REDUCTION OF STURM-LIOUVILLE BOUNDARY VALUE PROBLEM IN TO INTEGRAL EQUATION

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Notation

C=Continuous functions
C^1=Continuous Differentiable functions
C^2=Twice Continuous differentiable functions
C^∞=Infinitely differentiable functions
$\mathbb{C}$=complex number
R^n= n-tuple of Real number
supp$f$=Support of f
ϕ =Phi
ψ = Psi
ϱ = Rho
δ = Dirac Delta
α = Alpha
β = Beta
λ = Lambda
μ = Mu
$\mathcal{M}_L$ = Domain of the operator L
ϵ =Epsilon
$D(\Omega)$=The set of test functions
$D^{'}(\Omega)$= Distribution
∂=Partial derivatives
$\mathcal{L}_2$= Square integrable functions
$L[\mathcal{G}(.,.)](x,s)$ stands for $-\frac{\partial}{\partial x}[p(x)\frac{\partial \mathcal{G}}{\partial x}(x,s)] + q(x)\mathcal{G}(x,s)$

Abstract

In this paper we present a method of finding eigenvalue and eigenfunction for linear second order elliptic differential equation using the Fourier method. Then we present more specifically for one dimension space called sturm-lioville problems. Next Green's function to represent solution for sturm-lioville problem and its properties.Finally we present reduction of the Sturm-Liouville Problem to an Integral Equation and properties of sturm-lioville problem.

Introduction

In mathematics and its applications, a classical SturmLiouville equation, named after Jacques Charles Franois Sturm $(1803 - 1855)$ and Joseph Liouville $(1809 - 1882)$, is a real second-order linear differential equation of the form $-(py')' + qy = \lambda \varrho y$ where y is a function of the free variable x. In the simplest of cases all coefficients are continuous on the finite closed interval $[0, l]$, and p has continuous derivative. In addition, the unknown function y is typically required to satisfy some boundary conditions at 0 and l. The function $\varrho(x)$, which is sometimes called $r(x)$, is called the "weight" or "density" function. The value of λ is not specified in the equation; finding the values of λ for which there exists a non-trivial solution of the above satisfying the boundary conditions is part of the problem called the SturmLiouville (SL) problem. Under normal assumptions on the coefficient functions $p(x), q(x)$, and $\varrho(x)$ above, they induce a Hermitian differential operator in some function space defined by boundary conditions. The resulting theory of the existence and asymptotic behavior of the eigenvalues, the corresponding qualitative theory of the eigenfunctions and their completeness in a suitable function space became known as SturmLiouville theory. This is important in applied mathematics, where SturmLiouville problems occur very commonly, particularly when dealing with linear partial differential equations that are separable.In the theory of partial differential equations, elliptic operators are differential operators that generalize the Laplace operator.Elliptic operators are typical of potential theory, and they appear frequently in electrostatics and continuum mechanics.

This paper concerned about reduction of sturm-liouville boundary value problem into integral equation . The main objective of this paper is to solve Green's function for sturm-liouville problem,reduction of sturm-liouville problem to integral equation and properties sturm-liouville problem. This paper consists of three parts, in the first part I start with preliminaries Definitions, concepts and several results that are frequently used in the next parts. The second part of this paper is concerned with eigenvalue problem in elliptic type. Lastly sturm-liouville problem.

Chapter 1

Preliminaries

In this chapter we consider basic properties of function spaces. Several results and techniques of this chapter are frequently used in later chapters.

1.1 The Space of Functions $\mathcal{L}_2(G)$

Definition 1.1.0.1. *A function space is linear space whose points are functions.*

i.e.It is a space made of functions.

Let p be a real number and G be open set in R^n.
ThenL_p space is defined as follows:

Definition 1.1.0.2. *The L_p space is a set of real or complex-valued Lebesgue measurable functions $f(x)$on G that satisfy*

$$\int_G |f|^p d\mu < \infty,$$

for $1 \leq P < \infty$.

Example 1.1. $L_2[0,2]$ *consists of all functions $f(x)$ for which the integral*

$$\int_0^2 |f(x)|^2 dx < \infty$$

Definition 1.1.0.3. *A function $f : R^n \to R$ is called (Lebesgue-) integrable,if*

$$\int f dx < \infty$$

Let we define the norm of the L_p space by,
$||f||_{L_p}(G) = (\int_G f^p dx)^{\frac{1}{p}}$, for $1 \leq p < \infty$.
We define the Lebesgue space ,$L_P(G) = \{f : ||f||_{L_p}(G) < \infty\}$

Example 1.2. *$L_1(G), L_2(G)$, when $p = 1$ and $p = 2$*

Definition 1.1.0.4. *Let G be a region, $\mathcal{L}_2(G)$ will be used to denote the set of all functions f, for which the function $\mid f(x) \mid^2$ is integrable over the region G i.e*
$\mathcal{L}_2(G) = \{f : G \longrightarrow \mathbb{C} : \int |f(x)|^2 dx < \infty\}$,and functions in $\mathcal{L}_2(G)$ is called square integrable functions.

The set of functions $\mathcal{L}_2(G)$ is linear space.
In fact, if $f, g \in \mathcal{L}_2(G)$ and $\lambda, \mu \in \mathbb{C}$ it follows from inequality

$$|\lambda f + \mu g|^2 \leq 2|\lambda|^2|f|^2 + 2|\mu|^2|g|^2 \tag{1.1}$$

that is any linear combination $\lambda f + \mu g$ also belongs to $\mathcal{L}_2(G)$
Cauchy-Buniakowski Inequality: If $f, g \in \mathcal{L}_2(G)$, then

$$|\int_G f(x)g(x)dx| \leq (\int_G |f(x)|^2 dx)^{\frac{1}{2}} (\int_G |g(x)|^2 dx)^{\frac{1}{2}} \tag{1.2}$$

In fact, when $f, g \in \mathcal{L}_2(G)$ for all $\lambda, |f| + \lambda|g| \in \mathcal{L}_2(G)$ and by virtue of this
$0 \leq \int_G (|f(x)| + \lambda|g(x)|)^2 dx$
$= \int_G |f(x)|^2 dx + 2\lambda \int_G |f(x)g(x)|dx + \lambda^2 \int_G |g(x)|^2 dx$
consequently the discriminant of this quadratic form is non positive, that is

$$[\int_G |f(x)g(x)|dx]^2 - \int_G |f(x)|^2 dx \int_G |g(x)|^2 dx \leq 0$$

$$\Rightarrow \int_G |f(x)g(x)|dx \leq (\int_G |f(x)|^2 dx)^{\frac{1}{2}} (\int_G |g(x)|^2 dx)^{\frac{1}{2}}$$

If $f \in \mathcal{L}_2(G)$ and G is a bounded region,the function $f(x)$ is integrable over G.
In fact, by applying the cauchy-Buniakowski inequality with $g = 1$, we obtain

$\int_G |f(x)|dx \leq (\int_G |f(x)|^2 dx)^{\frac{1}{2}} (\int_G dx)^{\frac{1}{2}} < \infty$
on the set of functions $\mathcal{L}_2(G)$ we shall introduce a scalar product and norm according to the formulas

$$(f, g) = \int_G f(x)\overline{g(x)}dx \tag{1.3}$$

$$||f|| = \sqrt{(f, f)} = (\int_G |f(x)|^2 dx)^{\frac{1}{2}}$$

where $\overline{g(x)}$ is the complex conjugate of g(x). The above scalar product has the following properties:

- $$(f, g) = \overline{(g, f)} \tag{1.4}$$

- $$(\lambda f + \mu g, h) = \lambda(f, h) + \mu(g, h) \tag{1.5}$$

- $$|(f, g)| \leq |f||g| \quad f, g \in \mathcal{L}_2(G) \tag{1.6}$$

- $$(f, f) \geq 0, \quad (f, f) = 0 \Leftrightarrow f = 0$$

From the third inequality it follows *minkowski inequality:*

$$||f + g|| \leq ||f|| + ||g|| \qquad f, g \in \mathcal{L}_2(G) \tag{1.7}$$

In fact,

$$\begin{aligned} ||f + g||^2 &= (f + g, f + g) \\ &= (f, f) + (f, g) + (g, f) + (g, g) \\ &\leq ||f||^2 + |(f, g)| + |(g, f)| + ||g||^2 \leq ||f||^2 + 2||f||||g|| + ||g||^2 = (||f|| + ||g||)^2 \end{aligned}$$

Definition 1.1.0.5. The sequence of functions $f_k, k = 1,2...$, belonging to$\mathcal{L}_2(G)$ is said to converge to the function $f \in \mathcal{L}_2(G)$
(to converge in the mean in G) if $||f_k - f|| \longrightarrow 0$, as $k \to \infty$
$\Longrightarrow f_k \longrightarrow f$, as $k \to \infty$ in $\mathcal{L}_2(G)$

Definition 1.1.0.6. A sequence of functions $f_k(x)$ is said to be a cauchy sequence if
$\lim_{k,n\to\infty} \int_G |f_k(x) - f_n(x)|^2 dx = 0$ in $\mathcal{L}_2(G)$

Definition 1.1.0.7. An $\mathcal{L}_2(G)$ space is said to be complete if every cauchy sequence in the space converges in the mean to a function in the space.

i.e if the sequence of functions $f_k, k = 1, 2, ...$, belonging to $\mathcal{L}_2(G)$ converges in itself in $\mathcal{L}_2(G)$, that is, if $||f_k - f_p|| \to 0$ as $k \to \infty, p \to \infty$, then there is a function $f \in \mathcal{L}_2(G)$ such that $||f_k - f|| \to 0$ as $k \to \infty$. The space $\mathcal{L}_2(G)$ belonges to the class of so called Hilbert spaces.

Definition 1.1.0.8. *Inner product space (scalar product) is called Hilbert space if it is complete as a normed space.*

Example 1.3. *[2] Square summable sequence of complex numbers: ℓ^2 is the space of sequence of complex numbers*

$$x = \{x_n\}_{n=1}^{\infty} \ni / \sum_{n=1}^{\infty} |x_n|^2 < \infty$$

It is a Hilbert space with the inner product

$$(x, y) = \sum_{n=1}^{\infty} \overline{x_n} y_n \tag{1.8}$$

To show that ℓ^2 is Hilbert is showing ℓ^2 with the norm $||.||_2$ is complete.
Illustration

1. *Let $a_n \in \ell^2$ be cauchy. Let $a_n = (\alpha_1^n, \alpha_2^n ...)$*
2. *Let $\varepsilon > 0$. Then $\exists M$ such that for $n, m \geq M$,*

$$||a_n - a_m||^2 = \sum_{k=1}^{\infty} |\alpha_k^n - \alpha_k^m|^2 < \varepsilon^2$$

3. *So, for any $k \in N$, for $n, m \geq M$,*

$$|\alpha_k^n - \alpha_k^m| < \varepsilon$$

4. *so, α_k^n is a cauchy sequence for each $k \in N$*

5. *Let $\alpha_k = \lim_{n\to\infty} \alpha_k^n$ and $a = (\alpha_1, \alpha_2 ...)$*

6. *Claim:$a \in \ell^2$ and $a_n \to a$*

7. *We have, for any $n \geq M$*

$$||a_n - a||^2 < \varepsilon^2$$

8. *By Minkowski inequality $||a|| = ||a - a_n + a_n|| \leq ||a - a_n|| + ||a_n|| < \infty$ so, $a \in \ell^2$*

9. *Also $\lim_{n\to\infty} ||a_n - a|| = 0$ so, $a_n \to a$.*

Definition 1.1.0.9. A set of functions $\mathcal{M} \subset \mathcal{L}_2(G)$ is called dense in $\mathcal{L}_2(G)$ if for any $f \in \mathcal{L}_2(G)$ there is a sequence of functions belonging to $\mathcal{M}$ which converges to f in $\mathcal{L}_2(G)$.

