A Hecke Correspondence for Automorphic Integrals with Infinitely Many Poles

by

Haider Ebrahim Yesuf

Advisers: Professor Abdulkadir Hassen
and
Dr. Berhanu Bekele

A thesis submitted to
The Department of Mathematics
Presented in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy(Mathematics)

Department of Mathematics
Addis Ababa University
May 17, 2017

A Hecke Correspondence for Automorphic Integrals With Infinitely Many Poles

haider_ebrahim@yahoo.com
May, 2017

Declaration

I, Haider Ebrahim Yesuf, with student number $GSR/2972/05$, hereby declare that this thesis is my own work and that it has not been previously submitted for assessment or completion of any post graduate qualification to another university or for another qualification.

____________________________ Date ________________

Haider Ebrahim Yesuf

Addis Ababa University
Department of Mathematics

This is to certify that the thesis prepared by Haider Ebrahim Yesuf, entitled: **A Hecke Correspondence for Automorphic Integrals with Infinitely Many Poles** and submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy(in Mathematics) compiles with the regulations of the University and meets the accepted standards with respect to originality and quality.

Date: May 17, 2017

Signed by the Examining Committee:

External Examiner:
Prof. Hieu D. Nguyen ______________________________

Internal Examiner:
Dr. Hunduma Legesse Geleta ______________________________

Academic Supervisors:
1. Prof. Abdulkadir Hassen ______________________________

2. Dr. Berhanu Bekele ______________________________

Chairperson:
Dr. Tadesse Abdi Belete ______________________________
(Head, Department of Mathematics, Addis Ababa University, Ethiopia)

I dedicated this thesis to:

my father **Ebrahim Yesuf Mohammed**

and

my mother **Zemezem Aliye Kereta**

Abstract

A Hecke Correspondence for Automorphic Integrals With Infinitely Many Poles
Haider Ebrahim Yesuf
Addis Ababa University, 2017

In 1936 Hecke proved a correspondence theorem between Dirichlet series with functional equations and automorphic forms with certain growth conditions. In this correspondence, the Dirichlet series has at most simple poles at 0 and k. Since Hecke's original paper, many authors have generalized the correspondence theorem in different directions.

Among the generalizations, we shall be interested in is the one by Salomon Bochner in 1951. In Bochner's version, the correspondence is between automorphic integrals with finite log–polynomial sum and Dirichlet series with a functional equation. The most important feature of this generalization is the presence of the log–polynomial sum. Here a log-polynomial sum is a sum of the form $q(z) = \sum_{l=1}^{n} z^{\beta_l} \sum_{j=0}^{m(l)} \delta(l,j)(\log\, z)^j$, the coefficients, $\delta(l,j)$ and β_l are complex numbers, n, l, $m(l)$, and j are non negative integers. In Bochner's version, the Dirichlet series has a pole of order $m(l)+1$ at β_l.

Austin Daughton in 2012, extended Bochner's result to the Dirichlet series with infinitely many poles for the theta group and for weight $k \geq 0$.

This thesis will extend the Bochner correspondence theorem between automorphic integrals with infinite log-polynomial period function and of weight arbitrary real k on Hecke group which is generated by $S_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \quad and \quad T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. The theta group correspond to $\lambda = 2$. We will deal with the cases when $\lambda > 2$ and $\lambda = 2\cos\pi/p,\ p \in\ \mathbb{Z}$ and $p \geq 3$.

Acknowledgement

I am proud to express my deepest gratitude and deep sense of respect to my adviser, Professor Abdulkadir Hassen of Department of Mathematics, Rowan University, USA for his constant supervision, continuous inspiration, constructive comments and encouragement during my research work. He taught me several courses and introduced me Analytic Number Theory.

I express my sincere appreciation to my second adviser Dr. Berhanu Bekele who gave me the opportunity to join the Ph.D program and provide me valuable advises.

I want to thank Dr. Seid Mohhamed and Dr. Mengistu Goa they helped me during admission for Ph.D study and their valuable advises.

I want to thank Dr. Tadesse Abdi, Dr. Zelalem Teshome and Dr. Addisalem Abathun for their administrative support.

I want to thank Arba Minch University, Addis Ababa University, Department of Mathematics and ISP program of Sweden for their finical support.

I want to express my deep appreciation to all Mathematics Department staff of Addis Ababa University, for their comments and suggestions during seminar classes. In particular, to Dr. Hunduma Legesse and Dr. Berhanu Bekele.

I want thank you all my colleague Ph.D students, in particular, students of 2012/2013 batch and my friend Dr. Bogale Gebermariam of Arba Minch University for their help and moral support.

I would like to express my last but not least profound grateful to my father Ebrahim Yesuf, to my mother Zemezem Aliye and all my brothers and sisters for their love and support.

Thank you so much !!
Haider Ebrahim Yesuf
Addis Ababa, Ethiopia.

Contents

Notations

We will use the following notation.

The set of integers, the set of real numbers and the set of complex numbers are denoted by $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ respectively.

$$GL_2(\mathbb{R}) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,\ b,\ c,\ d\ \in\ \mathbb{R} \text{ and } ad - bc \neq 0 \right\}$$

$$SL_2(\mathbb{R}) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,\ b,\ c,\ d\ \in\ \mathbb{R} \text{ and } ad - bc = 1 \right\}$$

$\Gamma(1) \equiv SL_2(\mathbb{Z})$, the modular group

$G(\lambda) \equiv \left\langle \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$, for $\lambda > 0$, the Hecke groups

$\Gamma_\theta \equiv G(2)$, the theta group

$\mathcal{H} \equiv \{z \in \mathbb{C} : y = \Im(z) > 0\}$, the upper half-plane

$\mathcal{R}_\lambda \equiv \{z \in \mathcal{H} | |z| > 1, |\Re(z)| < \lambda/2\}$, the fundamental region for discrete Hecke groups $G(\lambda)$

$\mathcal{P} \equiv \{f : f$ is holomorphic in $\mathcal{H} \ni |f(z)| \leq K(|z|^\rho + y^{-\sigma}),\ \ y = \Im z > 0$, for some constants $K, \rho, \sigma > 0\}$

$\Gamma_\infty \equiv \langle S_\lambda \rangle$, where $S_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, $\lambda > 0$, the stabilizer of $i\infty$

$\Gamma_\lambda \equiv G(\lambda)$, for $\lambda = 2\cos\pi/p$, $p \in \mathbb{Z}$, $p \geq 3$

Chapter 1

Introduction

1.1 Definitions and Basic Concepts

The set of 2×2 matrices with real entries and nonzero determinant forms a group, and is denoted by $GL_2(\mathbb{R})$. $SL_2(\mathbb{R})$ is defined to be the subgroup of $GL_2(\mathbb{R})$ consisting of matrices of determinant 1.

We let $SL_2(\mathbb{R})$ act on the upper half-plane $\mathcal{H}$ by linear fractional transformations:

$$Mz = \frac{az+b}{cz+d}, \qquad \text{for } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}). \tag{1.1}$$

With this interpretation an element M is identified with its negative, $-M$. Note that $M \in SL_2(\mathbb{R})$ if and only if M preserves $\mathcal{H}$.

Linear fractional transformation are classified by their fixed points. The trace of a linear transformation M in (1.1) is given by $tr(M) = a + d$. Linear transformation (1.1) is called parabolic, elliptic, or hyperbolic if $|tr(M)| = |a + d|$ is equal to, less than, or greater than 2, respectively.

Let Γ be a subgroup of $SL_2(\mathbb{R})$. We say $\alpha \in \overline{\mathcal{H}}$ is a limit point of the group Γ if there is a $z \in \overline{\mathcal{H}}$ and a sequence of distinct $M_n \in \Gamma$ such that $M_n z \to \alpha$. If α is not a limit point, we say that it is an ordinary point of Γ. Γ is called a discontinuous group if and only if it has an ordinary point. A group is called *discrete* if it contains no convergent sequences of distinct matrices. The set of limit points of the full modular group $\Gamma(1)$(subgroup of $SL_2(\mathbb{R})$ its entries are integers) is $\mathbb{R}$.

A group Γ is discontinuous if and only if it is discrete (see *p.* 13, [15]). For a discrete group, if the set of limit points is all real number it is called group of *the first kind* or *horocyclic group* otherwise it is the second kind.

We call a group Γ an H–group if

i) Γ is finitely generated,

ii) Γ is discrete but discontinuous at no point of the real line,

iii) Γ contains translation.

We say that $z_1,\, z_2 \in \mathbb{C}$ are Γ- equivalent if and only if there is an $M \in \Gamma$ such that $Mz_1 = z_2$. This is an equivalence relation, which therefore partitions $\mathcal{H}$ into disjoint equivalence classes or orbits.

An orbit of z is the set consisting of all points that are equivalent to z. The orbit of z is denoted by Γz. Thus, $\Gamma z = \{Vz : V \in \Gamma\}$. An orbit consists entirely of real points or entirely of non-real points.

Definition 1.1.1. *Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$. An open subset R_Γ of $\mathcal{H}$ is called* a fundamental region *of Γ if it has the following two properties:*

1. *No two distinct points of R_Γ are equivalent under Γ.*
2. *If $\tau \in \mathcal{H}$ there is a point τ' in the closure of R_Γ such that τ' is equivalent to τ under Γ.*

A parabolic point (or parabolic cusp) for Γ in is any point $q \in \mathbb{R} \cup \{i\infty\}$ such that q is in the closure of R_Γ with respect to the topology of the Riemann sphere and q is fixed by a non identity parabolic transformation in Γ.

For $z \in \mathbb{C} \setminus \{0\}$ we fix the branch of $(cz+d)^k$ by convention to be $-\pi \leq \arg\, z < \pi$ and define

$$(cz+d)^k = |cz+d|^k \exp\{ki \arg(cz+d)\},$$

where k, c, d are real numbers.

$\Gamma(1)$ is generated by $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We note that

$$T^2 = (ST)^3 = I.$$

In general, the *Hecke groups*, $G(\lambda)$, is generated by

$$S_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \text{ and } T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{for } \lambda > 0.$$

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $G(\lambda)$ acts on $\mathcal{H}$ as a Möbius transformation: $Mz = (az+b)\,/\,(cz+d)$. $\Gamma(1) = G(1)$ and the theta group $\Gamma_\theta = G(2)$ are two important examples of Hecke groups. In 1936, Hecke in [4] proved the only values of λ, for which the Hecke groups discrete are :

$$\lambda = 2\cos \pi/p,\ \ p \in \mathbb{Z},\ p \geq 3 \tag{1.2}$$

and

$$\lambda \geq 2. \tag{1.3}$$

If $\lambda \geq 2, G(\lambda)$ has the single relation

$$T^2 = I. \tag{1.4}$$

On the other hand for λ given by (1.2), there is the second relation, namely

$$(S_\lambda T)^p = I. \tag{1.5}$$

When $G(\lambda)$ is discrete, the set

$$\mathcal{R}_\lambda = \{z \in \mathcal{H} \| z| > 1,\ |\Re(z)| < \lambda/2\} \tag{1.6}$$

is a fundamental region for $G(\lambda)$, [see for the proof [13]]. When $\lambda > 2$, $\mathcal{R}_\lambda$ has "free sides", and thus has infinite hyperbolic area.

Suppose Γ a discrete group and contain translation (in particular the Hecke discrete groups) acting on $\mathcal{H}$ and k a real number. Suppose F is a function analytic in $\mathcal{H}$ satisfying:

$$\overline{v}(M)(cz+d)^{-k}F(Mz) = F(z) + q_M(z), \tag{1.7}$$

where q_M is the period function (or cocycle) for each $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$. Here v is a *multiplier system* in weight k for Γ with $|v(M)| = 1$ for all $M\ \in \Gamma$ and satisfying the " consistency condition"

$$v(M_3)(c_3z+d_3)^k = v(M_1)v(M_2)(c_1M_2z+d_1)^k(c_2z+d_2)^k, \tag{1.8}$$

with $M_3 = M_1M_2$ and $M_i = \begin{pmatrix} * & * \\ c_i & d_i \end{pmatrix}, 1 \leq i \leq 3$.

Then F is called an *automorphic integral*. If $q_M = 0$ for all M, F is called an *automorphic form*.

Note that,

i. If $M_1 = M_2 = I$ in (1.8), we get $v(I) = 1$,

ii. If $M_1 = M_2 = -I$ in (1.8), we get $v(-I)(-1)^k = \pm 1$. For our purpose we consider only $v(-I)(-1)^k = 1$,

iii. If $M_1 = M_2 = T$ then $M_3 = T^2 = -I$, in (1.8), we get $v(T) = e^{-\pi ik/2}$.

Given a real number k and the multiplier system v in weight k, we define the *slash operator*

$$\left(F|_v^k M\right)(z) = \bar{v}(M)\,(cz+d)^{-k}\,F(Mz), \tag{1.9}$$

for $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$ and any function F defined in $\mathcal{H}$. Then we can rewrite (1.7) as

$$\left(F|_v^k M\right)(z) = F(z) + q_M(z).$$

A cocycle on Γ, for weight k and multiplier system v, is any collection of $\{q_M | M \in \Gamma\}$, defined on $\mathcal{H}$ and satisfying the "cocycle condition"

$$q_{M_1M_2} = q_{M_1}|_v^k M_2 + q_{M_2}, \quad M_1,\, M_2 \in \Gamma. \tag{1.10}$$

An easy exercise show that (1.7) and (1.8) gives

$$F|_v^k M_1M_2 = \left(F|_v^k M_1\right)|_v^k M_2, \tag{1.11}$$

where $M_1,\ M_2 \in G(\lambda)$.

Let $\mathcal{P}$ denote the collection of all functions g holomorphic in $\mathcal{H}$, satisfying the growth condition

$$|g(z)| \leq K\left(|z|^\rho + y^{-\sigma}\right), \quad y = \Im z > 0, \tag{1.12}$$

for some constants K, ρ, $\sigma > 0$. In [12], Austin Daughton proved that $\mathcal{P}$ is closed under addition, multiplication and slash operator for a group Γ.

1.2 Preliminaries

1.2.1 Stirling's Formula and Phragmén-Lidelöf Theorem

Now we present some analytic results which are required for the proofs in our results. First, let's recall that the gamma function is defined by

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt,\ \Re(s) > 0$$

Below we list some basic properties of $\Gamma(s)$ that we need in this work.

1. $\Gamma(s+1) = s\Gamma(s),\ \Gamma(1) = 1,\ \Gamma(1/2) = \sqrt{\pi}$.
2. $\Gamma(s)$ is never 0, and is meromorphic with simple poles at $s = -n$ of residue $\frac{(-1)^n}{n!}$, $n = 0, 1, 2, \cdots$.
3. $\Gamma(s) = \sqrt{2\pi} s^{\sigma-1/2} e^{-s+\mu(s)}$, where $\mu(s) \to 0$ as $|s| \to \infty$, uniformly in a half plane $\sigma \geq \sigma_o > 0$, where $s = \sigma + it$.
4. $\Gamma(s) \sim \sqrt{2\pi} t^{\sigma-1/2} e^{-\frac{\pi}{2}|t|}$, as $|t| \to \infty$, uniformly in $\sigma_1 \leq \sigma \leq \sigma_2$ (this follows from 3 above) when $\sigma_1 > 0$.

We shall require Stirling's formula and the Phragm$\acute{e}$-Lindel$\ddot{o}$ theorem for 'lacunary' vertical strip in the proof of the converse part of the correspondence theorems. Here we state them and give reference for the prove.

Theorem 1.2.1. *(Stirling's Formula I). For any $\delta > 0$*

$$\log \Gamma(s) = (s - 1/2)\log s - s + 1/2 \log(2\pi) + \mathcal{O}\left(\frac{1}{|s|}\right)$$

in $|\arg s| \leq \pi - \delta$, where the implied constant depends only on δ.

Applying the above theorem we can proof the following corollary, we state in (see for the prove [23]).

Corollary 1.2.2. *(Stirling's Formula II). For fixed real ρ,*

$$|\Gamma(\sigma + it)| \backsim \sqrt{2\pi} |t|^{\sigma-1/2} e^{-\pi|t|/2}, \text{ as } |t| \to \infty. \tag{1.13}$$

Thus for any $s = \sigma + it$ with $|t| \geq 1$ in the strip $S(a,\ b) = \{\sigma + it : a < \sigma < b\}$, we have

$$|\Gamma(s)| \leq K|t|^\rho e^{-\pi|t|/2}, \tag{1.14}$$

where K depends only on a and b while ρ depends only on a.

Theorem 1.2.3 (Phragmé-Linelöf). *Let $S(a, b)$ be the vertical strip $\{z \in\ \mathbb{C} : a < \Re(z) < b\}$, and let $\Omega \subseteq S(a, b)$ be open. Suppose that $f(z)$ is analytic on Ω and continuous on boundary of Ω such that*

(1) $|f(z)| \leq M$ for z in the boundary of Ω.

(2) $f(z) = \mathcal{O}\left(\exp\left(e^{\theta\pi|z|/(b-a)}\right)\right)$,

where $\theta > 1$ and the implied constant is independent of z (but may depend upon Ω and θ). Then $|f(z)| \leq M$ for all $z \in \Omega$.

The following consequence of Phragmé-Linelöf theorem will be used in subsequent discussions.

Corollary 1.2.4. *Let* Ω *be an open subset of the lacunary vertical strip* $S_\eta(a,b) = \{z = x+iy \in \mathbb{C} : a < x < b, \ |y| > \eta\}$ *for some* $\eta > 0$. *Suppose that* $f(z)$ *is analytic on* Ω *and continuous on the boundary of* Ω. *Also suppose that*

(1) $f(z) = \mathcal{O}\left(|y|^\alpha\right)$ *on the boundary of* Ω *for some* $\alpha \in \mathbb{R}^+$.

(2) $f(z) = \mathcal{O}\left(\exp\left(e^{\theta\pi|z|/(b-a)}\right)\right)$,

where $\theta > 1$ *and the implied constant is independent of* z. *Then* $f(z) = \mathcal{O}\left(|y|^\alpha\right)$, *uniformly in* Ω.

1.2.2 Dirichlet Series

A series of the form $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is called a *Dirichlet series*. We observe that if $a_n = \mathcal{O}(n^\rho), \ \rho > 0,$ as $n \to \infty$ then $\varphi(s)$ converges absolutely and uniformly in $\Re(s) \geq \rho + 1 + \epsilon$, for $\epsilon > 0$. Note that is $\varphi(s)$ dominated uniformly by $\sum_{n=1}^{\infty} n^{-1-\epsilon} < \infty$. Hence φ defines a holomorphic function in the half plane $\Re(s) > c+1$. Conversely, if φ converge for $s_o = \sigma_o + it_0$, then $a_n n^{-s_o}$ tends to 0 as n tends to ∞. In particular, $a_n = \mathcal{O}(n^{\sigma_o})$. Thus, $\varphi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges in a right half plane if and only if $a_n = \mathcal{O}(n^c)$. The following proposition gives a relation between the growth of the coefficient a_n of an exponential series and its growth near the real line.

Proposition 1.2.5. *Given* $f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n\tau/\lambda}$, *with the series converging in the upper half plane*

(a) If $a_n = \mathcal{O}(n^c)$, *then* $f(x+iy) = \mathcal{O}(y^{-c-1})$ *as* $y \to 0$ *uniformly in all real* x.

(b) If $f(x+iy) = \mathcal{O}(y^{-c})$ *as* $y \to 0$, *uniformly in* x, *then* $a_n = \mathcal{O}(n^c)$.

