Dynamics of Three-Level Laser Pumped by Electron Bombardment and with Spontaneous Emission

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DECLARATION

I hereby declare that this PhD dissertation is my original work and has not been presented for a degree in any other university, and that all sources of material used for the dissertation have been duly acknowledged.

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Abstract

In this PhD dissertation we have studied the statistical and squeezing properties of the cavity light generated by a three-level laser. In this quantum optical system, N three-level atoms available in an open cavity, coupled to a two-mode vacuum reservoir, are pumped to the top level by means of electron bombardment at constant rate. We have considered the case in which the three-level atoms and the cavity modes interact with the two-mode vacuum reservoir. We have carried out our analysis by putting the noise operators associated with the vacuum reservoir in arbitrary order. Applying the solutions of the equations of evolution for the expectation values of the atomic operators and the quantum Langevin equations for the cavity mode operators, we have calculated the mean and variance of the photon number as well as the quadrature squeezing for the cavity light. In addition, we have shown that the presence of the spontaneous emission process leads to a decrease in the mean and variance of the photon number. We have seen that the global mean photon numbers of the light modes emitted from the top and intermediate levels are the same both in the presence and absence of spontaneous emission, and are separately in a chaotic state. However, we have observed that the two-mode cavity light is in a squeezed state and the squeezing occurs in the minus quadrature. In addition, we have found that the effect of the vacuum reservoir noise is to increase the photon-number variance and to decrease the quadrature squeezing of the cavity light. However, the vacuum reservoir noise does not have any effect on the mean

photon number. Moreover, the maximum quadrature squeezing of the light generated by the laser, operating far below threshold, is found to be 37.5% below the vacuum-state level. In addition, our result indicates that the quadrature squeezing is greater for $\gamma = 0$ than that for $\gamma = 0.4$ for $0.01 < r_a < 0.35$ and is smaller for $\gamma = 0$ than that for $\gamma = 0.4$ for $0.35 < r_a < 1$. We have also noted that the local quadrature squeezing approaches the global quadrature squeezing as the frequency interval increases.

Furthermore, applying the density operator for a pair of superposed two- mode laser light beams, we have calculated the mean and variance of the photon number as well as the quadrature squeezing. We have found that both the mean photon number and the quadrature variance for the superposed two-mode laser light beams is the sum of the mean photon numbers and the quadrature variances of the constituent two-mode light beams. However, the variance of the photon number of the superposed two-mode laser light beams is not the sum of the variances of the photon numbers of the constituent two-mode light beams. Finally, our result shows that the quadrature squeezing of the superposed two-mode laser light beams is equal to the quadrature squeezing of one of the superposed the two-mode light beams. This implies that the superposition of the two-mode laser light beams does not affect the quadrature squeezing, but it increases the global mean photon number and the global variance of the photon number. Thus we note that the superposition of the two-mode laser light beams leads to a more bright squeezed light.

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1

Introduction

Squeezed states of light have played a crucial role in the development of quantum physics. Squeezing is one of the nonclassical features of light that has been extensively studied by several authors [1-18]. In squeezed light the noise in one quadrature is below the vacuum-state level at the expense of enhanced fluctuations in the other quadrature, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation [1]. Squeezed light has potential applications in low-noise optical communication and weak signal detection [1,7]. Squeezed light can be generated by various quantum optical processes such as subharmonic generations [1-5], four-wave mixing [2,3], resonance fluorescence [1,8], second harmonic generation [1-3], and three-level laser under certain conditions [1,16].

A three-level laser is a quantum optical system in which light is generated by three-level atoms inside a cavity usually coupled to a vacuum reservoir via a single-port mirror. When a three-level atom in a cascade configuration makes a transition from the top to the bottom level via the intermediate level, two photons are generated. If the two photons have the same frequency, then the three-level atom is called degenerate three-level atom otherwise it is called nondegenerate. The squeezing and statistical properties of the light produced by three-level lasers when the atoms are initially prepared in a coherent superposition of the top and bottom levels or when these levels are coupled by strong coherent light have been studied by several authors [19-37]. These authors have found that these quantum optical systems can generate squeezed light under certain conditions.

Moreover, Fesseha [16] has studied the squeezing and statistical properties of the light produced by a three-level laser with the atoms placed in a closed cavity and pumped by electron bombardment. He has shown that the maximum quadrature squeezing of the light generated by the laser, operating below threshold, is found to be 50% below the vacuum-state level. In addition, he has also found that the quadrature squeezing of the output light is equal to that of the cavity light. On the other hand, this study indicates that the local quadrature squeezing is greater than the global quadrature squeezing. He has also found that a large part of the total mean photon number is confined in a relatively small frequency interval. In addition, Fesseha [1] has studied the squeezing and statistical properties of the light produced by a three-level laser with the atoms placed in a closed cavity and pumped by coherent light. He has shown that the maximum quadrature squeezing is 43% below the vacuum-state level, which is slightly less than the result found with electron bombardment.

This PhD dissertation essentially has two parts. In the first part we wish to study the squeezing and statistical properties of the light generated by three-level atoms available in an open cavity and pumped to the top level by electron bombardment. We carry out our calculation by putting the noise operators associated with the vacuum reservoir in an arbitrary order. By taking into account the interaction of a three-level atom with a resonant cavity light and the damping of the cavity light by a vacuum reservoir, we first determine the master equation for a three-level atom in an open cavity coupled to a two-mode vacuum reservoir and the quantum Langevin equations for the cavity mode operators. In addition, employing the master equation and the large-time approximation scheme, we obtain equations of evolution of the expectation values of atomic operators. We then obtain the solutions of these equations and the quantum Langevin equations. Then applying the resulting solutions, we calculate the photon statistics and quadrature variances of the separate single-mode light beams. Furthermore, applying the same solutions, we obtain the mean and variance of the two-mode cavity light. Finally, we determine the global and local quadrature squeezing of the two-mode cavity light.

In the second part of this dissertation, we seek to analyze the squeezing and statistical properties of a pair of superposed two-mode light beams produced by three-level lasers in which the three-level atoms available in an open cavity are pumped to the top level by electron bombardment. To this end, we first determine for a two-mode laser light beam the Q function using the antinormally-ordered characteristic function, defined in the Heisenberg picture. Then employing the resulting Q function, we obtain the density operator for a pair of superposed light beams. Applying this density operator, we calculate the photon statistics and the quadrature squeezing of the superposed two-mode laser light beams

2

Operator Dynamics

In this chapter we first obtain the master equation for a three-level atom in an open cavity and the quantum Langevin equations for the cavity mode operators. In addition, employing the master equation and the large-time approximation scheme, we derive the equations of evolution of the expectation values of the atomic operators. Finally, we determine the steady-state solutions of the resulting equations. We carry out our calculation by putting the noise operators associated with the vacuum reservoir in arbitrary order.

2.1 Master equation

We consider here the case in which N three-level atoms, in a cascade configuration, are available in an open cavity. We denote the top, intermediate, and bottom levels of these atoms by $|a\rangle_k$, $|b\rangle_k$, and $|c\rangle_k$, respectively as shown in Fig. 2.1. We also consider the case in which a three-level atom is pumped from the bottom level $|c\rangle_k$ to the top level $|a\rangle_k$ at the rate of r_a by means of electron bombardment. A three-level atom may make a transition from the top level $|a\rangle_k$ to the intermediate level $|b\rangle_k$ and then from the intermediate level $|b\rangle_k$ to the bottom level $|c\rangle_k$ by emitting two photons of the same or different frequencies. Alternatively, the atoms may decay spontaneously from the top level $|a\rangle_k$ to the intermediate level $|b\rangle_k$ or from the

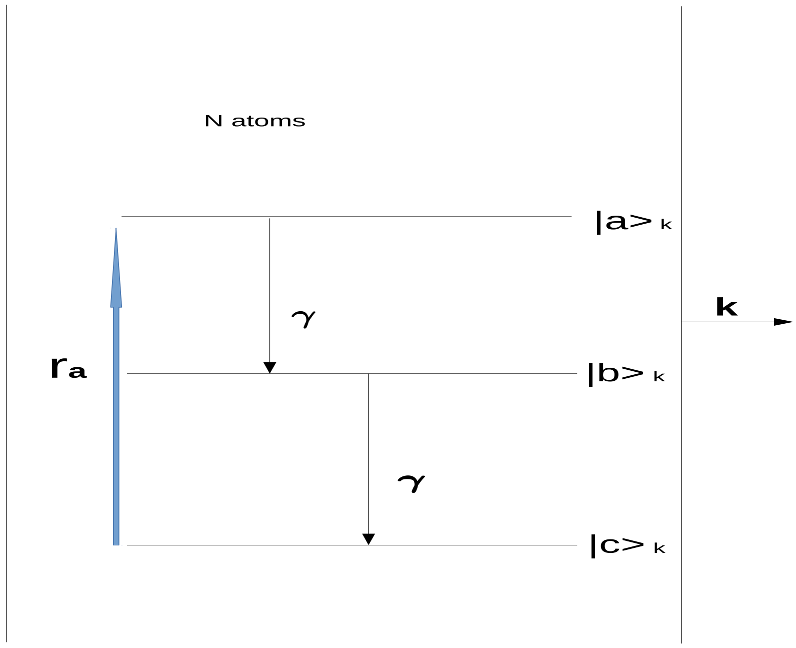

Figure 2.1: Schematic representation of a three-level laser coupled to a two-mode vacuum reservoir.

intermediate level $|b\rangle_k$ to the bottom level $|c\rangle_k$. We prefer to call the light emitted from the top level light mode a and the one emitted from the intermediate level light mode b. We carry out our analysis with light modes a and b having the same or different frequencies. In addition, we assume that light modes a and b to be at resonance with the two transitions $|a\rangle_k \to |b\rangle_k$ and $|b\rangle_k \to |c\rangle_k$, with direct transition between $|a\rangle_k$ and $|c\rangle_k$ to be electric-dipole forbidden. The interaction of a three-level atom with cavity modes a and b can be described at resonance by the Hamiltonian

$$\hat{H} = ig\left(\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{a}^\dagger\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b} - \hat{b}^\dagger\hat{\sigma}_b^k\right), \tag{2.1}$$

where

$$\hat{\sigma}_a^k = |b\rangle_{kk}\langle a| \tag{2.2}$$

and

$$\hat{\sigma}_b^k = |c\rangle_{kk}\langle b| \tag{2.3}$$

are lowering atomic operators, $\hat{a}$ and $\hat{b}$ are the annihilation operators for the cavity modes, g is the coupling constant between the atom and the cavity modes.

The quantum Langevin equations for the operators $\hat{a}$ and $\hat{b}$ are given by [1,16]

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - i[\hat{a}, \hat{H}] + \hat{F}_a(t), \tag{2.4}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - i[\hat{b}, \hat{H}] + \hat{F}_b(t), \tag{2.5}$$

where κ is the cavity damping constant considered to be the same for the two cavity modes and $\hat{F}_a(t)$ and $\hat{F}_b(t)$ are noise operators associated with the operators $\hat{a}$ and $\hat{b}$. These noise operators have the following correlation properties, when the cavity modes is coupled to a vacuum reservoir [5,9]:

$$\langle \hat{F}_a(t) \rangle = \langle \hat{F}_b(t) \rangle = 0, \tag{2.6}$$

$$\langle \hat{F}_a^\dagger(t)\hat{F}_a(t') \rangle = \langle \hat{F}_b^\dagger(t)\hat{F}_b(t') \rangle = 0, \tag{2.7}$$

$$\langle \hat{F}_a(t)\hat{F}_a^\dagger(t') \rangle = \langle \hat{F}_b(t)\hat{F}_b^\dagger(t') \rangle = \kappa\delta(t - t'), \tag{2.8}$$

$$\langle \hat{F}_a^\dagger(t)\hat{F}_a^\dagger(t') \rangle = \langle \hat{F}_b^\dagger(t)\hat{F}_b^\dagger(t') \rangle = \langle \hat{F}_a(t)\hat{F}_a(t') \rangle = \langle \hat{F}_b(t)\hat{F}_b(t') \rangle = 0. \tag{2.9}$$

With the aid of Eqs. (2.1), (2.4), and (2.5), one can easily establish that

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - g\hat{\sigma}_a^k + \hat{F}_a(t), \tag{2.10}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - g\hat{\sigma}_b^k + \hat{F}_b(t). \tag{2.11}$$

In addition, a three-level atom in an open cavity is coupled to a two-mode vacuum reservoir. The master equation for a three-level atom interacting with a two-mode vacuum reservoir is given by [1,37]

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}, \hat{\rho}] + \frac{\gamma}{2}\left[2\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} - \hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k + 2\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k} - \hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\hat{\rho} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\right], \tag{2.12}$$

where γ, considered to be the same for levels $|a\rangle$ and $|b\rangle$, is the spontaneous emission decay constant. We can rewrite Eq. (2.12) as

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}, \hat{\rho}] + \frac{\gamma}{2}\left[2\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} - \hat{\eta}_a^k\hat{\rho} - \hat{\rho}\hat{\eta}_a^k + 2\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k} - \hat{\eta}_b^k\hat{\rho} - \hat{\rho}\hat{\eta}_b^k\right], \tag{2.13}$$

where

$$\hat{\eta}_a^k = |a\rangle_{kk}\langle a|, \tag{2.14}$$

$$\hat{\eta}_b^k = |b\rangle_{kk}\langle b|. \tag{2.15}$$

Using Eq. (2.1), we can put Eq. (2.13) in the form

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= g\left[\hat{\sigma}_a^{\dagger k}\hat{a}\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{a} + \hat{\sigma}_b^{\dagger k}\hat{b}\hat{\rho} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{b} - \hat{a}^\dagger\hat{\sigma}_a^k\hat{\rho} - \hat{b}^\dagger\hat{\sigma}_b^k\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^k + \hat{\rho}\hat{b}^\dagger\hat{\sigma}_b^k\right] \\ &+ \frac{\gamma}{2}\left[2\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} - \hat{\eta}_a^k\hat{\rho} - \hat{\rho}\hat{\eta}_a^k + 2\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k} - \hat{\eta}_b^k\hat{\rho} - \hat{\rho}\hat{\eta}_b^k\right].\end{aligned} \tag{2.16}$$

This is the master equation for a three-level atom in an open cavity and coupled to a two-mode vacuum reservoir.

2.2 Equations of evolution of the atomic operators

Applying the master equation and the large-time approximation scheme, we seek to derive the equations of evolution of the expectation values of the atomic operators. To this end, employing the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr(\frac{d\rho}{dt}\hat{A}) \tag{2.17}$$

along with the master equation given by Eq. (2.16), we can easily establish that

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = -\gamma\langle\hat{\sigma}_a^k\rangle + g\left[\langle\hat{\eta}_b^k\hat{a}\rangle - \langle\hat{\eta}_a^k\hat{a}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_c^k\rangle\right], \tag{2.18}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle = -\frac{\gamma}{2}\langle\hat{\sigma}_b^k\rangle + g\left[\langle\hat{\eta}_c^k\hat{b}\rangle - \langle\hat{\eta}_b^k\hat{b}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_c^k\rangle\right], \tag{2.19}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^k\rangle = -\frac{\gamma}{2}\langle\hat{\sigma}_c^k\rangle + g\left[\langle\hat{\sigma}_b^k\hat{a}\rangle - \langle\hat{\sigma}_a^k\hat{b}\rangle\right], \tag{2.20}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -\gamma\langle\hat{\eta}_a^k\rangle + g\left[\langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle\right], \tag{2.21}$$

$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = \gamma\left[\langle\hat{\eta}_a^k\rangle - \langle\hat{\eta}_b^k\rangle\right] + g\left[\langle\hat{b}^\dagger\hat{\sigma}_b^k\rangle + \langle\hat{\sigma}_b^{\dagger k}\hat{b}\rangle - \langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle\right], \tag{2.22}$$

$$\frac{d}{dt}\langle\hat{\eta}_c^k\rangle = \gamma\langle\hat{\eta}_b^k\rangle - g\left[\langle\hat{b}^\dagger\hat{\sigma}_b^k\rangle + \langle\hat{\sigma}_b^{\dagger k}\hat{b}\rangle\right], \tag{2.23}$$

where

$$\hat{\sigma}_c^k = |c\rangle_{kk}\langle a| \tag{2.24}$$

and

$$\hat{\eta}_c^k = |c\rangle_{kk}\langle c|. \tag{2.25}$$

We see that Eqs. (2.18)-(2.23) are nonlinear differential equations and hence it is not possible to find exact time-dependent solutions of these equations. We intend to over come this problem by applying the large-time approximation [1,16]. Then using this approximation scheme, we get from Eqs. (2.10) and (2.11) the approximately valid relations

$$\hat{a} = -\frac{2g}{\kappa}\hat{\sigma}_a^k + \frac{2}{\kappa}\hat{F}_a(t), \tag{2.26}$$

$$\hat{b} = -\frac{2g}{\kappa}\hat{\sigma}_b^k + \frac{2}{\kappa}\hat{F}_b(t). \tag{2.27}$$

Evidently, these would turn out to be exact relations at steady state. Now on substituting Eqs. (2.26) and (2.27) into Eqs. (2.18)-(2.23), the equations of evolution of the atomic operators take the form

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = -\left[\gamma + \gamma_c\right]\langle\hat{\sigma}_a^k\rangle + \frac{2g}{\kappa}\left[\langle\hat{\eta}_b^k\hat{F}_a(t)\rangle - \langle\hat{\eta}_a^k\hat{F}_a(t)\rangle + \langle\hat{F}_b^\dagger(t)\hat{\sigma}_c^k\rangle\right], \tag{2.28}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle = -\left[\frac{\gamma}{2} + \frac{\gamma_c}{2}\right]\langle\hat{\sigma}_b^k\rangle + \frac{2g}{\kappa}\left[\langle\hat{\eta}_c^k\hat{F}_b(t)\rangle - \langle\hat{\eta}_b^k\hat{F}_b(t)\rangle - \langle\hat{F}_a^\dagger(t)\hat{\sigma}_c^k\rangle\right], \tag{2.29}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^k\rangle = -\left[\frac{\gamma}{2} + \frac{\gamma_c}{2}\right]\langle\hat{\sigma}_c^k\rangle + \frac{2g}{\kappa}\left[\langle\hat{\sigma}_b^k\hat{F}_a(t)\rangle - \langle\hat{\sigma}_a^k\hat{F}_b(t)\rangle\right], \tag{2.30}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -\left[\gamma + \gamma_c\right]\langle\hat{\eta}_a^k\rangle + \frac{2g}{\kappa}\left[\langle\hat{\sigma}_a^{\dagger k}\hat{F}_a(t)\rangle + \langle\hat{F}_a^\dagger(t)\hat{\sigma}_a^k\rangle\right], \tag{2.31}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{\eta}_b^k\rangle &= -\left[\gamma + \gamma_c\right]\langle\hat{\eta}_b^k\rangle + \left[\gamma + \gamma_c\right]\langle\hat{\eta}_a^k\rangle \\ &\quad + \frac{2g}{\kappa}\left[\langle\hat{F}_b^\dagger(t)\hat{\sigma}_b^k\rangle + \langle\hat{\sigma}_b^{\dagger k}\hat{F}_b(t)\rangle - \langle\hat{\sigma}_a^{\dagger k}\hat{F}_a(t)\rangle - \langle\hat{F}_a^\dagger(t)\hat{\sigma}_a^k\rangle\right],\end{aligned} \tag{2.32}$$

$$\frac{d}{dt}\langle\hat{\eta}_c^k\rangle = [\gamma + \gamma_c]\langle\hat{\eta}_b^k\rangle - \frac{2g}{\kappa}\left[\langle\hat{F}_b^\dagger(t)\hat{\sigma}_b^k\rangle + \langle\hat{\sigma}_b^{\dagger k}\hat{F}_b(t)\rangle\right], \tag{2.33}$$

where

$$\gamma_c = \frac{4g^2}{\kappa} \tag{2.34}$$

is the stimulated emission decay constant.

We next proceed to find the expectation value of the products involving a noise operator and an atomic operators that appear in Eq. (2.28). To this end, after removing the angular brackets, Eq. (2.31) can be rewritten as

$$\frac{d}{dt}\hat{\eta}_a^k = -[\gamma + \gamma_c]\hat{\eta}_a^k + \frac{2g}{\kappa}\left[\hat{\sigma}_a^{\dagger k}\hat{F}_a(t) + \hat{F}_a^\dagger(t)\hat{\sigma}_a^k\right] + \hat{f}_a(t), \tag{2.35}$$

where $\hat{f}_a(t)$ is the noise operator with vanishing mean. A formal solution of this equation can be written as

$$\hat{\eta}_a^k(t) = \hat{\eta}_a^k(0)e^{-(\gamma+\gamma_c)t} + \int_0^t e^{-(\gamma+\gamma_c)(t-t')}\left[\frac{2g}{\kappa}\left[\hat{\sigma}_a^{\dagger k}(t')\hat{F}_a(t') + \hat{F}_a^\dagger(t')\hat{\sigma}_a^k(t')\right] + \hat{f}_a(t')\right]dt'. \tag{2.36}$$

Multiplying Eq. (2.36) on the right by $\hat{F}_a(t)$ and taking the expectation value of the resulting equation, we have

$$\begin{aligned}\langle\hat{\eta}_a^k(t)\hat{F}_a(t)\rangle &= \langle\hat{\eta}_a^k(0)\hat{F}_a(t)\rangle e^{-(\gamma+\gamma_c)t} + \int_0^t e^{-(\gamma+\gamma_c)(t-t')}\\ &\left[\frac{2g}{\kappa}\left[\langle\hat{\sigma}_a^{\dagger k}(t')\hat{F}_a(t')\hat{F}_a(t)\rangle + \langle\hat{F}_a^\dagger(t')\hat{\sigma}_a^k(t')\hat{F}_a(t)\rangle\right] + \langle\hat{f}_a(t')\hat{F}_a(t)\rangle\right]dt'.\end{aligned} \tag{2.37}$$

Ignoring the noncommutativity of the atomic and noise operators as well as neglecting the correlation between $\hat{F}_a(t)$ and $\hat{\sigma}_a^k(t')$, assumed to be considerably small, one can write the approximately valid relations [10]

$$\langle\hat{\sigma}_a^{\dagger k}(t')\hat{F}_a(t')\hat{F}_a(t)\rangle = \langle\hat{\sigma}_a^{\dagger k}(t')\rangle\langle\hat{F}_a(t')\hat{F}_a(t)\rangle = 0, \tag{2.38}$$

$$\langle\hat{F}_a^\dagger(t')\hat{\sigma}_a^k(t')\hat{F}_a(t)\rangle = \langle\hat{\sigma}_a^k(t')\rangle\langle\hat{F}_a^\dagger(t')\hat{F}_a(t)\rangle = 0, \tag{2.39}$$

$$\langle\hat{f}_a(t')\hat{F}_a(t)\rangle = \langle\hat{f}_a(t')\rangle\langle\hat{F}_a(t)\rangle = 0. \tag{2.40}$$

Now on account of these approximately valid relations and in view of the fact that a noise operator $\hat{F}$ at a certain time should not affect an atomic variable at earlier time, Eq. (2.37) takes the form

$$\langle \hat{\eta}_a^k(t)\hat{F}_a(t)\rangle = 0. \tag{2.41}$$

Following a similar procedure, one can also check that

$$\langle \hat{\eta}_b^k(t)\hat{F}_a(t)\rangle = 0, \tag{2.42}$$

$$\langle \hat{\eta}_c^k(t)\hat{F}_b(t)\rangle = 0, \tag{2.43}$$

$$\langle \hat{\eta}_b^k(t)\hat{F}_b(t)\rangle = 0. \tag{2.44}$$

We also take

$$\langle \hat{F}_a^\dagger(t)\hat{\sigma}_c^k(t)\rangle = \langle \hat{F}_b^\dagger(t)\hat{\sigma}_c^k(t)\rangle = 0. \tag{2.45}$$