1.1.1 Orthonormal System

The functions f and g belonging to $\mathcal{L}_2(G)$ are called *orthogonal* if $(f, g) = 0$; the function belonging to $\mathcal{L}_2(G)$ is said to be *normalized* if $||f|| = 1$. The system of functions φ_k belonging to $\mathcal{L}_2(G)$ is said to be *orthonormal* in $\mathcal{L}_2(G)$ if $(\varphi_k, \varphi_i) = \delta_{ki}$, where δ_{ki} is the krönecker delta symbol $\delta_{ki} = 0, k \neq i, \delta_{ii} = 1$

Example 1.4. The trigonometric system $\varphi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$ is an orthonormal systemin $\mathcal{L}_2(-\pi, \pi)$

Definition 1.1.1.1. *A collection of vectors $A = \{x_j\}_{j=1}^k$ is linear independent if*

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_k x_k = 0$$

implies $\alpha_j = 0, \quad 1 \leq j \leq k$

Each orthonormal system φ_k consists of linearly independent functions. In fact, in the corresponding case involving a sequence of (complex) numbers c_k of which only a finite number are distinct from zero, we have the equation $\sum_k c_k \varphi_k = 0$, from which, by virtue of the orthonormality of the system φ_k, we obtain

$$0 = (\sum_k c_k \varphi_k, \varphi_i) = \sum_k (c_k \varphi_k, \varphi_i) = \sum_k c_k (\varphi_k, \varphi_i) = c_i$$

Each system of linearly independent functions ψ_1, ψ_2,..., belonging to $\mathcal{L}_2(G)$ may be converted into an orthonormal system φ_1, φ_2,..., by the following

Gram-Schmidt orthogonalization process:

$$\varphi_1 = \frac{\psi_1}{||\psi_1||}, \varphi_2 = \frac{\psi_2 - (\psi_2, \varphi_1)\varphi_1}{||\psi_2 - (\psi_2, \varphi_1)\varphi_1||}, ...,$$

$$\varphi_k = \frac{\psi_k - (\psi_k, \varphi_{k-1})\varphi_{k-1} - ... - (\psi_k, \varphi_1)\varphi_1}{||\psi_k - (\psi_k, \varphi_{k-1})\varphi_{k-1} - ... - (\psi_k, \varphi_1)\varphi_1||} \tag{1.9}$$

Example 1.5. . If in the space $\mathcal{L}_2(-1, 1)$ we can orthogonalize the power $f_n = x^n = 1, x, x^2, \quad , n =$ 0,1,2... by means of the Gram-Schmidt process, the system of normalized Legendre polynomials is obtained.

Definition 1.1.1.2. *Let a system of functions $\varphi_k, k =$ 1,2,..., be orthonormal in $\mathcal{L}_2(G)$ and let $f \in \mathcal{L}_2(G)$. Then the numbers (f, φ_k) are called Fourier coefficients and the formal series*

$$\sum_{k=1}^{\infty}(f, \varphi_k)\varphi_k \tag{1.10}$$

is called the Fourier series of the function f interms of the system φ_k.

If a system of functions φ_k, $k =$ 1,2,..., is orthonormal in $\mathcal{L}_2(G)$ then the equation

$$||f - \sum_{k=1}^{N} a_k\varphi_k||^2 = ||f - \sum_{k=1}^{N}(f, \varphi_k)\varphi_k||^2 + \sum_{k=1}^{N} |(f, \varphi_k) - a_k|^2 \tag{1.11}$$

is true for each $f \in \mathcal{L}_2(G)$ and any (complex)numbers $\alpha_1, \alpha_2, ..., \alpha_N, N = 1, 2,...$
In fact, writing

$$f_N = f - \sum_{k=1}^{N}(f, \varphi_k)\varphi_k, \qquad c_k = (f, \varphi_k) - a_k \tag{1.12}$$

we obtain when i=1,2,...,N

$$(f_N, \varphi_i) = (f - \sum_{k=1}^{N}(f, \varphi_k)\varphi_k, \varphi_i) = (f, \varphi_i) - \sum_{k=1}^{N}(f, \varphi_k)(\varphi_k, \varphi_i) = 0$$

consequently ,

$$\begin{aligned}
||f - \sum_{k=1}^{N} a_k\varphi_k||^2 &= ||f_N + \sum_{k=1}^{N} c_k\varphi_k||^2 \\
&= (f_N + \sum_{k=1}^{N} c_k\varphi_k, f_N + \sum_{k=1}^{N} c_k\varphi_k) \\
&= (f_N, f_N) + \sum_{k=1}^{N}(f_N, c_k\varphi_k) + \sum_{k=1}^{N}(c_k\varphi_k, f_N) + \sum_{k,i=1}^{N}(c_k\varphi_k, c_i\varphi_i) \\
&= ||f_N||^2 + \sum_{k,i=1}^{N} c_k\bar{c}_i(\varphi_k, \varphi_i) = ||f_N||^2 + \sum_{k=1}^{N} |c_k|^2
\end{aligned}$$

substituting f_N and c_k of equation (1.12) to the last equation we get

$$||f - \sum_{k=1}^{N} a_k\varphi_k||^2 = ||f - \sum_{k=1}^{N}(f, \varphi_k)\varphi_k||^2 + \sum_{k=1}^{N} |(f, \varphi_k) - a_k|^2$$

from equation (1.11) it follows the following inequality

$$||f - \sum_{k=1}^{N}(f, \varphi_k)\varphi_k||^2 \leq ||f - \sum_{k=1}^{N} a_k\varphi_k||^2 \tag{1.13}$$

Further, supposing in equation (1.11) that $a_k = 0$,k=1,2,...,N, we obtain the equation

$$||f - \sum_{k=1}^{N}(f, \varphi_k)\varphi_k||^2 = ||f||^2 - \sum_{k=1}^{N} |(f, \varphi_k)|^2 \tag{1.14}$$

From equation (1.14) there then follows the inequality

$$\sum_{k=1}^{N} |(f, \varphi_k)|^2 \leq ||f||^2 \tag{1.15}$$

which is known as Bessels inequality. Moreover, from equation (1.14) and from the convergence in $\mathcal{L}_2(G)$ we can say:
In order that the Fourier series (1.10) converge to the function f in $\mathcal{L}_2(G)$, it is necessary and sufficient that Parseval's equation (the equation of closure) be satisfied

$$\sum_{k=1}^{\infty} |(f, \varphi_k)|^2 = ||f||^2 \tag{1.16}$$

Complete Orthonormal Systems
Let the system of functions φ_l, φ_2 ,..., be orthonormal in $\mathcal{L}_2(G)$.

Definition 1.1.1.3. *If for any $f \in \mathcal{L}_2(G)$ its Fourier series in terms of the system φ_k converges to f in $\mathcal{L}_2(G)$, then this system is said to be complete (closed) in $\mathcal{L}_2(G)$. i,e the orthonormal φ_n in L_2 is compete if for all $f \in L_2$, the fourier series of f converge to f in L_2 ,$f = \sum_{n=1}^{\infty}(f, \varphi_n)\varphi_n$*

Remark 1. All that has been said of the space $\mathcal{L}_2(G)$ is true also of the space $\mathcal{L}_2(G; \varrho)$ and $\mathcal{L}_2(S)$ with the scalar products

$$(f, g)_\varrho = \int_G \varrho(x) f(x)\overline{g(x)}dx, f, g \in \mathcal{L}_2(G; \varrho)$$

$$(f, g) = \int_S f(x)\overline{g(x)}dS \quad f, g \in \mathcal{L}_2(S)$$

where the weight function $\varrho \in C(\bar{G}), \varrho(x) > 0, x \in \bar{G}$,and S is a piecewise sooth surface.

The scalar product with itself is called a norm squared and written as

$$(f, f)_{\varrho(x)} = \int \varrho(x) f^2(x)dx$$

The function $f(x)$ is called square integrable on the region G with respect to the weight function $\varrho(x)$ when $\int_G \varrho(x)f^2(x)dx < \infty$. A set or sequence of functions $f_1(x), f_2(x), ...f_n(x)...f_m(x)$ is said to be orthogonal over G with respect to a weight function $\varrho(x) > 0$ if for all integer values of m and n the scalar product of f_m with f_n satisfies

$$(f_m, f_n)_{\varrho(x)} = \int_G \varrho(x) f_m(x) f_n(x)dx = 0, \quad m \neq n$$

1.2 Linear Operator and Functionals

Definition 1.2.0.4. Let $\mathcal{M}$ and $\mathcal{N}$ be linear sets.
The operator L, transforming elements of set $\mathcal{M}$ into elements of set $\mathcal{N}$, is said to be linear if for any elements f and g belonging to $\mathcal{M}$, and complex numbers λ and μ the equation

$$L(\lambda f + \mu g) = \lambda Lf + \mu Lg$$

is true.

In this case the set $\mathcal{M} = \mathcal{M}_L$ is called the *domain of definition* of the operator L. If $Lf = f$ for all $f \in \mathcal{M}$, the operator L is called an identity(unit) operator and it is denoted by I.
Let the convergence of the elements be defined on the linear sets $\mathcal{M}$ and $\mathcal{N}$. The linear operator L , mapping $\mathcal{M}$ into $\mathcal{N}$ is said to be continuous from $\mathcal{M}$ to $\mathcal{N}$ if the convergence $Lf_k \to Lf$ as $k \to \infty$ in $\mathcal{N}$ follows from the convergence $f_k \to f$ as $k \to \infty$ in $\mathcal{M}$.
The linear operator L, mapping $\mathcal{M}$ into $\mathcal{N}$,is said to be bounded from $\mathcal{M}$ to $\mathcal{N}$ if there is a number $c > 0$ such that the inequality

$$||Lf||_{\mathcal{N}} \leq c||f||_{\mathcal{M}} \tag{1.17}$$

is true for $f \in \mathcal{M}$.

Linear functionals are a particular case of linear operators. If the linear operator l transforms the set of elements $\mathcal{M}$ into set of complex numbers $lf, f \in \mathcal{M}$ then l is said to be a linear functional on the set $\mathcal{M}$. We shall denote by (l, f) the effect of the functional l on element f-the complex number lf. In this way,by the continuity of the linear functional l we mean the following:
If $f_k \to f$ as $k \to \infty$ in $\mathcal{M}$, then the sequence of complex numbers (l, f_k) as $k \to \infty$ tends to (l, f).

