/BCH/Q/QS-algebras with BRK-Algebras.

Ravi kumar defined BRK-algebra as an algebra $X = (X,*,0)$ of type $(2,0)$ which satisfies the axioms (i) $x*0 = x$ and (ii) $(x*y)*x = 0*y$ for any $x, y \in X$. It is known that in any BRK-algebra X the following holds for any $x, y \in X$ (see [3]),

- $x*x = 0$
- $0*(x*y) = (0*x)*(0*y)$
- $x*y = 0$ implies $0*x = 0*y$

He also introduced the notion of positive implicative BRK-algebra as a BRK-algebra which satisfies the condition $((x*y)*y)*(0*y) = x*y$ and gave a necessary and sufficient condition for a BRK-algebra to be positive implicative.

In this paper we establish an isomorphism theorem for quotient BRK-Algebra determined by homomorphism. Furthermore we make use of weak positive implicative BRK-algebras and study their properties via right maps and left maps.

2. Quotient BRK-algebra determined by homomorphism

Definition 2.1. *Let* $X = (X,*,0)$ *and* $Y = (Y,*,0)$ *be BRK-algebras. A mapping* $f : X \to Y$ *is called a homomorphism from* X *into* Y *if* $f(x*y) = f(x)*f(y)$ *for all* $x, y \in X$.

A homorpism f is called a monomorphism (resp., epimorphism) if it is injective (resp., surjective). A bijective homomorphism is called an isomorpism. Two BRK-algebras X and Y are said to be isomorphic, written $X \cong Y$, if there exists an isomorphism $f : X \to Y$. For any homomorphism $f : X \to Y$ the set $\{x \in X : f(x) = 0\}$ is called kernel of f, denoted by $Kerf$ and the set $\{f(x) : x \in X\}$ is called the image of f, denoted by Imf .

Theorem 2.2. *Let* $f : X \to Y$ *be a homomorphism of BRK-algebras. Define a relation* $\sim$ *on* X *by* $x \sim y$ *if and only if* $f(x) = f(y)$ *for all* $x, y \in X$ *. Then* $\sim$ *is a congruence relation on* X *which is called the congruence relation determined by the homomorphism* f *.*

Proof: Clearly $\sim$ is an equivalence relation on X . Next suppose $x \sim y$ and $u \sim v$. Then $f(x) = f(y)$ and $f(u) = f(v)$. Now since $f(x*u) = f(x)*f(u) = f(y)*f(v) = f(y*v), x*u \sim y*v$. Thus $\sim$ is a congruence relation on X . ■

We denote the equivalence class of x determined by $\sim$ by $[x]_f$ and the set of all equivalence classes by X/f i.e $[x]_f = \{y \in X : x \sim y\}$ and $X/f = \{[x]_f : x \in X\}$.

Theorem 2.3. *Let* $f: X \to Y$ *be homomorphism on BRK-algebras. Define* $*$ *on* X/f *by* $[x]_f * [y]_f = [x*y]_f$*. Then* $(X/f, *, [0]_f)$ *is a BRK algebra. It is called the quotient BRK-algebra determined by the homomorphism* f *.*

Proof: Since $\sim$ is a congruence relation on X, * is well defined. Now for any $[x]_f, [y]_f \in X/f$ we have

1. $[x]_f * [0]_f = [x*0]_f = [x]_f$ and
2. $([x]_f * [y]_f) * [x]_f = [x*y]_f * [x]_f = [(x*y)*x]_f = [0*y]_f = [0]_f * [y]_f$.

Thus $(X/f, *, [0]_f)$ is a BRK-algebra. ■

Remark 2.4. *Clearly* $[0]_f = Kerf$.

Theorem 2.5. *Let* $f : X \to Y$ *be homomorphism of BRK-algebras. Then the image of* f *is isomorphic to the quotient BRK-algebra determined by* f *, i.e.* $X/f \cong Imf$.

