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Project

On

Sobolev Space and Application to Elliptic Partial Differential Equation

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Contents

Abstract

This project work contains two parts: sobolev spaces and application to elliptic partial differential equations. The theory of sobolev space has been originated by Russian mathematics
S.L sobolev around 1938. These spaces were not introduced for some theoretical purpose,
but for the need of the theory of partial differential equations.
This paper is then aimed at showing the applications of sobolev spaces defined in L_p spaces, solving Dirichlet problems for the poisson equations, and the Dirichlet problems with spectral parameters.

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Introduction

In mathematics, a sobolev space is a normed space of functions obtained by imposing on a function f and its derivatives up to some order l the condition of finite L_p norm, for given $p \geq 1$. It is named after Sergei L.Sobolev, In many problems of mathematical physics and variational calculus it is not sufficient to deal with the classical solution of differential equations. It is necessary to introduce the notion of weak derivatives and to work in the so called sobolev spaces.

Let us consider the simplest example the Dirichlet problem for Laplace equation in a bounded domain $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} \Delta u = 0, x \in \Omega \\ u(x) = \varphi(x), x \in \partial\Omega \end{cases} \qquad (*)$$

Where $\varphi(x)$ is a given function on the boundary $\partial\Omega$. It is known that the Laplace equation is the Euler equation for the functional

$$l(u) = \int_\Omega \sum_{j=1}^{n} \left| \frac{\partial u}{\partial x_j} \right| dx$$

We can consider $(*)$ as a variational problem: to find the minimum of $l(u)$ on the set of functions satisfying condition $u \setminus \partial\Omega = \varphi$.it is much easier to minimize this functional not in $C^1(\overline{\Omega})$,but in a larger class.

Namely, in the sobolev class $W_2^1(\Omega)$.

$W_2^1(\Omega)$ consists of all functions $u \in L_2(\Omega)$, having the *weak derivatives* $\partial_j u \in L_2(\Omega)$, $j = 1\ldots, n$.if the boundary $\partial\Omega$ is smooth , then the trace of $u(x)$ on $\partial\Omega$ is well defined and the relation $u \setminus \partial\Omega = \varphi$ makes sense. (This follows from the so called boundary trace theorem for sobolev spaces.)

If we consider $l(u)$ on $W_2^1(\Omega)$, it is easy to prove the existence and uniqueness of solution of our variational problem.

The function $u \in W_2^1(\Omega)$,that gives minimum to $l(u)$ under the condition $u \setminus \partial\Omega = \varphi$, is called the *weak solution* of the Dirichlet problem $(*)$.

Basic Definition and Notation

$\mathbb{N}$: The set of natural numbers

$\mathbb{N}_0$: The set of non-negative integer

$\mathbb{R}^n$: n - Dimensional Euclidean space and $x = (x_1, x_2, ...x_n)$ be a variable element of $\mathbb{R}^n$

Ω : denotes a bounded open three dimension region in $\mathbb{R}^3$

$\partial\Omega$: The boundary of Ω, is simply connected, closed infinitely smooth surface.

$\overline{\Omega}$: The closure of Ω $(\Omega \subset \mathbb{R}^n)$

We say that a domain of $\Omega' \subset \Omega \subset \mathbb{R}^n$ is a strictly interior sub domain of Ω and write $\Omega' \subset\subset \Omega$,if $\overline{\Omega'} \subset \Omega$ If Ω' is bounded and $\Omega' \subset\subset \Omega$,then dist$\{\Omega', \partial\Omega\} > 0$ We use the following notation

$$x = (x_1, x_2, ...x_n) \in \mathbb{R}^n, \partial_j u = \frac{\partial u}{\partial x_j}$$

$\alpha = (\alpha_1, \alpha_{2,} ...\alpha_n) \in \mathbb{N}_0^n$ is a multi-index

$$|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$$

$D^\alpha u = \dfrac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}$: the ordinary derivative of the function $u(x)$ of order α

where $\alpha \in \mathbb{N}_0^n$ and $\alpha \neq 0$

$D_w^\alpha u = \left(\dfrac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}} \right)_\omega$: the weak derivative $u(x)$ of order α

$$\nabla u = (\partial_1 u \partial_2 u ... \partial_n u), |\nabla u| = \left(\sum_{j=1}^{n} |\partial_j u|^2 \right)^{\frac{1}{2}}$$

