

GRADUATE PROJECT REPORT ON
UNIFORMLY CONVEX SPACES



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To The Memorizing Of My Mother.

Declaration

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associateship, Fellowship or any similar title to me.

Addis Ababa
Feb 2013

Gezahegn Anberber Tadesse

Permission

This is to certify that this project is compiled by **Mr. Gezahegn Anberber Tadesse** in the Department of Mathematics Addis Ababa University under my supervision. I here by also confirm that the project can be submitted for evaluation by examiners and eventual defence.

Addis Ababa
Feb 2013

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First of all I eulogize the Name of Almighty God with Saint Mary, Who gave Me Power, Health and Patience in every endeavour of my life.

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Abstract

In the study of solving operator equation, the geometric properties of underlying space has a great role. It is the purpose of this project to observe the geometric properties of Banach spaces that resemble Hilbert spaces. Mainly the project focuses on uniformly convex spaces, in particular on the geometric properties of sequential spaces l^p and the function spaces L^p ($1 \leq p \leq \infty$).

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Introduction

This project consists of an introduction and three chapters, the first chapter is the Preliminaries of Uniformly Convex spaces, the second chapter explore the definitions, examples and some theorems of Uniformly Convex spaces. The third chapter introduce the definition of the modulus of convexity, the characterization of Uniform Convexity Reflexivity of the Banach space and proved some Lemma, Theorems and proposition.

So these geometric identities which characterize inner product spaces make numerous problems posed in real Hilbert spaces more manageable than those posed in arbitrary real Banach spaces. Consequently, to extend some of the Hilbert space techniques to more general Banach spaces, analogues of these identities have to be developed in such Banach spaces.

In 1936, A.J. Clarkson published his famous paper on *uniform convexity*. This work signalled the beginning of extensive research efforts on the geometry of Banach spaces and its applications in functional analysis. And We begin with the notion of convex functions. A real-valued function f is called convex if it satisfies the following inequality:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),$$

for every $\lambda \in [0, 1]$ and every $x, y \in D(f)$, the domain of f , that we demand to be a *convex set*.

To motivate the definition of the modulus of convexity, we begin with some properties of inner product spaces. In an inner product space H , we consider the parallelogram law. For $x, y \in H$, and by identity

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

In the particular case $\|x\| = \|y\| = 1$, we get the expression

$$\left\| \frac{1}{2}(x + y) \right\|^2 = 1 - \frac{1}{4}\|x - y\|^2.$$

From this equality we can determine the distance between the midpoint of the line segment joining x and y from the unit sphere : $\mathbf{S} = \{x \in H : \|x\| = 1\}$ in H by:

$$1 - \left\| \frac{1}{2}(x + y) \right\| = 1 - \sqrt{1 - \frac{1}{4}\|x - y\|^2}.$$

Evidently this distance always lies between 0 and 1. If $\varepsilon \leq \|x - y\|$, then

$$1 - \left\| \frac{1}{2}(x + y) \right\| \geq 1 - \sqrt{1 - \frac{1}{4}\varepsilon^2}.$$

The idea behind these formulas is *the convexity of the unit ball* in an inner product space, i.e., if the distance between two points x and y in the unit sphere is larger than ε , then the midpoint of the segment joining x and y remains in the unit ball with

$$1 - \frac{\varepsilon^2}{4} \geq \left\| \frac{1}{2}(x + y) \right\|^2.$$

Motivated by this, we extend this notion to spaces, not with an inner product, but with a norm and study “*how much convex*” the unit ball is. All the results presented in these three chapters are well-known and standard and can be found in several books. Such as Chidume [1], Clarkson [3], Goebel and Kirk [5].

Chapter 1

PRELIMINARY NOTIONS

In this section we study special Banach spaces which possess an additional structure known as inner product. This enables us to generalize several geometric concepts; in particular the well known *parallelogram law*, *Cauchy-Schwartz Inequality*, *Hölder's Inequality*, *Minkowski's Inequality* and several other geometric relations of the plane.

1.1 Normed Spaces

Definition 1.1.1. A *linear (or vector) space* X over a field \mathfrak{R} of scalar is the set satisfying the following axiom:

1. X is closed under vector addition. For $x, y \in X, x + y \in X$
2. Vector addition is commutative and associative. For all $x, y, z \in X$

$$\begin{aligned}x + y &= y + x \\(x + y) + z &= x + (y + z)\end{aligned}$$

3. There is a zero element (denoted 0) in X , such that

$$x + 0 = x, \quad \forall x \in X$$

4. For each $x \in X$, there is an additive inverse $-x$, such that $x + (-x) = 0$ (Note. We usually written $x - x$ instead of $x + (-x)$)
5. X is closed under scalar multiplication: For $\alpha \in \mathfrak{R}, \alpha x \in X$
6. For any scalars $\alpha, \beta \in \mathfrak{R}$ and any vectors $x, y \in X$

$$\begin{aligned}\alpha(\beta x) &= (\alpha\beta)x \\(\alpha + \beta)x &= \alpha x + \beta x \\\alpha(x + y) &= \alpha x + \beta x\end{aligned}$$

7. $1x = x, \forall x \in X$

Definition 1.1.2. Let X be a vector space. Then the function $\|\cdot\| : X \rightarrow \mathfrak{R}$ is said to be **norm** ^[4] on X if and only if the following properties hold:

$$N1 . \|x\| \geq 0 \quad \text{and} \quad \|x\| = 0 \Leftrightarrow x = 0 \quad \forall x \in X$$

$$N2 . \|\lambda x\| = |\lambda| \cdot \|x\| \quad \forall x \in X, \forall \lambda \in \mathfrak{R}$$

$$N3 . \|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in X.$$

The pair $(X, \|\cdot\|)$ is said to be **normed linear space**.

Definition 1.1.3. A sequence $\{x_n\}$ of vectors in a normed vector space X is convergent, if there exist an $x \in X$, such that

$$\|x_n - x\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

We say that $\{x_n\}$ converges to x and we write $x_n \rightarrow x$ as $n \rightarrow \infty$.

Definition 1.1.4. A sequence $\{x_n\}$ of vectors in a normed vector space X is Cauchy-sequence, if there exist an $x \in X$, such that

$$\|x_m - x_n\| \rightarrow 0 \quad \text{as} \quad m, n \rightarrow \infty.$$

Definition 1.1.5. Let X be a normed linear space. X is complete if and only if every Cauchy sequence converges to a point in X . A complete normed linear space is called a **Banach space** ¹.

1

Example 1.1.1.

Let $X = l^p$ ², $1 \leq p < \infty$. The elements of l^p are the sequences $x := (\xi_n)$, where $\sum_{n=1}^{\infty} |\xi_n|^p < \infty$. Then we define the norm by ^[4] :

$$\|x\| := \|x\|_p = \left(\sum_{n=1}^{\infty} |\xi_n|^p \right)^{\frac{1}{p}}$$

The pair $(X, \|\cdot\|_p)$ is Banach space. ²

Proof. Let $x_k = (\xi_n^{(k)}) \in l^p$ be a Cauchy sequence and let $\varepsilon > 0$. Then there exist a $k_0 = k_0(\varepsilon)$ such that

$$\|x_k - x_j\| = \left(\sum_{n=1}^{\infty} |\xi_n^{(k)} - \xi_n^{(j)}|^p \right)^{\frac{1}{p}} \leq \varepsilon \quad \forall k, j \geq k_0.$$

Hence it follows that

$$|\xi_n^{(k)} - \xi_n^{(j)}| < \varepsilon \quad \forall k, j \geq k_0.$$

¹“Banach space”. These spaces are named in honor of Stefan Banach (1892-1945, ; Poland). Banach himself referred to the space as “espace-B”. However, shortly after Banach published his monumental book in, 1932 the terminology become standard. Karen Saxe, Bignning Functional Analysis, USA, April 2001, page 24

² l^p pronounced “little ell p”. Karen Saxe, page 9

From this we get that $(\xi_n^{(k)})$ is a Cauchy sequence in the complete \mathfrak{R} that convergent.

Let $\lim_{k \rightarrow \infty} \xi_n^{(k)} = \xi_n$, $n \in \mathbb{N}$. Moreover we have

$$\left(\sum_{n=1}^m |\xi_n^{(k)} - \xi_n^{(j)}|^p \right)^{\frac{1}{p}} \leq \varepsilon \quad \forall k, j \geq k_0, m \in \mathbb{N}$$

from which $\xi_n^k \rightarrow \xi_n$ as $k \rightarrow \infty$

$$\left(\sum_{n=1}^m |\xi_n - \xi_n^{(j)}|^p \right)^{\frac{1}{p}} < \varepsilon \quad \forall j \geq k_0, m \in \mathbb{N}$$

follows. From this we get for $m \rightarrow \infty$

$$\left(\sum_{n=1}^{\infty} |\xi_n - \xi_n^{(j)}|^p \right)^{\frac{1}{p}} < \varepsilon \quad \forall j \geq k_0, m \in \mathbb{N} \quad (1.1)$$

This implies $(\xi_1 - \xi_1^{(j)}, \xi_2 - \xi_2^{(j)}, \dots) \in l^p \quad \forall j \geq k_0$. Hence it follows (since l^p is vector space). $x = (\xi_1, \xi_2, \dots) = (\xi_1 - \xi_1^{(j)}, \xi_2 - \xi_2^{(j)}, \dots) + (\xi_1^{(j)}, \xi_2^{(j)}, \dots) \in l^p$. From this x we get from (1.1)

$$\|x - x_j\| \leq \varepsilon \quad \forall j \geq k_0 \quad \text{or} \quad x_j \rightarrow x \quad \text{as} \quad j \rightarrow \infty$$

Therefore, the Cauchy-sequence $(\xi_n^{(k)})$ has a limit point in l^p . □

If $p = \infty$ ¹ and $x := (\xi_n) \in X$, $\sup_{n=1}^{\infty} |\xi_n| < \infty$ then we define the norm by:

$$\|x\| := \|x\|_{\infty} = \sup_{n=1}^{\infty} \{|\xi_n|\}$$

then the space is complete.¹

Example 1.1.2. Let $X = L^p(\mu)$, $(1 \leq p < \infty)$ (the vector space of all functions $x : X \rightarrow \mathfrak{R}$ which is measurable on an arbitrary interval $\mu \subseteq \mathfrak{R}$ (μ can also $(-\infty, \infty)$.) For these function the Lebesgue integral $\int_a^b |x|^p d\mu$ exists, then, with the norm⁴:

$$\|x\| := \|x\|_p = \left(\int_a^b |x|^p d\mu \right)^{\frac{1}{p}}$$

the space X is complete. We have to interpret the norm in the set of all equivalence class \hat{x} , where \hat{x} is the set of all function x which have the same value on μ almost everywhere. Then $\|x\| = 0$ does not imply $x=0$ but only $x(t)=0$ almost everywhere.

Let $X = L^{\infty}(\mu)$ (vector space of all function $x : X \rightarrow \mathfrak{R}$ which are on μ measurable and essentially bounded.) Essentially bounded mean that for x there is a bounded M_x such that $|x(t)| \leq M_x$ almost everywhere. The least M_x is said to be the essential supremum of $|x(t)|$ and is denoted by **sup ess** $|x(t)|$. With the norm:

$$\|x\| := \|x\|_{\infty} = \sup \text{ess}_{t \in \mu} |x(t)|.$$

X will be a complete space.

¹ l^{∞} is the spaces of all bounded sequences

Example 1.1.3. Let $X = C(T)$ (particularly $C[a, b]$) (set of all continuous functions $x : T \rightarrow \mathfrak{R}$ on compact T). The norm is defined by ^[4] :

$$\|x\| = \max_{t \in T} \{|x(t)|\}$$

the space is complete.

1.2 Inner Product Spaces

Definition 1.2.1. Let \mathfrak{R} be a field and let \mathbf{X} be a vector space over the field \mathfrak{R} . An **inner product** on a vector space \mathbf{X} is a function $\langle \cdot, \cdot \rangle$ on the pair (x, y) of the vector in $X \times X$: such that, for all $x, y, z \in X$ and $\lambda \in \mathfrak{R}$, the following axioms are satisfied:

$$IP1 . \langle x, x \rangle \geq 0, \langle x, x \rangle = 0 \Leftrightarrow x = 0, \quad \forall x \in X$$

$$IP2 . \langle x, y \rangle = \langle y, x \rangle, \quad \forall x, y \in X$$

$$IP3 . \langle \lambda x, y \rangle = \lambda \langle x, y \rangle, \quad \forall x \in X, \lambda \in \mathfrak{R}$$

$$IP4 . \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle, \quad \forall x, y, z \in X.$$

We call the pair $(X, \langle \cdot, \cdot \rangle)$ is an **inner product space** or **pre-Hilbert space**. The function $\langle \cdot, \cdot \rangle$ is said to be a **scalar product** or an **inner product**.