Proof: Define a mapping $\theta : X/f \to Imf$ by $\theta([x]_f) = f(x)$. Then

1. θ is well defined. Indeed suppose $[x]_f = [y]_f$ but then $[x]_f = [y]_f \Rightarrow x \in [y]_f \Rightarrow f(x) = f(y)$. Thus $\theta([x]_f) = \theta([y]_f)$.
2. θ is homomorphism. Indeed for any $[x]_f, [y]_f \in X/f$ we have $\theta([x]_f * [y]_f) = \theta([x*y]_f) = f(x*y) = f(x)*f(y) = \theta([x]_f)*\theta([y]_f)$
3. Clearly θ is bijective.

Hence $X/f \cong Imf$. ■

3. Weak implicative BRK-algebra

Here we will define weak implicative BRK-algebra and investigate its properties.

Definition 3.1. *A BRK-algebra $X = (X,*,0)$ is said to be weak positive implicative if it satisfies $(x*y)*z = (x*z)*(y*z)$ for all x, y and $z \in X$*

Example 3.2. *Let $X = \{0,1,2,3\}$ be a set with the following cayley table:*

*	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	3	0

Then $(X,*,0)$ is a weak positive implicative BRK-algebra.

The next example shows the existence of weak implicative BRK algebra which is not BCK/BCI/BCH-algebra.

Example 3.3. *Let $\mathbb{Z}$ be the set of integers. Define * on $\mathbb{Z}$ by*

$$x*y = \begin{cases} x, & \text{if } y = 0 \\ 0, & \text{if } y \neq 0 \end{cases}$$

Then $(\mathbb{Z},*,0)$ is a weak positive implicative BRK-algebra which is not BCK/BCI/BCH-algebra.

Lemma 3.4. *In any weak positive implicative BRK-algebra X, the following hold for all $x, y \in X$.*

1. $0*x = 0$
2. $(x*y)*x = 0$
3. $x*y = (x*y)*y$
4. $(x*(x*y))*y = 0$

Proof. Let $x, y \in X$. Then

1. $0*x=(x*x)*x=(x*x)*(x*x)=0*0=0$
2. $(x*y)*x=0*y=0$
3. $x*y=(x*y)*0=(x*y)*(y*y)=(x*y)*y$
4. $(x*(x*y))*y=(x*y)*((x*y)*y)=(x*y)*(x*y)=0$. ■

Theorem 3.5. *Every weak positive implicative BRK-algebra is positive implicative.*
Proof. Let $X=(X,*,0)$ be a weak positive implicative BRK-algebra. For any $x,y\in X$, we have $((x*y)*y)*(0*y)=((x*y)*y)*0=(x*y)*y=x*y$.
Thus X is positive implicative BRK-algebra. ■

Remark 3.6. *The converse of the above theorem is not true.*

Example 3.7. *Let* $X=\{0,1,2\}$ *be a set with Cayley table:*

*	0	1	2
0	0	2	2
1	1	0	0
2	2	0	0

Then $(X,*,0)$ is a positive implicative BRK-algebra [see 3] which is not a weak positive implicative (as $(1*1)*1=2\neq 0=(1*1)*(1*1)$).

4. R-maps and L-maps in BRK-algebra
In this section we investigate the properties of R-maps and L-maps in weak positive implicative BRK-algebras.

Definition 4.1. *Let* $X=(X,*,0)$ *be a BRK-algebra and* $a\in X$ *be a fixed element. Then the map* $R_a:X\to X$ *given by* $R_a(x)=x*a$ *is called right map of* X *and the map* $L_a:X\to X$ *given by* $L_a(x)=a*x$ *is called left map of* X*. The set of all left maps is denoted by* $\mathcal{L}(X)$.

Definition 4.2. *A right map* R_a *is called idempotent if* $R_a\circ R_a=R_a$ *where* $\circ$ *is the usual composition of maps.*

Remark 4.3. *Clearly for any* a, R_a *is idempotent if and only if* $(x*a)*a=x*a$ *for all* $x\in X$.