Proof. By Sterling's formula, $\Gamma(x) \approx \sqrt{2\pi} x^{x-1/2} e^{-x}$, we see that,

$$\begin{aligned}(-1)^n \begin{pmatrix} -c-1 \\ n \end{pmatrix} &= \frac{(c+1)(c+2)\cdots,(c+n)}{n!} \\ &= \frac{\Gamma(c+n+1)}{\Gamma(c+1)\Gamma(n+1)} \\ &\approx An^c, \text{ where } A \text{ is some constant.}\end{aligned}$$

so if $a_n = \mathcal{O}(n^c)$, then $f(x+iy)$ is dominated by

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n \binom{-c-1}{n} e^{-2\pi y \frac{n}{\lambda}} &= \left(1-e^{-2\pi\frac{y}{\lambda}}\right)^{-c-1} \\ &= \mathcal{O}(y^{-c-1}).\end{aligned}$$

Conversely, if $|f(x+iy)| \leq By^{-c}$, then

$$\begin{aligned}|a_n| &= \left|\int_0^1 f\left(x+\frac{i}{n}\right) e^{-2\pi n\frac{(x+\frac{i}{n})}{\lambda}}\,d\,x\right| \\ &\leq Bn^c e^{\frac{2\pi}{\lambda}}.\end{aligned}$$

Therefore the conclusion follows. □

1.2.3 The Mellin transform and Its Inverse

The Mellin transform of $f(t)$ is defined by

$$\Phi(s) = \int_0^{\infty} f(t)t^{s-1}\,dt.$$

The substitution $t = e^{-x}$ transform it into

$$\begin{aligned}\Phi(s) &= \int_{-\infty}^{\infty} f(e^{-x})e^{-xs}d\,x \\ &= \int_0^{\infty} f(e^{-x})e^{-xs}d\,x + \int_0^{\infty} f(e^{x})e^{-x(-s)}d\,x.\end{aligned}$$

The second integrals are Laplace transform of parameter s and $-s$, then by applying the inverse Laplace transform, we get the inverse Mellin transform of $\Phi(s)$ as

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Phi(s)t^{-s}\,ds.$$

Here the line of integration is a vertical line with real part c.

Let $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda}$, $a_n = \mathcal{O}\left(n^{\rho}\right)$, for some $\rho > 0$. Then

$$\Phi_f(s) = \int_0^{\infty} \left(f(iy) - a_0\right) y^s \frac{dy}{y}$$

$$
\begin{aligned}
&= \int_0^\infty \left(\sum_{n=1}^\infty a_n e^{-2\pi n \frac{y}{\lambda}} \right) y^s \frac{dy}{y} \\
&= \sum_{n=1}^\infty a_n \int_0^\infty e^{-2\pi n \frac{y}{\lambda}} y^{s-1} dy \\
&= \left(\frac{2\pi}{\lambda} \right)^{-s} \Gamma(s) \sum_{n=1}^\infty \frac{a_n}{n^s} \\
&= \left(\frac{2\pi}{\lambda} \right)^{-s} \Gamma(s) \varphi_s(s).
\end{aligned}
$$

Thus, the Mellin transform of an exponential series is a general Dirichlet series. The Mellin transform of the exponential function and the inverse Mellin transform of the Gamma function play roles in demonstrating the equivalence of the modular relation and the functional equation.

1.2.4 Automorphic Form and Poincar$\acute{e}$ Series

Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$ and preserves $\mathcal{H}$. We shall see Poincaré construction of automorphic form. For $k \in \mathbb{Z}^+$, define

$$
g(z) = \sum_{M \in \Gamma} \left(\frac{d\,Mz}{dz} \right)^k .
$$

Then

$$
\begin{aligned}
g(Lz) &= \sum_{M \in \Gamma} \left(\frac{dM\,Lz}{dLz} \right)^k \\
&= \left(\frac{dz}{dLz} \right)^k \sum_{M \in \Gamma} \left(\frac{dMz}{dz} \right)^k \\
&= (cz+d)^{2k} g(z) \quad L = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma.
\end{aligned}
$$

Assuming g is absolutely convergent, then $g(z)$ is an automorphic form.

Here Poincaré used the analogy with the elliptic modular functions as a guiding principle. We recall that g_2, g_3 are defined by

$$
g_2(\tau) = \sum_{M \in \Gamma} (c\tau + d)^{-2}, \quad g_3(\tau) = \sum_{M \in \Gamma} (c\tau + d)^{-3}, \quad \tau \in \mathcal{H},
$$

for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Note that $\frac{dM}{dz} = (cz+d)^{-2}$ for the unimodular transformation $Mz = \frac{az+b}{cz+d}$, then $\frac{g_2^3}{g_3^2}$ is invariant under the modular group.

Let $H(z)$ be a rational function and define

$$F(z,H)(dz)^m = \sum_{M\in\Gamma} H(Mz)(dMz)^m, \tag{1.15}$$

where m is a positive integer and Γ is arbitrary Fuchsian group (a discontinuous group each element preserves $\mathcal{H}$).

Assume that the series converges absolutely then can be rearranged. We now have for $L \in \Gamma$:

$$\begin{aligned} F(Lz,H)(dLz)^m &= \sum_{M\in\Gamma} H(MLz)(dMLz)^m \\ &= \sum_{M\in\Gamma} H(Mz)(dMz)^m \\ &= F(z,H)(dz)^m. \end{aligned} \tag{1.16}$$

Note that when M runs over Γ, so does ML. We say the "differential" $F(z)(dz)^m$ is invariant under Γ. Poincaré called F a theta–Fuchsian series of order m, but later it was named Poincaré series.

If we write (1.16) as

$$F(Mz) = (cz+d)^{2m} F(z), \quad M \in \Gamma,$$

it becomes an automorphic form of weight $-2m$. The quotient of two Poincaré series of the same weight is an automorphic function. Since different H's give rise to different Poincaré series their ratio is a nontrivial automorphic function, this solves the existence problem. Poincaré proved the absolute convergence of (1.15) for $m \geq 2$ (see for the detail [15]), which permits us to rearrange the series.

Note that the series (1.15) may be identically zero. But for Poincare purpose it was sufficient to select $H(z)$ having a pole at a point of $\mathcal{H}$. Then $F(z)$ will have a pole at the same point and cannot be identically zero. Similarly if we choose H_1 and H_2 to have poles at different points, we can be sure the functions $F(z,H_1)$ and $F(z,H_2)$ will be linearly independent and the automorphic function $\frac{F(z,H_1)}{F(z,H_2)}$ will not be a constant. Note that F possess a definite value at a parabolic vertex.

The original Poincare series were automorphic form of negative integral weight ≤ -3. When we try to define them for groups acting on the upper half plane $\mathcal{H}$, we encounter a new difficulty. Consider

$$\sum_{M\in\Gamma}(c\tau+d)^{-k},\ k\in\mathbb{Z},\ \tau\ \in\mathcal{H},\quad M=\begin{pmatrix} a & b \\ c & d\end{pmatrix}, \tag{1.17}$$

where Γ contains translation generated by $S\tau = \tau+\lambda$. Every element fixing ∞ is a translation and we may therefore write $\{S^m,\ m\in\mathbb{Z}\}=\Gamma_\infty$. Since $S^mV\in\Gamma$ and also has (c,d) as its lower row, the term $(c\tau+d)^{-k}$, occurs infinitely often and the series (1.17) does not converge. Hecke got around the difficulty by summing over a set of representative of the cosets $\Gamma_\infty\backslash\Gamma$, i.e., a complete system of matrices of Γ with different lower row. We called the series

$$E(\tau)=\sum_{V\in\Gamma_\infty\backslash\Gamma}(c\tau+d)^{-k}\qquad k\geq 3$$

an Eisenstein series. It belongs to the class $\{\Gamma,-k,1\}$ and is analytic in $\mathcal{H}$ and at the parabolic cusps of Γ.

In the next chapter we review a Hecke correspondence theorem and we shall see how Bocheners developed this correspondence in [2], using residual function that were later called log-polynomial period sum as period function by Knopp in [7] and [20] .

In chapter three we shall discuss a correspondence theorem on infinite log-polynomial period functions for Hecke groups, in particular, for the theta group. In this chapter we will see back ground for the results in chapter four.

In chapter four we extend the results in [12] from a correspondence theorem on theta group for infinite log-polynomial period functions to a correspondence theorem on infinite log-polynomial period functions for $\lambda>2$ and arbitrary real weight k. This correspondence gives infinitely many poles for the generalized Dirichlet series.

In chapter five we generalize some of the results in [11] from finite log-polynomial sum period functions to infinite log-polynomial sum period functions. We shall characterize the infinite log-polynomial period functions for $\lambda=2\cos\pi/p,\ p\ \in\ \mathbb{Z},\ p\geq 3$ and for cases of multiplier system $v(S_\lambda)=1$ with weight $k>2$ and $v(S_\lambda)\neq 1$ with weight $k>2$.

Chapter 2

Riemann-Hecke-Bochner Correspondence

2.1 Riemann Functional Equation

The classical theta function, defined for $\Im(\tau) > 0$ by

$$\theta(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau},$$

satisfies the modular transformation law

$$\theta\left(\frac{-1}{\tau}\right) = \left(\frac{\tau}{i}\right)^{\frac{1}{2}} \theta\left(\tau\right). \tag{2.1}$$

Recall that the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \text{ for } \Re(s) > 1$$

and satisfies the functional equation:

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) = \pi^{-\frac{1}{2}(1-s)}\Gamma\left(\frac{1}{2}(1-s)\right)\zeta(1-s), \tag{2.2}$$

where $\Gamma(s)$ is the gamma function. Riemann proved (2.2) using (2.1) in [24].

2.2 Hecke Correspondence

In 1936, Hecke extended Riemann's work in [4] to a correspondence between a Dirichlet series with functional equation and an automorphic form under certain growth conditions, which we state in the following theorem.

Theorem 2.2.1. *(Hecke Correspondence Theorem)*
Let $\{a_n\}$ and $\{b_n\}$ be sequences of complex numbers such that $a_n, b_n =$

$\mathcal{O}(n^\rho)$, *as n tends to* ∞, *for some* $\rho > 0$. *Let* $\lambda > 0$, $k \in \mathbb{R}$, and $\gamma \in \mathbb{C}$. *For* $\sigma > c+1$, $s = \sigma + it$. *Let*

$$\varphi(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \qquad \psi(s) = \sum_{n=1}^{\infty} b_n n^{-s};$$

$$\Phi(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\varphi(s), \qquad \Psi(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\psi(s);$$

$$f(\tau) = \sum_{n=0}^{\infty} a_n e^{\frac{2\pi n i \tau}{\lambda}} \qquad \text{and} \qquad g(\tau) = \sum_{n=0}^{\infty} b_n e^{\frac{2\pi n i \tau}{\lambda}}, \quad \tau \in \mathcal{H}.$$

Then the following two assertions are equivalent.

(i) $f(\tau) = \gamma \left(\frac{\tau}{i}\right)^{-k} g\left(\frac{-1}{\tau}\right)$.

(ii) $\Phi(s) + \frac{a_0}{s} + \gamma\frac{b_0}{(k-s)}$ *has an analytic continuation to the entire complex plane and bounded in every vertical strip. Moreover,*

$$\Phi(s) = \gamma\Psi(k-s). \tag{2.3}$$

Remark 2.2.2. *1. Formulation of Theorem* 2.2.1 *deviate from Hecke's original statement in two ways. In Hecke's work there is a single Dirichlet series; that is* $\psi(s) = \varphi(s)$.

2. Boundedness condition in (ii) replaces a corresponding hypothesis of Hecke, who assume that $(s-k)\varphi(s)$ *is an entire function of finite genus, that is to say, there exists an* $A > 0$ *such that* $|(s-k)\varphi(s)| \leq \exp\{|s|^A\}$, *for all s in* $\mathbb{C}$.

3. Note that $\Phi(s)$ *is entire if and only if* $a_0 = 0$ *and* $b_0 = 0$. *If* $a_0 \neq 0$, $\Phi(s)$ *has a simple pole at* $s = k$. *Alternatively, one can say that* $\Phi(s)$ *is entire if and only if* $f(\tau)$ *vanishes at* $i\infty$.

4. Theorem 2.2.1 *has been generalized by several authors. We will review Bochener's generalization in the next sections.*

Remark 2.2.3. *Take* $\lambda = 2$, $k = \frac{1}{2}$, $\gamma = 1$, and $f(\tau) = g(\tau) = \theta(\tau)$. *Furthermore* $a_0 = b_0 = 1$ and $a_n = b_n = 2$, if $n = m^2, m \in \mathbb{Z}^+$; $a_n = b_n = 0$, otherwise. *Thus,*

$$\varphi(s) = \psi(s) = 2\sum_{m=1}^{\infty} \frac{1}{(m^2)^s} = 2\zeta(2s),$$

satisfies (2.3),*i.e.,*

$$\pi^{-s}\Gamma(s)\zeta(2s) = \pi^{(s-\frac{1}{2})}\Gamma(\frac{1}{2}-s)\zeta(1-2s),$$

replacing s by $\frac{s}{2}$, *we obtain* (2.2) .

2.3 Bochner's Correspondence Theorem

After Hecke generalized his correspondence theorem, significant generalization was given by then , S. Bochner, [2] in 1951. He established a correspondence between an automorphic integral with finite 'log-polynomial sum period function' and Dirichlet series with the classical functional equation and finitely many poles.

Here we state Bochner's correspondence theorem with slight reformulation of Bochner result as used by Knopp in [7] and [20] and since our main results in chapter IV essentially depend on the proof of the converse part of this correspondence, we include the proof of the converse part.

Theorem 2.3.1. *(Bochener's Correspondence Theorem)*
Let $\lambda_1, \lambda_2 > 0, k \in \mathbb{R}, \text{and}\, \gamma \in \mathbb{C}$. *Let*

$$f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi\, in\tau/\lambda_1} \quad \text{and} \quad g(\tau) = \sum_{n=0}^{\infty} b_n e^{2\pi\, in\tau/\lambda_2}, \tag{2.4}$$

be non-constant exponential series, such that the sequences $\{a_n\}$, $\{b_n\}$ *satisfying the growth condition*

$$a_n, b_n = \mathcal{O}(n^{\rho}), \qquad n \to \infty, \qquad \rho > 0.$$

As in theorem (2.2.1) , *put*

$$\Phi(s) = \left(\frac{2\pi}{\lambda_1}\right)^{-s} \Gamma(s) \sum_{n=1}^{\infty} a_n n^{-s} = \int_0^{\infty} \{f(iu) - a_0\} u^{s-1} \mathrm{d}\, u, \tag{2.5}$$

$$\Psi(s) = \left(\frac{2\pi}{\lambda_2}\right)^{-s} \Gamma(s) \sum_{n=1}^{\infty} b_n n^{-s} = \int_0^{\infty} \{g(iu) - b_0\} u^{s-1} \mathrm{d}\, u. \tag{2.6}$$

Then the following two statements are equivalent.

(A) $f(\tau)$ *and* $g(\tau)$ *satisfy the (generalized) modular transformation equation*

$$\left(\frac{\tau}{i}\right)^{-k} f(\frac{-1}{\tau}) = \gamma g(\tau) + q(\tau), \tag{2.7}$$

where $q(\tau) = \sum_{j=1}^{L} (\tau/i)^{-\alpha_j} \sum_{t=0}^{m(j)} \beta(j,t) \log^t (\tau/i)$ *is a log–polynomial sum(LPS).*

(B) $\Phi(s)$ *and* $\Psi(s)$ *have meromorphic continuation to the entire s-plane, each with at most a finite number of poles in* $\mathbb{C}$*. Furthermore,* $\Phi(s)$ *and* $\Psi(s)$ *satisfy the functional equation*

$$\Phi(k-s) = \gamma \Psi(s). \tag{2.8}$$

Finally, there exists $T_0 > 0$ *such that* $\Phi(s)$ *remains bounded in each "lacunary" vertical strip (LVS) of the form* $\sigma_1 \le \sigma \le \sigma_2$, $|t| \ge T_0$*. Here,* $s = \sigma + it$*.*

Proof. We want to prove (B) implies (A).

We begin by observing that for $y > 0$ and sufficiently large $d > 0$, we have both

$$g(iy) - b_0 = \frac{1}{2\pi i} \int_{(d)} \Psi(s) y^{-s} d\, s \tag{2.9}$$

and

$$f(iy) - a_0 = \frac{1}{2\pi i} \int_{(d)} \Phi(s) y^{-s} d\, s. \tag{2.10}$$

(Note that the equations (2.9) and (2.10) express $f - a_0$ and $g - b_0$ as the inverse Mellin transforms of Ψ and Φ, respectively.) By assumption, there exist

$$P_1(s) = \sum_{j=1}^{L} \sum_{t=1}^{N(j)} \frac{b(j,t)}{(s-\delta_j)^t},$$

$$P_2(s) = \sum_{j=1}^{L'} \sum_{t=1}^{N'(j)} \frac{b'(j,t)}{(s-\eta_j)^t},$$

with $\delta_j, \eta_j \in \mathbb{C}$, such that $\Phi(s) - P_1$ and $\Psi(s) - P_2(s)$ are entire.

We move the line of integration to $\sigma = -d$; (2.9) then implies that (see [1], Page 7)

$$g(iy) - b_0 = \frac{1}{2\pi i}\int_{(d)} \Psi(s) y^{-s} d\,s + \sum_{j=1}^{L'} y^{-\eta_j} \sum_{t=1}^{N'(j)} b'(j,t) \frac{(-\log y)^{t-1}}{(t-1)!}, \quad (2.11)$$

where as from (2.10) , we have

$$f(iy) - a_0 = \frac{1}{2\pi i}\int_{(d)} \Phi(s) y^{-s} d\,s + \sum_{j=1}^{L} y^{-\delta_j} \sum_{t=1}^{N(j)} b(j,t) \frac{(-\log y)^{t-1}}{(t-1)!}. \quad (2.12)$$

We should mention an additional restriction that we have imposed implicitly upon d : $d > 0$ is sufficiently large so that all poles of $\Phi(s)$ and $\Psi(s)$ lie in the vertical strip $|\Re(s)| < d$. It is also important to note that in the derivation of (2.11) and (2.12) we require both Stirling formula and the Phragmén-Lindelöf Theorem for a vertical strip.(See the proof of Theorem 2.1 in [1] Page 7-8 for more details.)

At this juncture we invoke the functional equation (2.8) in (2.11) to obtain

$$g(iy) - b_0 = \frac{\gamma^{-1}}{2\pi\, i}\int_{(-d)} \Phi(k-s) y^{-s} d\,s + p(y),$$

where $p(y)$ is the finite sum on the right-hand side of (2.11) . With a change of variables in the integral, we find that

$$\begin{aligned} g(iy) - b_0 &= \frac{\gamma^{-1} y^{-k}}{2\pi\, i}\int_{(k+d)} \Phi(s) y^{s} d\,s + p(y) \\ &= \frac{\gamma^{-1} y^{-k}}{2\pi\, i}\int_{(k+d)} \Phi(s) (\frac{1}{y})^{-s} d\,s + p(y) \\ &= \gamma^{-1} y^{-k} \{f(i/y) - a_0\} + p(y), \end{aligned}$$

by (2.10) . This can be rewritten as

$$(\tau/i)^{-k} f(-1/\tau) = \gamma g(\tau) + a_0 (\tau/i)^{-k} - \gamma b_0 - p(\tau/i), \quad (2.13)$$

valid for $\tau = iy$, $y > 0$. By analytic continuation, (2.13) holds for all τ in $\mathcal{H}$. Thus we have proved (2.7) with

$$q(\tau) = a_0 (\tau/i)^{-k} - \gamma b_0 - p(\tau/i),$$

an LPS. This completes the proof of the converse of the theorem. □

Remark 2.3.2. 1. *To compare Theorem* 2.2.1 *with Theorem* 2.3.1, *note that we have here interchanged the roles of* f *and* g, *and consequently the roles of* Φ *and* Ψ, *as well. In this generalization* λ_1, λ_2 *may be distinct positive numbers whereas in Theorem* 2.2.1, $\lambda = \lambda_1 = \lambda_2$, *even when* $f \neq g$.

2. *The most important feature of the generalization is the presence of the LPS* $q(\tau)$ *in* (2.7) *makes no change in the functional equation connecting* Φ *and* Ψ. *However,* $q(\tau)$ *is related to the poles of* Φ.

3. *In fact,* $q(\tau)$ *determines the exact locations and orders of the poles of* Φ *and* Ψ, *and vice verse.*

We make this precise for the special case $f = g$, $\Phi = \Psi$, in

Corollary 2.3.3. *Suppose* $f = g$, $\Phi = \Psi$, *as in Theorem* 2.3.1. *Then the term* $\beta\tau^{\beta}(\log\tau)^t$, $\beta \neq 0$, $t \in \mathbb{Z}$, $t \geq 0$, *occurring in* $q(\tau)$ *corresponds to poles of* $\Phi(s)$ *of order* $t+1$ *at the points* $s = \beta + k$ *and* $s = -\beta$. *The only possible further singularities of* Φ *are simple poles at* $s = 0$ *and* $s = k$.

Chapter 3

Infinite Log-Polynomial Sum Period (ILPSP) Functions

In this chapter, we summarize known results about LPSPFs.