Example 1.2.1. The linear space $X = L^2(a, b)$ spaces defined by ^[4] :

$$\langle x, y \rangle := \int_a^b x(t)y(t)dt$$

is defined an inner product space.

Example 1.2.2. The linear space $X = l^2$ spaces defined by ^[4] :

$$\langle x, y \rangle := \sum_{i=1}^{\infty} \xi_i \eta_i,$$

where $x = (\xi_1, \xi_2, \dots)$, $y = (\eta_1, \eta_2, \dots)$ is an inner product.

Definition 1.2.2. Let \mathbf{X} be a pre-Hilbert space. \mathbf{X} is said to be **Hilbert space** ¹ if and only if \mathbf{X} is **complete** with regard to the norm $\|x\| = \sqrt{\langle x, x \rangle}$ ¹

Examples of Hilbert spaces are $l^2, L^2(a, b)$ with the scalar products defined in previous examples.

Example 1.2.3. The space $X=C[a, b]$ where $\langle x, y \rangle := \int_a^b x(t)y(t)dt$ is pre-Hilbert space but **not a Hilbert space**.

¹“Hilbert space”. These spaces are named in honor of David Hilbert(1862-1943; Russia)

Proof. Let $C[a, b] = C[-1, 1]$ and it is not complete with regard to the norm

$$\|x\| = \sqrt{\int_{-1}^1 |x(t)|^2 dt}$$

We consider the sequence (x_n) , where

$$x_n(t) = \begin{cases} 0 & \text{if } -1 \leq t \leq -\frac{1}{n} \\ \frac{nt+1}{2} & \text{if } -\frac{1}{n} \leq t \leq \frac{1}{n} \\ 1 & \text{if } \frac{1}{n} \leq t \leq 1 \end{cases}$$

Since, $x_n \in C[-1, 1]$. Then we have for sufficiently great $m, n \in \mathbb{N}$ the inequalities $\frac{1}{n} < \varepsilon$, $\frac{1}{m} < \varepsilon$. Then we have

$$|x_n(t) - x_m(t)| \begin{cases} = 0 - 0 = 0 & \text{if } -1 \leq t \leq -\varepsilon \\ \leq 1 & \text{if } -\varepsilon \leq t \leq \varepsilon \\ = 1 - 1 = 0 & \text{if } \varepsilon \leq t \leq 1 \end{cases}$$

We set

$$\|x_n - x_m\|^2 = \int_{-1}^1 |x_n(t) - x_m(t)|^2 \leq \int_{-1}^{-\varepsilon} 0 dt + \int_{-\varepsilon}^{\varepsilon} 1 dt + \int_{\varepsilon}^1 dy = 2\varepsilon$$

So we have that (x_n) is a Cauchy-sequence. Now assume $x_n \rightarrow x \in C[-1, 1]$. Then

$$\int_{-1}^{-\varepsilon} |x_n(t) - x|^2 \leq \int_{-1}^1 |x_n(t) - x|^2 = \|x_n - x\|^2 \rightarrow 0, \text{ as } n \rightarrow \infty \text{ and } \int_{-1}^{-\varepsilon} |x_n(t) - 0|^2 = 0$$

for sufficient great $n \in \mathbb{N}$. Since the limit is unique, we have

$x(t) = 0$, for $t \in [-1, -\varepsilon]$, for each $\varepsilon > 0$ that means $x(t) = 0$, for $t \in [-1, 0)$

correspondingly, we can show that $x(t) = 0$, for $t \in (0, 1]$

Therefore, x can not be an element of $C[-1, 1]$ □

Depending on the above definitions we have the following important properties on the space.

1. *Hölder's Inequality*:- Let $p \geq 1$. For arbitrary $x = \{x_i\}$, $y = \{y_i\}$, $x \in l_p$, $y \in l_q$. We have ^[1]

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} |y_i|^q \right)^{\frac{1}{q}} \quad (1.2)$$

where $\frac{1}{p} + \frac{1}{q} = 1$

Proof. We first prove that $p \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then for arbitrary constant $a \geq b > 0$, we have

$$ab \leq \frac{a^p}{p} + \frac{b^p}{p} \quad (1.3)$$

So, we let $a > 0$, $b > 0$ be constant. Consider the function $y = x^{p-1}$ (or, equivalently $x = y^{q-1}$, since $\frac{1}{p-1} = q-1$)

By integration we obtain

$$ab \leq \int_0^a x^{p-1} dx + \int_0^b y^{q-1} dy = \frac{a^p}{p} + \frac{b^q}{q},$$

observe that if $a = 0$ or $b = 0$, this inequality still holds, let $\{u_i\}$, $\{v_i\}$ be such that;

$$\sum_{i=1}^{\infty} |u_i|^p = 1, \quad \sum_{i=1}^{\infty} |v_i|^q = 1 \quad (1.4)$$

Define $a = |u_i|$, $b = |v_i|$, and substitute in (1.2) to get

$$|u_i v_i| \leq \frac{|u_i|^p}{p} + \frac{|v_i|^q}{q}$$

Summing this inequality and using (1.3), we obtain;

$$\begin{aligned} \sum_{i=1}^{\infty} |u_i v_i| &\leq \frac{1}{p} \sum_{i=1}^{\infty} |u_i|^p + \frac{1}{q} \sum_{i=1}^{\infty} |v_i|^q = \frac{1}{p} + \frac{1}{q} = 1 \\ \sum_{i=1}^{\infty} |u_i v_i| &\leq 1. \end{aligned} \quad (1.5)$$

Now, let $x := \{x_i\}$, $y := \{y_i\}$, $x \in l_p$, $y \in l_q$, $x \neq 0$, $y \neq 0$. Define

$$u_i = \frac{x_i}{\left[\sum_{i=1}^{\infty} |x_i|^p \right]^{\frac{1}{p}}}, \quad v_i = \frac{y_i}{\left[\sum_{i=1}^{\infty} |y_i|^q \right]^{\frac{1}{q}}}$$

Since $\sum_{i=1}^{\infty} |u_i|^p = 1$, $\sum_{i=1}^{\infty} |v_i|^q = 1$. So that inequality (1.4) yields;

$$\sum_{i=1}^{\infty} |u_i v_i| \leq 1.$$

That is,

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \left[\sum_{i=1}^{\infty} |x_i|^p \right]^{\frac{1}{p}} \left[\sum_{i=1}^{\infty} |y_i|^q \right]^{\frac{1}{q}}.$$

Observe that if either x or y is zero this inequality still holds. This complete the proof of *Hölder's inequality*. \square

2. **Minkowski's Inequalities:-** For $1 \leq p \leq \infty$, we have

$$\left(\sum_{i=1}^{\infty} |f_i + g_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{\infty} |f_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} |g_i|^p \right)^{\frac{1}{p}}$$

and for $0 < p \leq 1$, we have that if f , g are nonnegative, then

$$\left(\sum_{i=1}^{\infty} |f_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} |g_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{\infty} |f_i + g_i|^p \right)^{\frac{1}{p}}$$

3. **Cauchy-Schwartz Inequality:-** For any elements x and y belonging to the inner product space X ,

$$|\langle x, y \rangle| \leq (\langle x, x \rangle)^{\frac{1}{2}} (\langle y, y \rangle)^{\frac{1}{2}} = \|x\| \|y\|$$

4. **Parallelogram law:-** For any elements x and y belonging to the inner product space X , we have:

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Theorem 1.2.1. Every inner product space X is a normed space with respect to the norm

$$\|x\| = \sqrt{\langle x, x \rangle}, \quad \forall x \in X$$

Proof. Since an inner product space is a vector space, by definition it is only required to verify axioms of norm.

Let $x, y \in X$, $\alpha \in \mathfrak{R}$ then we have

i . Since $\langle x, x \rangle \geq 0$ by (IP1) $\|x\| = \sqrt{\langle x, x \rangle} \geq 0$.

Moreover $\langle x, x \rangle = 0 \Leftrightarrow x = 0$. Therefore, condition (N1) is satisfied.

ii . Now $\|\alpha x\| = (\langle \alpha x, \alpha x \rangle)^{\frac{1}{2}} = [\alpha \bar{\alpha} \langle x, x \rangle]^{\frac{1}{2}}$, by (IP2) and since $\alpha \bar{\alpha} = |\alpha|^2$ is real, we have

$$\|\alpha x\| = [|\alpha|^2 \langle x, x \rangle]^{\frac{1}{2}} = |\alpha| (\langle x, x \rangle)^{\frac{1}{2}} = |\alpha| \|x\|$$

Therefore, condition (N2) is satisfied.

iii . Since

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \overline{\langle x, y \rangle} + \langle y, y \rangle \\ &= \langle x, x \rangle + 2\operatorname{Re}\langle x, y \rangle + \langle y, y \rangle \end{aligned}$$

By definition (1.2.1.) and by Cauchy-Schwartz Inequality

$$0 \leq \operatorname{Re}\langle x, y \rangle \leq |\langle x, y \rangle| \leq (\langle x, x \rangle)^{\frac{1}{2}} (\langle y, y \rangle)^{\frac{1}{2}} = \|x\| \|y\|$$

Therefore, we have

$$\begin{aligned} \|x + y\|^2 &\leq \langle x, x \rangle + 2 \left[(\langle x, x \rangle)^{\frac{1}{2}} (\langle y, y \rangle)^{\frac{1}{2}} \right] + \langle y, y \rangle \\ &= \left[(\langle x, x \rangle)^{\frac{1}{2}} + (\langle y, y \rangle)^{\frac{1}{2}} \right]^2 \\ &= [\|x\| + \|y\|]^2. \end{aligned}$$

Therefore, $\|x + y\| \leq \|x\| + \|y\|$. Hence (N3) is satisfied. From i, ii and iii, we have $\|x\| = (\langle x, x \rangle)^{\frac{1}{2}}$ is a norm on X and the pair $(X, \|\cdot\|)$ is a normed.

□

Chapter 2

UNIFORMLY CONVEX SPACES

2.1 Introduction

In this chapter, we introduce the classes of *uniformly convex* and *strictly convex spaces*. It is well known that if E is a real inner product space, then the following identities hold

- i. $\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2, \quad \forall x, y \in E,$
- ii. $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \quad \forall x, y \in E, \lambda \in (0, 1).$

These geometric identities which characterize inner product spaces make numerous problems posed in real Hilbert spaces more manageable than those posed in arbitrary real Banach spaces. Consequently, to extend some of the Hilbert space techniques to more general Banach spaces, analogues of these identities have to be developed in such Banach spaces.

In 1936, A. J. Clarkson published his famous paper on *uniform convexity* (defined below). This work signalled the beginning of extensive research efforts on the geometry of Banach spaces and its applications in functional analysis.

2.2 Uniformly Convex Spaces

Let E be a Banach space and $S_r(x_0)$, $B_r(x_0)$ denote the sphere and the open ball respectively centered at x_0 and with radius $r > 0$, i.e.

$$\begin{aligned} S_r(x_0) &= \{x \in E : \|x - x_0\| = r\}, \\ B_r(x_0) &= \{x \in E : \|x - x_0\| < r\}. \end{aligned}$$

Let E be a Banach space. The closed unit ball, denoted by B_E of E is defined by:

$$B_E = \{x \in E : \|x\| \leq 1\}.$$

The boundary of B_E is the unit sphere of E and is given by:

$$S_E = \{x \in E : \|x\| = 1\}.$$

Definition 2.2.1. A Banach space E is called **uniformly convex**^[1] if for any $\varepsilon \in (0, 2]$, there exist a $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$, then

$$\left\| \frac{1}{2}(x + y) \right\| \leq 1 - \delta. \tag{2.1}$$

Thus, E is uniformly convex if and only if for each two points x, y on the unit ball of E centered at the origin (*i.e.* $\|x\| \leq 1, \|y\| \leq 1$) which are such that they are at least a distance ε apart (*i.e.* $\|x - y\| \geq \varepsilon$), then the mid-point of x and y is *inside* the unit ball and at a distance δ from the boundary of the ball. (see, Fig 2.1.)