Example of linear operator:
(a) The linear operator of the form

$$Kf = \int_G \mathbb{K}(x, y)f(y)dy, x \in G \tag{1.18}$$

is called the *linear integral operator*, and the function $\mathbb{K}(x, y)$ is its kernel.
If the kernel $\mathbb{K} \in \mathcal{L}_2(G \times G)$,

$$\int_{G\times G} |\mathbb{K}(x, y)|^2 dxdy = C^2 < \infty \tag{1.19}$$

then the operator K is bounded (and, consequently, continuous) from $\mathcal{L}_2(G) = \mathcal{M}$ to $\mathcal{L}_2(G) = \mathcal{N}$. In fact, applying the Cauchy-Buniakowski inequality and the Fubini theorem and using (1.19), for all f $\in \mathcal{L}_2(G)$ we obtain the inequality
$||Kf||^2 = \int_G | \int_G \mathbb{K}(x, y)f(y)dy|^2 dx$

$$\leq \int_G (\int_G |\mathbb{K}(x, y)|^2 \int_G |f(y)|^2 dy)dx = C^2||f||^2$$

that is,

$$||Kf|| \leq C||f||, f \in \mathcal{L}_2(G) \quad (1.20)$$

which shows that the operator K is bounded in $\mathcal{L}_2(G)$.

1.3 Linear Equation

Let L be a linear operator with domain of definition $\mathcal{M}_L$. The equation

$$Lu = F \quad (1.21)$$

is called a *linear (inhomogeneous)* equation. In equation (1.21) the element F is called the *inhomogeneous term (free term, or right-hand side)*, and the unknown element u belonging to $\mathcal{M}_L$ is called the *solution* of this equation. If in equation (1.21) we assume that the inhomogeneous term F is equal to zero, then the equation obtained,

$$Lu = 0 \quad (1.22)$$

is called the *linear homogeneous equation* corresponding to equation (1.21). By virtue of the linearity of the operator L, the set of solutions of the homogeneous equation (1.22) forms a linear set;specifically, u =0 is always a solution of this equation.
Each solution u of the linear inhomogeneous equation (1.21) (if it exists) appears in the form of a sum of a particular solution U_0 of this equation and of the general solution u of the corresponding linear homogeneous equation (1.22),

$$u = u_0 + \tilde{u} \quad (1.23)$$

In fact, if u is an arbitrary solution of equation (1.21),$Lu = F$, $u \in \mathcal{M}_L$, while u_0 is a particular solution of this equation,$Lu_0 = F, u_0 \in \mathcal{M}_L$, then, by virtue of the linearity of operator L, their difference $u - u_0 = \tilde{u} \in \mathcal{M}_L$ also satisfies the homogeneous equation (1.22):

$$L\tilde{u} = L(u - u_0) = Lu - Lu_0 = F - F = 0$$

This proves the representation equation (1.23) for the solution u. So in order that the solution of equation (1.21) should be unique in $\mathcal{M}_L$, it is necessary and sufficient that the corresponding homogeneous equation (1.22) have only a zero solution in $\mathcal{M}_L$.
Let us consider the linear homogeneous equation

$$Lu = \lambda u \quad (1.24)$$

where λ is a complex parameter. This equation has a zero solution for all λ. It can happen that for some λ it has nonzero solutions belonging to $\mathcal{M}_L$ Those complex values λ for which equation (1.24) has nonzero solutions belonging to $\mathcal{M}_L$ are called the *eigenvalues* or *characteristic values* of the operator L and the corresponding solutions are the *eigenfunctions* or the *characteristic functions* corresponding to this eigenvalue.

1.4 Hermitian Operators

The linear operator L mapping $\mathcal{M}_L \subset \mathcal{L}_2$ into $\mathcal{L}_2$ is said to be *Hermitian* (or *self-adjoint in the Lagrangian sense*) if for any f and g belonging to $\mathcal{M}_L$ it is true that

$$(Lf, g) = (f, Lg) \tag{1.25}$$

The expressions (Lf, g) and (Lf, f) are called bilinear and quadratic forms, respectively, generated by the operator L.
In order that the operator L should be Hermitian, it is necessary and sufficient that the quadratic-form (Lf, f), $f \in \mathcal{M}_L$, generated by it should assume only real values.
In fact, if the operator L is Hermitian, then by virtue of (1.4) and (1.25)

$$(Lf, f) = (f, Lf) = \overline{(Lf, f)} = \overline{(f, Lf)}, f \in \mathcal{M}_L$$

so that the quadratic form (Lf, f) can assume only real values.
Conversely, if the quadratic form (Lf, f) assumes only real values, for all f and g belonging to $\mathcal{M}_L$ we have

$$Re[(Lg, f) - (Lf, g)] = Re\frac{1}{i}[(L(f + ig), f + ig) - (Lf, f) - (Lg, g)] = 0$$

$$Im[(Lg, f) + (Lf, g)] = Im[(L(f + g), f + g) - (Lf, f) - (Lg, g)] = 0$$

and therefore,

$$\begin{aligned}(Lf, g) &= Re(Lf, g) + iIm(Lf, g) \\ &= Re(Lg, f) - iIm(Lg, f) \\ &= \overline{(Lg, f)} \\ &= (f, Lg)\end{aligned}$$

so that the operator L is Hermitian. A linear operator is said to be positive if the quadratic form (Lf, f), $f \in \mathcal{M}_L$, generated by it assumes only non-negative values. From the assertion which has been proved it follows that

each positive operator is Hermitian.
Leibnizs formula:

$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)} u(x,t)dx\right) = \int_{a(t)}^{b(t)} \frac{d}{dt}u(x,t)dx + u(b(t),t)b'(t) - u(a(t),t)a'(t)$$

proof:Let $G(t,a,b) = \int_a^b u(x,t)dx$ where $a = a(t),\ \ b = b(t)$
The chain rule now gives
$\frac{dG}{dt} = G_t^{'}(t,a,b) + G_a^{'}(t,a,b)a'(t) + G_b^{'}(t,a,b)b^{'}(t)$
$= \int_a^b u_t^{'}dx + u(b(t),t)b'(t) - u(a(t),t)a'(t)$

1.5 Distribution and The Delta Function

Definition 1.5.0.5. *The space of infinitely differentiable functions with compact support (set of basic functions) in the open set $\Omega \subseteq \Re^n$ is defined as $D(\Omega) = \{f : \Omega \longrightarrow \mathbb{C}; f \in C^{\infty}(\Omega)$ and $supp(f)$ is compact in $\Omega\} = C_c^{\infty}(\Omega)$ the element of $D(\Omega)$ are called test functions. The set of test functions, the support of which are contained in the given region Ω, is denoted by $D(\Omega)$. where $suppf = \overline{\{x \in \Omega : f(x) \neq 0\}}$ the closure set where f does not vanish.*

Definition 1.5.0.6. A distribution in $D'(\Omega)$ is a class of continuous linear functional that maps a set of test function in $D(\Omega)$ into the real(complex) numbers.
$D'(\Omega) = \{f : D(\Omega) \longrightarrow \mathbb{C}\}$, where f is linear and continuous in the region Ω.

Definition 1.5.0.7. A distribution $f \in D'$ on a non-empty open set $\Omega \subseteq \Re^n$ is any continuous linear functional on the space of basic functions $D(\Omega)$.

- We write the value of functional f on the basic function $\varphi \in D$ as (f, φ) which is a (complex)number.
- A distribution $f \in D'$ is a linear functional on the space of basic functions D, that is , if $\varphi, \psi \in D$ and $\lambda, \mu \in$, then

$$(f, \lambda\varphi + \mu\psi) = \lambda(f, \varphi) + \mu(f, \psi)$$

- A distribution $f \in D'$ is a continuous functional on D, i.e. if $\varphi_k \longrightarrow \varphi$ in D as $k \longrightarrow \infty$, then $(f, \varphi_k) \longrightarrow (f, \varphi)$

The Delta Function

There is a great need in differential equations to define objects that arise as limits of functions and behave like functions under integration but they are not properly speaking, functions themselves. The most basic one of these is the so-called $\delta-$function, it is an example of distribution.

Definition 1.5.0.8. *The Dirac delta function $\delta(x)$ is defined by the following properties:*

1.
$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x=0 \end{cases}$$

2. $\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$

3. $\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$ *where $f(x)$ is any continuous function on an open interval containing $x = 0$ as well as $x = a$.*

Chapter 2

Eigenvalue Problem

2.1 Formulation of the Eigenvalue problem

Definition 2.1.0.9. A linear elliptic differential operator has the form

$$LU = -\sum_{i,j=1}^{n} \partial_i(a^{ij}\partial_j U) + \sum_{i=1}^{n} \partial_i(b_i U) + cU, a^{ij} = a^{ji} \tag{2.1}$$

acting on functions $U : G \longrightarrow R$ where $x = (x_1, x_2, \ldots, x_n)$ lies in a domain G which is open subset of R^n , $n \geq 2$, $U \in C^2(G)$.

L is elliptic at a point $x \in G$ if the coefficient matrix $[a^{ij}(x)]$ is positive definite.

Consider the following linear homogeneous boundary value problem for an equation of elliptic type

$$-div(p\ \ grad\ \ u) + qu = \lambda u, x \in G \tag{2.2}$$

$$\alpha u + \beta \frac{\partial u}{\partial n}|_S = 0 \tag{2.3}$$

we suppose $p \in C^1(\bar{G})$, $q \in C(\bar{G})$, $p(x) > 0$, $q(x) \geq 0$, $x \in \bar{G}$

$$\alpha \in C(S), \beta \in C(S), \alpha(x) \geq 0, \beta(x) \geq 0, \alpha(x) + \beta(x) > 0, x \in S \tag{2.4}$$

Let S_o be that part of S on which $\alpha(x) > 0$ and $\beta(x) > 0$ are simultaneously true.

The problem (2.2)-(2.3) is to find a function $u(x)$ of the class $C^2(G) \cap C^1(\bar{G})$ which satisfies Equation (2.2) in the region G and the boundary conditions

(2.3) on the boundary S. Obviously problem (2.2)-(2.3) always has a zero solution, and this solution is of no interest. We must therefore consider problem (2.2)-(2.3) as an eigenvalue problem for the operator

$$L = -div(p\ \ grad\,) + q \tag{2.5}$$

All the functions $f(x)$ of the class $C^2(G) \cap C^1(\bar{G})$ satisfying the boundary condition (2.3) and the condition that $Lf \in \mathcal{L}_2(G)$ will be related to the domain of definition $\mathcal{M}_L$ of the operator L.
So the problem (2.2)-(2.3) is to find those values λ (the eigenvalues of the operator L) for which the equation

$$Lu = \lambda u \tag{2.6}$$

has a nonzero solution $u(x)$ belonging to the domain of definition $\mathcal{M}_L$ (the eigenfunctions which correspond to this eigenvalue).

2.2 Green's Formulas

2.2.1 Green's First Formula

If $u \in C^2(G) \cap C^1(\bar{G})$ and $v \in C^1(\bar{G})$, then Green's first formula holds :

$$\int_G vLudx = \int_G p\sum_{i=1}^{n} \frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i}dx - \int_S pv\frac{\partial u}{\partial n}ds + \int_G quvdx \tag{2.7}$$

ILLUSTRATION
To prove the above equation take an arbitrary region G' with a piecewise smooth boundary S', strictly lying in the region G
Since $u \in C^2(G)$, then $u \in C^2(\bar{G}')$ and so

$$\begin{aligned}
\int_{G'} vLudx &= \int_{G'} v[-div(p\ \ grad\ \ u) + qu]dx \\
&= -\int_{G'} vdiv(p\ \ grad\ \ u)dx + \int_{G'} quvdx \\
&= -\int_{G'} div(pv\ \ grad\ \ u)dx + \int_{G'} p\sum_{i=1}^{n} \frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i}dx + \int_{G'} quvdx
\end{aligned}$$

Then using the Gauss-Ostrogradski formula, we obtain

$$\int_{G'} vLudx = \int_{G'} p\sum_{i=1}^{n} \frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i}dx - \int_{S'} pv\frac{\partial u}{\partial n}dx + \int_{G'} quvdx$$

Allowing G' to tend to G in this equation, and using the fact that u and $v \in C^1(G)$, we conclude that the limit of the right-hand side exists, so that there is a limit of the left-hand side and the Green's first formula is true. For this the integral on the left in Green's first formula must be understood to be nonsingular.