3.1 Finitely Many Essential Singularities

3.1.1 Finite Log-Polynomial Sum Period Functions

A log–polynomial sum $q(z) = \sum_{l=1}^{n} z^{\beta_l} \sum_{j=0}^{m(l)} \delta(l,j)(\log\, z)^j$, where $\beta_1, \cdots, \beta_n$ and the coefficients $\delta(l,j)$ are complex numbers, is said to be a log–polynomial period function of weight k and multiplier system v for the Hecke group $G(\lambda)$, if there exists a function F defined and holomorphic in $\mathcal{H}$ such that:

$$e^{-2\pi\kappa i}F(z+\lambda) = F(z), \tag{3.1}$$

$$\overline{v}(T)z^{-k}F\left(\frac{-1}{z}\right) = F(z) + q(z), \tag{3.2}$$

where $e^{2\pi\kappa i} = v(S_\lambda)$, $0 \leq \kappa < 1$.

Such a function F satisfying (3.1) and (3.2) is called an automorphic integrals of weight k and multiplier system v for $G(\lambda)$, if it has an exponential series expansion:

$$F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i(n+\kappa)\frac{z}{\lambda}}, \tag{3.3}$$

where $a_n \in \mathbb{C}$ satisfy the growth condition $a_n = \mathcal{O}(n^\rho)$ as $n \to \infty$, $\rho > 0$. In this case we say that q is the log–polynomial period function of the automorphic integral F.

The log–polynomial sum which occur as period function for (entire) automorphic integral of weight k have completely characterized in the cases $k > 2, v(S_\lambda) = 1$ and $k > 0, v(S_\lambda) \neq 1$ in [11].

3.1.2 Estimation of Infinite Log–Polynomial Sum (First Version)

Letting $m(l) \to \infty$ in the second sum, we define the first version of "infinite" log–polynomial sum of the form

$$q(z) = \sum_{l=1}^{n} \left(\frac{z}{i}\right)^{-\beta_l} \sum_{j=0}^{\infty} \delta(l,j) \left(\log \frac{z}{i}\right)^j, \tag{3.4}$$

where the β_l are all distinct and finitely many complex numbers, $\delta(l,j) \in \mathbb{C}$, and $z \in \mathcal{H}$.

This version of infinite log–polynomial sum was necessary to produce finitely many essential singularities, under certain conditions the convergence and the associated correspondence theorem is proved in [12]. For the sake of comparison we state the estimation of (3.4) and the associated correspondence theorem.

Proposition 3.1.1. *Let* $q(z) = \sum_{l=1}^{n} \left(\frac{z}{i}\right)^{-\beta_l} \sum_{j=0}^{\infty} \delta(l,j) \left(\log \frac{z}{i}\right)^j$ *suppose that*

(i) $|\Re(\beta_l)| \le \beta \qquad \text{for all } l, \text{where } \beta > 0$

(ii) $\sum_{j=1}^{\infty} \left(\frac{j}{e}\right)^j |\delta(l,j)| < \infty \quad \text{for } l = 1, \cdots, n.$

Then $q(z)$ *converges absolutely for all* $z \in \mathcal{H}$ *and uniformly on compact subset of* $\mathcal{H}$*, and* $q \in \mathcal{P}$*. Furthermore, the series for* $q(-1/z)$ *also converges absolutely and uniformly on compact subset of* $\mathcal{H}$*.*

For the next theorem we assume that the condition (i) of proposition 3.1.1 and further we assume, $\sum_{l=1}^{n} \sum_{j=0}^{\infty} j!|\delta(l,j)|\epsilon^j < \infty$, for every $\epsilon > 0$. Note that this is stronger than (ii). Also, assume $v(S_\lambda) = 1$.

Theorem 3.1.2. *Let* $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi inz/\lambda}$ *be holomorphic in* $\mathcal{H}$ *with* $a_n = \mathcal{O}(n^\rho)$ *for some* $\rho > 0$*, as* $n \to \infty$*. Let*

$$q(z) = \sum_{l=1}^{n} \left(\frac{z}{i}\right)^{-\beta_l} \sum_{j=0}^{\infty} \delta(l,j) \left(\log \frac{z}{i}\right)^j$$

and define the functions

$$\varphi_f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

$$\begin{aligned} \Phi_f(s) &= \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\varphi_f(s), \\ Q(s) &= \bar{v}(T) \sum_{l=1}^{n} \sum_{j=0}^{\infty} \frac{(-1)^{j+1} j! \delta(l,j)}{(s-\beta_l)^{j+1}} + a_0 \left(\frac{i^k v(T)}{s-k} - \frac{1}{s} \right). \end{aligned}$$

Then the following are equivalent:

(A) $f(z)$ satisfies the transformation law

$$z^{-k} f\left(\frac{-1}{z}\right) = v(T) f(z) + q(z). \tag{3.5}$$

(B) $\Phi_f(s) - Q(s)$ has an analytic continuation to the entire s-plane that is bounded in vertical strips, and $\Phi_f(k-s) = i^k v(T) \Phi_f(s)$.

3.2 Infinitely Many Poles

3.2.1 Estimation of Infinite Log–Polynomial Sum (Second Version)

To investigate Dirichlet series with infinitely many poles of finite order, we consider the second version of "infinite log–polynomial sum" of the form

$$q(z) = \sum_{l=1}^{\infty} \left(\frac{z}{i}\right)^{-\beta_l} \sum_{j=0}^{m(l)} \delta(l,j) \left(\log \frac{z}{i}\right)^j, \tag{3.6}$$

where the $\beta_l's$ are all distinct complex numbers, $\delta(l,j) \in \mathbb{C}$, and $z \in \mathcal{H}$. This infinite log–polynomial sum is essential to the correspondence theorem with Dirichlet series having infinitely many poles. As we shall see, the Dirichlet series will have singularities at β_l, which are poles of order $m(l)+1$.

We now show that $q(z)$ converges absolutely for $z \in \mathcal{H}$ and uniformly in the compact subset of $\mathcal{H}$. In fact $q(z)$ is in $\mathcal{P}$ under the assumption (i) and (iii) below. With slight difference in the second condition, but no change has to be made in the proof. We have the following proposition, which is Proposition 4.1 in [12].

Proposition 3.2.1. *Let q(z) be in* (3.6)*. Suppose that*

(i) $|\Re(\beta_l)| \le \beta$ for $l \ge 1$, where $\beta > 0$;

(iii) $\sum_{l=1}^{\infty}\sum_{j=9}^{m(l)} e^{|\Im(\beta_l)|\frac{\pi}{2}}\left(\frac{j}{e}\right)^j |\delta(l,j)| < \infty$ *and*
$\sum_{l=1}^{\infty}\sum_{j=0}^{8} e^{|\Im(\beta_l)|\frac{\pi}{2}}|\delta(l,j)| < \infty.$

Then $q(z)$ converges absolutely and uniformly on compact subsets of $\mathcal{H}$, and $q \in \mathcal{P}$, that is $|q| < K\left(|z|^\rho + y^{-\sigma}\right)$ for some constants K, ρ and $\sigma > 0$. Furthermore, the series $q(-1/z)$ also converges absolutely $z \in \mathcal{H}$ and uniformly on compact subsets of $\mathcal{H}$.

Proof. Let $z \in \mathcal{H}$. By argument convention we have $\left|\arg\left(\frac{z}{i}\right)\right| < \frac{\pi}{2}$, which implies

$$\left|\left(\frac{z}{i}\right)^{-\beta_l}\right| = \left|\frac{z}{i}\right|^{-\Re(\beta_l)} e^{\Im(\beta_l)\arg(z/i)} \;\leq\; |z|^{-\Re(\beta_l)} e^{|\Im(\beta_l)|\frac{\pi}{2}}.$$

From (i), we conclude

$$|z|^{-\Re(\beta_l)} \leq \left(|z|^\beta + y^{-\beta}\right).$$

Hence

$$\left|\left(\frac{z}{i}\right)^{-\beta_l}\right| \leq e^{|\Im(\beta_l)|\frac{\pi}{2}}\left(|z|^\beta + y^{-\beta}\right). \tag{3.7}$$

See [[12], PP. 17-18] for the proof of

$$\left|\log\left(\frac{z}{i}\right)\right|^j \leq \begin{cases} K\left(\frac{j}{e}\right)^j\left(|z|^4 + y^{-4}\right), & \text{if } j \geq 9 \\ 2^j\left(|z| + y^{-1}\right)^j, & \text{if } 0 \leq j < 9 \end{cases} \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6), we have

$$\begin{aligned}
|q(z)| &\leq \left(|z|^\beta + y^{-\beta}\right)\sum_{l=1}^{\infty}\left(\sum_{j=0}^{8} e^{|\Im(\beta_l)|\frac{\pi}{2}}|\delta(l,j)|2^j\left(|z| + y^{-1}\right)^j\right.\\
&\qquad\left. + \mathrm{K}\left(|z|^4 + y^{-4}\right)\sum_{j=9}^{m(l)} e^{|\Im(\beta_l)|\frac{\pi}{2}}\left(\frac{j}{e}\right)^j|\delta(l,j)|\right)\\
&\leq \left(|z|^\beta + y^{-\beta}\right)\left(2^8\left(|z| + y^{-1}\right)^8\sum_{l=1}^{\infty}\sum_{j=0}^{8} e^{|\Im(\beta_l)|\frac{\pi}{2}}|\delta(l,j)|\right.\\
&\qquad\left. + \mathrm{K}\left(|z|^4 + y^{-4}\right)\sum_{l=1}^{\infty}\sum_{j=9}^{m(l)} e^{|\Im(\beta_l)|\frac{\pi}{2}}\left(\frac{j}{e}\right)^j|\delta(l,j)|\right).
\end{aligned}$$

By (iii), both the above series converge. Therefore, by the Weierstrass M–test $q(z)$ converges absolutely and uniformly on compact subset of $\mathcal{H}$. This estimation also shows that $q \in \mathcal{P}$. To complete the proof of the proposition, we note that $\log\left(\frac{-1}{iz}\right) = -\log\left(\frac{z}{i}\right)$, and $\left|\left(\frac{-1}{iz}\right)^{-\beta_l}\right| \leq \mathrm{K}\left(\frac{l}{e}\right)^l\left(|z|^4 + y^{-4}\right)$ for $l \geq 9$. Thus, with the same argument $q\left(-1/z\right)$ has the properties stated. □

For the next correspondence theorems , we apply a strong condition on the coefficients. We use an explicit estimate for $|q(z)|$ it is proved in [[12] P. 18].

Corollary 3.2.2. *Let*

$$q(z) = \sum_{l=1}^{\infty}\left(\frac{z}{i}\right)^{-\beta_l}\sum_{j=0}^{m(l)}\delta(l,j)\left(\log\frac{z}{i}\right)^j,$$

and suppose that

(i) $|\Re(\beta_l)| \leq \beta$ *for* $l \geq 1$, *where* $\beta > 0$;

(iii)* $\sum_{l=1}^{\infty}\sum_{j=0}^{m(l)} e^{|\Im(\beta_l)|\frac{\pi}{2}} j!|\delta(l,j)|\epsilon^j < \infty$, *for every* $\epsilon > 0$.

Then $q \in \mathcal{P}$ *with*

$$|q(z)| \leq \mathrm{K}''\left(|z|^{2\max(\beta,4)} + y^{-2\max(\beta,4)}\right), \tag{3.9}$$

where $\mathrm{K}'' = \mathrm{k}'\sum_{l=1}^{\infty}\sum_{j=0}^{m(l)} e^{|\Im(\beta_l)|\frac{\pi}{2}} j!\left|\delta(l,j)\right|$, *for some constant* K'.

In [12] the following theorem was proved by using the construction of automorphic integrals by the generalized Poincaré series for an H groups of the form

$$\Psi\left(q_M,\ m,\ v; z\right) = \Psi(z) = \sum_{M\ \in\ \mathcal{M}}\frac{q_M(z)}{v(M)\left(cz+d\right)^m},$$

where $\mathcal{M}$ is a complete set of coset representatives for $\left(\Gamma_\theta\right)_\infty \setminus \Gamma_\theta$ and $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_\theta$. This sum is over all distinct lower rows $c,\ d$ of the elements in Γ_θ. Here Γ_θ is the theta group, i.e., $\lambda = 2$ and $\left(\Gamma_\theta\right)_\infty$ is the stabilizer of $i\infty$. Then the construction involves division by Eisenstein series (1.17) attached to Γ_θ. It is known that the Eisenstein series has finite number of zeros in the fundamental region for $\lambda \leq 2$, (see [15], P.274). To avoid those poles, the zeros of the Eisenstein series, applied a

Mittag-Leffler principle for automorophic form on Γ_θ. See the construction in [12] and [20]. But this construction does not hold for $\lambda > 2$. We need different construction we shall see in the next Chapter .

One of the main objective of this thesis is to extend the following theorem, from $\lambda = 2$ and $k \geq 0$ to arbitrary real weight k and arbitrary $\lambda > 2$. We will do so in the next chapter in the main correspondence theorem.

Theorem 3.2.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n e^{\pi i n z}$ be holomorphic in $\mathcal{H}$ with $a_n = \mathcal{O}(n^\gamma)$ for some $\gamma > 0$, as $n \to \infty$. Let*

$$q(z) = \sum_{l=1}^{\infty} \left(\frac{z}{i}\right)^{-\beta_l} \sum_{j=0}^{m(l)} \delta(l,j) \left(\log \frac{z}{i}\right)^j$$

be a log–polynomial sum satisfying (i), and (iii). Additionally assume that (iv) : There exist $\epsilon > 0$ and a sequence of positive numbers $T_n \to \infty$ such that $|\Im(\beta_l) - T_n| \geq \epsilon$ for all n and l.*
Define the functions

$$\begin{aligned} \varphi_f(s) &= \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \\ \Phi_f(s) &= \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\varphi_f(s), \\ Q(s) &= \bar{v}(T) \sum_{l=1}^{\infty} \sum_{j=0}^{m(l)} \frac{(-1)^{l+j} j! \delta(l,j)}{(s-\beta_l)^{j+1}} + a_0 \left(\frac{i^k v(T)}{s-k} - \frac{1}{s}\right). \end{aligned}$$

Suppose that $k \geq 0$ and $v(T) = i^{-k}$ or that $k > 2$ and $v(T) = -i^{-k}$. Then the following are equivalent.

(A) $f(z)$ satisfies the transformation law $z^{-k} f(1/z) = v(T) f(z) + q(z)$.

(B) $\Phi_f(s) - Q(s)$ has an analytic continuation to the entire s-plane that is bounded in vertical strips, and $\Phi_f(k-s) = i^k v(T) \Phi_f(s)$.

Chapter 4

A Correspondence on ILPSP Functions for Hecke Groups for $\lambda > 2$

In this chapter, we shall formally define automorphic integrals with ILPSPFs of weight k and multiplier system v for $G(\lambda)$. We shall prove the Hecke Correspondence Theorem between Dirichlet series with functional equation having infinite poles of finite order and automorphic integrals with ILPSPFs for the Hecke group $G(\lambda)$ with $\lambda > 2$. These results will extend A. Daughton of [12].

4.1 Construction of Automorphic Integrals for Hecke Groups $G(\lambda)$, $\lambda > 2$

Eichler's [4] arguments consist of setting up two Poincaré series, the quotient of which is an automorphic integral F. We construct an automorphic integral function F of Theorem 4.1.1 below using a "generalized Poincaé series," as first applied by Eichler [4], latter by Knopp[6] and Lehener [22] for an H-group. Recall that an H-group is finitely generated discrete group of real linear transformation, that contain translations and have the entire real line as their limit set.

Some of the arguments in the proof is valid for the general discrete Hecke groups $G(\lambda)$. Since we require the construction here only for $G(\lambda)$, $\lambda > 2$, we shall restrict ourselves to this case throught this chapter. We denote this groups by $G = G(\lambda)$, $\lambda > 2$.

Theorem 4.1.1. *Let k be any real number and v be a multiplier system of weight k for G, Suppose $\{q_M | M \in G\}$ is a parabolic cocycle of weight k such that $q_{S_\lambda} = 0$, $q_M \in \mathcal{P}$. (Note that q_M satisfies* (1.10)*). Then there exists a function F, analytic in $\mathcal{H}$, with $F \in \mathcal{P}$ such that*

$$F|_v^k M - F = q_M, \tag{4.1}$$

for all $M \in G$ and having an expansion at the parabolic cusp $z = i\infty$ of the form

$$F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda}, \quad z \in \mathcal{H} \;\; \lambda > 2, \tag{4.2}$$

where

$$a_n = \mathcal{O}\left(n^{\rho}\right), \quad \rho > 0,\, n \to \infty. \tag{4.3}$$

Proof. Let $\{v(M)|M \in G\}$ be a multiplier system in weight k for G. Since $G(\lambda) = \langle S_\lambda, T\rangle$, for any $M \in G$, $v(M)$ is determined by the values of $v(S_\lambda)$ and $v(T)$ using the consistency condition (1.8) . For $\lambda > 2$, the only relation in G is (1.4) . Hence $v(S_\lambda)$ is unrestricted and using (1.8) $v(T)$ must be one of the four complex numbers $\pm e^{-\pi i k/2}$, $\pm i e^{-\pi i k/2}$. But $v(T) = \pm i e^{-\pi i k/2}$ gives trivial space of automorphic integral of weight k on G. For our purpose here we consider only those multiplier system satisfying $v(S_\lambda) = 1$ and $v(T) = e^{-\pi i k/2}$.

A cocycle for G is determined from q_{S_λ} and q_T using the cocycle condition (1.10) . Note that (1.4) implies that q_T must satisfy

$$q_T|_v^k T + q_T = 0. \tag{4.4}$$

Therefore, we can generate a cocycle $\{q_M|M \in G\}$ with the choices

$$q_{S_\lambda} = 0, \text{ and } q_T = q, \tag{4.5}$$

where $q(z)$ is

$$q(z) = \sum_{l=1}^{L} z^{\beta_l} \sum_{j=0}^{m(l)} \delta(l,j) \left(\log z\right)^j, \tag{4.6}$$

where $\beta_1, \cdots, \beta_L$ are complex numbers, as are the coefficients $\delta(l,j)$. The $j's$, l and $m(l)$ are nonnegative integers.

Note that we are fixing a log-polynomial sum $q(z)$ satisfying (4.4) . For example we can take $q = \varphi|T - \varphi$ with φ is any log-polynomial sum.

The Eichler's generalized Poincaré series required for our proof is defined by

$$\Psi(s) = \Psi\left(\{q_M\}, f, m, z\right) = \sum_M \frac{q_M(z) f(Mz)}{\left[(az+b) + i(cz+d)\right]^{2m}}, \tag{4.7}$$

where the summation is over all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. We assume that f is bounded in $\mathcal{H}$ and $m \in \mathbb{Z}^+$ is a "large" integer to ensure absolute convergence. Here $\{q_M\}$ is the cocycle generated by (1.10) and (4.5) from the log-polynomial sum $q(z)$ given by (4.6) . We shall show below that for m sufficiently large, the series (4.7) converges absolutely for $z \in \mathcal{H}$ and uniformly on compact subset of $\mathcal{H}$. For such m we show that $\Psi \in \mathcal{P}$.

Assuming (4.7) converges absolutely for $z \in \mathcal{H}$, for now, by rearrangement of terms, it follow that

$$\left(\Psi|_v^k M\right)(z) = (cz+d)^{2m}\,\Psi(z) - (cz+d)^{2m}\,\vartheta_m(z) q_M(z) \tag{4.8}$$

for $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G$, where

$$\vartheta_m(z) = \vartheta_m(f;z) = \sum_M \frac{f(Vz)}{[(az+b)+i(cz+d)]^{2m}}, \tag{4.9}$$

where the summation is over all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$.

With similar arguments as in (4.7), for large $m \in \mathbb{Z}^+$, the series in (4.9) converges absolutely and uniformly on compact subset of $\mathcal{H}$. Therefore, $\vartheta_m(z)$ is holomorphic automorphic form on G of weight $2m$ and multiplier system $\equiv 1$:

$$(cz+d)^{-2m}\,\vartheta_m(Mz) = \vartheta_m(z) \tag{4.10}$$

for all $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G$.

Thus, putting $F(z) = -\frac{\Psi(z)}{\vartheta_m(z)}$ and applying (4.8), we get that

$$F|_v^k M = -\frac{(\Psi|_v^k Mz)}{\vartheta_m(Mz)} = -\frac{\Psi(z)}{\vartheta_m(z)} + q_M(z),$$

for $M \in G$. That is

$$F|_v^k M = F(z) + q_M(z), \;\; M \in G. \tag{4.11}$$

In [7], Knopp and Sheingorn stated without proof the absolute convergence of the series (4.7) . However, they have supplied references [6] , [7], and [22] for the proof. However, the proofs in the references are for the H-groups. Since our group G is not an H-group and this construction is essential for the proof of the main correspondence Theorem 4.3.1, we shall give a proof below. The proof of Proposition 4.1.6, below treated differently from [19]. We will break the construction of the proof in to several lemmas.