With the aid of Eqs. (2.41)-(2.45), we can rewrite Eqs. (2.28) and (2.29) as

$$\frac{d}{dt}\langle \hat{\sigma}_a^k\rangle = -\left[\gamma + \gamma_c\right]\langle \hat{\sigma}_a^k\rangle, \tag{2.46}$$

$$\frac{d}{dt}\langle \hat{\sigma}_b^k\rangle = -\left[\frac{\gamma}{2} + \frac{\gamma_c}{2}\right]\langle \hat{\sigma}_b^k\rangle. \tag{2.47}$$

We next proceed to find the expectation value of the products involving a noise operator and an atomic operator that appear in Eq. (2.31). To this end, after removing the angular brackets, Eq. (2.28) can be rewritten as

$$\frac{d}{dt}\hat{\sigma}_a^k = -\left[\gamma + \gamma_c\right]\hat{\sigma}_a^k + \frac{2g}{\kappa}\left[\hat{\eta}_b^k\hat{F}_a(t) - \hat{\eta}_a^k\hat{F}_a(t) + \hat{F}_b^\dagger(t)\hat{\sigma}_c^k\right] + \hat{g}_a(t), \tag{2.48}$$

where $\hat{g}_a(t)$ is the noise operator with vanishing mean. A formal solution of this equation can be written as

$$\begin{aligned}\hat{\sigma}_a^k(t) = \hat{\sigma}_a^k(0)e^{-(\gamma+\gamma_c)t} + \int_0^t e^{-(\gamma+\gamma_c)(t-t')}\\ \left[\frac{2g}{\kappa}\left[\hat{\eta}_b^k(t')\hat{F}_a(t') - \hat{\eta}_a^k(t')\hat{F}_a(t') + \hat{F}_b^\dagger(t')\hat{\sigma}_c^k(t')\right] + \hat{g}_a(t')\right] dt'.\end{aligned} \tag{2.49}$$

Multiplying this equation from the left by $\hat{F}_a^\dagger(t)$ and taking the expectation value of the resulting equation, we have

$$\langle \hat{F}_a^\dagger(t)\hat{\sigma}_a^k(t)\rangle = \langle \hat{F}_a^\dagger(t)\hat{\sigma}_a^k(0)\rangle e^{-(\gamma+\gamma_c)t} + \int_0^t e^{-(\gamma+\gamma_c)(t-t')} \left[\frac{2g}{\kappa}\left[\langle \hat{F}_a^\dagger(t)(\hat{\eta}_b^k(t') - \hat{\eta}_a^k(t'))\hat{F}_a(t')\rangle + \langle \hat{F}_a^\dagger(t)\hat{F}_b^\dagger(t')\hat{\sigma}_c^k(t')\rangle\right] + \langle \hat{F}_a^\dagger(t)\hat{g}_a(t')\rangle\right] dt'. \tag{2.50}$$

Applying the approximately valid relations

$$\langle \hat{F}_a^\dagger(t)\big(\hat{\eta}_b^k(t') - \hat{\eta}_a^k(t')\big)\hat{F}_a(t')\rangle = \langle \hat{\eta}_b^k(t') - \hat{\eta}_a^k(t')\rangle\langle \hat{F}_a^\dagger(t)\hat{F}_a(t')\rangle = 0, \tag{2.51}$$

$$\langle \hat{F}_a^\dagger(t)\hat{F}_b^\dagger(t')\hat{\sigma}_c^k(t')\rangle = \langle \hat{\sigma}_c^k(t')\rangle\langle \hat{F}_a^\dagger(t)\hat{F}_b^\dagger(t')\rangle = 0, \tag{2.52}$$

$$\langle \hat{F}_a^\dagger(t)\rangle\langle \hat{g}_a(t')\rangle = 0 \tag{2.53}$$

and taking into account the fact that a noise operator $\hat{F}$ at certain time should not affect the atomic variable at earlier time, Eq. (2.50) can be written as

$$\langle \hat{F}_a^\dagger(t)\hat{\sigma}_a^k(t)\rangle = 0. \tag{2.54}$$

It can also be established in a similar manner that

$$\langle \hat{F}_b^\dagger(t)\hat{\sigma}_b^k(t)\rangle = 0. \tag{2.55}$$

Using Eqs. (2.54), (2.55), and their complex conjugates, Eqs. (2.31), (2.32), and (2.33) can be rewritten as

$$\frac{d}{dt}\langle \hat{\eta}_a^k\rangle = -\left[\gamma + \gamma_c\right]\langle \hat{\eta}_a^k\rangle, \tag{2.56}$$

$$\frac{d}{dt}\langle \hat{\eta}_b^k\rangle = -\left[\gamma + \gamma_c\right]\langle \hat{\eta}_b^k\rangle + \left[\gamma + \gamma_c\right]\langle \hat{\eta}_a^k\rangle, \tag{2.57}$$

$$\frac{d}{dt}\langle \hat{\eta}_c^k\rangle = \left[\gamma + \gamma_c\right]\langle \hat{\eta}_b^k\rangle. \tag{2.58}$$

We note that Eqs. (2.46), (2.47), (2.56), (2.57), and (2.58) represent the equations of evolution for the atomic operators in the absence of the pumping process. We thus want to include the effect of the pumping process. The pumping process must surely affect the dynamics of $\langle\hat{\eta}_a^k\rangle$ and $\langle\hat{\eta}_c^k\rangle$. We seek here to pump the atoms by electron bombardment. If r_a represents the rate at which a single atom is pumped from the bottom to the top level, then $\langle\hat{\eta}_a^k\rangle$ increases at the rate of $r_a\langle\hat{\eta}_c^k\rangle$ and $\langle\hat{\eta}_c^k\rangle$ decreases by the same rate. In view of this, we can rewrite Eqs. (2.56) and (2.58) as [1]

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -\left[\gamma+\gamma_c\right]\langle\hat{\eta}_a^k\rangle + r_a\langle\hat{\eta}_c^k\rangle, \tag{2.59}$$

$$\frac{d}{dt}\langle\hat{\eta}_c^k\rangle = \left[\gamma+\gamma_c\right]\langle\hat{\eta}_b^k\rangle - r_a\langle\hat{\eta}_c^k\rangle. \tag{2.60}$$

We next sum Eqs. (2.46), (2.47), (2.57), (2.59), and (2.60) over the N three-level atoms, so that

$$\frac{d}{dt}\langle\hat{m}_a\rangle = -\left[\gamma+\gamma_c\right]\langle\hat{m}_a\rangle, \tag{2.61}$$

$$\frac{d}{dt}\langle\hat{m}_b\rangle = -\frac{1}{2}\left[\gamma+\gamma_c\right]\langle\hat{m}_b\rangle, \tag{2.62}$$

$$\frac{d}{dt}\langle\hat{N}_a\rangle = -\left[\gamma+\gamma_c\right]\langle\hat{N}_a\rangle + r_a\langle\hat{N}_c\rangle, \tag{2.63}$$

$$\frac{d}{dt}\langle\hat{N}_b\rangle = -\left[\gamma+\gamma_c\right]\langle\hat{N}_b\rangle + \left[\gamma+\gamma_c\right]\langle\hat{N}_a\rangle, \tag{2.64}$$

$$\frac{d}{dt}\langle\hat{N}_c\rangle = \left[\gamma+\gamma_c\right]\langle\hat{N}_b\rangle - r_a\langle\hat{N}_c\rangle, \tag{2.65}$$

in which

$$\hat{m}_a = \sum_{k=1}^{N}\hat{\sigma}_a^k, \tag{2.66}$$

$$\hat{m}_b = \sum_{k=1}^{N}\hat{\sigma}_b^k, \tag{2.67}$$

$$\hat{N}_a = \sum_{k=1}^{N}\hat{\eta}_a^k, \tag{2.68}$$

$$\hat{N}_b = \sum_{k=1}^{N}\hat{\eta}_b^k, \tag{2.69}$$

$$\hat{N}_c = \sum_{k=1}^{N}\hat{\eta}_c^k, \tag{2.70}$$

with the operators $\hat{N}_a$, $\hat{N}_b$, and $\hat{N}_c$ representing the number of atoms in the top, intermediate, and bottom levels. In addition, employing the completeness relation

$$\hat{\eta}_a^k + \hat{\eta}_b^k + \hat{\eta}_c^k = \hat{I}, \tag{2.71}$$

we easily arrive at

$$\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle = N. \tag{2.72}$$

Furthermore, applying the definition given by Eq. (2.2) and setting for any k

$$\hat{\sigma}_a^k = |b\rangle\langle a|, \tag{2.73}$$

we have

$$\hat{m}_a = N|b\rangle\langle a|. \tag{2.74}$$

Following the same procedure, one can also check that

$$\hat{m}_b = N|c\rangle\langle b|, \tag{2.75}$$

$$\hat{m}_c = N|c\rangle\langle a|, \tag{2.76}$$

$$\hat{N}_a = N|a\rangle\langle a|, \tag{2.77}$$

$$\hat{N}_b = N|b\rangle\langle b|, \tag{2.78}$$

$$\hat{N}_c = N|c\rangle\langle c|, \tag{2.79}$$

where

$$\hat{m}_c = \sum_{k=1}^{N} \hat{\sigma}_c^k. \tag{2.80}$$

Moreover, using the definition

$$\hat{m} = \hat{m}_a + \hat{m}_b \tag{2.81}$$

and taking into account Eqs. (2.74)-(2.79), it can be readily established that

$$\hat{m}^\dagger \hat{m} = N(\hat{N}_a + \hat{N}_b), \tag{2.82}$$

$$\hat{m}\hat{m}^{\dagger} = N(\hat{N}_b + \hat{N}_c), \tag{2.83}$$

$$\hat{m}^2 = N\hat{m}_c. \tag{2.84}$$

In the presence of N three-level atoms, we can rewrite Eq. (2.10) as [1]

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \lambda\hat{m}_a + \beta\hat{F}_a(t), \tag{2.85}$$

in which λ and β are constants whose values remain to be fixed. Using Eqs. (2.26) and (2.54), we get

$$[\hat{a}, \hat{a}^{\dagger}]_k = \frac{4g^2}{\kappa^2}(\hat{\eta}_b^k - \hat{\eta}_a^k) + \frac{4}{\kappa^2}[F_a, F_a^{\dagger}] \tag{2.86}$$

and on summing over all atoms, we have

$$[\hat{a}, \hat{a}^{\dagger}] = \frac{4g^2}{\kappa^2}(\hat{N}_b - \hat{N}_a) + \frac{4N}{\kappa^2}[F_a, F_a^{\dagger}], \tag{2.87}$$

where

$$[\hat{a}, \hat{a}^{\dagger}] = \sum_{k=1}^{N}[\hat{a}, \hat{a}^{\dagger}]_k \tag{2.88}$$

stands for the commutator of $\hat{a}$ and $\hat{a}^{\dagger}$ when light mode a is interacting with all the N three-level atoms. On the other hand, applying the large-time approximation to Eq. (2.85), we get

$$\hat{a} = \frac{2\lambda}{\kappa}\hat{m}_a + \frac{2\beta}{\kappa}F_a(t). \tag{2.89}$$

In view of this result, one can easily verify that

$$[\hat{a}, \hat{a}^{\dagger}] = N\frac{4\lambda^2}{\kappa^2}(\hat{N}_b - \hat{N}_a) + \frac{4\beta^2}{\kappa^2}[F_a, F_a^{\dagger}]. \tag{2.90}$$

Thus on account of Eqs. (2.87) and (2.90), we see that

$$\lambda = \pm\frac{g}{\sqrt{N}}, \tag{2.91}$$

$$\beta = \sqrt{N}. \tag{2.92}$$

In view of Eqs. (2.91) and (2.92), Eq. (2.85) can be written as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \frac{g}{\sqrt{N}}\hat{m}_a + \sqrt{N}\hat{F}_a(t). \tag{2.93}$$

Following a similar procedure, one can also readily establish that

$$[\hat{b}, \hat{b}^\dagger] = \frac{4g^2}{\kappa^2}(\hat{N}_c - \hat{N}_b) + \frac{4N}{\kappa^2}[F_b, F_b^\dagger], \tag{2.94}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} + \frac{g}{\sqrt{N}}\hat{m}_b + \sqrt{N}\hat{F}_b(t). \tag{2.95}$$

Furthermore, in order to include the effect of the pumping process, we rewrite Eqs. (2.61) and (2.62) as [1,16]

$$\frac{d}{dt}\hat{m}_a = -\frac{\mu}{2}\hat{m}_a + \hat{G}_a(t), \tag{2.96}$$

$$\frac{d}{dt}\hat{m}_b = -\frac{\mu}{2}\hat{m}_b + \hat{G}_b(t), \tag{2.97}$$

in which $\hat{G}_a(t)$ and $\hat{G}_b(t)$ are noise operators with vanishing mean and μ is a parameter whose value remains to be determined. Employing the relation

$$\frac{d}{dt}\left\langle \hat{m}_a^\dagger \hat{m}_a \right\rangle = \left\langle \frac{d\hat{m}_a^\dagger}{dt}\hat{m}_a \right\rangle + \left\langle \hat{m}_a^\dagger \frac{d\hat{m}_a}{dt} \right\rangle \tag{2.98}$$

along with Eq. (2.96), we easily find

$$\frac{d}{dt}\langle \hat{m}_a^\dagger \hat{m}_a\rangle = -\mu\langle \hat{m}_a^\dagger \hat{m}_a\rangle + \langle \hat{m}_a^\dagger \hat{G}_a(t)\rangle + \langle \hat{G}_a^\dagger(t)\hat{m}_a\rangle, \tag{2.99}$$

from which follows

$$\frac{d}{dt}\langle \hat{N}_a\rangle = -\mu\langle \hat{N}_a\rangle + \frac{1}{N}\left[\langle \hat{m}_a^\dagger \hat{G}_a(t)\rangle + \langle \hat{G}_a^\dagger(t)\hat{m}_a\rangle\right]. \tag{2.100}$$

Applying the large-time approximation scheme to Eq. (2.64), we get

$$\langle \hat{N}_b\rangle = \langle \hat{N}_a\rangle. \tag{2.101}$$

Therefore, with the aid of Eqs. (2.72) and Eq. (2.101), one can rewrite Eq. (2.63) as

$$\frac{d}{dt}\langle \hat{N}_a\rangle = -\left[\gamma + \gamma_c + 2r_a\right]\langle \hat{N}_a\rangle + Nr_a. \tag{2.102}$$

Hence comparison of Eqs. (2.100) and (2.102) shows that

$$\mu = \gamma + \gamma_c + 2r_a \tag{2.103}$$

and

$$\left\langle \hat{m}_a^\dagger \hat{G}_a(t) \right\rangle + \left\langle \hat{G}_a^\dagger(t) \hat{m}_a \right\rangle = r_a N^2. \tag{2.104}$$

We observe that Eq. (2.104) is equivalent to

$$\left\langle \hat{G}_a^\dagger(t) \hat{G}_a(t') \right\rangle = r_a N^2 \delta(t - t'). \tag{2.105}$$

One can also easily verify that

$$\left\langle \hat{G}_a(t) \hat{G}_a^\dagger(t') \right\rangle = (\gamma + \gamma_c) N^2 \delta(t - t'). \tag{2.106}$$

Furthermore, adding Eqs. (2.61) and (2.62), we have

$$\frac{d}{dt}\langle \hat{m} \rangle = -\frac{1}{2}\left[\gamma + \gamma_c\right]\langle \hat{m} \rangle - \frac{1}{2}\left[\gamma + \gamma_c\right]\langle \hat{m}_a \rangle, \tag{2.107}$$

where $\hat{m}$ is given by Eq. (2.81). Upon casting Eq. (2.107) into the form

$$\frac{d}{dt}\hat{m} = -\frac{\mu}{2}\hat{m} - \frac{\mu}{2}\hat{m}_a + \hat{G}(t), \tag{2.108}$$

where $\hat{G}(t)$ is the noise operator with vanishing mean, one can also easily verify that μ has the value given by Eq. (2.103) and

$$\left\langle \hat{G}^\dagger(t) \hat{G}(t') \right\rangle = r_a N^2 \delta(t - t'), \tag{2.109}$$

$$\left\langle \hat{G}(t) \hat{G}^\dagger(t') \right\rangle = (\gamma + \gamma_c) N^2 \delta(t - t'). \tag{2.110}$$

Finally, we seek to determine the steady-state solutions of the expectation values of the atomic operators. To this end, the steady-state solution of Eq. (2.102) is expressible as

$$\langle \hat{N}_a \rangle = \frac{r_a N}{\gamma + \gamma_c + 2r_a}. \tag{2.111}$$

In view of Eq. (2.101), we have

$$\langle \hat{N}_b \rangle = \frac{r_a N}{\gamma + \gamma_c + 2r_a}. \tag{2.112}$$

Using the steady-state solution of Eq. (2.65) and taking into account Eq. (2.101), we get

$$\langle \hat{N}_c \rangle = \frac{\gamma + \gamma_c}{r_a} \langle \hat{N}_a \rangle. \tag{2.113}$$

Now on account of Eq. (2.111), Eq. (2.113) takes the form

$$\langle \hat{N}_c \rangle = \frac{(\gamma + \gamma_c) N}{\gamma + \gamma_c + 2r_a}. \tag{2.114}$$

We note that Eqs. (2.111),(2.112), and (2.114) represent the steady-state solutions of the equations of evolution of the atomic operators. Furthermore, upon setting $\gamma = 0$, for the case in which spontaneous emission is absent, the steady-state solutions described by Eqs. (2.111),(2.112), and (2.114) take the form

$$\langle \hat{N}_a \rangle = \frac{r_a N}{\gamma_c + 2r_a}, \tag{2.115}$$

$$\langle \hat{N}_b \rangle = \frac{r_a N}{\gamma_c + 2r_a}, \tag{2.116}$$

$$\langle \hat{N}_c \rangle = \frac{\gamma_c N}{\gamma_c + 2r_a}. \tag{2.117}$$

These represent the steady-state solutions of the equations of evolution of the expectation values of the atomic operators when the three-level laser is not interact with the two-mode vacuum reservoir. The results described by Eqs. (2.115)-(2.117) are exactly the same as the one obtained by Fesseha [1,16]. In addition, we note that for $r_a \gg \gamma + \gamma_c$, Eqs. (2.111), (2.112), and (2.114) reduce to

$$\langle \hat{N}_a \rangle = \frac{N}{2}, \tag{2.118}$$

$$\langle \hat{N}_b \rangle = \frac{N}{2}, \tag{2.119}$$

$$\langle \hat{N}_c \rangle = 0. \tag{2.120}$$

Again, for $r_a = \gamma + \gamma_c$, Eqs. (2.111), (2.112), and (2.114) become

$$\langle \hat{N}_a \rangle = \frac{N}{3}, \tag{2.121}$$

$$\langle \hat{N}_b \rangle = \frac{N}{3}, \tag{2.122}$$

$$\langle \hat{N}_c \rangle = \frac{N}{3}. \tag{2.123}$$

Finally, for $r_a = 0$, the steady-state solutions described by Eqs. (2.111), (2.112), and (2.114) take the form

$$\langle \hat{N}_a \rangle = 0, \tag{2.124}$$

$$\langle \hat{N}_b \rangle = 0, \tag{2.125}$$

$$\langle \hat{N}_c \rangle = N. \tag{2.126}$$

3

Photon statistics

In this chapter we wish to study the statistical properties of the cavity light modes produced by the three-level laser under consideration. Applying the solutions of the equations of evolution of the expectation values for the atomic operators and the quantum Langevin equations for the cavity mode operators, we obtain the global photon statistics for light modes a and b. In addition, we determine the global and local photon statistics of the two-mode cavity light.

3.1 Single-mode photon statistics

In this section we seek to obtain the global mean and variance of the photon numbers for light modes a and b.

3.1.1 Global mean photon number

Here we seek to calculate the mean photon number for light modes a and b in the entire frequency interval. To this end, using the relation

$$\frac{d}{dt}\left\langle \hat{a}^{\dagger}(t)\hat{a}(t)\right\rangle = \left\langle \frac{d\hat{a}^{\dagger}(t)}{dt}\hat{a}(t)\right\rangle + \left\langle \hat{a}^{\dagger}(t)\frac{d\hat{a}(t)}{dt}\right\rangle \tag{3.1}$$

along with Eq. (2.93), we readily find

$$\frac{d}{dt}\langle\hat{a}^\dagger t)\hat{a}(t)\rangle = -\kappa\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle + \frac{g}{\sqrt{N}}\left[\langle\hat{a}^\dagger(t)\hat{m}_a(t)\rangle + \langle\hat{m}_a^\dagger(t)\hat{a}(t)\rangle\right]$$
$$+\sqrt{N}\left[\langle\hat{F}_a^\dagger(t)\hat{a}(t)\rangle + \langle\hat{a}^\dagger(t)\hat{F}_a(t)\rangle\right]. \tag{3.2}$$

Next we seek to evaluate $\langle\hat{a}^\dagger(t)\hat{m}_a(t)\rangle$. To this end, applying the large-time approximation scheme to Eq. (2.93), we get

$$\hat{a} = \frac{2g}{\sqrt{N}\kappa}\hat{m}_a + \frac{2\sqrt{N}}{\kappa}\hat{F}_a(t). \tag{3.3}$$

Multiplying the adjoint of Eq. (3.3) on the right by $\hat{m}_a(t)$ and taking the expectation value of the resulting expression, we have

$$\langle\hat{a}^\dagger(t)\hat{m}_a(t)\rangle = \frac{2g\sqrt{N}}{\kappa}\langle\hat{N}_a(t)\rangle + \frac{2\sqrt{N}}{\kappa}\langle\hat{F}_a^\dagger(t)\hat{m}_a(t)\rangle. \tag{3.4}$$

We now proceed to find the expectation value of the product involving a noise operator and an atomic operator that appears in Eq. (3.4). To this end, a formal solution of Eq. (2.96) can be written as

$$\hat{m}_a(t) = \hat{m}_a(0)e^{-\frac{\mu}{2}t} + \int_0^t e^{-\frac{\mu}{2}(t-t')}\hat{G}_a(t')dt'. \tag{3.5}$$

Multiplying Eq. (3.5) on the left by $\hat{F}_a^\dagger(t)$ and taking the expectation value of the resulting expression, we have

$$\langle\hat{F}_a^\dagger(t)\hat{m}_a(t)\rangle = \langle\hat{F}_a^\dagger(t)\hat{m}_a(0)\rangle e^{-\frac{\mu}{2}t} + \int_0^t e^{-\frac{\mu}{2}(t-t')}\langle\hat{F}_a^\dagger(t)\hat{G}_a(t')\rangle dt'. \tag{3.6}$$

Taking into account the fact that a noise operator $\hat{F}$ at a certain time should not affect an atomic variable at earlier time, Eq. (3.6) can be put in the form

$$\langle\hat{F}_a^\dagger(t)\hat{m}_a(t)\rangle = \int_0^t e^{-\frac{\mu}{2}(t-t')}\langle\hat{F}_a^\dagger(t)\hat{G}_a(t')\rangle dt'. \tag{3.7}$$

Assuming the atomic and cavity mode noise operators are independent, one can write

$$\langle\hat{F}_a^\dagger(t)\hat{G}_a(t')\rangle = \langle\hat{F}_a^\dagger(t)\rangle\langle\hat{G}_a(t')\rangle = 0. \tag{3.8}$$

On account of Eq. (3.8), Eq. (3.7) takes the form

$$\langle \hat{F}_a^\dagger(t)\hat{m}_a(t)\rangle = 0. \tag{3.9}$$

In view of this result, Eq. (3.4) becomes

$$\langle \hat{a}^\dagger(t)\hat{m}_a(t)\rangle = \frac{2g\sqrt{N}}{\kappa}\langle \hat{N}_a(t)\rangle. \tag{3.10}$$