2.2.2 Green's Second Formula

If $u, v \in C^2(G) \cap C^1(\bar{G})$, then Green's second formula holds :

$$\int_G (vLu - uLv)dx = \int_S p(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})ds \tag{2.8}$$

ILLUSTRATION
To prove formula (2.8) we interchange u and v in Green's first formula (2.7)

$$\int_G uLvdx = \int_G p\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx - \int_S pu\frac{\partial v}{\partial n}ds + \int_G quvdx$$

Subtracting this equation from equation (2.7), then we obtain Green's second formula (2.8).

Specifically, for $p = 1$, $q = 0$ Green' s formulas (2.7) and (2.8) are transformed into the following

$$\int_G v\triangle udx = -\int_G \sum_{i=1}^{n}\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i}dx + \int_S v\frac{\partial u}{\partial n}dS$$

$$\int_G (v\triangle u - u\triangle v)dx = \int_S (v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})dS$$

2.3 Properties of the Operator L

The operator L is Hermitian (self adjoint in the sense of Lagrange),

$$(Lf, g) = (f, Lg), \quad f, g \in \mathcal{M}_L \tag{2.9}$$

ILLUSTRATION
The functions f and $\bar{g}$ belong to the region $\mathcal{M}_L$, then $Lf \in \mathcal{L}_2(G)$ and $L\bar{g} = \overline{Lg} \in \mathcal{L}_2(G)$ and Green's second formula (2.8) with $u = f$ and $v = \bar{g}$ takes the form:

$$\int_G (\bar{g}Lf - f\overline{Lg})dx = \int_G (\bar{g}Lf - fL\bar{g})dx$$

$$(Lf, g) - (f, Lg) = \int_S p(f\frac{\partial \bar{g}}{\partial n} - \bar{g}\frac{\partial f}{\partial n})ds \tag{2.10}$$

Moreover the functions f and g satisfy boundary condition (2.3):

$$\alpha f + \beta\frac{\partial f}{\partial n}|_S = 0, \alpha\bar{g} + \beta\frac{\partial \bar{g}}{\partial n}|_S = 0 \tag{2.11}$$

By supposition $\alpha + \beta > 0$ over S. Therefore the homogeneous system of linear algebraic equations (2.11) has a nonzero solution (α, β) and so its determinant is equal to zero , that is,

$$\begin{vmatrix} f & \frac{\partial f}{\partial n} \\ \bar{g} & \frac{\partial \bar{g}}{\partial n} \end{vmatrix} = f\frac{\partial \bar{g}}{\partial n} - \bar{g}\frac{\partial f}{\partial n}|_S = 0$$

Taking this into equation (2.10) , then we obtain $(Lf, g) - (f, Lg) = 0$ This implies that

$$(Lf, g) = (f, Lg)$$

This means that the operator is Hermitian.
Let $f \in M_L$, setting $u = f$ and $v = \bar{f}$ in Green's first formula and take into consideration $Lf \in \mathcal{L}_2(G)$,

$$\int_G vLudx = \int_G p\sum_{i=1}^{n}\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i}dx - \int_S pv\frac{\partial u}{\partial n}ds + \int_G quvdx$$

$$\begin{aligned} \int \bar{f}Lfdx &= (Lf, f) \\ &= \int p\sum_{i=1}^{n}\frac{\partial \bar{f}}{\partial x_i}\frac{\partial f}{\partial x_i}dx - \int p\bar{f}\frac{\partial f}{\partial n}ds + \int_G qf\bar{f}dx \\ &= \int p|gradf|^2dx - \int p\bar{f}\frac{\partial f}{\partial n}ds + \int q|f|^2dx \end{aligned}$$

Therefore,

$$\int \bar{f}Lfdx = \int p|gradf|^2dx - \int p\bar{f}\frac{\partial f}{\partial n}ds + \int q|f|^2dx \tag{2.12}$$

From the boundary condition (2.3)

$$\alpha f + \beta\frac{\partial f}{\partial n}|_s = 0 \Rightarrow \frac{\partial f}{\partial n} = -\frac{\alpha}{\beta}f, x \in S, \beta(x) > 0$$

$f = 0$ if $\beta(x) = 0, x \in S$

Substituting these results, then we obtain an expression for quadratic form

$$(Lf, f) = \int p|gradf|^2 dx + \int p\bar{f}\frac{\alpha}{\beta} f ds + \int q|f|^2 dx$$

$\Rightarrow$

$$(Lf, f) = \int_G (p|gradf|^2 + \int_G q|f|^2) dx + \int_{S_o} p\frac{\alpha}{\beta}|f|^2 ds, f \in \mathcal{M}_L \qquad (2.13)$$

where S_o is that part of S on which $\alpha(x), \beta(x) > 0$.

The quadratic form $(Lf, f), f \in \mathcal{M}_L$, is known as the energy integral. By virtue of suppositions (2.4), all three terms in the right-hand side of (2.13) are nonnegative. Therefore , disregarding the second and third terms and underestimating the first term , we obtain the inequality

$$(Lf, f) \geq \int_G p|gradf|^2 dx \geq \min_{x \in \bar{G}} p(x) \int_G |gradf|^2 dx$$

That is ,

$$(Lf, f) \geq P_o|| \ |gradf| \ ||^2, f \in \mathcal{M}_L \qquad (2.14)$$

where $P_o = \min p(x)$; since the function p is continuous and positive over $\bar{G}$, $P_o > 0$.

It follows from inequality (2.14) that the operator L is positive , that is,

$$(Lf, f) \geq 0, \quad f \in \mathcal{M}_L \qquad (2.15)$$

From this, specifically, it follows once more that the operator L is Hermitian.(sec 1.4)

2.4 Properties of Eigenvalues and Eigenfunctions of the operator L

All eigenvalues of the operator L are nonnegative because the operator is positive and the eigenfunctions of the operator L corresponding to the different eigenvalues are orthogonal and can be stated by the following theorem.

Theorem 2.1. *[1]* If the operator L is Hermitian(positive), all its eigenvalues are real(non-negative) and its eigenfunctions, corresponding to different eigenvalues are orthogonal.

Proof. Let λ_o be an eigenvalue and u_0 a corresponding normalized eigenfunction of the Hermitian operator L, $Lu_o = \lambda_o u_o$. Scalar multiplication of this equation by u_0 will give

$$(Lu_o, u_o) = (\lambda_o u_o, u_o) = \lambda_o(u_o, u_o) = \lambda_o||u_o||^2 = \lambda_o \tag{2.16}$$

But for a Hermitian(positive) operator the quadratic form (Lf, f) assumes only real (non-negative) values and consequently, by virtue of (2.15), λ_o is real(non-negative) number. And also we can prove that any eigenfunctions u_1 and u_2 corresponding to different eigenvalues λ_1 and λ_2 are orthogonal. From the results $Lu_1 = \lambda_1 u_1$, $Lu_2 = \lambda_2 u_2$ and from the Hermitian properties of the operator L we obtain the sequence of equations:

$$\begin{aligned} \lambda_1(u_1, u_2) &= (\lambda_1 u_1, u_2) \\ &= (Lu_1, u_2) \\ &= (u_1, Lu_2) \\ &= (u_1, \lambda_2 u_2) \\ &= \lambda_2(u_1, u_2) \end{aligned}$$

That is, $(\lambda_1 - \lambda_2)(u_1, u_2) = 0$
From this as $\lambda_1 \neq \lambda_2$, it follows that $(u_1, u_2) = 0$ □

The eigenfunction of the operator L may be chosen to be real, since the operator L is real.

Verification: Let λ_o be an eigenvalue and u_o the corresponding eigenfunction of the operator L,

$$Lu_0 = \lambda_0 u_0, \ \ u_0 \in \mathcal{M}_L \tag{2.17}$$

Separating the real and imaginary parts of them in equation (2.17), we find that the real and imaginary parts of the eigenfunction $u_0 = u_1 + iu_2$ which are distinct from zero are also eigenfunction corresponding to the eigenvalue λ_0;

$$Lu_j = \lambda_0 u_j, j = 1, 2$$

That is $Lu_1 = \lambda_0 u_1, Lu_2 = \lambda_0 u_2$

Lemma 2.1. *[1]* In order that $\lambda = 0$ be an eigenvalue of the operator L, it is necessary and sufficient that $q = 0$ and $\alpha = 0$. For this $\lambda = 0$ is a simple eigenvalue and $u_0 = constant$ is the corresponding eigenfunction.

Proof. Let $\lambda = 0$ be the eigenvalue of the operator L and let u_0 be the corresponding eigenfunction, so that $Lu_0 = 0$, $u_0 \in \mathcal{M}_L$, then using

$$(Lf, f) = \int_G (p|grad\ f|^2 + q|f|^2)dx + \int_{S_o} p\frac{\alpha}{\beta}|f|^2 ds$$

we obtain

$$0 = (Lu_0, u_0) = \int_G (p|grad\ u_0|^2 + q|u_0|^2)dx + \int_{S_o} p\frac{\alpha}{\beta}|u_0|^2 ds$$

From which, taking supposition (2.4) into account , we get

$$p\ grad\ u_0 = 0, qu_0 = 0, x \in G$$

That is, $u_0 = constant$ and $q = 0$. It follows from boundary condition (2.2) for the eigenfunction $u_0 = constant$ that $\alpha = 0$. The necessity of the conditions is proved. Moreover, it is established that $u_0 = constant$ is a unique eigenfunction corresponding to the eigenvalue $\lambda = 0$; that is the eigenvalue is simple.

If $q = 0$ and $\alpha = 0$, then by virtue of (2.4), $\beta > 0$ and the problem (2.1-2.2) becomes:

$$-div(p\ grad\ u) = \lambda u, \frac{\partial u}{\partial n}|_s = 0$$

for which $u_0 = constatnt$ is the eigenfunction corresponding to the eigenvalue $\lambda = 0$. □

Theorem 2.2. *[6]* The system of eigenfunctions of the operator L is complete in $\mathcal{L}_2(G)$.

