On the length of D-modules over Hyperplane Arrangements in Space

Smegnsh Demelash
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Department of Mathematics

The undersigned hereby certify that we have read and recommended to the graduate program of Addis Ababa University for acceptance of the thesis entitled **On the length of D-modules over Hyperplane Arrangements in Space** by Smegnsh Demelash in partial fulfillment of the requirements for the Degree of Master of Science in Mathematics.

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Advisor: ________________________

Tilahun Ababaw (Ph.D)

Examining committee: ________________________

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Contents

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Abstract

A D-module is a module over a ring of Differential Operators. The major interest of such D-modules is as an approach to the theory of linear partial differential equations. Algebraic D-modules are modules over the Weyl algebra A_n over the field $\mathbb{C}$ of complex numbers. The main purpose of this thesis to work on the A_3-module $\mathbb{C}[x, y, z]_\alpha$, where α is the product of linear forms. In this thesis we computed the decomposition factors and hence the length of A_3 -module $\mathbb{C}[x, y, z]_\alpha$ using the method of partial fractions of rational functions.

Notations

- K denotes a field of characteristic zero.
- $\mathbb{C}$ denotes a field of complex numbers.
- $\mathbb{N}_0$ denotes the set of natural numbers containing zero.
- $\mathbb{Z}$ is the set of integers.
- $K[x_1, x_2, ..., x_n]$ denotes the ring of polynomials in n variables over a field K.
- $\mathbb{C}[X]$ is the ring of polynomials in n variables over $\mathbb{C}$.
- $End_K(K[X])$ denotes the set of endomorphism from $K[X]$ to $K[X]$.
- $c(M)$ Denotes length of an R-module M.

Chapter 1

Introduction

A D-module is a module over a ring of Differential Operators. The major interest of such D-modules is as an approach to the theory of linear partial differential equations. Algebraic D-modules are modules over the Weyl algebra A_n over a field K of characteristic zero.

Let $\alpha_i : \mathbb{C}^n \to \mathbb{C}, i = 1, 2, ..., m.$ such that

$$\alpha_i(x_1, x_2, ..., x_n) = \sum_{j=1}^{n} \alpha_{ij} x_j,$$

$\alpha_{ij} \in \mathbb{C}$ be linear forms. The the localization of $\mathbb{C}[x_1, x_2, ..., x_m]_\alpha$, where $\alpha = \prod_{i=1}^{m} \alpha_i$, is an A_n-module. The A_n-module $\mathbb{C}[x_1, x_2, ..., x_m]_\alpha$ has finite length.

The main purpose of this thesis to work on the A_n-module $\mathbb{C}[x_1, x_2, ..., x_m]_\alpha$, and mainly focuses on composition series and on the computations of the decomposition factors of the A_n module $\mathbb{C}[x_1, x_2, ..., x_m]_\alpha$.

This thesis is divided into two parts. The first part is concerned with preliminary; definitions and basic properties of Weyl algebra and modules over Weyl algebra. We will discuss on the degree of an operator like the degree of polynomial. Most importantly,it is shown that the only invertible elements of A_n are non-zero constant. In the second part we will focus on the computations of the decomposition factors of the A_n module $\mathbb{C}[x_1, x_2, ..., x_m]_\alpha$, especially when $n = 1, 2, 3$ using the idea of partial fractions.

Chapter 2

Preliminaries

In this chapter we will define the Weyl Algebra and present its basic properties. The Weyl algebra is introduced as a ring of operators and it is shown that the Weyl algebra is a simple domain, the only invertible elements of A_n are non-zero constants and see modules over Weyl algebra.

2.1 Definition and some properties of the Weyl Algebra

In this section we will define the Weyl algebra and see some important properties of Weyl algebra.

Definition 1. *An algebra A over a field F is a non-empty set A together with three operations, namely Addition (+),multiplication (.) and scalar multiplication such that the following are satisfied.*

(i) A *is a vector space over* F.

(ii) $(A, +, .)$ *is a ring.*

(iii) If $\alpha \in F$ *and* $a, b \in A$, *then* $\alpha(a.b) = (\alpha.a)b = a(\alpha.b)$

Example 1. *If* K *is a field of characteristic zero, the ring of polynomials* $K[x_1, ..., x_n]$ *is a vector space over* K. *Thus* $K[x_1, ..., x_n]$ *forms a* K*- algebra called polynomial algebra.*

Definition 2. *Let K be a field of characteristic zero. A subalgebra of a $K-$ algebra A is a nonempty subset S of A that is also an algebra of the same type when the algebraic operations on A are restricted to S.*

Let K be a field of characteristic zero and $End_K K[X]$ be the set of endomorphisms of $K[X]$ over K, where $X = (x_1, x_2, ..., x_n)$. The algebra operations in the endomorphism ring; addition and multiplication (in this case it will be composition) of operators are defined as follows:

(i) $\varphi + \theta : K[X] \to K[X]$ by$(\varphi + \theta)(f) = \varphi(f) + \theta(f)$ and

(ii) $(\varphi o \theta)(f) : K[X] \to K[X]$ by $(\varphi \circ \theta)(f) = \varphi(\theta(f))$
where $f \in K[X], \varphi, \theta \in End_K K[X]$.

Here $+$ is addition of endomorphisms and $\circ$ is the composition of endomorphisms.

(iii) $\alpha(\varphi) : K[X] \to K[X]$ by $\alpha(\varphi) = \alpha.\varphi$

Thus, we have the following lemma.

Lemma 1. $(End_K K[X], +, \circ)$ *is a a $K-$algebra.*

Proof. [7] □

Proposition 1. *Let $x_1, x_2, ..., x_n$ and $\partial_1, \partial_2, ..., \partial_n$ be operators on $K[X]$ defined on an element $f \in K[X]$ by:*

$$x_i(f) = x_i \cdot f$$

(read as x_i acts on f)

$$\partial_i f = \frac{\partial f}{\partial x_i}$$

(read as ∂_i acts on f.)
The operators $x_1, x_2, ..., x_n$ and $\partial_{x_1}, \partial_{x_2}, ..., \partial_{x_n}$ are linear operators on $K[X]$.

Proof. Let $f, f_1, f_2 \in K[X]$ and $\alpha \in K$. Then clearly the operators x_i, ∂_i for $i = 1, \cdots, n$ are well-defined. We want to show that the operators x_i, ∂_i for $i = 1, \cdots, n$ are linear operators.

1. Thus

$$\begin{aligned} x_i(f_1+f_2) &= x_i.(f_1+f_2) \\ &= x_i.f_1 + x_i.f_2(\text{ as multiplication is distributive over addition}) \\ &= x_i(f_1)+x_i(f_2). \end{aligned}$$

2. And also

$$\begin{aligned} x_i(\alpha f) &= x_i.(\alpha f) \\ &= \alpha x_i.f \\ &= \alpha x_i f \end{aligned}$$

3. Similaly

$$\begin{aligned} \partial_{x_i}(f_1+f_2) &= \frac{\partial(f_1+f_2)}{\partial x_i} \\ &= \frac{\partial f_1}{\partial x_i} + \frac{\partial(f_2)}{\partial x_i} \\ &= \partial_{x_i}(f_1) + \partial_{x_i}(f_1). \end{aligned}$$

4. and also

$$\begin{aligned} \partial_{x_i}(\alpha f) &= \frac{\partial(\alpha f)}{\partial x_i} \\ &= \frac{\alpha\partial(f)}{\partial x_i} \\ &= \alpha(\partial_{x_i}(f)). \end{aligned}$$

Therefore x_i and ∂_i are linear operators. □

Definition 3. *Let $n \geq 1$. Then the n^{th} Weyl Algebra, A_n is a K subalgebra of $End_K K[X]$ generated by operators $x_1, x_2, ..., x_n$ and $\partial_1, \partial_2, ..., \partial_n$.*

Taking $n = 3$, the 3^{rd} Weyl Algebra A_3 is the K-subalgebra of $End_K(K[x, y, z])$ generated by the operators $x, y, z, \partial_x, \partial_y, \partial_z$.