For a measurable non-empty set $\Omega \subset \mathbb{R}^n$ we shall denote by:

$L_p(\Omega) : (1 \leq p < \infty)$: is the set of all measurable function $u(x)$ in Ω such that

the norm $\|u\|_{p,\Omega} = \left(\int_\Omega |u(x)|^p \, dx \right)^{\frac{1}{p}}$ is finite

$L_p(\Omega)$: is a Banach space we'll use the following property

Let $u \in L_p(\Omega), 1 \le p < \infty$ we denote $J_\rho(u, L_p) = \sup\limits_{|z|=\rho} \left(\int\limits_{\mathbb{R}^n} |u(x+z) - u(x)|^p \, dx \right)^{\frac{1}{p}}$

Here $u(x)$ is extended by zero on $\mathbb{R}^n \setminus \Omega$. $J_\rho(u, L_p)$ is called the modulus of continuity of a function $u(x)$ in $L_p(\Omega)$ then $J_\rho(u, L_p) \to 0$ as $\rho \to 0$

$L_\infty(\Omega)$: is the set of all bounded measurable function in Ω, the norm is defined

$$\|u\|_{\infty,\Omega} = ess \sup_{x \in \Omega} |u(x)|$$

$L_{p,loc}(\Omega)(1 \le p \le \infty)$: the set of functions defined on Ω such that for each compact $k \subset \Omega, f \in L_p(k)$

$w_p^l(\Omega)(l \in \mathbb{N}, 1 \le p < \infty)$: sobolev space, which is the Banach space of functions $f \in L_p(\Omega)$ with the norm

$$\|f\|_{w_p^l(\Omega)} = \|f\|_{L_p(\Omega)} + \sum_{|\alpha|=l} \|D_w^\alpha f\|_{L_p(\Omega)}$$

For any arbitrary non-empty set $\Omega \subset \mathbb{R}^n$ we shall denoted by

$c(\Omega)$: the space of functions continuous on Ω

$c_0^\infty(\Omega)$: is the class of function $u(x)$ in Ω such that

a) $u(x)$ is infinitely smooth, which means that $\partial^\alpha u$ is uniformly continuous in $\overline{\Omega}, \forall \alpha$

b) $u(x)$ is compactly supported: supp u is compact subset of Ω

$u \in c_0^\infty(\Omega) \Leftrightarrow u \in c^\infty(\overline{\Omega})$ and supp $u \subset \Omega$

$c^l(\overline{\Omega})$: is the Banach space of all functions in $\overline{\Omega}$ such that $u(x)$ and $\partial^\alpha u(x)$ with $|\alpha| \le l$ are Uniformly continuous in $\overline{\Omega}$ and the norm

$$\|u\|_{c^l(\overline{\Omega})} = \sum_{|\alpha| \le l} \sup_{x \in \Omega} |\partial^\alpha u(x)| \text{ is finite}$$

If $l = 0$, we denote $c^0\left(\overline{\Omega}\right) = c\left(\overline{\Omega}\right)$

$c^\infty(\Omega) = \bigcap_{l=0}^{\infty} c^1(\Omega)$: the space of infinitely continuously differentiable on Ω

for Ω be a measurable set and $1 \le p \le \infty$

Holder inequality*:*

Suppose $\frac{1}{p} + \frac{1}{p'} = 1$ that is $p' = \frac{p}{p-1}$ for $1 < p < \infty$, $p' = \infty$ for $p = 1$ and $p' = 1$ for $p = \infty$. if $f \in L_p(\Omega)$ and $g \in L_p(\Omega)$, then $fg \in L_1(\Omega)$ and $\|fg\|_{L_1(\Omega)} \le \|f\|_{L_p(\Omega)} \|g\|_{L_P(\Omega)}$

Minkowski's inequality:

If $f, g \in L_p(\Omega)$, then $f + g \in L_p(\Omega)$, and $\|f + g\|_{L_p(\Omega)} \le \|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)}$

PART-I

Sobolev space

The analysis of PDE's naturally involves functions space that is not only defined in terms of the properties of the function itself, but also in terms of the properties of its derivatives. Sobolev spaces are useful tools in this analysis. Let Ω be an open subset of $\mathbb{R}^n$, with smooth boundary $\partial\Omega$ then now we give the general definition of sobolev space, based on L_p spaces

1.1 Mollification of function.

1.1.1 Definition of mollification

The procedure of the mollification allows us to approximate function $u \in L_p(\Omega)$ by smooth functions.