We next seek to evaluate $\langle \hat{F}_a^\dagger(t)\hat{a}(t)\rangle$. To this end, a formal solution of Eq. (2.93) can be written as

$$\hat{a}(t) = \hat{a}(0)e^{-\frac{\kappa}{2}t} + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\left[\frac{g}{\sqrt{N}}\hat{m}_a(t') + \sqrt{N}\hat{F}_a(t')\right]dt'. \tag{3.11}$$

Multiplying Eq. (3.11) on the left by $\hat{F}_a^\dagger(t)$ and taking the expectation value of the resulting expression, we get

$$\langle \hat{F}_a^\dagger(t)\hat{a}(t)\rangle = \langle \hat{F}_a^\dagger(t)\hat{a}(0)\rangle e^{-\frac{\kappa}{2}t} + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\left[\frac{g}{\sqrt{N}}\langle \hat{F}_a^\dagger(t)\hat{m}_a(t')\rangle + \sqrt{N}\langle \hat{F}_a^\dagger(t)\hat{F}_a(t')\rangle\right]dt'. \tag{3.12}$$

In view of Eqs. (2.7) and (3.9) and the fact that a noise operator $\hat{F}$ at a certain time should not affect the cavity mode operator at earlier time, Eq. (3.12) reduces to

$$\langle \hat{F}_a^\dagger(t)\hat{a}(t)\rangle = 0. \tag{3.13}$$

Now on account of Eqs. (3.10) and (3.13) along with their complex conjugates, Eq. (3.2) becomes

$$\frac{d}{dt}\left\langle \hat{a}^\dagger(t)\hat{a}(t)\right\rangle = -\kappa\left\langle \hat{a}^\dagger(t)\hat{a}(t)\right\rangle + \frac{4g^2}{\kappa}\langle \hat{N}_a(t)\rangle. \tag{3.14}$$

The steady-state solution of this equation has the form

$$\langle \hat{a}^\dagger\hat{a}\rangle = \frac{\gamma_c}{\kappa}\langle \hat{N}_a\rangle. \tag{3.15}$$

Furthermore, employing the relation

$$\frac{d}{dt}\left\langle \hat{a}(t)\hat{a}^\dagger(t)\right\rangle = \left\langle \frac{d\hat{a}(t)}{dt}\hat{a}^\dagger(t)\right\rangle + \left\langle \hat{a}(t)\frac{d\hat{a}^\dagger(t)}{dt}\right\rangle \tag{3.16}$$

along with Eq. (2.93), we readily find

$$\frac{d}{dt}\langle \hat{a}(t)\hat{a}^{\dagger}(t)\rangle = -\kappa\langle \hat{a}(t)\hat{a}^{\dagger}(t)\rangle + \frac{g}{\sqrt{N}}\left[\langle \hat{a}(t)\hat{m}_a^{\dagger}(t)\rangle + \langle \hat{m}_a(t)\hat{a}^{\dagger}(t)\rangle\right] + \sqrt{N}\left[\langle \hat{F}_a(t)\hat{a}^{\dagger}(t)\rangle + \langle \hat{a}(t)\hat{F}_a^{\dagger}(t)\rangle\right]. \tag{3.17}$$

Next we seek to evaluate $\langle \hat{m}_a(t)\hat{a}^{\dagger}(t)\rangle$. Multiplying the adjoint of Eq. (3.3) on the left by $\hat{m}_a(t)$ and taking the expectation value of the resulting expression, we get

$$\langle \hat{m}_a(t)\hat{a}^{\dagger}(t)\rangle = \frac{2g\sqrt{N}}{\kappa}\langle \hat{N}_b(t)\rangle + \frac{2\sqrt{N}}{\kappa}\langle \hat{m}_a(t)\hat{F}_a^{\dagger}(t)\rangle. \tag{3.18}$$

We now proceed to find the expectation value of the product involving a noise operator and an atomic operator that appears in Eq. (3.18). Multiplying Eq. (3.5) on the right by $\hat{F}_a^{\dagger}(t)$ and taking the expectation value of the resulting expression, we have

$$\langle \hat{m}_a(t)\hat{F}_a^{\dagger}(t)\rangle = \langle \hat{m}_a(0)\hat{F}_a^{\dagger}(t)\rangle e^{-\frac{\mu}{2}t} + \int_0^t e^{-\frac{\mu}{2}(t-t')}\langle \hat{G}_a(t')\hat{F}_a^{\dagger}(t)\rangle dt'. \tag{3.19}$$

Using Eq. (3.8) and taking into account the fact that a noise operator $\hat{F}$ at a certain time should not affect an atomic variable at earlier time, Eq. (3.19) can be written as

$$\langle \hat{m}_a(t)\hat{F}_a^{\dagger}(t)\rangle = 0. \tag{3.20}$$

Upon substituting this result into Eq. (3.18), we obtain

$$\langle \hat{m}_a(t)\hat{a}^{\dagger}(t)\rangle = \frac{2g\sqrt{N}}{\kappa}\langle \hat{N}_b(t)\rangle. \tag{3.21}$$

We next seek to evaluate $\langle \hat{a}(t)\hat{F}_a^{\dagger}(t)\rangle$. Multiplying Eq. (3.11) on the right by $\hat{F}_a^{\dagger}(t)$ and taking the expectation value of the resulting expression, we have

$$\langle \hat{a}(t)\hat{F}_a^{\dagger}(t)\rangle = \langle \hat{a}(0)\hat{F}_a^{\dagger}(t)\rangle e^{-\frac{\kappa}{2}t} + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\left[\frac{g}{\sqrt{N}}\langle \hat{m}_a(t')\hat{F}_a^{\dagger}(t)\rangle + \sqrt{N}\langle \hat{F}_a(t')\hat{F}_a^{\dagger}(t)\rangle\right] dt'. \tag{3.22}$$

With the aid of Eqs. (2.8) and (3.20) along with the fact that a noise operator $\hat{F}$ at a certain time should not affect the cavity mode operator at earlier time, one can easily get

$$\langle \hat{a}(t)\hat{F}_a^\dagger(t)\rangle = \frac{\sqrt{N}\kappa}{2}. \tag{3.23}$$

In view of Eqs. (3.21), (3.23), and their complex conjugates, Eq. (3.17) takes the form

$$\frac{d}{dt}\left\langle \hat{a}(t)\hat{a}^\dagger(t)\right\rangle = -\kappa\left\langle \hat{a}(t)\hat{a}^\dagger(t)\right\rangle + \frac{4g^2}{\kappa}\langle \hat{N}_b(t)\rangle + N\kappa. \tag{3.24}$$

The steady-state solution of this equation can be written as

$$\langle \hat{a}\hat{a}^\dagger\rangle = \frac{\gamma_c}{\kappa}\langle \hat{N}_b\rangle + N. \tag{3.25}$$

It can also be established in a similar manner that

$$\langle \hat{b}^\dagger\hat{b}\rangle = \frac{\gamma_c}{\kappa}\langle \hat{N}_b\rangle, \tag{3.26}$$

$$\langle \hat{b}\hat{b}^\dagger\rangle = \frac{\gamma_c}{\kappa}\langle \hat{N}_c\rangle + N. \tag{3.27}$$

Now using Eqs. (3.15) and (3.25), one can easily establish that

$$[\hat{a}, \hat{a}^\dagger] = \frac{\gamma_c}{\kappa}\left[\hat{N}_b - \hat{N}_a\right] + N. \tag{3.28}$$

Similarly, using Eqs. (3.26) and (3.27), one can readily obtain

$$[\hat{b}, \hat{b}^\dagger] = \frac{\gamma_c}{\kappa}\left[\hat{N}_c - \hat{N}_b\right] + N. \tag{3.29}$$

On account of Eqs. (3.15) and (2.111), the steady-state mean photon number of light mode a has the form

$$\bar{n}_a = \frac{\gamma_c}{\kappa}\left[\frac{r_a N}{\gamma + \gamma_c + 2r_a}\right]. \tag{3.30}$$

We also find the steady-state mean photon number of light mode b to be

$$\bar{n}_b = \frac{\gamma_c}{\kappa}\left[\frac{r_a N}{\gamma + \gamma_c + 2r_a}\right]. \tag{3.31}$$

From Eqs. (3.30) and (3.31), we see that

$$\bar{n}_a = \bar{n}_b. \tag{3.32}$$

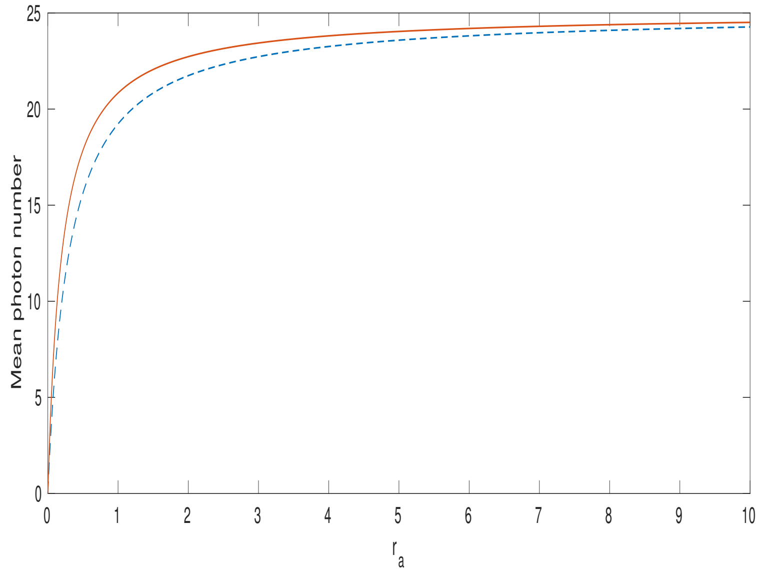

Figure 3.1: Plots of the mean photon number of light mode a at steady state, [Eq. (3.30)](dashed curve) and [Eq. (3.33)](solid curve) for $\kappa = 0.8$, $\gamma_c = 0.4$, $\gamma = 0.2$, and $N = 100$.

We would like to point out that the result given by Eq. (3.32) is in complete agreement with the one obtained in references [1,16]. The plots in Fig. 3.1 indicate that the effect of the spontaneous emission is to decrease the mean photon number.

Moreover, in the absence of spontaneous emission $(\gamma = 0)$, the mean photon number of light modes a and b has the form

$$\bar{n}_a = \frac{\gamma_c}{\kappa}\left[\frac{Nr_a}{\gamma_c + 2r_a}\right], \tag{3.33}$$

$$\bar{n}_b = \frac{\gamma_c}{\kappa}\left[\frac{Nr_a}{\gamma_c + 2r_a}\right]. \tag{3.34}$$

In addition, the mean photon number of light modes a and b reduces to

$$\bar{n}_a = \bar{n}_b = \frac{\gamma_c}{\kappa}\left[\frac{N}{2}\right] \tag{3.35}$$

for $r_a \gg \gamma + \gamma_c$ and to

$$\bar{n}_a = \bar{n}_b = \frac{\gamma_c}{\kappa}\left[\frac{N}{3}\right] \tag{3.36}$$

for $r_a = \gamma + \gamma_c$. From Eqs. (3.35) and (3.36), we see that the mean photon number of light modes a and b is less for $r_a = \gamma + \gamma_c$ than that for $r_a \gg \gamma + \gamma_c$.

Furthermore, employing the relation

$$\frac{d}{dt}\left\langle \hat{a}(t)\hat{a}(t)\right\rangle = \left\langle \frac{d\hat{a}(t)}{dt}\hat{a}(t)\right\rangle + \left\langle \hat{a}(t)\frac{d\hat{a}(t)}{dt}\right\rangle \tag{3.37}$$

along with Eq. (2.93), we readily find

$$\frac{d}{dt}\langle\hat{a}(t)\hat{a}(t)\rangle = -\kappa\langle\hat{a}(t)\hat{a}(t)\rangle + \frac{g}{\sqrt{N}}[\langle\hat{a}(t)\hat{m}_a(t)\rangle + \langle\hat{m}_a(t)\hat{a}(t)\rangle] + \sqrt{N}\big[\langle\hat{F}_a(t)\hat{a}(t)\rangle + \langle\hat{a}(t)\hat{F}_a(t)\rangle\big]. \tag{3.38}$$

Next we seek to evaluate $\langle\hat{a}(t)\hat{m}_a(t)\rangle$. Multiplying Eq. (3.3) on the right by $\hat{m}_a(t)$ and taking the expectation value of the resulting expression, we get

$$\langle\hat{a}(t)\hat{m}_a(t)\rangle = \frac{2\sqrt{N}}{\kappa}\langle\hat{F}_a(t)\hat{m}_a(t)\rangle. \tag{3.39}$$

We now proceed to find the expectation value of the product involving a noise operator and an atomic operator that appears in Eq. (3.39). To this end, multiplying Eq. (3.5) on the left by $\hat{F}_a(t)$ and taking the expectation value of the resulting expression, we have

$$\langle\hat{F}_a(t)\hat{m}_a(t)\rangle = \langle\hat{F}_a(t)\hat{m}_a(0)\rangle e^{-\frac{\mu}{2}t} + \int_0^t e^{-\frac{\mu}{2}(t-t')}\langle\hat{F}_a(t)\hat{G}(t')\rangle dt'. \tag{3.40}$$

Assuming the atomic and cavity mode noise operators are not correlated along with the fact that a noise operator $\hat{F}$ at a certain time should not affect the atomic variable at earlier time, Eq. (3.40) takes the form

$$\langle\hat{F}_a(t)\hat{m}_a(t)\rangle = 0. \tag{3.41}$$

Upon substituting Eq. (3.41) into Eq. (3.39), we obtain

$$\langle\hat{a}(t)\hat{m}_a(t)\rangle = 0. \tag{3.42}$$

Following a similar procedure, one can also check that

$$\langle\hat{m}_a(t)\hat{a}(t)\rangle = 0. \tag{3.43}$$

We next proceed to evaluate $\langle \hat{F}_a(t)\hat{a}(t)\rangle$. Multiplying Eq. (3.11) on the left by $\hat{F}_a(t)$ and taking the expectation value of the resulting expression, we have

$$\langle \hat{F}_a(t)\hat{a}(t)\rangle = \langle \hat{F}_a(t)\hat{a}(0)\rangle e^{-\frac{\kappa}{2}t} + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\left[\frac{g}{\sqrt{N}}\langle \hat{F}_a(t)\hat{m}_a(t')\rangle + \sqrt{N}\langle \hat{F}_a(t)\hat{F}_a(t')\rangle\right] dt'. \tag{3.44}$$

Applying Eqs. (2.9) and (3.41) along with the fact that a noise operator $\hat{F}$ at a certain time should not affect the cavity mode operator at earlier time, we easily obtain

$$\langle \hat{F}_a(t)\hat{a}(t)\rangle = 0. \tag{3.45}$$

It can also be established in a similar manner that

$$\langle \hat{a}(t)\hat{F}_a(t)\rangle = 0. \tag{3.46}$$

Now on account of Eqs. (3.45), (3.46) ,(3.42), and (3.43), Eq. (3.38) takes the form

$$\frac{d}{dt}\langle \hat{a}(t)\hat{a}(t)\rangle = -\kappa\langle \hat{a}(t)\hat{a}(t)\rangle. \tag{3.47}$$

The steady-state solution of this equation turns out to be

$$\langle \hat{a}^2\rangle = 0. \tag{3.48}$$

On the other hand, assuming the atoms to be initially in the bottom level, the expectation value of Eq. (3.5) happens to be

$$\langle \hat{m}_a(t)\rangle = 0. \tag{3.49}$$

Furthermore, using Eq. (3.49) along with Eq. (2.6) and the assumption that the cavity light is initially in a vacuum state, the expectation value of Eq. (3.11) takes the form

$$\langle \hat{a}(t)\rangle = 0. \tag{3.50}$$

In view of Eqs. (2.93) and (3.50), we claim that $\hat{a}(t)$ is a Gaussian variable with zero mean. One can also easily verify that

$$\langle \hat{b}(t) \rangle = 0. \tag{3.51}$$

Then on account of Eqs. (2.95) and (3.51), we realize that $\hat{b}(t)$ is a Gaussian variable with zero mean.

3.1.2 Global photon number variance

In this section we wish to calculate the variance of the photon number for light modes a and b in the entire frequency interval. To this end, the variance of the photon number for light mode a can be written as

$$(\Delta n_a)^2 = \langle (\hat{a}^\dagger \hat{a})^2 \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2. \tag{3.52}$$

Using the fact that $\hat{a}$ is a Gaussian variable with zero mean, we readily get

$$(\Delta n_a)^2 = \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a} \hat{a}^\dagger \rangle + \langle \hat{a}^{\dagger 2} \rangle \langle \hat{a}^2 \rangle. \tag{3.53}$$

On account of Eqs. (3.15), (3.25), (2.101), and (3.48), we arrive at

$$(\Delta n_a)^2 = \bar{n}_a^2 + N \bar{n}_a. \tag{3.54}$$

This represents the variance of the photon number for chaotic light.

One can also establish in a similar manner that the photon number variance for light mode b has the form

$$(\Delta n_b)^2 = \frac{\gamma + \gamma_c}{r_a} \bar{n}_b^2 + N \bar{n}_b. \tag{3.55}$$

We note that Eq. (3.55) takes the form

$$(\Delta n_b)^2 = N \bar{n}_b \tag{3.56}$$

for $\gamma + \gamma_c \ll r_a$ and

$$(\Delta n_b)^2 = \bar{n}_b^2 + N\bar{n}_b \tag{3.57}$$

for $\gamma + \gamma_c = r_a$. Furthermore, inspection of Eqs. (3.54) and (3.55) indicates that

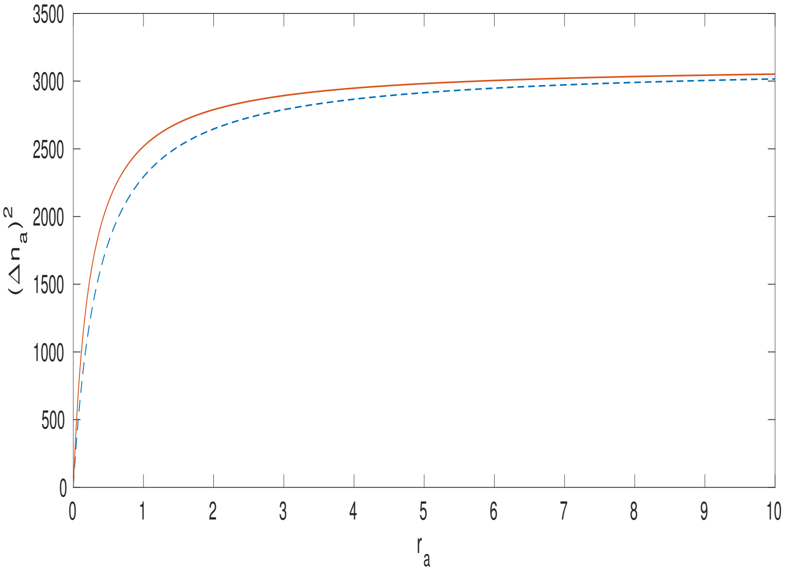

Figure 3.2: Plots of the photon number variance of light mode a at steady state, [Eq. (3.54)] for $\kappa = 0.8$, $\gamma_c = 0.4$, $\gamma = 0$(solid curve), $\gamma = 0.2$(dashed curve), and $N = 100$.

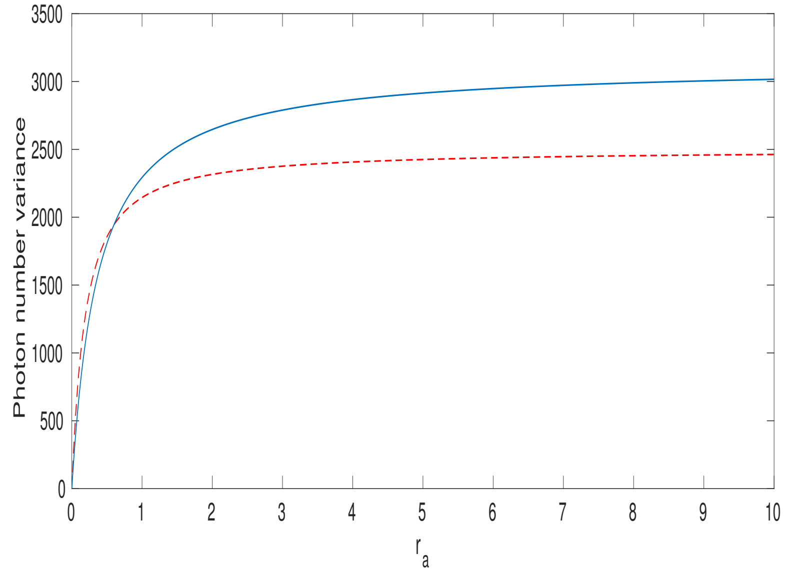

Figure 3.3: Plots of the photon number variance of light modes a and b at steady state, [Eq. (3.54)](solid curve) and [Eq. (3.55)](dashed curve) for $\kappa = 0.8$, $\gamma_c = 0.4$, $\gamma = 0.2$, and $N = 100$.

$(\Delta n_a)^2 > \bar{n}_a$ and $(\Delta n_b)^2 > \bar{n}_b$ and hence the photon statistics of each light mode is super-Poissonian. The plots in Fig. 3.2 indicate that the photon number variance of light mode a is less for $\gamma = 0.2$ than that for $\gamma = 0$. Moreover, the plots in Fig. 3.3 indicate that the photon number variance of light mode a is less than that of light mode b for $0.01 < r_a < 0.6$ and greater for $0.6 < r_a < 10$.

3.2 Two-mode photon statistics

In this section we seek to determine the global mean photon number and variance of the two-mode cavity light. In addition, we obtain the local mean photon number and variance of the two-mode cavity light.

