SPECTRAL THEORY OF COMPACT OPERAT0RS AND ITS APPLICATIONS

M.Sc. Thesis

BY

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In partial fulfillment of the requirements for the award of the degree of Master of Science in Mathematics

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ADVISOR'S APPROVAL SHEET

This is to officially state that the Thesis work titled "**Spectral theory of compact operators** and **applications**" done under my supervision and guidance, is a genuine work that has been done by "Tetemike Belachew " for partial fulfillment of the requirements for the award of the degree of Master of Science in mathematics from Addis Ababa University. Therefore, I recommend and would accept it as fulfilling the Thesis requirements.

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EXAMINERS APPROVAL SHEET

We, the under signed members of the board of examiners of the final open defense by Tetemike Belachew, have read and evaluated this Thesis entitled "Spectral theory of compact operators and its applications" and examined the candidate. This is therefore to certify that the Thesis work has been accepted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

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My heartfelt gratitude goes to GOD for the strength, wisdom and direction in this journey of completing this work. All glory to GOD.

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NOTATION AND SYMBOLS

σ_p(T): the point spectrum of T

σ_c(T): the continuous spectrum of T

σ (T) : Spectrum of operator of T

$\langle x, y \rangle$: Inner product of x and y

$\sigma_r \ (T)$: the residual spectrumof T

$\|f\|$: Norm of a bounded linear function f

T^* : adjoint of operator T

$M^{\perp}$: 0rthogonal complement of M

$B(H)$: the space of all bounded linear operators on ahilbert spaceH

$K(x, y)$: kernel of the operator

$K(H_{1,} H_2)$: the space of all compact operators from a Hilbert space $H_1$to a Hilbert space H_2

B(x, y): $bounded\ bilinear\ operator$

B(X): the space of all bounded linear operators from Banach space X to itself

K$(\boldsymbol{X})$: the space of all compact linear operators from $XtoX$

$B\ [a, b]$: the Space of all bounded functions on $[a, b]$

Abstract

This Thesis presents a systematic investigation into the spectral theory of compact linear operators and demonstrates its profound utility across several key areas of mathematics. The central motivation for this work is the challenge of extending the well-understood spectral properties of matrices in finite-dimensional linear algebra to operators on infinite-dimensional Banach and Hilbert spaces. While general bounded operators can exhibit complex spectral behavior, compact operators form a special class whose spectral properties are remarkably tractable and analogous to the finite-dimensional case. The theoretical core of the work is dedicated to establishing the foundational structure theorems for this class of operators. We develop the Riesz-Schauder theorem, which asserts that the spectrum of a general compact operator on a Banach space is a discrete set of eigenvalues with finite multiplicity that can only accumulate at zero. For the more specialized case of compact self-adjoint operators on a Hilbert space, we present the Spectral Theorem, a powerful result guaranteeing the existence of a complete orthonormal basis of eigenvectors. This theorem provides an infinite-dimensional analogue of matrix diagonalization, which is the key to its wide-ranging applicability. The utility of this abstract framework is demonstrated through three cornerstone applications. First, in the theory of integral equations, we show that Fredholm integral operators are compact and apply the Fredholm Alternative to derive a comprehensive theory for the existence and uniqueness of solutions. Second, we address Sturm-Liouville eigenvalue problems for unbounded differential operators by analyzing their compact, self-adjoint inverse operators (Green's functions), thereby proving the existence of a discrete spectrum and a complete basis of eigenfunctions.

I. INTRODUCTION

1.1Background of the Study

Functional analysis, a cornerstone of modern mathematics, extends the concepts of linear algebra and calculus to infinite-dimensional spaces. A central and particularly fruitful area within this field is the spectral theory of compact operators. This theory generalizes the familiar notions of eigenvalues and eigenvectors from finite-dimensional linear algebra to the broader context of operatorsonBanachandHilbertspaces.

While the spectral theory of general bounded linear operators can be quite intricate, there exists a special class of operators, known as compact linear operators, whose spectral properties are remarkably well-structured and more analogous to the finite-dimensional case. A compact linear operator is one that maps bounded sets to precompact sets (sets whose closures are compact). This property, while abstract, has profound consequences for the operator's spectrum.

The study of compact operators is deeply connected with the theory of integral equations. Many integral operators, such as those of the Fredholm type, are compact. The spectral theory of these operators provides a powerful framework for analyzing and solving such equations, transforming complex analytical problems into a more manageable algebraic form. Furthermore, this theory is instrumental in the study of certain differential equations and has significant applications in quantummechanics.

This Thesis aims to provide a rigorous and self-contained development of the spectral theory of compact linear operators and to demonstrate its power through key applications.

1.2Statement of the Problem

The primary challenge in extending spectral theory from finite to infinite dimensions lies in the complexity of the spectrum of a general linear operator. Unlike the finite-dimensional case where the spectrum consists solely of eigenvalues, the spectrum of an operator on an infinite-dimensional space can be much more varied. The central problem to be addressed is to develop a comprehensive understanding of the spectral properties and to apply this theoretical framework to solve concrete problems in analysis, particularly in the theory of integral equations.

1.3Objectives of the Study

The main objective of this Thesis is to present a detailed study of the spectral theory of compact linear operators and its applications.

• To review the necessary background material from functional analysis, including the theory of Banach and Hilbert spaces, bounded linear operators, and general spectral theory. Introduce the conceptofacompactlinearoperatorandtostudyitsfundamentalproperties.