Theorem 4.4. *If a BRK-algebra* $X=(X,*,0)$ *is weak positive implicative, then every right map on* X *is idempotent.*
Proof. For any $a\in X$, $R_a(x)=x*a=(x*a)*a=R_a(R_a(x))=(R_a\circ R_a)(x)$ for all

$x \in X$. Hence $R_a \circ R_a = R_a$. ■

Theorem 4.5. *A BRK-algebra* $X = (X,*,0)$ *is a weak positive implicative if and only if every right map is a homomorphism.*

Proof. Suppose X is a weak positive implicative BRK-algebra. Then for each $a \in X$, $R_a(x*y) = (x*y)*a = (x*a)*(y*a) = R_a(x)*R_a(y)$ .Thus R_a is a homomorphism. For the converse suppose every right map is a homomorphism. Now for any $x, y, z \in X$ we have $(x*y)*z = R_z(x*y) = R_z(x)*R_z(y) = (x*z)*(y*z)$. Hence X is weak positive implicative. ■

Theorem 4.6. *In any BRK-algebra* $X = (X,*,0)$ *, if* L_a *is a homomorphism, then* $a = 0$.

Proof. Suppose L_a is a homomorphism. But then

$a = a*0 = L_a(0) = L_a(0*0) = L_a(0)*L_a(0) = (a*0)*(a*0) = a*a = 0$. ■

For a BRK-algebra $X = (X,*,0)$ we define a binary operation $\otimes$ on $\mathsf{L}(X)$ by $(L_a \otimes L_b)(x) := L_a(x)*L_b(x)$ for any $L_a, L_b \in \mathsf{L}(X)$. We have the following Lemma.

Lemma 4.7. *Let* $X = (X,*,0)$ *be weak positive implicatice BRK-algebra. For any* $L_a, L_b, L_c \in \mathsf{L}(X)$ *, we have*

i. $L_a \otimes L_b = L_{a*b}$ i.e. $L_a \otimes L_b \in \mathsf{L}(X)$.

ii. $(L_a \otimes L_b) \otimes L_c = (L_a \otimes L_c) \otimes (L_a \otimes L_c)$.

Proof. For any $x \in X$ we have

i. $(L_a \otimes L_b)(x) = L_a(x)*L_b(x) = (a*x)*(b*x) = (a*b)*x = L_{a*b}(x)$ and so $L_a \otimes L_b = L_{a*b}$.

ii. $(L_a \otimes L_b) \otimes L_c = L_{a*b} \otimes L_c = L_{(a*b)*c}$

$= L_{(a*c)*(b*c)} = L_{a*c} \otimes L_{b*c}$

$= (L_a \otimes L_c) \otimes (L_a \otimes L_c)$ ■

Theorem 4.8 *If* $X = (X,*,0)$ *is a weak positive implicative BRK-algebra, then* $\mathsf{L}(X) = (\mathsf{L}(X), \otimes, L_0)$ *is a weak positive implicative BRK-algebra.*

Proof. It is enough to show that $\mathsf{L}(X) = (\mathsf{L}(X), \otimes, L_0)$ is a BRK-algebra. Now for any $L_a, L_b \in \mathsf{L}(X)$ we have

1. $L_a \otimes L_0 = L_{a*0} = L_a$, and
2. $(L_a \otimes L_b) \otimes L_a = L_{(a*b)*a} = L_{0*b} = L_0 \otimes L_b$.

Therefore $\mathsf{L}(X)$ is a BRK-algebra and hence by the above lemma it is weak positive implicative. ■

Corollary 4.9. *Let* $X = (X,*,0)$ *be a weak positive implicative BRK-algebra. Then the map* $f : X \rightarrow L(X)$ *given by* $f(x) = L_x$ *is an epimorphism and* $X/f \cong L(X)$ *where* X/f *is the quotient BRK-algebra determined by the homomorphism* f *.*

5. Conclusion

In this paper, we have introduced the notion of weak positive implicative BRK-algebra and showed that the set of all left maps on weak positive implicative BRK-algebra is also weak positive implicative BRK-algebra. We have also investigated the conditions under which right maps and left maps becomes a homomorphsm.

Acknowledgment. We thank the referees for their valuable comments and suggestions. Moreover the second author would like to thank the authorities of Dr. Lankapalli Bullyya College for providing the necessary facilities to carry out the research.