First we make the following assumptions:

A_1 . The Hermitian operator L on $\mathcal{L}_2$ has domain $\mathcal{M}_{\mathcal{L}}$ that is dense in $\mathcal{L}_2$

A_2 . The operator L bounded from below, so we assume that

$$L \geq 0, \ \textit{or equivalently} \ \ F(f) \geq 0, \quad f \in \mathcal{L}_2 \tag{2.18}$$

A_3 . The operator L has an orthonormal set of eigenfunctions u_j, for j=1,2,... with λ_j the corresponding eigenvalues. We order the eigenvalues so $\lambda_1 \leq \lambda_2 \leq$ ..... Also we make the following two properties

P_1 . The minimax property:

$$\lambda_{N+1} = \inf_{f \perp u_1, u_2, .. u_n,\ f \in \mathcal{M}_{\mathcal{L}}} F(f) = \inf_{f \perp u_1, u_2, .. u_n,\ f \in \mathcal{M}_{\mathcal{L}}} \frac{(f, Lf)}{(f, f)}$$

P_2 . The unboundedness property:
$0 \leq \lambda_N \to +\infty$ as $N \to \infty$. For any $f \in \mathcal{L}_2$ define the partial sum f_N of the first N eigenfunctions,

$$f_N = \sum_{j=1}^{N} (u_j, f) u_j$$

The assumption $A_1 - A_2$ ensures that the $\{u_j\}$ are a basis for $\mathcal{L}_2$.

Proof. First note that for any $f \in \mathcal{L}_2$ the approximation f_N satisfies the Bessel's inequality(1.15):

$$||f_N|| \leq ||f|| \tag{2.19}$$

Second remark that it is sufficient to prove that $\lim_{N\to\infty} ||f - f_N|| = 0$ holds for f in a dense subset $\mathcal{M} \subset \mathcal{L}_2$. For in that case, given $\varepsilon > 0$ and $f \in \mathcal{L}_2$ there exists $g \in \mathcal{M}$ with $||f - g|| < \varepsilon$. Then

$$||f - f_N|| = ||(f - g) + (g - g_N) + (g_N - f_N)|| \leq ||f - g|| + ||g - g_N|| + ||g_N - f_N||$$

But $(g_N - f_N) = (g - f)_N$, so using Bessel's inequality we infer

$$||f - f_N|| \leq 2||f - g|| + ||g - g_N|| \leq 2\varepsilon + ||g - g_N||$$

As the the approximation property $\lim_{N\to\infty} ||f - f_N|| = 0$ holds for $g \in \mathcal{M}$, there is $N_0 = N_0(\varepsilon, g)$ Sufficiently large such that $||g - g_N|| < \varepsilon$, for $N > N_0$. Therefore

$$||f - f_N|| \leq 3\varepsilon, \;\; forall \;\; N > N_0 \tag{2.20}$$

This is arbitrary small, so $\lim_{N\to\infty} ||f - f_N|| = 0$ holds for all $f \in \mathcal{L}_2$ as claimed. We now complete the proof by showing that $\lim_{N\to\infty} ||f - f_N|| = 0$ holds for all f in the dense set $\mathcal{M} = \mathcal{M}_{\mathcal{L}}$, the domain of L. For $f \in \mathcal{M}_{\mathcal{L}}$, we claim that$(f - f_N, f - f_N)$ satisfies the upper and lower bounds;

$$\lambda_{N+1} ||f - f_N||^2 \leq (f - f_N, L(f - f_N)) \leq (f, Lf) \tag{2.21}$$

Assuming (2.21), we obtain the following results; as $||f - f_N||^2 \leq \frac{(f,Lf)}{\lambda_{N+1}}$ property P_1 ensures $\lambda_{N+1} \to \infty$ showing $||f - f_N|| \to 0$. So we need only establish (2.21) The lower bound is just a restatement of the variational property P_1, namely

$$\lambda_{N+1} \leq F(f - f_N) = \frac{(f - f_N, L(f - f_N))}{(f - f_N, f - f_N)}$$

The upper bound follows from the equality $(f_N, Lf) = (f_N, Lf_N) = (f, Lf_N)$.For then

$$(f - f_N, L(f - f_N)) = (f, Lf) - (f, Lf_N) = (f, Lf) - (f_N, Lf_N) \leq (f, Lf)$$

In the last step we appeal to the positivity of L, which ensures that $-(f_N, Lf_N) \leq 0$ □

Note: The results obtained may be extended to the boundary value problem involving the eigenvalues

$$Lu = \lambda \varrho u, \alpha u + \beta \frac{\partial u}{\partial n}|_S = 0$$

where the weight $\varrho(x) > 0$ is a continuous function over $\bar{G}$ if this problem is considered in the space $\mathcal{L}_2(G; \varrho)$.

2.5 The Fourier Method(Separation of Variables)

The Fourier method may be used to define the eigenvalues and eigenfunctions of a many-dimensional elliptic operator which permits separation of its variables. The essence of this method is as follows. We divide the independent variables into two groups, $x = (x_1, x_2, ...x_n)$ and $y = (y_1, y_2, ..., y_m)$, and let $G \subset R^n$ be the region of variation of x and $D \subset R^m$ be the region of variation of y. We shall use S and Γ to denote the boundaries of the region G and D, respectively. Then $(S \times \bar{D}) \cup (\Gamma \times \bar{G})$ is the boundary of the region $G \times D \subset R^{n+m}$. In the region $G \times D$ we shall examine the following boundary value problem involving the eigenvalues of an equation of elliptic type

$$Lu + Mu = \lambda u \tag{2.22}$$

$$\alpha u + \beta \frac{\partial u}{\partial n}|_{S \times \bar{D}} = 0, \quad \gamma u + \delta \frac{\partial u}{\partial n}|_{\Gamma \times \bar{G}} = 0 \tag{2.23}$$

Where L and M are elliptic operators not depending on y and x, respectively; the functions α, β do not depend on y and the functions γ, δ do not depend

on x.
We shall seek the eigenfunctions of problem (2.22)-(2.23) in the form of the product $X(x)Y(y)$,

$$u(x, y) = X(x)Y(y) \tag{2.24}$$

Substituting this expression into equation(2.22), we obtain

$$Y(y)LX(x) + X(x)MY(y) = \lambda X(x)Y(y)$$

from which

$$\frac{LX(x)}{X(x)} = \lambda - \frac{MY(y)}{Y(y)} \tag{2.25}$$

The left hand side of equation(2.25) does not depend on y, nor the right hand side on x. Therefore these expressions do not depend either on x or y; that is, they are equal to a constant. Denoting this constant by μ and writing $v = \lambda - \mu$, from (2.25) we obtain two equations:

$$LX = \mu X \tag{2.26}$$

$$MY = vY \tag{2.27}$$

In this way, equation(2.22) has split in to two equations (2.26) and (2.27), or as it is said, the variable has been separated;in addition, an unknown parameter μ has appeared. To deduce the boundary conditions for the functions $X(x)$ and $Y(y)$ we shall substitute the product $X(x)Y(y)$ in the boundary conditions (2.23). As a result,after abbreviation,we obtain

$$\alpha X + \beta \frac{\partial X}{\partial n}|_S = 0 \tag{2.28}$$

$$\gamma Y + \delta \frac{\partial Y}{\partial n}|_\Gamma = 0 \tag{2.29}$$

So the boundary value problem involving the eigenvalues (2.22)(2.23) becomes two boundary value problems involving the eigenvalues (2.26)-(2.28) and (2.27)-(2.29) with a smaller number of independent variables. We shall denote by $\mu_k, X_k(x), k = 1, 2, ...$, and $v_j, Y_j(y), j = 1, 2, ...$, all the eigenvalues and the eigenfunctions of the operators L and M, respectively. By virtue of (2.24),

$$\lambda_{kj} = \mu_k + v_j, \quad u_{kj} = X_k(x)Y_j(y), \quad k, j = 1, 2, ... \tag{2.30}$$

are the eigenvalues and the eigenfunctions of the initial boundary value problem (2.22)-(2.23).

Example 2.1. *Consider a boundary value problem involving the eigenvalues for a rectangle* $\Pi = (0,l) \times (0,m)$ *with the boundary* L

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = \lambda u, \quad u|_L = 0 \tag{2.31}$$

Solution: In according with the scheme set out in section(2.5), this problem may be divided into two one-dimensional boundary value problems:

$$-X'' = \mu X, \quad X(0) = X(l) = 0 \tag{2.32}$$

$$-Y'' = vY, \quad Y(0) = Y(m) = 0 \tag{2.33}$$

The eigenvalues and eigenfunctions of these boundary value problem are easily calculated. We write out the general solution of the differential equation (2.32)

$$X(x) = c_1 \sin\sqrt{\mu}x + c_2 \cos\sqrt{\mu}x$$

and select the arbitrary constants c_1 *and* c_2 *and the parameter* μ *so as to satisfy the boundary conditions (2.32) and the normalization condition* $||X|| = 1$. *For this it is necessary to put* $c_2 = 0$ *and* $\sqrt{\mu}l = k\pi, k = \pm 1, \pm 2, ...$, *so that*

$$X(x) = c_1 \sin\frac{k\pi x}{l}$$

The normalization condition

$$1 = \int_0^l X^2(x)dx = c_1^2 \int_0^l \sin^2\frac{k\pi x}{l}dx = \frac{l}{2}c_1^2$$

gives $c_1 = \sqrt{\frac{2}{l}}$ *and , therefore,*

$$\mu_k = (\frac{k\pi}{l})^2, \quad X_k(x) = \sqrt{\frac{2}{l}}\sin\frac{k\pi x}{l}, \quad k = 1, 2, ... \tag{2.34}$$

Analogously for problem (2.33) we have

$$v_j = (\frac{j\pi}{m})^2, \quad Y_j(y) = \sqrt{\frac{2}{m}}\sin\frac{j\pi y}{m}, \quad j = 1, 2, ... \tag{2.35}$$

From (2.34) and (2.35), in accordance with equation (2.30) we obtain the following eigenvalues and eigenfunctions of the boundary value problem (2.31):

$$\lambda_{kj} = \pi^2(\frac{k^2}{l^2} + \frac{j^2}{m^2}), \quad u_{kj}(x,y) = \frac{2}{\sqrt{lm}}\sin\frac{k\pi x}{l}\sin\frac{j\pi y}{m}, k, j = 1, 2, .. \tag{2.36}$$

Chapter 3

The Sturm-Liouville Problem

For $n = 1$ the problem involving the eigenvalues (2.2)-(2.3) is known as the *Sturm-Liouville problem*,

$$Ly \equiv -(py')' + qy = \lambda y, 0 < x < l \tag{3.1}$$

$$h_1 y(0) - h_2 y'(0) = 0, H_1 y(l) + H_2 y'(l) = 0 \tag{3.2}$$

According to the conditions (2.4) we consider

$$p \in C^1([0, l]), q \in C([0.l]), p(x) > 0, q(x) \geq 0$$

$$h_1 \geq 0, h_2 \geq 0, H_1 \geq 0, H_2 \geq 0, h_1 + h_2 > 0, H_1 + H_2 > 0$$

We recall that the domain of definition $\mathcal{M}_L$ of the operator L consists of the functions $y(x)$ of the class $C^2(0, l) \cap C^1([0, l]), y'' \in \mathcal{L}_2(0, l)$ satisfying the boundary conditions (3.2). Finding the value of λ for which there exists a nontrivial solution of (3.1) satisfying the boundary conditions is called the Sturm-Liouville problem. Expression (2.13) for the quadratic form (Lf, f),$f \in \mathcal{M}_L$,takes the following form:

$$(Lf, f) = \int_0^l (p|f'|^2 + q|f|^2)dx + \frac{h_1}{h_2} p(0)|f(0)|^2 + \frac{H_1}{H_2} p(l)|f(l)|^2$$

(the last terms to be excluded if $h_2 = 0$ or $H_2 = 0$,respectively).