First we estimate the cocycles $\{q_M | M \in G\}$ based upon a series of lemmas. We use m_1, m_2, m_3, ... to denote arbitrary constants. Our first lemma has appeared in [[6], Lemma 4] and [[22], Lemma 4] we include the proof here for sake of completeness.

Lemma 4.1.2. *For real numbers c, d and $z = x + iy$, we have*

$$\left(\frac{y^2}{1+4|z|^2}\right)\left((c^2+d^2\right) \leq |cz+d|^2 \leq 2\left(|z|^2+y^{-2}\right)\left(c^2+d^2\right).$$

Proof. By Schwarz's inequality, we have
$|cz + d|^2 \leq (|z|^2 + 1)(c^2 + d^2)$. For $z \in \mathcal{H}$, $|z|^2 + 1 \leq 2(|z|^2 + y^{-2})$, which imply that

$$|cz+d|^2 \leq 2(|z|^2+y^{-2})(c^2+d^2)$$

and

$$|cz+d|^2 \geq c^2y^2 \text{ and } \left|\overline{z}|^2|cz+d\right|^2 = |c|z|^2+dz|^2 \geq d^2y^2.$$

Thus,

$$|cz+d|^2 \geq \frac{y^2(c^2+d^2)}{(1+|z|^2)} \geq \frac{y^2(c^2+d^2)}{(1+4|z|^2)}.$$

□

Since $G_\lambda = \langle S_\lambda, T\rangle$, with S_λ pararabolic and T nonparabolic generators, any $M \in G$ has a factorization of M into sections [[22], PP.156 − 157],

$$M = C_1C_2\cdots C_r,$$

where r is the number of sections. In fact one can express as

$$M = S_\lambda^{a_1}TS_\lambda^{a_2}T\cdots TS_\lambda^{a_n}, a_i \in \mathbb{Z}, n \geq 2, a_2, \cdots, a_{n-1} \neq 0. \tag{4.12}$$

Each section C_i is either the nonparabolic generator of G or a power of the parabolic generator of G. Eichler in [17] has proved factorization into sections is possible for an H-groups. He showed that, for any M in an H-group, $M = C_1 \cdots C_r$, that $r \leq m_1 \log\mu(M) + m_2$, where $m_1, m_2 > 0$ are independent of M where, $\mu(M) = a^2 + b^2 + c^2 + d^2$ for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Note that $\mu(M) \geq 2$ and $\mu(MN) \leq \mu(M)\mu(N)$ for matrices of determinant one.

But G is not an H-group. However, an upper bound of the minimum length r was established by Knopp in [7]. In fact in (4.12) if $wl(M)$ denotes word length of M given by $wl(M) = n - 1 + \sum_{i=1}^{n} |a_i|$ and $fl(M) = n-1+l$, l is the length of non zero $a_i, i = 1, 2, ..., n$, then there are constants α_λ and β_λ such that

$$\begin{aligned} wl(M) &\leq \alpha_\lambda \mu(M), \\ fl(M) &\leq \beta_\lambda \log \mu(M). \end{aligned}$$

Thus, for our groups, the factorization into section has finite word length too.

Lemma 4.1.3. *Assume the cocycle* $\{q_M | M \in G\}$ *is in* $\mathcal{P}$ *and* $2\sigma > -k$, $\rho > k$, *where*

$$|q_M| < K \left(|z|^\rho + y^{-\sigma} \right). \tag{4.13}$$

Then there exists m_3 *depending only on* G *and* $\{q_M\}$ *such that*

$$|q_{C_h}|_k^v C_{h+1} \cdots C_r| \leq m_3 \mu(M)^e \left[|z|^{6e-2k} + y^{-6e+2k} \right], \tag{4.14}$$

for $1 \leq h \leq r$, *where* $e = \max(\rho/2,\, \sigma + k/2)$ *and* $M = C_1 ... C_r$.

Proof. First we consider the case when C_h is a nonparabolic generator. Let $V = C_{h+1} \cdots C_r = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then, by (4.13),

$$\begin{aligned} |q_{C_h}|V| &= |(\gamma z + \delta)|^k \, |q_{C_h}(Vz)| < |(\gamma z + \delta)|^k K \left[|Vz|^\rho + y^{-\sigma} |\gamma z + \delta|^{2\sigma} \right] \\ &= K |\alpha z + \beta|^\rho |\gamma z + \delta|^{k-\rho} + K |\gamma z + \delta|^{2\sigma + k} y^{-\sigma}. \end{aligned}$$

Then by the Lemma (4.1.2),

$$\begin{aligned} |\alpha z + \beta|^\rho &\leq 2^{\rho/2} \left(|z|^2 + y^{-2} \right)^{\rho/2} \left(\alpha^2 + \beta^2 \right)^{\rho/2}, \\ |\gamma z + \delta|^{2\sigma+k} &\leq 2^{\sigma+k/2} \left(|z|^2 + y^{-2} \right)^{\sigma+k/2} (\gamma^2 + \delta^2)^{\sigma+k/2}, \\ &\text{and} \\ |\gamma z + \delta|^{k-\rho} &\leq \left(\frac{y^2}{1 + 4|z|^2} \right)^{\frac{k-\rho}{2}} (\gamma^2 + \delta^2)^{\frac{k-\rho}{2}}. \end{aligned}$$

Hence,

$$\begin{aligned} |q_{C_h}|V| \;&<\; K 2^{\rho/2} \left(|z|^2 + y^{-2} \right)^{\rho/2} \left(\alpha^2 + \beta^2 \right)^{\rho/2} \left(\frac{1 + 4|z|^2}{y^2} \right)^{\frac{\rho-k}{2}} (\gamma^2 + \delta^2)^{\frac{k-\rho}{2}} \\ &\quad + K 2^{\sigma+k/2} \left(|z|^2 + y^{-2} \right)^{\sigma+k/2} (\gamma^2 + \delta^2)^{\sigma+k/2} y^{-\sigma}. \end{aligned}$$

For a discrete group that has translations, if $\gamma \neq 0,\ C_{h+1}...C_r = \begin{pmatrix} * & * \\ \gamma & * \end{pmatrix} \in G$, then there exist a constantm_4 such that $|\gamma| \geq m_4 > 0$, see ([16], $P.35$). It follow that $\gamma^2 + \delta^2$ has a positive lower bound. Thus

$$\begin{aligned} |q_{C_h}|V| &< m_5 \left(|z|^2 + y^{-2}\right)^{\rho/2} \left(\alpha^2 + \beta^2\right)^{\rho/2} \left(\frac{1 + 4|z|^2}{y^2}\right)^{\frac{\rho-k}{2}} \\ &\quad + m_5' \left(|z|^2 + y^{-2}\right)^{\sigma+k/2} (\gamma^2 + \delta^2)^{\sigma+k/2} y^{-\sigma}. \end{aligned}$$

By Lemma 5 of [7] we have

$$\mu(C_h \cdots C_r) \leq m_6 \mu(C_1 \cdots C_r), \quad \text{for } 1 \leq h \leq r.$$

Thus, $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \mu(V) \leq m_7 \mu(M)$, consequently

$$\begin{aligned} |q_{C_h}|V| &\leq m_8 \mu(M)^{\frac{\rho}{2}} \left(|z|^2 + y^{-2}\right)^{\frac{\rho}{2}} y^{k-\rho} (1 + 4|z|^2)^{\frac{\rho-k}{2}} \\ &\quad + m_8' \mu(M)^{\sigma+k/2} y^{-\sigma} (|z|^2 + y^{-2})^{\sigma+k/2}. \end{aligned}$$

Letting $e = \max(\rho/2, \sigma + k/2)$, we get

$$\begin{aligned} |q_{C_h}|V| &\leq m_9 \mu(M)^e \left(|z|^2 + y^{-2}\right)^e \left\{ y^{k-\rho} \left(1 + 4|z|^2\right)^{(\rho-k)/2} + y^{-\sigma} \right\} \\ &\leq m_9 \mu(M)^e \left(|z|^2 + y^{-2}\right)^e \left\{ 1/2 y^{2k-2\rho} + 1/2 \left(1 + 4|z|^2\right)^{\rho-k} + y^{-\sigma} \right\}. \end{aligned}$$

Let $\sigma \leq e - k/2$ and $\rho - k \leq 2 - k$, we have

$$\begin{aligned} |q_{C_h}|V| &\leq m_9' \mu(M)^e (|z| + y^{-2})^e \left[|z|^{4e-2k} + y^{-4e+2k} \right] \\ &\leq m_3 \mu(M)^e \left[|z|^{6e-2k} + y^{-6e+2k} \right]. \end{aligned}$$

Next, we consider the case when C_h is a parabolic section, i.e., $C_h = S_\lambda^m$. In this case $q_{S_\lambda} = q'|S_\lambda - q',\ q' \in \{q_M | M \in G\}$. By the cocycle condition (1.10), $q_{C_h} = q'|C_h - q'$. This implies that,

$$q_{C_h}|C_{h+1} \cdots C_r = q'|C_h \cdots C_q - q'|C_{h+1}...C_r.$$

By the previous case, we have

$$|q_{C_h}|C_{h+1} \cdots C_r| \leq m_3 \mu(M)^e \left(|z|^{6e-2k} + y^{-6e+2k}\right).$$

This completes the proof. □

By repeatedly applying the cocycle condition for $M = C_1 \cdots C_r$, we get

$$q_M = q_{C_1 \cdots C_r} = q_{C_1}|C_2 \cdots C_r + q_{C_2}|C_3 \cdots C_r + \cdots + q_{C_r}, \tag{4.15}$$

with terms on the right hand side bound by $wl\, M \leq \alpha_\lambda \mu(M)$ and $fl\, M \leq \beta_\lambda \log \mu(M)$, gives, $r \leq f_\lambda \mu(M)$. Hence, by (4.13) and Lemma 4.1.3, we have the estimate of the cocycle for $M \in G = G(\lambda)$, $|q_M(z)| \leq m_{10}\mu(M)^e \left(|z|^\eta + y^{-\eta}\right) q$. That is,

$$|q_M(z)| \leq m_{11}\mu(M)^{e+1} \left(|z|^\eta + y^{-\eta}\right), \tag{4.16}$$

where $\eta = 6e - 2k$. The following lemma is Lemma 4 of [22]. We introduce the region E_α, $\alpha > 0$:

$$E_\alpha = \{z \in \mathbb{C} : |x| \leq \frac{1}{\alpha},\ y > \alpha,\ z = x + iy\}. \tag{4.17}$$

Note that every compact subset of $\mathcal{H}$ is contained in some E_α.

Lemma 4.1.4. *Let c, d be real and $\tau \in E_\alpha$. We have*

$$m_{12}(c^2 + d^2) \leq |c\tau + d|^2 \leq m_{13}|\tau|^2(c^2 + d^2).$$

Proof. When $c = 0$, it holds for $m_{13} \geq \alpha^{-2}$. When $c \neq 0$, we have

$$\begin{aligned} \left|\frac{c\tau + d}{ci + d}\right|^2 &= \left|\frac{\tau + d/c}{i + d/c}\right|^2 \leq \sup_{-\infty < u < \infty} \left|\frac{\tau + u}{i + u}\right|^2 = \sup \frac{(x+u)^2 + y^2}{1 + u^2} \\ &\leq 2 \sup \frac{|\tau|^2 + u^2}{1 + u^2} \leq 2|\tau|^2 + \frac{u^2}{1 + u^2}(1 - |\tau|^2) \\ &\leq 2|\tau|^2 + 1 \leq (2 + \alpha^{-2})|\tau|^2 = m_{13}|\tau|^2, \end{aligned}$$

and $\left|\frac{c\tau+d}{ci+d}\right|^2 \geq \inf \frac{(x+u)^2+y^2}{1+u^2} = \inf_u \phi(u)$.
For $|u| \leq 2/\alpha$, $\phi(u) \geq \frac{y^2}{1+u^2)} \geq \frac{\alpha^2}{(1+4\alpha^{-2})}$, where as, for $|u| > 2/\alpha$,

$$\phi(u) > \frac{u^2}{4(1 + u^2)} \geq \frac{\alpha^{-2}}{(1 + 4\alpha^{-2})}.$$

Hence, $\inf_u \phi(u) \geq (1 + 4\alpha^{-2})^{-1} \min(\alpha^2, \alpha^{-2}) = m_{12}$. This completes the proof of the lemma. □

For the following lemma, we use the Ford fundamental region R_λ [see [16], *P*.58] or [[15], *P*.139]. For $G = G(\lambda)$, $\lambda > 2$, has ∞ as a fixed point. The stabilizer of ∞, denoted by Γ_∞, is a cyclic and $\Gamma_\infty = \langle S_\lambda \rangle$. A fundamental region R_∞ of Γ_∞ is the strip:

$$z = x + iy, -\lambda/2 < x < \lambda/2,\ y > 0.$$

Each of the elements of Γ_∞ fixes ∞, and Γ_∞ a subgroup of G. Let $\triangle = G - \Gamma_\infty$. Then $\triangle$ consists of elements that have isometric circles. $\overline{R}(\triangle)$ denote the set of closed isometric disks of elements of $\triangle$. Each transformation of Γ_∞ maps $\overline{R}(\triangle)$ on to itself (for detail see [[15], P.142]. $R_\lambda = \{z \in \mathcal{H} : \Re(z) < \lambda/2$ and $|cz+d| > 1$ for all $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G - \Gamma_\infty\}$. Then there exist $y_o > 0$ with $w_o = iy_o \in R_\lambda$ such that $M \in G - \Gamma_\infty$, $|\Re(Mw_o)| \leq \lambda/2$. Now we prove the following lemma and we may assume $c \neq 0$, otherwise $M = \pm I$.

Lemma 4.1.5. *If* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \triangle$, *and* $|\Re(Mw_o)| \leq \lambda/2$, *then* $\mu(M) \leq m_{14}(c^2+d^2)$.

Proof. First observe that $0 < \Im(M(w_o)) = \frac{y_o}{|cw_o+d|^2} = \frac{y_o}{c^2y_o^2+(cx_o+d)^2} \leq \frac{1}{c^2y_o^2}$. If $c = 0$, then $d = 1$ and $\Im(M(w_o)) = y_o$. For $c \neq 0$, then there exist $m_{15} > 0$ such that $|c| \geq m_{15}$, thus, $0 < \Im(M(w_o)) \leq \frac{1}{m_{15}^2 y_o}$.
Hence $|(M(w_o))|^2 \leq m_{16}$, which with Lemma 4.1.3 gives,

$$\begin{aligned} m_{17}(a^2+b^2) &\leq |aw_o+b|^2 \leq m_{16}|cw_o+d|^2 \\ &\leq m'_{16}|w_o|^2(c^2+d^2) \leq m''_{16}(c^2+d^2). \end{aligned}$$

Since $\mu(M) = a^2+b^2+c^2+d^2$, it follow that $\mu(M) \leq m_{14}(c^2+d^2)$. □

From (4.16) and Lemma 4.1.5, we have the estimate of the cocycle $\{q_M,\, M \in G\}$

$$|q_M| \leq m'_{11}(c^2+d^2)^{e+1}\left(|z|^\eta + y^{-\eta}\right). \tag{4.18}$$

We use (4.18), to prove the following proposition using isometric circles, with different treatment of the proof given in [19].

Proposition 4.1.6. *For m sufficiently large ($m \geq e+2$) the Eichler's Generalized Poincaré series* (4.7), *converges absolutely for $z \in \mathcal{H}$ and uniformly on compact subset of $\mathcal{H}$. For such m, we also have $\Psi \in \mathcal{P}$.*

Proof. By assumption f is bounded in $\mathcal{H}$. Note then that we need to estimate the absolute value of the series (4.7) . That is,

$$|\Psi(s)| = \left| \sum_M \frac{q_M(z)f(Mz)}{[(az+b)+i(cz+d)]^{2m}} \right|$$

$$\leq Km'_{11}(|z|^{\eta}+y^{-\eta})\sum_{M\in G}\frac{(c^2+d^2)^{e+1}}{\left|[(az+b)+i(cz+d)]^{2m}\right|}$$
$$\leq K'm'_{11}(|z|^{\eta}+y^{-\eta})\sum_{M}\frac{|cz+d|^{2e+2}}{\left|[(az+b)+i(cz+d)]^{2m}\right|}$$
$$= K'm'_{11}(|z|^{\eta}+y^{-\eta})\sum_{M}\frac{|cz+d|^{2e+2}}{|(Mz+i)^{2m}(cz+d)^{2m}|}$$
$$= K'm'_{11}(|z|^{\eta}+y^{-\eta})\sum_{M}\frac{1}{|(Mz+i)^{2m}|\,|(cz+d)^{2m-2e-2}|},$$

where we have used the bound of f, Lemma 4.1.4 and (4.18). Let

$$H(z)=\sum_{M}\frac{1}{(Mz+i)^{2m}(cz+d)^{2m-2e-2}}, \tag{4.19}$$

where the sum run over $M=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in G$. Next, we show that (4.19) converges. With E_α as in (4.18) , since all $M \in G$ preserves $\mathcal{H}$, then $Mz \neq -i$, for $z\in E_\alpha$, $M\in G$. Suppose $|c|>r_o=\max(2, 2/\alpha)$. Then E_α lies out side the isometric circles $|cz+d|$ of radius $\frac{1}{|c|}<\alpha/2$. Hence, $M(E_\alpha)$ is inside the isometric circle of the inverse transformation, $|-cz+b|$. In $\mathcal{R}_\lambda$, the largest radius is the unit disc corresponding to $c=\pm 1$. Any other isometric circle has radius $<1/2$. It follows that, for all such c, $|Mz+i|>\frac{1}{2}$.

For case of finitely many c in $|c|\leq r_o$, we have $|Mz+i|>0$. Therefore, there is a constant M^* depending on E_α such that

$$|Mz+i|>M^*,\quad z\in E_\alpha,\ M\in G. \tag{4.20}$$

Also, $|cz+d|=|c||z+d/c|$. If D is a compact subset of E_α, the center of the isometric circles $\{-d/c\}$ is bounded above and $|z+d/c|\geq r>0$, where $r>0$ is a constant depending on D. Thus,

$$|cz+d|\geq r|c|,\quad z\in D,\ M\in G. \tag{4.21}$$

Using (4.20) and (4.21) in (4.19), we get

$$|H(z)|=\left|\sum_{M}\frac{1}{(Mz+i)^{2m}(cz+d)^{2m-2e-2}}\right|$$

$$\le \sum_M \frac{1}{|(Mz+i)^{2m}|\,|(cz+d)|^{2m-2e-2}}$$
$$\le \left(\frac{1}{M^*}\right)^{2m}\left(\frac{1}{r}\right)^{2m-2e-2}\sum_M |c|^{2e-2m+2}.$$

Consider the last series $\sum_M |c|^{-2t}$, $t = m - e - 1$. Since G is finitely generated Fuchsian group of the second kind acting on $\mathcal{H}$, $H(z)$ converges for $t = m-e-1 \ge 1$. (see [15] PP. 177-179). Note that the series diverge for $t = 1$ for Hecke group of $\lambda \le 2$ (finitely generated Fuchsian group of first kind), see ([15], P. 181). Thus, H converge absolutely and uniformly on E_α, it follows that, H is holomorphic on E_α. Hence,

$$|\Psi(z)| \le K' m'_{11} m^* \left(|z|^\eta + y^{-\eta}\right), \tag{4.22}$$

where m^* is a constant depending on m. Therefore, $\Psi(z)$ converge absolutely and uniformly on a compact subset of $\mathcal{H}$. Also Ψ is in $\mathcal{P}$. This completes the proof. □

Now returning to the proof of Theorem 4.1.1, note that (4.11) , is not quite enough to call $F(z)$ an automorphic integral, for F may have poles in $\mathcal{H}$ and $\vartheta_m(z)$ can be identically zero. Which imply that $F(z)$ fail to be in $\mathcal{P}$. For this we choose f, bounded and analytic in $\mathcal{H}$, so that $\vartheta_m(f; z)$ is bounded away from zero in $\mathcal{H}$, by applying the "Modified Main Lemma" Page 14 of [19], that can be applied for the group G, (is a finitely generated Fuchsian group of the second kind). Since the result is applied to any region which is conformaly equivalent to the unit disk. For the region $\mathcal{H}$, we consider the mapping $g(z) = \frac{(-z+i)}{z+i)}$ of $\mathcal{H}$ onto the unit disk and the group G as follow:

Lemma 4.1.7. *Suppose $m \in \mathbb{Z}$, with $m \ge 2$ and $g(z) = \frac{(-z+i)}{z+i)}$. Then there exist a polynomial $P = P_m$ such that*

(i) the Poincaré series

$$\vartheta_m\left(P_m o g; z\right) = \sum_{V\in G} \frac{P_m\left(\frac{-Vz+i}{Vz+i}\right)}{\left[(\alpha z+\beta)+i(\gamma z+\delta)\right]^{2m}}, \quad V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

has an expansion at $i\infty$ of the form

$$\vartheta_m\left(P_m o g; z\right) = \sum_{n=1}^{\infty} c_n e^{2\pi i n z/\lambda}, \quad c_1 \ne 0, \tag{4.23}$$

valid for all $z \in \mathcal{H}$;

(ii) $|g'(z)|^{-m} |\vartheta_m (P_m og; z)|$ *is bounded away from zero in the "truncated fundamental region"* $\mathcal{R}_\lambda(y_0) = \{z \in \mathcal{R}_\lambda | \Im z < y_0\}$, *for each* $y_0 > 0$.