Let $\omega(x)$, $x \in \mathbb{R}^n$,bea function such that

$\omega \in c_0^\infty(\mathbb{R}^n)$, $\omega(x) \geq 0$, $\omega(x) = 0$ if $|x| \geq 1$, and

$$\int_{\mathbb{R}^n} \omega(x)dx = 1 \tag{1}$$

For example, we may take

$$\begin{cases} c\exp\left\{\dfrac{-1}{1-|x|^2}\right\} if\ |x| < 1 \\ 0 if\ |x| \geq 1 \end{cases}$$

Where constant c is chosen so that condition (1) is satisfied.

For $\rho > 0$ we put

$$\omega_\rho(x) = \rho^{-n}\omega\left(\frac{x}{\rho}\right), x \in \mathbb{R}^n \tag{2}$$

Then $\omega_\rho \in c_0^\infty(\mathbb{R}^n)$, $\omega_\rho(x) \geq 0$

$$\omega_\rho(x) = 0 \text{ if } |x| \geq \rho \tag{3}$$

$$\int_{\mathbb{R}^n} \omega_\rho(x)dx = 1 \tag{4}$$

Definition:

ω_ρ is called a mollifier

Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $u \in L_p(\Omega)$ with some $1 \le p \le \infty$

We extend $u(x)$ by zero on $\mathbb{R}^n \setminus \Omega$ and consider the convolution $\omega_\rho * u =: u_\rho$

$$u_\rho(x) = \int_{\mathbb{R}^n} \omega_\rho(x-y)u(y)\,dy \tag{5}$$

Definition:

$u_\rho(x)$ is called a mollification or regularization of $u(x)$

1.1.2: properties of mollification

1. $u_\rho \in c^\infty(\mathbb{R}^n)$, and $\partial^\alpha u_\rho(x) = \int_{\mathbb{R}^n} \partial_x^\alpha \omega_\rho(x-y)u(y)\,dy$ this follows from $\omega_\rho \in c^\infty(\mathbb{R}^n)$
2. $u_\rho(x) = 0$ if $\operatorname{dist}\{x:\Omega\} \ge \rho$, since $\omega_\rho(x-y) = 0, y \in \Omega$
3. Let $u \in L_p(\Omega)$ with some $p \in [1,\infty]$ then $\|u_\rho\|_{p,\mathbb{R}^n} \le \|u\|_{p,\Omega}$ (6)

Proof:

Case-1: $1 < p < \infty$

Let $\frac{1}{p} + \frac{1}{p'} = 1$ by Holder inequality and (4) we have

$$|u_\rho(x)| = \left| \int_{\mathbb{R}^n} \omega_\rho(x-y)u(y)\,dy \right|$$

$$|u_\rho(x)| = \left| \int_{\mathbb{R}^n} \omega_\rho(x-y)u(y)\,dy \right| \le \int_{\mathbb{R}^n} |\omega_\rho(x-y)u(y)|\,dy$$

$$= \int_{\mathbb{R}^n} \omega_\rho(x-y)|u(y)|\,dy$$

$$= \int_{\mathbb{R}^n} \omega_\rho(x-y)^{\frac{1}{p}} \omega_\rho(x-y)^{\frac{1}{p'}} |u(y)|\,dy$$

$$=\underbrace{\left(\int_{\mathbb{R}^n}\omega_\rho(x-y)\right)^{\frac{1}{p'}}}_{=1}\left(\int_{\mathbb{R}^n}\omega_\rho(x-y)^{\frac{1}{p}}\left|u(y)\right|dy\right)$$