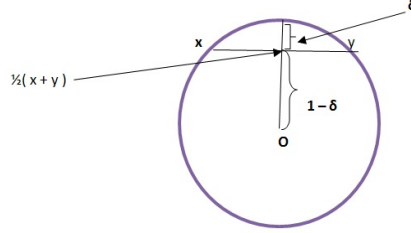


Fig 2.1.

Remark 2.2.1. Since $x, y \in B_1(0)$, the maximum value of $\varepsilon > 0$ is 2. Hence the restriction $\varepsilon \in (0, 2]$. Geometrically, a Banach space E is uniformly convex if and only if the unit ball centered at origin is “Uniformly round”. To fix these idea, consider the following examples;

Example 2.2.1. Every Hilbert space H is uniformly convex ^[1].

Proof. Recall the following identity (the parallelogram law), which is valid in any inner product space. For each $x, y \in H$.

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2). \quad (2.2)$$

Let $\varepsilon \in (0, 2]$, be given and let $x, y \in H$ with $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$. Then from equation (2.2) we have:

$$\begin{aligned} \|x + y\|^2 + \|x - y\|^2 &= 2(\|x\|^2 + \|y\|^2) \leq 2(1 + 1) = 4 \\ \Rightarrow \left\| \frac{x + y}{2} \right\|^2 + \frac{1}{4}\|x - y\|^2 &\leq 1 \\ \Rightarrow \left\| \frac{x + y}{2} \right\|^2 &\leq 1 - \frac{1}{4}\varepsilon^2, \text{ since } \|x - y\| \geq \varepsilon \\ \Rightarrow \left\| \frac{x + y}{2} \right\| &\leq \left[1 - \frac{1}{4}\varepsilon^2 \right]^{1/2}, \end{aligned}$$

$$\text{choose } \delta = 1 - \left[1 - \frac{1}{4}\varepsilon^2 \right]^{1/2}.$$

$$\text{Hence, } \left\| \frac{(x + y)}{2} \right\| \leq 1 - \delta.$$

Therefore, every Hilbert space H is uniformly convex. □

Example 2.2.2. The space l_1 is not uniformly convex space.

Proof. Let $\varepsilon = 1$ and $x = (1, 0, 0, \dots), y = (0, -1, 0, \dots)$ since $x, y \in l_1$ and

$$\begin{aligned} \|x\|_1 &= \|(1, 0, 0, \dots)\|_1 = |1| + |0| + |0| + \dots = 1 \\ \|y\|_1 &= \|(0, -1, 0, \dots)\|_1 = |0| + |-1| + |0| + \dots = 1 \\ \text{and } \|x - y\|_1 &= \|(1, 0, 0, \dots) - (0, -1, 0, \dots)\|_1 = \\ &= |1 - 0| + |0 + 1| + |0 + 0| + \dots = 2 \\ \|x - y\|_1 &= 2 \geq \varepsilon = 1 \\ \text{and } \left\| \frac{x + y}{2} \right\|_1 &= \left\| \frac{(1, 0, 0, \dots) + (0, -1, 0, \dots)}{2} \right\|_1 = \left| \frac{1}{2} \right| + \left| \frac{-1}{2} \right| + 0 + \dots = 1 \\ \Rightarrow \left\| \frac{x + y}{2} \right\|_1 &= 1 \end{aligned}$$

So that, $\left\| \frac{x+y}{2} \right\|_1 \leq 1 - \delta$, for $\delta > 0$ is not satisfied.

Hence, the space l_1 is not uniformly convex space. □

Example 2.2.3. *The space l_∞ is not uniformly convex space.*

Proof. Let $\varepsilon = 1$ and choose $u = (1, 1, 0, \dots), v = (1, -1, 0, \dots)$ and

$$\begin{aligned} \|u\|_\infty &= \|(1, 1, 0, \dots)\|_\infty = \max\{|1|, |1|, |0|, \dots\} = 1 \\ \|v\|_\infty &= \|(1, -1, 0, \dots)\|_\infty = \max\{|1|, |-1|, |0|, \dots\} = 1 \\ \|u - v\|_\infty &= \|(1 - 1, 1 + 1, 0 - 0, \dots)\|_\infty = \max\{|0|, |2|, |0|, \dots\} = 2 \geq \varepsilon = 1 \\ \Rightarrow \|u - v\|_\infty &\geq \varepsilon \text{ and} \\ \left\| \frac{u + v}{2} \right\|_\infty &= \left\| \left[\frac{(1 + 1)}{2}, \frac{(1 - 1)}{2}, \frac{0 + 0}{2}, \dots \right] \right\|_\infty \\ \|(1, 0, 0, \dots)\|_\infty &= \max\{|1|, |0|, |0|, \dots\} = 1 \Rightarrow \left\| \frac{u + v}{2} \right\|_\infty = 1 \end{aligned}$$

but, $\left\| \frac{u+v}{2} \right\|_\infty \leq 1 - \delta$, for $\delta > 0$ is not satisfied.

Hence the space l_∞ is not uniformly convex space. □

Example 2.2.4. *The space $C[0, 1]$ of all real valued-continuous function on the compact interval $[0, 1]$ endowed with the “sup norm” is not uniformly convex.*

Proof. Choose two function $f, g \in C[0, 1]$ as follows:

$$f(t) = 1, \quad \forall t \in [0, 1], \quad g(t) = 1 - t \text{ for each } t \in [0, 1].$$

Take $\varepsilon = \frac{1}{2}$. Since $f, g \in C[0, 1]$, such that

$$\begin{aligned} \|f\| &= \sup_{t \in [0, 1]} |f(t)| = \sup_{t \in [0, 1]} |1| = 1 \\ \|g\| &= \sup_{t \in [0, 1]} |g(t)| = \sup_{t \in [0, 1]} |1 - t| = 1 \\ \|f - g\| &= \sup_{t \in [0, 1]} |f - g| = \sup_{t \in [0, 1]} |1 - (1 - t)| = \sup_{t \in [0, 1]} |t| = 1 \geq \varepsilon = \frac{1}{2} \\ \Rightarrow \|f - g\| &\geq \varepsilon. \text{ Also} \\ \left\| \frac{1}{2}(f + g) \right\| &= \sup_{t \in [0, 1]} \left| \frac{1}{2}[1 + (1 - t)] \right| = \sup_{t \in [0, 1]} \left| 1 - \frac{t}{2} \right| = 1 \\ \Rightarrow \left\| \frac{1}{2}(f + g) \right\| &= 1. \end{aligned}$$

But $\|\frac{1}{2}(f + g)\| \leq 1 - \delta$, for $\delta > 0$ is not satisfied.

Hence, $C[0, 1]$ is not uniformly convex space. □

Example 2.2.5. The Banach space $E = L_1[0, 1]$ is not uniformly convex space.

Proof. Now, $E = L_1[0, 1]$ the space of measurable function such that

$$\int_0^1 |f(t)|dt < \infty$$

$$\text{with the norm given by : } \|f\| = \int_0^1 |f(t)|dt$$

Then take two functions f and g such that

$$f(t) = 1, \quad g(t) = \frac{3}{2} - t, \quad \forall t \in [0, 1] \quad \text{and} \quad \varepsilon = \frac{1}{8}$$

Then, $f, g \in L_1[0, 1]$ with

$$\|f\| = \int_0^1 |f(t)|dt = \int_0^1 1dt = t \Big|_0^1 = 1 \quad \text{and}$$

$$\begin{aligned} \|g\| &= \int_0^1 |g(t)|dt = \int_0^1 \left| \frac{3}{2} - t \right| dt = \left(\frac{3t}{2} - \frac{t^2}{2} \right) \Big|_0^1 \\ &= \frac{3}{2} - \frac{1}{2} = 1. \end{aligned}$$

$$\begin{aligned} \text{and } \|f - g\| &= \int_0^1 \left| 1 - \left(\frac{3}{2} - t \right) \right| dt = \int_0^1 \left| t - \frac{1}{2} \right| dt \\ &= \int_0^{\frac{1}{2}} \left(\frac{1}{2} - t \right) dt + \int_{\frac{1}{2}}^1 \left(t - \frac{1}{2} \right) dt = \frac{1}{4} \geq \varepsilon = \frac{1}{8}. \end{aligned}$$

So that $\|f - g\| \geq \varepsilon$, and also

$$\begin{aligned} \left\| \frac{1}{2}(f + g) \right\| &= \frac{1}{2} \int_0^1 \left| 1 + \left(\frac{3}{2} - t \right) \right| dt = \frac{1}{2} \int_0^1 \left| \frac{5}{2} - t \right| dt \\ &= \frac{1}{2} \left(\frac{5}{2}t - \frac{t^2}{2} \right) \Big|_0^1 = 1 \end{aligned}$$

Which shows that $\left\| \frac{1}{2}(f + g) \right\| = 1$.

So we have no $\delta > 0$, such that, $\left\| \frac{1}{2}(f + g) \right\| \leq 1 - \delta$.

Hence, $E = L_1[0, 1]$ is not uniformly convex space. □

The following facts shows that a large class of Banach spaces is uniformly convex. We might now attempt to prove the uniform convexity of space l^p by extending the argument of Example 2.2.1 to an infinite number of factors. We prefer to prove this fact and the corresponding statement for L^p by exhibiting a set of inequalities for these spaces which have close analogy with identity (2.2).

Lemma 2.2.1. (Clarkson, [3]). For spaces l^p or L^p with $2 \leq p < \infty$, the following inequalities between the norm of two arbitrary elements x and y of the spaces are valid [3]. (here q is the conjugate index, $q = \frac{p}{p-1}$)
For $2 \leq p < \infty$

$$2\left(\|x\|^p + \|y\|^p\right) \leq \|x + y\|^p + \|x - y\|^p \leq 2^{p-1}\left(\|x\|^p + \|y\|^p\right), \quad (2.3)$$

$$2\left(\|x\|^p + \|y\|^p\right)^{q-1} \leq \|x + y\|^q + \|x - y\|^q, \quad (2.4)$$

$$\|x + y\|^p + \|x - y\|^p \leq 2\left(\|x\|^q + \|y\|^q\right)^{p-1}. \quad (2.5)$$

For $1 < p \leq 2$ these inequalities hold in the reverse sense.

Proof. If $p = 2$ the right-hand side of (2.3) is equivalent to the left-hand side. That is

$$\begin{aligned} 2(\|x\|^p + \|y\|^p) &= 2^{p-1}(\|x\|^p + \|y\|^p), & (p = 2) \\ 2(\|x\|^2 + \|y\|^2) &= 2^{2-1}(\|x\|^2 + \|y\|^2) \\ 2(\|x\|^2 + \|y\|^2) &= 2^1(\|x\|^2 + \|y\|^2) \\ 2(\|x\|^2 + \|y\|^2) &= 2(\|x\|^2 + \|y\|^2). \end{aligned}$$

While (2.4) is equivalent to (2.5); to see this, set $x + y = \xi$, $x - y = \eta$ substitute these value in right-hand side of (2.4) and left-side of (2.5) we have

$$\|\xi\|^q + \|\eta\|^q = \|\xi\|^p + \|\eta\|^p, \quad (p = 2).$$

Since q is the conjugate index of p , the equality holds for $p = 2 = q$.

i.e. $\frac{1}{p} + \frac{1}{q} = 1 = \frac{1}{2} + \frac{1}{2}$ and reduce. We make full use of this fact in proving the theorem.

proof of (2.4)

Case I. Consider $1 < p \leq 2$. Then we need to show that:

$$\|x + y\|^q + \|x - y\|^q \leq 2(\|x\|^p + \|y\|^p)^{q-1}.$$

We commence by showing that for any two complex numbers x, y we have

$$|x + y|^q + |x - y|^q \leq 2(|x|^p + |y|^p)^{q-1}. \quad (2.6)$$

To prove this, assume $|x| \geq |y|$, and divide (2.6) by $|x|^q \neq 0$, reduce (2.6) to

$$\begin{aligned} \frac{|x + y|^q}{|x|^q} + \frac{|x - y|^q}{|x|^q} &\leq 2 \frac{(|x|^p + |y|^p)^{q-1}}{|x|^q} \\ \left|\frac{x}{x} + \frac{y}{x}\right|^q + \left|\frac{x}{x} - \frac{y}{x}\right|^q &\leq 2\left(\left|\frac{x}{x}\right|^p + \left|\frac{y}{x}\right|^p\right)^{q-1} \end{aligned}$$

$$|1 + c|^q + |1 - c|^q \leq 2(1 + |c|^p)^{q-1}. \quad (2.7)$$

Where $c = \frac{y}{x}$, $|c| \leq 1$. Setting $c = \rho e^{i\theta}$, we see by elementary calculus method, that it suffices to consider $\theta = 0$; i.e. $0 \leq c \leq 1$. (2.7) is established, for $c = 0$ or $c = 1$. i.e. $c = 0$, if $y = 0, x \neq 0$ and $c = 1$, if $x = y, x \neq 0$.