Proposition 2. *A Weyl algebra A_n is not commutative.*

Proof. Consider the operator $\partial_i x_i$ and acting on an element $f \in K[x]$.

$$\begin{aligned}\partial_i x_i(f) =& \partial_i(x_i(f)) \\ =& \partial_i(x_i.f) \\ =& f.\partial_i(x_i) + x_i\partial_i(f) \\ =& \frac{f.\partial(x_i)}{\partial x_i} + x_i\partial_i(f) \\ =& f + x_i\partial_i(f) \\ =& (1 + x_i\partial_i)(f)\end{aligned}$$

This implies $\partial_i x_i = 1 + x_i\partial_i$.
Thus,∂_i and x_i are not commutating and hence A_n is not commutative. □

We denote the $n-$th Weyl Algebra by

$$A_n = K < x_1, x_2, ..., x_n, \partial_1, \partial_2, ..., \partial_n.$$

Remark 1. *If $P, L \in A_n$, then their commutator is the operator $PL - LP$ and denoted by $[P, L]$.*

In general, the commutators of the generators on $n - th$ Weyl algebra A_n are given by

$$\begin{cases} [\partial_i, \partial_j] = [x_i, x_j] = 0, & \text{if } 1 \leq i, j \leq n \\ [\partial_i, x_j] = 1, & \text{for } i = j \\ [\partial_i, x_j] = 0, & \text{otherwise.} \end{cases}.$$

2.2 Canonical form(as a vector space)

In this section, we will construct a basis of the Weyl algebra A_n (as K-vector space),which is called a **canonical basis** of A_n with respect to this basis every non zero elements of A_n possesses a unique representation which helps us to define degree of any polynomial in A_n.It is easier to describe the canonical basis if we use a multi-index notation.
A multi index notation α an element of $\mathbb{N}^n$: say, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $x^\alpha = x^{\alpha_1}x^{\alpha_2}...x^{\alpha_n}$and also degree of monomial x^α is $\mid \alpha \mid= \alpha_1 + \alpha_2 + ... + \alpha_n$
A pair (α, β) of multi indices in $\mathbb{N}^n$ is it self a multi index in $\mathbb{N}^n \times \mathbb{N}^n$.The factorial of a multi index $\alpha \in \mathbb{N}^n$ is $\alpha! = \alpha_1!\alpha_2!...\alpha_n!$

Lemma 2. *Let $\delta, \beta \in \mathbb{N}^n$ and assume $\mid \delta \mid \leq \mid \beta \mid$. Then*

$$\begin{cases} \partial^\beta x^\delta = \beta!, & \textit{if } \beta = \delta \\ \partial^\beta x^\delta = 0, & \textit{otherwise.} \end{cases}$$

Proof. Assume $\delta = \beta$,then

$$\begin{aligned} \partial^\beta x^\delta =& \partial^\beta x^\beta \\ =&(\partial_1^{\beta_1}\partial_2^{\beta_2}...\partial_n^{\beta_n})(x_1^{\beta_1}x_2^{\beta_2}...x_n^{\beta_n}) \\ =&(\partial_1^{\beta_1}\partial_2^{\beta_2}...\partial_{n-1}^{\beta_{n-1}})\partial_n^{\beta_n}(x_1^{\beta_1}x_2^{\beta_2}...x_n^{\beta_n}) \\ =&(\partial_1^{\beta_1}\partial_2^{\beta_2}...\partial_{n-1}^{\beta_{n-1}})(x_1^{\beta_1}x_2^{\beta_2}...x_{n-1}^{\beta_{n-1}}\partial_n^{\beta_n}x_n^{\beta_n}) \\ =&(\partial_1^{\beta_1}\partial_2^{\beta_2}...\partial_{n-1}^{\beta_{n-1}})(x_1^{\beta_1}x_2^{\beta_2}...x_{n-1}^{\beta_{n-1}})\beta_n! \\ =&(\partial_1^{\beta_1}\partial_2^{\beta_2}...\partial_{n-2}^{\beta_{n-2}})(x_1^{\beta_1}x_2^{\beta_2}...x_{n-2}^{\beta_{n-2}})\beta_{n-1}!\beta_n! \\ &. \\ &. \\ &. \\ =&\beta_1!\beta_2!...\beta_n! \end{aligned}$$

If $\delta \neq \beta$ and $\mid \delta \mid < \mid \beta \mid$,then $\delta_i \leq \beta_i$ for some $i \in \{1, 2, ..., n\}$.Thus,

$$\begin{aligned} \partial^\beta x^\delta =&(\partial_1^{\beta_1}\partial_2^{\beta_2}...\partial_n^{\beta_n})(x_1^{\beta_1}x_2^{\beta_2}...x_n^{\beta_n}) \\ =&0 \end{aligned}$$

This completes the proof. □

Proposition 3. *The set $B = \{x^\alpha\partial^\beta : \alpha, \beta \in \mathbb{N}^n\}$ is a basis of A_n as a vector space over K.*

Proof. To show that elements of B generates the Weyl Algebra as a vector space, consider a monomial on the generators of A_n.If $f \in K[X]$, then using the commutators we have $\partial_i f - f\partial_i = \frac{\partial f}{\partial x_i}$. This implies all powers of $x's$ in an element will be moved to the left of all $\partial's$. Thus, the monomials can be written as a linear combination of elements of B.

Now,it remains to show B is linearly independent.
Let $D \in A_n$ and $D = \sum c_{\alpha\beta}x^\alpha\partial^\beta$ be a finite linear combination.

Suppose $D = 0$, then we want to show $C_{\alpha\beta} = 0$.
Since 0 is an operator for every $f \in K[x], D(f) = 0(f) = 0$.
In particular, for $f = x^\beta = x_1^{\beta_1} x_2^{\beta_2} ... x_n^{\beta_n}$, where $\mid \beta \mid = \beta_1 + \beta_2 + ... + \beta_n$ and $X = (x_1, ..., x_n)$,we have $D(f) = \sum c_{\alpha\beta} x^\alpha \partial^\beta (f) = 0 = 0(f)$

$$
\begin{aligned}
&\Rightarrow \sum c_{\alpha\beta} x^\alpha \partial^\beta (x^\beta) = 0 \\
&\Rightarrow \sum c_{\alpha\beta} x^\alpha \beta! = 0 \text{.........by lemma 2} \\
&\Rightarrow \beta! \sum c_{\alpha\beta} x^\alpha = 0
\end{aligned}
$$

Since $x^\alpha \neq 0$,now we have $c_{\alpha\beta} = 0 \qquad \forall \alpha, \beta \in \mathbb{N}^n$
Thus,$\sum c_{\alpha\beta} x^\alpha \partial^\beta = 0 \Rightarrow c_{\alpha\beta} = 0$.
This implies, B is linearly independent. Therefore, B is a basis of A_n.

□

Definition 4. *The set B is called canonical basis of A_n.*

Example 2. *Find the canonical form of elements of*
$\partial_2^3 x_1 \partial_3 x_3 + x_3 \partial_1 x_1$.

Solution.

$$
\begin{aligned}
\partial_2^3 x_1 \partial_3 x_3 + x_3 \partial_1 x_1 &= \partial_2^3 x_1 (x_3 \partial_3 + 1) + x_3 (x_1 \partial_1 + 1) \\
&= \partial_2^3 (x_1 x_3 \partial_3 + x_1) + x_3 x_1 \partial_1 + x_3 \\
&= x_1 x_3 \partial_2^3 \partial_3 + x_1 \partial_2^3 + x_1 x_3 \partial_1 + x_3
\end{aligned}
$$

2.3 The degree of an operator

A domain is a non zero ring R in which $ab = 0 \Rightarrow a = 0$or $b = 0$, $\forall a, b \in R$. Or, domain is a ring whose only proper two sided ideal is zero.In this section we discussed on degree of an operator will be used to show Weyl algebra is a domain.

Definition 5. *Let $D \in A_n$ such that $D = \sum c_{\alpha\beta} x^\alpha \partial^\beta$.*

$$
deg(D) = \max_{\alpha,\beta | C_{\alpha\beta} \neq 0} (|\alpha| + |\beta|)
$$

The notation deg(D) denotes degree of an operator D
Degree of zero operator(like zero polynomial)is negative infinity.
For example,

. degree of $2x_1^2\partial_1^3 + 6x_1\partial_1^2 + 6\partial_1$ is 5.

. degree of $2x_2^3\partial_1^3 + 6x_3\partial_2^4 + 11\partial_3$ is 6.

Theorem 1. *The degree function satisfies the following properties.For* $D, D' \in A_n$

(1) $\deg(D + D') \leq \max\{\deg D, \deg D'\}$

(2) $\deg(DD') = \deg D + \deg D'$

(3) $\deg[D + D'] \leq \deg D + \deg D' - 2$

Proof. [2]

□

Corollary 1. *The algebra* A_n *is a domain.*

Proof. Let $D, D' \in A_n$ with $D \neq 0, D' \neq 0$. Then we want to show $DD' \neq 0$
Let $\deg D = n$ and $\deg D' = m$ for some positive integers $n, m \geq 0$.
Then $\deg D + \deg D' = m + n \geq 0$. This implies $DD' \neq 0$.

Thus,$DD' = 0 \Rightarrow D = 0$ or $D' = 0$ and hence A_n is a domain. □

Corollary 2. *The only elements of* A_n *that have an inverse are constants.*

Proof. Let $D \in A_n, D \neq 0$ be an invertible element. Then there exists nonzero $D' \in A_n$ such that $DD' = 1$ (an identity operator) This implies $\deg DD' = \deg 1 = 0$ and hence $\deg D + \deg D' = 0 \Rightarrow \deg D = \deg D' = 0$ as D and D' are nonzero operators.

Hence, D is constant.