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Annex-B: Second Paper

Quotient BRK-Algebras

Journal of Hyperstructures 7 (2) (2018), 94-103.
ISSN: 2322-1666 print/2251-8436 online

QUOTIENT BRK-ALGEBRAS

K. VENKATESWARLU AND GIRUM AKLILU

Abstract. In this paper we introduce the notion of quotient BRK-algebras and also investigate the properties of these algebras. Further we establish first isomorphism theorem for the special subclass of BRK-algebras namely anti-symmetric BRK-algebras.

Key Words: BRK-algebra, congrunces, translation ideal, anti-symmetric BRK- algebra.
2010 Mathematics Subject Classification: Primary: 06F35; Secondary 08A35.

1. Introduction

Two classes of abstract algebras namely BCK and BCI algebras were initially introduced by Y. Imai and K. Iseki [1, 2]. It is known that BCK is a proper subclass of BCI algebras. Subsequently many researchers introduced and studied extensively on generalizations of BCK/BCI-algebras namely BCH-algebras by Hu and LI [3], Q- algebras by J.Negger and etl [4], BRK-algebras by Ravi Kumar in [5] . The order of generalization is as follows BCK/BCI/BCH/Q/BRK-algebras.

In this paper, we investigate the study of BRK- algebras by introducing the notion of homomorphism, congruence and translation ideals. The process of constructing a quotient BRK-algebra is in the usual way but not with simply ideal. A stronger condition has been introduced on ideal called translation ideal to obtain a quotient BRK-Algebra. Further we gave an example (see 4.6) that an ordinary ideal does not give a congruence. Also we introduce a sub class of BRK- algebra called anti -symmetric BRK-algebra which is a common subclass of the two

Received: 12 January 2018, Accepted: 11 April 2018. Communicated by Irina Cristea;
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distinct algebras namely BRK-algebra and BH-algebra. Finally we conclude this paper by establishing the first isomorphism theorem for the anti-symmetric BRK-algebras, which are a subclass of BRK-algebras.

2. Preliminaries

We collect certain definitions and examples from the existing literature.

Definition 2.1. Let $X = (X, *, 0)$ be an algebra of type $(2, 0)$. Then X is called:

(1) BCH-algebra ([3]) if it satisfies
 (1) $x * x = 0$,
 (2) $x * y = 0$ and $y * x = 0$ imply $x = y$,
 (3) $(x * y) * z = (x * z) * y$
(2) a Q-algebra ([4]) if it satisfies (1), (3) and
 (4) $x * 0 = x$.
(3) a BH-algebra ([6]) if it satisfies (1), (2) and (4).

Definition 2.2. ([5]). An algebra $(X, *, 0)$ of type $(2, 0)$ is called a BRK-algebra if it satisfies the following axioms
 (1) $x * 0 = x$,
 (2) $(x * y) * x = 0 * y$
for all $x, y \in X$.

Example 2.3. ([5]) Let $X = R \setminus \{-n\}$, $0 \neq n \in Z^+$ where R is the set of real numbers and Z^+ is the set of positive integers.If we define a binary operation $*$ on X by

$$x * y := \frac{n(x - y)}{n + y},$$

then $(X, *, 0)$ is a BRK algebra.

Example 2.4. ([5]) Let $X = \{0, 1, 2\}$ be a set with the following Cayley table:

*	0	1	2
0	0	2	2
1	1	0	0
2	2	0	0

Then $(X, *, 0)$ is a BRK algebra.

Theorem 2.5. *([5]) In any BRK-algebra X the following holds for any $x, y \in X$,*

(1) $x * x = 0$
(2) $0 * (x * y) = (0 * x) * (0 * y)$
(3) $x * y = 0$ *implies* $0 * x = 0 * y$

Definition 2.6. Let X be a BRK-algebra and let I be nonempty subset of X. Then

(1) I is called a subalgebra of X if $x * y \in I$ for all $x, y \in I$
(2) I is called an ideal of X if for any $x, y \in X$:
 (i) $0 \in I$,
 (ii) $x * y \in I$ and $y \in I$ imply $x \in I$.
(3) I is called closed ideal of X if it is both an ideal and a subalgebra.