•To develop the spectral theory of compact operators culminating in the Riesz-Schauder theorem and the Fredholm alternative.

• To present the spectral theorem for compact self-adjoint operators on Hilbert spaces.

• To apply the developed theory to the analys is of Fredholm integral equations.

Significance of the Study

The spectral theory of compact operators is a fundamental topic in functional analysis with wide-ranging applications in both pure and applied mathematics. A thorough understanding of this theory is essential for graduate students in mathematics and is a prerequisite for more advanced topics in operator theory and differential equations. This Thesis will provide a clear and comprehensive exposition of this important subject, making it a valuable resource for students and researchers.

1.4Scope and Limitation of the Study

This Thesis will focus on the spectral theory of compact linear operators on Banach and Hilbert spaces. The primary application to be considered will be the theory of Fredholm integral equations. The study will not delve into the spectral theory of more general classes of operators, such as unbounded operators, which require a more advanced mathematical concept.

II. PRELIMINARIES

2.1 Definitions

The purpose of this section is to refresh our minds on some basic fundamentals required to facilitate a smooth understding of the study of spectral theory of compact operators and its applications

Definition 2.1 A vector space over F is a set V together with the operations of addition V×V→V and scalar multiplication F×V→V satisfying each of the following properties.

1.Commutativity: $u + v = v + u$ for all $u, v \in V$;

2.Associativity: $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and $a, b \in F$;

3.Additive identity: There exists an element 0∈V such that 0+v=v for all $v \in V$;

4.Additive inverse: For every $v \in V$, there exists an element $w \in V$ such that $v + w = 0$;

5.Multiplicative identity: 1v=v for $all\ v \in V$;

6.Distributivity : $a(u + v) = au + av\ and\ (a + b)u = au + bu\ for\ all\ u, v \in V\ and\ a, b \in F.$

A vector space over R is usually called a real vector space, and a vector $spaceover\ C$ is similarly called a complex vector space. The elements v∈V of a vector space are called vectors.

Definition 2.2 A metric space is a pair (X,d), where X is a set and d is a function from X×X to R such that the following conditions hold for every x,y,z ∈ X.

1. Non-negativity: $d(x, y) \geq 0$.
2. Symmetry: $d(x, y) = d(y, x)$.
3. $Triangle\ inequality\ d(x, y) + d(y, z) \geq d(x, z)$
4. $d(x, y) = 0\ if\ and\ only\ if\ x = y$ Elements of X are called points of the metric space, and d **is** called a metric or distance function on X.

Definition 2.3 A non-negative function $\|.\|$ on a vector space X is called a *norm* on X if and only if

i$\|x\| \geq 0$ for every x ∈ X (Positivity).

ii)) $\|x\| = 0$ if and only if x = 0 (Non degeneracy).

`iii)) $\|\lambda x\| = |\lambda|\|x\|$ for every x ∈ X (Homogeneity).

iv) $\|x + y\| \geq \|x\| + \|y\|$ for every x, y ∈ X(subadditivity)

A vector space X with a norm $\|.\|$ is denoted by (X,$\|.\|$) and is called a normed linear space (or just a normed space).

$A\ sequence\ (x_n)_n \in \mathbb{N}\ of\ elements\ in\ a\ normed\ linear\ space\ X\ is\ called\ Cauchy\ if\ \forall\epsilon > 0, \exists N \in \mathbb{N}, such\ that\ for\ m,n \geq N, ||x_m - x_n|| \leq \epsilon.$

A ***Banach space*** is a normed linear space $(X, \|.\|)$that is complete in the canonical metric defined $by\ \rho(x,y) = ||x - y||$ for x, y ∈ X i.e every Cauchy sequence in X for the metric ρ converges to some point in X.

example of a Banach space is the space of continuous functions defined on a closed interval [a, b], denoted by C[a, b].

Remark: Every normed linear space has a completion.

Definition 2,4 (Inner Product)

Let H be a vector space over a field K, where K is either the real numbers (ℝ) or the complex numbers (ℂ). An inner product on H is a function $\langle \cdot\ \cdot \rangle : H \times H \to K$ that satisfies the following axioms for all vectors x, y, z ∈ H and all scalars α, β ∈ K:

i. Conjugate Symmetry:

⟨x, y⟩ = ⟨y, x⟩

(Note: If the field K is ℝ, this simplifies to symmetry: ⟨x, y⟩ = ⟨y, x⟩.)

ii. Linearity in the first argument:

⟨αx + βy, z⟩ = α⟨x, z⟩ + β⟨y, z⟩

iii. Positive-Definiteness**:**

⟨x, x⟩ ≥ 0 and ⟨x, x⟩ = 0 if and only if x = 0.