3.1 Green's Function

In this section we derive an integral representation for the solution to boundary value problem $Ly \equiv -(py')' + qy = f(x), y \in \mathcal{M}_L$. Namely, we show that the solution can be expressed in the form

$$y(x) = \int_0^l \mathcal{G}(x, s) f(s) ds$$

where the function $\mathcal{G}(x, s)$ is called a Green's function.

Lemma 3.1. *[1] If $\lambda = 0$ is not an eigenvalue of the operator L,then the solution of the boundary value problem*

$$Ly \equiv -(py')' + qy = f(x), y \in \mathcal{M}_L \tag{3.3}$$

exists and is expressed by the equation

$$y(x) = \int_0^l \mathcal{G}(x, s) f(s) ds$$

where

$$\mathcal{G}(x, s) = -\frac{1}{C} \begin{cases} v_1(x)v_2(s), & 0 \leq x \leq S \\ v_2(x)v_1(s), & S \leq x \leq l \end{cases} \tag{3.4}$$

And C=p(x)W(X)=constant

Proof. Let us suppose that $\lambda = 0$ is not an eigenvalue of the operator L;this means,by vitue of the lemma(2.1)that either $q \neq 0$,or $h_1 \neq 0$, or $H_1 \neq 0$. Consider the boundary value problem

$$Ly \equiv -(py')' + qy = f(x), y \in \mathcal{M}_L \tag{3.5}$$

where $f \in C(0, l) \cap \mathcal{L}_2(0, l)$. Since $\lambda = 0$ is not an eigenvalue of the operator L, the solution of the boundary value problem (3.5) in the class $\mathcal{M}_L$ is unique. We shall construct the solution of this problem.
Let v_1 and v_2 be non-zero(real) solution of the homogeneous equation $Lv = o$, satisfying the conditions

$$h_1 v_1(0) - h_2 {v_1}'(0) = 0, H_1 v_2(l) + H_2 v_2'(l) = 0 \tag{3.6}$$

It follows from the theory of ordinary differential equations that such solutions always exist and belong to the class $C^2([0, l])$. The solution v_1 and v_2 are linearly independent. In fact, in the opposite case $v_1(x) = cv_2(x)$ and therefore, by virtue of (3.6), the solution v_1 also satisfies the second boundary condition (3.2). This means that v_1 is an eigenfunction of the operator L corresponding to the eigenvalue $\lambda = 0$, despite the supposition. Therefore the Wronskian determinant

$$W(x) = \begin{vmatrix} v_1(x) & v_2(x) \\ v_1'(x) & v_2'(x) \end{vmatrix} \neq 0, x \in [0, l]$$

and it holds the Ostrogradski-Liouville identity :

$$p(x)W(x) = C, C = constant \ x \in [0, l] \tag{3.7}$$

We shall seek the solution of problem (3.5) by the method of variation of parameter,but we first write the non-homogenous equation (3.5) in the standard form

$$y'' + \frac{p'}{p}y' - \frac{q}{p}y = -\frac{f}{p}$$

and its solution is given by

$$y(x) = C_1(x)v_1(x) + C_2(x)v_2(x) \tag{3.8}$$

In accordance with this method,the functions C_1 and C_2 must satisfy the system of linear differential equations

$$C_1'v_1 + C_2'v_2 = 0, C_1'V_1' + C_2'v_2' = -\frac{f}{p} \tag{3.9}$$

with the determinant $w(x) \neq 0$. when we solve this system using cramer's rule and identity (3.7), we shall obtain

$$C_1' = \frac{1}{W}\begin{vmatrix} 0 & V_2 \\ -\frac{f}{p} & v_2' \end{vmatrix} = \frac{f(x)v_2(x)}{C}$$

$$C_2' = \frac{1}{W}\begin{vmatrix} v_1 & 0 \\ v_1' & -\frac{f}{p} \end{vmatrix} = -\frac{f(x)v_1(x)}{C} \tag{3.10}$$

For $y(x)$to satisfy boundary conditions (3.2), we put $C_2(0) = 0, C_1(l) = 0$. Integrating (3.10) using the conditions $C_1(l) = 0, C_2(0) = 0$, and Since we are free to pick the constants in the antiderivatives for C_1' and C_2', it will turn out to be convenient to choose

$$C_1(x) = -\frac{1}{C}\int_x^l f(s)v_2(s)ds$$

$$C_2(x) = -\frac{1}{C}\int_0^x f(s)v_1(s)ds$$

If we substitute those expressions into (3.8), we find the required solution of problem (3.5) in the form

$$y(x) = -\frac{1}{C}[v_2(x)\int_0^x f(s)v_1(y)ds + v_1(x)\int_x^l f(s)v_2(s)ds]$$

or

$$y(x) = \int_0^l \mathcal{G}(x,s)f(s)ds \tag{3.11}$$

where

$$\mathcal{G}(x,s) = -\frac{1}{C}\begin{cases} v_1(s)v_2(x), & 0 \le s \le x \\ v_2(s)v_1(x), & x \le s \le l \end{cases} \tag{3.12}$$

The function $\mathcal{G}(x,s)$ is known as the Green's function of the boundary value problem (3.5), or of the operator L. $y(x)$ also satisfies the boundary condition (3.6). □

To check that $p(x)W(x) = c$ let $Ly_1 = \lambda y_1$ and $Ly_2 = \lambda y_2$, multiply the first and the second by y_2 and y_1 respectively. Then subtracting the second from the first we obtain $(p(y_1y_2' - y_2y_1'))' = 0$ that is as required.

Example 3.1. The Green's function of the boundary value problem

$$-y'' = f(x), u(0) = u(1) = 0$$

has the form

$$\mathcal{G}(x,s) = \begin{cases} s(1-x), & 0 \le s \le x \\ (1-s)x, & x \le s \le 1 \end{cases}$$

Solution: The general solution of the homogenous problem $y'' = 0$ is $y_h(x) = Ax + B$,so $v_1(x)$ and $v_2(x)$ must be of this form. To get $v_1(x)$ we want to choose A and B so that $v_1(0) = B = 0$. Since A is arbitrary we can set it equal to 1. Hence we take $v_1(x) = x$. To get $v_2(x)$ we want to choose A and B so that $v_2(1) = A + B = 0$. Thus $B = -A$ taking $A = 1$, we get $v_2(x) = x - 1$. Compute

$$p(x)W[v_1, v_2](x) = 1(v_2'v_1 - v_1'v_2) = 1$$

Next is

$$\mathcal{G}(x,s) = -\frac{1}{p(x)W[v_1v_2' - v_1'v_2](x)}\begin{cases} v_1(s)v_2(x), & 0 \le s \le x \\ v_2(s)v_1(x), & x \le s \le 1 \end{cases}$$

substituting the values we get

$$\mathcal{G}(x,s) = \begin{cases} s(1-x), & 0 \le s \le x \\ x(1-s), & x \le s \le 1 \end{cases}$$

Properties of Green's function

Theorem 3.1. *[4] Let $\mathcal{G}(x,s)$ be the Green's function defined by (3.4) for the boundary value problem (3.5). Then*

a . *$\mathcal{G}(x,s)$ is continuous on the square $[0,l] \times [o,l]$. For fixed s the partial derivatives $(\frac{\partial \mathcal{G}}{\partial x})(x,s)$ and $(\frac{\partial^2 \mathcal{G}}{\partial x^2})(x,s)$ are continuous functions of x for $x \ne s$.*

b . *At $x = s$ the partial derivative $\frac{\partial \mathcal{G}}{\partial x}$ has a jump discontinuity:*

$$\lim_{x \to s^+} \frac{\partial \mathcal{G}}{\partial x}(x,s) - \lim_{x \to s^-} \frac{\partial \mathcal{G}}{\partial x}(x,s) = \frac{-1}{p(s)} \tag{3.13}$$

c . *For each fixed s, the function $\mathcal{G}(x,s)$ satisfies the corresponding homogenous problem for $x \neq s$; that is,*

$$L[\mathcal{G}(x,s)] = 0, for x \neq s; \tag{3.14}$$

$$h_1\mathcal{G}(0,s) - h_2\frac{\partial \mathcal{G}}{\partial x}(0,s) = 0, \quad H_1\mathcal{G}(l,s) + H_2\frac{\partial \mathcal{G}}{\partial x}(l,s) = 0. \tag{3.15}$$

d . *There is only one function satisfying properties (a)-(c).*

e . *$\mathcal{G}(x,s)$ is symmetric ; that is ,*

$$\mathcal{G}(x,s) = \mathcal{G}(s,x)$$

.

Proof. **a** . Since v_1 and v_2 are solutions to L[y]=0 on [0,l], they are continuous and have continuous first and second derivatives. Hence, it follows from the formula for $\mathcal{G}(x,s)$ that $\frac{\partial \mathcal{G}}{\partial x}$ and $\frac{\partial^2 \mathcal{G}}{\partial x^2}$ are continuous functions of x for $x \neq s$.

b . Using p(x)W(x)=C ,C=constant we find

$$\begin{aligned} \lim_{x \to s^+} \frac{\partial \mathcal{G}}{\partial x}(x,s) - \lim_{x \to s^-} \frac{\partial \mathcal{G}}{\partial x}(x,s) &= \frac{-v_1(s)v_2'(s)}{c} - \frac{-v_1'(s)v_2(s)}{c} \\ &= \frac{-W[v_1,v_2](s)}{c} \\ &= \frac{-1}{p(s)} \end{aligned}$$

c . For fixed s,the function $\mathcal{G}(x,s)$ is a constant multiple of $v_1(x)$ when $x < s$ and a constant multiple of $v_2(x)$ when $s < x$. Since both v_1 and v_2 satisfy $L[y] = 0$,then so does $\mathcal{G}(.,s)$ for $x \neq s$. Next we verify (3.15). When $x < s$, we have $\mathcal{G}(x,s) = (\frac{-v_2(s)}{C})v_1(x)$. Since $v_1(x)$ satisfies the first of (3.2), we have $h_1\mathcal{G}(0,s) - h_2\frac{\partial \mathcal{G}}{\partial x}(0,s) = \frac{-v_2(s)}{C}[h_1v_1(0) - h_2v_1'(0)] = 0$. Since $v_2(x)$ satisfies the second (3.2), we find $H_1\mathcal{G}(l,s) + H_2\frac{\partial \mathcal{G}}{\partial x}(l,s) = (\frac{-v_1(s)}{C})[H_1v_2(l) + H_2v_2'(l)] = 0$ thus, for fixed s,$\mathcal{G}(.,s)$ satisfies (3.14)-(3.15).