3.2.1 Global mean photon number

Here we wish to calculate the steady-state mean photon number for the two-mode cavity light in the entire frequency interval. To this end, adding Eqs. (3.50) and (3.51), we obtain

$$\langle \hat{c} \rangle = 0, \tag{3.58}$$

where

$$\hat{c} = \hat{a} + \hat{b}. \tag{3.59}$$

In addition, adding Eqs. (2.93) and (2.95), we get

$$\frac{d\hat{c}}{dt} = -\frac{\kappa}{2}\hat{c} + \frac{g}{\sqrt{N}}\hat{m} + \sqrt{N}\hat{F}_c(t), \tag{3.60}$$

where

$$\hat{F}_c(t) = \hat{F}_a(t) + \hat{F}_b(t) \tag{3.61}$$

and $\hat{m}$ is given by Eq. (2.81). In view of Eqs. (3.58) and (3.60), we see that $\hat{c}$ is a Gaussian variable with zero mean. One can also easily check that

$$\langle \hat{F}_c(t) \rangle = 0, \tag{3.62}$$

$$\langle \hat{F}_c^\dagger(t)\hat{F}_c(t')\rangle = 0, \tag{3.63}$$

$$\langle \hat{F}_c^\dagger(t)\hat{F}_c^\dagger(t')\rangle = \langle \hat{F}_c(t)\hat{F}_c(t')\rangle = 0, \tag{3.64}$$

$$\langle \hat{F}_c(t)\hat{F}_c^\dagger(t')\rangle = 2\kappa\delta(t-t'). \tag{3.65}$$

Furthermore, using the relation

$$\frac{d}{dt}\left\langle \hat{c}^\dagger(t)\hat{c}(t)\right\rangle = \left\langle \frac{d\hat{c}^\dagger(t)}{dt}\hat{c}(t)\right\rangle + \left\langle \hat{c}^\dagger(t)\frac{d\hat{c}(t)}{dt}\right\rangle \tag{3.66}$$

along with Eq. (3.60), we readily find

$$\begin{aligned}\frac{d}{dt}\langle \hat{c}^\dagger(t)\hat{c}(t)\rangle = -\kappa\langle \hat{c}^\dagger(t)\hat{c}(t)\rangle + \frac{g}{\sqrt{N}}\left[\langle \hat{c}^\dagger(t)\hat{m}(t)\rangle + \langle \hat{m}^\dagger(t)\hat{c}(t)\rangle\right] \\ + \sqrt{N}\left[\langle \hat{F}_c^\dagger(t)\hat{c}(t)\rangle + \langle \hat{c}^\dagger(t)\hat{F}_c(t)\rangle\right].\end{aligned} \tag{3.67}$$

Next we seek to evaluate $\langle \hat{c}^\dagger(t)\hat{m}(t)\rangle$. Applying the large-time approximation, one gets from Eq. (3.60)

$$\hat{c}(t) = \frac{2g}{\kappa\sqrt{N}}\hat{m} + \frac{2\sqrt{N}}{\kappa}\hat{F}_c(t). \tag{3.68}$$

Multiplying the adjoint of Eq. (3.68) on the right by $\hat{m}(t)$ and taking the expectation value of the resulting expression, we get

$$\langle \hat{c}^\dagger(t)\hat{m}(t)\rangle = \frac{2g\sqrt{N}}{\kappa}\left[\langle \hat{N}_a(t)\rangle + \langle \hat{N}_b(t)\rangle\right] + \frac{2\sqrt{N}}{\kappa}\langle \hat{F}_c^\dagger(t)\hat{m}(t)\rangle. \tag{3.69}$$

We now proceed to find $\langle \hat{F}_c^\dagger(t)\hat{m}(t)\rangle$. To this end, a formal solution of Eq. (2.108) can be written as

$$\hat{m}(t) = \hat{m}(0)e^{-\frac{\mu}{2}t} + \int_0^t e^{-\frac{\mu}{2}(t-t')}[-\frac{\mu}{2}\hat{m}_a(t') + \hat{G}(t')]dt'. \tag{3.70}$$

Multiplying Eq. (3.70) on the left by $\hat{F}_c^\dagger(t)$ and taking the expectation value of the resulting expression, we have

$$\langle \hat{F}_c^\dagger(t)\hat{m}(t)\rangle = \langle \hat{F}_c^\dagger(t)\hat{m}(0)\rangle e^{-\frac{\mu}{2}t} + \int_0^t e^{-\frac{\mu}{2}(t-t')}[-\frac{\mu}{2}\langle \hat{F}_c^\dagger(t)\hat{m}_a(t')\rangle + \langle \hat{F}_c^\dagger(t)\hat{G}(t')\rangle]dt'. \tag{3.71}$$

In view of the fact that a noise operator $\hat{F}$ at a certain time should not affect an atomic variable at earlier time, Eq. (3.71) reduces to

$$\langle \hat{F}_c^\dagger(t)\hat{m}(t)\rangle = \int_0^t e^{-\frac{\mu}{2}(t-t')}\langle \hat{F}_c^\dagger(t)\hat{G}(t')\rangle dt'. \tag{3.72}$$

Assuming the atomic and cavity mode noise operators are independent, one can write

$$\langle \hat{F}_c^\dagger(t)\hat{G}(t')\rangle = \langle \hat{F}_c^\dagger(t)\rangle\langle \hat{G}(t')\rangle = 0. \tag{3.73}$$

Upon substituting Eq. (3.73) into Eq. (3.72), we have

$$\langle \hat{F}_c^\dagger(t)\hat{m}(t)\rangle = 0. \tag{3.74}$$

On account of this result, Eq. (3.69) takes the form

$$\langle \hat{c}^\dagger(t)\hat{m}(t)\rangle = \frac{2g\sqrt{N}}{\kappa}[\langle \hat{N}_a(t)\rangle + \langle \hat{N}_b(t)\rangle]. \tag{3.75}$$

We next seek to evaluate $\langle \hat{F}_c^\dagger(t)\hat{c}(t)\rangle$. To this end, a formal solution of Eq. (3.60) can be written as

$$\hat{c}(t) = \hat{c}(0)e^{-\frac{\kappa}{2}t} + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\left[\frac{g}{\sqrt{N}}\hat{m}(t') + \sqrt{N}\hat{F}_c(t')\right]dt'. \tag{3.76}$$

Multiplying Eq. (3.76) on the left by $\hat{F}_c^\dagger(t)$ and taking the expectation value of the resulting expression, we have

$$\langle \hat{F}_c^\dagger(t)\hat{c}(t)\rangle = \langle \hat{F}_c^\dagger(t)\hat{c}(0)\rangle e^{-\frac{\kappa}{2}t} + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\left[\frac{g}{\sqrt{N}}\langle \hat{F}_c^\dagger(t)\hat{m}(t')\rangle + \sqrt{N}\langle \hat{F}_c^\dagger(t)\hat{F}_c(t')\rangle\right]dt'. \tag{3.77}$$

In view of Eqs. (3.74) and (3.63) along with the fact that a noise operator $\hat{F}$ at a certain time should not affect the cavity mode operator at earlier time, Eq. (3.77) becomes

$$\langle \hat{F}_c^\dagger(t)\hat{c}(t)\rangle = 0. \tag{3.78}$$

Now with the aid of Eqs. (3.75) and (3.78) along with their complex conjugates, we can rewrite Eq. (3.67) as

$$\frac{d}{dt}\left\langle \hat{c}^\dagger(t)\hat{c}(t)\right\rangle = -\kappa\left\langle \hat{c}^\dagger(t)\hat{c}(t)\right\rangle + \frac{4g^2}{\kappa}\left[\langle \hat{N}_a(t)\rangle + \langle \hat{N}_b(t)\rangle\right]. \tag{3.79}$$

The steady-state solution of this equation is expressible as

$$\langle \hat{c}^\dagger\hat{c}\rangle = \frac{\gamma_c}{\kappa}\left[\langle \hat{N}_a\rangle + \langle \hat{N}_b\rangle\right]. \tag{3.80}$$

Following a similar procedure, one can also get

$$\langle \hat{c}\hat{c}^\dagger\rangle = \frac{\gamma_c}{\kappa}\left[\langle \hat{N}_c\rangle + \langle \hat{N}_b\rangle\right] + 2N. \tag{3.81}$$

In view of Eqs. (2.101) and (2.111), Eq. (3.80) can be rewritten as

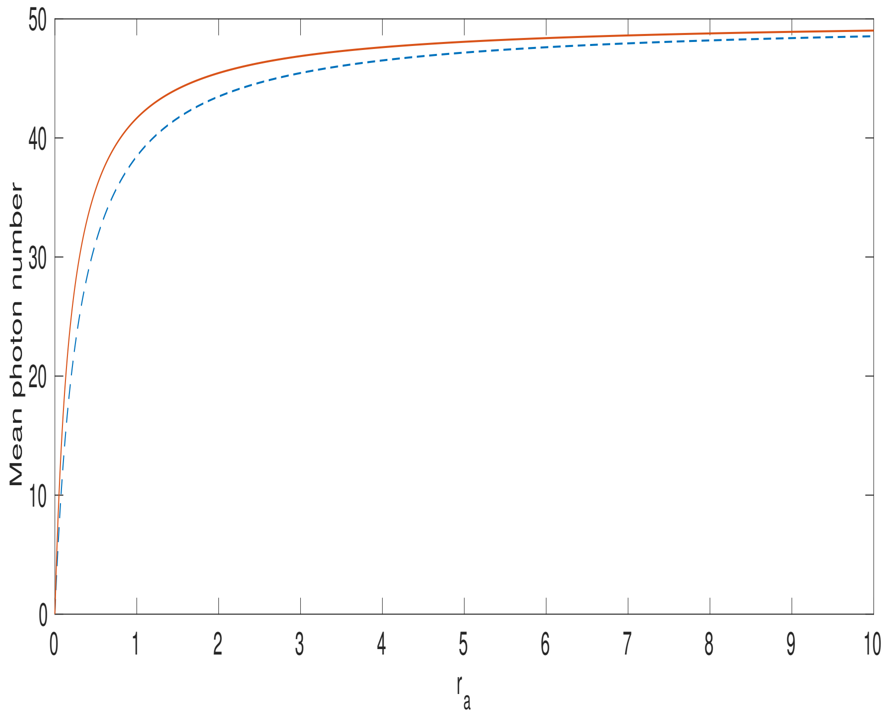

Figure 3.4: Plots of the mean photon number for the two-mode cavity light at steady state [Eq. (3.82)] for $\kappa = 0.8$, $\gamma_c = 0.4$, $\gamma = 0.2$ (dashed curve), $\gamma = 0$ (solid curve), and $N = 100$.

$$\langle \hat{c}^\dagger\hat{c}\rangle = \frac{\gamma_c}{\kappa}\left[\frac{2Nr_a}{\gamma + \gamma_c + 2r_a}\right]. \tag{3.82}$$

Similarly on account of Eqs. (2.111) and (2.113), Eq. (3.81) takes the form

$$\langle \hat{c}\hat{c}^{\dagger} \rangle = \frac{\gamma_c}{\kappa}\left[\frac{\gamma + \gamma_c + r_a}{\gamma + \gamma_c + 2r_a}\right] N + 2N. \tag{3.83}$$

We note from Eqs. (3.30) and (3.82) that the mean photon number of the two-mode cavity light is just the sum of the mean photon numbers of the single-mode light beams. It can be seen from the plots in Fig. 3.4 that the effect of the spontaneous emission is to decrease the mean photon number.

Moreover, in the absence of the spontaneous emission ($\gamma = 0$), the mean photon number of the two-mode cavity light becomes

$$\bar{n} = \frac{\gamma_c}{\kappa}\left(\frac{2Nr_a}{\gamma_c + 2r_a}\right). \tag{3.84}$$

In addition, the mean photon number of the two-mode cavity light reduces to

$$\bar{n} = \frac{\gamma_c}{\kappa} N \tag{3.85}$$

for $r_a \gg \gamma + \gamma_c$ and to

$$\bar{n} = \frac{\gamma_c}{\kappa}\left[\frac{2N}{3}\right] \tag{3.86}$$

for $r_a = \gamma + \gamma_c$. From Eqs. (3.85) and (3.86), we see that the mean photon number of the two-mode cavity light is less for $r_a = \gamma + \gamma_c$ than that for $r_a \gg \gamma + \gamma_c$.

3.2.2 Global photon number variance

In this section we seek to calculate the variance of the photon number for the two-mode cavity light. To this end, the variance of the photon number for two-mode cavity light is expressible as

$$(\Delta n)^2 = \langle (\hat{c}^{\dagger}\hat{c})^2 \rangle - \langle \hat{c}^{\dagger}\hat{c} \rangle^2. \tag{3.87}$$

Using the fact that $\hat{c}$ is a Gaussian variable with zero mean, we readily get

$$(\Delta n)^2 = \langle \hat{c}^{\dagger}\hat{c} \rangle \langle \hat{c}\hat{c}^{\dagger} \rangle + \langle \hat{c}^{2\dagger} \rangle \langle \hat{c}^2 \rangle. \tag{3.88}$$

Furthermore, employing the relation

$$\frac{d}{dt}\left\langle \hat{c}(t)\hat{c}(t)\right\rangle = \left\langle \frac{d\hat{c}(t)}{dt}\hat{c}(t)\right\rangle + \left\langle \hat{c}(t)\frac{d\hat{c}(t)}{dt}\right\rangle \tag{3.89}$$

along with Eq. (3.60), we easily obtain

$$\begin{aligned}\frac{d}{dt}\langle \hat{c}(t)\hat{c}(t)\rangle &= -\kappa\langle \hat{c}(t)\hat{c}(t)\rangle + \frac{g}{\sqrt{N}}\left[\langle \hat{c}(t)\hat{m}(t)\rangle + \langle \hat{m}(t)\hat{c}(t)\rangle\right] \\ &+ \sqrt{N}\left[\langle \hat{F}_c(t)\hat{c}(t)\rangle + \langle \hat{c}(t)\hat{F}_c(t)\rangle\right].\end{aligned} \tag{3.90}$$

Next we seek to evaluate $\langle \hat{c}(t)\hat{m}(t)\rangle$. Multiplying Eq. (3.68) on the right by $\hat{m}(t)$ and taking the expectation value of the resulting expression, we get

$$\langle \hat{c}(t)\hat{m}(t)\rangle = \frac{2g\sqrt{N}}{\kappa}\langle \hat{\sigma}_c^k\rangle + \frac{2\sqrt{N}}{\kappa}\langle \hat{F}_c(t)\hat{m}(t)\rangle. \tag{3.91}$$

We now proceed to find the expectation value of the product involving a noise operator and an atomic operator that appears in Eq. (3.91). To this end, multiplying Eq. (3.70) on the left by $\hat{F}_c(t)$ and taking the expectation value of the resulting expression, we have

$$\langle \hat{F}_c(t)\hat{m}(t)\rangle = \langle \hat{F}_c(t)\hat{m}(0)\rangle e^{-\frac{\mu}{2}t} + \int_0^t e^{-\frac{\mu}{2}(t-t')}[-\frac{\mu}{2}\langle \hat{F}_c(t)\hat{m}_a(t')\rangle + \langle \hat{F}_c(t)\hat{G}(t')\rangle]dt'. \tag{3.92}$$

In view of the fact that a noise operator $\hat{F}$ at a certain time should not affect an atomic variable at earlier time, Eq. (3.92) takes the form

$$\langle \hat{F}_c(t)\hat{m}(t)\rangle = \int_0^t e^{-\frac{\mu}{2}(t-t')}\langle \hat{F}_c(t)\hat{G}(t')\rangle dt'. \tag{3.93}$$

Assuming the atomic and cavity mode noise operators are independent, one can write

$$\langle \hat{F}_c(t)\hat{G}(t')\rangle = \langle \hat{F}_c(t)\rangle\langle \hat{G}(t')\rangle = 0. \tag{3.94}$$

On account of this result, Eq. (3.93) takes the form

$$\langle \hat{F}_c(t)\hat{m}(t)\rangle = 0. \tag{3.95}$$

Upon substituting Eq. (3.95) into Eq. (3.91), we obtain

$$\langle \hat{c}(t)\hat{m}(t)\rangle = \frac{2g\sqrt{N}}{\kappa}\langle \hat{\sigma}_c^k\rangle. \tag{3.96}$$

Following a similar procedure, one can also check that

$$\langle \hat{m}(t)\hat{c}(t)\rangle = \frac{2g\sqrt{N}}{\kappa}\langle \hat{\sigma}_c^k\rangle. \tag{3.97}$$

We next proceed to evaluate $\langle \hat{F}_c(t)\hat{c}(t)\rangle$. Multiplying Eq. (3.76) on the left by $\hat{F}_c(t)$ and taking the expectation value of the resulting expression, we have

$$\langle \hat{F}_c(t)\hat{c}(t)\rangle = \langle \hat{F}_c(t)\hat{c}(0)\rangle e^{-\frac{\kappa}{2}t} + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\left[\frac{g}{\sqrt{N}}\langle \hat{F}_c(t)\hat{m}(t')\rangle + \sqrt{N}\langle \hat{F}_c(t)\hat{F}_c(t')\rangle\right]dt'. \tag{3.98}$$

In view of the fact that a noise operator $\hat{F}$ at a certain time should not affect the cavity mode operator at earlier time, Eq. (3.98) becomes

$$\langle \hat{F}_c(t)\hat{c}(t)\rangle = \int_0^t e^{-\frac{\kappa}{2}(t-t')}\left[\frac{g}{\sqrt{N}}\langle \hat{F}_c(t)\hat{m}(t')\rangle + \sqrt{N}\langle \hat{F}_c(t)\hat{F}_c(t')\rangle\right]dt'. \tag{3.99}$$

On account of Eqs. (3.95) and (3.64), Eq. (3.99) takes the form

$$\langle \hat{F}_c(t)\hat{c}(t)\rangle = 0 \tag{3.100}$$

It can also be established in a similar manner that

$$\langle \hat{c}(t)\hat{F}_c(t)\rangle = 0. \tag{3.101}$$

Now in view of Eqs. (3.100), (3.101) ,(3.96), and (3.97), Eq. (3.90) takes the form

$$\frac{d}{dt}\langle \hat{c}(t)\hat{c}(t)\rangle = -\kappa\langle \hat{c}(t)\hat{c}(t)\rangle + \frac{4g^2}{\kappa}\langle \hat{\sigma}_c^k\rangle. \tag{3.102}$$

The steady-state solution of this equation turns out to be

$$\langle \hat{c}^2\rangle = \frac{\gamma_c}{\kappa}\langle \hat{\sigma}_c^k\rangle. \tag{3.103}$$

We now proceed to calculate the expectation value of the atomic operator $\hat{\sigma}_c$ following the approach presented in [1]. To this end, applying the identity given by Eq. (2.71), the state vector of a three-level atom can be put in the form

$$|\psi_k\rangle = c_a|a_k\rangle + c_b|b_k\rangle + c_c|c_k\rangle, \tag{3.104}$$

in which

$$c_a = \langle a_k|\psi_k\rangle, \tag{3.105}$$

$$c_b = \langle b_k|\psi_k\rangle, \tag{3.106}$$

$$c_c = \langle c_k|\psi_k\rangle. \tag{3.107}$$

The state vector described by Eq. (3.104) can be used to determine the expectation value of an atomic operator formed by a pair of identical energy levels or by two distinct energy levels between which transition with the emission of a photon is dipole forbidden. One can thus readily establish that

$$\langle\hat{\eta}_a^k\rangle = c_a c_a^*, \tag{3.108}$$

$$\langle\hat{\eta}_c^k\rangle = c_c c_c^*, \tag{3.109}$$

and

$$\langle\hat{\sigma}_c^k\rangle = c_a c_c^*. \tag{3.110}$$

We then see that

$$|\langle\hat{\sigma}_c^k\rangle|^2 = \langle\hat{\eta}_a^k\rangle\langle\hat{\eta}_c^k\rangle, \tag{3.111}$$

and on taking $|\langle\hat{\sigma}_c^k\rangle|$ to be real, we see that

$$\langle\hat{\sigma}_c^k\rangle = \sqrt{\langle\hat{\eta}_a^k\rangle\langle\hat{\eta}_c^k\rangle} \tag{3.112}$$

so that on substituting this into Eq. (3.103), we get

$$\langle\hat{c}^2\rangle = \frac{\gamma_c}{\kappa}\sqrt{\langle\hat{\eta}_a^k\rangle\langle\hat{\eta}_c^k\rangle}. \tag{3.113}$$

Upon summing over k from 1 up to N and taking into account Eqs. (2.68) and (2.70), we have

$$\langle \hat{c}^2 \rangle = \frac{\gamma_c}{\kappa}\sqrt{\langle \hat{N}_a \rangle \langle \hat{N}_c \rangle}. \tag{3.114}$$

Now using Eq. (2.113), Eq. (3.114) can be put in the form

$$\langle \hat{c}^2 \rangle = \frac{\gamma_c}{\kappa}\sqrt{\frac{\gamma+\gamma_c}{r_a}}\langle \hat{N}_a \rangle. \tag{3.115}$$

In view of Eqs. (3.80), (3.81), and (3.115), Eq. (3.88) becomes

$$\begin{aligned}(\Delta n)^2 = {} & \frac{\gamma_c}{\kappa}\Big(\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle\Big)\left(\frac{\gamma_c}{\kappa}\Big(\langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle\Big) + 2N\right) \\ & + \left(\frac{\gamma_c}{\kappa}\sqrt{\frac{\gamma+\gamma_c}{r_a}}\langle \hat{N}_a \rangle\right)^2 .\end{aligned} \tag{3.116}$$

We observe from Eq. (3.116) that the photon number variance of the two-mode

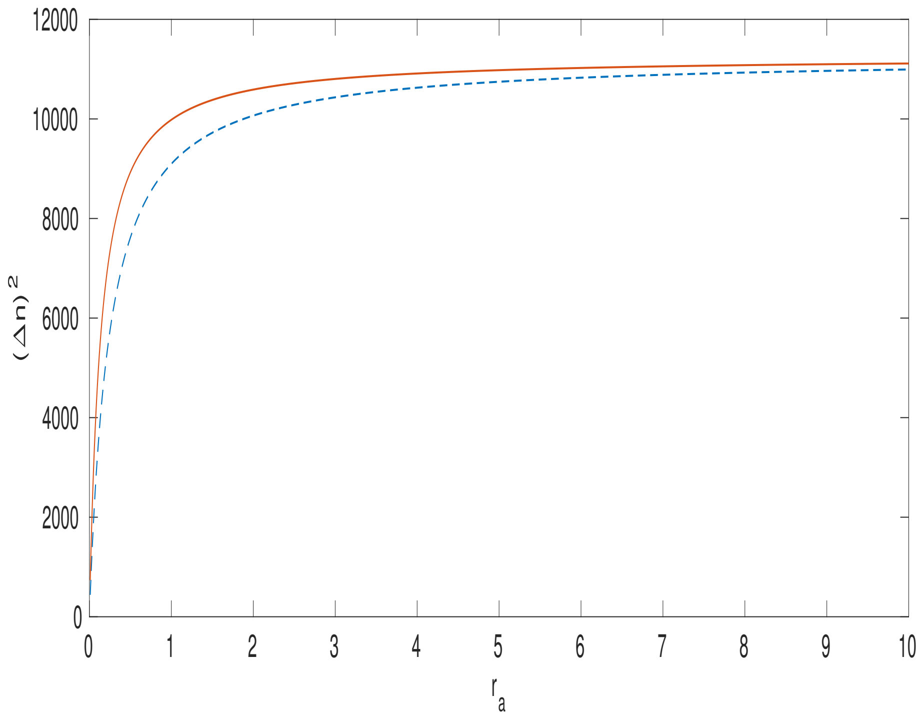

Figure 3.5: Plots of the photon number variance of two-mode cavity light at steady state, [Eq. (3.117)] for $\kappa = 0.8$, $\gamma_c = 0.4$, $\gamma = 0$(solid curve), $\gamma = 0.2$(dashed curve), and $N = 100$.

cavity light does not happen to be the sum of the photon number variance of the separate single-mode light beams given by Eqs. (3.54) and (3.55). Finally, on account of Eqs. (2.101), (2.111), (2.113), and (3.82), Eq. (3.116) takes the form

$$(\Delta n)^2 = \frac{1}{4}\bar{n}^2\left(\frac{3(\gamma+\gamma_c)}{r_a}+2\right)+2N\bar{n}. \tag{3.117}$$

This can be rewritten as

$$(\Delta n)^2 = \frac{1}{4}\bar{n}^2\left(3\eta+2\right)+2N\bar{n}, \tag{3.118}$$

where

$$\eta = \frac{\gamma+\gamma_c}{r_a} \tag{3.119}$$

and $\bar{n}$ is given by Eq. (3.82). Our result shows that the photon number variance of the two-mode cavity light is greater than the one obtained by Fesseha [1,16]. This must be due to the reservoir noise operators. Furthermore, inspection of Eq. (3.118) indicates that $(\Delta n)^2 > \bar{n}$ and hence the photon statistics of the two-mode cavity light is super-Poissonian. The plots in Fig. 3.5. indicate that the photon number variance of the two-mode cavity light is less for $\gamma = 0.2$ than that for $\gamma = 0$. In other words, the effect of spontaneous emission is to decrease the variance of the photon number. Moreover, we observe that the photon number variance of the two-mode cavity light increases with r_a.

3.2.3 Local mean photon number

We now proceed to obtain the mean photon number of the two-mode cavity light in a given frequency interval. To determine the local mean photon number of the two-mode cavity light, we need to obtain the power spectrum of the two-mode cavity light when both light modes a and b have the same frequency. The power spectrum of the two-mode cavity light is expressible as [1]

$$P(\omega) = \frac{1}{\pi}Re\int_0^{\infty} d\tau e^{i(\omega-\omega_0)\tau}\langle\hat{c}^{\dagger}(t)\hat{c}(t+\tau)\rangle_{ss}, \tag{3.120}$$

in which ω_0 is the central frequency of light mode a or b. Upon integrating both sides of Eq. (3.120) over ω, we readily get [38]

$$\int P(\omega)d\omega = \bar{n}, \tag{3.121}$$

in which $\bar{n}$ is the steady-state mean photon number for the two-mode cavity light. From this result, we observe that $P(\omega)d\omega$ is the steady-state mean photon number in the interval between ω and $\omega + d\omega$.