A vector space H equipped with an inner product is called a Pre-Hilbert Space or an Inner Product Space

Definition 2.5 Let X and Y be K-linear spaces ($K = \mathbb{R}\ or\ K = \mathbb{C}$)$.\ A\ map\ T: X \to$ Y is called linear if

$T\ (\lambda f + \mu g) = \lambda T\ (f) + \mu T(g)\ for\ all\ f, g \in X, and\ all\ \lambda, \mu \in K.$

Definition 2.6 A mapping $T: X \rightarrow Y$ is called a continuous linear operator if T is $linear\ and$ is continuous at each point in $a \in X$, that is
$lim_{\top}(x \rightarrow a)\ (Tx) = Ta\ for\ all\ a \in X. The\ space$ of continuous linear operators from a Banach space X into a Banach Space Y is denoted by B(X, Y).
Proposition 2.1 Every bounded linear map between normed linear spaces has a unique extension between their completions

Definition 2.7 A map T defined from a Banach space X into a Banach space Y is called closed if its graph
$G(T) = \{(x, y) : x \in D(T)\}$is closed in X × Y.
In other words, T is closed if whenever $x_n \longrightarrow x$ and $Tx_n \longrightarrow y$, we have x ∈ D(T) and Tx = y.
Remark Every T ∈ B(X, Y) is closed.

Definition 2.8 A map $T \in B(X, Y)$ is invertible if there is a bounded linear map $T^{-1} \in B(Y, X)$ such that $T^{-1}T = I_x$ (the identity of X) and $TT^{-1} = I_Y$ (the identity operator of Y).

Corollary 2.1 Every continuous bijection between Banach spaces has a continuous inverse.
Proof
$Let\ T \in B(X, Y). Then\ G(T) = \{(x, Tx) : x \in X\}\ is\ closed\ in\ X \times Y, G(T^{-1}) = \{(Tx, x) : x \in X\}$ is closed in Y × X. T^{-1} is a closed linear operator mapping Y into X. Therefore T^{-1} is bounded by the closed graph theorem.

Definition 2.9 ($A\ Hilbert\ space$)
A Hilbert space is a complete inner product space. This means it is a vector space with a defined notion of distance and angle that behaves like 3D Euclidean space but can have any number of dimensions (including infinite). The "completeness" property ensures that it has no "holes" or "missing points.

Definition 2.10 $A\ Pre - Hilbertian\ space\ (H, \langle\cdot,\cdot\rangle)$ is called a Hilbert space if it is complete when equipped with the corresponding canonical norm.

$\boldsymbol{Definition\ 2.11}$ $Let\ H\ be\ a\ Hilbert\ space\ and\ x, y \in H$. We say that x is orthogonal to y denoted by $x \perp y\ if\ \langle x, y\rangle = 0$.
$For\ M \subset H$, we say that x is orthogonal to M and write x ⊥ M, if x is orthogonal to every vector y in M.
The subset of vectors of H orthogonal to M is denoted by

$$M^\wedge \perp = \{x \in H : x \perp M\}$$

and is called the orthogonal complement of M in H.

Proposition 2.2 Let M and N be arbitrary subspaces of a Hilbert space H. Then the following holds:

i) $M^{\perp}$ is a closed subspace of H
ii) M ⊂ $M^{\perp\perp}$,
iii) if M ⊂ N then $N^{\perp} \subset M^{\perp}$
iv) $(M^{\perp})^{\perp}$ = M.

Definition 2.12Anadjoint operator denoted as $T^{\star}$is a linear operator that reverses the order of an inner product such that for any vectors x and y in the inner product space the inner product of Tx with y is equal to the inner product of x withTy $\langle Tx, y\rangle = \langle x, Ty\rangle$.

Theorem 2.1 The projection Theorem
Let H be a Hilbert space and M be a closed subspace of H. For an arbitrary given vector x ∈ H, there exists a unique vector m* ∈ M such that

$$||x - m*|| \leq ||x - m|| \, for \, all \, m \in M.$$

Furthermore, z ∈ M is the unique vector m* if and only if
(x - z) ⊥ M.

Corollary 2.2 Direct Sum Decomposition

Let M be a closed subspace of a Hilbert Space, H. Then H = M $\bigoplus$ $M^{\perp}$

Definition 2.13 A Hilbert space H is **se**parable if H contains a countable dense subset.

Equivalently, a Hilbert space H is said to be separable if there exists a sequence of vectors v_1, v_2, ..., v_k, ... which span a dense subspace of H.

Theorem 2.2

A Hilbert space admits a countable orthonormal basis if and only if it is separable.

Definition 2.14 $Let\ X\ and\ Y\ be\ Banach$ spaces and T ∈ B(X, Y), define the dual (also called $adjoint$) operator as a map T* : Y* → X* defined by

Tf = f ∘T_n *that is,(T*f)(x) = f(T(x)) for all x ∈ X.

T* is called the (topological) dual or adjoint operator of T.

Remark T* ∈ B(Y*, X*)

Definition 2.15 Ad joint operators on Hilbert spaces

Let $T \in B(H_1, H_2)$, the adjoint of T is the unique map $T^* : H_2 \to H_1$ such that

$\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in H_1$ and all $y \in H_2$.

Definition2.16 Symmetric operator is a linear operator A on a Hilbert space whose domain $D(A)$ is dense in the space and it satisfies the condition (Ax,y)= (x,Ay) for all x and y in$D(A)$.

Definition 2.17 A continuous linear operator is defined as a bijective map \(T: N \to X \) between normed spaces that maintains linearity and has a sequentially continuous inverse, ensuring the continuity of the operator.