From (ii) we see that, $|z+i|^{2m}|\vartheta_m (P_m og; z)|$ is bounded away from zero in $\mathcal{R}_\lambda(y_0)$, for each $y_0 > 0$. It immediately follows that $|\vartheta_m (P_m og; z)|$ is bounded away from zero in $\mathcal{R}_\lambda(y_0)$, for each $y_0 > 0$.
Now, the required automorphic integrals function be constructed using the function $\Psi(z) \equiv \Psi(\{q_M\}, P_m og, m; z)$, with sufficiently large $m \geq m_0$, where m_0 is as in Proposition 4.1.6, Ψ hods (4.8). We redefine once again the function:

$$F(z) = \frac{-\psi(\{q_M\}, P_m og, m; z)}{\vartheta_m (P_m og; z)}. \tag{4.24}$$

With the choice of $f = P_m og$, F satisfies (4.11). Since Ψ is uniformly convergent, we have

$$\begin{aligned}\lim_{\Im(z)\to\infty} |\Psi(z)| &\leq \lim_{\Im(z)\to\infty} Km'_{11}(|z|^\eta + y^{-\eta}) \sum_{M\in G} \frac{(c^2+d^2)^{e+1}}{\left|[(az+b)+i(cz+d)]^{2m}\right|} \\ &= \sum_{M\in G} \lim_{\Im(z)\to\infty} Km'_{11}(|z|^\eta + y^{-\eta}) \frac{(c^2+d^2)^{e+1}}{\left|[(az+b)+i(cz+d)]^{2m}\right|} \\ &= 0,\end{aligned}$$

thus, Ψ is analytic in $\mathcal{H}$ and by (4.8), $\Psi(z+\lambda) = \Psi(z)$. Consequently,

$$\Psi(z) = \sum_{n=1}^{\infty} d_n e^{2\pi i n z/\lambda}, \quad \lambda > 2, \quad z \in \mathcal{H}. \tag{4.25}$$

By Lemma 4.1, $\vartheta_m (P_m og; z)$ is bounded away from zero in $\mathcal{H}$, with the exponential series (4.23), (4.25) and by assumption $q_M \in \mathcal{P}$ for all $M \in G$, by the result of [[21] , PP. 149-150], we have $F \in \mathcal{P}$. This with the assumption $q_{S_\lambda} = 0$, and the transformation law

$$\left(F|_v^k M\right)(z) = F(z) + q_M,\ M \in G_\lambda,\ M = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix} \in G$$

and $F(z)$ is analytic at $i\infty$ imply

$$F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda},\ z \in \mathcal{H},\ \lambda > 2. \tag{4.26}$$

Furthermore, from the result of Knopp [[21], P. 151], it follow that

$$a_n = \mathcal{O}(n^\rho) \quad \text{for} \quad \text{some}\, \rho > 0,\ n \to \infty. \tag{4.27}$$

This completes the proof of Theorem 4.1.1. □

4.2 Estimation of Automorphic Integrals

In this section, we provide a technique to approximate automorphic integrals with infinite log-polynomial sum period functions by automorphic integrals with finite log-polynomial sum period functions. Which is essential to apply in the proof of the main correspondence theorem with Dirichlet series having infinitely many poles, for Hecke group for $\lambda > 2$.

Lemma 4.2.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi niz/\lambda},\ \lambda > 2$ be holomorphic in $\mathcal{H}$ with $a_n = \mathcal{O}(n^\gamma),\ as\ n \to \infty$, for some $\gamma > 0$. Suppose that $f(z)$ satisfies the transformation law $\bar{v}(T) z^{-k} f\left(\frac{-1}{z}\right) = f(z) + q(z)$, where*

$$q(z) = \sum_{l=1}^{\infty} \left(\frac{z}{i}\right)^{-\beta_l} \sum_{j=0}^{m(l)} \delta(l,j) \left(\log \frac{z}{i}\right)^j$$

is an infinite log–polynomial sum satisfying the conditions in Corollary 3.2.2 and $|\Im(\beta_l)| \to \infty$ as $l \to \infty$. Then there exist automorphic integrals $f_N(z)$ on $G = G(\lambda)$, $\lambda > 2$ with finite log–polynomial period functions such that $f_N \to f$ as $N \to \infty$ uniformly on compact subset of $\mathcal{H}$.

Proof. We use the result of construction of automorphic integrals in the previous section, to proof the existence of automorphic integrals $f_N(z)$ on $G = G(\lambda)$, $\lambda > 2$ with finite log–polynomial period functions such that $f_N \to f$ as $N \to \infty$ uniformly on compact subset of $\mathcal{H}$.

We have that $q|_T + q = 0$ from the group relation. This means that both $-\beta_l$ and $k - \beta_l$ appear in pair as exponents in the log–polynomial sum. By rearranging, we can make $-\beta_l$ and $k - \beta_l$ appear consecutively, we can assume that the function

$$r_N(z) = \sum_{l=2N+1}^{\infty} \left(\frac{z}{i}\right)^{-\beta_l} \sum_{j=0}^{m(l)} \delta(l,j) \left(\log \frac{z}{i}\right)^j,$$

satisfies $r_N|_T + r_N = 0$ for all N. For every l, we have $|\Re(\beta_l)| \leq \beta$, then $r_N(z)$ satisfies the first condition of the Corollary 3.2.2 with the same β for each N. It is easy to see $|\Im(\beta_l)| \to \infty$ as $l \to \infty$ holds for r_N. Since

$$\sum_{l=2N+1}^{\infty} \sum_{j=0}^{m(l)} e^{|\mathrm{Im}(\beta_l)|\frac{\pi}{2}} j!\, |\delta(l,j)|\, \epsilon^j \ \leq\ \sum_{l=1}^{\infty} \sum_{j=0}^{m(l)} e^{|\mathrm{Im}(\beta_l)|\frac{\pi}{2}} j!\, |\delta(l,j)|\, \epsilon^j < \infty$$

and so the second condition of the corollary holds for each function r_N. Thus, $r_N \in \mathcal{P}$.

From (3.9), taking both $A = B = 2\max(\beta, 4)$, we get

$$\begin{aligned} |r_N(z)| &\le K' \sum_{l=2N+1}^{\infty} \sum_{j=0}^{m(l)} e^{|\mathrm{Im}(\beta_l)|\frac{\pi}{2}} j!\, |\delta(l,j)| \left(|z|^A + y^{-B}\right) \\ &\le \left(K' \sum_{l=1}^{\infty} \sum_{j=0}^{m(l)} e^{|\mathrm{Im}(\beta_l)|\frac{\pi}{2}} j!\, |\delta(l,j)| \right) \left(|z|^A + y^{-B}\right). \end{aligned} \tag{4.28}$$

Therefore, $r_N(z)$ is the tail of a convergent sum, it implies that, $r_N(z) \to 0$ uniformly on compact subset of $\mathcal{H}$ as $N \to \infty$. Uniformly because in (4.28) in the last inequality the constants A, B, and

$$K = K' \sum_{l=1}^{\infty} \sum_{j=0}^{m(l)} e^{|\mathrm{Im}(\beta_l)|\frac{\pi}{2}} j!\, |\delta(l,j)|$$

are all independent of N. Now we generate parabolic cocycles $\{(r_N)_V(z) : V \in G\}$ with the assumption $(r_N)_S \equiv 0$ and $(r_N)_T(z) = r_N(z)$ using the cocycle condition. Let us define the Poincaré series as in the previous section using this parabolic cocycles:

$$\psi_N(z) = \sum_V \frac{(r_N)_V(z) f(Vz)}{[(\alpha z + \beta) + i(\gamma z + \delta)]^{2m}}, \tag{4.29}$$

where the summation is over all $V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$, we assume f is bounded in $\mathcal{H}$ and $m \in \mathbb{Z}^+$ is large integers.
Then

$$|\psi_N(z)| \le K \sum_V \frac{|(r_N)_V(z)|}{\left|[(\alpha z + \beta) + i(\gamma z + \delta)]^{2m}\right|},$$

and from (4.28)

$$|r_N(z)| \le K' \sum_{l=2N+1}^{\infty} \sum_{j=0}^{m(k)} e^{|\mathrm{Im}(\beta_l)|\frac{\pi}{2}} j!\, |\delta(l,j)| \left(|z|^A + y^{-B}\right).$$

Thus,

$$|\Psi_N(z)| \ \le \ KK' \sum_{l=2N+1}^{\infty} \sum_{j=0}^{m(k)} e^{|\Im(\beta_l)|\frac{\pi}{2}} j!\, |\delta(l,j)| \left(|z|^A + y^{-B}\right)$$

$$\left(\sum_{V\in G} \frac{1}{\left|(\alpha z+\beta)+i(\gamma z+\delta)\right|^{2m}}\right)$$
$$\le \; KK' \sum_{l=1}^{\infty}\sum_{j=0}^{m(k)} e^{|\Im(\beta_l)|\frac{\pi}{2}} j!\,|\delta(l,j)|\left(|z|^A+y^{-B}\right)$$
$$\left(\sum_{V\in G} \frac{1}{\left|(\alpha z+\beta)+i(\gamma z+\delta)\right|^{2m}}\right).$$

Now, from the previous section , applying equation (4.20) and (4.21) to the last series, we get

$$\begin{aligned}\sum_{V\in G} \frac{1}{|(\alpha z+\beta)+i(\gamma z+\delta)|^{2m}} &= \sum_{V\in G} \frac{1}{|(Vz+i)^{2m}||(\gamma z+\delta)|^{2m}} \\ &\le \left(\frac{1}{M^*}\right)^{2m}\left(\frac{1}{r}\right)^{2m}\sum_{V}|\gamma|^{-2m},\end{aligned}$$

thus, the last series converges absolutely and uniformly for $m \ge 1$, in a compact subset of $\mathcal{H}$, (see [15] PP. 177-179). Hence, we have

$$|\Psi_N(z)| \le KK'K'' \sum_{l=1}^{\infty}\sum_{j=0}^{m(k)} e^{|\mathrm{Im}(\beta_l)|\frac{\pi}{2}} j!\,|\delta(l,j)|\left(|z|^A+y^{-B}\right),$$

where K'' is a positive constant depending only on m, and all the constants are independent of N.

Therefore, $\Psi_N(z)$ is a tail of a uniformly convergent series on the compact subset of $\mathcal{H}$. It follows that $\Psi_N(z) \to 0$ uniformly on compact subset of $\mathcal{H}$ as $N \to \infty$. Now define

$$F_N(z) = \frac{-\Psi_N(r_N,\, f,\, m;\, z)}{\vartheta_m(f; z)}. \tag{4.30}$$

Since we are using the construction of previous section, then $\vartheta_m(f; z)$ holomorphic automorphic form with multiplier system 1 and bounded away from zero with a suitable choice of f. It follows that $F_N(z) \to 0$ uniformly on compact subset of $\mathcal{H}$ as $N \to \infty$.

Following similar procedure as in the construction in the previous section, $F_N(z)$ is analytic in $\mathcal{H}$, as well as

$$F_N|T = F_N + (r_N)_T\,, \tag{4.31}$$

which implies that, $F_N(z)$ is an automorphic integral with period function $(r_N)_T$.
Let $f_N(z) = f(z) - F_N(z)$, then $f_N(z)$ is analytic in $\mathcal{H}$ and

$$f_N(z)|T = f|T - F_N|T = [f(z) + q(z)] - [F_N + (r_N)_T].$$

Thus,

$$f_N(z)|T = f_N(z) + \sum_{l=1}^{2N} \left(\frac{z}{i}\right)\right)^{-\beta_l} \sum_{j=0}^{m(l)} \delta(l,j) \left(\log \frac{z}{i}\right)^j, \tag{4.32}$$

which implies that, $f_N(z)$ is an automorphic integrals with finite log-polynomial sum period functions. Also, $f_N(z) \to f$ uniformly on a compact subset of $\mathcal{H}$ as $N \to \infty$.
Therefore, $f_N(z)$ is the required automorphic integrals with finite log-polynomial sum period functions, which converge to a function f.

□

4.3 The Main Correspondence Theorem

In this section, we use the infinite log-polynomial sum period function discussed in chapter 3 and the approximation in Lemma 4.2.1. We will assume in addition to the necessary condition to for the convergence of the infinite log-polynomial sum period function, in particular Corollary 3.2.2. First let us state some of the conditions necessary for the next correspondence theorem.

(A) $|\Im(\beta_l)| \to \infty$, as $l \to \infty$ and

(B) There exist $\epsilon > 0$ and a sequence of positive numbers $T_n \to \infty$ such that $|\Im(\beta_l) - T_n| \geq \epsilon$ for all n and l.

We can observe that condition (B) implies that condition (A). Condition (B) means, β_l have uniform gaps in the vertical strip. We shall all assume due to the construction of the previous section that the multiplier system v on $G = G(\lambda)$, $\lambda > 2$ of weight k with $v(S_\lambda) = 1$.

Theorem 4.3.1. *Let k be real, $v(T) = e^{-\pi ik/2}$ and assume that*

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi inz/\lambda}, \quad z \in \mathcal{H}, \quad \lambda > 2$$

is holomorphic in $\mathcal{H}$ *with* $a_n = \mathcal{O}(n^\rho)$, $\rho > 0$, $n \to \infty$. *Let*

$$q(z) = \sum_{l=1}^{\infty} \left(\frac{z}{i}\right)^{-\beta_l} \sum_{j=0}^{m(l)} \delta(l,j) \left(\log \frac{z}{i}\right)^j$$

be a log-polynomial sum satisfying the conditions in Corollary3.2.2, (A) *and* (B). *Define the functions*

$$\varphi_f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

$$\Phi_f(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\varphi_f(s),$$

$$Q(s) = \overline{v}(T) \sum_{l=1}^{\infty} \sum_{j=0}^{m(l)} \frac{(-1)^{j+1} j! \delta(l,j)}{(s-\beta_l)^{j+1}} + a_o \left(\frac{i^k v(T)}{s-k} - \frac{1}{s}\right).$$

Then,
$f(z)$ *satisfies the transformation law*

$$z^{-k} f\left(\frac{-1}{z}\right) = v(T)f(z) + q(z), \tag{4.33}$$

is equivalent to the following three statements:

(a) $\Phi_f(s) - Q(s)$ *has analytic continuation to the entire s- plane,*

(b) $\Phi_f(s)$ *satisfies the functional equation*

$$\Phi_f(k-s) = i^k v(T)\Phi_f(s), \tag{4.34}$$

(c) $\Phi_f(s) - Q(s)$ *is bounded in 'lacunary' vertical strip, i.e.,* $\sigma_1 \leq \sigma \leq \sigma_2$ *and* $|t| \geq t_o > 0$.

Remark 4.3.2. *1. Since the proof of the theorem depends in an essential way upon the construction and approximation of automorphic integral done in the previous sections, we cannot drop the assumption* $\lambda > 2$, *i.e., we are working on the Hecke group* $G = G(\lambda)$, $\lambda > 2$.

2. The direct part of the theorem holds true for the general discrete Hecke group $G(\lambda)$, $\lambda > 0$.

3. Because of the functional equation (4.34), *and the need to have a Dirichlet series with infinitely many poles, which converges in some right half plane, we cannot relax the first condition in Corollary 3.2.2, that is, the poles lie in vertical strip is crucial.*

Proof of The Direct Part of The Theorem

Proof. By the Mellin transform of $f(iy) - a_0$

$$\begin{aligned}\Phi_f(s) &= \int_0^\infty (f(iy) - a_0)\, y^s \frac{dy}{y} \\ &= \int_0^\infty \left(\sum_{n=1}^\infty a_n e^{-2\pi n \frac{y}{\lambda}}\right) y^s \frac{dy}{y} \\ &= \sum_{n=1}^\infty a_n \int_0^\infty e^{-2\pi n \frac{y}{\lambda}} y^{s-1} dy \\ &= \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) \sum_{n=1}^\infty \frac{a_n}{n^s} \\ &= \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) \varphi_s(s).\end{aligned}$$

Since interchange of the order of summation and integration by absolute convergence, for $\sigma = \Re(s) > \gamma + 1$.

$$\begin{aligned}\Phi_f(s) &= \int_0^\infty (f(iy) - a_0)\, y^{s-1} dy \\ &= \int_0^1 [f(iy) - a_0]\, y^{s-1} dy + \int_1^\infty [f(iy) - a_0]\, y^{s-1} dy \\ &= \int_1^\infty \left[f\left(\frac{i}{y}\right) - a_0\right] y^{-s-1} dy + \int_1^\infty [f(iy) - a_0]\, y^{s-1} dy \\ &= I + II.\end{aligned}$$

Using (4.33) with $z = iy, y > 0$, we find that, for $\sigma > \sup(\gamma + 1, k)$

$$\begin{aligned}I = i^k v(T) &\int_1^\infty [f(iy) - a_0] y^{k-s-1} dy + i^k v(T) \int_1^\infty a_0 y^{k-s-1} dy - \int_1^\infty a_o y^{-s-1} dy \\ &+ i^k \int_1^\infty q(iy) y^{k-s-1} dy \\ = i^k v(T) &\int_1^\infty [f(iy) - a_0] y^{k-s-1} dy + a_0 \left(\frac{i^k v(T)}{s-k} - \frac{1}{s}\right) + i^k \int_1^\infty q(iy) y^{k-s-1} dy.\end{aligned}$$

Thus,

$$\Phi_f(s) = E(s) + R(s) + L(s),$$

where

$$E(s) = \int_1^\infty [f(iy) - a_0] y^{s-1} dy + i^k v(T) \int_1^\infty [f(iy) - a_0] y^{k-s-1} dy$$

$$R(s) = a_0 \left(\frac{i^k v(T)}{s-k} - \frac{1}{s} \right)$$
$$L(s) = i^k \int_1^\infty q(iy) y^{k-s-1} dy.$$

Now we examine first $E(s)$. Since it is an exponential decay, i.e., $f(iy) - a_0 = \mathcal{O}(e^{-2\pi \frac{y}{\lambda}})$ as y tends to ∞, it immediately follows that $E(s)$ is an entire function. Since $(i^k v(T))^2 = 1$, we replace s by $k-s$ in $E(s)$, we get, $E(k-s) = i^k v(T) E(s)$. Now we show that $E(s)$ is bounded in each set of the form, $\sigma_1 \le \sigma \le \sigma_2$, $|t| \ge t_0 > 0$ and $s = \sigma + it$.

$$E(s) \le \int_1^\infty |f(iy) - a_0| y^{\sigma-1} dy + \int_1^\infty |f(iy) - a_0| y^{k-\sigma-1} dy.$$

Here the right hand side of the above inequality is independent of the value of t.

$$E(s) \le \int_1^\infty |f(iy) - a_0| \left(y^{\sigma-1} + y^{k-\sigma-1} \right) dy.$$