Remark 2.7. In general an ideal of a BRK-algebra may not be a subalgebra and vice-versa.

Example 2.8. Let $X = (Z, -, 0)$ be the BRK-algebra of set of integers under subtraction. Then the set I of all non negative integer forms an ideal which is not a subalgebra. Indeed $0 \in I$ and if $x, y - x \in I$ then $0 \leq x$ and $0 \leq y - x$ which imply $0 \leq y$ and hence $y \in I$ (in this case the relation $\leq$ is the usual order of real numbers. Thus I is an ideal of X. Clearly I is not a subalgebra as it is not closed under subtraction.

3. Translation Ideals, Homomorphisms

We introduce the notion of translation ideals in a BRK-algebra.
Definition 3.1. An ideal I of a BRK algebra X is called translation ideal if it satisfies the condition $x * y, y * x \in I \Rightarrow (x * z) * (y * z), (z * x) * (z * y) \in I$.

Example 3.2. Let $X = (Z, -, 0)$ be the BRK-algebra of set of integers under the usual subtraction of real numbers. Then the set of all non negative integers forms a translation ideal of X.

The following two examples demonstrate the existence of an ideal which is not translation.

Example 3.3. Let $X = \{0, 1, 2, 3\}$ be a set with the following Cayley table:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	0	1	0

Then $(X, *, 0)$ is a BRK-algebra. Clearly $I = \{0\}$ is an ideal but not translation since $3 * 1, 1 * 3 \in I$ but $(3 * 2) * (1 * 2) = 1 \notin I$.

Example 3.4. Let $X = \{0, 1, 2, 3\}$ be a set with the following Cayley table:

*	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	3	0	2
3	3	3	0	0

Then $(X, *, 0)$ is a BRK-algebra and $I = \{0, 1\}$ is an ideal of X which is not translation (as $0 * 1, 1 * 0 \in I$ but $(2 * 0) * (2 * 1) = 2 \notin I$).

Remark 3.5. In general a translation ideal may not be closed (see Example 3.2).

Definition 3.6. Let X and Y be BRK-algebras. A mapping $f : X \longrightarrow Y$ is called a homomorphism from X into Y if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

A homomorpism f is called a monomorphism (resp., epimorphism) if it is injective (resp., surjective). A bijective homomorphism is called an isomorpism. Two BRK-algebras X and Y are said to be isomorphic, written $X \cong Y$, if there exists an isomorphism $f : X \longrightarrow Y$. For any homomorphism $f : X \longrightarrow Y$ the set $\{x \in X : f(x) = 0\}$ is called kernel of f, denoted by $Ker(f)$ and the set $\{f(x) : x \in X\}$ is called the image of f, denoted by Imf.

Lemma 3.7. *Let $f : X \longrightarrow Y$ be homomorphism of BRK-algebras. Then*

(1) $f(0) = 0$
(2) $x * y = 0$ *implies* $f(x) * f(y) = 0$.

Proof.
(1) $f(0) = f(0 * 0) = f(0) * f(0) = 0$.
(2) If $x * y = 0$, then $f(x * y) = f(0)$ which implies $f(x) * f(y) = 0$. □

Theorem 3.8. *Let $f : X \longrightarrow Y$ be homomorphism of BRK-algebras.*

i. *If S is a subalgebra of X, then $f(S)$ a subalgebra of Y*
ii. *If K is a subalgebra of Y, then $f^{-1}(K)$ is a subalgebra of X containing $Kerf$*
iii. *If I is an ideal of X and f is injective, then $f(I)$ is an ideal of $f(X)$.*
iv. *If J is an ideal of Y, then $f^{-1}(J)$ is an ideal of X.*
v. *If I is a translation ideal of X and f is injective, then $f(I)$ is a translation ideal of $f(X)$.*
vi. *If J is a translation ideal of Y, then $f^{-1}(J)$ is a translation ideal of X.*