III .SPECTRAL THEORY OF COMPACT OPERATORS

3.1. SPECTRAL THEORY FOR FINITE DIMENTIONAL SPACES

Definition3.1 $Let\ V\ be\ afinite\ dimensional$ vector space over the field of complex numbers and let T: $VtoV$ be $alinear\ operator$ on $V. The\ spectrum\ of\ T\ denoted\ by\ sigma(T)$is the set of all complex numbers **λ** such that the operator T-**λ**I is not invertible.where I is the identity operator on V

Definition 3.2 A compact operator is abounded linear operator between normed vector spaces that maps abounded subset of the domain to a relatively compact subset of the codomain.
Let X and Y be arbitrary Banach Spaces. A linear operator,$T : X \to Y$ is called compact if the image of the closed unit ball $B_x(0,1) = \{x \in X : \|X\|_x \leq 1\}$ by T is a relatively compact subset of Y. In other words, T is $compact\ if\ T(B_x)$ is compact.
This definition is equivalent to each of the following properties.
i) For each bounded B ⊂ X, the image T(B) is relatively compact in Y.**)**.Every bounded sequence $\{x_n\}_n \in \mathbb{N} \subset X$, $\{Tx_n\}_n \in \mathbb{N}$ has a convergent subsequence in Y**.**
We introduce the following notations
K(X, Y) := {T : X → Y | T is linear and compact}
K(X) = K(X, X)
Lemma 3,1 K(X, Y) ⊂ B(X, Y)
Proof. Suppose T ∈ K(X, Y), we show that T ∈ B(X, Y) i.e for all x ∈ X there exist M > 0 such that $\|Tx\| \leq \|M\|\ \|x\|$
$Let\ x \in X. if\ x = 0, it\ holds\ trivially. Assume\ x \neq 0, then\ x/||x|| \in B_x$
$T(B_x)\ is\ compact\ since\ T \in K(X,Y), so\ T(B_x)\ is\ bounded.$
Therefore there exist $M > 0\ such\ that\ ||T(x/||x||)|| \leq M$, which implies $that\ ||Tx||\ \ M||x||$.
Remark Recall that Riesz theorem characterizes the compactness of the closed unit ball of a

Banach space X by the finiteness of the dimension of X.

Thus for an infinite dimensional $Banach\ space\ X$, we have I ∈ B(X) \ K(X) and so the inclusion $K(X) \subset B(X)$ is strict, that is,

K(X) ⊂ B(X) whenever $dimX = +\infty$.

Definition 3.3 Let A be a subset of a Banach space X. A is said to be precompact if any sequence of A has a Cauchy subsequence. A is totally bounded if for every $\epsilon > 0$, there exists a finite cover for A of open balls of same radius ϵ.

Definition 3.4 A is said to be compact if any sequence of A has a subsequence that converges to some point of A.

Remark A set is said to be pre-compact in a Banach X if and only if its closure is compact in X.

Fredholm's theorem3.1 in linear algebra(matrix).In linear algebra for agiven matrix M,Fredholms theorem states that the orthogonal complement of the row space of M is the null space of M and conversely the orthogonal complement of the column spaceof M is the null space of it's a djoint(or transpose in the real case).

Mathematically if *M* is a matrix, then the orthogonal complement of the row space of *M* is the null space of *M* is expressed as $(rowM)^{\perp} = kerM$.

Similarly, the orthogonal complement of the column space of *M* is the null space of the adjoint:$(colM)^{\perp}$=ker$M^{\star}$. More generally the Fredholm alternatives states that for the matrix equation Ax=b either the homogenous equation Ax=0 has only the trivial solution(x=0) in which case Ax=b has a unique solution for any b or the homogeneous equation has non trivial solutions and the inhomogeneous equation Ax=b has a solution if and only if b is orthogonal to the null space of $A^{\star}(the\ adjoint\ of A)$

Fredholm's theorem for integral equations is expressed as follows.

Let K(X,Y) be an integral kernel, and consider the homogeneous equations

$\int_a^b k(x,y)\phi(y)dy = \lambda\phi(X)$and its complex adjoint

$$\int_a^b \phi(x)\overline{k(x,y)}\, dx=\bar{\lambda}\phi(\mathrm{y})$$

Here, $\bar{\lambda}$ denotes the complex conjugate of the complex number **λ** and similarly

for $\overline{k(x,y)}$ denotes the conjugate of k(x,y).Then, Fredholm's theorem is that, for any fixed value of **λ**, these equations have either the trivial solutions ψ(x)=ϕ(x)=0 or have the same

number of linearly independent solutions $\phi_{1(x)},\ldots,\phi_{n(x)}$, $\psi_{1(y)}$,…, ψ_n(y).

A sufficient condition for this theorem to hold is for k(x,y) to be square integrable on the rectangle [a,b]x[a,b] (where *a* and/or *b* may be minus or plus infinity).

Here, the integral is expressed as a one-dimensional integral on the real number line.
In Fredholm theory, this result generalizes to integral operators on multi-dimensional spaces.

One of Fred holm's theorems, closely related to the Fredholm alternative, concerns the existence of solutions to the inhomogeneous Fredholm equation

$$\boldsymbol{\lambda}\Phi(x)-\int_a^b k(x,y)\phi(y)dy = f(x).$$

Solutions to this equation exist if and only if the function f(x) is orthogonal to the complete set of solutions $\{\psi_n(x)\}$ of the corresponding homogeneous adjoint equation. $\int_a^b \overline{\psi_{n(x)}}$ f(x)dx=0.