□

2.4 Modules over the Weyl algebra

In this section we state the correspondence theorem which is important for our work. Also, by writing $\mathbb{C}[x]$ as a polynomial ring $\mathbb{C}[x_1, x_2, ..., x_n]$, where $\mathbb{C}$ is the set of complex number and A_n is a submodule of $End_{\mathbb{C}}\mathbb{C}[x]$ and we define action of elements of A_n to elements of $\mathbb{C}[x]$ to make $\mathbb{C}[x]$ is left A_n-module. Let us start by giving a formal definition of a module.

Definition 6. *Let $(M, +)$ be an abelian group and $(R, +, .)$ be a ring with unity. Then M is called a unitary left R-module if there exists a function*

$$\eta : R \times M \longrightarrow M,$$

where the image of (r, m) is given by $\eta(r, m) = rm$ such that the following conditions are satisfied. For all $x, y \in M$ and all $a, b \in R$:

(i) $a(x + y) = ax + ay$

(ii) $(ab)x = a(bx)$

(iii) $(a + b)x = ax + bx$

(iv) $1x = x$, *where 1 is the multiplicative identity in R.*

The most common examples of modules are vector spaces over certain fields.
Submodules of modules are subsets that are modules with respect to the operations defined in the given module and a formal definition is given below.

Definition 7. *A nonempty subset N of M is called a submodule of M if*

(i) $x, y \in N \Rightarrow x - y \in N.$

(ii) $a \in R, x \in N \Rightarrow ax \in R.$

An R-module M is said to be simple or irreducible if there is no submodule proper submodule other than the zero module; that is, if N is a submodule of M such that $\{0\} \subset N \subset M$, then $\{0\} = N$ or $N = M$. .

Definition 8. *Let M and N be modules over a ring R. A function*

$$f : M \longrightarrow N$$

is said to be an $R-$module homomorphism if it satisfies the following:

(i) $f(x + y) = f(x) + f(y), \forall x, y \in M$

(ii) $f(ax) = af(x)$ $\forall a \in R$ *and* $\forall x \in M$

An $R-$module homomorphism

$$f : M \longrightarrow N$$

is said to be:

(a) monomorphism if it is injective.

(b) epimorphism if it is surjective.

(c) endomorphism if $M = N$.

(d) automorphism if $f : M \longrightarrow M$ is isomorphism.

Theorem 2. *Let M be a unitary R module. If for any $m \in M, m \neq 0, Rm = M$, then M is simple.*

Proof. Suppose $M \neq \{0\}$ and M is cyclic with a nonzero element as its generator, that is, $Rm = M$ for some $m \in M$ and $m \neq 0$. Then consider a nonzero submodule N of M and let $q \in N, q \neq 0$. Since N is a submodule, then $Rq \subset N$. Since M is generated by any nonzero element $q \in M$, we have $Rq = M$. This implies $M \subset N$ and hence $M = N$.
Therefore, M is irreducible(or simple). □

Let M be an R- module, N be a submodule of M and

$$M/N = \{x + N : x \in M\}.$$

Define an R-module structure on M/N by: $(x+N)+(y+N) = (x+y)+N$ and $r(x + N) = rx + N$ for all $x, y \in M$ and all $r \in R$. Then M/N is an R module called the quotient module of M by N.

The map $\pi : M \rightarrow M/N$ defined by $\pi(m) = m + N$ is an R-module homomorphism and it is called the canonical quotient homomorphism.

Let us state the Isomorphism Theorems for modules. All the results are taken from [3].

Theorem 3 (First Isomorphism Theorem)**.** *Let $f : M \rightarrow M'$ be an R-module homomorphism and $N = Ker f$. Then*

$$M/N \cong Im(f).$$

Proof. [3] □

Theorem 4 (Second isomorphism theorem)**.** *Let K, L be R-submodules of an R-module M. Then*

$$K/(K \cap L) \cong (K + L)/L.$$

Proof. [3] □

Theorem 5 (Third isomorphism theorem)**.** *Let $K \subseteq L$ be R-submodules of an R-module M. Then L/K is an R-submodule of M/K and*

$$M/L \cong (M/K)/(L/K).$$

Proof. [3] □

Theorem 6 (Correspondence Theorem)**.** *If R is a ring and N is a submodule of an R module M, then there is a one to one correspondence between the set of all submodules M of containing N and the set of all sub modules of M/N,given by*

$$C \mapsto C/N.$$

Hence every submodule of $M/N$is of the form C/N, where C is a submodule of M contains N.

Proof. [3] □

Having all these as a background information, let us start our objects of interest, the modules of the Weyl Algebra. Now, consider the polynomial ring $\mathbb{C}[x_1, x_2, ..., x_n] = \mathbb{C}[x]$ and the subalgebra A_n of $End_{\mathbb{C}}\mathbb{C}[x_1, x_2, ..., x_n]$. Consider the actions of ∂_i and x_i on $\mathbb{C}[x]$, where

$$x_i : \mathbb{C}[x] \longrightarrow \mathbb{C}[x]$$

given by $x_i f = x_i.f$ and

$$\partial_i : \mathbb{C}[x] \longrightarrow \mathbb{C}[x]$$

given by

$$\partial_i(f) = \frac{\partial f}{\partial_{x_i}}$$

for $i = 1, \cdots, n$.

Proposition 4. $\mathbb{C}[x]$ *is simple left A_n module.*

Proof. First we want to show $\mathbb{C}[x]$ is left A_n module.
We know that $(\mathbb{C}[X], +)$ is an abelian group and $(A_n, +, o)$ is a ring. Let $P, Q \in A_n$ and $f, g \in C[x]$

(i) $P(f+g) = Pf + Pg$ (by linearity of P)

(ii) $(PQ)f = P(Q(f)) = P(Qf)$ (by definition)

(iii) $(P+Q)f = P(f) + Q(f)$ (by definition)

(iv) $1f = f$, for all $f \in C[x]$, where 1 stands for identity operator in A_n.

Moreover for all $i, j = 1, 2, ..., n$:

(a) $[\partial_{x_i}, x_i](f) = (\partial_{x_i} x_i - x_i \partial_{x_i})(f) = 1f = f$

(b) $[\partial_{x_i}, \partial_{x_j}](f) = (\partial_{x_i}\partial_{x_j} - \partial_{x_j}\partial_{x_i})(f) = 0(f) = 0$

(c) $[x_i, x_j](f) = (x_i x_j - x_j x_i)(f) = 0(f) = 0.$

Therefore, $\mathbb{C}[x]$ is a left A_n- module.
Then, it remains to show $\mathbb{C}[x]$ is simple, that is,

$$\mathbb{C}[x] = A_n f.$$

for every nonzero $f \in \mathbb{C}[x]$.
Let $f \in \mathbb{C}[x_1, x_2, ..., x_n]$. Then

$$f = \sum^{\text{finite}} c_{i_1 i_2,\ldots i_n} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}.$$

Consider the operator $\partial_{x_1}^{i_1^*} \partial_{x_2}^{i_2^*} \ldots \partial_{x_n}^{i_n^*} \in A_n$ where $i_1^*, i_2^*, ..., i_n^*$ is maximum degree such that $c_{i_1 i_2,\ldots i_n} \neq 0$. Then

$$\begin{aligned}
\partial_{x_1}^{i_1^*}\partial_{x_2}^{i_2^*}\ldots\partial_{x_n}^{i_n^*} f =& \partial_{x_1}^{i_1^*}\partial_{x_2}^{i_2^*}\ldots\partial_{x_n}^{i_n^*}\Big(\sum_{i_1,i_2,\ldots,i_n=0}^{n} C_{i_1 i_2,\ldots i_n} x_1^{i_1} x_2^{i_2}\ldots x_n^{i_n}\Big)\\
=& \partial_{x_1}^{i_1^*}\partial_{x_2}^{i_2^*}\ldots\Big(\partial_{x_n}^{i_n^*}\Big(\sum_{i_1,i_2,\ldots,i_n=0}^{n} C_{i_1 i_2,\ldots i_n} x_1^{i_1} x_2^{i_2}\ldots x_n^{i_n}\Big)\Big)\\
=& \partial_{x_1}^{i_1^*}\partial_{x_2}^{i_2^*}\ldots\Big(\Big(\sum_{i_1,i_2,\ldots,i_n=0}^{n} C_{i_1 i_2,\ldots i_n} x_1^{i_1} x_2^{i_2}\ldots\partial_{x_n}^{i_n^*}(x_n^{i_n})\Big)\Big)\\
=& \partial_{x_1}^{i_1^*}\partial_{x_2}^{i_2^*}\ldots\Big(\Big(\sum_{i_1,i_2,\ldots,i_n=0}^{n} C_{i_1 i_2,\ldots i_n} x_1^{i_1} x_2^{i_2}\ldots\partial_{x_{n-1}}^{i_{n-1}^*}(x_{n-1}^{i_{n-1}})\Big)\Big) i_n!\\
&.\\
&.\\
&.\\
=& \sum_{i_1,i_2,\ldots,i_n=0}^{n} C_{i_1 i_2,\ldots i_n} i_n! i_{n-1}!\ldots i_1!\\
=& C_{i_1 i_2,\ldots i_n} i_n! i_{n-1}!\ldots i_1!
\end{aligned}$$

$$1 = \frac{1}{C_{i_1 i_2,\ldots i_n} i_n! i_{n-1}!\ldots i_1!}\partial_{x_1}^{i_1^*}\partial_{x_2}^{i_2^*}\ldots\partial_{x_n}^{i_n^*} f \in A_n f$$

This implies $\mathbb{C}[x] = A_n f$ and hence $\mathbb{C}[x]$ is simple A_n-module. □

Theorem 7. *Every simple A_n module is cyclic.*

Proof. Let M be a simple A_n- module. For each $a \in M, a \neq 0$, consider $A_n a$ which is submodule of M. Since, M is simple $A_n a = 0$ or $A_n a = M$. But $A_n a \neq 0$, then $A_n a = M$
Therefore, M is cyclic.