Proof. Straightforward □

Corollary 3.9. *If $f : X \longrightarrow Y$ is homomorphism of BRK algebras, then $Kerf$ is a closed ideal of X and Imf is a subalgebra of Y.*

Remark 3.10. For any BRK-homomorphism f,

(1) $kerf = \{0\}$ may not imply f is injective.
For instance, let $X = (X, *, 0)$ be the BRK-algebra where $X = \{0, 1, 2\}$ and $*$ is given by the Cayley table:

*	0	1	2
0	0	2	2
1	1	0	0
2	2	0	0

Let $f : X \longrightarrow X$ be defined by $f(0) = 0$ and $f(1) = f(2) = 2$. clearly f is a homomorphism with $kerf = \{0\}$ but f is not injective.

(2) $kerf$ may not be a translation ideal of X.
For instance, consider the BRK-algebra X in example 3.3. Clearly the identity map $id : X \longrightarrow X$ is a homomorphism, but $kerf = \{0\}$ is not a translation ideal.

4. Quotient BRK-Algebra

In this section we will study the quotient algebra of BRK-algebra. We define congruence relation on BRK-algebra as in the usual way.

Definition 4.1. An equivalence relation θ on a BRK-algebra X is called a congruence relation if it has a compatibility property:

$(x, y) \in \theta$ and $(u, v) \in \theta$ imply $(x*u, y*v) \in \theta$ for all $x, y, u, v \in X$.

Given a congruence relation θ on a BRK-algebra X, we use the notation θ_x for the equivalence class determined by x i.e. $\theta_x = \{y \in X : (y, x) \in \theta\}$ and X/θ for the quotient set $\{\theta_x : x \in X\}$.

Theorem 4.2. *Let X be a BRK-algebra. Define $*$ on the quotient set X/θ by $\theta_x * \theta_y = \theta_{x*y}$. Then $(X/\theta, *, \theta_0)$ is a BRK-algebra which is called a quotient BRK-algebra induced by the congruence θ.*

Proof. Since θ is a congruence relation $*$ is well defined. For any $\theta_x, \theta_y \in X/\theta$ we have

(1) $\theta_x * \theta_0 = \theta_{x*0} = \theta_0$, and

(2) $(\theta_x * \theta_y) * \theta_x = \theta_{x*y} * \theta_x = \theta_{(x*y)*x} = \theta_{0*y} = \theta_0 * \theta_y$.

Thus $(X/\theta, *, \theta_0)$ is a BRK-algebra. □

Theorem 4.3. *Let $X/\theta = (X/\theta, *, \theta_0)$ be the quotient BRK-algebra induced by a congruence θ. Then θ_0 is a closed ideal of X.*

Proof. Since $(0, 0) \in \theta$, $0 \in \theta_0$. Suppose $x, y * x \in \theta_0$, then $(x, 0) \in \theta$ and $(y * x, 0) \in \theta$. Now from $(y, y) \in \theta$ and $(x, 0) \in \theta$ we have $(y * x, y) \in \theta$. Also from $(y * x, y) \in \theta$ and $(y * x, 0) \in \theta$ we get $(0, y) \in \theta$ and hence by symmetry we have $(y, 0) \in \theta$ i.e. $y \in \theta_0$. Therefor θ_0 is an ideal of X. Next, if $x, y \in \theta_0$ then $(x, 0) \in \theta$ and $(y, 0) \in \theta$ and hence $(x * y, 0) \in \theta$ i.e. $x * y \in \theta_0$. Thus it is closed. □

Now we construct a congruence relation on X via translation ideal.