Where, $\overline{\psi_n}(x)$ is the complex conjugate of $\psi_n(x)$ and the former is one of the complete set of solutions to $\overline{\lambda\phi(y)} - \int_a^b \overline{\phi(x)}$ k(x,y)dx=0.

A sufficient condition for this theorem to hold is for k(x,y) to be square integrable on the rectangle [a,b]x[a,b] .

Fredholm Alternatives is a fundamental theorem in functional analysis stating that for an operator equation (I-K)x =y where k is a compact operator on aBanach space either the equation has unique solution for every right hand side y or the homogeneous equation(I-K)x=0 has nontrivial solutions.

More generally for **λ**$\neq 0$ the equation **λ**x-Ax=z has a solution if and only if z is in the orthogonal complement of the null space of **λ**-A.

3.2 Properties of compact linear maps

The set of compact operators is not dense in the space of all bounded linear operators on an infinite dimensional space**.**
Compact linear operators play a crucial role in functional analysis, particularly in the study of infinite-dimensional spaces. Below, we will explore each of the properties you've mentioned in detail, along with examples to illustrate these concepts.

1. Every Compact Linear Operator is Bounded (Continuous).
Property: If $T: X \to Y$ is a compact linear operator between normed spaces X and Y, then T is bounded. This means there exists a constant $C > 0$ such that $\|Tx\| \leq C\|x\|$ for all $x \in X$.
Example: Consider the compact operator $T: L^2[0, 1] \to L^2[0, 1]$ defined by $(Tf)(x) = \int_0^1 k(x, y) f(y)dy$,where $k(x, y)$ is a continuous function on $[0, 1] \times [0, 1]$. By the continuity of the integral operator and the boundedness of $k(x, y)$, it follows that T is bounded.

2. Compact Operators are Norm Limit of Finite Rank Operators.
Property: Every compact operator can be approximated in the operator norm by finite-rank operators. Example: Let $T: L^2[0, 1] \to L^2[0, 1]$ be the compact operator defined as above. We can approximate T by a sequence of finite-rank operators T_n, which can be constructed by truncating the integral to a finite-dimensional subspace. For instance, if we take an orthonormal basis $\{e_1, e_2,\}$ in $L^2[0, 1]$, then we can define$T_n f = \sum_{i=1}^n \langle f, e_i \rangle e_i$,which is a finite-rank operator.
As $n \to \infty$, T_n T *in operator norm.*

3. Compact Operators are Not Invertible Except for Zero.
Property: A compact operator can only have a bounded inverse if it is the zero operator.
Example: *Let* $T: l^2 \to l^2$ *be defined by* $T(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$.This operator is compact because it sends sequences to sequences with their first element zero. The only way T could have an inverse is if it were bijective and bounded; however, it is clearly not injective (as it maps many sequences to the same output). Thus, compact operators are not invertible unless they are the zero operators.

4. Compact Operators are Not Surjective Unless They are Invertible.
Property: A compact operator cannot be surjective unless it is invertible.
Example: Consider the compact operator $T: l^2 \to l^2$ defined by$T(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$.
As shown earlier, this operator is not injective and thus cannot be surjective. In fact, any compact operator on an infinite-dimensional space cannot map onto the entire space unless it is invertible.

5. Compact Operators Preserve Compact Sets.
Property: If $K \subset X$ is a compact set in a normed space and $T: X \to Y$ is a compact operator, then $T(K)$ is also compact in Y.
Example: Let $K = [0, 1] \subset L^2[0, 1]$. The image under the compact operator defined above will yield a set that can be shown to be totally bounded and closed in the image space.

6. Compact Operators are Not Dense in the Space of Bounded Operators
Property: The set of compact operators is not dense in the space of all bounded linear operators on an infinite-dimensional space.
Example: Consider the identity operator $I: l^2 \to l^2$. The identity operator is not compact because it does not map bounded sets to relatively compact sets (the image of the unit ball under the identity is itself). Since compact operators cannot approximate the identity operator in norm (as any approximation would need to be compact), we conclude that compact operators are not dense in the space of bounded operators.
The properties of compact linear operators illustrate their unique characteristics and significance in functional analysis. They bridge finite-dimensional intuition with infinite-dimensional spaces and provide a framework for understanding convergence and continuity in various mathematical contexts.

Proposition 3.1 *Let* $T_n : X \to Y$ be a compact linear operator for each $n \geq 1$. Assume that $(T_n)_n$ converges to some T in B(X, Y). Then T is compact.

Proposition 3.2 Let $T \in K(X, Y)$ be an injective compact linear operator. Then T^{-1} is not continuous unless X is finite dimensional.

Proof

First, we recall that for Banach spaces X, Y and Z, if $S \in K(Y, Z)$, and $T \in B(X, Y)$ then $ST \in K(X, Z)$. Assume that $T^{-1} : R(T) \to X$ is bounded. It follows immediately that $T^{-1}T = I : X \; X$ is compact, which is possible only if X is finite dimensional (by Riesz theorem).