So we need only consider $0 < c < 1$. Then setting again c by:

$$c = \frac{1-z}{1+z} \quad (\text{so that } 0 < z < 1),$$

we reduce (2.7) to the form

$$\mathbf{S} = \frac{1}{2} \left[(1+z)^p + (1-z)^p \right] - (1+z^q)^{p-1} \geq 0$$

we need to show this, $|1+c|^q + |1-c|^q \leq 2(1+|c|^p)^{q-1}$,
since $c = \frac{1-z}{1+z}$, ($0 < z < 1$)

$$\begin{aligned} &= \left[1 + \left(\frac{1-z}{1+z} \right)^q \right] + \left[1 - \left(\frac{1-z}{1+z} \right)^q \right] \leq 2 \left[1 + \left(\frac{1-z}{1+z} \right)^p \right]^{q-1} \\ &= \left[\frac{(1+z) + (1-z)}{1+z} \right]^q + \left[\frac{(1+z) - (1-z)}{1+z} \right]^q \leq 2 \left[\frac{(1+z)^p - (1-z)^p}{(1+z)^p} \right]^{q-1} \\ &= \left(\frac{2}{1+z} \right)^q + \left(\frac{2z}{1+z} \right)^q \leq 2 \frac{\left[(1+z)^p + (1-z)^p \right]^{q-1}}{\left[(1+z)^p \right]^{\frac{q}{p}}} \end{aligned}$$

since $\frac{1}{p} = 1 - \frac{1}{q} = \frac{q-1}{q} \Rightarrow \frac{q}{p} = q-1$

$$\begin{aligned} &= \left(\frac{2}{1+z} \right)^q + \left(\frac{2z}{1+z} \right)^q \leq 2 \frac{\left[(1+z)^p + (1-z)^p \right]^{q-1}}{(1+z)^q} \\ &= 2^q + 2^q z^q \leq 2 \left[(1+z)^p + (1-z)^p \right]^{q-1} \\ &= 2^q (1+z^q) \leq 2 \left[(1+z)^p + (1-z)^p \right]^{q-1} \\ &= 2^{q-1} (1+z^q) \leq \left[(1+z)^p + (1-z)^p \right]^{q-1} \\ &= \frac{\left[(1+z)^p + (1-z)^p \right]^{q-1}}{2^{q-1}} \geq (1+z^q) \\ &= \frac{\left[(1+z)^p + (1-z)^p \right]}{2} \geq \left[(1+z^q) \right]^{\frac{1}{q-1}}, \text{ since } \frac{1}{q-1} = p-1 \\ &= \frac{\left[(1+z)^p + (1-z)^p \right]}{2} - \left[(1+z^q) \right]^{p-1} \geq 0 \end{aligned}$$

$$\therefore \mathbf{S} = \frac{1}{2} \left[(1+z)^p + (1-z)^p \right] - (1+z^q)^{p-1} \geq 0.$$

Expanding each terms of \mathbf{S} in its Taylor's series ^[6], we have :

$$\begin{aligned} \frac{1}{2} \left[(1+z)^p + (1-z)^p \right] &= 1 + \frac{p(p-1)}{2!} z^2 + \frac{p(p-1)(2-p)(3-p)}{4!} z^4 + \dots \\ &+ \frac{p(p-1)(2-p) \dots (2k-1-p)}{(2k)!} z^{2k} + \dots \end{aligned}$$

$$\begin{aligned}
(1+z^q)^{p-1} &= 1 + (p-1)z^q - \frac{(p-1)(2-p)}{2!}z^{2q} + \dots \\
&+ \frac{(p-1)(2-p)\dots(2k-1-p)}{(2k-1)!}z^{(2k-1)q} \\
&- \frac{(p-1)(2-p)\dots(2k-p)}{2k!}z^{2kq} + \dots
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbf{S} &= \sum_{k=1}^{\infty} \left[\frac{p(p-1)(2-p)\dots(2k-1-p)}{(2k)!}z^{2k} - \frac{(p-1)(2-p)\dots(2k-1-p)}{(2k-1)!}z^{(2k-1)q} \right. \\
&\quad \left. + \frac{(p-1)(2-p)\dots(2k-p)}{(2k)!}z^{2kq} \right] \\
&= \sum_{k=1}^{\infty} \frac{(2-p)(3-p)\dots(2k-p)}{(2k-1)!}z^{2k} \left[\frac{1-z^{\frac{(2k-1)q}{p-1}}}{\frac{2k-1}{p-1}} - \frac{1-z^{\frac{2kq}{p-1}}}{\frac{2k}{p-1}} \right].
\end{aligned}$$

But $\frac{1-z^t}{t}$, for $t > 0$ and $0 < z < 1$, is a decreasing function of t , hence the series for \mathbf{S} has non-negative terms, because the first term is greater than the second term and (2.7) is established.

Turning now to the proof of (2.4), we consider first space l^p .

Let the two elements considered to be

$$x = (x_1, x_2, \dots), \quad y = (y_1, y_2, \dots),$$

then, from (2.6) we have the following relation:

$$\left[\sum_{i=1}^{\infty} |x_i + y_i|^p \right]^{\frac{q}{p}} + \left[\sum_{i=1}^{\infty} |x_i - y_i|^p \right]^{\frac{q}{p}} \leq 2 \left[\sum_{i=1}^{\infty} (|x_i|^p + |y_i|^p) \right]^{q-1}. \quad (2.8)$$

Now one form of Minkowski's Inequality states that if A_i, B_i are any two sets of non-negative numbers, finite or infinite in number, and $0 < s \leq 1$, then

$$\left(\sum A_i^s \right)^{\frac{1}{s}} + \left(\sum B_i^s \right)^{\frac{1}{s}} \leq \left[\sum (A_i + B_i)^s \right]^{\frac{1}{s}}.$$

Proper account being taken of convergence. Setting,

$$\frac{p}{q} = s, \quad |x_i + y_i|^q = A_i, \quad |x_i - y_i|^q = B_i$$

Then we have the following results:

$$\left[\sum_{i=1}^{\infty} (|x_i + y_i|^q)^{\frac{p}{q}} \right]^{\frac{q}{p}} + \left[\sum_{i=1}^{\infty} (|x_i - y_i|^q)^{\frac{p}{q}} \right]^{\frac{q}{p}} \leq \left[\sum_{i=1}^{\infty} (|x_i + y_i|^q + |x_i - y_i|^q)^{\frac{p}{q}} \right]^{\frac{q}{p}}.$$

Since,

$$\begin{aligned}
\sum_{i=1}^{\infty} (|x_i + y_i|^q)^{\frac{p}{q}} &= (\|x + y\|_q)^{\frac{p}{q}} \\
\sum_{i=1}^{\infty} (|x_i - y_i|^q)^{\frac{p}{q}} &= (\|x - y\|_q)^{\frac{p}{q}}, \text{ and}
\end{aligned}$$

$$\sum_{i=1}^{\infty} (|x_i + y_i|^q + |x_i - y_i|^q)^{\frac{p}{q}} = (\|x + y\|_q + \|x - y\|_q)^{\frac{p}{q}}.$$

From (2.6), that is:

$$|x + y|^q + |x - y|^q \leq 2(|x|^p + |y|^p)^{q-1},$$

we get the following:

$$\begin{aligned} \sum_{i=1}^{\infty} (|x_i + y_i|^q) + \sum_{i=1}^{\infty} (|x_i - y_i|^q) &\leq 2 \left[\sum_{i=1}^{\infty} (|x_i|^p + |y_i|^p) \right]^{\frac{q}{p}} \\ \|x + y\|^q + \|x - y\|^q &\leq 2(\|x\|^p + \|y\|^p)^{\frac{q}{p}} \end{aligned}$$

Since, $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \frac{1}{p} = 1 - \frac{1}{q} \Rightarrow \frac{1}{p} = \frac{q-1}{q} \Rightarrow \frac{q}{p} = q - 1$, $(1 < p \leq 2)$

$$\therefore \|x + y\|^q + \|x - y\|^q \leq 2(\|x\|^p + \|y\|^p)^{q-1} \quad (1 < p \leq 2).$$

Hence, this is the desired result of (2.4) for $(1 < p \leq 2)$

Case II. Now consider (2.4) for $2 \leq p < \infty$.

We need to show that:

$$2(\|x\|^p + \|y\|^p)^{q-1} \leq \|x + y\|^q + \|x - y\|^q.$$

Again let x, y be any two elements of l^p , the relation which we must prove is (2.8) with the sense *reversed*. i.e.

$$2 \left[\sum_{i=1}^{\infty} (|x_i|^p + |y_i|^p) \right]^{q-1} \leq \left[\sum_{i=1}^{\infty} |x_i + y_i|^p \right]^{\frac{q}{p}} + \left[\sum_{i=1}^{\infty} |x_i - y_i|^p \right]^{\frac{q}{p}}.$$

Letting A_i, B_i and s have the same value as above, i.e.

$$\frac{p}{q} = s, |x_i + y_i|^q = A_i, |x_i - y_i|^q = B_i,$$

and again applying Minkowski's Inequality, which now *reversed* in sense, since s exceed 1 ($s > 1$), then

$$\left[\sum (A_i + B_i)^s \right]^{\frac{1}{s}} \leq \left(\sum A_i^s \right)^{\frac{1}{s}} + \left(\sum B_i^s \right)^{\frac{1}{s}},$$

then we have the following results;

$$\left[\sum_{i=1}^{\infty} (|x_i + y_i|^q + |x_i - y_i|^q)^{\frac{p}{q}} \right]^{\frac{q}{p}} \leq \left[\sum_{i=1}^{\infty} (|x_i + y_i|^p)^{\frac{p}{q}} \right]^{\frac{q}{p}} + \left[\sum_{i=1}^{\infty} (|x_i - y_i|^p)^{\frac{p}{q}} \right]^{\frac{q}{p}}. \quad (2.9)$$

Now (2.5) (or its equivalent (2.4)) has already been proved for $1 < p \leq 2$, hence employing it for complex number we have that for $2 \leq p < \infty$,

$$2(|x_i|^p + |y_i|^p)^{q-1} \leq |x_i + y_i|^q + |x_i - y_i|^q.$$

From this we have that (2.9) is

$$\begin{aligned}
\left[\sum_{i=1}^{\infty} (2\{|x_i|^p + |y_i|^p\}^{q-1})^{\frac{p}{q}} \right]^{\frac{q}{p}} &\leq \left[\sum_{i=1}^{\infty} (|x_i + y_i|^q)^{\frac{p}{q}} \right]^{\frac{q}{p}} + \left[\sum_{i=1}^{\infty} (|x_i - y_i|^q)^{\frac{p}{q}} \right]^{\frac{q}{p}} \\
\left[\sum_{i=1}^{\infty} 2^{\frac{p}{q}} \{|x_i|^p + |y_i|^p\} \right]^{\frac{q}{p}} &\leq \left[\sum_{i=1}^{\infty} (|x_i + y_i|^q)^{\frac{p}{q}} \right]^{\frac{q}{p}} + \left[\sum_{i=1}^{\infty} (|x_i - y_i|^q)^{\frac{p}{q}} \right]^{\frac{q}{p}} \\
2 \left[\sum_{i=1}^{\infty} (|x_i|^p + |y_i|^p) \right]^{q-1} &\leq \left[\sum_{i=1}^{\infty} (|x_i + y_i|^q)^{\frac{p}{q}} \right]^{\frac{q}{p}} + \left[\sum_{i=1}^{\infty} (|x_i - y_i|^q)^{\frac{p}{q}} \right]^{\frac{q}{p}} \\
2(\|x\|^p + \|y\|^p)^{q-1} &\leq \|x + y\|^q + \|x - y\|^q. \\
\therefore 2(\|x\|^p + \|y\|^p)^{q-1} &\leq \|x + y\|^q + \|x - y\|^q.
\end{aligned}$$

Hence this is the desired results of (2.4), for l^p , ($2 \leq p < \infty$).