We now proceed to calculate the two-time correlation function that appears in Eq. (3.120). To this end, we realize that the solution of Eq. (3.60) can be written as

$$\hat{c}(t+\tau) = \hat{c}(t)e^{-\frac{\kappa}{2}\tau} + \int_0^\tau e^{-\frac{\kappa}{2}(\tau-\tau')}\left[\frac{g}{\sqrt{N}}\hat{m}(t+\tau') + \sqrt{N}\hat{F}_c(t+\tau')\right]d\tau'. \tag{3.122}$$

On the other hand, the solution of Eq. (2.108) is expressible as

$$\hat{m}(t+\tau) = \hat{m}(t)e^{-\frac{\mu}{2}\tau} + \int_0^\tau e^{-\frac{\mu}{2}(\tau-\tau')}\left[-\frac{\mu}{2}\hat{m}_a(t+\tau') + \hat{G}(t+\tau')\right]d\tau'. \tag{3.123}$$

Applying the large-time approximation scheme to Eq. (2.96), we get

$$\hat{m}_a(t+\tau) = \frac{2}{\mu}\hat{G}_a(t+\tau), \tag{3.124}$$

so that on introducing this into Eq. (3.123), there follows

$$\hat{m}(t+\tau) = \hat{m}(t)e^{-\frac{\mu}{2}\tau} + \int_0^\tau e^{-\frac{\mu}{2}(\tau-\tau')}\left[-\hat{G}_a(t+\tau') + \hat{G}(t+\tau')\right]d\tau'. \tag{3.125}$$

Now combination of Eqs. (3.122) and (3.125) yields

$$\begin{aligned}\hat{c}(t+\tau) &= \hat{c}(t)e^{-\frac{\kappa}{2}\tau} + \sqrt{N}\int_0^\tau e^{-\frac{\kappa}{2}(\tau-\tau')}\hat{F}_c(t+\tau')d\tau' + \frac{g}{\sqrt{N}}\hat{m}(t)e^{-\frac{\kappa}{2}\tau}\int_0^\tau d\tau' e^{\frac{1}{2}(\kappa-\mu)\tau'} \\ &+ \frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}\tau}\int_0^\tau d\tau' e^{\frac{1}{2}(\kappa-\mu)\tau'}\int_0^\tau d\tau'' e^{\frac{1}{2}\mu\tau''}\left[-\hat{G}_a(t+\tau'') + \hat{G}(t+\tau'')\right]\end{aligned} \tag{3.126}$$

and carrying out the second integration, we find

$$\hat{c}(t+\tau) = \hat{c}(t)e^{-\frac{\kappa}{2}\tau} + \int_0^\tau e^{-\frac{\kappa}{2}(\tau-\tau')}\hat{F}_c(t+\tau')d\tau' + \frac{2g\hat{m}(t)}{\sqrt{N}(\kappa-\mu)}\left[e^{-\frac{1}{2}\mu\tau} - e^{-\frac{1}{2}\kappa\tau}\right]$$
$$+ \frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}\tau}\int_0^\tau d\tau' e^{\frac{1}{2}(\kappa-\mu)\tau'} \times \int_0^\tau d\tau'' e^{\frac{1}{2}\mu\tau''}\left[-\hat{G}_a(t+\tau'') + \hat{G}(t+\tau'')\right]. \tag{3.127}$$

Now multiplying this equation from the left by $\hat{c}^\dagger(t)$ and taking the expectation value of the resulting expression, we get

$$\langle\hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle = \langle\hat{c}^\dagger(t)\hat{c}(t)\rangle e^{-\frac{\kappa}{2}\tau} + \frac{2g}{\sqrt{N}(\kappa-\mu)}\langle\hat{c}^\dagger(t)\hat{m}(t)\rangle\left[e^{-\frac{1}{2}\mu\tau} - e^{-\frac{1}{2}\kappa\tau}\right]. \tag{3.128}$$

Applying once more the large-time approximation, one gets from Eq. (3.60)

$$\hat{m}(t) = \frac{\kappa\sqrt{N}}{2g}\hat{c}(t) - \frac{N}{g}\hat{F}_c(t) \tag{3.129}$$

and with this substituted into Eq. (3.128), there emerges

$$\langle\hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle = \langle\hat{c}^\dagger(t)\hat{c}(t)\rangle e^{-\frac{\kappa}{2}\tau} + \left[\frac{\kappa}{\kappa-\mu}\langle\hat{c}^\dagger(t)\hat{c}(t)\rangle - \frac{2\sqrt{N}}{\kappa-\mu}\langle\hat{c}^\dagger(t)\hat{F}_c(t)\rangle\right]$$
$$\times\left[e^{-\frac{1}{2}\mu\tau} - e^{-\frac{1}{2}\kappa\tau}\right]. \tag{3.130}$$

In view of the complex conjugate of Eq. (3.78), Eq. (3.130) takes the form

$$\langle\hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle = \langle\hat{c}^\dagger(t)\hat{c}(t)\rangle\left[\frac{\kappa}{\kappa-\mu}e^{-\frac{1}{2}\mu\tau} - \frac{\mu}{\kappa-\mu}e^{-\frac{1}{2}\kappa\tau}\right]. \tag{3.131}$$

Hence on substituting this into Eq. (3.120) and carrying out the integration, we get

$$P(\omega) = \frac{\bar{n}\kappa}{\kappa-\mu}\left[\frac{\frac{\mu}{2\pi}}{(\omega-\omega_0)^2 + (\frac{\mu}{2})^2}\right] - \frac{\bar{n}\mu}{\kappa-\mu}\left[\frac{\frac{\kappa}{2\pi}}{(\omega-\omega_0)^2 + (\frac{\kappa}{2})^2}\right]. \tag{3.132}$$

We recall that the mean photon number in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$ is expressible as [1]

$$\bar{n}_{\pm\lambda} = \int_{-\lambda}^{\lambda} P(\omega')d\omega', \tag{3.133}$$

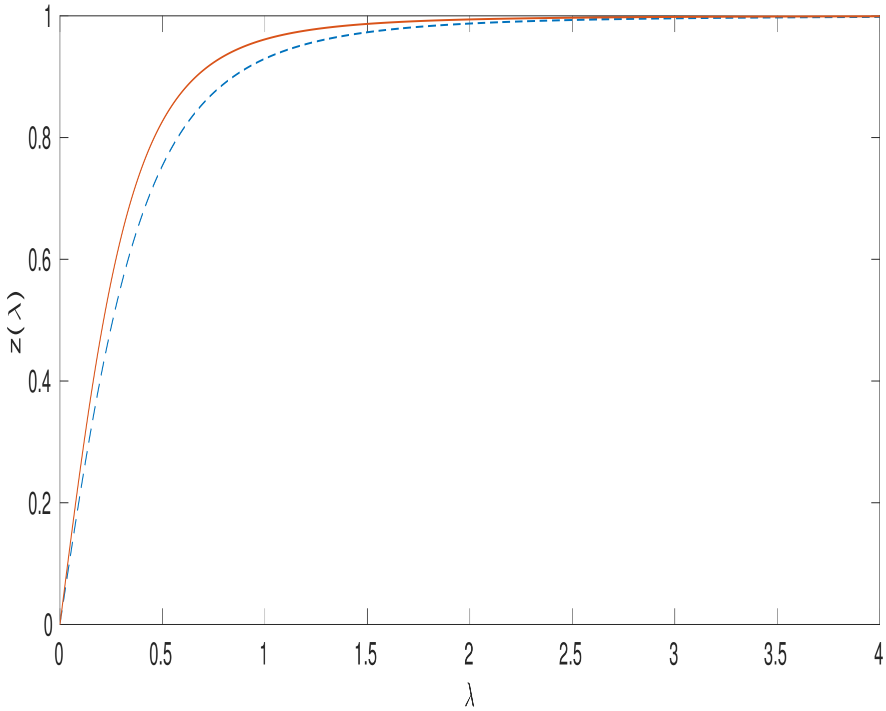

Figure 3.6: Plots of [Eq. (3.136)] for $\kappa = 0.8$, μ=2 (dashed curve), and μ=1.2 (solid curve).

in which $\omega'= \omega - \omega_0$. Therefore, upon inserting Eq. (3.132) into Eq. (3.133) and carrying out the integration, applying the relation described by

$$\int_{-\lambda}^{\lambda} \frac{dx}{x^2 + a^2} = \frac{2}{a} tan^{-1}\left(\frac{\lambda}{a}\right), \tag{3.134}$$

we easily obtain

$$\bar{n}_{\pm\lambda} = \bar{n} z(\lambda) \tag{3.135}$$

where $z(\lambda)$ is given by

$$z(\lambda) = \frac{\frac{2\kappa}{\pi}}{\kappa - \mu} tan^{-1}\left(\frac{2\lambda}{\mu}\right) - \frac{\frac{2\mu}{\pi}}{\kappa - \mu} tan^{-1}\left(\frac{2\lambda}{\kappa}\right). \tag{3.136}$$

From the plots in Fig. 3.6, we easily find $z(0.5) = 0.66$, $z(1) = 0.86$, $z(2) = 0.96$. Then combination of these results with Eq. (3.135) yields $\bar{n}_{\pm 0.5} = 0.66\bar{n}$, $\bar{n}_{\pm 1} = 0.86\bar{n}$, $\bar{n}_{\pm 2} = 0.96\bar{n}$. In addition, we see that as λ increases the local mean photon number of the two-mode cavity light approaches to the global mean photon number of the two-mode cavity light. We therefore observe that a large part of the total mean photon number is confined in a relatively small frequency interval. Furthermore, from the plots in Fig. 3.6, we see that $z(\lambda)$ in the absence of spontaneous emission is greater than in the presence of spontaneous emission.

3.2.4 Local photon number variance

We now proceed to obtain the photon number variance of the two-mode cavity light in a given frequency interval. To determine the local photon number variance of the two-mode cavity light, we need to obtain the spectrum of the photon number fluctuations for the two-mode cavity light when both light modes a and b have the same frequency. We define this spectrum by

$$J(\omega) = \frac{1}{\pi} Re \int_0^\infty d\tau e^{i(\omega-\omega_0)\tau} \langle \hat{n}(t), \hat{n}(t+\tau)\rangle_{ss}, \tag{3.137}$$

in which ω_0 is the central frequency of light mode a or b and $\hat{n}(t)$ is given by

$$\hat{n}(t) = \hat{c}^\dagger(t)\hat{c}(t) \tag{3.138}$$

and

$$\hat{n}(t+\tau) = \hat{c}^\dagger(t+\tau)\hat{c}(t+\tau). \tag{3.139}$$

Upon integrating both sides of Eq. (3.137) over ω, we get

$$\int_{-\infty}^\infty J(\omega)d\omega = (\Delta n)^2, \tag{3.140}$$

in which

$$(\Delta n)^2 = \langle \hat{n}(t), \hat{n}(t)\rangle_{ss} \tag{3.141}$$

is the variance of the photon number for the two-mode cavity light at steady state. On the basis of the result given by Eq. (3.140), we assert that $J(\omega)d\omega$ is the steady-state variance of the photon number for the two-mode cavity light in the interval between ω and $\omega + d\omega$. Furthermore, we see that

$$\langle \hat{n}(t), \hat{n}(t+\tau)\rangle = \langle \hat{n}(t)\hat{n}(t+\tau)\rangle - \langle \hat{n}(t)\rangle\langle \hat{n}(t+\tau)\rangle. \tag{3.142}$$

In view of Eqs. (3.138) and (3.139), Eq. (3.142) takes the form

$$\langle \hat{n}(t), \hat{n}(t+\tau)\rangle = \langle \hat{c}^{\dagger}(t)\hat{c}(t)\hat{c}^{\dagger}(t+\tau)\hat{c}(t+\tau)\rangle - \langle \hat{c}^{\dagger}(t)\hat{c}(t)\rangle\langle \hat{c}^{\dagger}(t+\tau)\hat{c}(t+\tau)\rangle. \tag{3.143}$$

Using the fact that $\hat{c}(t)$ is a Gaussian variable with zero mean, one can write Eq. (3.143) as

$$\langle \hat{n}(t), \hat{n}(t+\tau)\rangle = \langle \hat{c}^{\dagger}(t)\hat{c}^{\dagger}(t+\tau)\rangle\langle \hat{c}(t)\hat{c}(t+\tau)\rangle + \langle \hat{c}^{\dagger}(t)\hat{c}(t+\tau)\rangle\langle \hat{c}(t)\hat{c}^{\dagger}(t+\tau)\rangle. \tag{3.144}$$

We now proceed to determine the two-time correlation function that appears in Eq. (3.144). To this end, multiplying the adjoint of Eq. (3.127) from the left by $\hat{c}(t)$ and taking the expectation value of the resulting expression, we have

$$\begin{aligned}\langle \hat{c}(t)\hat{c}^{\dagger}(t+\tau)\rangle &= \langle \hat{c}(t)\hat{c}^{\dagger}(t)\rangle e^{-\frac{\kappa}{2}\tau} + \int_0^{\tau} e^{-\frac{\kappa}{2}(\tau-\tau')}\langle \hat{c}(t)\hat{F}_c^{\dagger}(t+\tau')\rangle d\tau' \\ &+ \frac{2g\langle \hat{c}(t)\hat{m}^{\dagger}(t)\rangle}{\sqrt{N}(\kappa-\mu)}\left[e^{-\frac{1}{2}\mu\tau} - e^{-\frac{1}{2}\kappa\tau}\right] + \frac{g}{\sqrt{N}}e^{-\frac{\kappa}{2}\tau}\int_0^{\tau} d\tau' e^{\frac{1}{2}(\kappa-\mu)\tau'} \\ &\times \int_0^{\tau} d\tau'' e^{\frac{1}{2}\mu\tau''}\left[-\langle \hat{c}(t)\hat{G}_a^{\dagger}(t+\tau'')\rangle + \langle \hat{c}(t)\hat{G}^{\dagger}(t+\tau'')\rangle\right].\end{aligned} \tag{3.145}$$

Taking into account the fact that a noise operator $\hat{F}$ at certain time should not affect the cavity mode operator at earlier time, one can write

$$\langle \hat{c}(t)\hat{F}_c^{\dagger}(t+\tau')\rangle = \langle \hat{c}(t)\hat{G}_a^{\dagger}(t+\tau'')\rangle = \langle \hat{c}(t)\hat{G}^{\dagger}(t+\tau'')\rangle = 0. \tag{3.146}$$

In view of this result, Eq. (3.145) takes the form

$$\langle \hat{c}(t)\hat{c}^{\dagger}(t+\tau)\rangle = \langle \hat{c}(t)\hat{c}^{\dagger}(t)\rangle e^{-\frac{\kappa}{2}\tau} + \frac{2g\langle \hat{c}(t)\hat{m}^{\dagger}(t)\rangle}{\sqrt{N}(\kappa-\mu)}\left[e^{-\frac{1}{2}\mu\tau} - e^{-\frac{1}{2}\kappa\tau}\right]. \tag{3.147}$$

We next seek to evaluate $\langle \hat{c}(t)\hat{m}^{\dagger}(t)\rangle$. To this end, multiplying the adjoint of Eq. (3.129) from the left by $\hat{c}(t)$ and taking the expectation value of the resulting expression, we get

$$\langle\hat{c}(t)\hat{m}^{\dagger}(t)\rangle = \frac{\kappa\sqrt{N}}{2g}\langle\hat{c}(t)\hat{c}^{\dagger}(t)\rangle - \frac{N}{g}\langle\hat{c}(t)\hat{F}_c^{\dagger}(t)\rangle. \tag{3.148}$$

On the other hand, multiplying Eq. (3.76) from the right by $\hat{F}_c^{\dagger}(t)$ and taking the expectation value of the resulting expression, we have

$$\langle\hat{c}(t)\hat{F}_c^{\dagger}(t)\rangle = \langle\hat{c}(0)\hat{F}_c^{\dagger}(t)\rangle e^{-\frac{\kappa}{2}t} + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\left[\frac{g}{\sqrt{N}}\langle\hat{m}(t')\hat{F}_c^{\dagger}(t)\rangle + \sqrt{N}\langle\hat{F}_c(t')\hat{F}_c^{\dagger}(t)\rangle\right]dt' \tag{3.149}$$

Using Eq. (3.65) along with the fact that a noise operator $\hat{F}$ at certain time should not affect an atomic and cavity mode operators at earlier time, one readily obtains

$$\langle\hat{c}(t)\hat{F}_c^{\dagger}(t)\rangle = \kappa\sqrt{N}. \tag{3.150}$$

Hence with the aid of Eqs. (3.147), (3.148), and (3.150), we easily get

$$\begin{aligned}\langle\hat{c}(t)\hat{c}^{\dagger}(t+\tau)\rangle &= \langle\hat{c}(t)\hat{c}^{\dagger}(t)\rangle\left[\frac{\kappa}{\kappa-\mu}e^{-\frac{1}{2}\mu\tau} - \frac{\mu}{\kappa-\mu}e^{-\frac{1}{2}\kappa\tau}\right]\\ &+ \frac{2N\kappa}{\kappa-\mu}\left[e^{-\frac{1}{2}\kappa\tau} - e^{-\frac{1}{2}\mu\tau}\right].\end{aligned} \tag{3.151}$$

Following a similar procedure, one can also readily establish that

$$\langle\hat{c}(t)\hat{c}(t+\tau)\rangle = \langle\hat{c}^2(t)\rangle\left[\frac{\kappa}{\kappa-\mu}e^{-\frac{1}{2}\mu\tau} - \frac{\mu}{\kappa-\mu}e^{-\frac{1}{2}\kappa\tau}\right]. \tag{3.152}$$

Now combination of Eqs. (3.131), (3.151), (3.152), and (3.144) results in

$$\begin{aligned}\langle\hat{n}(t),\hat{n}(t+\tau)\rangle &= \frac{(\Delta n)^2}{(\kappa-\mu)^2}\left[\kappa^2 e^{-\mu\tau} + \mu^2 e^{-\kappa\tau} - 2\kappa\mu e^{-\frac{1}{2}(\mu+\kappa)\tau}\right]\\ &+ \frac{2\bar{n}N\kappa}{(\kappa-\mu)^2}\left[(\kappa+\mu)e^{-\frac{1}{2}(\mu+\kappa)\tau} - \mu e^{-\kappa\tau} - \kappa e^{-\mu\tau}\right].\end{aligned} \tag{3.153}$$

On introducing Eq. (3.153) into Eq. (3.137) and carrying out the integration, we get

$$\begin{aligned}J(\omega) = &\frac{(\Delta n)^2}{(\kappa-\mu)^2}\left[\kappa^2\frac{\frac{\mu}{\pi}}{(\omega-\omega_0)^2+\mu^2} + \mu^2\frac{\frac{\kappa}{\pi}}{(\omega-\omega_0)^2+\kappa^2} - 2\kappa\mu\frac{\frac{\kappa+\mu}{2\pi}}{(\omega-\omega_0)^2+(\frac{\kappa+\mu}{2})^2}\right]\\ &+ \frac{2\bar{n}N\kappa}{(\kappa-\mu)^2}\left[(\kappa+\mu)\frac{\frac{\kappa+\mu}{2\pi}}{(\omega-\omega_0)^2+(\frac{\kappa+\mu}{2})^2} - \mu\frac{\frac{\kappa}{\pi}}{(\omega-\omega_0)^2+\kappa^2} - \kappa\frac{\frac{\mu}{\pi}}{(\omega-\omega_0)^2+\mu^2}\right],\end{aligned} \tag{3.154}$$

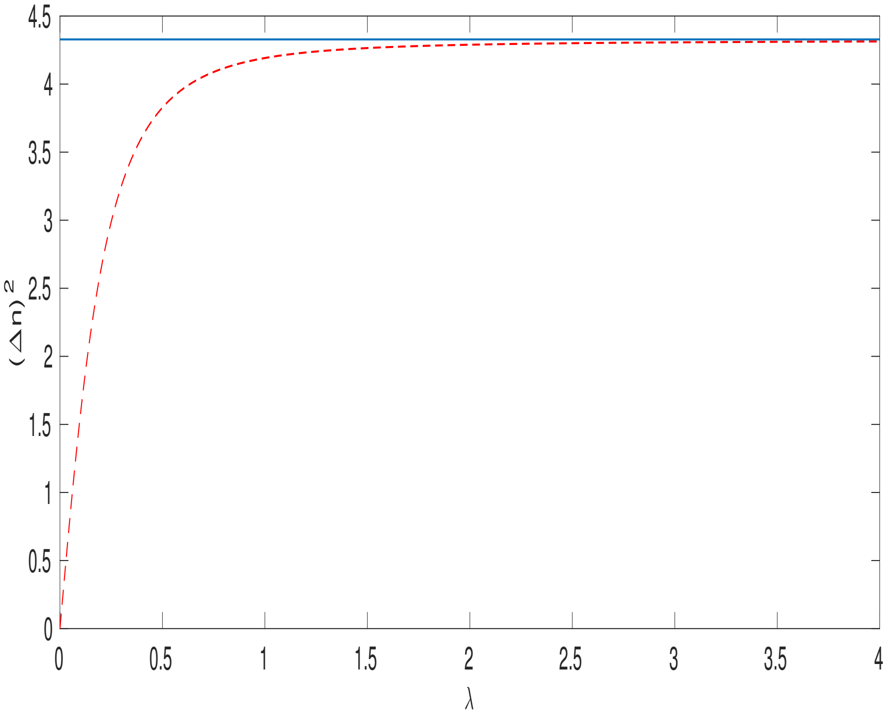

Figure 3.7: Plots of [Eq. (3.156)] for $\kappa = 0.2$, $\gamma = 0.2$, $\gamma_c = 1.2$, $\mu = 1.404$, $N = 5$, and $r_a = 0.002$.

in which $(\Delta n)^2$ is given by Eq. (3.118). We realize that the variance of the photon number in the interval between $\omega' = -\lambda$ and $\omega' = \lambda$ is expressible as

$$(\Delta n)^2_{\pm\lambda} = \int_{-\lambda}^{\lambda} J(\omega')d\omega' \tag{3.155}$$

in which $\omega' = \omega - \omega_0$. Substituting Eq. (3.154) into Eq. (3.155) and carrying out the integration, we readily get

$$(\Delta n)^2_{\pm\lambda} = \frac{(\Delta n)^2}{(\kappa-\mu)^2}\left[\frac{2\kappa^2}{\pi}tan^{-1}\left(\frac{\lambda}{\mu}\right) + \frac{2\mu^2}{\pi}tan^{-1}\left(\frac{\lambda}{\kappa}\right) - \frac{4\mu\kappa}{\pi}tan^{-1}\left(\frac{2\lambda}{\mu+\kappa}\right)\right] + \frac{2\bar{n}N\kappa}{(\kappa-\mu)^2}\left[\frac{2}{\pi}(\mu+\kappa)tan^{-1}\left(\frac{2\lambda}{\mu+\kappa}\right) - \frac{2\mu}{\pi}tan^{-1}\left(\frac{\lambda}{\kappa}\right) - \frac{2\kappa}{\pi}tan^{-1}\left(\frac{\lambda}{\mu}\right)\right]. \tag{3.156}$$

The plots in Fig. 3.7 indicate that as λ increases the local photon number variance of two-mode cavity light approaches to the global photon number variance of the two-mode cavity light. We observe that a large part of the total variance of photon number is confined in a relatively small frequency interval.

4

Quadrature Squeezing

In this chapter we seek to study the quadrature variance of single-mode light beams and the quadrature squeezing of the two-mode cavity light, produced by the system under consideration. Applying the steady-state solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langevin equations for the cavity mode operators, we obtain the global quadrature variance for light modes a and b. In addition, we determine the global and local quadrature squeezing of the two-mode cavity light.