We note that,

$$\begin{aligned} |f(iy) - a_0| &= \left| \sum_{n=1}^\infty a_n e^{-2\pi n \frac{y}{\lambda}} \right| \le \sum_{n=1}^\infty |a_n| e^{-2\pi n \frac{y}{\lambda}} \\ &\le M \sum_{n=1}^\infty n^\gamma e^{-2\pi n \frac{y}{\lambda}} \le M e^{-2\pi \frac{y}{\lambda}} \sum_{n=1}^\infty n^\gamma e^{-2\pi (n-1) \frac{y}{\lambda}} \\ &\le M e^{-2\pi \frac{y}{\lambda}} \sum_{n=1}^\infty n^\gamma e^{-\frac{2\pi(n-1)}{\lambda}}, \quad \text{for} \quad y \ge 1 \\ &\le M' e^{-2\pi \frac{y}{\lambda}}. \end{aligned}$$

Therefore, we show that

$$|E(s)| \le \int_1^\infty M' e^{-2\pi \frac{y}{\lambda}} \left(y^{\sigma-1} + y^{k-\sigma-1} \right) dy \le M' M'',$$

where M' is independent of σ and $y^{\sigma-1} + y^{k-\sigma-1}$ is bounded uniformly for $\sigma_1 \le \sigma \le \sigma_2$. Since for large y, $y \ge y_0$ and $\sigma_1 \le \sigma \le \sigma_2$,

$$\left(y^{\sigma-1} + y^{k-\sigma-1} \right) e^{-\frac{\pi y}{\lambda}} \le 1.$$

Hence,

$$|E(s)| \le M' \int_1^{y_0} \left(y^{\sigma-1} + y^{k-\sigma-1} \right) e^{-2\pi \frac{y}{\lambda}} dy + M' \int_{y_0}^\infty e^{-\pi \frac{y}{\lambda}} dy.$$

But $M' \int_{y_0}^{\infty} e^{-\pi \frac{y}{\lambda}} dy < \infty$, we have proved that $E(s)$ is entire function by analytic continuation. It is easy to see that $R(s)$ is meromorphic on $\mathbb{C}$. Since $R(s) \to 0$ as $|t| \to \infty$, then $R(s)$ is bounded on the strip and

$$\begin{aligned} R(k-s) = a_0 \left(\frac{i^k v(T)}{-s} - \frac{1}{k-s} \right) &= a_0 \left(\frac{1}{s-k} - \frac{i^k v(T)}{s} \right) \\ &= i^k v(T) a_0 \left(\frac{i^k v(T)}{s-k} - \frac{1}{s} \right) \\ &= i^k v(T) R(s). \end{aligned}$$

Thus, $R(s)$ satisfied the functional equation. Finally, we examine the remaining term

$$L(s) = i^k \int_1^{\infty} q(iy) y^{k-s-1} dy. \tag{4.35}$$

If $f(z)$ satisfies the transformation law (4.33), applying (4.33) twice and using $(i^k v(T))^2 = 1$, then $q(z)$ satisfies the equation

$$z^{-k} q \left(\frac{-1}{z} \right) + v(T) q(z) = 0, \quad z \in \mathcal{H}. \tag{4.36}$$

By (4.36) , we have an alternative representation

$$L(s) = -\bar{v}(T) \int_1^{\infty} q \left(\frac{i}{y} \right) y^{-s-1} dy. \tag{4.37}$$

Since the integrand of $L(s)$ is an infinite series, we need to justify term by term integration. A single term in $q(iy)y^{k-s-1}$ is the form $\delta y^{\alpha} (\log y)^j$ with $\delta, \alpha \in \mathbb{C}$, $\Re(\alpha) < -1$ and $j = 0, 1, 2, \ldots$. Using integration by parts j times or applying induction, we have

$$\int_1^{\infty} \delta y^{\alpha} (\log y)^j dy = (-1)^{j+1} \delta j! (\alpha + 1)^{-j-1}. \tag{4.38}$$

For $y \geq 1$, we have the estimation, $\log y \leq \frac{j}{e} y^{\frac{1}{j}}$, for any $j \geq 1$. Then $|\log y|^j \leq j! y$ for $j \geq 0$. Now, we can estimate the integrand (4.35) ,

$$\begin{aligned} \left| q(iy) y^{k-s-1} \right| &= \left| \sum_{l=1}^{\infty} y^{-\beta_l} \sum_{j=0}^{m(l)} \delta(l,j) (\log y)^j y^{k-s-1} \right| \\ &\leq \sum_{l=1}^{\infty} \sum_{j=0}^{m(l)} y^{k-1-\Re(s)-\Re(\beta_l)} |\delta(l,j)| |\log y|^j \end{aligned}$$

$$\leq \sum_{l=1}^{\infty}\sum_{j=0}^{m(l)} y^{k-1-\Re(s)+\beta}|\delta(l,j)|j!y$$
$$= \sum_{l=1}^{\infty}\sum_{j=0}^{m(l)} |\delta(l,j)|j!y^{k+\beta-\Re(s)}.$$

Since the last double sum dominates the integrand in (4.35) and each term is non negative, by the Monotone Convergence Theorem, it is integrable for $\Re(s) \geq |k| + \beta + 1$. Therefore, the problem of integrating the series (4.35) term by term is now solved by the second assumption in the Corollary and the Lebesgue Dominated Convergence Theorem. By applying (4.38), for a single term of (4.35), we get

$$\int_1^{\infty} \delta(l,j)y^{-\beta_l+k-s-1}(\log y)^j dy = \delta(l,j)(-1)^{j+1}j!(k-s-\beta_l)^{-j-1}.$$

Thus, $L(s)$ of (4.35) becomes

$$L(s) = i^k \sum_{l=1}^{\infty}\sum_{j=0}^{m(l)} \delta(l,j)(-1)^{j+1}j!(k-s-\beta_l)^{-j-1}. \tag{4.39}$$

Let $K \subset \mathbb{C}\setminus\{k-\beta_l\}$ be compact. Then there exists $\epsilon > 0$, such that $|s-(k-\beta_l)| \geq \epsilon$ for all l and $s \in K$. From (4.39), we get

$$\left| i^k \sum_{l=1}^{\infty}\sum_{j=0}^{m(l)} j!(-1)^{j+1}\delta(l,j)(k-s-\beta_l)^{-j-1} \right|$$
$$\leq \sum_{l=1}^{\infty}\sum_{j=0}^{m(l)} j!|\delta(l,j)|\epsilon^{-j-1} < \infty.$$

We have applied condition two of the Corollary and therefore, $L(s)$ in (4.39) is holomorphic in $\mathbb{C}\setminus\{k-\beta_l\}$.

Following similar procedure, we can integrate term by term in (4.37). By applying (4.38) for a single term of (4.37), we find that

$$\int_1^{\infty} \delta(l,j)y^{s-\beta_l-1}(-\log y)^j dy = \delta(l,j)(-1)^{j+2}j!(s-\beta_l)^{-j-1},$$

which imply that, $L(s)$ of (4.37), becomes

$$L(s) = \bar{v}(T)\sum_{l=1}^{\infty}\sum_{j=1}^{m(l)} \delta(l,j)(-1)^{j+1}j!(s-\beta_l)^{-j-1}. \tag{4.40}$$

With similar reason, $L(s)$ of (4.40) is holomorphic in $\mathbb{C} \setminus \{\beta_l\}$, since $q|_v^k T + q = 0$ both $\{\beta_l\}$ and $\{k - \beta_l\}$ appear on pair with the same domain. Therefore, the representation of $L(s)$ on (4.39) and (4.40) are holomorphic on the same domain. From (4.39), we find that

$$L(k-s) = i^k \sum_{l=1}^{\infty} \sum_{j=0}^{m(l)} \delta(l,j)(-1)^{j+1} j!(s-\beta_l)^{-j-1}.$$

Since $(i^k v(T))^2 = 1$, comparing (4.39) and (4.40), we obtain

$$L(k-s) = i^k v(T) L(s).$$

Therefore, each of the decomposition from $\Phi_f(s) = E(s) + R(s) + L(s)$ satisfies (4.34) and hence (b) is proved. Now, rewriting $\Phi_f(s)$ using (4.40), we get

$$\Phi_f(s) = E(s) + a_0 \left(\frac{i^k v(T)}{s-k} - \frac{1}{s} \right) + \bar{v}(T) \sum_{l=1}^{\infty} \sum_{j=0}^{m(l)} \delta(l,j)(-1)^{j+1} j!(s-\beta_l)^{-j-1}.$$

Then $\Phi_f(s) - Q(s) = E(s)$ is bounded in the strip and entire on s-plane. Therefore, (a) and (c) holds true. This completes the proof of the direct part of the theorem.

Proof of The Converse Part of The Theorem

Conversely, we assume that (a), (b) and (c) holds. By the Cahen-Mellin formula,

$$e^{-x} = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \Gamma(s) x^{-s} \, ds, \quad \text{for } x, \ d > 0.$$

Putting $x = 2\pi ny/\lambda$ with $n, \ y > 0$, it follow that

$$e^{-2\pi ny/\lambda} = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \Gamma(s) \left(2\pi ny/\lambda\right)^{-s} \, ds.$$

Multiplying both sides by a_n and summing on n, we deduce that, for $d > \gamma + 1$,

$$\begin{aligned} f(iy) - a_o &= \sum_{n=1}^{\infty} a_n \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \Gamma(s) \left(2\pi ny/\lambda\right)^{-s} \, ds. \\ &= \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \Phi_f(s) y^{-s} \, ds, \end{aligned}$$

the interchange of summation and integration is justified by the absolute and uniform convergence of $\varphi_f(s)$ on the line $\sigma = d$, for, $d > \beta+\gamma+1+|k|$, so that all the poles lie in the vertical strip.

Following Hecke and Bochner, we next move the path of integration to the line $\sigma = -d$. We shall do this by integrating around a rectangle with vertices $\pm d \pm iT$, $T > 0$, applying the residue theorem, and then showing that the integrals along the horizontal sides tend to 0 as T tends to ∞. This done by use of Stirling formula [23] and Phragm$\acute{e}n$-Lindel$\ddot{o}f$ Theorem [10] to get

$$\varphi_f(s) = \mathcal{O}\left(|t|^A\right), \ s = \sigma + it, \tag{4.41}$$

uniformly on $-d \leq \sigma \leq d$ and $|t| \geq t_o > 0$ as $|t| \to \infty$ for some t_o, where A is a constant depending on k and d. Which gives the estimate

$$\Phi_f(s) = \mathcal{O}\left(|t|^{\rho+A} e^{-\pi|t|/2}\right),$$

where ρ is coming from Stirling formula and it depend on d. But in our case $\Phi_f(s)$ has infinitely many poles in the strip, so there is no such bound (4.41) near each poles. We prefer to use the automorphic integrals constructed and approximated in Lemma 4.2.1. Let

$$F_N(z) = \sum_{n=0}^{\infty} c_n(N) e^{2\pi i n z/\lambda}, \quad \lambda > 2,$$

be the automorphic integral with period function

$$r_N(z) = \sum_{l=2N+1}^{\infty} \left(\frac{z}{i}\right)^{-\beta_l} \sum_{j=0}^{m(l)} \delta(l,j) \left(\log \frac{z}{i}\right)^j,$$

From (4.30) , we have seen $F_N(z) \to 0$ uniformly on compact subset of $\mathcal{H}$. $F_N(z)$ is analytic in $\mathcal{H}$ as well as, with $f_N(z) = f(z) - F_N(z)$, where $f_N(z)$ is an automorphic integral with finite log-polynomial periods and $f_N(z) \to f$ uniformly on a compact subset of $\mathcal{H}$ as $N \to \infty$.

Now, we use the Mellin transform $F_N(iy) - c_0(N)$ as in the direct part of the proof, we get

$$\Phi_{F_N}(s) = E_N(s) + R_N(s) + L_N(s),$$

where,

$$E_N(s) = \int_1^{\infty} \left(F_N(iy) - c_0(N)\right) y^{s-1}\, dy + i^k v(T) \int_1^{\infty} \left(F_N(iy) - c_0(N)\right) y^{k-s-1}\, dy,$$

$$R_N(s) = c_0(N)\left(\frac{i^k v(T)}{s-k} - \frac{1}{s}\right), \text{ and}$$

$$L_N(s) = \overline{v}(T) \sum_{l=2N+1}^{\infty} \sum_{j=0}^{m(l)} \delta(l,j)(-1)^{j+1} j!\,(s-\beta_l)^{-j-1}.$$

Now, we show that $\Phi_{F_N}(s) \to 0$ uniformly on $V = \cup_{n=1}^{\infty}\{s : -d \leq \sigma \leq d,\ t = T_n,\ s = \sigma + it\}$, to show this, we show that each of the decomposition of $\Phi_{F_N}(s)$ tends to zero on V uniformly. First

$$\begin{aligned} |E_N(s)| &= \left| \int_1^{\infty} (F_N(iy) - c_0(N))\, y^{s-1}\, d\,y + i^k v(T) \int_1^{\infty} (F_N(iy) - c_0(N))\, y^{k-s-1}\, d\,y \right| \\ &\leq \int_1^{\infty} |F_N(iy) - c_0(N)|\, y^{\sigma-1}\, d\,y + \int_1^{\infty} |F_N(iy) - c_0(N)|\, y^{k-\sigma-1}\, d\,y. \end{aligned}$$

Then we show that each of the above integrals tends to zero as N tends to infinity, by applying the Lebesgue Dominated convergence Theorem, we find bound of an integrable function.

$$\begin{aligned} |F_N(iy) - c_0(N)|\, y^{\sigma-1} &= y^{\sigma-1} \left| \sum_{n=1}^{\infty} c_n(N) e^{2\pi i n z/\lambda} \right| \\ &\leq y^{\sigma-1} \sum_{n=1}^{\infty} \left| c_n(N) e^{2\pi i n z/\lambda} \right| \\ &\leq M y^{\sigma-1} \sum_{n=1}^{\infty} n^{\gamma} e^{-2\pi n y/\lambda} \\ &\leq M_1 y^{\sigma-1} e^{-2\pi y/\lambda} \sum_{n=1}^{\infty} n^{\gamma} e^{\frac{-2\pi}{\lambda}(n-1)} \\ &\leq M_1 M_2 y^{\sigma-1} e^{-2\pi y/\lambda}, \end{aligned}$$

M_1 and M_2 are constants independent of N, where we have used, $y \geq 1$ and since by construction, F_N is in $\mathcal{P}$, for each N, then $c_n(N) = \mathcal{O}(n^{\gamma})$ for some $\gamma > 0$, by [21] and $F_N \in \mathcal{P}$ is independent of N. and $\int_1^{\infty} y^{\sigma-1} e^{-2\pi y/\lambda} d\,y < \infty$. Thus,

$$\begin{aligned} &\lim_{N\to\infty} \int_1^{\infty} |F_N(iy) - c_0(N)|\, y^{\sigma-1}\, d\,y \\ &= \int_1^{\infty} \lim_{N\to\infty} |F_N(iy) - c_0(N)|\, y^{\sigma-1}\, d\,y = 0, \end{aligned}$$

because $F_N(iy) \to 0, c_0(N) \to 0$ (see next) as $N \to \infty$. Similarly, the second integral also tends to zero as N tends to infinity.

Hence, $E_N(s) \to 0$ as $N \to \infty$, uniformly on $-d \le \sigma \le d$, on V. Next,

$$\begin{aligned} |R_N(s)| &= \left| c_0(N) \left(\frac{i^k v(T)}{s-k} - \frac{1}{s} \right) \right| \\ &\le M_3 \left| c_0(N) \right|, \end{aligned}$$

for $-d \le \sigma \le d$, and using $F_N(z) = \sum_{n=0}^{\infty} c_n(N) e^{2\pi i n z/\lambda}$, by interchanging the order of summation and integration, where the integration is along a horizontal path, and τ_o is an arbitrary point in $\mathcal{H}$, we find that

$$\begin{aligned} |c_n(N)| &= \left| \frac{1}{\lambda} \int_{\tau_o}^{\tau_o+\lambda} F_N(\tau) e^{-2\pi i n \tau/\lambda} d\,\tau \right| \\ &\le \frac{1}{\lambda} \int_{\tau_o}^{\tau_o+\lambda} \left| F_N(\tau) \right| e^{2\pi n y/\lambda} d\,\tau, \ \tau = x + iy \ \in \ \mathcal{H}, \end{aligned}$$

since $F_N(\tau) \to 0$ as $N \to 0$ uniformly, then $c_0(N) \to 0$ as $N \to \infty$ uniformly on the strip. Hence, $R_N(s) \to 0$ as $N \to \infty$ uniformly on the strip. Finally,

$$\begin{aligned} |L_N(s)| &= \left| \overline{v}(T) \sum_{l=2N+1}^{\infty} \sum_{j=0}^{m(l)} \delta(l,j)(-1)^{j+1} j! \left(s - \beta_l\right)^{-j-1} \right| \\ &\le \sum_{l=2N+1}^{\infty} \sum_{j=0}^{m(l)} \left| \delta(l,j) \right| j! \epsilon^{-j-1}, \text{ for } |s - \beta_l| \ge \epsilon, \ s \in \ V, \text{ all } j \\ &\le \sum_{l=1}^{\infty} \sum_{j=0}^{m(l)} \left| \delta(l,j) \right| j! \epsilon^{-j-1} < \infty. \end{aligned}$$

The last inequality is by the second condition in the Corollary. Hence, $L_N(s) \to 0$ as $N \to \infty$ uniformly on the strip. Therefore, $\Phi_{F_N}(s) \to 0$ as $N \to \infty$ uniformly on V.

Now, $f_N(s) = f(s) - F_N(s)$ is an automorphic integral with finite log-polynomial periods, for any fixed N. By construction, $F_N(s)$ satisfies the transformation law 4.33 , then (a), (b) and (c) holds for each N with $Q(s)$ defined accordingly. Which immediately follow that $f_N = f - F_N$ satisfies for each N the following conditions:

(a') $\Phi_{f_N}(s) - Q'(s)$ has analytic continuation to the entire plane, where, $Q'(s) = \overline{v}(T) \sum_{l=1}^{2N} \sum_{j=0}^{m(l)} \frac{(-1)^{j+1} j! \delta(l,j)}{(s-\beta_l)^{j+1}} + c_o(N) \left(\frac{i^k v(T)}{s-k} - \frac{1}{s} \right)$

(b') $\Phi_{f_N}(s)$ satisfies the functional equation $\Phi_{f_N}(k-s) = i^k v(T) \Phi_{f_N}(s)$,

(c') $\Phi_{f_N}(s) - Q'(s)$ is bounded in a vertical strip.

Since $f_N(s)$ is an automorphic integral with finite log-polynomial periods, for any fixed N, by the converse part of Theorem 7.1 [[1], PP.87-96], after using Stiriling formula and Phragm$\acute{e}$n-Lindel$\ddot{o}$f theorem, to get along the horizontal sides of $\pm d \pm iT_n,\ T_n > 0,$ say $r \pm t_n$ tends to zero as T_n tends to infinity. Thus, for any fixed N

$$\lim_{n\to\infty} \int_{r\pm t_n} \Phi_{f_N}(s) y^{-s} d\, s = 0.$$

Since $f_N \to f$ as $N \to \infty$ uniformly, we have

$$\lim_{N\to\infty} \int_{r\pm t_n} \Phi_{f_N}(s) y^{-s} d\, s = \int_{r\pm t_n} \Phi_f(s) y^{-s} d\, s,$$

uniformly in n because $\Phi_{f_N}(s) \to \Phi_f(s)$ (or $\Phi_{FN}(s) \to 0$) uniformly on V. Thus, we can interchange the order of limit in n and N. That is

$$\begin{aligned}
\lim_{n\to\infty} \int_{r\pm t_n} \Phi_f(s) y^{-s} d\, s &= \lim_{n\to\infty} \lim_{N\to\infty} \int_{r\pm t_n} \Phi_{f_N}(s) y^{-s} d\, s \\
&= \lim_{N\to\infty} \lim_{n\to\infty} \int_{r\pm t_n} \Phi_{f_N}(s) y^{-s} d\, s \\
&= 0,
\end{aligned}$$

which proves our assertion the integral along horizontal path tends to zero by applying the residue theorem over the rectangle. Thus, we have

$$f(iy) - a_0 = \frac{1}{2\pi i} \int_{-d-i\infty}^{-d+i\infty} \Phi_f(s) y^{-s} d\, s + \sum \text{Res}\ (\Phi_f(s)), \tag{4.42}$$

where the summation run over all poles of Φ_f. Next, let's calculate the residues over the poles. Since by (a), $\Phi_f(s) - Q(s)$ is an entire function on s-plane, then Φ_f has singularities at $0,\ k,$ and β_l for all l.