d . Assume that both $\mathcal{G}(x,s)$ and $\mathcal{H}(x,s)$ satisfy properties (3.1)-(3.2). We will show that

$$\mathcal{K}(x,s) = \mathcal{G}(x,s) - \mathcal{H}(x,s)$$

is identically zero on $[0,l] \times [0,l]$. For this purpose, fix $s = s_0$. By property (a), $\mathcal{K}(x,s_0)$ is continuous on $[0,l]$, and $(\frac{\partial \mathcal{K}}{\partial x})(x,s_0)$ and $(\frac{\partial^2 \mathcal{K}}{\partial x^2})(x,s_0)$ are continuous for $x \neq s_0$. More over since property (b) says that $(\frac{\partial \mathcal{G}}{\partial x})(x,s_0)$ and $(\frac{\partial \mathcal{H}}{\partial x})(x,s_0)$ have equal jumps at $x = s_0$, we see that $(\frac{\partial \mathcal{K}}{\partial x})(x,s_0) = (\frac{\partial \mathcal{G}}{\partial x})(x,s_0) - (\frac{\partial \mathcal{H}}{\partial x})(x,s_0)$ exists and continuous (has no jump) at $x = s_0$. Hence $\mathcal{K}(x,s_0)$ and $(\frac{\partial \mathcal{K}}{\partial x})(x,s_0)$ are continuous on $[0,l]$. From property (c) equation (3.14), it follows that for $x \neq s_0$, $L[\mathcal{K}(.,s_0)] = 0$. Solving this equation for $(\frac{\partial^2 \mathcal{K}}{\partial x^2})(x,s_0)$ we find that $(\frac{\partial^2 \mathcal{K}}{\partial x^2})(x,s_0)$ is equal to a function that is continuous at $x = s_0$. Consequently $(\frac{\partial^2 \mathcal{K}}{\partial x^2})(x,s_0)$ is continuous on $[0,l]$. Thus $\mathcal{K}(x,s_0)$ is a solution to the homogenous boundary value problem on the whole interval $[0,l]$. But this problem has only the trivial solution, so $\mathcal{K}(x,s_0) \equiv 0$. Finally, since s_0 is arbitrary,$\mathcal{K}(x,s) \equiv 0$ on $[0,l] \times [0,l]$,that is, the Green's function is unique.

e . The symmetry follows from the definition of $\mathcal{G}(x,s)$ in (3.12); simply interchanging x and s in the formula.

□

The Green's function can be characterized using the dirac delta function. If we assume that there exists a function $\mathcal{G}(x,s)$ such that $y(x) = \int_0^l \mathcal{G}(x,s)f(s)ds$ is the solution to (3.2)-(3.5), then operating on both sides with L, and assuming we can interchange differentiation and integration, gives

$$L[y](x) = \int_0^l L[\mathcal{G}(x,s)]f(s)ds = f(x) \tag{3.16}$$

Using the propert y of the delta-function this can be written as

$$L[y] = \int [L[\mathcal{G}(x,s)] - \delta(x-s)]f(s)ds = 0$$

For this to hold for any function f , it must be the case that $L[\mathcal{G}(x,s)] = \delta(x-s)$

3.2 Reduction of the Sturm-Liouville Problem to an Integral Equation

We can show that the Sturm-Liouville problem may be reduced to a Fredholm integral equation with a symmetric,and continuous kernel $\mathcal{G}(x,s)$.

Definition 3.2.0.1. *An integral equation is an equation whose unknown appears under an integral sign. That is it is an equation of the form*

$$\psi(x) = f(x) + \lambda \int_0^l \mathcal{G}(x,s)\psi(s)ds \tag{3.17}$$

where f and $\mathcal{G}$ are known functions defined on $[0,l]$ and $[0,l]\times[0,l]$ respectively and ψ is an unknown function. The function $\mathcal{G}$ is called the kernel of the integral equation.

Definition 3.2.0.2. *Fredholm integral equation(FIE) is an integral equation of the form*

$$\psi(x) = f(x) + \lambda \int_0^l \mathcal{G}(x,s)\psi(s)ds \quad (FIE\ of\ the\ second\ kind) \tag{3.18}$$

$$f(x) = \int_0^l \mathcal{G}(x,s)\psi(s)ds \quad (FIE\ of\ the\ first\ kind)$$

where 0 and l are fixed numbers,f and $\mathcal{G}$ are as in (3.17) and ψ is the required(unknown) function.

Example 3.2. $\phi(x) = 3x - x^3 + \int_0^1 (2x - 5t - 1)\phi(t)dt$ *is FIE of the second kind. Where as* $3x - x^3 = \int_0^1 (2x - 5t - 1)\phi(t)dt$ *is a FIE of the first kind.*

Theorem 3.2. *[1]* The boundary value problem

$$Ly = \lambda y + f, y \in \mathcal{M}_L, f \in C(0,l) \cap \mathcal{L}_2(0,l) \tag{3.19}$$

$$h_1 y(0) - h_2 y'(0) = 0, H_1 y(l) + H_2 y'(l) = 0$$

for which $\lambda = 0$ is not an eigenvalue of the operator L is equivalent to the integral equation

$$y(x) = \lambda \int_0^l \mathcal{G}(x,s)y(s)ds + \int_0^l \mathcal{G}(x,s)f(s)ds, y \in C([0,l]) \tag{3.20}$$

where $\mathcal{G}(x,s)$is the Green's function of the operator L.

Proof. If $y(x)$ is the solution of the boundary value problem (3.19) then applying the lemma (3.1) with a change of to $\lambda y + f$,we obtain

$$y(x) = \int_0^l \mathcal{G}(x,s)[\lambda y(s) + f(s)]ds$$

that is, the function $y(x)$ satisfies integral equation(3.20). Conversely,let the function $y_0 \in C([0,l])$ satisfies the integral equation (3.20). Consider the boundary value problem

$$Ly = \lambda y_0 + f, y \in \mathcal{M}_L$$

By the above lemma and theorem(3.1-d) the unique solution of this problem is given by the equation

$$y(x) = \int_0^l \mathcal{G}(x,s)[\lambda y_0(s) + f(s)]ds = y_0(x)$$

from which it follows that $y_0 \in \mathcal{M}_L$ and satisfies the equation

$$Ly_0 = \lambda y_0 + f$$

that is y_0 is the solution of the boundary value problem (3.19).

When $f = 0$ the boundary value problem (3.19) is the Sturm-Liouville problem,and therefore the Sturm-Liouville problem (3.1)-(3.2) is equivalent to a problem involving the eigenvalue of the homogeneous integral equation

$$y(x) = \lambda \int_0^l \mathcal{G}(x,s)y(s)ds \tag{3.21}$$

provided that $\lambda = 0$ is not an eigenvalue of the operator L. □

We can eliminate the assumption that $\lambda = 0$ is not an eigenvalue of the operator L. For this we note that,by virtue of the lemma (2.1) $\mu = 0$ is not an eigenvalue of the Sturm-Liouville problem

$$L_1 y = -(py')' + (q+1)y = \mu y \tag{3.22}$$

$$h_1 y(0) - h_2 y'(0) = H_1 y(l) + H_2 y'(l) = 0 \tag{3.23}$$

But $\mathcal{M}_L = \mathcal{M}_{L_1}$ and therefore problem(3.22)-(3.23)is equivalent to problem (3.1)-(3.2) for $\mu = \lambda + 1$. Therefore the Sturm-Liouville problem (3.1)-(3.2) is equivalent to the integral equation

$$y(x) = (\lambda + 1)\int_0^l \mathcal{G}_1(x,s)y(s)ds \tag{3.24}$$

where $\mathcal{G}_1(x,s)$is the Green's function of the operator L_1.

Example 3.3. *The integral equation*

$$y(x) = \lambda \int_0^1 \mathcal{K}(x,t)y(t)dt$$

where

$$\mathcal{K}(x,t) == \begin{cases} x(1-t), & x \leq t \leq 1 \\ t(1-x), & 0 \leq t \leq x \end{cases}$$

is equivalent to the boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = y(1) = 0$$

solution: Substituting $\mathcal{K}(x,t)$ we have
$y(x) = \lambda \int_0^x t(1-x)y(t)dt + \lambda \int_x^1 x(1-t)y(t)dt$
If we differentiate $y(x)$ we get (using Lebiniz's formula)
$y'(x) = \lambda \int_0^x -ty(t)dt + \lambda x(1-x)y(x) + \lambda \int_x^1 (1-t)y(t)dt - \lambda x(1-x)y(x)$
$= \lambda \int_0^x -ty(t)dt + \lambda \int_x^1 (1-t)y(t)dt$
and one further differntiation gives us

$$y''(x) = -\lambda xy(x) - \lambda(1-x)y(x) = -\lambda y(x)$$

furthermore we see that $y(0) = y(1) = 0$. Thus the above integral equation is equivalent to the boundary value problem

$$\begin{cases} y''(x) + \lambda y(x) = 0 \\ y(0) = y(1) = 0 \end{cases}$$

On the other hand the solution of $y''(x) + \lambda y(x) = 0, \;\; y(0) = y(1) = 0$ is equivalent to to the above integral equation. To see this let $f = \lambda y$ then it becomes $-y''(x) = f$ which is as an example of the above green's function of the boundary value problem. so $y(x) = \lambda \int_0^1 \mathcal{K}(x,t)y(t)dt$ where $\mathcal{K}(x,t) = \mathcal{G}(x,t)$.

3.3 properties of Eigenvalues and Eigenfunctions

We have established that the Sturm-Liouville problem (3.1)-(3.2) is equivalent to the problem involving the eigenvalues of the homogeneous integral equation (3.24) with a symmetrical(and therefore Hermitian) continuous kernel $\mathcal{G}_1(x,s)$. For this the eigenvalues λ of the problem (3.1)-(3.2) are linked

with the characteristic numbers μ of the kernel $\mathcal{G}_1(x,s)$ by the equation $\mu = \lambda + 1$,and the eigenfunctions corresponding to them coincide.Therefore all the statements of the theory of integral equations with a symmetrical continuous kernel are also valid for the SturmLiouville problem.
The Sturm-Liouville problem has a number of properties:

1. Their Eigenvalues are real and non negative.

   Proof. Let λ be any eigenvalue of $(p\phi')' - q\phi + \lambda\varrho\phi = 0$, with associate eigenfunction $\phi(x)$. If λ is complex, then $\lambda = \lambda_r + i\lambda_i$ and its complex conjugate is $\bar{\lambda} = \lambda_r - i\lambda_i$, with associated eigenfunction $\psi(x) = \overline{\phi(x)}$. Since $\{\lambda, \phi\}$ satisfies

   $$(p\phi')' - q\phi + \lambda\phi\varrho = 0, on(0,1), h_1\phi(0) - h_2\phi'(0) = 0, H_1\phi(l) + H_2\phi'(l) = 0 \quad (3.25)$$

   then, by taking the complex conjugation of the equation (3.25), and noting that p, q and ϱ are real functions, $\bar{\lambda}, \psi$ satisfies

   $$(p\psi')' - q\psi + \bar{\lambda}\varrho\psi = 0, \;\; h_1\psi(0) - h_2\psi'(0) = 0, \;\; H_1\psi(l) + H_2\psi'(l) = 0 \quad (3.26)$$

   So, multiply equation (3.25) by ψ and multiply (3.26) by ϕ, then subtract the two resulting equations. This gives us

   $$\psi(p\phi')' - \phi(p\psi')' + (\lambda - \bar{\lambda})\varrho\psi\phi = 0$$

   Now integrate:

   $$\int_0^l [\psi(p\phi')' - \phi(p\psi')']dx + (\lambda - \bar{\lambda})\int_0^l \varrho\psi\phi dx = 0$$

   By integration-by-parts and substituting the boundary conditions,

   $$\int_0^l [\psi(p\phi')' - \phi(p\psi')']dx =$$

   $$\psi p\phi'|_0^l - \int_0^l \psi' p\phi' dx - \{\phi p\psi'|_0^l - \int_0^l \psi' p\phi' dx\} = 0$$