Therefore, (2.4) holds for $1 < p < \infty$, from two cases.

Proof of (2.3). For $2 \leq p < \infty$, and consider the right-hand side of this inequality. That is,

$$2(\|x\|^p + \|y\|^p) \leq \|x + y\|^p + \|x - y\|^p \leq 2^{p-1}(\|x\|^p + \|y\|^p),$$

and consider $2^{p-1}(\|x\|^p + \|y\|^p)$. And we need to show that

$$2(\|x\|^q + \|y\|^q)^{p-1} \leq 2^{p-1}(\|x\|^p + \|y\|^p).$$

Then, for $a, b \geq 0$, we have

$$2(a^q + b^q)^{p-1} \leq 2^{p-1}(a^p + b^p). \quad (2.10)$$

For suppose $a \leq b > 0$, which entails no real loss of generality; dividing (2.10) by

$$\begin{aligned}
\frac{b^{q(p-1)}}{2(a^q + b^q)^{p-1}} &= \frac{b^p}{2^{p-1}(a^p + b^p)} \quad \text{we obtain} \\
&\leq \frac{b^p}{b^p} \\
\Rightarrow 2 \left[\left(\frac{a}{b} \right)^q + 1 \right]^{p-1} &\leq 2^{p-1} \left[\left(\frac{a}{b} \right)^p + 1 \right],
\end{aligned}$$

we obtain set $c = \frac{a}{b}$, since $a \leq b > 0$, i.e. $b \neq 0$, then $0 \leq c \leq 1$

$$\Rightarrow 2(c^q + 1)^{p-1} \leq 2^{p-1}(c^p + 1).$$

($0 \leq c \leq 1$), $c = 0$, if $a = 0$, $c = 1$, if $a = b \neq 0$

$$\Rightarrow 2^{p-2} \frac{(c^p + 1)}{(c^q + 1)^{p-1}} \geq 1,$$

which, being raised to the power $\frac{1}{p}$ gives

$$\begin{aligned}
2^{\frac{p-2}{p}} \frac{(c^p + 1)^{\frac{1}{p}}}{(c^q + 1)^{\frac{p-1}{p}}} &= 2^{\frac{p-2}{p}} \frac{(c^p + 1)^{\frac{1}{p}}}{(c^q + 1)^{\frac{1}{q}}} \geq 1, \\
H(c) &= 2^{\frac{p-2}{p}} \frac{(c^p + 1)^{\frac{1}{p}}}{(c^q + 1)^{\frac{1}{q}}} \geq 1.
\end{aligned}$$

$$\begin{aligned}
\text{Then, } H(1) &:= 2^{\frac{p-2}{p}} \frac{(1^p + 1)^{\frac{1}{p}}}{(1^q + 1)^{\frac{1}{q}}} \\
&= 2^{\frac{p-2}{p}} \frac{(2)^{\frac{1}{p}}}{(2)^{\frac{1}{q}}} \\
&= 2^{\frac{p-2}{p}} (2)^{\frac{1}{p}} (2)^{-\left(\frac{1}{q}\right)} \\
&= 2^{\frac{p-2}{p}} (2)^{\frac{1}{p} - \frac{1}{q}} \\
&= 2^{\frac{p-2}{p}} (2)^{\frac{1}{p} - \frac{p-1}{p}} \\
&= 2^{\frac{p-2}{p}} (2)^{\frac{2-p}{p}} \\
&= 2^{\frac{p-2}{p} + \frac{2-p}{p}} \\
&= 2^0 = 1. \\
\therefore H(1) &= 1
\end{aligned}$$

Hence, if $c = 1$ the equality is hold.

$$\text{If } c = 0 \text{ then, } H(0) = 2^{\frac{p-2}{p}} \geq 1, \quad (2 \leq p < \infty).$$

$$\text{Thus } H(c) = 2^{\frac{p-2}{p}} \frac{(c^p+1)^{\frac{1}{p}}}{(c^q+1)^{\frac{1}{q}}} \geq 1, \quad \forall c \in [0, 1].$$

Therefore, from (2.10) it follows that:

$$2(\|x\|^q + \|y\|^q) \leq 2^{p-1}(\|x\|^p + \|y\|^p), \quad (2 \leq p < \infty)$$

and hence

$$\|x + y\|^p + \|x - y\|^p \leq 2^{p-1}(\|x\|^p + \|y\|^p)$$

This is the desired result of (2.3) for $2 \leq p < \infty$.

For $1 < p \leq 2$, the inequality (2.10) holds in the reverse sense, the proof being identical. This complete the proof of Lemma 2.2.1. \square

Remark 2.2.2. From Lemma 2.2.1. If $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and for each pair $x, y \in E$ we will use the following inequalities in the next theorems:

- i. $\|x + y\|^q + \|x - y\|^q \leq 2(\|x\|^p + \|y\|^p)^{q-1}, \quad \text{for } 1 < p \leq 2$
- ii. $\|x + y\|^p + \|x - y\|^p \leq 2(\|x\|^q + \|y\|^q)^{p-1}, \quad \text{for } 2 \leq p < \infty$
- iii. $\|x + y\|^p + \|x - y\|^p \leq 2^{p-1}(\|x\|^p + \|y\|^p), \quad \text{for } 2 \leq p < \infty.$

Theorem 2.2.1. l^p spaces ¹, ($1 < p < \infty$) are uniformly convex spaces ^[1].

1

¹ l^p -spaces are spaces of sequences and we denote $l^p = \left\{ (\xi_n)_{n \in \mathbb{N}} \subseteq \mathbb{R} : \sum_{n=1}^{\infty} |\xi_n|^p < \infty \right\}$

Proof. let $\varepsilon \in (0, 2]$ be given and let $x = (\xi_i), y = (\eta_i) \in l^p$ such that,

$$\|x\| = \left(\sum_{i=1}^{\infty} |\xi_i|^p \right)^{\frac{1}{p}} \leq 1, \quad \|y\| = \left(\sum_{i=1}^{\infty} |\eta_i|^p \right)^{\frac{1}{p}} \leq 1 \text{ and } \|x - y\| \geq \varepsilon. \text{ Two cases arise:}$$

Case I . For $1 < p \leq 2$

In this case use the identity (i) of Remark 2.2.2,i.e.

$$\begin{aligned} \|x + y\|^q + \|x - y\|^q &\leq 2(\|x\|^p + \|y\|^p)^{q-1} \leq 2(1 + 1)^{q-1} = 2^q \\ \|x + y\|^q + \|x - y\|^q &\leq 2^q \\ \frac{\|x + y\|^q}{2^q} + \frac{\|x - y\|^q}{2^q} &\leq 1 \\ \left\| \frac{1}{2}(x + y) \right\|^q + \left(\frac{\varepsilon}{2} \right)^q &\leq 1, \quad \text{since } \|x - y\| \geq \varepsilon \\ \left\| \frac{1}{2}(x + y) \right\|^q &\leq 1 - \left(\frac{\varepsilon}{2} \right)^q \\ \left\| \frac{1}{2}(x + y) \right\| &\leq \left[1 - \left(\frac{\varepsilon}{2} \right)^q \right]^{\frac{1}{q}}. \end{aligned}$$

Choose $\delta = 1 - \left[1 - \left(\frac{\varepsilon}{2} \right)^q \right]^{\frac{1}{q}} > 0$.

Hence, $\left\| \frac{1}{2}(x + y) \right\| \leq 1 - \delta$ and $E = l^p$ is uniformly convex space, where $1 < p \leq 2$

Case II . For $2 < p < \infty$

In this case use the identity (ii) or (iii) of Remark 2.2.2, we have

$$\begin{aligned} \|x + y\|^p + \|x - y\|^p &\leq 2^{p-1}(\|x\|^p + \|y\|^p) \leq 2^{p-1}(1 + 1) = 2^p \\ \|x + y\|^p + \|x - y\|^p &\leq 2^p \\ \frac{\|x + y\|^p}{2^p} + \frac{\|x - y\|^p}{2^p} &\leq 1 \\ \left\| \frac{1}{2}(x + y) \right\|^p + \left(\frac{\varepsilon}{2} \right)^p &\leq 1, \quad \text{since } \|x - y\| \geq \varepsilon \\ \left\| \frac{1}{2}(x + y) \right\|^p &\leq 1 - \left(\frac{\varepsilon}{2} \right)^p \\ \left\| \frac{1}{2}(x + y) \right\| &\leq \left[1 - \left(\frac{\varepsilon}{2} \right)^p \right]^{\frac{1}{p}} \\ \text{choose } \delta &= 1 - \left[1 - \left(\frac{\varepsilon}{2} \right)^p \right]^{\frac{1}{p}} > 0. \end{aligned}$$

Hence, $\left\| \frac{1}{2}(x + y) \right\| \leq 1 - \delta$ and $E = l^p$ is uniformly convex space, where $2 \leq p < \infty$.

Therefore, $l^p(1 < p < \infty)$ spaces are uniformly convex space.

□

Theorem 2.2.2. $L^p(\mu)$ spaces¹, ($1 < p < \infty$) are uniformly convex Banach spaces.

1

¹ L^p -spaces are spaces of measurable functions and we denote $L^p = \left\{ f : E \rightarrow \mathfrak{R} \mid f \text{ is } \mu\text{-measurable and } \int_E |f|^p d\mu < \infty \right\}$

Proof. Choose $f, g, \in L^p(\mu)$ such that $f, g \in B_E$ and for any $\varepsilon \in (0, 2]$, such that $\|f\| = \left(\int |f|^p d\mu \right)^{\frac{1}{p}} \leq 1$, $\|g\| = \left(\int |g|^p d\mu \right)^{\frac{1}{p}} \leq 1$ and $\|f - g\| \geq \varepsilon$, then two cases arise:

Case I . For $1 < p \leq 2$,

then by identity (i) of Remark 2.2.2 we have:

$$\begin{aligned} \|f + g\|^q + \|f - g\|^q &\leq 2(\|f\|^p + \|g\|^p)^{q-1}, \quad \text{for } 1 < p \leq 2 \\ \|f + g\|^q + \|f - g\|^q &\leq 2(1 + 1)^{q-1} = 2^q \\ \frac{\|f + g\|^q}{2^q} + \frac{\|f - g\|^q}{2^q} &\leq 1 \\ \left\| \frac{1}{2}(f + g) \right\|^q + \left(\frac{\varepsilon}{2} \right)^q &\leq 1 \\ \left\| \frac{1}{2}(f + g) \right\|^q &\leq 1 - \left(\frac{\varepsilon}{2} \right)^q \\ \left\| \frac{1}{2}(f + g) \right\| &\leq \left[1 - \left(\frac{\varepsilon}{2} \right)^q \right]^{\frac{1}{q}}. \\ \text{Choose } \delta &= 1 - \left[1 - \left(\frac{\varepsilon}{2} \right)^q \right]^{\frac{1}{q}} > 0. \end{aligned}$$

$$\text{Hence, } \left\| \frac{1}{2}(f + g) \right\| \leq 1 - \delta.$$

Therefore, $L^p(\mu)$, for $1 < p \leq 2$ is uniformly convex.

Case II .

For $2 < p < \infty$, then by identity (ii) or (iii) of Remark 2.2.2 we have:

$$\begin{aligned} \|f + g\|^p + \|f - g\|^p &\leq 2(\|f\|^q + \|g\|^q)^{p-1} \leq 2(1 + 1)^{p-1} = 2^p \\ \|f + g\|^p + \|f - g\|^p &\leq 2^p \\ \frac{\|f + g\|^p}{2^p} + \frac{\|f - g\|^p}{2^p} &\leq 1 \\ \left\| \frac{1}{2}(f + g) \right\|^p + \left(\frac{\varepsilon}{2} \right)^p &\leq 1 \\ \left\| \frac{1}{2}(f + g) \right\|^p &\leq 1 - \left(\frac{\varepsilon}{2} \right)^p \\ \left\| \frac{1}{2}(f + g) \right\| &\leq \left[1 - \left(\frac{\varepsilon}{2} \right)^p \right]^{\frac{1}{p}}. \\ \text{Choose } \delta &= 1 - \left[1 - \left(\frac{\varepsilon}{2} \right)^p \right]^{\frac{1}{p}} > 0. \end{aligned}$$

$$\text{Hence, } \left\| \frac{1}{2}(f + g) \right\| \leq 1 - \delta.$$

Therefore, $L^p(\mu)$, for $2 \leq p < \infty$ is uniformly convex.