□

Chapter 3

Decomposition Factors of Modules

In this chapter we will see the computations of decomposition factor of modules over the Weyl Algebra in the case for $n = 3$.

3.1 Basic Properties of Modules

Let us see some basic properties of modules.

Definition 9. *Let M be an R module. If $0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$ is a composition series of M, then the set $DF(M) := \{M_i/M_{i-1}\}_{i=1}^n$ of R module is the set of decomposition factors of M.*

We use the idea of decomposition factors of modules to define length of a module.

Definition 10. *Let M be an R module. Then we define the length of M over R by $c(M) = 0 if\ M = \{0\}$ and $c(M) = n$ for $M \neq \{0\}$ if there exists composition series $0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$ such that M_i/M_{i-1} is simple and nonzero.*

Remark 2. *If M is simple, $c(M) = 1$.*

Theorem 8. *Let M be an R module and N be a submodule of M. Then*

(i) $DF(M) = DF(N) \cup DF(M/N)$

(ii) $c(M) = c(N) + c(M/N)$

Proof. Give an R-module M and a submodule N:

(i) Consider the following composition series
$0 = N_0 \subset N_1 \subset \cdots \subset N_n = N$ and $0 = P_0 \subset P_1 \subset \cdots \subset P_m = M/N$
By correspondence theorem we can set $P_i = M_i/N, i = 1, ..., m$.
Put $M_i = q^{-1}(P_i)$, where q is canonical quotient map,$q : M \to M/N$
By second isomorphism theorem we get

$$P_i/P_{i-1} \cong (M_i/N)\big/(M_{i-1}/N) \cong M_i/M_{i-1}$$

and since P_i/P_{i-1} has no non trivial submodule (i.e $P_0 = N = N/N$), we obtain the following composition series

$$0 = N_0 \subset N_1 \subset \cdots \subset N_n = N \subset M_1 \subset \cdots \subset M_m = M.$$

Therefore $DF(M) = DF(N) \cup DF(M/N)$

(ii) Following the proof of (i) we get
$c(M) = c(N) + c(M/N) = n + m$

□

Let M be an R module and let $N_1, ..., N_n$ be submodules of M. The sum of $N_1, ..., N_n$ is the set of all finite sums of elements from the $N_i^{\prime}s$.

$$N_1 + ... + N_n = \{a_1 + ... + a_n | a_i \in N_i, \text{ for all } i = 1, 2, ..., n\}$$

Then $N_1 + \cdots + N_n$ is a submodule of M.

Proposition 5. *Let $N_1, ..., N_n$ be submodules of an R module M. Then the following are equivalent.*

(1) The map $\pi : N_1 \times ... \times N_n \to N_1 + ... + N_n$ by

$$\pi(a_1, ..., a_n) = a_1 + ... + a_n$$

is isomorphism.

(2) $N_j \cap (N_1 + ... + N_{j-1} + N_{j+1} + ... + N_n) = 0, \forall j \in \{1, 2, ..., n\}$

(3) *Every* $x \in N_1 + ... + N_n$ *written uniquely in the form* $a_1 + ... + a_n$ *with* $a_i \in N_i$

Proof. $(1 \Rightarrow 2)$ Suppose for some j,(2) fails to hold.
Let $a_j \in N_j \cap (N_1 + ... + N_{j-1} + N_{j+1} + ... + N_n)$with $a_j \neq 0$.Then

$$a_j = a_1 + ... + a_{j-1} + a_{j+1} + ... + a_n$$

for $a_i \in N_i$

$$\Rightarrow 0 \neq (a_1, ..., a_j, a_{j+1}, ..., a_n) \in ker(\pi)$$

Which is a contradiction to the fact that π is an isomorphism (or injective).
$(2 \Rightarrow 3)$ Assume that (2) holds. If for some $a_i, b_i \in N_i$,we have

$$a_1 + ... + a_n = b_1 + ... + b_n$$

Then for each j we have

$$a_j - b_j = (b_1 - a_1) + ...(b_{j-1} - a_{j-1}) + (b_{j+1} - a_{j+1}) + ... + (b_n - a_n).$$

This implies

$$a_j - b_j \in N_j$$

and

$$(b_1-a_1)+...(b_{j-1}-a_{j-1})+(b_{j+1}-a_{j+1})+...+(b_n-a_n) \in N_1+...+N_{j-1}+N_{j+1}+...+N_n$$

$$\Rightarrow a_j - b_j \in N_j \cap N_1 + ... + N_{j-1} + N_{j+1} + ... + N_n = \{0\}.$$

$$\Rightarrow a_j = b_j$$

$(3 \Rightarrow 1)$ Define the map $\pi : N_1 \times ... \times N_n \to N_1 + ... + N_n$ by

$$\pi(a_1, ..., a_n) = a_1 + ... + a_n.$$

We want to show π is isomorphism.

Let $(a_1, ..., a_n), (b_1, ..., b_n) \in N_1 \times ... \times N_n$ and $\alpha \in R$.

(i) Then

$$\begin{aligned}\pi((a_1, ..., a_n) + (b_1, ..., b_n)) =& \pi(a_1 + b_1, ..., a_n + b_n)\\ =& a_1 + b_1 + ... + a_n + b_n\\ =& a_1 + ... + a_n + b_1 + ... + b_n\\ =& \pi((a_1, ..., a_n)) + \pi((b_1, ..., b_n))\end{aligned}$$

and we also have

$$\begin{aligned}\pi(\alpha(a_1, ..., a_n)) =& \pi(\alpha a_1, ..., \alpha a_n) \\ =& \alpha a_1 + ... + \alpha a_n \\ =& \alpha(a_1 + ... + a_n) \\ =& \alpha\pi(a_1, ..., a_n)\end{aligned}$$

Therefore π is a homomorphism.

(ii) Let $x \in N_1 + ... + N_n$ such that $x = a_1 + ... + a_n$.Then there exists $y = (a_1, ..., a_n) \in N_1 \times ... \times N_n$ such that $\pi(y) = x$.

This implies π is surjective.

(iii) Suppose $\pi(a_1, ..., a_n) = \pi(b_1, ..., b_n)$. Then

$$a_1 + ... + a_n = b_1 + ... + b_n.$$

This implies $a_i = b_i$ for $i = 1, \ldots, n$, because of uniqueness. This implies π is one-to-one.

Therefore π is isomorphism.

□

Definition 11. *If an R module $M = N_1 + ... + N_n$ is the sum of sub modules $N_1, ..., N_n$ of M satisfying the equivalent conditions of the above proposition then M is said to be the direct sum of $N_1, ..., N_n$, written as*

$$M = N_1 \oplus ... \oplus N_n.$$

Now let us consider the decomposition factor of a direct sum of modules.

Theorem 9. *Let M_1, M_2 be submodules of an R module M. Then*

$$c(M_1 \oplus M_2) = c(M_1) + c(M_2).$$

Proof. [6] □

3.2 Decomposition Factors of Modules over the Weyl Algebra

In this section we will compute the decomposition factors of modules over the Weyl algebra in the cases for $n = 1, 2, 3$ over hyperplane arrangements.