4.1 Single-mode quadrature variance

In this section we wish to obtain the global quadrature variance of light modes a and b.

4.1.1 Global quadrature variance

Here we seek to calculate the quadrature variance of light modes a and b in the entire frequency interval. To this end, we define the quadrature operators for light mode a by

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a} \tag{4.1}$$

and

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}), \tag{4.2}$$

where $\hat{a}_+$ and $\hat{a}_-$ are Hermitian operators representing physical quantities called the plus and minus quadratures. Applying Eqs. (3.28), (4.1), and (4.2), one readily gets

$$[\hat{a}_-, \hat{a}_+] = 2i\frac{\gamma_c}{\kappa}(\hat{N}_a - \hat{N}_b) - 2iN. \tag{4.3}$$

It then follows that [8]

$$\Delta a_+ \Delta a_- \geq \left|\frac{\gamma_c}{\kappa}(\langle\hat{N}_a\rangle - \langle\hat{N}_b\rangle) - N\right|. \tag{4.4}$$

In view of Eq. (2.101), Eq. (4.4) takes the form

$$\Delta a_+ \Delta a_- \geq N. \tag{4.5}$$

Next we proceed to calculate the quadrature variance of light mode a. The variance of the plus and minus quadrature operators are defined by

$$(\Delta a_+)^2 = \langle\hat{a}_+^2\rangle - \langle\hat{a}_+\rangle^2 \tag{4.6}$$

$$(\Delta a_-)^2 = \langle\hat{a}_-^2\rangle - \langle\hat{a}_-\rangle^2. \tag{4.7}$$

On account of Eqs. (4.1) and (4.2), Eqs. (4.6) and (4.7) take the form

$$(\Delta a_+)^2 = \langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{a}^{\dagger 2}\rangle + \langle\hat{a}^2\rangle - \langle\hat{a}^\dagger\rangle^2 - \langle\hat{a}\rangle^2 - 2\langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle \tag{4.8}$$

and

$$(\Delta a_-)^2 = \langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}\hat{a}^\dagger\rangle - \langle\hat{a}^{\dagger 2}\rangle - \langle\hat{a}^2\rangle + \langle\hat{a}^\dagger\rangle^2 + \langle\hat{a}\rangle^2 - 2\langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle. \tag{4.9}$$

With the aid of Eqs. (4.8), (4.9), (3.48), and (3.50), the variance of the quadrature operators is expressible as

$$(\Delta a_+)^2 = \langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}\hat{a}^\dagger\rangle, \tag{4.10}$$

$$(\Delta a_-)^2 = \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a} \hat{a}^\dagger \rangle. \tag{4.11}$$

Now employing Eqs. (3.15), (3.25), and (2.101), we arrive at

$$(\Delta a_\pm)^2 = 2\bar{n}_a + N, \tag{4.12}$$

which is the quadrature variance of a light mode in a chaotic state. This indicates that light mode a is in a chaotic state. Following the same procedure, one can readily verify that the quadrature variance of light mode b has the form

$$(\Delta b_\pm)^2 = \bar{n}_b + \frac{\gamma + \gamma_c}{r_a}\bar{n}_b + N. \tag{4.13}$$

This reduces to

$$(\Delta b_\pm)^2 = \bar{n}_b + N \tag{4.14}$$

for $\gamma + \gamma_c \ll r_a$ and to

$$(\Delta b_\pm)^2 = 2\bar{n}_b + N \tag{4.15}$$

for $\gamma + \gamma_c = r_a$.

One can easily show that

$$[\hat{b}_-, \hat{b}_+] = 2i\frac{\gamma_c}{\kappa}(\hat{N}_b - \hat{N}_c) - 2iN. \tag{4.16}$$

On account of this commutation relation [8], we have

$$\Delta b_+ \Delta b_- \geq \left| \frac{\gamma_c}{\kappa}(\langle \hat{N}_b \rangle - \langle \hat{N}_c \rangle) - N \right|. \tag{4.17}$$

This uncertainity relation takes the form

$$\Delta b_+ \Delta b_- \geq \bar{n}_b - N \tag{4.18}$$

for $\gamma + \gamma_c \ll r_a$ and

$$\Delta b_+ \Delta b_- \geq N \tag{4.19}$$

for $\gamma + \gamma_c = r_a$. We observe from Eqs. (3.57) and (4.15) that light mode b is in a chaotic state for $\gamma + \gamma_c = r_a$. On the other hand, in the absence of cavity mode noise operators, Eqs. (4.14) and (4.18) reduces to [16]

$$(\Delta b_{\pm})^2 = \bar{n}_b \tag{4.20}$$

and

$$\Delta b_+ \Delta b_- \geq \bar{n}_b. \tag{4.21}$$

This indicates that light mode b is in coherent light for $\gamma + \gamma_c \ll r_a$.

4.2 Two-mode quadrature squeezing

In this section we seek to study the global and local quadrature squeezing of the two-mode cavity light.

4.2.1 Global quadrature squeezing

The squeezing properties of the two-mode cavity light are described by two quadrature operators defined by

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c} \tag{4.22}$$

and

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}), \tag{4.23}$$

where $\hat{c}_+$ and $\hat{c}_-$ are Hermitian operators representing physical quantities called the plus and minus quadratures. Using Eq. (3.59), one can readily get

$$[\hat{c}, \hat{c}^\dagger] = \frac{\gamma_c}{\kappa}[\hat{N}_c - \hat{N}_a] + 2N. \tag{4.24}$$

With the aid of Eqs. (4.22), (4.23), and (4.24), we can show that the plus and minus quadrature operators satisfy the commutation relation

$$[\hat{c}_-, \hat{c}_+] = 2i\frac{\gamma_c}{\kappa}(\hat{N}_a - \hat{N}_c) - 4iN. \tag{4.25}$$

It then follows that [1]

$$\Delta c_+ \Delta c_- \geq \left| \frac{\gamma_c}{\kappa}(\langle \hat{N}_c \rangle - \langle \hat{N}_a \rangle) + 2N \right|. \tag{4.26}$$

Upon setting $r_a = 0$, we see that

$$\Delta c_+ \Delta c_- \geq \frac{\gamma_c}{\kappa} N + 2N, \tag{4.27}$$

where Δc_+ and Δc_- are the uncertainties in the plus and minus quadratures.

Next we proceed to calculate the quadrature variance of the two-mode cavity light. To this end, the variance of the plus and minus quadrature operators of the two-mode cavity light are defined by

$$(\Delta c_+)^2 = \langle \hat{c}_+^2 \rangle - \langle \hat{c}_+ \rangle^2 \tag{4.28}$$

and

$$(\Delta c_-)^2 = \langle \hat{c}_-^2 \rangle - \langle \hat{c}_- \rangle^2. \tag{4.29}$$

On account of Eqs. (4.22) and (4.23), Eqs. (4.28) and (4.29) take the form

$$(\Delta c_+)^2 = \langle \hat{c}^\dagger \hat{c} \rangle + \langle \hat{c}\hat{c}^\dagger \rangle + \langle \hat{c}^{\dagger 2} \rangle + \langle \hat{c}^2 \rangle - \langle \hat{c}^\dagger \rangle^2 - \langle \hat{c} \rangle^2 - 2\langle \hat{c}^\dagger \rangle \langle \hat{c} \rangle \tag{4.30}$$

and

$$(\Delta c_-)^2 = \langle \hat{c}^\dagger \hat{c} \rangle + \langle \hat{c}\hat{c}^\dagger \rangle - \langle \hat{c}^{\dagger 2} \rangle - \langle \hat{c}^2 \rangle + \langle \hat{c}^\dagger \rangle^2 + \langle \hat{c} \rangle^2 - 2\langle \hat{c}^\dagger \rangle \langle \hat{c} \rangle. \tag{4.31}$$

With the aid of Eqs. (4.30), (4.31), and (3.58), the variance of the quadrature operators is expressible as

$$(\Delta c_+)^2 = \langle \hat{c}^\dagger \hat{c} \rangle + \langle \hat{c}\hat{c}^\dagger \rangle + \langle \hat{c}^{\dagger 2} \rangle + \langle \hat{c}^2 \rangle, \tag{4.32}$$

$$(\Delta c_-)^2 = \langle \hat{c}^\dagger \hat{c} \rangle + \langle \hat{c}\hat{c}^\dagger \rangle - \langle \hat{c}^{\dagger 2} \rangle - \langle \hat{c}^2 \rangle. \tag{4.33}$$

Now employing Eqs. (3.80), (3.81), (3.115), and the complex conjugate of (3.115), we arrive at

$$(\Delta c_+)^2 = \frac{\gamma_c}{\kappa}\left(N + \langle \hat{N}_a \rangle + 2\sqrt{\frac{\gamma + \gamma_c}{r_a}} \langle \hat{N}_a \rangle \right) + 2N, \tag{4.34}$$

$$(\Delta c_-)^2 = \frac{\gamma_c}{\kappa}\left(N + \langle \hat{N}_a \rangle - 2\sqrt{\frac{\gamma + \gamma_c}{r_a}}\langle \hat{N}_a \rangle\right) + 2N. \tag{4.35}$$

Moreover, on setting $r_a = 0$ in Eqs. (4.34) and (4.35), we get

$$(\Delta c_+)^2_v = (\Delta c_-)^2_v = \frac{\gamma_c}{\kappa}N + 2N. \tag{4.36}$$

This represents the quadrature variance of a two-mode cavity vacuum state. From Eqs. (4.27) and (4.36), we see that the two-mode cavity vacuum light is in a minimum uncertainty state.

Next we seek to calculate the quadrature squeezing of the two-mode cavity light in the entire frequency interval relative to the quadrature variance of the two-mode cavity vacuum state. We then define the quadrature squeezing of the two-mode cavity light by

$$S = \frac{(\Delta c_-)^2_v - (\Delta c_-)^2}{(\Delta c_-)^2_v}. \tag{4.37}$$

Now employing Eqs. (4.35) and (4.36), one can put Eq. (4.37) in the form

$$S = \frac{\frac{\gamma_c}{\kappa}\left(2\sqrt{\frac{\gamma+\gamma_c}{r_a}}\langle \hat{N}_a \rangle - \langle \hat{N}_a \rangle\right)}{\frac{\gamma_c}{\kappa}N + 2N}. \tag{4.38}$$

On account of Eq. (2.111), Eq. (4.38) takes the form

$$S = \frac{\gamma_c}{\gamma_c + 2\kappa}\left(\frac{2\sqrt{\frac{\gamma+\gamma_c}{r_a}} - 1}{\frac{\gamma+\gamma_c}{r_a} + 2}\right). \tag{4.39}$$

Finally, we consider the case in which spontaneous emission is absent $(\gamma = 0)$. Then the quadrature squeezing for this case takes the form

$$S = \frac{\gamma_c}{\gamma_c + 2\kappa}\left(\frac{2\sqrt{\frac{\gamma_c}{r_a}} - 1}{\frac{\gamma_c}{r_a} + 2}\right). \tag{4.40}$$

We note that, unlike the mean photon number and variance of the photon number, the quadrature squeezing does not depend on the number of atoms. This implies that the quadrature squeezing of the cavity light is independent of the number

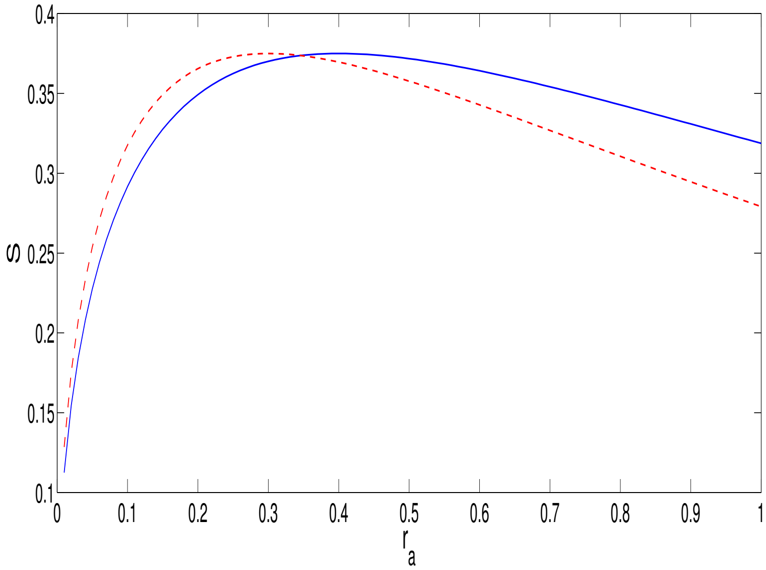

Figure 4.1: Plots of the quadrature squeezing at steady state, [Eq. (4.39)] versus r_a for $\kappa = 0.2$, $\gamma_c = 1.2$, $\gamma = 0$ (dashed curve), and for $\gamma = 0.4$ (solid curve).

of photons. The plots in Fig. 4.1 indicates that for small values of r_a, the degree of squeezing of the two-mode cavity light increases with r_a and for large values of r_a the quadrature squeezing decreases as r_a increases. The maximum quadrature squeezing is 37.5% both for $\gamma = 0$ and $\gamma = 0.4$. This occurs when the three-level laser is operating at $r_a = 0.30$ and $r_a = 0.40$ respectively. In addition, we can easily see from the same plots that the quadrature squeezing is greater for $\gamma = 0$ than that for $\gamma = 0.4$ for $0.01 < r_a < 0.35$ and is lesser for $\gamma = 0$ than that for $\gamma = 0.4$ for $0.35 < r_a < 1$. Our result shows that the maximum quadrature squeezing is less than the one obtained by Fesseha [16]. This must be due to the reservoir noise operators.

4.2.2 Local quadrature squeezing

We finally seek to obtain the local quadrature squeezing of the two-mode cavity light when both light modes a and b have the same frequency. To this end, we first determine the spectrum of quadrature fluctuations of the two-mode cavity light.

We define this spectrum by

$$S_{\pm}(\omega) = \frac{1}{\pi} Re \int_0^{\infty} d\tau e^{i(\omega-\omega_0)\tau} \langle \hat{c}_{\pm}(t), \hat{c}_{\pm}(t+\tau)\rangle_{ss}, \tag{4.41}$$

in which ω_0 is the central frequency of light mode a or b. Upon integrating both sides of Eq. (4.41) over ω, we get

$$\int_{-\infty}^{\infty} S_{\pm}(\omega) d\omega = (\Delta \hat{c}_{\pm}(t))^2, \tag{4.42}$$

in which

$$(\Delta \hat{c}_{\pm}(t))^2 = \langle \hat{c}_{\pm}(t), \hat{c}_{\pm}(t)\rangle_{ss} \tag{4.43}$$

is the quadrature variance of the light mode at steady state. On the basis of the result given by Eq. (4.42), we assert that $S_{\pm}(\omega)d\omega$ is the steady-state quadrature variance of the light mode in the interval between ω and $\omega + d\omega$ [1].

Furthermore, we recall that

$$\langle \hat{c}_{\pm}(t), \hat{c}_{\pm}(t+\tau)\rangle = \langle \hat{c}_{\pm}(t)\hat{c}_{\pm}(t+\tau)\rangle - \langle \hat{c}_{\pm}(t)\rangle\langle \hat{c}_{\pm}(t+\tau)\rangle. \tag{4.44}$$

Now using Eqs. (3.58), (4.22), and (4.23), one easily gets

$$\langle \hat{c}_{\pm}(t)\rangle = \langle \hat{c}_{\pm}(t+\tau)\rangle = 0. \tag{4.45}$$

On account of this result, Eq. (4.44) takes the form

$$\langle \hat{c}_{\pm}(t), \hat{c}_{\pm}(t+\tau)\rangle = \langle \hat{c}_{\pm}(t)\hat{c}_{\pm}(t+\tau)\rangle. \tag{4.46}$$

We now proceed to determine the two-time correlation function that appears in Eq. (4.46) for the cavity light. To this end, using Eqs. (4.22) and (4.23), one readily obtains

$$\langle \hat{c}_{+}(t)\hat{c}_{+}(t+\tau)\rangle = \langle \hat{c}^{\dagger}(t)\hat{c}^{\dagger}(t+\tau)\rangle + \langle \hat{c}(t)\hat{c}(t+\tau)\rangle + \langle \hat{c}^{\dagger}(t)\hat{c}(t+\tau)\rangle + \langle \hat{c}(t)\hat{c}^{\dagger}(t+\tau)\rangle \tag{4.47}$$

and

$$\langle \hat{c}_-(t)\hat{c}_-(t+\tau)\rangle = \langle \hat{c}^\dagger(t)\hat{c}(t+\tau)\rangle + \langle \hat{c}(t)\hat{c}^\dagger(t+\tau)\rangle - \langle \hat{c}^\dagger(t)\hat{c}^\dagger(t+\tau)\rangle - \langle \hat{c}(t)\hat{c}(t+\tau)\rangle. \quad (4.48)$$

Using Eqs. (3.131), (3.151), (3.152), (4.47), (4.48), and (4.46), we arrive at

$$\langle \hat{c}_\pm(t), \hat{c}_\pm(t+\tau)\rangle = (\Delta c_\pm)^2 \left[\frac{\kappa}{\kappa-\mu}e^{-\frac{1}{2}\mu\tau} - \frac{\mu}{\kappa-\mu}e^{-\frac{1}{2}\kappa\tau}\right] + \frac{2N\kappa}{\kappa-\mu}\left[e^{-\frac{1}{2}\kappa\tau} - e^{-\frac{1}{2}\mu\tau}\right]. \quad (4.49)$$

Now on introducing Eq. (4.49) into Eq. (4.41) and carrying out the integration, we find the spectrum of the minus quadrature fluctuations for the cavity light to be

$$\begin{aligned} S_-(\omega) = (\Delta c_-)^2 &\left[\frac{\kappa}{\kappa-\mu}\frac{\mu/\pi}{(\omega-\omega_0)^2+(\frac{\mu}{2})^2}\right] - \left[\frac{\mu}{\kappa-\mu}\frac{\kappa/2\pi}{(\omega-\omega_0)^2+(\frac{\kappa}{2})^2}\right] \\ &+ \frac{2N\kappa}{\kappa-\mu}\left[\frac{\frac{\kappa}{2\pi}}{(\omega-\omega_0)^2+(\frac{\kappa}{2})^2} - \frac{\frac{\mu}{2\pi}}{(\omega-\omega_0)^2+(\frac{\mu}{2})^2}\right], \end{aligned} \quad (4.50)$$

in which $(\Delta c_-)^2$ is given by Eq. (4.35).

We realize that the variance of the minus quadrature in the interval between ω' = $-\lambda$ and ω' = λ is expressible as

$$(\Delta c_-)^2_{\pm\lambda} = \int_{-\lambda}^{\lambda} S_-(\omega')d\omega', \quad (4.51)$$

in which ω'= $\omega - \omega_0$. Hence applying Eq. (4.50) and taking into account Eq. (4.35) along with Eq. (2.103), we easily get

$$\begin{aligned} (\Delta c_-)^2_{\pm\lambda} = &\left[\frac{2\kappa/\pi}{\kappa-(\gamma+\gamma_c+2r_a)}tan^{-1}\left(\frac{2\lambda}{\gamma+\gamma_c+2r_a}\right) - \frac{2(\gamma+\gamma_c+2r_a)/\pi}{\kappa-(\gamma+\gamma_c+2r_a)}tan^{-1}\left(\frac{2\lambda}{\kappa}\right)\right] \\ &\times\left(\frac{\gamma_c}{\kappa}\left(N+\langle\hat{N}_a\rangle - 2\sqrt{\frac{\gamma+\gamma_c}{r_a}}\langle\hat{N}_a\rangle\right)+2N\right) \\ &+\frac{4N\kappa/\pi}{\kappa-(\gamma+\gamma_c+2r_a)}\left[tan^{-1}\left(\frac{2\lambda}{\kappa}\right) - tan^{-1}\left(\frac{2\lambda}{(\gamma+\gamma_c+2r_a)}\right)\right]. \end{aligned} \quad (4.52)$$

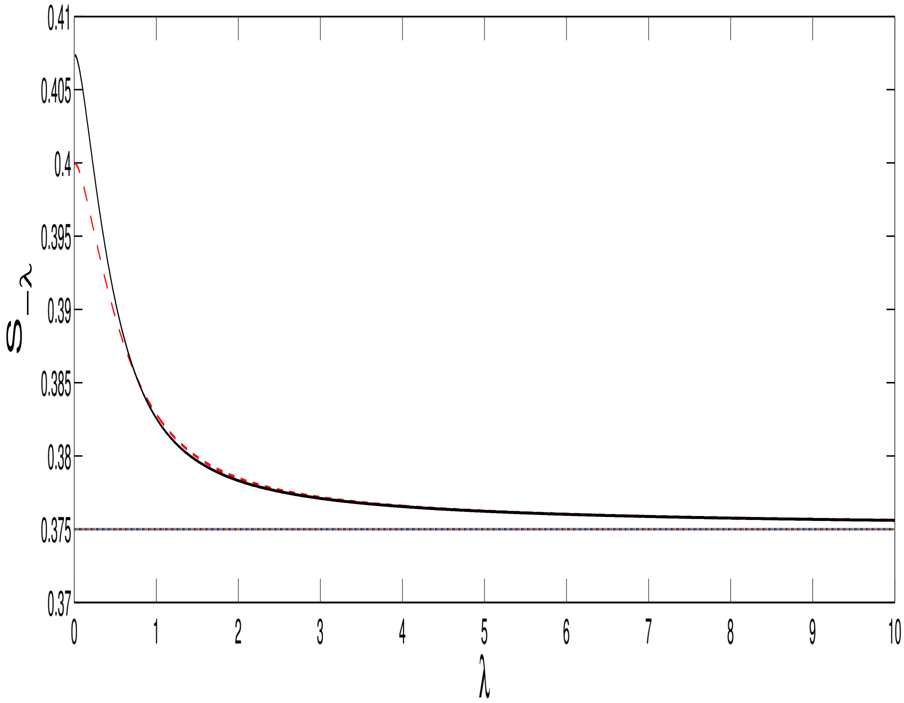

Figure 4.2: Plots of [Eq. (4.55)]versus λ for [$\gamma_c = 1.2$, $r_a = 0.30$] (solid curve), [$\gamma = 0.4$, $r_a = 0.40$](dashed curve), and $\kappa = 0.2$.

On account of Eq. (2.111), Eq. (4.52) can be rewritten as

$$(\Delta c_-)^2_{\pm\lambda} = \left[\frac{2\kappa/\pi}{\kappa-(\gamma+\gamma_c+2r_a)}tan^{-1}\left(\frac{2\lambda}{\gamma+\gamma_c+2r_a}\right) - \frac{2(\gamma+\gamma_c+2r_a)/\pi}{\kappa-(\gamma+\gamma_c+2r_a)}tan^{-1}\left(\frac{2\lambda}{\kappa}\right)\right]$$
$$\times\left(\frac{\gamma_c}{\kappa}\left(1+\frac{r_a-2\sqrt{(\gamma+\gamma_c)r_a}}{\gamma+\gamma_c+2r_a}\right)+2\right)N$$
$$+\frac{4N\kappa/\pi}{\kappa-(\gamma+\gamma_c+2r_a)}\left[tan^{-1}\left(\frac{2\lambda}{\kappa}\right)-tan^{-1}\left(\frac{2\lambda}{(\gamma+\gamma_c+2r_a)}\right)\right]. \tag{4.53}$$

Furthermore, upon setting $r_a = 0$ in Eq. (4.53), we find the local quadrature variance of the two-mode vacuum state to be

$$(\Delta c_-)^2_{v\pm\lambda} = \left[\frac{2\kappa/\pi}{\kappa-(\gamma+\gamma_c)}tan^{-1}\left(\frac{2\lambda}{\gamma+\gamma_c}\right) - \frac{2(\gamma+\gamma_c)/\pi}{\kappa-(\gamma+\gamma_c)}tan^{-1}\left(\frac{2\lambda}{\kappa}\right)\right]$$
$$\times\left(\frac{\gamma_c}{\kappa}N+2N\right)+\frac{4N\kappa/\pi}{\kappa-(\gamma+\gamma_c)}\left[tan^{-1}\left(\frac{2\lambda}{\kappa}\right)-tan^{-1}\left(\frac{2\lambda}{(\gamma+\gamma_c)}\right)\right]. \tag{4.54}$$

We define the quadrature squeezing of the cavity light in the $\lambda_\pm$ frequency interval by

$$S_{\pm\lambda} = \frac{(\Delta c_-)^2_{v\pm\lambda} - (\Delta c_-)^2_{\pm\lambda}}{(\Delta c_-)^2_{v\pm\lambda}}. \tag{4.55}$$

The plots in Fig. 4.2 indicate that the maximum local quadrature squeezing is 40.74% and 40% for $\gamma = 0$ and $\gamma = 0.4$, respectively and happens to be in the $\lambda = \pm 0.01$ frequency interval. Moreover, the plots in Fig. 4.2 indicate that the effect of the spontaneous emission is to decrease the local quadrature squeezing. We also notice that as λ increases the local quadrature squeezing in both cases approaches the global quadrature squeezing. In addition, we have also seen that the local quadrature squeezing of the cavity light is greater than the global quadrature squeezing.