Thus, from Case I and Case II $L^p(\mu)$ spaces, $1 < p < \infty$ spaces are uniformly convex spaces. □

The following propositions are consequences of the definition of uniform convexity;

Proposition 2.2.1. *Let E be a uniformly convex space. Then for any $d > 0, \varepsilon > 0$, and arbitrary vectors $x, y \in E$ with $\|x\| \leq d, \|y\| \leq d$ and $\|x - y\| \geq \varepsilon$, there exist a $\delta > 0$ such that ^[2]*

$$\left\| \frac{1}{2}(x + y) \right\| \leq \left[1 - \delta \left(\frac{\varepsilon}{d} \right) \right] d. \quad (2.11)$$

Proof. Let $\varepsilon > 0, d > 0; x, y \in E$ with $\|x\| \leq d, \|y\| \leq d$ and $\|x - y\| \geq \varepsilon$ be given.

Let $z_1 = \frac{x}{d}, z_2 = \frac{y}{d}$ and, suppose we set $\bar{\varepsilon} = \frac{\varepsilon}{d}$. Since, $\varepsilon > 0, d > 0$. Hence, $\bar{\varepsilon} > 0$. Moreover, $\|z_1\| \leq 1, \|z_2\| \leq 1$ and, $\|z_1 - z_2\| = \left\| \frac{x}{d} - \frac{y}{d} \right\| = \left\| \frac{x-y}{d} \right\| = \frac{1}{d} \|x - y\| \geq \frac{1}{d} \varepsilon = \bar{\varepsilon}$, since $\|x - y\| \geq \varepsilon \Rightarrow \|z_1 - z_2\| \geq \bar{\varepsilon}$. Now, by uniform convexity we have:

$$\begin{aligned} \left\| \frac{1}{2}(z_1 + z_2) \right\| &\leq 1 - \delta(\bar{\varepsilon}) \\ \text{i.e. } \left\| \frac{1}{2} \left(\frac{x}{d} + \frac{y}{d} \right) \right\| &\leq 1 - \delta \left(\frac{\varepsilon}{d} \right) \\ \Rightarrow \left\| \frac{1}{2} \left(\frac{x+y}{d} \right) \right\| &\leq 1 - \delta \left(\frac{\varepsilon}{d} \right) \\ \Rightarrow \frac{1}{d} \left\| \frac{1}{2}(x+y) \right\| &\leq 1 - \delta \left(\frac{\varepsilon}{d} \right) \\ \Rightarrow \left\| \frac{1}{2}(x+y) \right\| &\leq \left[1 - \delta \left(\frac{\varepsilon}{d} \right) \right] d. \text{ This complete the proof.} \end{aligned}$$

□

Proposition 2.2.2. *Let E be a uniformly convex space and let $\alpha \in (0, 1)$ and $\varepsilon > 0$,. Then for any $d > 0$ and $x, y \in E$, such that $\|x\| \leq d, \|y\| \leq d, \|x - y\| \geq \varepsilon$, then there exist $\delta = \delta \left(\frac{\varepsilon}{d} \right) > 0$ such that ^[2]*

$$\|\alpha x + (1 - \alpha)y\| \leq \left[1 - 2\delta \left(\frac{\varepsilon}{d} \right) \min\{\alpha, 1 - \alpha\} \right] d. \quad (2.12)$$

Proof. Let $d > 0, \varepsilon > 0, x, y \in E$ with $\|x\| \leq d, \|y\| \leq d, \|x - y\| \geq \varepsilon$ be given. Without loss of generality, we may take $\alpha \in (0, \frac{1}{2}]$, then we have two cases;

Case I . Let $\alpha \in (0, \frac{1}{2})$. Then we get;

$$\begin{aligned} \|\alpha x + (1 - \alpha)y\| &= \|\alpha x + \alpha y - \alpha y + (1 - \alpha)y\| \\ &= \|\alpha(x + y) + (1 - 2\alpha)y\| \leq 2\alpha \left\| \frac{1}{2}(x + y) \right\| + (1 - 2\alpha)\|y\|. \end{aligned}$$

$$\text{Thus, } \|\alpha x + (1 - \alpha)y\| \leq 2\alpha \left\| \frac{1}{2}(x + y) \right\| + (1 - 2\alpha)\|y\|. \quad (2.13)$$

By Proposition 2.2.1. There exist $\delta = \delta \left(\frac{\varepsilon}{d} \right) > 0$, such that

$$\left\| \frac{1}{2}(x + y) \right\| \leq \left[1 - \delta \left(\frac{\varepsilon}{d} \right) \right] d.$$

Substituting in (2.13), we have

$$\begin{aligned}
\|\alpha x + (1 - \alpha)y\| &\leq 2\alpha \underbrace{\left\|\frac{1}{2}(x + y)\right\|}_{\leq [1 - \delta(\frac{\varepsilon}{d})]d} + (1 - 2\alpha) \underbrace{\|y\|}_{\leq d} \\
&\leq 2\alpha[1 - \delta(\frac{\varepsilon}{d})]d + (1 - 2\alpha)d, \quad \text{since } \|y\| \leq d \\
&= 2\alpha d - 2\alpha\delta(\frac{\varepsilon}{d})d + d - 2\alpha d = d - 2\alpha\delta(\frac{\varepsilon}{d})d = \left[1 - 2\alpha\delta(\frac{\varepsilon}{d})\right]d \\
\therefore \|\alpha x + (1 - \alpha)y\| &\leq \left[1 - 2\delta(\frac{\varepsilon}{d})\alpha\right]d.
\end{aligned}$$

Put by choice of $\alpha \in (0, \frac{1}{2})$, we have

$$\begin{aligned}
\alpha < \frac{1}{2} &\Rightarrow 1 - \alpha > \frac{1}{2} > \alpha \Rightarrow \alpha \geq \min\{\alpha, 1 - \alpha\}, \\
\text{hence, } \|\alpha x + (1 - \alpha)y\| &\leq [1 - 2\delta(\frac{\varepsilon}{d})\min\{\alpha, 1 - \alpha\}]d
\end{aligned}$$

Case II . Let $\alpha = \frac{1}{2}$ we are done by the Proposition (2.2.1.), i.e.

$$\|\alpha x + (1 - \alpha)y\| \leq [1 - 2\delta(\frac{\varepsilon}{d})\alpha]d \Rightarrow \left\|\frac{1}{2}(x + y)\right\| \leq [1 - \delta(\frac{\varepsilon}{d})]d \quad (\alpha = \frac{1}{2})$$

Hence from the above two cases, we have:

$$\|\alpha x + (1 - \alpha)y\| \leq [1 - 2\delta(\frac{\varepsilon}{d})\min\{\alpha, 1 - \alpha\}]d.$$

□

2.3 Strictly Convex Spaces

Definition 2.3.1. A Banach space E is said to be strictly convex ^[2], if for any x and y in S_E , $x \neq y$, $\|x\| = 1 = \|y\|$ and $\lambda \in (0, 1)$, we have:

$$\|\lambda x + (1 - \lambda)y\| < 1. \tag{2.14}$$

Theorem 2.3.1. Every uniformly convex space is strictly convex ^[2] .

Proof. Let E be uniformly convex space, for every $x, y \in E$, $x \neq y$, $\|x\| = 1 = \|y\|$, let $\lambda \in (0, \frac{1}{2})$ and (2.12) yields with $\alpha = \lambda$, $d = 1$, then we get:

$$\begin{aligned}
\|\alpha x + (1 - \alpha)y\| &\leq [1 - 2\delta(\frac{\varepsilon}{d})\min\{\alpha, 1 - \alpha\}]d \\
\|\lambda x + (1 - \lambda)y\| &\leq [1 - 2\delta(\frac{\varepsilon}{d})\min\{\lambda, 1 - \lambda\}]d \\
\|\lambda x + (1 - \lambda)y\| &\leq [1 - 2\delta(\varepsilon)\min\{\lambda, 1 - \lambda\}] < 1, \quad \text{since } d = 1, \text{ and } \frac{\varepsilon}{d} = \varepsilon \\
\text{because } \varepsilon \in (0, 2] &\Rightarrow \delta \in (0, 1], \lambda \in (0, \frac{1}{2}). \text{ Then, } \|\lambda x + (1 - \lambda)y\| < 1
\end{aligned}$$

$\therefore E$ is strictly convex space.

Hence, every uniformly convex space is strictly convex space. □

Theorem (2.3.1) gives a large class of strictly convex spaces. However, we shall see later that some well known Banach spaces are not strictly convex. We first give two examples of Banach spaces which are strictly convex but not uniformly convex.

Example 2.3.1. (Goebel and Kirk, [5]). Fix $\mu > 0$ and let $C[0, 1]$ be endowed with the norm $\|\cdot\|_\mu$ defined as follows:

$$\|x\|_\mu = \|x\|_0 + \mu \left(\int_0^1 x^2(t) dt \right)^{\frac{1}{2}},$$

where $\|\cdot\|_0$ is the usual supremum norm. Then,

$$\|x\|_0 \leq \|x\|_\mu \leq (1 + \mu)\|x\|_0, \quad x \in C[0, 1],$$

and the two norms are equivalent with $\|x\|_\mu$ near $\|x\|_0$ for small μ . However, $(C[0, 1], \|\cdot\|_0)$ is not strictly convex while for any $\mu > 0$, $(C[0, 1], \|\cdot\|_\mu)$ is strictly convex. On the other hand, for any $\varepsilon \in (0, 2]$ there exist functions $x, y \in C[0, 1]$ with $\|x\|_\mu = 1 = \|y\|_\mu$, $\|x - y\|_\mu = \varepsilon$, and $\|\frac{1}{2}(x + y)\|$ arbitrarily near 1. Thus, $(C[0, 1], \|\cdot\|_\mu)$ is not uniformly convex.

Example 2.3.2. (Goebel and Kirk, [5]). Let $\mu > 0$ and let $c_0 = c_0(\mathbb{N})$ be given the norm $\|\cdot\|_\mu$ defined for $x = \{x_n\} \in c_0$ by:

$$\|x\|_\mu = \|x\|_{c_0} + \mu \left[\sum_{n=1}^{\infty} \left(\frac{x_n}{n} \right)^2 \right]^{\frac{1}{2}}$$

where $\|\cdot\|_{c_0}$ is the usual l_∞ norm. As in Example 2.3.1. the spaces $(c_0, \|\cdot\|_\mu)$ for $\mu > 0$ are strictly convex but not uniformly convex, while c_0 with its usual norm is not strictly convex.

We now return to uniformly convex spaces. Although Theorem 2.2.3. and Theorem 2.2.4. provides examples of large classes of spaces which are uniformly convex, some well known spaces are not uniformly convex. We, in fact, show that these spaces are not strictly convex.

Example 2.3.3. The space $l_1(2)$ is not strictly convex [2]. To see this, take $\varepsilon = 1$ and choose $x = (1, 0)$, $y = (0, -1)$. Since, $x, y \in l_1(2)$ and $\|x\|_1 = 1 = \|y\|_1$, $\|x - y\|_1 = 2 > \varepsilon$.

However, $\|\frac{1}{2}(x + y)\|_1 = 1$, showing that $l_1(2)$ is not strictly convex.

Example 2.3.4. The space $l_\infty(2)$ is not strictly convex [2]. Consider $u = (1, 1)$, and $v = (-1, 1)$. Both $u, v \in l_\infty(2)$.

Take $\varepsilon = 1$. Then $\|u\|_\infty = 1 = \|v\|_\infty$ and $\|u - v\|_\infty = 2 > \varepsilon$.

However, $\|\frac{1}{2}(u + v)\|_\infty = 1$ and so $l_\infty(2)$ is not strictly convex.

Example 2.3.5. Consider $C[a, b]$, the space of real-valued continuous functions on the compact interval $[a, b]$, with the "sup norm". Then $C[a, b]$ is not strictly convex [2]. To see this, choose two functions $f, g \in C[a, b]$ defined as follows:

$$f(t) := 1, \quad \forall t \in [a, b], \quad g(t) := \frac{b-t}{b-a}, \quad \forall t \in [a, b].$$

Take $\varepsilon = \frac{1}{2}$.