Lemma 3. *Let $\alpha_1, \alpha_2, ..., \alpha_{k+1}$ be a nonzero linear forms with $\alpha_1 = \sum_{j=2}^{k+1} c_j \alpha j$. Then we have*

$$\frac{1}{\prod_{j=1}^{k+1} \alpha_j} = \sum_{j=2}^{k+1} \frac{\alpha_j}{\alpha_1^2 \prod_{i=2}^{k+1} \alpha_i}$$

Proof. $\frac{1}{\prod_{j=1}^{k+1} \alpha_j} = \frac{1}{\alpha_1 \prod_{i=2}^{k+1} \alpha_i}$

$$\begin{aligned} &= \frac{\alpha_1}{\alpha_1^2 \prod_{i=2}^{k+1} \alpha_i} \\ &= \sum_{j=2}^{k+1} c_j \frac{\alpha_j}{\alpha_1^2 \prod_{i=2}^{k+1} \alpha_i} \\ &= \sum_{j=2}^{k+1} c_j \frac{1}{\alpha_1^2 \prod_{i=1}^{j-1} \alpha_i \prod_{i=j+1}^{k+1} \alpha_i} \end{aligned}$$

□

Theorem 10. *Let $\alpha = x \prod_{i=2}^{m}(x + i)$. Then the number of decomposition factors of the A_1-module $\mathbb{C}[x]_\alpha$ is $m + 1$.*

Proof. Let $\alpha = x \prod_{i=2}^{m}(x+i)$. Consider the following sequence of A_1-modules,

$$0 \subset \mathbb{C}[x] \subset \mathbb{C}[x]_\alpha.$$

Then by using the concept of decomposition by partial fractions we can write

$$\frac{1}{\alpha} = \frac{1}{x \prod_{i=2}^{m}(x + i)}$$

as follows:

$$\frac{1}{\alpha} = \frac{B_1}{x} + \frac{B_2}{x+2} + \cdots + \frac{B_m}{x+m},$$

where $B_1, B_2, \ldots, B_m$ are constants. This leads us to the following decomposition of the the A_1-module $\mathbb{C}[x]_\alpha$ as a sum of modules:

$$\mathbb{C}[x]_\alpha = \mathbb{C}[x]_x + \mathbb{C}[x]_{x+2} + \cdots + \mathbb{C}[x]_{(x+m)}.$$

Then we have the following decomposition of the quotient A_1- module $\mathbb{C}[x]_\alpha/\mathbb{C}[x]$,

$$\mathbb{C}[x]_\alpha/\mathbb{C}[x] = \oplus_{j=1}^{m} R_{s_j},$$

where R_1 is the A_1-module generated by $e_{s_1} = \frac{1}{x}(\text{mod } \mathbb{C}[x])$ and R_j is the A_1 module generated by $e_{s_j} = \frac{1}{x+j}(\text{mod}\mathbb{C}[x])$ for $j = 2, 3, ..., m$. Then it remain to show that each R_j is simple for $j = 1, 2, \ldots, m$.
Let $t \in \mathbb{C}[x]_x/\mathbb{C}[x]$ and $t \neq 0$. Then

$$t = \frac{f}{x^r} + \mathbb{C}[x],$$

where $f \in \mathbb{C}[x], r \in \mathbb{N}_0$ and we can write t as:

$$t = \frac{\sum_{i=1}^{n} c_i x^i}{x^r} + \mathbb{C}[x],$$

where $c_1, \ldots, c_n \in \mathbb{C}$ and $c_n \neq 0$.
If $n \geq r$, then, using long division, we can write $t = q + \frac{h}{x^r} + \mathbb{C}[x]$, where $q, h \in \mathbb{C}[x]$ and the degree of h is less than r. Therefore $t = \frac{h}{x^r} + \mathbb{C}[x]$.

Therefore we mainly focuss on $n < r$.
Let c_k be nonzero coefficient of the smallest power of x in f. Then $t = \frac{\sum_{i=1}^{n} c_i x^i}{x^r} + \mathbb{C}[x]$.

$$t = (\frac{c_k}{x^r}x^k + \frac{c_{k+1}}{x^r}x^{k+1} + ... + \frac{c_n}{x^r}x^n) + \mathbb{C}[x].$$

Let $D = \frac{1}{c_k}x^{r-(k+1)} \in A_1$. Then we have:

$$\begin{aligned} D(t) =& (\frac{1}{c_k}x^{r-(k+1)})(\frac{c_k}{x^r}x^k + \frac{c_{k+1}}{x^r}x^{k+1} + ... + \frac{c_n}{x^r}x^n) + \mathbb{C}[x] \\ =& \frac{1}{x} + \frac{c_{k+1}}{c_k} + \frac{c_{k+2}}{c_k}x + ... + \frac{c_{m-1}}{c_k}x^{m-1} + \mathbb{C}[x] \\ =& \frac{1}{x} + \mathbb{C}[x] \end{aligned}$$

Now, it remains to show $\frac{1}{x}(\text{mod}\mathbb{C}[x])$ is the generator of the A_1-module $\mathbb{C}[x]_x/\mathbb{C}[x]$. Let's consider $D \in A_1, D = \frac{f\partial_x^k}{k!(-1)^k}$, where $k = r - 1$

$$
\begin{aligned}
D(\frac{1}{x} + \mathbb{C}[x]) =& \frac{f}{k!(-1)^k}(\partial_x^k(\frac{1}{x} + \mathbb{C}[x])) \\
=& \frac{f}{k!(-1)^k}\frac{((-1)^k k!)}{x^{k+1}} + \mathbb{C}[x], \quad \text{where } m = n - k \\
=& \frac{f}{x^{k+1}} + \mathbb{C}[x] \\
=& t
\end{aligned}
$$

Since, t is nonzero arbitrary element of the quotient $\mathbb{C}[x]_x/\mathbb{C}[x]$ and $\frac{1}{x}(\text{mod}\mathbb{C}[x])$ is its generator, then $\frac{1}{x}\text{mod}\mathbb{C}[x]$ is simple A_1 module.
Similarly we can show that the A_1-module

$$\mathbb{C}[x]_{(x+j)}/\mathbb{C}[x]$$

for $j = 2, \cdots, m$ is simple.
Therefore the length of the module $\mathbb{C}[x]_\alpha/\mathbb{C}[x]$ is, $c(\mathbb{C}[x]_\alpha/\mathbb{C}[x]) = m$ and we also know that

$$c(\mathbb{C}[x]_\alpha) = c(\mathbb{C}[x]_\alpha/\mathbb{C}[x]) + c(\mathbb{C}[x]).$$

On the other hand the A_1-module $\mathbb{C}[x]$ is simple and hence its length is 1. Therefore

$$c(\mathbb{C}[x]_\alpha) = c(\mathbb{C}[x]_\alpha/\mathbb{C}[x]) + c(\mathbb{C}[x]) = m + 1.$$

□

Theorem 11. *Let $\alpha = xy\prod_{i=3}^m(x + a_i y)$, where $a_i's$ are nonzero constants and $a_i \neq a_j$ for $i \neq j$. Then the number of decomposition factors of the A_2-module $\mathbb{C}[x, y]_\alpha$ is $2m$.*

Proof. Let $\alpha = xy\prod_{i=3}^m(x + a_i y)$. Using the method of partial fractions we have the following:

(i) $\frac{1}{y(x+a_iy)} = \frac{1}{xy} - \frac{a_i}{x(x+a_iy)}$

(ii) $\frac{1}{(x+a_iy)(x+a_jy)} = \frac{a_j}{(a_j-a_i)x(x+a_jy)} - \frac{a_i}{(a_j-a_i)x(x+a_jy)}$.

Then we have the following sequence of A_2-modules:

$$0 \subset \mathbb{C}[x,y] \subset R_1 \subset R_2,$$

where

$$R_1 = \mathbb{C}[x,y]_x + \mathbb{C}[x,y]_y + \mathbb{C}[x,y]_{(x+a_3y)} + ... + \mathbb{C}[x,y]_{(x+a_my)}$$

and

$$R_2 = \mathbb{C}[x,y]_{xy} + \mathbb{C}[x,y]_{x(x+a_3y)} + ... + \mathbb{C}[x,y]_{x(x+a_my)}.$$

Then the A_2-module

$$R_1/\mathbb{C}[x,y] = \oplus_{j=1}^{m} R_{s_j}$$

where R_{s_1} is the A_2- module generated by

$$e_{s_1} = \frac{1}{x}(\text{mod } \mathbb{C}[x,y]),$$

R_{s_2} is the A_2-module generated by

$$e_{s_2} = \frac{1}{y}(\text{mod } \mathbb{C}[x,y])$$

and R_{s_j} is the A_2- module generated by

$$e_{s_j} = \frac{1}{x+a_jy}(\text{mod } \mathbb{C}[x,y])$$

for $j = 3, ..., m$. To show the above statement is true it suffice to show only for $e_{s_1} = \frac{1}{x}(\text{mod } \mathbb{C}[x,y])$, is generator for $\mathbb{C}[x,y]_x/\mathbb{C}[x,y]$.

Let $f \in \mathbb{C}[x,y], t \in \mathbb{C}[x,y]_x/\mathbb{C}[x,y] = \{\frac{f}{x^\delta} + R_0 | \delta \in \mathbb{N}_0\}, t = \frac{f}{x^r} + \mathbb{C}[x,y], r \in \mathbb{N}_0$. Let $D \in A_2, D = \frac{f\partial_x^m}{m!(-1)^m},$ for $m = r-1$.

$$\begin{aligned} D(\frac{1}{x} + \mathbb{C}[x,y]) =& \frac{f\partial_x^m}{m!(-1)^m}(\frac{1}{x} + \mathbb{C}[x,y]) \\ =& \frac{f}{m!(-1)^m}(\frac{(-1^m m!)}{x^{m+1}} + \mathbb{C}[x,y]) \\ =& \frac{f}{x^r} + \mathbb{C}[x,y] \end{aligned}$$

Thus, $D(\frac{1}{x} + \mathbb{C}[x, y] = t$
Therefore, $\frac{1}{x}\text{mod}\mathbb{C}[x, y]$ is generator.