The power series expansion of y^{-s} with center at β_l is

$$y^{-s} = y^{-\beta_l} \sum_{n=0}^{\infty} \frac{(\log y)^n (-1)^n (s - \beta_l)^n}{n!}.$$

Then the expansion of $\Phi_f(s)y^{-s}$ at β_l is given by

$$y^{-\beta_l}\left(\sum_{n=0}^{\infty}\frac{(-\log y)^n (s-\beta_l)^n}{n!}\right)\left(\overline{v}(T)\sum_{j=0}^{m(l)}\frac{(-1)^{j+1}j!\delta(l,j)}{(s-\beta_l)^{j+1}}\right),$$

thus,

$$\text{Res}\ \left(\Phi_f(s)y^{-s},\ \beta_l\right) = \overline{v}(T)y^{-\beta_l}\sum_{j=0}^{m(l)}(-1)^{j+1}\delta(l,j)\left(-\log y\right)^j,$$

$$\text{Res}\ \left(\Phi_f(s)y^{-s},\ 0\right) = -a_0,$$

and

$$\text{Res}\ \left(\Phi_f(s)y^{-s}, k\right) = y^{-k}a_0 i^k v(T).$$

Then (4.42) become,

$$\begin{aligned} f(iy)-a_0 &= \frac{1}{2\pi i}\int_{-d-i\infty}^{-d+i\infty}\Phi_f(s)y^{-s}d\,s + a_0\left(y^{-k}i^k v(T)-1\right) \\ &\quad + \overline{v}(T)\sum_{l=1}^{\infty}y^{\beta_l}\sum_{j=0}^{m(l)}(-1)^{j+1}\delta(l,j)\left(-\log y\right)^j, \end{aligned}$$

which implies that

$$f(iy)-a_0 = \frac{1}{2\pi i}\int_{-d-i\infty}^{-d+i\infty}\Phi_f(s)y^{-s}d\,s + a_0\left(y^{-k}i^k v(T)-1\right) - \overline{v}(T)q(iy). \tag{4.43}$$

Applying the functional equation, 4.34 and replacing s by $k-s$ in the integral, we get

$$\begin{aligned} \frac{1}{2\pi i}\int_{-d-i\infty}^{-d+i\infty}\Phi_f(s)y^{-s}d\,s &= \frac{1}{2\pi i\left(i^k v(T)\right)}\int_{-d-i\infty}^{-d+i\infty}\Phi_f(k-s)y^{-s}d\,s \\ &= \frac{\overline{v}(T)}{(iy)^k}\frac{1}{2\pi i}\int_{k+d-i\infty}^{k+d+i\infty}\Phi_f(s)\left(\frac{1}{y}\right)^{-s}d\,s \\ &= \frac{\overline{v}(T)}{(iy)^k}\left(f\left(\frac{i}{y}\right)-a_0\right), \end{aligned}$$

where $k+d > \beta+\gamma+1$. Then (4.43) become
$f(iy)-a_0 = \frac{\overline{v}(T)}{(iy)^k}\left(f\left(\frac{i}{y}\right)-a_0\right) + a_0\left(y^{-k}i^k v(T)-1\right) - \overline{v}(T)q(iy).$
Then by analytic continuation, we get

$$f(z)-a_0 = \frac{\overline{v}(T)}{(z)^k}\left(f\left(\frac{-1}{z}\right)-a_0\right) + a_0\left(\left(\frac{z}{i}\right)^{-k}i^k v(T)-1\right) - \overline{v}(T)q(z),$$

multiplying by, $v(T)$ $(|v(T)| = 1)$ and using $\left(i^k v(T)\right)^2$, we get the required transformation (4.33). This completes the proof of the converse part of the theorem. □

Chapter 5

Infinite Log-Polynomial Period Functions on Discrete Hecke Groups for $0 < \lambda < 2$

5.1 Introductions

Let us denote Γ_λ, for the Hecke group $G(\lambda)$ for $\lambda = 2\cos\pi/p,\ p\ \in\ \mathbb{Z}$, $p \geq 3$. For this group we have two group relations (1.4) and (1.5) . If F satisfies (1.7) for the Hecke group Γ_λ, we have

$$e^{-2\pi\,\kappa i}F(z+\lambda) = F(z) + q_{S_\lambda}(z) \tag{5.1}$$

$$\overline{v}(T)z^{-k}F\left(\frac{-1}{z}\right) = F(z) + q_T(z), \tag{5.2}$$

where $e^{2\pi\kappa i} = v(S_\lambda)$, $0 \leq \kappa < 1$.

Let F be an *automorphic integral* of weight k and multiplier system v for the Heck group Γ_λ. Here q_{S_λ} and q_T are not completely arbitrary. The second relation $(S_\lambda T) = I$, imposes a restriction on $\{q_M\}$. Applying the cocycle condition (1.10) , we can generate q_M for arbitrary $M\ \in\ \Gamma_\lambda$, after writing M as word in S_λ and T. With $q_{S_\lambda} = 0,\ q_T = q$ is finite LPS. If an automorphic integral F of weight k and multiplier system v for Γ_λ exists, then $q(z)$ is said to be a *log–polynomial period function* of weight k and multiplier system v for the Hecke group Γ_λ.

Since Γ_λ is generated by S_λ and T, a multiplier system for Γ_λ can be completely determined from its values on S_λ and T and the consistency condition (1.8) .

Let $\Gamma_\infty = \langle S_\lambda \rangle$ be a subgroup of Γ_λ, the stabilizer of $i\infty$. If F is analytic and satisfies (5.1) with the assumption $q_{S_\lambda} = 0$, then the Fourier series expansion of F at the parabolic cusp $z = i\infty$ has the form

$$F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi\, i(n+\kappa)\frac{z}{\lambda}}, \tag{5.3}$$

where $a_n \in \mathbb{C}$ satisfy the growth condition $a_n = O(n^\rho)$ as $n \to \infty$, $\rho > 0$.

Now we rewrite the infinite log-polynomial sum by letting $n \to \infty$ in the first sum, that is

$$q(z) = \sum_{l=1}^{\infty} z^{\beta_l} \sum_{j=0}^{m(l)} \delta(l,j) (\log z)^j , \tag{5.4}$$

where the β_l are all distinct complex numbers, $\delta(l,j) \in \mathbb{C}$, and $z \in \mathcal{H}$. For the convergence of $q(z)$ we have Proposition 3.2.1. (See Proposition 4.1 in [12]).

A. Hassen in [11] determined the automorphic integrals of positive and negative weight for the full modular group and some Hecke groups, for finite log-polynomial period function q.

In the next sections we will determine the necessary conditions for the existence of an infinite log-polynomial sum period function (ILPSPF), q as in (5.4) for Hecke group $G(\lambda)$, $\lambda = 2\cos \pi/p$, and an integer $p \geq 3$. Then with these groups we will investigate the ILPSPFs for $k > 2$ with $v(S_\lambda) = 1$. Also we will characterize for the case $k > 0$, $v(S_\lambda) \neq 1$.

5.2 Preliminaries

If F is analytic in the upper half plane and rewriting (5.1) and (5.2) in slash form for Γ_λ, we have

$$F|_v^k S_\lambda = F \quad \text{and} \quad F|_v^k T = F + q, \tag{5.5}$$

here assuming $q_{S_\lambda} = 0$ and $q = q_T$ given by (5.4) with the necessary conditions in Proposition 3.2.1 is an *ILPSPFs of weight k and multiplier system v for* Γ_λ. Form now on we use $F|M$ for $F|_v^k M$ for $M \in \Gamma_\lambda$.

To prove the main results of the next sections first we need to prove the existence of such an F with q as in (5.4) . If v is a multiplier system of weight k for Γ_λ with $v(-I)(-1)^k = 1$, from (1.4) and (5.5) , we obtain

$$q + q|T = 0. \tag{5.6}$$

Let us define $V_1 = S_\lambda T = \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix}$, for λ given by (1.2) . V_1 is elliptic (its trace is $\lambda < 2$), the fixed point of V_1 is in $\mathcal{H}$, consequently V_1 has finite order. Let

$$V_m = (S_\lambda T)^m = \begin{pmatrix} \alpha_m & \beta_m \\ \gamma_m & \delta_m \end{pmatrix}, \quad m \in \mathbb{Z}^+.$$

By simple induction on m, we get

$$V_m = \frac{1}{\sin \pi/p} \begin{pmatrix} \sin(1+m)\pi/p & -\sin m\pi/p \\ \sin m\pi/p & \sin(1-m)\pi/p \end{pmatrix}. \tag{5.7}$$

If $m = p$, then $V_p = -I$, which is the second group relation (1.5) for Γ_λ. This extra relation imposes the additional restrictions on the multiplier system, the weight or the automorphic integrals with respect to Γ_λ, when $\lambda < 2$. By repeated application of the cocycle condition (1.10) and obviously $q_{-I} = q_I = 0$, we have

$$q + q|(S_\lambda T) + q|(S_\lambda T)^2 + \cdots + q|(S_\lambda T)^{p-1} = 0. \tag{5.8}$$

One of our objective of the next section is to fix an ILPSPFs q which holds the conditions in Proposition 3.2.1 satisfying both (5.6) and (5.8). Now combining (5.6) and (5.8), we get

$$-q|T + q|(S_\lambda T) + q|(S_\lambda T)^2 + \cdots + q|(S_\lambda T)^{p-1} = 0.$$

Applying (1.11) to the above equation, we get

$$-q + q|(S_\lambda T)T + q|(s_\lambda T)^2 T + \cdots + q|(S_\lambda T)^{p-1}T = 0. \tag{5.9}$$

To apply the results of [11] from (5.7), we have

$$\begin{aligned} M_m = V_m T &= \frac{1}{\sin \pi/p} \begin{pmatrix} -\sin m\pi/p & -\sin(1+m)\pi/p \\ \sin(1-m)\pi/p & -\sin m\pi/p \end{pmatrix} \\ &= \begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix}, \quad \text{for } m = 1, 2, \cdots, p. \end{aligned} \tag{5.10}$$

Our q is absolutely convergence for all $z \in \mathcal{H}$ and uniformly convergence in every compact subset of $\mathcal{H}$ under the conditions in Proposition 3.2.1. Therefore, interchange of limit and summation is justifiable. So, now we can use the following two equations from [11]:

$$\lim_{z \to \infty} \left(\overline{v}(S_\lambda) q(z+\lambda) - q(z) \right) = 0. \tag{5.11}$$

And

$$\lim_{z \to \infty} \left(v(S_\lambda) \left(\frac{z+\lambda}{z} \right)^{-k} q\left(\frac{1}{z+\lambda} \right) - q\left(\frac{1}{z} \right) \right) = L, \tag{5.12}$$

where, $L = -\sum_{m=1}^{p-2} \overline{v}(M_m) d_m^{-k} q\left(\frac{b_m}{d_m} \right)$.

In the next section, first we determine an existence of an automorphic integrals F which satisfies (5.5) for the group Γ_λ. Then we apply (5.11) and (5.12) to study an ILPSPFs for this group with $v(S_\lambda) = 1$ and $k > 2$; $v(S_\lambda) \neq 1$ and of positive weight.

5.3 Existence of an ILPSPFs for Γ_λ

In this section we determine the existence of an automorphic integrals corresponding to q as in proposition 3.2.1 or an ILPSPFs q.

The cocycles $\{q_M : M \in \Gamma_\lambda\}$, the cocycle of period functions of the automorphic integrals F is said to be *parabolic cocycles*, if q_M is in the space $\mathcal{P}$ for all $M \in \Gamma_\lambda$ and there exists q' in $\mathcal{P}$ such that $q_{S_\lambda} = q'|S_\lambda - q'$. We need to restrict all cocycles to the space $\mathcal{P}$, this applies, of course, to the q_M occurring in $F|M = F + q_M$, for all $M \in \Gamma_\lambda$. Since $\Gamma_\lambda = \langle S_\lambda,\ T\rangle$, for $M \in \Gamma_\lambda$, we can write as word in the generator S_λ and T, that is

$$M = S_\lambda^{a_1} T S_\lambda^{a_2} T \cdots T S_\lambda^{a_n}, \quad a_i \in \mathbb{Z},\ n \geq 2,\ a_2, \cdots, a_n \neq 0. \tag{5.13}$$

(Eichler [16], Theorem 3). Applying the cocycle condition (1.10) repeatedly, we have

$$\begin{aligned} q_M &= q_{S_\lambda^{a_1} T S_\lambda^{a_2} T \cdots T S_\lambda^{a_n}} \\ &= q_{S_\lambda^{a_1}} | T S_\lambda^{a_2} T \cdots T S_\lambda^{a_n} + q_T | S_\lambda^{a_2} T \cdots T S_\lambda^{a_n} + \cdots + q_{S_\lambda^{a_n}}. \end{aligned} \tag{5.14}$$

Here M is written as a factorization into S_λ^n, $n \in \mathbb{Z}$ as parabolic generator and T non-parabolic generator. Defining $q = q_T$, where q is given in Proposition 3.2.1 which is in $\mathcal{P}$ and since the space $\mathcal{P}$ is closed under the slash operator, if q_{S_λ} is in $\mathcal{P}$, then $q_{S_\lambda^n}$ is in $\mathcal{P}$. Therefore, q_M is in $\mathcal{P}$. Also for the space $\mathcal{P}$, by (Proposition 9, [6]), for the parabolic transformation $S_\lambda \in \Gamma_\lambda$ and if q_{S_λ} is in $\mathcal{P}$, then there exists q' in $\mathcal{P}$ such that $q'|S_\lambda - q' = q_{S_\lambda}$. Thus, $\{q_M : M \in \Gamma_\lambda\}$ is a parabolic cocycle for the group. Then by (Theorem 3, [6]), we have the following theorem.

Theorem 5.3.1. *Let k be any real number and v a multiplier system of weight $2k$. Suppose $\{q_M : M \in \Gamma_\lambda\}$ is a parabolic cocycle of wight $2k$ in $\mathcal{P}$. Then there exists a function F, analytic in $\mathcal{H}$, such that*

$$F|M - F = q_M, \quad \textit{for all } M \in \Gamma_\lambda,$$

and with expansion at the parabolic cusp $z = i\infty$ of the form

$$F(z) = q'(z) + \sum_{n+\kappa \geq 0}^{\infty} a_n \exp\left(\frac{2\pi i (n+\kappa) z}{\lambda}\right), \quad 0 \leq \kappa < 1,\ v(S_\lambda) = e^{2\pi i \kappa}.$$

Now, let $\{q_M\}$ be a parabolic cocycle in $\mathcal{P}$ for the groups Γ_λ, of weight k and multiplier system v. For $M \in \Gamma_\lambda$, put $r_M = q_M - (q'|M - q')$, then

$\{r_M\}$ is aparabolic cocycle in $\mathcal{P}$ and $r_{S_\lambda} = q'|s_\lambda - q' - (q'|S_\lambda - q') = 0$. By the above theorem, there exist a function G analytic and $G|M - G = r_M$, for $M \in \Gamma_\lambda$. Define $F(z) = G(z) + q'(z)$, which implies that $F|M - F = r_M + q'|M - q' = q_M$, for $M \in \Gamma_\lambda$. Therefore, with out loss of generality, we can assume that $q_{S_\lambda} = 0$. Then we redefine the parabolic cocycle $\{q_M : M \in \Gamma_\lambda\}$ with this additional assumption. We have ensured the existence of automorphic integrals for our group Γ_λ with period function q given as in Proposition 3.2.1 by the Theorem 5.3.1.

5.4 ILPSPFs for Γ_λ of Positive Weight

In this section first we will investigate the ILPPFs for the group Γ_λ and weight $k > 2$ with $v(S_\lambda) = 1$. Next, we will study the period function q for the case $v(S_\lambda) \neq 1$ and of positive weight. Before we state and prove our main results. We need the following lemma under the the assumptions of Proposition 3.2.1, if $q = 0$.

Lemma 5.4.1. *Let $\{\beta_l\}_{l=1}^{\infty}$ be a sequence of distinct complex numbers. Suppose $\Re(\beta_l) \leq \beta$, for all l, for some $\beta > 0$ and $\Im(\beta_l) \to \infty$ as $l \to \infty$. If*

$$\sum_{l=1}^{\infty} \sum_{j=0}^{m_l} \delta(l,j) z^{\beta_l} (\log z)^j = 0, \tag{5.15}$$

then $\delta(l,j) = 0$ for all l and j.

Proof. Let $p_l(z) = \sum_{j=0}^{m_l} \delta(l,j) z^j$, then (5.15) , becomes

$$\sum_{l=1}^{\infty} p_l(z) e^{\beta_l z} = 0. \tag{5.16}$$

Now, let us estimate the term $e^{\beta_l z}$, $|e^{\beta_l z}| = e^{\Re(\beta_l z)} = e^{\Re(\beta_l)x - \Im(\beta_l)y}$, $z = x + iy \in \mathcal{H}$. Since $y > 0$ and $\Im(\beta_l) \to \infty$ as $l \to \infty$, then there is a constant $M > 0$, for large $n_0 \in \mathbb{Z}^+$, such that $e^{-\Im(\beta_l)y} \leq M$ for $l > n_0$. Thus,

$$|e^{\beta_l z}| \leq M e^{\Re(\beta_l)x} \leq M e^{\beta x} \leq M e^{\beta}, \quad \text{for } |x| < \lambda/2 \text{ and } \lambda < 2. \tag{5.17}$$

Therefore, for some constant M' we have

$$0 < |e^{\beta_l z}| < M', \quad \text{for all } l > n_0.$$

Using this from (5.16), we have $\lim_{l\to\infty} p_l(z) = 0$, which implies that there exist a constant M'' such that $|p_l(z)| < M''$, for $l > n_1$. That is, each polynomials p_l for $l > n_1$ are constants, i.e., $p_l(z) = K_l$ for $l > n_1$, here $K_l = \delta(l, 0), \quad l > n_1$. Note that possibly only finitely many polynomials are non constant polynomials. First we want to show that $p_l \equiv 0$ for all l. If only finitely many polynomials are assumed to be non zero, then (5.16), becomes $\sum_l^n p_l(z)e^{\beta_l z} = 0$, with distinct $\beta_l's$. By (Lemma 3.6., [11]), we are done. With out loss of generality, let us assume that $p_l \not\equiv 0$ for all l. Let m be the largest degree of the polynomials $p_l's$.

$$\begin{aligned}\left|\sum_{l=n_*}^{\infty} p_l(z)e^{\beta_l z}\right| &\leq M''' e^{\beta x} \sum_{l=n_*}^{\infty} e^{-\Im(\beta_l)y} \\ &\leq M''' e^{\beta x} \sum_{l=n_*}^{\infty} e^{-\Im(\beta_l)y_0},\end{aligned}$$

where $n_* = \max(n_0, n_1)$ and for $y \geq y_0 > 0$ and some constant M'''. Which implies that this infinite series converges absolutely and uniformly on a compact subset of $\mathcal{H}$ by Weierstrass M-Test. Therefore, we can differentiate both sides of (5.16) m times, we get

$$\sum_{l=1}^{\infty}\left(\sum_{t=0}^{m}\binom{m}{t}\beta_l^t p_l^{(m-t)}(z)\right)e^{\beta_l z} = 0,$$

which is written as

$$\sum_{l=1}^{\infty}\left(\sum_{t=0}^{m} c_t^l p_l^{(m-t)}(z)\right)e^{\beta_l z} = 0, \tag{5.18}$$

where $c_t^l = \binom{m}{t}\beta_l^t \neq 0$ for all t and l. Since $0 < |e^{\beta_l z}| < M'$, for all $l > n_0$, from (5.18), we get

$$\lim_{l\to\infty}\sum_{t=0}^{m} c_t^l p_l^{(m-t)}(z) = 0. \tag{5.19}$$

Here the $(m-t)$th derivatives, $p_l^{(m-t)}(z) = 0, \quad$ for $0 < t < m$. If $p_l^{(m)}(z)$ is non zero constants, then we obtain

$$\lim_{l\to\infty}\sum_{t=0}^{m} c_t^l p_l^{(m-t)}(z) = c_0^l \lim_{l\to\infty} p_l^{(m)}(z) + c_m^l \lim_{l\to\infty} p_l(z) = c_0^l \lim_{l\to\infty} p_l^{(m)}(z) \neq 0,$$

which is a contradiction, and hence $p_l \equiv 0$ for all l. Therefore, from p_l before (5.16) , $\delta(l, j) = 0$ for all l and j. The case if all or infinitely many of the polynomials $p_l(z)$ are nonzero constants, (5.16) becomes

$$\sum_{l=0}^{\infty} K_l e^{\beta_l z} = 0.$$

That means we need to show that $\{e^{\beta_l z}\}$ with distinct $\beta_l' s$ is infinitely linearly independent. To show infinitely linearly independent, we required an infinite collection of functions $\{e^{\beta_l z}\}$ with distinct $\beta_l' s$. For any arbitrary distinct $\beta_i' s$, for $1 \leq i \leq n$, $b_i \in \mathbb{C}$, in the set we show that

$$\sum_{i=0}^{n} b_i e^{\beta_i z} = 0, \quad \text{implies that} \quad b_i = 0, \text{ for all } i = 1, \cdots, n.$$

But this holds by Lemma 3.6. in [11].This completes the proof of the lemma. □

Note that from the second condition of Proposition 3.2.1. We have

$$\sum_{l=1}^{\infty} \sum_{j=0}^{m_l} |\delta(l, j)| < \infty,$$

which implies that

$$\lim_{l \to \infty} \sum_{j=0}^{m_l} |\delta(l, j)| = 0.$$

Then

$$|p_l(z)| \leq \sum_{j=0}^{m_l} |\delta(l, j)| \, |z|^j \leq M \sum_{j=0}^{m_l} |\delta(l, j)| \, ,$$

where $M = \max_{0 \leq j \leq m_l} |z|^j, \ z \in \mathcal{H}$.