Since $f, g \in C[a, b]$, $\|f\| = \|g\| = 1$ and $\|f - g\| = 1 \geq \varepsilon$.

Also $\|\frac{1}{2}(f + g)\| = 1$ and so $C[a, b]$ is not strictly convex.

Example 2.3.6. The spaces L_1, L_∞ are not strictly convex.

Chapter 3

THE MODULUS OF CONVEXITY

3.1 Introduction

In this section, we shall define a function called the *modulus of convexity* of a normed space E (**denoted by** δ_E , $\delta_E : (0, 2] \rightarrow (0, 1]$) and prove three important properties of the function that will be used in the sequel, namely:

1. (Lemma 3.1.1): For every normed space, the function $\frac{\delta_E(\varepsilon)}{\varepsilon}$ is non-decreasing on $(0, 2]$.
2. (Theorem 3.1.1): The modulus of convexity of a normed space E , δ_E , is a convex and continuous function.
3. (Corollary 3.2.1): In a uniformly convex space E , the modulus of convexity, δ_E , is a strictly increasing function.

We begin with the notion of convex functions. A real-valued function f is called convex if it satisfies the following inequality:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),$$

for every $\lambda \in [0, 1]$ and every $x, y \in D(f)$, the domain of f , that we demand to be a *convex set*.

To motivate the definition of the modulus of convexity, we begin with some properties of inner product spaces. In an inner product space H , we consider the parallelogram law. For $x, y \in H$, and by identity (2.2)

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

In the particular case $\|x\| = \|y\| = 1$, we get the expression

$$\left\|\frac{1}{2}(x + y)\right\|^2 = 1 - \frac{1}{4}\|x - y\|^2.$$

From this equality we can determine the distance between the midpoint of the line segment joining x and y from the unit sphere :

$\mathbf{S} = \{x \in H : \|x\| = 1\}$ in H by:

$$1 - \left\|\frac{1}{2}(x + y)\right\| = 1 - \sqrt{1 - \frac{1}{4}\|x - y\|^2}.$$

Evidently this distance always lies between 0 and 1. If $\varepsilon \leq \|x - y\|$, then

$$1 - \left\| \frac{1}{2}(x + y) \right\| \geq 1 - \sqrt{1 - \frac{1}{4}\varepsilon^2}.$$

The idea behind these formulas is *the convexity of the unit ball* in an inner product space, i.e., if the distance between two points x and y in the unit sphere is larger than ε , then the midpoint of the segment joining x and y remains in the unit ball with

$$1 - \frac{\varepsilon^2}{4} \geq \left\| \frac{1}{2}(x + y) \right\|^2.$$

Motivated by this, we extend this notion to spaces, not with an inner product, but with a norm and study “*how much convex*” the unit ball is.

Definition 3.1.1. *Let E be a Banach space with $\dim E \geq 2$. The modulus of convexity of E is the function $\delta_E : (0, 2] \rightarrow (0, 1]$ defined by:*

$$\delta_E(\varepsilon) = \inf \left\{ 1 - \left\| \frac{1}{2}(x + y) \right\| : x, y \in B_E, \varepsilon \leq \|x - y\| \right\}$$

In the particular case an inner product space H , we have

$$\delta_H(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$$

Lemma 3.1.1. *For every Banach space E , the function $\frac{\delta_E(\varepsilon)}{\varepsilon}$ is non-decreasing on $(0, 2]$.*

Proof. Fix $0 < \eta \leq 2$ with $\eta \leq \varepsilon \in (0, 2]$ and $x, y \in E$, such that $\|x\| = 1 = \|y\|$ and $\|x - y\| = \varepsilon$.

Suffices to prove $\frac{\delta_E(\eta)}{\eta} \leq \frac{\delta_E(\varepsilon)}{\varepsilon}$.

Consider $u = \frac{\eta}{\varepsilon}x + \left(1 - \frac{\eta}{\varepsilon}\right)\frac{x+y}{\|x+y\|}$, and $v = \frac{\eta}{\varepsilon}y + \left(1 - \frac{\eta}{\varepsilon}\right)\frac{x+y}{\|x+y\|}$.

Then $u - v = \frac{\eta}{\varepsilon}(x - y)$. So that $\|u\| \leq 1$, $\|v\| \leq 1$.

$$\Rightarrow \|u - v\| = \left\| \frac{\eta}{\varepsilon}(x - y) \right\| = \frac{\eta}{\varepsilon}\|x - y\| = \eta, \quad (\text{since } \|x - y\| = \varepsilon)$$

$$\Rightarrow \|u - v\| = \eta \quad \text{and}$$

$$\frac{u + v}{2} = \frac{x + y}{\|x + y\|} \left(1 - \frac{\eta}{\varepsilon} + \frac{\eta\|x + y\|}{2\varepsilon} \right), \quad \text{then } \left\| \frac{u + v}{2} \right\| = 1 - \frac{\eta}{\varepsilon} + \frac{\eta\|x + y\|}{2\varepsilon}$$

which implies that

$$\begin{aligned} \left\| \frac{x + y}{\|x + y\|} - \frac{u + v}{2} \right\| &= \left\| \frac{x + y}{\|x + y\|} - \frac{x + y}{\|x + y\|} \left(1 - \frac{\eta}{\varepsilon} + \frac{\eta\|x + y\|}{2\varepsilon} \right) \right\| \\ &= \left\| \frac{x + y}{\|x + y\|} \right\| \left[1 - \left(1 - \frac{\eta}{\varepsilon} + \frac{\eta\|x + y\|}{2\varepsilon} \right) \right] \\ &= \underbrace{\left[1 - \left(1 - \frac{\eta}{\varepsilon} + \frac{\eta\|x + y\|}{2\varepsilon} \right) \right]}_{= \left\| \frac{1}{2}(u + v) \right\|} = 1 - \left\| \frac{1}{2}(u + v) \right\|, \end{aligned}$$

$$\begin{aligned}
& \text{since } \left\| \frac{x+y}{\|x+y\|} \right\| = 1 \\
\text{Note that } & \left\| \frac{x+y}{\|x+y\|} - \frac{x+y}{2} \right\| = \|x+y\| \left(\frac{1}{\|x+y\|} - \frac{1}{2} \right) = 1 - \frac{\|x+y\|}{2} \\
\text{Now } & \frac{\left\| \frac{x+y}{\|x+y\|} - \frac{u+v}{2} \right\|}{\|u-v\|} = \frac{1}{\eta} \left(\frac{\eta}{\varepsilon} - \frac{\eta\|x+y\|}{2\varepsilon} \right) = \frac{1}{\varepsilon} \left(1 - \frac{\|x+y\|}{2} \right) = \frac{\left\| \frac{x+y}{\|x+y\|} - \frac{x+y}{2} \right\|}{\|x-y\|} \\
\text{Hence, } & \frac{\left\| \frac{x+y}{\|x+y\|} - \frac{u+v}{2} \right\|}{\|u-v\|} = \frac{\left\| \frac{x+y}{\|x+y\|} - \frac{x+y}{2} \right\|}{\|x-y\|} \\
\text{and then } & \frac{\delta_E(\eta)}{\eta} \leq \frac{1 - \frac{\|u+v\|}{2}}{\|u-v\|} = \frac{\left\| \frac{x+y}{\|x+y\|} - \frac{u+v}{2} \right\|}{\|u-v\|} = \frac{\left\| \frac{x+y}{\|x+y\|} - \frac{x+y}{2} \right\|}{\|x-y\|} = \frac{1 - \|\frac{1}{2}(x+y)\|}{\|x-y\|} \\
& = \frac{1 - \|\frac{1}{2}(x+y)\|}{\varepsilon} \\
& \Rightarrow \frac{\delta_E(\eta)}{\eta} \leq \frac{1 - \|\frac{1}{2}(x+y)\|}{\varepsilon}.
\end{aligned}$$

By taking the infimum over all possible x and $y \in E$ with $\varepsilon = \|x-y\|$, and

$$\|x\| = 1 = \|y\|, \quad \left[\text{i.e. } \delta_E(\varepsilon) \leq 1 - \|\frac{1}{2}(x+y)\| \right].$$

$$\text{We obtain that } \frac{\delta_E(\eta)}{\eta} \leq \frac{\delta_E(\varepsilon)}{\varepsilon}, \quad \text{whenever } 0 < \eta \leq \varepsilon \leq 2$$

Therefore, the function $\frac{\delta_E(\varepsilon)}{\varepsilon}$ is non-decreasing on $(0, 2]$, for every normed space E . \square

Theorem 3.1.1. *The modulus of convexity of a Banach space E is convex and continuous function.*

Proof. Let E be a Banach space and we define the set

$$S_{u,v} = \left\{ (x, y) : x, y \in B_E, x-y = au, x+y = bv, \text{ for some } u, v \in E \quad a, b > 0 \right\}$$

and the function

$$\delta_{u,v}(\varepsilon) = \inf \left\{ 1 - \|\frac{1}{2}(x+y)\| : x, y \in S_{u,v}, \|x-y\| \geq \varepsilon \right\}$$

$$\text{Note that : } \delta_{u,v}(0) = 0.$$

First we need to show that $\delta_E(\varepsilon)$ is convex function.

Let $0 < \varepsilon_1 < \varepsilon_2 \leq 2$ and $\eta > 0$ be given, we can choose $(x_i, y_i) \in S_{u,v}$ such that

$$\|x_i - y_i\| \geq \varepsilon_i \quad \text{and} \quad 1 - \|\frac{1}{2}(x_i + y_i)\| \leq \delta_{u,v}(\varepsilon_i) + \frac{\eta}{2}, \quad \text{for } i = 1, 2.$$

$$\text{Now for } t \in [0, 1], \quad x_3 = tx_1 + (1-t)x_2 \quad \text{and} \quad y_3 = ty_1 + (1-t)y_2.$$

Because $x_i, y_i \in B_E$ for $i = 1, 2$; it follows that:

$$\|x_3\| = \|tx_1 + (1-t)x_2\| \leq 1 \quad \text{and} \quad \|y_3\| = \|ty_1 + (1-t)y_2\| \leq 1$$

implies $x_3, y_3 \in B_E$. If $(x_i, y_i) \in S_{u,v}$, there exist a positive constant $a_i, b_i > 0$ with $i = 1, 2$, such that

$$x_i - y_i = a_i u \quad \text{and} \quad x_i + y_i = b_i v.$$

$$\text{Set } \alpha = ta_1 + (1-t)a_2 \quad \text{and} \quad \beta = tb_1 + (1-t)b_2 \quad (\alpha, \beta > 0).$$

$$\begin{aligned} \text{Then, } x_3 - y_3 &= t \underbrace{(x_1 - y_1)}_{=a_1 u} + (1-t) \underbrace{(x_2 - y_2)}_{=a_2 u} \\ &= ta_1 u + (1-t)a_2 u = \underbrace{[ta_1 + (1-t)a_2]u}_{=\alpha} \end{aligned}$$

$$\text{Thus, } x_3 - y_3 = \alpha u, \quad \text{since } \alpha = ta_1 + (1-t)a_2 > 0 \text{ for some } u \in E$$

$$\begin{aligned} \text{and, } x_3 + y_3 &= t \underbrace{(x_1 + y_1)}_{=b_1 v} + (1-t) \underbrace{(x_2 + y_2)}_{=b_2 v} \\ &= tb_1 v + (1-t)b_2 v = \underbrace{[tb_1 + (1-t)b_2]v}_{=\beta} \end{aligned}$$

$$\text{Thus, } x_3 + y_3 = \beta v, \quad \text{since } \beta = tb_1 + (1-t)b_2 > 0 \text{ for some } v \in E.$$

Hence, $(x_3, y_3) \in S_{u,v}$.