Then it remains to show that each A_2-module R_j for $j = 1, \cdots, m$ is simple.
From our previous discussions, we know that $\mathbb{C}[x, y]$ is a simple A_2-module and hence its length is 1.

Let $T \in \mathbb{C}[x, y]_x/\mathbb{C}[x, y]$ and $T \neq 0$. Then

$$T = \frac{p(x, y)}{x^r} + \mathbb{C}[x, y]$$

for some $r > 0$ and $p(x, y) \in \mathbb{C}[x, y]$. We can write $p(x, y) = \sum_{k=0}^{m} q_k(y)x^k$, where $q_k(y) \in \mathbb{C}[y]$ for $k = 0, \ldots, m$ and $q_m(y)$ in a nonzero polynomial. Here we assume that r is greater than m. Let $D_1 = x^{r-(k+1)} \in A_2$,where k is smallest power of x. Then

$$D_1(T) = \frac{q_m(y)}{x} + \mathbb{C}[x, y].$$

Let $q_k(y) = \sum_{i=0}^{n} a_i y^i$ and a_n not equal to zero. Then for $D_2 = \frac{\partial_y^n}{n!a_n}$

$$D_2(D_1(T)) = \frac{1}{x} + \mathbb{C}[x, y].$$

This implies the A_2-module R_{s_1} is simple and similarly it can be shown that each R_{s_j} is simple for $j = 2, \ldots, m$.
Therefore the length of the module $R_1/\mathbb{C}[x, y]$ is, $c(R_1/\mathbb{C}[x, y]) = m$ and we know that

$$c(R_1) = c(R_1/\mathbb{C}[x, y]) + c(\mathbb{C}[x, y]).$$

Therefore

$$c(R_1) = c(R_1/\mathbb{C}[x, y]) + c(\mathbb{C}[x, y]) = m + 1.$$

The quotient A_2-module $\mathbb{C}[x, y]_\alpha/R_1 = \oplus_{j=2}^{m} R_j$, where R_2 is the A_2-module generated by $e_{s_2} = \frac{1}{xy}(\text{mod } R_1)$ and R_j is the A_2-module generated by $e_{s_j} = \frac{1}{x+a_j y}(\text{mod } R_1)$ for $j = 3, ..., m$. Then it remains to show that R_{s_j} for $j = 2, \ldots, m$ is simple A_2-module and it is as follows.

Let $T \in \mathbb{C}[x, y]_{xy}/R_1$ and $T \neq 0$. Then

$$T = \frac{p(x, y)}{(xy)^r} + R_1$$

for some $r > 0$ and $p(x, y) \in \mathbb{C}[x, y]$. We can write $p(x, y) = \sum_{k=0}^{m} q_k(y)x^k$, where $q_k(y) \in \mathbb{C}[y]$ for $k = 0, \ldots, m$ and $q_m(y)$ in a nonzero polynomial of y. Here we assume that r is greater than m. Let $D_1 = x^{r-(m+1)} \in A_2$. Then

$$D_1(T) = \frac{q_m(y)}{xy^r} + \mathbb{C}[x, y].$$

Let $q_m(y) = \sum_{i=0}^{n} a_i y^i$ and a_n not equal to zero and and also assume that $n < r$. Then for $D_2 = \frac{y^{r-(n+1)}}{a_n} \in A_2$, we have:

$$D_2(D_1(T)) = \frac{1}{xy} + R_1.$$

Let $D = \frac{f \partial_x^r \partial_y^r}{(r!)^2} \in A_2$. Then

$$D(\frac{1}{xy} + R_1) = \frac{f}{x^r y^r} + R_1.$$

This implies the A_2-module R_{s_1} is simple and similarly it can be shown that each R_{s_j} is simple for $j = 2, \ldots, m$.

This implies $c(\mathbb{C}[x, y]_\alpha / R_1) = m - 1$.

Therefore

$$c(\mathbb{C}[x, y]_\alpha) = c(\mathbb{C}[x, y]_\alpha / R_1) + c(R_1/\mathbb{C}[x, y]) + c(\mathbb{C}[x, y]) = (m-1) + m + 1 = 2m.$$

□

Now, we state and prove the main result of this work.

Theorem 12. *Let $\alpha = xyz \prod_{i=4}^{m}(x + a_i y + b_i z)$ where $a_i, b_i's$ are constants and $(1, a_i, b_i) \neq (1, a_j, b_j)$ for $i \neq j$. Then the number of decomposition factors of the A_3-module $\mathbb{C}[x, y]_\alpha$ is $6m - 10$.*

Proof. Let $\alpha = xyz \prod_{i=4}^{m}(x + a_i y + b_i z)$. Using method of partial fraction we have the following result:

(1)

$$\frac{1}{yz(x + a_i y + b_i z)} = \frac{1}{xyz} - \frac{b_i}{xy(x + a_i y + b_i z)} - \frac{a_i}{xz(x + a_i y + b_i z)}.$$

That is, we can write

$$\frac{1}{yz(x+a_iy+b_iz)}$$

as a linear combinations of

$$\frac{1}{xyz}, \quad \frac{1}{xy(x+a_iy+b_iz)} \text{ and } \frac{1}{xz(x+a_iy+b_iz)}$$

(2)

$$\begin{aligned}\frac{1}{y(x+a_iy+b_iz)(x+a_jy+b_jz)} &= \frac{b_j}{(b_j-b_i)xy(x+a_jy+b_jz)} \\ &-\frac{b_i}{(b_j-b_i)xy(x+a_iy+b_iz)} \\ &-\frac{(b_ja_i-b_ia_j)}{(b_j-b_i)x(x+a_iy+b_iz)(x+a_jy+b_jz)}.\end{aligned}$$

That is, we can write

$$\frac{1}{y(x+a_iy+b_iz)(x+a_jy+b_jz)}$$

as a linear combination of

$$\frac{1}{xy(x+a_jy+b_jz)}, \quad \frac{1}{xy(x+a_iy+b_iz)} \quad \text{and} \quad \frac{1}{x(x+a_iy+b_iz)(x+a_jy+b_jz)}.$$

(3)

$$\begin{aligned}\frac{1}{z(x+a_iy+b_iz)(x+a_jy+b_jz)} &= \frac{a_j}{(a_j-a_i)xz(x+a_jy+b_jz)} \\ &+\frac{a_i}{(a_j-a_i)xz(x+a_iy+b_iz)} \\ &+\frac{(a_jb_i-a_ib_j)}{(a_j-a_i)x(x+a_iy+b_iz)(x+a_jy+b_jz)}.\end{aligned}$$

That is,

$$\frac{1}{z(x+a_iy+b_iz)(x+a_jy+b_jz)}$$

is written as a linear combinations of:

$$\frac{1}{xz(x+a_jy+b_jz)}, \quad \frac{1}{xz(x+a_iy+b_iz)} \text{ and } \frac{1}{x(x+a_iy+b_iz)(x+a_jy+b_jz)}.$$

Then we have the following sequence of A_3-modules.

$$0 \subset \mathbb{C}[x,y,z] \subset R_1 \subset R_2 \subset R_3 := \mathbb{C}[x,y,z]_\alpha,$$

where

$$R_1 = \mathbb{C}[x,y,z]_x + \mathbb{C}[x,y,z]_y + \mathbb{C}[x,y,z]_z + \mathbb{C}[x,y,z]_{(x+a_4y+b_4z)} + \cdots + \mathbb{C}[x,y,z]_{(x+a_my+b_mz)},$$

$$\begin{aligned} R_2 \;=\;& \mathbb{C}[x,y,z]_{xy} + \mathbb{C}[x,y,z]_{xz} + \mathbb{C}[x,y,z]_{x(x+a_4y+b_4z)} + \cdots \\ &+ \mathbb{C}[x,y,z]_{x(x+a_my+b_mz)} + \mathbb{C}[x,y,z]_{yz} + \mathbb{C}[x,y,z]_{y(x+a_4y+b_4z)} + \cdots \\ &+ \mathbb{C}[x,y,z]_{y(x+a_my+b_mz)} + \mathbb{C}[x,y,z]_{z(x+a_4y+b_4z)} \cdots \\ &+ \mathbb{C}[x,y,z]_{z(x+a_my+b_mz)} \end{aligned}$$

and

$$\begin{aligned} R_3 \;=\;& \mathbb{C}[x,y,z]_{xyz} + \mathbb{C}[x,y,z]_{xy(x+a_4y+b_4z)} + \cdots + \mathbb{C}[x,y,z]_{xy(x+a_my+b_mz)} \\ & \mathbb{C}[x,y,z]_{xy(x+a_4y+b_4z)} + \cdots + \mathbb{C}[x,y,z]_{xy(x+a_my+b_mz)} \\ & \mathbb{C}[x,y,z]_{xz(x+a_4y+b_4z)} + \cdots + \mathbb{C}[x,y,z]_{xz(x+a_my+b_mz)} \end{aligned}$$