Therefore, the assumption strength $\lim_{l \to \infty} p_l(z) = 0$, similarly for equation (5.19) .

Now let us consider the infinite log-polynomial sum

$$q(z) = \sum_{l=1}^{\infty} z^{\beta_l} \sum_{j=0}^{m(l)} \delta(l, j) \, (\log z)^j \, ,$$

with the necessary conditions in Proposition 3.2.1, where the β_l are all distinct complex numbers, $\delta(l, j) \in \mathbb{C}$, and $z \in \mathcal{H}$. Since it is absolutely and uniformly convergent in a compact subset of $\mathcal{H}$. By simple rearrangement, the real part of $\beta_l's$ can be written as increasing sequence, i.e.,

$$\cdots \Re(\beta_{m_0-2}) \leq \Re(\beta_{m_0-1}) \leq \Re(\beta_{m_0}) \leq 0 < \Re(\beta_{m_0+1}) \leq \Re(\beta_{m_0+2}) \leq \cdots,$$

with the condition $m(j) \leq m(t)$ whenever $j < t$ and $\Re(\beta_j) = \Re(\beta_t)$. Thus, an infinite log-polynomial sum can be expressed in the form:

$$q(z) = \sum_{l=1}^{\infty}\sum_{j=0}^{m(l)} a(l,j) z^{-\beta_l} (\log z)^j + \sum_{s=1}^{\infty}\sum_{t=0}^{n(s)} b(s,t) z^{\gamma_s} (\log z)^t, \tag{5.20}$$

where,

$$\begin{aligned} &0 \leq \Re(\beta_1) \leq \Re(\beta_2) \leq \cdots; m(l) \leq m(j) \text{ if } \Re(\beta_l) = \Re(\beta_j)\ (l < j);\\ &0 < \Re(\gamma_1) \leq \Re(\gamma_2) \leq \cdots; n(l) \leq n(j) \text{ if } \Re(\gamma_l) = \Re(\gamma_j)\ (l < j). \end{aligned} \tag{5.21}$$

Also for the next results we will assume $\Im(\beta_l)$ are distinct for l.

Theorem 5.4.2. *Let $q(z)$ as in Proposition* 3.2.1 *and given by the form* (5.20) *and* (5.21) *is an ILPPFs of weight k, $k > 0$ and multiplier system v for Γ_λ with $v(S_\lambda) = 1$. Suppose $\Im(\beta_l)$, $\Im(\gamma_l) \to \infty$ as $l \to \infty$. If $k > 2$, then $q(z)$ is of the form*

$$q(z) = a(1 - \overline{v}(T) z^{-k}).$$

Proof. Suppose $v(S_\lambda) = 1$. Then (5.11) becomes

$$\lim_{z\to\infty} \{q(z+\lambda) - q(z)\} = 0, \tag{5.22}$$

here each terms of (5.22) is the form

$$\alpha\,(z+\lambda)^\gamma\,(\log(z+\lambda))^t - \alpha z^\gamma (\log z)^t,$$

where $\gamma,\ \alpha \in \mathbb{C}$, $\alpha \neq 0$ and $t \geq 0,\ t \in \mathbb{Z}$. The limit at infinity of this term holds for

$$\begin{aligned} &\alpha \lim_{z\to\infty} \left((z+\lambda)^\gamma (\log(z+\lambda))^t - z^\gamma (\log z)^t\right)\\ &= \begin{cases} 0, \text{ if } \Re(\gamma) < 1, t \geq 0 \text{ or } \Re(\gamma) = 1,\ t < 0;\\ \infty, \text{ if } \Re(\gamma) > 1,\ t \geq 0 \text{ or } \Re(\gamma) = 1,\ t > 0. \end{cases} \end{aligned}$$

The case $t = 0$ and $\Re(\gamma) = 1$, then we have

$$\lim_{z\to\infty} \alpha\left[(z+\lambda)^{1+ic} - z^{1+ic}\right] = \alpha \lim_{z\to\infty} \frac{\left(\frac{z+\lambda}{z}\right)^{1+ic} - 1}{z^{-1-ic}}$$

$$= \alpha \lim_{z\to\infty} \lambda(z+\lambda)^{ic}$$
$$= \begin{cases} \alpha\lambda & \text{if } c = 0; \\ \nexists \text{ if } c \neq 0. \end{cases}$$

Therefore, (5.22) does not holds. Thus, $\Re(\gamma) < 1$ and $t \geq 0$. Which implies that from (5.21) , we get

$$\Re(\gamma_s) < 1, \text{ for all } s = 1, 2, 3, \cdots . \tag{5.23}$$

For $v(S_\lambda) = 1$, (5.12) becomes

$$\lim_{z\to\infty} \left(\left(\frac{z+\lambda}{z}\right)^{-k} q\left(\frac{1}{z+\lambda}\right) - q\left(\frac{1}{z}\right) \right) = L, \tag{5.24}$$

each terms of (5.24) is the form

$$\alpha \left(\frac{z+\lambda}{z}\right)^{-k} (z+\lambda)^{\gamma} \left(\log(z+\lambda)\right)^{t} - \alpha z^{\gamma} (\log z)^{t},$$

where $\alpha,\ \gamma\ \in\ \mathbb{C},\ \alpha \neq 0$ and $t \geq 0$. The limit at infinity of this term holds for

$$\lim_{z\to\infty} \alpha \left[\left(\frac{z+1}{z}\right)^{-k} (z+\lambda)^{\gamma} \left(\log(z+\lambda)\right)^{t} - z^{\gamma}(\log z)^{t} \right]$$
$$= \begin{cases} 0 & \text{if } \Re(\gamma) < 1,\ t \geq 0 \text{ or } \Re(\gamma) = 1,\ t < 0; \\ \infty, & \text{if } \Re(\gamma) > 1,\ t \geq 0 \text{ or } \Re(\gamma) = 1,\ t > 0. \end{cases}$$

The case $t = 0$ and $\Re(\gamma) = 1$, then we obtain

$$\lim_{z\to\infty} \alpha \left[\left(\frac{z+1}{z}\right)^{-k} (z+\lambda)^{1+ic} - z^{1+ic} \right] = \lim_{z\to\infty} \alpha \frac{\left(\frac{z+\lambda}{z}\right)^{1+ic-k} - 1}{z^{-1-ic}}$$
$$= \alpha \lim_{z\to\infty} \lambda \frac{(1+ic-k)}{1+ic} \frac{(z+\lambda)^{ic-2k}}{z^{-k}}$$
$$= \begin{cases} \alpha\lambda(1-k), & \text{if } c = 0; \\ \nexists, & \text{if } c \neq 0. \end{cases}$$

Which implies, (5.24) holds if $t = 0$ and $\gamma = 1$, from (5.22) we have

$$\Re(\beta_l) < 1 \text{ or } \Re(\beta_l) = 1 \text{ with } m(l) = 0 \text{ for all } l. \tag{5.25}$$

We can observe that (5.24) holds for

$$t = 0 \text{ and } \gamma = k \tag{5.26}$$

Using (5.23), (5.25) and (5.26), then (5.20) becomes the form:

$$\begin{aligned} q(z) &= a + cz^{-k} + \sum_{l=1}^{\infty} a(l,0) z^{-1+iy_l} + \sum_{l=1}^{\infty}\sum_{j=0}^{m(l)} a(l,j) z^{-\beta_l} (\log z)^j \\ &+ \sum_{s=1}^{\infty}\sum_{t=0}^{n(s)} b(s,t) z^{\gamma_s} (\log z)^t, \end{aligned} \tag{5.27}$$

where, $0 \le \Re(\beta_1) \le \Re(\beta_2) \le \cdots < 1$; $0 < \Re(\gamma_1) \le \Re(\gamma_2) \le \cdots < 1$ and $y_1, y_2, y_3, \cdots$ are real numbers.

Now using (5.6), $q(z) + \overline{v}(T) z^{-k} q\left(\frac{-1}{z}\right) = 0$ and (5.27), we have

$$\begin{aligned} &(a + (-1)^{-k} c\overline{v}(T)) + (a\overline{v}(T) + c) z^{-k} + \sum_{l=1}^{\infty} a(l,0) z^{-1+iy_l} + \\ &\sum_{l=1}^{\infty} a(l,0)\overline{v}(T)(-1)^{-1+iy_l} z^{1-iy_l-k} + \sum_{l=1}^{\infty}\sum_{j=0}^{m(l)} a(l,j) z^{-\beta_l} (\log z)^j + \\ &\sum_{l=1}^{\infty}\sum_{j=0}^{m(l)} a(l,j)\overline{v}(T)(-1)^{-\beta_l+j} z^{\beta_l-k} (\log(-z))^j + \sum_{s=1}^{\infty}\sum_{t=0}^{n(s)} b(s,t) z^{\gamma_s} (\log z)^t + \\ &\sum_{s=1}^{\infty}\sum_{t=0}^{n(s)} b(s,t)\overline{v}(T)(-1)^{\gamma_s+t} z^{-\gamma_s-k} (\log(-z))^t = 0. \end{aligned} \tag{5.28}$$

Which is the sum of the form $\sum_{l=1}^{\infty}\sum_{j=0}^{m_l} \delta(l,j) z^{\alpha_l} (\log z)^j = 0$, with all conditions in Lemma 5.4.1 satisfied if $\alpha_l's$ are all distinct. Which are

$$0, \quad -k, \quad -1+iy_l, \quad 1-iy_l-k, \quad -\beta_l, \quad \beta_l - k, \quad \gamma_s, \quad -\gamma_s - k. \tag{5.29}$$

Since $0 \le \Re(\beta_l) < 1$, for all l and $0 < \Re(\gamma_s) < 1$, for all s, also $\beta_l's$ and $\gamma_s's$ are distinct. Then for $k > 2$ all the powers of z in (5.29) are distinct. Therefore, by Lemma 5.4.1 all the coefficients in (5.28) are zero. We obtain

$$\begin{gathered} a + (-1)^{-k} c\overline{v}(T) = 0, \ a\overline{v}(T) + c = 0, \\ a(l,j) = 0 \text{ for all } l \text{ and } j, b(s,t) = 0 \text{ for all } s \text{ and } t. \end{gathered} \tag{5.30}$$

Therefore, from (5.27) we get $q(z) = a + cz^{-k} = a(1 - \overline{v}(T) z^{-k})$. This completes the proof of the theorem. □

Remark 5.4.3. *For an integer weight k on the full modular group, i.e., $\lambda = 1$. This result coincides with the rational period functions of weight k*

in [8] and [9]. Which are the only rational period functions with rational pole at 0 *for weight* $k > 2$.

Theorem 5.4.4. *Let* $q(z)$ *as in Proposition* 3.2.1 *and given by the form* (5.20) *and* (5.21) *is an ILPSPFs of weight* $k,\ k > 0$ *and multiplier system* v *for* Γ_λ *with* $v(S_\lambda) \neq 1$. *Suppose* $\Im(\beta_l),\ \Im(\gamma_l) \to \infty$ *as* $l \to \infty$. *Then* $q(z) = 0$.

Proof. Suppose $v(S_\lambda) \neq 1$, then each terms of (5.11) is the form

$$\overline{v}(S_\lambda)\alpha\,(z+\lambda)^{\gamma}\,(\log(z+\lambda))^{t} - \alpha z^{\gamma}(\log z)^{t},$$

where $\gamma,\ \alpha\ \in\ \mathbb{C},\ \alpha \neq 0$ and $t \geq 0,\ t\ \in\ \mathbb{Z}$. The limit at infinity of this term holds for

$$\alpha \lim_{z\to\infty}\left[\overline{v}(S_\lambda)\,(z+\lambda)^{\gamma}\,(\log(z+\lambda))^{t} - z^{\gamma}(\log z)^{t}\right]$$
$$= \begin{cases} 0, & \text{if } \Re(\gamma) < 0,\ t \geq 0 \ \text{ or } \Re(\gamma) = 0,\ t < 0; \\ \infty, & \text{if } \Re(\gamma) > 0,\ t \geq 0 \ \text{ or } \Re(\gamma) = 0,\ t > 0. \end{cases}$$

Which implies that $\Re(\gamma_s) < 0$, for all $s = 1,\ 2,\ 3,\ \cdots$. But this contradicts (5.20) . Therefore, $b(s,t) = 0$ for all s and t.

For $v(S_\lambda) \neq 1$, each terms of (5.12) becomes

$$\alpha\overline{v}(S_\lambda)\left(\frac{z+\lambda}{z}\right)^{-k}(z+\lambda)^{\gamma}\,(\log(z+\lambda))^{t} - \alpha z^{\gamma}(\log z)^{t},$$

where $\gamma,\ \alpha\ \in\ \mathbb{C},\ \alpha \neq 0$ and $t \geq 0$. The limit at infinity of this term holds for

$$\alpha \lim_{z\to\infty}\left[\overline{v}(S_\lambda)\left(\frac{z+1}{z}\right)^{-k}(z+\lambda)^{\gamma}\,(\log(z+\lambda))^{t} - z^{\gamma}(\log z)^{t}\right]$$
$$= \begin{cases} 0, & \text{if } \Re(\gamma) < 0, t \geq 0 \text{ or } \Re(\gamma) = 0,\ t < 0; \\ \infty, & \text{if } \Re(\gamma) > 0, t \geq 0 \text{ or } \Re(\gamma) = 0,\ t > 0. \end{cases}$$

The case $t = 0$ and $\Re(\gamma) = 0$, then we obtain

$$\alpha \lim_{z\to\infty}\left[\overline{v}(S_\lambda)\left(\frac{z+1}{z}\right)^{-k}(z+\lambda)^{ic} - z^{ic}\right] = \alpha \lim_{z\to\infty}\frac{\overline{v}(S_\lambda)\left(\frac{z+\lambda}{z}\right)^{1+ic-k} - 1}{z^{-1-ic}}$$
$$= \begin{cases} \alpha\overline{v}(S_\lambda)\lambda(1-k), & \text{if } c = 0; \\ \nexists, & \text{if } c \neq 0. \end{cases}$$

Which implies, (5.12) holds if $t = 0$ and $\gamma = 0$, then we have

$$\Re(\beta_l) < 0 \text{ or } \Re(\beta_l) = 0 \text{ with } m(l) = 0.$$

From (5.20) we have $\Re(\beta_l) = 0$ with $m(l) = 0$. Then (5.20) becomes the form

$$q(z) = \sum_{l=1}^{\infty} a_l z^{iy_l},$$

where y_l are real numbers for all l. Using (5.6) , we get

$$q + q|T = \sum_{l=1}^{\infty} a_l z^{iy_l} + \sum_{l=1}^{\infty} a_l \overline{v}(T)(-1)^{iy_l} z^{-iy_l-k} = 0.$$

By Lemma 5.4.1 all the coefficients are zero, i.e., $a_l = 0$ for all l. That gives $q(z) = 0$. This completes the proof of the theorem. □

Remark 5.4.5. *In Theorem 5.4.3, $q(z) = 0$, that means the corresponding analytic function F is an automorphic form. Therefore, we can apply a Hecke correspondence, Theorem 2.2.1. With $f = g$, then the generalized Dirichlet series has at most simple poles at $s = 0$ and $s = k$.*

Remark 5.4.6. *Since q given by the Proposition* 3.2.1 *is analytic and absolutely and uniformly convergent in every compact subset of $\mathcal{H}$. Then term by term differentiation is possible, we can observe that the n^{th} derivatives, $q^{(n)}(z)$ exists and also an infinite log-polynomial sum for $n \in \mathbb{Z}^+$. Therefore, it is possible to extend Theorem* 5.4.2 *and Theorem* 5.4.3 *to ILPSPFs of negative integral weight by applying G.Bol's theorem as in (Theorem 3.4, [11]).*

References

[1] Berndt, Bruce C. and Knopp, Marvin I. *Hecke's Theory of Modular Forms and Dirichlet Series.* World Scientific, 2008.

[2] Bochner, Saloman. *Some Properties of Modular Relation.* Ann. Math., 53:332-363, 1951.

[3] Hawkins, John H. and Knopp, Marvin I. *A Hecke Correspondence Theorem for Automorphic Integrals with Rational Period Functions.* Illinois J. Math.,36:178-207, 1992.

[4] Hecke, Erich. *Über die Bestimmung Dirichletscher Reihen durch ihre Functionalgleichung.* Math. Ann.,112:664-669, 1936.

[5] Knopp, Marvin I. *Modular Functions in Analytic Number Theory.* AMS Chelsea Publishing, 1970.

[6] Knopp, Marvin I. *Some New Results on the Eichler Cohomology of Automorphic Forms.* Bulletin of the American Mathematical Society, 80:607-633, 1974.

[7] Knopp, Marvin I. and Sheingorn, Mark *On Dirichlet Series and Hecke Triangle Groups of Infinite Volume.* Acta Arithmatica, 76:227-244, 1996.

[8] Knopp, Marvin I. *Rational Period Functions of the Modular Group II.* Glasgow Math. J.,22:185-197, 1981.

[9] Knopp, Marvin I. *Rational Period Functions of the Modular Group I.* Duke Math., J.45:47-62, 1978.

[10] Whittaker, E.T. and Watson, G.N. *A Course of Modern Analysis.* Cambridge University Press, 4th Edition, 1927.

[11] A. Hassen, *Log-Polynomial Period Functions for Hecke Groups* Ramamujan J.3, NO. 2, 119-151, 1999.

[12] A. Daughton. *A Hecke Correspondence for Automorphic Integrals with Infinite Log-Polynomial Periods.* A Dissertation to Temple University, 2012.

[13] Evans Ronald. *A Fundamental Region for Hecke's Modular Groups.* J. of Number Theory,5:108-115, 1973.

[14] Bol, G. *Invarianten Differentialgleichungen.* Abh. Math. Sem. Univ. Hamburg,16:1-28, 1949.

[15] Lehner, Joseph, *Discontinuous Groups and Automorphic Functions.*Providence: American Math Soc. 1964.

[16] Lehner, Joseph, *A Short Course in Automorphic Functions.*Providence: University of Maryland. 1966.

[17] M. Eichler, *Grenzkreisgruppen and kettenbruchartige Algorithmen*, Acta Arith. 11 (1965), 169-180.

[18] E. Hecke, *Herleitung des Euler-Produktes der Zetafunktion und einiger L-Reihen ous ihr Funktionalgleichung*, Math. Ann. 199 (1944), 266-287.

[19] M. Knopp, *A corona theorem for automorphic forms and related results*, Amer. J. Math. 91 (1969), 599-618.

[20] M. Knopp, *On Dirichlet series satisfying Riemann's functional equation*, Invent. Math. 117 (1994), 361-372.

[21] M. Knopp, *On the growth of entire automorphic integrals*, Result in Math. 8 (1985), 146-152.

[22] J. Lehner, *Automorphic forms with preassigned periods*, J. Res. Nat. Bur. Standards Sect. B 73 (1969), 153-161.

[23] Copson, E.T. *Theory of Functions of Complex Variables*, Clarendon Press, Oxford, 1935.

[24] B. Riemann, *Über die Ansahl der Primzahel unter einer gegebenen Grösse*, Monastber. Berlin. Akad (1859),671-680.