Theorem 4.4. *Let I be a translation ideal of a BRK-algebra X. Define a relation $\backsim$ on X by $x \backsim y \Leftrightarrow x * y, y * x \in I$. Then $\backsim$ is a congruence relation on X which is called the congruence relation of X induced by a translation ideal I.*

Proof. For any $x \in X$, since $x * x = 0 \in I$, we have $x \backsim x$ hence it is reflexive. From the definition of $\backsim$ it is clear that $\backsim$ is symmetric. If $x \backsim y$ and $y \backsim z$ then $x * y, y * x, y * z, z * y \in I$. Now $x * y, y * x \in I$ implies $(x * z) * (y * z) \in I$. But then since I is an ideal and $y * z \in I$, $x * z \in I$. Similarly $z * x \in I$. Thus $\backsim$ is transitive. Hence $\backsim$ is an equivalence relation. Next suppose $x \backsim y$ and $u \backsim v$. But then since I is a translation ideal we have $x * y, y * x \in I \Rightarrow (x * u) * (y * v) \in I \Rightarrow ((x * u) * (y * v)) * ((y * u) * (y * v)) \in I$ and hence $(x * u) * (y * v) \in I$ (as I is an ideal and $(y * u) * (y * v) \in I$). Similarly we can show that $(y * v) * (x * u) \in I$. Thus $\backsim$ is a congruence relation. □

Remark 4.5. In the above theorem if we take an arbitrary ideal instead of translation , the relation may not be congruence.

Example 4.6. Consider the BRK-algebra X and its ideal I in Example3.4. Define $\sim$ on X by $x \sim y \Leftrightarrow x * y, y * x \in I$. But then $2 \sim 2$ and $0 \sim 1$ but $2 * 0 = 2 \nsim 3 = 2 * 1$ as $2 * 3 = 2 \notin I$ hence not a congruence relation.

Now for any translation ideal I of a BRK-algebra X we use the notation I_x for the equivalence class determined by x and X/I for the set of all equivalence classes of X for the congruence relation of X induced by I. Clearly $I_x = \{y \in X : x \sim y\}$ and $X/I = \{I_x : x \in X\}$.

Corollary 4.7. *Let X be a BRK-algebra and I be a translation ideal of X. Define* $*$ *on* X/I *by* $I_x * I_y = I_{x*y}$*, for all* $x, y \in X$*. Then* $(X/I, *, I_0)$ *is a BRK algebra.*

Definition 4.8. $(X/I, *, I_0)$ is called the quotient BRK-algebra of X determined by a translation ideal I

Remark 4.9. Let X be a BRK-algebra. Then

(1) in general for a translation ideal I of X, I_0 may not be equal to I and
(2) a translation ideal I of X is closed if and only if $I_0 = I$.

5. ANTI-SYMMETRIC BRK-ALGEBRA

In this section we introduce a new subclass of BRK-algebra called anti-symmetric BRK-algebra and we will estabilish the homomorpism theorem for this subclass.

Definition 5.1. A BRK-algebra X is called an anti-symmetric BRK-algebra if it satisfies (2): $x * y = 0$ and $y * x = 0$ imply $x = y$ for any $x, y \in X$.

The following examples illustrate such algebra exist.

Example 5.2. Let $X = \{0, 1, 2\}$ be a set with Cayley table:

*	0	1	2
0	0	2	2
1	1	0	0
2	2	0	0

Then $(X, *, 0)$ is a BRK- algebra which is not anti-symmetric BRK-algebra as $2 * 1 = 1 * 2 = 0$ but $1 \neq 2$.

Example 5.3. Let $X = \{0, 1, 2, 3\}$ be a set with Cayley table:

*	0	1	2	3
0	0	1	0	1
1	1	0	1	0
2	2	1	0	1
3	3	2	3	0

Then $(X, *, 0)$ is an anti-symmetric BRK-algebra which is not BCH as $(3 * 1) * 1 = 0 \neq 2 = (3 * 2) * 1$.

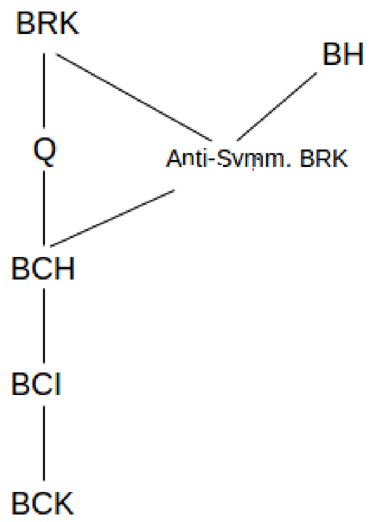
Example 5.4. Let $X = \{0, 1, 2, 3\}$ be a set with Cayley table:

*	0	1	2	3
0	0	3	0	2
1	1	0	0	0
2	2	2	0	3
3	3	3	1	0

Then $(X, *, 0)$ is a BH-algebra which is not an anti-symmetric BRK-algebra as $(2 * 3) * 2 = 1 \neq 2 = 0 * 3$.