5

Superposed Two-mode Laser Light Beams

In this chapter we seek to study the squeezing and statistical properties of a pair of superposed two-mode laser light beams. To this end, we first obtain the Q function with the aid of the antinormally-ordered characteristic function defined in the Heisenberg picture for a two-mode laser light beam. Then using the resulting Q function, we obtain an expression for the density operator for a pair of superposed two-mode laser light beams. Applying this density operator, we calculate the photon statistics and quadrature squeezing.

5.1 The Q function

With the aid of the completeness relation given by [1]

$$\hat{I} = \frac{\lambda}{\pi}\int d^2\beta|\beta\rangle\langle\beta|, \tag{5.1}$$

the antinormally-ordered characteristic function for a two-mode laser light beam, defined by

$$\phi_a(z,t) = Tr\left(\hat{\rho}e^{-z^*\hat{c}(t)}e^{z\hat{c}^\dagger(t)}\right), \tag{5.2}$$

can be rewritten as

$$\phi_a(z,t) = \frac{\lambda}{\pi}\int d^2\beta Tr\left(\hat{\rho}e^{-z^*\hat{c}(t)}|\beta\rangle\langle\beta|e^{z\hat{c}^\dagger(t)}\right). \tag{5.3}$$

Moreover, employing the relation [1]

$$\hat{c}|\beta\rangle = \lambda\beta|\beta\rangle, \tag{5.4}$$

we obtain

$$\phi_a(z,t) = \int d^2\beta \lambda Q(\lambda\beta) exp(z\lambda\beta^* - z^*\lambda\beta), \tag{5.5}$$

where $Q(\lambda\beta)$ is the Q function. Introducing the variable α= $\lambda\beta$, we easily find

$$\phi_a(z,t) = \int d^2\alpha \frac{Q(\alpha)}{\lambda} exp(z\alpha^* - z^*\alpha). \tag{5.6}$$

Since $\frac{Q(\alpha)}{\lambda}$ is the inverse Fourier transform of the characteristic function, we see that

$$Q(\alpha) = \frac{\lambda}{\pi^2} \int d^2z \phi_a(z) exp(z^*\alpha - z\alpha^*). \tag{5.7}$$

Upon integrating both sides of Eq. (5.7) over α and taking into account the fact that

$$\frac{1}{\pi^2} \int d^2\alpha exp(z^*\alpha - z\alpha^*) = \delta^2(z), \tag{5.8}$$

we arrive at

$$\int d^2\alpha Q(\alpha) = \lambda \int d^2z Tr\left(\hat{\rho} e^{-z^*\hat{c}(t)} e^{z\hat{c}^\dagger(t)}\right) \delta^2(z), \tag{5.9}$$

from which follows

$$\int d^2\alpha Q(\alpha) = \lambda. \tag{5.10}$$

This shows that the Q function is normalized to λ.

We now proceed to obtain the explicit form of the antinormally-ordered characteristic function for the two-mode laser light beam. Upon replacing the atomic operators that appear in Eq. (4.24) by their expectation values, the commutation relation for the light generated by the three-level laser can be written as

$$[\hat{c}, \hat{c}^\dagger] = \lambda, \tag{5.11}$$

in which

$$\lambda = \frac{\gamma_c}{\kappa}[\langle\hat{N}_c\rangle - \langle\hat{N}_a\rangle] + 2N. \tag{5.12}$$

Now applying the Baker-Hausdorff identity

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}, \tag{5.13}$$

which holds for

$$[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0, \tag{5.14}$$

we see that

$$e^{-z^*\hat{c}(t)}e^{z\hat{c}^\dagger(t)} = e^{-z^*\hat{c}(t)+z\hat{c}^\dagger(t)-\frac{1}{2}z^*z\lambda}. \tag{5.15}$$

Then substituting (5.15) into Eq. (5.2), we obtain

$$\phi_a(z,t) = e^{-\frac{1}{2}z^*z\lambda}\langle e^{-z^*\hat{c}(t)+z\hat{c}^\dagger(t)}\rangle. \tag{5.16}$$

Since $\hat{c}(t)$ is a Gaussian variable with zero mean, we can rewrite Eq. (5.16) as [1]

$$\phi_a(z,t) = e^{-\frac{1}{2}z^*z\lambda}exp\left[\frac{1}{2}\big\langle(z\hat{c}^\dagger(t) - z^*\hat{c}(t))^2\big\rangle\right]. \tag{5.17}$$

It then follows that

$$\phi_a(z,t) = exp\left[\frac{1}{2}\bigg(z^2\langle\hat{c}^{\dagger 2}(t)\rangle + z^{*2}\langle\hat{c}^2(t))\rangle - z^*z\big(\lambda + \langle\hat{c}^\dagger(t)\hat{c}(t)\rangle + \langle\hat{c}(t)\hat{c}^\dagger(t)\rangle\big)\bigg)\right]. \tag{5.18}$$

Since Eq (3.115) shows that $\langle\hat{c}^2(t)\rangle$ is real , we see that $\langle\hat{c}^{\dagger 2}(t)\rangle$=$\langle\hat{c}^2(t)\rangle$. Thus Eq (5.18) takes the form

$$\phi_a(z,t) = exp\left[\frac{1}{2}\bigg(\langle\hat{c}^2(t)\rangle(z^2 + z^{*2}) - z^*z\big(\lambda + \langle\hat{c}^\dagger(t)\hat{c}(t)\rangle + \langle\hat{c}(t)\hat{c}^\dagger(t)\rangle\big)\bigg)\right]. \tag{5.19}$$

Now with the aid of Eqs. (3.80), (3.81), (3.115), and (5.12), the antinormally-ordered characteristic function can be put in the form

$$\phi_a(z,t) = exp\left[\frac{1}{2}\frac{\gamma_c}{\kappa}\sqrt{\frac{\gamma+\gamma_c}{r_a}}\langle\hat{N}_a\rangle(z^2 + z^{*2}) - z^*z\big(\frac{\gamma_c}{\kappa}(\langle\hat{N}_b\rangle + \langle\hat{N}_c\rangle) + 2N\big)\right]. \tag{5.20}$$

This can be rewritten as

$$\phi_a(z,t) = exp\left[-Tz^*z + R\left(\frac{z^2}{2} + \frac{z^{*2}}{2}\right)\right], \tag{5.21}$$

in which

$$T = \frac{\gamma_c}{\kappa}(\langle\hat{N}_b\rangle + \langle\hat{N}_c\rangle) + 2N \tag{5.22}$$

and

$$R = \frac{\gamma_c}{\kappa}\sqrt{\frac{\gamma+\gamma_c}{r_a}}\langle\hat{N}_a\rangle. \tag{5.23}$$

Now introducing Eq. (5.21) into Eq. (5.7), the Q function for the two-mode laser light beam can be written as

$$Q(\alpha) = \frac{\lambda}{\pi^2}\int d^2z exp\left[-Tz^*z + z^*\alpha - z\alpha^* + R\left(\frac{z^2}{2} + \frac{z^{*2}}{2}\right)\right]. \tag{5.24}$$

Carrying out the integration, using the relation

$$\int \frac{d^2z}{\pi}e^{-az^*z+bz+cz^*+Az^2+Bz^{*2}} = \sqrt{\frac{1}{a^2-4AB}}exp\left[\frac{abc+Ac^2+Bb^2}{a^2-4AB}\right], a>0, \tag{5.25}$$

we get

$$Q(\alpha) = \frac{\lambda}{\pi}\left[u^2-v^2\right]^{1/2}exp\left[-u\alpha^*\alpha + v\left(\frac{\alpha^2}{2}+\frac{\alpha^{*2}}{2}\right)\right], \tag{5.26}$$

in which

$$u = \frac{T}{T^2-R^2} \tag{5.27}$$

and

$$v = \frac{R}{T^2-R^2}. \tag{5.28}$$

We note that Eq. (5.26) is the Q function for the two-mode laser light beam.

According to Ref. [1], the density operator for the pair of superposed two-mode laser light beams has the form

$$\hat{\rho} = \lambda^2\int d^2\beta d^2\gamma Q(\lambda\beta^*, \lambda\beta+\partial/\partial\beta^*)Q(\lambda\gamma^*, \lambda\gamma+\partial/\partial\gamma^*)|\beta+\gamma\rangle\langle\gamma+\beta|. \tag{5.29}$$

Suppose $\hat{d}$ and $\hat{d}^\dagger$ represent the annihilation and creation operators for the superposed two-mode light beams. Then the expectation value of the annihilation operator for the superposed two-mode light beams can be expressed as

$$\langle \hat{d} \rangle = Tr(\hat{\rho}\hat{d}). \tag{5.30}$$

Now applying the density operator given by Eq. (5.29), Eq. (5.30) can be rewritten as

$$\langle \hat{d} \rangle = \lambda^2 \int d^2\beta d^2\gamma Q(\lambda\beta^*, \lambda\beta + \partial/\partial\beta^*)Q(\lambda\gamma^*, \lambda\gamma + \partial/\partial\gamma^*)Tr(\hat{d}|\beta + \gamma\rangle\langle\gamma + \beta|). \tag{5.31}$$

We then see that

$$\langle \hat{d} \rangle = \lambda^3 \int d^2\beta d^2\gamma Q(\lambda\beta^*, \lambda\beta + \partial/\partial\beta^*)Q(\lambda\gamma^*, \lambda\gamma + \partial/\partial\gamma^*)(\beta + \gamma). \tag{5.32}$$

Now introducing the variables $\alpha_1 = \lambda\beta$ and $\alpha_2 = \lambda\gamma$, Eq. (5.32) takes the form

$$\langle \hat{d} \rangle = \frac{1}{\lambda^2} \int d^2\alpha_1 d^2\alpha_2 Q(\alpha_1^*, \alpha_1 + \lambda\partial/\partial\alpha_1^*)Q(\alpha_2^*, \alpha_2 + \lambda\partial/\partial\alpha_1^*)(\alpha_1 + \alpha_2). \tag{5.33}$$

This can be rewritten as

$$\begin{aligned} \langle \hat{d} \rangle &= \frac{1}{\lambda} \int d^2\alpha_2 Q(\alpha_2^*, \alpha_2 + \lambda\partial/\partial\alpha_2^*) \\ &\times \frac{1}{\lambda} \int d^2\alpha_1 Q(\alpha_1^*, \alpha_1 + \lambda\partial/\partial\alpha_1^*)\alpha_1 \\ &+ \frac{1}{\lambda} \int d^2\alpha_1 Q(\alpha_1^*, \alpha_1 + \lambda\partial/\partial\alpha_1^*) \\ &\times \frac{1}{\lambda} \int d^2\alpha_2 Q(\alpha_2^*, \alpha_2 + \lambda\partial/\partial\alpha_2^*)\alpha_2. \end{aligned} \tag{5.34}$$

Using the fact that the Q function is normalized to λ, we have

$$\begin{aligned} \langle \hat{d} \rangle &= \frac{1}{\lambda} \int d^2\alpha_1 Q(\alpha_1^*, \alpha_1 + \lambda\partial/\partial\alpha_1^*)\alpha_1 \\ &+ \frac{1}{\lambda} \int d^2\alpha_2 Q(\alpha_2^*, \alpha_2 + \lambda\partial/\partial\alpha_2^*)\alpha_2. \end{aligned} \tag{5.35}$$

Furthermore, following the discussion presented in [1], one can express the expectation value of a given operator function $A(\hat{c}^\dagger, \hat{c})$ in the form

$$\langle \hat{A} \rangle = \int \frac{d^2\xi}{\lambda} Q(\xi^*, \xi + \lambda\partial/\partial\xi^*)A_n(\xi^*, \xi), \tag{5.36}$$

where $A_n(\xi^*, \xi)$ is the c-number function corresponding to $\hat{A}$ in the normal order, and $\xi = \lambda\beta$. On account of Eq. (5.36), Eq. (5.35) takes the form

$$\langle \hat{d} \rangle = \langle \hat{c}_1 \rangle + \langle \hat{c}_2 \rangle. \tag{5.37}$$

Now in view of Eq. (3.60), the equations of evolution of the cavity mode operators for the two-mode laser light beams can be expressed as

$$\frac{d\hat{c}_1}{dt} = -\frac{\kappa}{2}\hat{c}_1 + \frac{g}{\sqrt{N}}\hat{m}_1 + \sqrt{N}\hat{F}_1(t) \tag{5.38}$$

and

$$\frac{d\hat{c}_2}{dt} = -\frac{\kappa}{2}\hat{c}_2 + \frac{g}{\sqrt{N}}\hat{m}_2 + \sqrt{N}\hat{F}_2(t). \tag{5.39}$$

Then adding Eqs. (5.38) and (5.39), we get

$$\frac{d\hat{d}}{dt} = -\frac{\kappa}{2}\hat{d} + \frac{g}{\sqrt{N}}\hat{m}' + \sqrt{N}\hat{F}(t), \tag{5.40}$$

where

$$\hat{d} = \hat{c}_1 + \hat{c}_2, \tag{5.41}$$

$$\hat{m}' = \hat{m}_1 + \hat{m}_2, \tag{5.42}$$

and

$$\hat{F} = \hat{F}_1 + \hat{F}_2. \tag{5.43}$$

Furthermore, using Eq. (3.58) along with Eq. (5.37) , we have

$$\langle \hat{d} \rangle = 0. \tag{5.44}$$

In view of the linear equation described by Eq. (5.40) and the result given by Eq. (5.44), we claim that $\hat{d}(t)$ is a Gaussian variable with zero mean [1]. Moreover, using Eq. (4.24) along with Eq. (5.41), one can easily establish that

$$[\hat{d}, \hat{d}^\dagger] = 2\lambda, \tag{5.45}$$

where λ is given by Eq. (5.12).

5.2 Photon statistics

In this section, applying the density operator for the pair of superposed two-mode laser light beams, we seek to calculate the mean and variance of the photon number.

5.2.1 Mean photon number

Now applying the density operator given by Eq. (5.29), the mean photon number for the pair of superposed two-mode laser light beams can be expressed as

$$\bar{n}_s = \lambda^2 \int d^2\beta d^2\gamma Q(\lambda\beta^*, \lambda\beta + \partial/\partial\beta^*)Q(\lambda\gamma^*, \lambda\gamma + \partial/\partial\gamma^*)Tr(\hat{d}^\dagger\hat{d}|\beta+\gamma\rangle\langle\gamma+\beta|). \quad (5.46)$$

We then see that

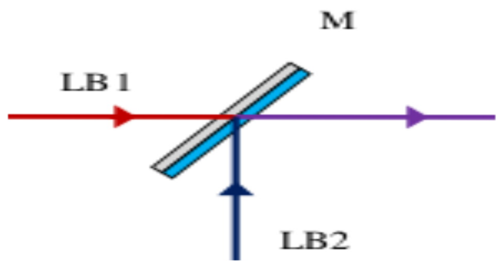

Figure 5.1: The superposed two-mode laser light beams, with $\kappa = 1$ and $\kappa = 0$ for the upper and lower surfaces of the mirror, respectively.

$$\bar{n}_s = \lambda^4 \int d^2\beta d^2\gamma Q(\lambda\beta^*, \lambda\beta + \partial/\partial\beta^*)Q(\lambda\gamma^*, \lambda\gamma + \partial/\partial\gamma^*)\Big(\beta\beta^* + \gamma\gamma^* + \beta^*\gamma + \gamma^*\beta\Big). \quad (5.47)$$

Next introducing the variables $\alpha_1 = \lambda\beta$ and $\alpha_2 = \lambda\gamma$, Eq. (5.47) can be rewritten as

$$\begin{aligned}\bar{n}_s = \frac{1}{\lambda^2}\int d^2\alpha_1 d^2\alpha_2 Q(\alpha_1^*, \alpha_1 + \lambda\partial/\partial\alpha_1^*)Q(\alpha_2^*, \alpha_2 + \lambda\partial/\partial\alpha_2^*) \\ \times\left(\alpha_1\alpha_1^* + \alpha_2\alpha_2^* + \alpha_1^*\alpha_2 + \alpha_2^*\alpha_1\right).\end{aligned} \tag{5.48}$$

This can be put in the form

$$\begin{aligned}\bar{n}_s = &\frac{1}{\lambda}\int d^2\alpha_2 Q(\alpha_2^*, \alpha_2 + \lambda\partial/\partial\alpha_2^*) \\ &\times \frac{1}{\lambda}\int d^2\alpha_1 Q(\alpha_1^*, \alpha_1 + \lambda\partial/\partial\alpha_1^*)\alpha_1^*\alpha_1 \\ &+ \frac{1}{\lambda}\int d^2\alpha_1 Q(\alpha_1^*, \alpha_1 + \lambda\partial/\partial\alpha_1^*) \\ &\times \frac{1}{\lambda}\int d^2\alpha_2 Q(\alpha_2^*, \alpha_2 + \lambda\partial/\partial\alpha_2^*)\alpha_2^*\alpha_2 \\ &+ \frac{1}{\lambda}\int d^2\alpha_1 Q(\alpha_1^*, \alpha_1 + \lambda\partial/\partial\alpha_1^*)\alpha_1^* \\ &\times \frac{1}{\lambda}\int d^2\alpha_2 Q(\alpha_2^*, \alpha_2 + \lambda\partial/\partial\alpha_2^*)\alpha_2 \\ &+ \frac{1}{\lambda}\int d^2\alpha_2 Q(\alpha_2^*, \alpha_2 + \lambda\partial/\partial\alpha_2^*)\alpha_2^* \\ &\times \frac{1}{\lambda}\int d^2\alpha_1 Q(\alpha_1^*, \alpha_1 + \lambda\partial/\partial\alpha_1^*)\alpha_1.\end{aligned} \tag{5.49}$$

Using Eq. (5.36) along with the fact that the Q function is normalized to λ, we see that

$$\bar{n}_s = \langle\hat{c}_1^\dagger\hat{c}_1\rangle + \langle\hat{c}_2^\dagger\hat{c}_2\rangle + \langle\hat{c}_1^\dagger\rangle\langle\hat{c}_2\rangle + \langle\hat{c}_2^\dagger\rangle\langle\hat{c}_1\rangle. \tag{5.50}$$

Taking into account the fact that the Q functions of the two identical two-mode laser light beams have exactly the same form, we observe that

$$\langle\hat{c}_1^\dagger\hat{c}_1\rangle = \langle\hat{c}_2^\dagger\hat{c}_2\rangle \tag{5.51}$$

and

$$\langle\hat{c}_1\rangle = \langle\hat{c}_2\rangle. \tag{5.52}$$

Hence the mean photon number for the pair of superposed two-mode laser light beams is expressible as

$$\bar{n}_s = 2\langle\hat{c}_1^\dagger\hat{c}_1\rangle + 2\langle\hat{c}_1^\dagger\rangle\langle\hat{c}_1\rangle. \tag{5.53}$$

We next proceed to evaluate the expectation values involved in this expression. Thus with the aid of Eq. (5.26) along with Eq. (5.36), one can write

$$\langle\hat{c}_1\rangle = \left(u^2 - v^2\right)^{1/2} \int \frac{d^2\alpha_1}{\pi} exp\left[- u\alpha_1^*\alpha_1 + v\left(\frac{\alpha_1^2}{2} + \frac{\alpha_1^{*2}}{2}\right)\right] \times exp\left[- u\lambda\alpha_1^*\partial/\partial\alpha_1^* + v\lambda\alpha_1\partial/\partial\alpha_1^* + \frac{v}{2}\lambda^2\partial^2/\partial\alpha_1^{*2}\right]\alpha_1. \quad (5.54)$$

Now upon expanding the exponential function $e^{-u\lambda\alpha_1^*\partial/\partial\alpha_1^*+v\lambda\alpha_1\partial/\partial\alpha_1^*+\frac{v}{2}\lambda^2\partial^2/\partial\alpha_1^{*2}}$ in a power series, we have

$$e^{-u\lambda\alpha_1^*\partial/\partial\alpha_1^*+v\lambda\alpha_1\partial/\partial\alpha_1^*+\frac{v}{2}\lambda^2\partial^2/\partial\alpha_1^{*2}} = 1-u\lambda\alpha_1^*\partial/\partial\alpha_1^*+v\lambda\alpha_1\partial/\partial\alpha_1^*+\frac{v}{2}\lambda^2\partial^2/\partial\alpha_1^{*2}+... \quad (5.55)$$

We then easily find

$$\langle\hat{c}_1\rangle = \left(u^2 - v^2\right)^{1/2} \int \frac{d^2\alpha_1}{\pi} exp\left[- u\alpha_1^*\alpha_1 + v\left(\frac{\alpha_1^2}{2} + \frac{\alpha_1^{*2}}{2}\right)\right]\alpha_1 \quad (5.56)$$

and on carrying out the integration, we arrive at

$$\langle\hat{c}_1\rangle = 0. \quad (5.57)$$

In view of this result, Eq. (5.53) reduces to

$$\bar{n}_s = 2\langle\hat{c}_1^\dagger\hat{c}_1\rangle. \quad (5.58)$$

Moreover, applying Eq. (5.26) together with Eq. (5.36), the expectation value of $\hat{c}_1^\dagger\hat{c}_1$ can be put in the form

$$\langle\hat{c}_1^\dagger\hat{c}_1\rangle = \left(u^2 - v^2\right)^{1/2} \int \frac{d^2\alpha_1}{\pi} exp\left[- u\alpha_1^*\alpha_1 + v\left(\frac{\alpha_1^2}{2} + \frac{\alpha_1^{*2}}{2}\right)\right] \times exp\left[- u\lambda\alpha_1^*\partial/\partial\alpha_1^* + v\lambda\alpha_1\partial/\partial\alpha_1^* + \frac{v}{2}\lambda^2\partial^2/\partial\alpha_1^{*2}\right]\alpha_1^*\alpha_1. \quad (5.59)$$

Hence applying the power series given by Eq. (5.55), we readily obtain

$$\langle\hat{c}_1^\dagger\hat{c}_1\rangle = \frac{1}{\pi}\left(u^2 - v^2\right)^{1/2} \int d^2\alpha_1 exp\left[-u\alpha_1^*\alpha_1 + v\left(\frac{\alpha^2}{2} + \frac{\alpha^{*2}}{2}\right)\right] \times \left[(1 - u\lambda)\alpha_1^*\alpha_1 + v\lambda\alpha_1^2\right]. \quad (5.60)$$

This can be rewritten as

$$\begin{aligned}\langle\hat{c}_1^\dagger\hat{c}_1\rangle = &\left(u^2-v^2\right)^{1/2}(1-\lambda u)\frac{\partial^2}{\partial z\partial z^*}\int\frac{d^2\alpha_1}{\pi}\\&\times exp\left[-u\alpha_1^*\alpha_1+z^*\alpha_1+z\alpha_1^*+v\left(\frac{\alpha_1^2}{2}+\frac{\alpha_1^{*2}}{2}\right)\right]\Bigg|_{z=z^*=0}\\&+\left(u^2-v^2\right)^{1/2}(v\lambda)\frac{\partial^2}{\partial z^{*2}}\int\frac{d^2\alpha_1}{\pi}\\&\times exp\left[-u\alpha_1^*\alpha_1+z^*\alpha_1+z\alpha_1^*+v\left(\frac{\alpha_1^2}{2}+\frac{\alpha_1^{*2}}{2}\right)\right]\Bigg|_{z=z^*=0},\end{aligned} \tag{5.61}$$

so that carrying out the integration over α_1, we get

$$\begin{aligned}\langle\hat{c}_1^\dagger\hat{c}_1\rangle = &(1-\lambda u)\frac{\partial^2}{\partial z\partial z^*}exp\left[\frac{rzz^*+v(\frac{z^2}{2}+\frac{z^{*2}}{2})}{u^2-v^2}\right]\Bigg|_{z=z^*=0}\\&+v\lambda\frac{\partial^2}{\partial z^{*2}}exp\left[\frac{uzz^*+v(\frac{z^2}{2}+\frac{z^{*2}}{2})}{u^2-v^2}\right]\Bigg|_{z=z^*=0}.\end{aligned} \tag{5.62}$$

Now performing the differentiation and applying the condition $z = z^* = 0$, we have

$$\langle\hat{c}_1^\dagger\hat{c}_1\rangle = \frac{u}{u^2-v^2}(1-\lambda u)+\frac{v^2\lambda}{u^2-v^2}. \tag{5.63}$$

Hence on account of Eqs. (5.27) and (5.28), there follows

$$\langle\hat{c}_1^\dagger\hat{c}_1\rangle = T-\lambda \tag{5.64}$$

and in view of Eqs. (5.12) and (5.22), we see that

$$\langle\hat{c}_1^\dagger\hat{c}_1\rangle = \frac{\gamma_c}{\kappa}(\langle\hat{N}_a\rangle+\langle\hat{N}_b\rangle). \tag{5.65}$$

Now with the aid of Eq. (5.58), the mean photon number for the pair of superposed two-mode laser light beams is found to be

$$\bar{n}_s = 2\frac{\gamma_c}{\kappa}(\langle\hat{N}_a\rangle+\langle\hat{N}_b\rangle). \tag{5.66}$$

This can be expressed as

$$\bar{n}_s = 2\bar{n}, \tag{5.67}$$

where $\bar{n}$ is the mean photon number for a two-mode laser light beam given by Eq. Eq. (3.80). We note that the mean photon number for the pair of superposed two-mode laser light beams is the sum of the mean photon numbers of the constituent two-mode laser light beams. Moreover, on account of Eqs. (2.101) and (2.111), the mean photon number given by Eq. (5.66) can be put in the form

$$\bar{n}_s = 4N\frac{\gamma_c}{\kappa}\left[\frac{r_a}{\gamma+\gamma_c+2r_a}\right]. \tag{5.68}$$

This is the steady-state mean photon number for the pair of superposed two-mode laser light beams.