Let $\alpha = \alpha_1\alpha_2...\alpha_m$.Where $\alpha_1 = x, \alpha_2 = y, \alpha_3 = z, \alpha_j = x + a_jy + b_jz$, for $j = 4, 5, ..., m$
We want to show $\frac{1}{\alpha_i}\text{mod}R_0$ generates $\mathbb{C}[x,y,z]_{\alpha_i}/R_0, i = 1, 2, ..., m$.
Let $f \in \mathbb{C}[x,y,z], t \in \mathbb{C}[x,y,z]_{\alpha_1}/R_0 = \{\frac{f}{x^\beta} + R_0 | \beta \in \mathbb{N}_0\}, t = \frac{f}{x^r} + R_0, r \in \mathbb{N}_0$,where $R_0 = \mathbb{C}[x,y,z]$ Let $D \in A_3, D = \frac{f\partial_x^m}{m!(-1)^m}$, for $m = r - 1$.

$$\begin{aligned} D(\frac{1}{x} + R_0) =& \frac{f\partial_x^m}{m!(-1)^m}(\frac{1}{x} + R_0) \\ =& \frac{f}{m!(-1)^m}(\frac{(-1^m m!)}{x^{m+1}} + R_0) \\ =& \frac{f}{x^r} + R_0 \end{aligned}$$

Thus, $D(\frac{1}{x}+R_0)=t$
Therefore, $\frac{1}{x}\text{mod}R_0$ is a generator.
Since, each α_i is linear,we can show also for others for $k=2,3,...,m$ simply by substitution.
Therefore $\frac{1}{\alpha_k}\text{mod}R_0$ generates $\mathbb{C}[x,y,z]_{\alpha_k}/R_0$.
Then the A_3 module

$$R_1/R_0=\oplus_{j=1}^{m}R_{s_j},$$

where R_{s_1} is the A_3 module generated by

$$e_{s_1}=\frac{1}{x}\text{mod}R_0$$

R_{s_2} is A_3 module generated by

$$e_{s_2}=\frac{1}{y}\text{mod}R_0$$

R_{s_3} is A_3 module generated by

$$e_{s_3}=\frac{1}{z}\text{mod}R_0$$

and R_{s_j} is A_3 module generated by

$$e_{s_j}=\frac{1}{x+a_jy+b_jz}\text{mod}R_0$$

for $j=4,5,...,m$.Then it remains to show $\mathbb{C}[x,y,z]_{\alpha_i}/R_0$ is simple.
Let $P\in\mathbb{C}[x,y,z]_{\alpha_i}/R_0$ and $P\neq 0$.Then

$$P=\frac{f(x,y,z)}{x^r}+R_0$$

for some $r\geq 0$ and $f(x,y,z)\in\mathbb{C}[x,y,z]$. Then we can write $f(x,y,z)=\sum_{k=0}^{n}Q_k(y,z)x^k$ where $0\neq Q_k(y,z)\in\mathbb{C}[y,z]$, for $k=0,1,...,n$.
Let's assume for $r\geq n$. Let $D'=x^{r-(i+1)}$ where i is the smallest power of x appearing in f with non zero coefficient.Then

$$D'(P)=\frac{Q_i(y,z)}{x}+R_0$$

Let $Q_i(y,z) = \sum_{j=0}^{m} H_j(z)y^j$, where $H_j(z) \in \mathbb{C}[z], for j = 0,1,...,m$. Then for $D'' = \frac{\partial_y^m}{m!}$

$$D''(D'(P)) = \frac{H_j(z)}{x} + R_0$$

Let $H_j(z) = \sum_{k=0}^{l} C_k z^k$. Also for $D''' = \frac{\partial_z^l}{C_l l!}$

$$D'''(D''(D'))(P) = \frac{1}{x} + R_0$$

The A_3 module R_{s_1} is simple we can also show that each R_{s_j} is simple for $j = 2,3,...m$. Thus, the length of R_1/R_0 is m.Since

$$c(R_1) = c(R_1/R_0) + c(R_0)$$

we have

$$c(R_1) = m + 1 \qquad (1)$$

Consider $\mathbb{C}[x,y,z]_{xy}/R_1 = \{\frac{f}{(xy)^\gamma} + R_1 | f \in \mathbb{C}[x,y,z], \gamma \in \mathbb{N}_0\}$. Let $t = \frac{f}{(xy)^r} + R_1, r \in \mathbb{N}_0$ and let $D \in A_3, D = \frac{f\partial_y^m \partial_x^m}{m!^2(-1)^{2m}}$, for $m = r-1$.

$$\begin{aligned} D(\frac{1}{xy} + R_1) =& \frac{f\partial_y^m \partial_x^m}{m!^2(-1)^{2m}}(\frac{1}{xy} + R_1) \\ =& \frac{f\partial_y^m}{m!^2(-1)^{2m}}(\frac{(-1^m m!)}{yx^{m+1}} + R_1) \\ =& \frac{f}{y^r x^r} + R_1 \end{aligned}$$

Thus, $D(\frac{1}{xy} + R_1) = t$

Therefore $\frac{1}{xy}\text{mod}R_1$ is a generator.

Since, each α_i is linear,we can get $D \in A_3$ and show also for others for $k = 2,3,...,m$ simply by substitution.

Then the quotient module $R_2/R_1 = (\oplus_{j=2}^{m} R_{s_j}) \oplus (\oplus_{j'=3}^{m} R_{s'_{j'}}) \oplus (\oplus_{j''=4}^{m} R_{s''_{j''}})$ where R_{s_2} is A_3 module generated by

$$e_{s_2} = \frac{1}{xy}\text{mod}R_1$$

R_{s_3} is A_3 module generated by

$$e_{s_3} = \frac{1}{xz}\text{mod}R_1$$

R_{s_j} is A_3 module generated by

$$e_{s_j} = \frac{1}{x(x + a_j y + b_j z)} \text{mod} R_1$$

for $j = 4, 5, ..., m$,

$R_{s'_3}$ is A_3 module generated by

$$e_{s'_3} = \frac{1}{yz} \text{mod} R_1$$

$R_{s'_{j'}}$ is A_3 module generated by

$$e_{s'_{j'}} = \frac{1}{y(x + a_{j'} y + b_{j'} z)} \text{mod} R_1$$

for $j' = 4, 5, ..., m$,
and

$R_{s''_{j''}}$ is A_3 module generated by

$$e_{s''_{j''}} = \frac{1}{z(x + a_{j''} y + b_{j''} z)} \text{mod} R_1$$

for $j'' = 4, 5, ..., m$,
Now it remains to show $\mathbb{C}[x, y, z]_{xy}/R_1$ is simple. Let $T \in \mathbb{C}[x, y, z]_{xy}/R_1$. Then $T = \frac{f}{(xy)^r} + R_1, f \in \mathbb{C}[x, y, z], r \in \mathbb{N}_0$. Then we can write $f(x, y, z) = \sum_{k=0}^{n} Q_k(y, z)x^k$ where $0 \neq Q_k(y, z) \in \mathbb{C}[y, z]$, for $k = 0, 1, ..., n$.
Let's assume for $r \geq n$. Let $D_1 = x^{r-(i+1)}$ where i is the smallest power of x in f with non zero coefficient.Then

$$D_1(T) = \frac{Q_i(y, z)}{xy^r} + R_1$$

Let $Q_i(y, z) = \sum_{k=0}^{m} H_k(z)y^k$, where $H_k(z) \in \mathbb{C}[z], fork = 0, 1, ..., m$. Then for $D_2 = y^{r-(j+1)}$ where j is also the smallest power of y in f with non zero coefficient.

$$D_2(D_1(T)) = \frac{H_j(z)}{xy} + R_1$$

Let $H_j(z) = \sum_{k=0}^{l} C_k z^k$. Also for $D_3 = \frac{\partial_z^l}{C_l l!}$

$$D_3(D_2(D_1))(T) = \frac{1}{xy} + R_1$$

The A_3 module R_{s_1} is simple we can show also for each R_{s_j} for $j = 2, 3, ..., m$. Similarly,its true for $R_{s'_{j'}}$ and $R_{s''_{j''}}$ for $j' = 3, ..., m$, and $j'' = 4, ..., m$. Thus the length is given by:

$$\begin{aligned} c(R_2/R_1) =& c((\oplus_{j=2}^{m} R_{s_j}) \oplus (\oplus_{j'=3}^{m} R_{s'_{j'}}) \oplus (\oplus_{j''=4}^{m} R_{s''_{j''}})) \\ =& c(\oplus_{j=2}^{m} R_{s_j}) + c(\oplus_{j'=3}^{m} R_{s'_{j'}}) + c(\oplus_{j''=4}^{m} R_{s''_{j''}}) \\ =& (m-1) + (m-2) + (m-3) = 3m - 6 \end{aligned} \tag{2}$$