Remark 5.5. Every anti-symmetric BRK-algebra is a BH- algebra but not the converse.

The following figure shows the relationship of the algebras.



Theorem 5.6. *Let X and Y be anti-symmetric BRK-algebra and $f : X \longrightarrow Y$ be BRK-homomorhism. Then f is injective if and only if $kerf = \{0\}$.*

Proof. Obviously if f is injective then clearly $Kerf = \{0\}$. On the other hand, suppose that $x, y \in X$ and $f(x) = f(y)$. Then $f(x * y) = f(x) * f(y) = f(x) * f(x) = 0$. Hence $x * y \in Kerf$ and so $x * y = 0$. Similarly we have $y * x = 0$. Thus $x = y$ and hence f is injective. □

Theorem 5.7. *Let X and Y be anti-symmetric BRK-algebra and $f : X \longrightarrow Y$ be BRK-homomorhism. Then $Kerf$ is a translation ideal of X.*

Proof. Since $Kerf$ is an ideal of X, it is enough to show that it is translation. If $x * y, y * x \in kerf$, then $f(x) * f(y) = f(y) * f(x) = 0 \Rightarrow f(x) = f(y)$. But then for any $z \in X$ $f((x * z) * (y * z)) = (f(x) * f(z)) * (f(y) * f(z)) = (f(x) * f(z)) * ((f(x) * f(z)) = 0$ which implies $(x * z) * (y * z) \in kerf$. Similarly $(z * x) * (z * y) \in kerf$. Thus $Kerf$ is a translation ideal. □

Theorem 5.8. *Let X and Y be anti-symmetric BRK-algebra and $f : X \longrightarrow Y$ be a BRK homomorphism. If $I = Kerf$, then $X/I \cong Imf$.*

Proof. As I is a translation ideal of X, X/I is a BRK algebra. Define a mapping $\alpha : X/I \longrightarrow Imf$ by $\alpha(I_x) = f(x)$. Then

(1) α is well defined. Suppose $I_x = I_y$ for some $I_x, I_y \in X/I$. Then
$I_x = I_y \Rightarrow x * y, y * x \in I \Rightarrow f(x * y) = f(y * x) = 0$
$\Rightarrow f(x) * f(y) = f(y) * f(x) = 0 \Rightarrow f(x) = f(y) \Rightarrow \alpha(I_x) = \alpha(I_y)$.
Thus α is well defined.

(2) α a homomorpism. For any $I_x, I_y \in X/I$ we have
$\alpha(I_x * I_y) = \alpha(I_{x*y}) = f(x * y) = f(x) * f(y) = \alpha(I_x) * \alpha(I_y)$.
Therefor α is a homomorphism.

(3) α is injecive. Let $\alpha(I_x) = \alpha(I_y)$ for some $I_x, I_y \in X/I$. Then
$\alpha(I_x) = \alpha(I_y) \Rightarrow f(x) = f(y) \Rightarrow f(x) * f(y) = f(y) * f(x) = 0$
$\Rightarrow f(x * y) = f(y * x) = 0 \Rightarrow x * y, y * x \in I \Rightarrow I_x = I_y$.
Hence α is one to one.

(4) α is onto. Indeed let y be any element in Imf. But then there exists $x \in X$ such that $f(x) = y$. Now $I_x \in X/I$ and

$\alpha(I_x) = f(x) = y$ and hence α is onto.

Hence α is an isomorphism and $X/I \cong Imf$. □

Acknowledgments

The authors thank the referee for his/her valuable comments and suggestions in improving this paper. Moreover the second author would like to thank the authorities of Dr. Lankapalli Bullyya College for providing the necessary facilities to carry out the research.

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