Some examples of compact linear operators

Finite rank operators is a linear operator whose image is finite dimensional such operators are always compact

Example Now let $X = C[a, b]$ and $G \in C([a, b] \times [a, b])$ the operator

$T : C[a, b] \to C[a, b]$ by

$(Tf)(x) = \int_a^b G(x,t)f(t)dt$ *is also compact.*

Example Compact embedding theorems

suppose that Ω is C^1 - bounded open subset of $\mathbb{R}^n$. Let $k \geq 1$ and $1 \leq p \leq \infty$, the following embeddings are compact.

i) if $kp < n$, then $H^{k,p} \hookrightarrow L^q(\Omega)$, for all $1 \leq q < np/(n - kp)$,

ii) if $kp = n$, then $H^{k,p} \hookrightarrow L^q(\Omega)$, for all $q \in [1, \infty)$,

iii) if $kp > n$, then $H^{k,p} \hookrightarrow C(\Omega)$.

Theorem 3.2 Let H_1 and H_2 be Hilbert spaces.

An operator $T \in B(H_1, H_2)$ is compact if and only if its adjoint T^* is compact.

Proof

Suppose $T \in K(H_1, H_2)$.

Observe that, $T^* \in B(H_2, H_1)$ so $TT^* \in K(H_2)$ by (2.1.1).

Let $\{x_n\} \subset H_2$ such that $\|x_n\| = 1$. We have that $\{Tx_n\}$ *is bounded in* H_1. *Therefore there exists* $\{Tx_{nk}\}$ subsequence of $\{Tx_n\}$ *such that* $\{TTx_{nk}\}$ converges in H_2. since $T \in K(H_1, H_2)$.

$\|Tx_{nk} - Tx_{nl}\|^2 = \langle Tx_{nk} - Tx_{nl}, Tx_{nk} - Tx_{nl}\rangle$

$= \langle TT^*(x_{nk} - x_{nl}), x_{nk} - x_{nl}\rangle$

$\leq \|TT^*(x_{nk} - x_{nl})\|\|x_{nk} - x_{nl}\|$

$\leq 2\|TT^*(x_{nk} - x_{nl})\| \to 0,\ k, l \to \infty$ since $\{TTx_{nk}\}$ *converges.*

Consequently $\{Tx_{nk}\}$ is Cauchy in H_1, its convergence is ensured by the completeness of H_1.

Hence, T^* is compact.

Conversely, Suppose T^* is compact. Then by the forward direction of this theorem, $T^{**} = T$ is compact

Theorem 3.3 Fredholm Alternatives

Let X be a Banach space and $T \in K(X)$. Then for any $\lambda \neq 0$, the following holds,

i) $Ker(\lambda I - T)$ is finite dimensional

ii) $R(\lambda I - T)$ is closed

iii) $Ker(\lambda I - T) = \{0\} \Leftrightarrow R(\lambda I - T) = X$

3.3 Spectrum of Linear compact operators

Definition3.5 Let X and Y be normed vector spaces (or more generally Banach spaces).A linear operator T:X to Y is said to be of finite rank if the imageof Tdenoted by im(T)=T(x) is afinite dimensional subspace of Y.

Definition3.6 Let X be a Banach space over a scalar field K for $T \in B(X, Y)$, the spectrum $\sigma(T)$ of T is defined by

$\sigma(T) = \{\lambda \in K : \lambda I - T \text{ is not invertible in } B(X)\}$

The resolvent set $\rho(T)$ is defined by

$\rho(T) = K \setminus \sigma(T)$.

The points of $\rho(T)$ are called the regular values of T.

If $\lambda \in \rho(T)$ then

$R\lambda(T) = (\lambda I - T)^{-1}$ is called the resolvent of T at λ.

The spectrum is decomposed into the disjoint union of the following three sets.

a) The point spectrum of T:

$\sigma_p(T) = \{\lambda \in K \mid Ker(\lambda I - T) \neq \{0\}\}$.

b) The continuous spectrum of T:

$\sigma c(T) = \{\lambda \in K : Ker(\lambda I - T) = \{0\}, \overline{R(\lambda I - T)} = X \text{ but } R(\lambda I - T) \neq X\}$.

c) The residual spectrum of T:

$\sigma_r(T) = \{\lambda \in K \mid Ker(\lambda I - T) = \{0\}, \overline{R(\lambda I - T)} \neq X\}$.

Remark: An element of $\sigma_p(T)$ is called an eigenvalue of T and a non-zero vector f such that $Tf = \lambda f$ is called an eigenvector of T associated to the eigenvalue λ.

Theorem 3.4 Let X be a Banach space over $\mathbb{C}$ and $T \in B(X)$. Then the following holds.

i) The spectrum, $\sigma(T)$ is a closed subset of $\mathbb{C}$

ii) The spectrum, $\sigma(T) \subset B(0, ||T||$_{B(X)})

iii) The spectrum, $\sigma(T)$ is a compact subset of $\mathbb{C}$

Proof.

i) It suffices to show that $\rho(T)$ is open. Let $\lambda_o \in \rho(T)$.

$R\lambda_o(T) = (\lambda_o I - T)^{-1}$

$R\lambda_o(T)^{-1} = \lambda_o I - T \Rightarrow T = \lambda_o I - R\lambda_o(T)^{-1}$

$\Rightarrow \lambda I - T = R\lambda_o(T)^{-1}[I - (\lambda_o - \lambda)R\lambda_o]$ (2.3.1)

If $|\lambda_o - \lambda| < 1/||R\lambda_o||$, then $I - (\lambda_o - \lambda)R\lambda_o(T)$ is invertible.