$$=\int_{\mathbb{R}^n}\omega_\rho(x-y)^{\frac{1}{p}}\left(\left|u(y)\right|^p\right)^{\frac{1}{p}}dy$$

$$\left|u_\rho(x)\right|\le\left(\int_{\mathbb{R}^n}\omega_\rho(x-y)\left|u(y)\right|^p dy\right)^{\frac{1}{p}}$$

$$\left|u_\rho(x)\right|^p\le\int_{\mathbb{R}^n}\omega_\rho(x-y)\left|u(y)\right|^p dy$$

$$\int_{\mathbb{R}^n}\left|u_\rho(x)\right|^p dx\le\int_{\mathbb{R}^n}dx\int_{\mathbb{R}^n}\omega_\rho(x-y)\left|u(y)\right|^p dy$$

$$\le\underbrace{\int_{\mathbb{R}^n}\omega_\rho(x-y)dx}_{=1}.\int_{\mathbb{R}^n}\left|u(y)\right|^p dy$$

$$\le\int_{\mathbb{R}^n}\left|u(y)\right|^p dy$$

$$\left\|u_\rho\right\|_{p,\mathbb{R}^n}\le\left\|u\right\|_{p,\mathbb{R}^n}$$

$$\Rightarrow\left\|u_\rho\right\|_{p,\mathbb{R}^n}\le\left\|u\right\|_{p,\Omega}$$

Case 2: when $p=\infty$,we have

$$\left|u_\rho(x)\right|\le\int_{\mathbb{R}^n}\omega_\rho(x-y)\left|u(y)\right|dy\le\left\|u\right\|_\infty\underbrace{\int_{\mathbb{R}^n}\omega_\rho(x-y)dy}_{=1}$$

$$\Rightarrow\left\|u_\rho\right\|_{\infty,\mathbb{R}^n}\le\left\|u\right\|_{\infty,\Omega}$$

Case3: when $p=1$

We integrate the inequality

$$\left|u_\rho(x)\right|\le\int_{\mathbb{R}^n}\omega_\rho(x-y)\left|u(y)\right|dy=\int_{\mathbb{R}^n}\omega_\rho(x-y)dx.\int_{\mathbb{R}^n}\left|u(y)\right|dy$$

$$\le\int_{\mathbb{R}^n}\left|u(y)\right|dy$$

$$\Rightarrow\left\|u_\rho\right\|_{1,\mathbb{R}^n}\le\left\|u\right\|_{1,\Omega}$$

4. Let $u \in L_p(\Omega), 1 \le p < \infty$, then

$$\left\| u_\rho - u \right\|_{p,\mathbb{R}^n} \to 0 \text{ as } \rho \to 0 \tag{7}$$

Consequently.

$$\left\| u_\rho - u \right\|_{p,\Omega} \to 0 \text{ as } \rho \to 0$$

Proof

The proof is based on the following property if $u \in L_p(\Omega)$ and $u(x)$ is extended by zero, then

$$\sup_{|z| \le \rho} \left(\int_{\mathbb{R}^n} \left| u(x+z) - u(x) \right|^p dx \right)^{\frac{1}{p}} = J_\rho\left(u, L_p\right) \to 0 \text{ as } \rho \to 0$$

$J_\rho\left(u, L_p\right)$ is called the modulus of continuity of u in L_p

Case 1: $1 < p < \infty$ and by (4) and (5), we have

$$\left| u_\rho(x) - u(x) \right| \le \int_{\mathbb{R}^n} \omega_\rho(x-y) \left| u(y) - u(x) \right| dy$$

$$= \int_{\mathbb{R}^n} \omega_\rho(x-y)^{\frac{1}{p'}} \omega_\rho(x-y)^{\frac{1}{p}} \left| u(y) - u(x) \right| dy$$

Then by Holders inequality, it follows that

$$\left| u_\rho(x) - u(x) \right| \le \underbrace{\left(\int_{\mathbb{R}^n} \omega_\rho(x-y) dy \right)^{\frac{1}{p'}}}_{=1} \left(\int_{\mathbb{R}^n} \omega_\rho(x-y) \left| u(y) - u(x) \right|^p dy \right)^{\frac{1}{p}}$$