   Then

   $$(\lambda - \bar{\lambda})\int_0^l \varrho\psi\phi dx = (\lambda - \bar{\lambda})\int_0^l \varrho|\phi|^2 dx = 0$$

   Since $|\phi|^2 = \phi\psi = \phi\bar{\phi} > 0$, then $\lambda = \bar{\lambda}$ which implies λ is real. Let λ, ϕ be any eigenvalue-eigenfunction pair, then by (3.25),$\phi(p\phi')' - q\phi^2 + \lambda\varrho\phi^2 = 0$ on (0,l). So,

   $$\int_0^l \phi(p\phi')'dx - \int_0^l q\phi^2 dx + \int_0^l \lambda\varrho\phi^2 = 0$$

By integration-by-parts, the first integral, after applying the boundary conditions, is $-\frac{H_1}{H_2}p(l)(\phi(l))^2 - \frac{h_1}{h_2}p(0)(\phi(0))^2 - \int_0^l p(\phi')^2$ thus

$$\lambda = \frac{\int_0^l p(\phi')^2 dx + \int_0^l q\phi^2 dx + \frac{h_1}{h_2}p(0)(\phi(0))^2 + \frac{H_1}{H_2}p(l)(\phi(l))^2}{\int_0^l \varrho\phi^2 dx} \geq 0$$

Because of the positivity conditions on $p, q, \varrho, h_1, h_2, H_1, H_2$ and the fact the ϕ is a non-zero function, the nominator is positive, so $\lambda > 0$. □

2. Eigenfunctions corresponding to different eigenvalues are orthogonal with respect to ϱ.

   Proof. Let λ, ϕ, μ, ψ be two arbitrary eigenvalue-eigenfunction pairs as solutions to (3.25), with $\lambda \neq \mu$. Thus,

$$(p\phi')' - q\phi + \lambda\phi\varrho = 0, \quad h_1\phi(0) - h_2\phi'(0) = 0, \quad H_1\phi(l) + H_2\phi'(l) = 0$$

$$(p\psi')' - q\psi + \mu\psi\varrho = 0, \quad h_1\psi(0) - h_2\psi'(0) = 0, \quad H_1\psi(l) + H_2\psi'(l) = 0$$

   Multiply the first equation by ψ, the second equation by ϕ, subtract and integrate:

$$\int_0^l [\psi(p\phi')' - \phi(p\psi')']dx + (\lambda - \mu)\int_0^l \varrho\psi\phi dx = 0$$

   The first integral is 0 via integration-by-parts and boundary conditions. Since $\lambda \neq \mu$, then $\int_0^l \varrho\psi\phi dx = 0$, which was to be proved. □

3. Sturm-Liouville operator is self-adjoint operator or Hermitian.

   Proof. Note:

$$\begin{aligned}(f, Lg) &= \int_0^l \overline{f(x)} Lg(x) dx \\ &= \int_0^l \overline{f(x)}[-(p(x)g'(x))' + q(x)g(x)]dx \\ &= -p(x)g'(x)\overline{f(x)}|_0^l + \int_0^l \overline{f(x)'} p(x)g'(x)dx + \int_0^l q(x)g(x)\overline{f(x)}dx\end{aligned}$$

We get that by integration by part and Similarly,

$$\begin{aligned}(Lf,g) &= \int_0^l L\overline{f(x)}g(x)dx \\ &= -\int_0^l (p(x)\overline{f'(x)})'g(x)dx + \int_0^l q(x)g(x)\overline{f(x)}dx \\ &= -p(x)\overline{f'(x)}g(x)|_0^l + \int_0^l \overline{f'(x)}g'(x)p(x)dx + \int_0^l g(x)q(x)\overline{f(x)}dx\end{aligned}$$

Thus

$$\begin{aligned}(f,Lg)-(Lf,g) &= -p(x)g'(x)\overline{f(x)}|_0^l + p(x)\overline{f'(x)}g(x)|_0^l \\ &= -p(l)g'(l)\overline{f(l)} + p(0)g'(0)\overline{f(0)} + p(l)g(l)\overline{f'(l)} - p(0)g(0)\overline{f'(0)}\end{aligned}$$

Now, since both f, and g obey boundary conditions,

$$h_1f(0) - h_2f'(0) = 0 \;\Rightarrow\; h_1\overline{f(0)} - h_2\overline{f'(0)} = 0$$

$H_1f(l) + H_2f'(l) = 0 \;\Rightarrow\; H_1\overline{f(l)} + H_2\overline{f'(l)} = 0$ and

$$h_1g(0) - h_2g'(0) = 0,\;\; H_1g(l) + H_2g'(l) = 0$$

It is easy to see by substituting the boundary condition that $(Lf,g) = (f,Lg)$.

□

4. Each eigenvalue is simple.
Verification:
In fact,let y_1 and y_2 be eigenfunctions corresponding to the eigenvalue λ_0. This means that those functions satisfy Equation(3.1) for $\lambda = \lambda_0$ and satisfy boundary condition (3.2). From the first boundary condition(3.2)

$$h_1y_1(0) - h_2y_1'(0) = 0, h_1y_2(0) - h_2y_2'(0) = 0$$

it follows, by virtue of the supposition that $h_1 + h_2 > 0$, that

$$\begin{vmatrix} y_1(0) & -y_1'(0) \\ y_2(0) & -y_2' \end{vmatrix} = -\begin{vmatrix} y_1(0) & y_2(0) \\ y_1'((0) & y_2'(0) \end{vmatrix} = 0$$

that is, the wronskian determinant of the solutions $y_1(x)$ and $y_2(x)$ of Equation (3.1) for $\lambda = \lambda_0$ becomes zero at the point $x = 0$. So these solutions are linearly dependent. This also means that λ_0 is a simple eigenvalue of the Sturm-Liouville problem (3.1)-(3.2).

3.4 Finding Eigenvalue and Eigenfunctions

We set out the process of calculating eigenvalues and eigenfunctions of the Sturm-Liouville problem (3.1)-(3.2). Let $y_1(x;\lambda)$ and $y_2(x,\lambda)$ be the solution of Equation (3.1) satisfying the initial conditions:

$$y_1(0;\lambda) = 1, y_1'(0;\lambda) = 0; y_2(0;\lambda) = 0, y_2'(0;\lambda) = 1$$

Then the function

$$y(x;\lambda) = h_2 y_1(x;\lambda) + h_1 y_2(x;\lambda) \tag{3.27}$$

satisfies Equation (3.1) and the first of the boundary conditions (3.2). To satisfy the second of the boundary conditions (3.2), it is necessary to require

$$H_1 h_2 y_1(l;\lambda) + H_1 h_1 y_2(l;\lambda) + H_2 h_2 y_1'(l;\lambda) + H_2 h_1 y_2'(l;\lambda) = 0$$

the roots λ_1, λ_2, ..., of the transcendental equation obtained give all the eigenvalues of the Sturm-Lioville problem (3.1)-(3.2). The corresponding eigenfunctions y_k are defined according to Equation (3.27) for $\lambda = \lambda_k$,

$$y_k(x) = y(x;\lambda_k) = h_2 y_1(x;\lambda_k) + h_1 y_2(x;\lambda_k), k = 1, 2, ...$$

From equation (3.21) we have seen that the Sturm-Liouville problem (3.1)-(3.2) is equivalent to a problem involving the eigenvalue of the homogenous integral equation $y(x) = \lambda \int_0^l \mathcal{G}(x,s)y(s)ds$. This is a homogenous Fredholm integral equation of the second kind. Consider the equation

$$y(x) = f(x) + \lambda \int_0^l \mathcal{G}(x,s)y(s)ds \tag{3.28}$$

A number λ is called a characteristic value of the integral equation (3.28) if there exist nontrivial solutions of the corresponding homogeneous equation (with $f(x) = 0$). The nontrivial solutions themselves are called the eigenfunctions of the integral equation corresponding to the characteristic value λ. If λ is a characteristic value, the number $\frac{1}{\lambda}$ is called an eigenvalue of the integral equation (3.28).Sometimes the characteristic values and the eigenfunctions of a Fredholm integral equation are called the characteristic values and the eigenfunctions of the kernel $\mathcal{G}(x,s)$. Assume that the kernel $\mathcal{G}(x,s)$ is separable which means that it can be written as $\mathcal{G}(x,s) = \sum_{j=1}^n \alpha_j(x)\beta_j(s)$. If we insert this into (3.28) we get

$$y(x) = f(x) + \lambda \int_0^l \sum_{j=1}^n \alpha_j(x)\beta_j(s)y(s)ds$$

$$y(x) = f(x) + \lambda \sum_{j=1}^{n} \alpha_j(x) \int_0^l \beta_j(s)y(s)ds$$

$$y(x) = f(x) + \lambda \sum_{j=1}^{n} C_j\alpha_j(x) \tag{3.29}$$

observe that $y(x)$ as in (3.29) gives us a solution to (3.28)as soon as we know the coefficient C_j. To find C_j multiply (3.29) with $\beta_i(x)$ and integrating gives

$$\int_0^l y(x)\beta_i(x)dx = \int_0^l f(x)\beta_i(x)dx + \lambda \sum_{j=1}^{n} C_j \int_0^l \alpha_j(x)\beta_i(x)dx$$

Or equivalently

$$C_i = f_i + \lambda \sum_{j=1}^{n} C_j a_{ij} \tag{3.30}$$

. Thus we have a linear system with n unknown variables:$C_1, ...C_n$ and n equations $C_i = f_i + \lambda \sum_{j=1}^{n} C_j a_{ij} \quad 1 \le i \le n$. In matrix form we can write this as $(I - \lambda A)C = f$ where

$$A = \begin{pmatrix} a_{11} & . & . & . & a_{1n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1} & . & . & . & a_{nn} \end{pmatrix}, f = \begin{pmatrix} f_1 \\ . \\ . \\ . \\ f_n \end{pmatrix}, C = \begin{pmatrix} C_1 \\ . \\ . \\ . \\ C_n \end{pmatrix}$$

We seek the characteristic value and eigenfunction (when $f(x) = 0$) then the matrix form becomes $(I - \lambda A)C = 0$. If λ is a characteristic value then every solution of the homogeneous equation with degenerate kernel has the form $y(x) = \sum_{k=1}^{p} b_k y_k(x)$ where the b_k are arbitrary constants and the $y_k(x)$ are linearly independent eigenfunctions of the kernel corresponding to the characteristic value λ:$y_k(x) = \sum_{j=1}^{n} C_{j(k)}\alpha_j(x)$. Here the constants $C_{j(k)}$ form p ($p \le n$) linearly independent solutions of the homogeneous system of algebraic equations Equ(3.30)with $f_i = 0 : C_{i(k)} - \lambda \sum_{j=1}^{n} C_{j(k)} a_{ij} = 0 \quad k = 1, ...p, i = 1, ..., n$

3.5 Conclusion

The eigenvalues of Sturm-Liouville problem are real numbers and non negative. Eigenfunctions of Sturm-Liouville problem corresponding to different eigenvalues are orthogonal and the set of all eigenfunctions is complete in the sense that every square integrable function f can be expanded in terms of the eigenfunctions. Sturm-Liouville problem is reduced into homogenous Fredholm integral equation of the second kind.

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