Therefore $(\lambda I - T)^{-1} = [I - (\lambda_o - \lambda)R\lambda_o]^{-1}R\lambda_o(T)$

$R\lambda(T) = (\lambda I - T)^{-1} = R\lambda_o(T) \sum_{k=0}^{\infty} (\lambda - \lambda_o)^k R\lambda_o(T)^k = \sum_{k=0}^{\infty} (\lambda - \lambda_o)^k R\lambda_o(T)^{k+1}$

$\Rightarrow R\lambda(T) = \sum_{k=0}^{\infty} (\lambda - \lambda_o)^k R\lambda_o(T)^{k+1}$ (2.3.2)

This implies that $R\lambda$ is invertible in the neighbourhood of $\lambda = \lambda_o$ such that

$\{|\lambda - \lambda_o| < 1/||R\lambda_o||\} \in \rho(T)$. Thus $B(\lambda_o, 1/||R\lambda_o||) \subset \rho(T)$.

Hence $\rho(T)$ is open, it follows immediately that $\sigma(T)$ is closed.

ii) Let $\lambda \in \mathbb{C} : |\lambda| > ||T||$

$||T/\lambda|| < 1$, therefore $(I - T/\lambda)^{-1}$ exist and $(I - T/\lambda)^{-1} = \sum_{k=0}^{\infty} (T/\lambda)^k$

$(\lambda I - T)^{-1} = (1/\lambda)(I - T/\lambda)^{-1} = \sum_{k=0}^{\infty} T^k/\lambda^{k+1}$

This implies that $(\lambda I - T)$ is invertible, thus $\lambda \in \rho(T)$.

Observe that we have shown that $B(0, ||T||)^c \subset \rho(T) \Rightarrow \sigma(T) \subset B(0, ||T||)$.

iii) It follows immediately from i) and ii) that the spectrum, $\sigma(T)$ is a closed and bounded subset of $\mathbb{C}$. Therefore it is compact.

Remark: For real Banach space X with $A \in B(X)$, the spectrum $\sigma(T)$ may be empty.

For example, with $X = \mathbb{R}^2$ and $A = [[1], [-1, 0]]$, we have $\lambda I - A$ is invertible for all $\lambda \in \mathbb{R}$.

However the spectrum of the complexification of T is nonempty. In fact $\sigma(T_{\mathcal{C}}) = \{\lambda \in \mathbb{C} \mid \lambda I - T$ is not invertible$\} = \{-i, i\}$.

Theorem holds.

i) $0 \in \sigma(T)$

ii) $\sigma(T) \setminus \{0\}$ consist of eigenvalues of finite multiplicity i.e the dimension of the λ-eigenspace($Ker(\lambda I - T)$) has finite dimension $\forall\lambda \in \sigma \setminus \{0\}$

iii) $\sigma(T) \setminus \{0\}$ is either empty, finite or a sequence converging to 0 (i.e., it is a discrete set with no limit point other than 0).

Spectral theory of compact linear self-adjoint operator

Definition 3.7 Let T: $\mathbb{R}^n \to \mathbb{R}^n$. Then the following fact is well known: If T is a real symmetric matrix, then it has real eigenvalues and there exists an orthonormal basis $\{e_1, e_2 ... e_n\}$ of $\mathbb{R}^n$ such that e_i is an eigenvector of T. In other words, T is diagonalisable over an orthonormal basis.
$\langle Tx, y\rangle = \langle x, Ty\rangle \ \forall x, y \in H$
Remark. TT* and T*T are self-ad joint for any $T\epsilon B(H)$.

Definition 3.8 Let H is a Hilbert space. An operator T ∈ B (H) is called self-adjoint or hermitian if T* = T

Definition3.9 A projection is a linear transformation P (or matrix P corresponding to this transformation in an appropriate basis) from a vector space to itself such that p^2=P. That is, whenever P is applied twice to any vector, it gives the same result as if it were applied once (idempotent)
The Thesision theorem states that every vector in a Hilbert space can be uniquely expressed as a linear combination of components, where one component lies in a closed subspace and the other is orthogonal to that subspace

Example1 let x be a(real or complex)linear space and P,Q;XtoX projection show that I-P is the projection on to kerP along ranP

Proof To show that I-P is aprojection we need to show that$(I-P)^2$=I-P .It is easy to see that (I-P)(x)=x-Px,and there fore$(I-P)^2$(x)=(I-P)(x-Px)=I(x-Px)-P(x-Px)=x-Px-Px+P^2(x)=x-Px-Px+Px=x-Px=(I-P)(x) .Note that we used that P is aprojection which by definitionmeans in particular that P^2=P.We still have to prove I-P is the projection on to kerP along ranP.

Let $x\epsilon$kerP.Then Px=0, So (I-P) x=x-Px=x

Let $y\epsilon$ren (I-P).Then y= (I-P)v=v-Pv for some $v\epsilon X$,So Py=P(v-Pv)=0.

There fore ren (I-P)=kerP.