Observe that, by choice of $x_i, y_i \in B_E$ and $a_i, b_i > 0$ for $i = 1, 2$

$$\begin{aligned} \|x_3 - y_3\| = \|\alpha u\| &= \|[ta_1 + (1-t)a_2]u\| = [ta_1 + (1-t)a_2]\|u\| \\ &= ta_1\|u\| + (1-t)a_2\|u\| = t\|a_1 u\| + (1-t)\|a_2 u\| \\ &= t\|x_1 - y_1\| + (1-t)\|x_2 - y_2\| \end{aligned}$$

$$\text{since } a_1 u = x_1 - y_1 \quad \text{and} \quad a_2 u = x_2 - y_2$$

$$\Rightarrow \|x_1 - y_1\| + (1-t)\|x_2 - y_2\| \geq t\varepsilon_1 + (1-t)\varepsilon_2,$$

$$\text{since } \|x_i - y_i\| \geq \varepsilon_i, \text{ for } i = 1, 2$$

$$\therefore \|x_3 - y_3\| \geq t\varepsilon_1 + (1-t)\varepsilon_2.$$

$$\begin{aligned} \text{And } \|x_3 + y_3\| = \|\beta v\| &= \|[tb_1 + (1-t)a_2]v\| = [tb_1 + (1-t)b_2]\|v\| \\ &= tb_1\|v\| + (1-t)b_2\|v\| = t\|b_1 v\| + (1-t)\|b_2 v\| \\ &= t\|x_1 + y_1\| + (1-t)\|x_2 + y_2\| \end{aligned}$$

$$\text{since, } b_1 v = x_1 + y_1 \quad \text{and} \quad b_2 v = x_2 + y_2.$$

$$\therefore \|x_3 + y_3\| = t\|x_1 + y_1\| + (1-t)\|x_2 + y_2\|.$$

By definition of $\delta_{u,v}(\cdot)$

$$\begin{aligned} \delta_{u,v}(t\varepsilon_1 + (1-t)\varepsilon_2) \leq 1 - \left\| \frac{1}{2}(x_3 + y_3) \right\| &= 1 - \left[t \left\| \frac{1}{2}(x_1 + y_1) \right\| + (1-t) \left\| \frac{1}{2}(x_2 + y_2) \right\| \right] \\ &= 1 - t \left\| \frac{1}{2}(x_1 + y_1) \right\| - (1-t) \left\| \frac{1}{2}(x_2 + y_2) \right\| \\ &= \underbrace{t \left(1 - \left\| \frac{1}{2}(x_1 + y_1) \right\| \right)}_{\leq \delta_{u,v}(\varepsilon_1) + \frac{\eta}{2}} + (1-t) \underbrace{\left(1 - \left\| \frac{1}{2}(x_2 + y_2) \right\| \right)}_{\leq \delta_{u,v}(\varepsilon_2) + \frac{\eta}{2}} \\ &\leq t \left[\delta_{u,v}(\varepsilon_1) + \frac{\eta}{2} \right] + (1-t) \left[\delta_{u,v}(\varepsilon_2) + \frac{\eta}{2} \right] \\ &= t\delta_{u,v}(\varepsilon_1) + (1-t)\delta_{u,v}(\varepsilon_2) + \frac{\eta}{2} \end{aligned}$$

$$\begin{aligned}\Rightarrow \delta_{u,v}(t\varepsilon_1 + (1-t)\varepsilon_2) &\leq t\delta_{u,v}(\varepsilon_1) + (1-t)[\delta_{u,v}(\varepsilon_2)] + \frac{\eta}{2}, \\ \forall t \in [0, 1], \quad \forall 0 < \varepsilon_1 < \varepsilon_2 \leq 2, \eta > 0\end{aligned}$$

Since, η is arbitrary, it follow that:

$$\delta_{u,v}(t\varepsilon_1 + (1-t)\varepsilon_2) \leq t\delta_{u,v}(\varepsilon_1) + (1-t)[\delta_{u,v}(\varepsilon_2)], \quad \forall t \in [0, 1], 0 < \varepsilon_1 < \varepsilon_2 \leq 2$$

Therefore, $\delta_{u,v}(\varepsilon)$ is a convex function of ε .

Note that $\delta_E(\varepsilon) \leq \delta_{u,v}(\varepsilon)$, $\forall x, y \in B_E$ and $(x, y) \in S_{u,v}$ with $\|x\| \leq 1, \|y\| \leq 1$ and for some $u, v \in E$; hence, $\delta_E(\varepsilon) = \inf\{\delta_{u,v}(\varepsilon) : u, v \in E \setminus \{0\}\}$

$\therefore \delta_E(\varepsilon)$ is convex.

Secondly we need to show that $\delta_E(\varepsilon)$ is continuous function.

Now, for any real number $\varepsilon > 0$, there exist $u, v \in E$, such that, $\delta_E(\varepsilon_1) \leq \delta_{u,v}(\varepsilon_1) + \varepsilon$.

$$\text{Suppose } \varepsilon_2 = \left(\frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}\right)2 + \left(1 - \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}\right)\varepsilon_1$$

and $\delta_E(\varepsilon)$ is a convex function, then

$$\delta_{u,v}(\varepsilon_2) = \delta_{u,v}\left[\left(\frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}\right)2 + \left(1 - \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}\right)\varepsilon_1\right] \leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}\delta_{u,v}(2) + \left(1 - \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}\right)\delta_{u,v}(\varepsilon_1)$$

which implies that

$$\begin{aligned}\delta_{u,v}(\varepsilon_2) - \delta_{u,v}(\varepsilon_1) &\leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}\delta_{u,v}(2) - \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}\delta_{u,v}(\varepsilon_1) \\ \Rightarrow \delta_{u,v}(\varepsilon_2) - \delta_{u,v}(\varepsilon_1) &\leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}\left(\delta_{u,v}(2) - \delta_{u,v}(\varepsilon_1)\right) \leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}\left(1 - \delta_E(\varepsilon_1)\right) \\ \Rightarrow \delta_{u,v}(\varepsilon_2) - \delta_{u,v}(\varepsilon_1) &\leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}\left(1 - \delta_E(\varepsilon_1)\right).\end{aligned}$$

Then, we have

$$\delta_E(\varepsilon_2) - \delta_E(\varepsilon_1) \leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}\left(1 - \delta_E(\varepsilon_1)\right) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\begin{aligned}\delta_E(\varepsilon_2) - \delta_E(\varepsilon_1) &\leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}\left(1 - \delta_E(\varepsilon_1)\right) \\ \text{Since } \delta_E(\varepsilon_1) &\geq 0 \text{ we have} \\ \delta_E(\varepsilon_2) - \delta_E(\varepsilon_1) &\leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}\left(1 - \delta_E(\varepsilon_1)\right) \leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1} \\ \Rightarrow \delta_E(\varepsilon_2) - \delta_E(\varepsilon_1) &\leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}.\end{aligned}$$

Let $M = \frac{1}{2 - \varepsilon_1} > 0$, then

$$\begin{aligned}\delta_E(\varepsilon_2) - \delta_E(\varepsilon_1) &\leq M(\varepsilon_2 - \varepsilon_1) \\ \Rightarrow \|\delta_E(\varepsilon_2) - \delta_E(\varepsilon_1)\| &\leq \|M(\varepsilon_2 - \varepsilon_1)\| \\ \|\delta_E(\varepsilon_2) - \delta_E(\varepsilon_1)\| &\leq M\|\varepsilon_2 - \varepsilon_1\| \\ \Rightarrow \delta_E(\varepsilon) &\text{ is bounded.}\end{aligned}$$

Now, for every $\varepsilon > 0$, there exist a $\delta > 0$ such that

$$\|\delta_E(\varepsilon_2) - \delta_E(\varepsilon_1)\| < \varepsilon, \quad \text{whenever } \|\varepsilon_2 - \varepsilon_1\| < \delta.$$

Choose $\delta = \frac{\varepsilon}{M}$.

Hence, $\delta_E(\varepsilon)$ is continuous on $(0, 2]$.

Therefore, the modulus of convexity of a normed space E is convex and continuous function. \square

3.2 Characterization Of Uniform Convexity, Strict Convexity and Reflexivity

The following result characterizes the uniformly convex spaces interms of $\delta_E(\varepsilon)$.

Theorem 3.2.1. *A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.*

Proof. (\Rightarrow)

Suppose E is uniformly convex space, given $\varepsilon \in (0, 2]$, there exists $\delta > 0$, such that:

$$\delta(\varepsilon) \leq 1 - \left\| \frac{1}{2}(x + y) \right\|,$$

for every x and y in E such that

$$\|x\| = \|y\| = 1 \quad \text{and} \quad \varepsilon \leq \|x - y\|.$$

$$\therefore \delta_E(\varepsilon) > 0, \forall \varepsilon \in (0, 2].$$

(\Leftarrow)

Assume $0 < \delta_E(\varepsilon)$, for every $\varepsilon \in (0, 2]$.

Fix $\varepsilon \in (0, 2]$ and take $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\varepsilon \leq \|x - y\|$, then

$$0 < \delta_E(\varepsilon) \leq 1 - \left\| \frac{1}{2}(x + y) \right\|$$

Therefore,

$$\left\| \frac{1}{2}(x + y) \right\| \leq 1 - \delta,$$

with $\delta = \delta_E(\varepsilon)$ which does not depend on x or y . \square

Corollary 3.2.1. *In a uniformly convex space E , the modulus of convexity is a strictly increasing function.*

Proof. By Theorem 3.1.1. for $0 < s < t \leq 2$, we have $t\delta_E(s) \leq s\delta_E(t)$, thus we get

$$\begin{aligned} t\delta_E(s) &\leq s\delta_E(t) < t\delta_E(t) \\ \Rightarrow \frac{t\delta_E(s)}{t} &\leq \frac{s\delta_E(t)}{t} < \frac{t\delta_E(t)}{t} \\ \Rightarrow \delta_E(s) &< \delta_E(t). \end{aligned}$$

Therefore, $\delta_E(s) < \delta_E(t)$ whenever $0 < s < t \leq 2$, and the modulus of the convexity is a strictly increasing function. \square

Definition 3.2.1. Let E be a normed linear space and J be a canonical embedding of E into E^{**} . If J is onto, then E is called **reflexive**.

An interesting result about uniform convexity is the Milman-Petti's Theorem.

Theorem 3.2.2. (Milman-Petti's Theorem). If E is uniformly convex space, then E is reflexive.

Proof. Suppose that E is non-reflexive, uniformly convex space. Then for some $\varepsilon > 0$ there exists $x^{**} \in E^{**}$, with $\|x^{**}\| = 1$ and such that the distance between x^{**} and B_E , (the closed unit ball of E) is 2ε .

Let $\delta > 0$ be chosen such that if x and y are in E with $\|x\| \leq 1, \|y\| \leq 1$ and $2 - \delta \leq \|x + y\|$, then $\|x - y\| \leq \varepsilon$. Take $x^* \in E^*$, with $\|x^*\| = 1$ such that $\langle x^{**}, x^* \rangle = 1$.

Let V be the weak-star neighborhood of x^{**} given by:

$$V = \left\{ u^{**} \in E^{**} : |\langle x^*, u^{**} \rangle - 1| < \frac{\delta}{2} \right\}.$$

If x and y are in the closed unit ball of E belonging to V (under identification by the canonical injection) then $|\langle x^*, x + y \rangle| > 2 - \delta$, so $2 - \delta \leq \|x + y\|$. Hence $\|x - y\| \leq \varepsilon$. Fixing x we conclude that $V \cap B_E \subseteq x + \varepsilon B_{E^{**}}$. By Goldstein's Theorem (Goldstein's Theorem says that " $J(B_E)$ is weak-star dense in $B_{E^{**}}$ ", where J is the canonical injection of E into E^{**}), but $V \cap B_E$ is weak-star dense in $V \cap B_{E^{**}}$ which, since $x + \varepsilon B_{E^{**}}$ is weak-star closed, yields x^{**} belongs to $x + \varepsilon B_{E^{**}}$. This means that the distance between x^{**} and B_E is less than or equal to ε , contradicting our choice of x^{**} . \square

Bibliography

- [1] C. E. Chidume. Applicable Analysis. International Centre for Theoretical Physics Trieste, Italy. July, 2006
- [2] C. E. Chidume. Geometric properties of Banach spaces and Non-linear Iterations, Lecture Notes in Mathematics, Springer-Verlag London Limited, 2009
- [3] A. J. Clarkson.; Uniformly Convex space., Trans. Amer. Math. Soc.,; 40, 396-414, 1936
- [4] R. Deumlich, Functional Analysis I, Addis Ababa University, 1997
- [5] K. Goebel and W. A. Kirk; Topics in Metric Fixed Point Theory, Cambridge studies in Advanced Mathematics 28, CUP. Cambridge, New York, Poort Chester, Melbourne, Sydney, 1990
- [6] R. Goldberg, Methods of Real Analysis (1sted), The University of Iowa, 1964
- [7] H. L. Royden, Real Analysis (3rd ed), Macmillan, New York, 1988