Consider $\mathbb{C}[x, y, z]_{xyz}/R_2 = \{\frac{f}{(xyz)^\gamma} + R_2 | f \in \mathbb{C}[x, y, z], \gamma \in \mathbb{N}_0\}$. Let $F = \frac{f}{(xyz)^r} + R_2, r \in \mathbb{N}_0$. Let $D \in A_3, D = \frac{f\partial_z^m \partial_y^m \partial_x^m}{m!^3(-1)^{3m}},$ for $m = r - 1$.

$$\begin{aligned} D(\frac{1}{xyz} + R_2) =& \frac{f\partial_z^m \partial_y^m \partial_x^m}{m!^3(-1)^{3m}} (\frac{1}{xyz} + R_2) \\ =& \frac{f\partial_z^m \partial_y^m}{m!^3(-1)^{3m}} (\frac{(-1^m m!)}{x^{m+1}yz} + R_2) \\ =& \frac{f\partial_z^m}{m!^3(-1)^{3m}} (\frac{(-1^{2m} m!^2)}{x^{m+1}y^{m+1}z} + R_2) \\ =& \frac{f}{m!^3(-1)^{3m}} (\frac{(-1^{3m} m!^3)}{x^{m+1}y^{m+1}z^{m+1}} + R_2) \end{aligned}$$

Thus, $D(\frac{1}{xyz} + R_2) = F$
Therefore $\frac{1}{xyz} \mathrm{mod} R_2$ is generator.

Then the quotient module $R_3/R_2 = (\oplus_{j=2}^{m} R_{s_j}) \oplus (\oplus_{j'=3}^{m} R_{s'_{j'}})$ where R_{s_2} is A_3 module generated by

$$e_{s_2} = \frac{1}{xyz} \mathrm{mod} R_2$$

R_{s_j} is A_3 module generated by

$$e_{s_j} = \frac{1}{xy(x + a_j y + b_j z)} \mathrm{mod} R_2$$

for $j = 4, 5, ..., m$,

$R_{s'_{j'}}$ is A_3 module generated by

$$e_{s'_{j'}} = \frac{1}{xz(x + a_{j'} y + b_{j'} z)} \mathrm{mod} R_2$$

for $j' = 4, 5, ..., m$,

Now we want to show $\mathbb{C}[x, y, z]_{xyz}/R_2$ is simple.
For $f \in \mathbb{C}[x, y, z]$ can be written as $f(x, y, z) = \sum_{k=0}^{n} A_k(y, z)x^k$. Let $B \in \mathbb{C}[x, y, z]_{xyz}/R_2$. Then $B = \frac{f(x,y,z)}{(xyz)^r} + R_2$.
Let for $D_{1'} = x^{r-(i+1)}$, where i is the smallest power of x with non zero coefficient appearing in f.Then

$$D_{1'}(B) = \frac{A_i(y, z)}{x(yz)^r} + R_2$$

where $A_i(y, z) = \sum_{t=0}^{m} V_t y^t$ Let for $D_{2'} = y^{r-(j+1)}$, where j is the smallest power of y with non zero coefficient appearing in A_i.Then

$$D_{2'}(D_{1'}(B)) = \frac{V_i(z)}{xy(z)^r} + R_2$$

where $V_i(z) = \sum_{t=0}^{l} C_t z^t$. And for Let for $D_{3'} = \frac{z^{r-(k+1)}}{C_k}$, where k is the smallest power of z with non zero coefficient appearing in V_i.Then

$$D_{3'}(D_{2'}(D_{1'}(B))) = \frac{1}{xyz} + R_2$$

Thus $\mathbb{C}[x, y, z]_{xyz}/R_2$ is simple.
The quotient A_3-module

$$R_3/R_2 = (\oplus_{j=3}^{m} R_{s_j}) \oplus (\oplus_{j=4}^{m} R_{s'_j}),$$

where R_{s_3} is the A_3- module generated by $e_{s_3} = \frac{1}{xyz}(\text{mod } R_2)$, R_{s_j} is the A_3-module generated by $e_{s_j} = \frac{1}{xy(x+a_jy+b_jz)}(\text{mod } R_2)$, R_{s_3} is the A_3-module generated by $e_{s'_j} = \frac{1}{xz(x+a_jy+b_jz)}(\text{mod } R_2)$ for $j = 3, \ldots, m$ and each of these modules are simple.
Hence, the length of the the A_3-module R_3/R_2 is given by:

$$c(R_3/R_2) = c(\oplus_{j=3}^{m} R_{s_j}) + c(\oplus_{j=4}^{m} R_{s'_j}) = (m-2) + (m-3) = 2m-5 \qquad (3).$$

Therefore,from (1), (2), (3) the length of the A_3-module $\mathbb{C}[x, y, z]_\alpha$, where

$$\alpha = xyz \prod_{i=4}^{m} (x + a_i y + c_i z)$$

is given by

$$\begin{aligned} c(\mathbb{C}[x, y, z]_\alpha) &= c(R_3/R_2) + c(R_2/R_1) + c(R_1/\mathbb{C}[x, y, z]) + c(\mathbb{C}[x, y, z]) \\ &= (2m-5) + (3m-6) + m + 1 \\ &= 6m - 10. \end{aligned}$$

□

Example 3. *Let* $\{x, y, z, x + a_1y + b_1z\}$ *be linear forms.Then* $\alpha = xyz(x + a_1y + b_1z)$*. Calculate length of* $\mathbb{C}[x, y, z]_\alpha$*.*

Solution. *Consider the following sequence of* A_3 *submodules of* $\mathbb{C}[x, y, z]_\alpha$*.*

$$0 \subset \mathbb{C}[x, y, z]_x \subset R_1\big(= \mathbb{C}[x, y, z]_x + \mathbb{C}[x, y, z]_y + \mathbb{C}[x, y, z]_z + \mathbb{C}[x, y, z]_{x+a_1y+b_1z}\big)$$

$$\subset R_2\big(= \mathbb{C}[x, y, z]_{xy} + \mathbb{C}[x, y, z]_{xz} + \mathbb{C}[x, y, z]_{x(x+a_1y+b_1z)} + \mathbb{C}[x, y, z]_{yz} + \mathbb{C}[x, y, z]_{y(x+a_1y+b_1z)} + \mathbb{C}[x, y, z]_{z(x+a_1y+b_1z)}\big)$$

$$\subset R_3\big(= \mathbb{C}[x, y, z]_{xyz} + \mathbb{C}[x, y, z]_{xy(x+a_1y+b_1z)} + \mathbb{C}[x, y, z]_{xy(x+a_1y+b_1z)}\big)$$

$$DF(\mathbb{C}[x, y, z]_\alpha) = D(R_3/R_2 \cup R_2/R_1 \cup R_1/\mathbb{C}[x, y, z] \cup \mathbb{C}[x, y, z])$$

$$c(\mathbb{C}[x, y, z]_\alpha) = c(R_3/R_2) + c(R_2/R_1) + c(R_1/\mathbb{C}[x, y, z]) + c(\mathbb{C}[x, y, z])$$

$$= 3 + 6 + 4 + 1 = 14$$

However,by using result above we can calculate length of $\mathbb{C}[x, y, z]_\alpha$*. which is :*
$m = 4$ *substitute to the equation* $6m - 10$*.Then you get result 14.*

3.3 Conclusion

In our discussions, given the A_n-module $\mathbb{C}[x_1, x_2, ..., x_n]_\alpha$, where A_n is the n^{th} Weyl Algebra and α is a product of linear forms we have the following main points.

- Every simple module is cyclic.
- If every nonzero element of a module M is generator then M is simple.
- The length of simple module is one.
- The A_n-module $\mathbb{C}[x_1, x_2, ..., x_n]_\alpha$ has finite length.
- The number of decomposition factors of A_3-module $\mathbb{C}[x, y, z]_\alpha$, where $\alpha = \alpha_1\alpha_2...\alpha_m$ and each α_i for $i = 1, \ldots, n$ is a linear form is:

$$c(\mathbb{C}[x,y,z]_\alpha) = \begin{cases} m+1, & \text{if } n = 1 \\ 2m, & \text{if } n = 2 \\ 6m-10, & \text{for} n = 3. \end{cases}$$

For future work we compute the number of decomposition factors of the A_n-module $\mathbb{C}[x_1, x_2, ..., x_n]_\alpha$, where $\alpha = \alpha_1\alpha_2...\alpha_m$ and each α_i is a linear combination of $x_j's$ for $i = 1, 2, ..., m$, $j = 1, 2, ..., n$. And also, we want to associate the number of decomposition factors of the module and with the number of no broken circuits of the hyperplane arrangements determined by the linear forms.

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