Example 2 The projection P,Q are orthogonal written P⊥Q if PQ=QP=0,Show that P+Q is aprojection if and only if P⊥Q

Proof First let assme that P⊥Q we want to prove that P+Q is aprojection and for this we will use that P^2=P and that $Q^2 = Q$

$(P + Q)^2$(x)=(P+Q)(Px+Qx)

=P(Px+Qx)+Q(Px+Qx)

=P^2(x)+PQ(x)+QP(x)+Q^2(x)

=P(x)+Q(x)

Therefore P+Q is projection

V. APPLICATIONS OF THE SPECTRAL THEORY OF COMPACT LINEAR OPERATORS

The Spectral Theory of Compact Linear Operators is a cornerstone of functional analysis because it provides a powerful bridge between the familiar, well-behaved world of finite-dimensional linear algebra (matrices) and the complex, infinite-dimensional world of analysis. Compact operators, while acting on infinite-dimensional spaces (like spaces of functions), have a spectrum that is "tame" and remarkably similar to that of a matrix. This "tameness" is the key that unlocks solutions to a vast range of problems in mathematics and data science.

4.1Theory of Integral Equations (The Classic Application)

This is the historical birthplace and most direct application of the theory.

The Problem: Solving Fredholm integral equations of the second kind, which are of the form:

$$\boldsymbol{f(x) - \lambda \int [a,b]\, K(x,t) f(t)\, dt = g(x)}$$

Here, g(x) and the kernel K(x, t) are known functions, λ is a scalar, and f(x) is the unknown function we want to find.

The Connection**:** We define an integral operator T on a function space (like $L^2[a,b]$) by:

(Tf)(x) = ∫[a,b] K(x, t)f(t) dt

Under mild conditions (e.g., if the kernel K is square-integrable), this operator T is compact.

The integral equation can then be written in the abstract form:

(I - λT)f = g

The Solution: The Fredholm Alternative, a direct consequence of the Riesz- Schauder theorem, gives a complete answer about the solvability of this equation for $\lambda \neq 0$:

Either**:** The equation (I - λT)f = g has a unique solution for any choice of g.

This happens if and only if the number 1/λ is not an eigenvalue of T.or The number 1/**λ is** an eigenvalue of T. In this case, the homogeneousequation $f - \lambda Tf = 0$ has non-trivial solutions. The original equation $(I - \lambda T)f = g$ has a solution only if g satisfies specific orthogonality conditions, and if a solution exists, it is not unique.

4.2 Numerical Analysis

The theory provides the theoretical underpinning for many numerical methods used to solve operator equations.

The Problem**:** We often cannot find an exact analytical solution to an equation like $Tx = y$. Instead, we need to approximate the solution numerically.

The Connection**:** Many numerical methods, like the Galerkin method or the Finite Element Method, work by approximating the infinite-dimensional space with a sequence of finite-dimensional subspaces and the operator T with a sequence of finite-rank operators T_n. Finite-rank operators are the simplest type of compact operators.

The Solution: The space of compact operators $K(X)$ is a closed subspace of the space of bounded operators $B(X)$. This means that if our sequence of finite-rank approximations T_n converges to T in the operator norm, then the limit operator T is also guaranteed to be compact. Furthermore, the spectral properties of T_n (which are just matrix eigenvalue problems) will approximate the spectral properties of T.

4.3 Data Analysis and Statistics

This application extends the idea of Principal Component Analysis (PCA) to infinite-dimensional data like stochastic processes.

The Problem: In PCA, we find the best orthogonal basis to represent finite-dimensional data by finding the eigenvectors of the covariance matrix. How can we do this for a continuous signal or a stochastic process (which can be seen as a function)?

The Connection**:** For a stochastic process, the role of the covariance matrix is played by a covariance function $K(s, t)$. This function serves as the kernel of a self-adjoint integral operator T, which is often compact.

The Solution: The Karhunen-Loève Theorem is a direct application of the Spectral Theorem for. compact self-adjoins operators to this covariance operator. It states that the eigenfunctions of the covariance operator form the optimal orthonormal basis for representing the process, in the sense that it minimizes the mean squared error of any finite-term approximation. The Spectral Theory of Compact Operators provides a profound and unifying framework. It shows that many seemingly intractable problems in continuous systems (integral and differential equations) can be solved by converting them into an operator framework where the essential structure is discrete and countable, just like in finite-dimensional linear algebra

CONCLUSION

The spectral theory of compact linear operators provides a profound and unifying framework within modern mathematics. It demonstrates that many seemingly intractable problems in continuous, infinite-dimensional systems can be understood by converting them into an operator framework where the essential structure is discrete and countable, much like in the familiar setting of finite-dimensional linear algebra. This Thesis has sought to build this theory from its foundations and, more importantly, to illustrate its remarkable power through a series of critical applications that bridge the gap between abstract analysis and concrete scientific results. Our investigation began with a review of the essential preliminaries from functional analysis, establishing the concepts of Banach and Hilbert spaces which serve as the stage for operator theory. We then formally introduced compact linear operators, showing them to be a special, well-behaved subset of bounded operators. The theoretical heart of this work lay in the development of their spectral properties. For general compact operators on Banach spaces, the Riesz-Schauder theorem revealed a surprisingly simple spectral structure: a discrete set of eigenvalues of finite multiplicity, with zero as the only possible accumulation point. This result, through the Fredholm Alternative, provided a powerful dichotomy for the solvability of operator equations involving compact perturbations of the identity. For the more structured environment of a Hilbert space, we showed that for a compact and self-adjoint operator, the theory culminates in the Spectral Theorem—a powerful decomposition result that guarantees the existence of a complete orthonormal basis of eigenvectors. We emphasized that neither compactness nor self-adjointness alone is sufficient to guarantee such a basis, but their combination provides the key to "diagonalizing" an infinite-dimensional operator.

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