We next proceed to consider the case in which spontaneous emission is absent ($\gamma = 0$). Then the mean photon number for this case takes the form

$$\bar{n}_s = 4N\frac{\gamma_c}{\kappa}\left[\frac{r_a}{\gamma_c+2r_a}\right]. \tag{5.69}$$

This can be expressed as

$$\bar{n}_s = 2\bar{n}, \tag{5.70}$$

where $\bar{n}$ is the mean photon number for a two-mode laser light beam given by Eq. (3.84). In addition, the mean photon number for the pair of superposed two-mode laser light beams reduces to

$$\bar{n} = 2N\frac{\gamma_c}{\kappa} \tag{5.71}$$

for $r_a \gg \gamma + \gamma_c$ and to

$$\bar{n} = \frac{\gamma_c}{\kappa}\left[\frac{4N}{3}\right] \tag{5.72}$$

for $r_a = \gamma + \gamma_c$. From Eqs. (5.71) and (5.72), we see that the mean photon number for the pair of superposed two-mode laser light beams is less for $r_a = \gamma + \gamma_c$ than that for $r_a \gg \gamma + \gamma_c$.

5.2.2 Photon-number variance

Applying the density operator given by Eq. (5.29), we calculate the photon number variance for the pair of superposed two-mode laser light beams. The variance of the photon number for the pair of superposed two-mode laser light beams is expressible as

$$(\Delta n)_s^2 = \langle \hat{d}^\dagger \hat{d} \hat{d}^\dagger \hat{d} \rangle - \langle \hat{d}^\dagger \hat{d} \rangle^2. \tag{5.73}$$

Using the fact that $\hat{d}$ is a Gaussian variable with zero mean, we readily get

$$(\Delta n)_s^2 = \langle \hat{d}^\dagger \hat{d} \rangle \langle \hat{d} \hat{d}^\dagger \rangle + \langle \hat{d}^{\dagger 2} \rangle \langle \hat{d}^2 \rangle. \tag{5.74}$$

This can be rewritten as

$$(\Delta n)_s^2 = \bar{n}_s \langle \hat{d} \hat{d}^\dagger \rangle + \langle \hat{d}^{\dagger 2} \rangle \langle \hat{d}^2 \rangle, \tag{5.75}$$

in which $\bar{n}_s$ is the mean photon number for the pair of superposed two-mode laser light beams given by Eq. (5.68).

We next proceed to calculate the expectation values involved in Eq. (5.75). Thus employing the density operator described by Eq. (5.29), we can write

$$\langle \hat{d}^2 \rangle = \lambda^2 \int d^2\beta d^2\gamma Q(\lambda\beta^*, \lambda\beta + \partial/\partial\beta^*) Q(\lambda\gamma^*, \lambda\gamma + \partial/\partial\gamma^*) Tr\big(\hat{d}^2 |\beta + \gamma\rangle\langle\gamma + \beta|\big), \tag{5.76}$$

so that with the aid of Eq. (5.4), one can put Eq. (5.76) in the form

$$\langle \hat{d}^2 \rangle = \lambda^4 \int d^2\beta d^2\gamma Q(\lambda\beta^*, \lambda\beta + \partial/\partial\beta^*) Q(\lambda\gamma^*, \lambda\gamma + \partial/\partial\gamma^*) \bigg(\beta^2 + \gamma^2 + 2\gamma\beta\bigg). \tag{5.77}$$

Using the variables $\alpha_1 = \lambda\beta$ and $\alpha_2 = \lambda\gamma$, we find

$$\langle \hat{d}^2 \rangle = \frac{1}{\lambda^2} \int d^2\alpha_1 d^2\alpha_2 Q(\alpha_1^*, \alpha_1 + \lambda\partial/\partial\alpha_1^*) Q(\alpha_2^*, \alpha_2 + \lambda\partial/\partial\alpha_2^*) \bigg(\alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_2\bigg). \tag{5.78}$$

This can be rewritten as

$$\begin{aligned}\langle \hat{d}^2 \rangle = & \frac{1}{\lambda} \int d^2\alpha_2 Q(\alpha_2^*, \alpha_2 + \lambda\partial/\partial\alpha_2^*) \\ & \times \frac{1}{\lambda} \int d^2\alpha_1 Q(\alpha_1^*, \alpha_1 + \lambda\partial/\partial\alpha_1^*)\alpha_1^2 \\ & + \frac{1}{\lambda} \int d^2\alpha_1 Q(\alpha_1^*, \alpha_1 + \lambda\partial/\partial\alpha_1^*) \\ & \times \frac{1}{\lambda} \int d^2\alpha_2 Q(\alpha_2^*, \alpha_2 + \lambda\partial/\partial\alpha_2^*)\alpha_2^2 \\ & + 2\frac{1}{\lambda} \int d^2\alpha_1 Q(\alpha_1^*, \alpha_1 + \lambda\partial/\partial\alpha_1^*)\alpha_1 \\ & \times \frac{1}{\lambda} \int d^2\alpha_2 Q(\alpha_2^*, \alpha_2 + \lambda\partial/\partial\alpha_2^*)\alpha_2.\end{aligned} \tag{5.79}$$

In view of Eq. (5.36) along with the fact that the Q function is normalized to λ, we see that

$$\langle \hat{d}^2 \rangle = \langle \hat{c}_1^2 \rangle + \langle \hat{c}_2^2 \rangle + 2\langle \hat{c}_1 \rangle \langle \hat{c}_2 \rangle. \tag{5.80}$$

On account of Eq. (5.57), this result takes the form

$$\langle \hat{d}^2 \rangle = \langle \hat{c}_1^2 \rangle + \langle \hat{c}_2^2 \rangle. \tag{5.81}$$

On account of the fact that

$$\langle \hat{c}_1 \rangle = \langle \hat{c}_2 \rangle, \tag{5.82}$$

we see that

$$\langle \hat{d}^2 \rangle = 2\langle \hat{c}_1^2 \rangle. \tag{5.83}$$

We now proceed to evaluate the expectation value involved in this expression. Thus with the aid of Eq. (5.36) along with Eq. (5.26), one can write

$$\begin{aligned}\langle \hat{c}_1^2 \rangle = & \left(u^2 - v^2\right)^{1/2} \int \frac{d^2\alpha_1}{\pi} exp\left[- u\alpha_1^*\alpha_1 + v\left(\frac{\alpha_1^2}{2} + \frac{\alpha_1^{*2}}{2}\right)\right] \\ & \times exp\left[- u\lambda\alpha_1^*\partial/\partial\alpha_1^* + v\lambda\alpha_1\partial/\partial\alpha_1^* + \frac{v}{2}\lambda^2\partial^2/\partial\alpha_1^{*2}\right]\alpha_1^2.\end{aligned} \tag{5.84}$$

Hence applying the power series given by Eq. (5.55), we readily get

$$\langle \hat{c}_1^2 \rangle = \left[u^2 - v^2\right]^{1/2} \frac{\partial^2}{\partial z^{*2}} \int \frac{d^2\alpha_1}{\pi} exp\left[-u\alpha_1^*\alpha_1 + z^*\alpha_1 + z\alpha_1^* + v\left(\frac{\alpha_1^2}{2} + \frac{\alpha_1^{*2}}{2}\right)\right] \Big|_{z=z^*=0}, \tag{5.85}$$

so that carrying out the integration over α_1 , we obtain

$$\langle\hat{c}_1^2\rangle = \frac{\partial^2}{\partial z^{*2}} exp\bigg[\frac{uzz^* + v(\frac{z^2}{2} + \frac{z^{*2}}{2})}{u^2 - v^2}\bigg]\bigg|_{z=z^*=0}. \tag{5.86}$$

Now performing the differentiation and applying the condition $z = z^* = 0$, we have

$$\langle\hat{c}_1^2\rangle = \frac{v}{u^2 - v^2}. \tag{5.87}$$

On account of Eq. (5.87), Eq. (5.83) can be put in the form

$$\langle\hat{d}^2\rangle = \frac{2v}{u^2 - v^2}. \tag{5.88}$$

With the help of Eqs. (5.27) and (5.28), we see that

$$\langle\hat{d}^2\rangle = 2R \tag{5.89}$$

and in view of Eq. (5.23), there follows

$$\langle\hat{d}^2\rangle = 2\frac{\gamma_c}{\kappa}\sqrt{\frac{\gamma + \gamma_c}{r_a}}\langle\hat{N}_a\rangle. \tag{5.90}$$

Following a similar procedure, one can also check that

$$\langle\hat{d}\hat{d}^\dagger\rangle = 2\frac{\gamma_c}{\kappa}\big(\langle\hat{N}_c\rangle + \langle\hat{N}_b\rangle\big) + 4N. \tag{5.91}$$

Upon substituting Eqs. (5.66), (5.90), (5.91), and the complex conjugate of Eq. (5.90) into Eq. (5.75), the photon number variance for the pair of superposed two-mode laser light beams is found to be

$$(\Delta n)_s^2 = 2\frac{\gamma_c}{\kappa}\bigg(\langle\hat{N}_a\rangle + \langle\hat{N}_b\rangle\bigg)\bigg(2\frac{\gamma_c}{\kappa}\big(\langle\hat{N}_c\rangle + \langle\hat{N}_b\rangle\big) + 4N\bigg) + 4\bigg(\frac{\gamma_c}{\kappa}\bigg)^2\frac{\gamma + \gamma_c}{r_a}\langle\hat{N}_a\rangle^2. \tag{5.92}$$

We then easily see that

$$(\Delta n)_s^2 = 4(\Delta n)^2, \tag{5.93}$$

where $(\Delta n)^2$ is the variance of the photon number for the two-mode laser light beam given by Eq. (3.116). We note that the variance of the photon number for the pair of superposed two-mode light beams is not the sum of photon number variances of the constituent two-mode laser light beam. However, it turns out to be four times that of a two-mode laser light beam. In addition, on account of Eqs. (2.101), (2.111), and (2.113), Eq. (5.92) can be put in the form

$$(\Delta n)_s^2 = \bar{n}^2\left(\frac{3(\gamma+\gamma_c)}{r_a} + 2\right) + 8N\bar{n}. \tag{5.94}$$

This is the steady-state variance of photon number for the pair of superposed two-mode laser light beams.

We next proceed to consider the case in which spontaneous emission is absent $(\gamma = 0)$. Then the variance of the photon number for this case takes the form

$$(\Delta n)_s^2 = \bar{n}^2\left(\frac{3\gamma_c}{r_a} + 2\right) + 8N\bar{n}, \tag{5.95}$$

where $\bar{n}$ is given by Eq. (3.84).

5.3 Quadrature squeezing

In this section, employing the density operator for the pair of superposed two-mode laser light beams, we obtain the quadrature variance and the quadrature squeezing. The squeezing properties of the superposed two-mode laser light beams are described by two quadrature operators defined by

$$\hat{d}_+ = \hat{d}^\dagger + \hat{d} \tag{5.96}$$

and

$$\hat{d}_- = i(\hat{d}^\dagger - \hat{d}), \tag{5.97}$$

where $\hat{d}_+$ and $\hat{d}_-$ are Hermitian operators representing physical quantities called plus and minus quadratures. On account of Eqs. (5.96), (5.97), and (5.45), one can show that the two quadrature operators satisfy the commutation relation

$$[\hat{d}_-, \hat{d}_+] = 4i\bigg(\frac{\gamma_c}{\kappa}(\hat{N}_a - \hat{N}_c) - 2N\bigg). \tag{5.98}$$

The product of the uncertainities in the plus and minus quadratures can be expressed as [1]

$$\Delta d_+ \Delta d_- \geq \bigg|2\frac{\gamma_c}{\kappa}(\langle\hat{N}_c\rangle - \langle\hat{N}_a\rangle) + 4N\bigg|. \tag{5.99}$$

In addition, applying Eqs. (2.111) and (2.114), one put Eq. (5.99) in the form

$$\Delta d_+ \Delta d_- \geq \bigg|2N\bigg(\frac{\gamma_c}{\kappa}\bigg(\frac{\gamma + \gamma_c - r_a}{\gamma + \gamma_c + 2r_a}\bigg) + 2\bigg)\bigg|. \tag{5.100}$$

Moreover, on setting $r_a = 0$ in Eq. (5.100), we get

$$\Delta d_+ \Delta d_- \geq 2N\bigg(\frac{\gamma_c}{\kappa} + 2\bigg). \tag{5.101}$$

This represents the uncertainty relation for the superposition of two-mode vacuum state.

Next we proceed to calculate the quadrature variance for the superposed two-mode laser light beams. The variance of the plus and minus quadrature operators for the superposed two-mode laser light beams are defined by

$$(\Delta d_+)^2 = \langle\hat{d}_+^2\rangle - \langle\hat{d}_+\rangle^2 \tag{5.102}$$

and

$$(\Delta d_-)^2 = \langle\hat{d}_-^2\rangle - \langle\hat{d}_-\rangle^2. \tag{5.103}$$

On account of Eqs. (5.96) and (5.97), Eqs. (5.102) and (5.103) take the form

$$(\Delta d_+)^2 = \langle\hat{d}^\dagger\hat{d}\rangle + \langle\hat{d}\hat{d}^\dagger\rangle + \langle\hat{d}^{\dagger 2}\rangle + \langle\hat{d}^2\rangle - \langle\hat{d}^\dagger\rangle^2 - \langle\hat{d}\rangle^2 - 2\langle\hat{d}^\dagger\rangle\langle\hat{d}\rangle \tag{5.104}$$

and

$$(\Delta d_-)^2 = \langle\hat{d}^\dagger\hat{d}\rangle + \langle\hat{d}\hat{d}^\dagger\rangle - \langle\hat{d}^{\dagger 2}\rangle - \langle\hat{d}^2\rangle + \langle\hat{d}^\dagger\rangle^2 + \langle\hat{d}\rangle^2 - 2\langle\hat{d}^\dagger\rangle\langle\hat{d}\rangle. \tag{5.105}$$

With the aid of Eqs. (5.104), (5.105), and (5.44), the variance of the quadrature operators is expressible as

$$(\Delta d_+)^2 = \langle \hat{d}^\dagger \hat{d} \rangle + \langle \hat{d}\hat{d}^\dagger \rangle + \langle \hat{d}^{\dagger 2} \rangle + \langle \hat{d}^2 \rangle, \tag{5.106}$$

$$(\Delta d_-)^2 = \langle \hat{d}^\dagger \hat{d} \rangle + \langle \hat{d}\hat{d}^\dagger \rangle - \langle \hat{d}^{\dagger 2} \rangle - \langle \hat{d}^2 \rangle. \tag{5.107}$$

Now employing Eqs. (5.66), (5.90), (5.91), and the complex conjugate of Eq. (5.90) , we arrive at

$$(\Delta d_+)^2 = 2\frac{\gamma_c}{\kappa}\left(N + \langle \hat{N}_a \rangle + 2\sqrt{\frac{\gamma + \gamma_c}{r_a}}\langle \hat{N}_a \rangle\right) + 4N, \tag{5.108}$$

$$(\Delta d_-)^2 = 2\frac{\gamma_c}{\kappa}\left(N + \langle \hat{N}_a \rangle - 2\sqrt{\frac{\gamma + \gamma_c}{r_a}}\langle \hat{N}_a \rangle\right) + 4N. \tag{5.109}$$

The quadrature variance of the superposed two-mode cavity light beams is the sum of the quadrature variances of the constituent two-mode laser light beams. Moreover, on setting $r_a = 0$ in Eqs. (5.108) and (5.109), we get

$$(\Delta d_+)^2_v = (\Delta d_-)^2_v = 2N\big(\frac{\gamma_c}{\kappa} + 2\big). \tag{5.110}$$

This can be rewritten as

$$(\Delta d_\pm)^2_v = 2(\Delta c_\pm)^2_v, \tag{5.111}$$

where $(\Delta c_\pm)^2_v$ is the quadrature variance of a two-mode cavity vacuum state given by Eq. (4.36). From Eqs. (5.101) and (5.110), we see that the uncertainty in the plus and minus quadratures are equal and satisfy the minimum uncertainty relation.

We now seek to obtain the quadrature squeezing for the superposed two-mode laser light beams in the entire frequency interval relative to the quadrature variance of a pair of superposed two-mode vacuum states. We then define the quadrature squeezing of the superposed two-mode laser light beams by [16]

$$S_s = \frac{(\Delta d_-)^2_v - (\Delta d_-)^2}{(\Delta d_-)^2_v}. \tag{5.112}$$

Now employing Eqs. (5.109) and (5.110), one can put Eq. (5.112) in the form

$$S_s = \frac{\frac{\gamma_c}{\kappa}\left(2\sqrt{\frac{\gamma+\gamma_c}{r_a}}\langle\hat{N}_a\rangle - \langle\hat{N}_a\rangle\right)}{\frac{\gamma_c}{\kappa}N + 2N}. \tag{5.113}$$

In view of Eq. (2.111), Eq. (5.113) takes the form

$$S_s = \frac{\gamma_c}{\gamma_c + 2\kappa}\left(\frac{2\sqrt{\frac{\gamma+\gamma_c}{r_a}} - 1}{\frac{\gamma+\gamma_c}{r_a} + 2}\right). \tag{5.114}$$

We then note that

$$S_s = S, \tag{5.115}$$

where S is the quadrature squeezing of a two-mode light beam given by Eq. (4.39). Therefore, we realize that the quadrature squeezing of the superposed two-mode laser light beams is equal to the quadrature squeezing of one of the superposed two- mode laser light beams.

6

Conclusion

In this PhD dissertation we have studied the statistical and squeezing properties of the cavity light generated by a three-level laser. In this quantum optical system, N three-level atoms available in an open cavity, coupled to a two-mode vacuum reservoir, are pumped to the top level by means of electron bombardment at constant rate. We have considered the case in which the three-level atoms and the cavity modes interact with the two-mode vacuum reservoir. We have carried out our analysis by putting the noise operators associated with the vacuum reservoir in arbitrary order.

We have first obtained the master equation for a three-level atom in an open cavity and the quantum Langevin equations for the cavity mode operators. In addition, employing the master equation and the large-time approximation scheme, we have obtained equations of evolution of the expectation values of the atomic operators. We have then obtained the solutions of these equations and the quantum Langevin equations. Applying the resulting solutions, we have calculated the mean photon number, the variance of the photon number, and the quadrature variance for separate light modes and for the two-mode cavity light. We have seen that the global mean photon numbers of the light modes emitted from the top and intermediate levels are the same both in the presence and absence of spontaneous emission, and are separately in a chaotic state.

Moreover, we have shown that the mean photon number of the two-mode light beam is the sum of the mean photon numbers of the separate single-mode light beams. However, we have observed that the photon number variance of the two-mode light beam does not happen to be the sum of the photon number variance of the separate single-mode light beams. Furthermore, the global photon number variance calculated by taking the noise operators in arbitrary order turns out to be greater than that obtained by putting the noise operators in normal order. In addition, we have established that a large part of the total mean photon number and variance of the photon number are confined in relatively small frequency intervals. In addition, we have shown that the presence of the spontaneous emission process leads to a decrease in the mean and variance of the photon number.

Our calculation shows that the two-mode cavity light is in squeezed state and the squeezing occurs in the minus quadrature. It so turns out that the maximum quadrature squeezing of the two-mode cavity light is 37.5% for $\gamma = 0$ and $\gamma = 0.4$ below the vacuum-state level. The presence of the vacuum reservoir noise has the effect of decreasing the quadrature squeezing of the cavity light. However, the presence of the vacuum reservoir noise does not have any effect on the mean photon number. Unlike the mean photon number and the quadrature variance, the quadrature squeezing does not depend on the number of atoms. This implies that the quadrature squeezing of the two-mode light beam is independent of the number of photons. In addition, we have also seen that the local quadrature squeezing of the cavity light is greater than the global quadrature squeezing.

Furthermore, employing the density operator for a pair of superposed two-mode laser light beams, we have calculated the mean and variance of the photon number as well as the quadrature variance and the quadrature squeezing. We have found that both the mean photon number and the quadrature varinace for the pair of superposed two-mode laser light beams is the sum of the mean photon numbers and the quadrature variances of the constituent two-mode laser light beams. However, the variance of the photon number for the superposed two-mode light beams is not the sum of the variances of the photon numbers of the constituent two-mode laser light beams. It turns out to be four times that of a two-mode laser light beam. Finally, we have seen that the quadrature squeezing of the superposed two-mode laser light beams is equal to the quadrature squeezing of one of the superposed two-mode laser light beams. This implies that the superposition of the two-mode laser light beams does not affect the quadrature squeezing, but it increases the global mean photon number and the global variance of the photon number. Thus we note that the superposition of the two-mode laser light beams leads to a more bright squeezed light.

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