Integrating both sides

$$\int_{\mathbb{R}^n} \left| u_\rho(x) - u(x) \right|^p dx \le \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} \omega_\rho(x-y) \left| u(y) - u(x) \right|^p dy$$

$$\overset{x-y=z}{=} \int_{|z|<\rho} \omega_\rho(z) dz \int_{\mathbb{R}^n} \left| u(y+z) - u(y) \right|^p dy$$

$$\le \sup_{|z| \le \rho} \int_{\mathbb{R}^n} \left| u(y+z) - u(y) \right|^p dy \underbrace{\int_{|z|=\rho} \omega_\rho(z) dz}_{=1}$$

$$= J_\rho\left(u, L_p\right)^p$$

$$\Rightarrow \left\| u_\rho - u \right\|_{p,\mathbb{R}^n} \le J_\rho\left(u, L_p\right) \to 0 \text{ as } \rho \to 0$$

Case-2: $p = 1$,we have

$$\left| u_\rho(x) - u(x) \right| \le \int_{\mathbb{R}^n} \omega_\rho(x-y)\left| u(y) - u(x) \right| dy$$

$$\Rightarrow \int_{\mathbb{R}^n} \left| u_\rho(x) - u(x) \right| dx \le \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} \omega_\rho(x-y)\left| u(y) - u(x) \right| dy$$

$$\overset{x-y=z}{=} \int_{|z|\le\rho} \omega_\rho(z) dz \int_{\mathbb{R}^n} \left| u(y+z) - u(y) \right| dy$$

$$\le J_\rho\left(u, L_1\right) \to 0 \text{ as } \rho \to 0$$

Remark:

If $p = \infty$,there is no such property, since L_∞ limit of smooth functions $u_\rho(x)$ must be continuous function. If $u \in c\left(\overline{\Omega}\right)$ and extend $u(x)$ by zero, then we may loose continuity.

5. If $u \in c\left(\overline{\Omega}\right)$, $\overline{\Omega} \subset\subset \Omega$ and Ω' is bounded, then

$$\left\| u_\rho - u \right\|_{c\left(\overline{\Omega'}\right)} \to 0 \text{ as } \rho \to 0$$

Proof: Let $\rho < dist\{\Omega'; \partial\Omega\}$. Then

$$u_\rho(x) - u(x) = \int_{\mathbb{R}^n} \omega_\rho(x-y)\left(u(y) - u(x)\right) dy$$

$$\overset{x-y=z}{=} \int_{\mathbb{R}^n} \omega_\rho(z)\left(u(x-z) - u(x)\right) dz$$

$$\Rightarrow \sup_{x\in\overline{\Omega'}} \left| u_\rho(x) - u(x) \right| \le \sup_{x\in\overline{\Omega'}} \sup_{|z|\le\rho} \left| u(x-z) - u(x) \right| \to 0 \text{ as } \rho \to 0$$

(Since $u(x)$ is continuous in $\overline{\Omega}$)

1.2: class $C_0^\infty(\Omega)$

By $C_0^\infty(\Omega)$ we denote the class of infinitely smooth functions in Ω with compact support:

$$u \in C_0^\infty(\Omega) \Leftrightarrow u \in C^\infty(\overline{\Omega}) \text{ and } \sup u \subset \Omega$$

Theorem 1:

$$c_0^\infty(\Omega) \text{ is dense in } L_p(\Omega),\ 1 \le p < \infty$$

Proof:

Let $u \in L_p(\Omega)$ and $\varepsilon > 0$.

Let Ω' be a bounded domain, $\Omega' \subset\subset \Omega$ and $\|u\|_{p,\Omega\setminus\Omega'} \le \frac{\varepsilon}{2}$

We put $u^{(\varepsilon)}(x) = \begin{cases} u(x) if x \in \Omega' \\ 0 if x \in \Omega \setminus \Omega' \end{cases}$

Then $\|u - u^{\varepsilon}\|_{p,\Omega} \le \frac{\varepsilon}{2}$

Let $u_\rho^\varepsilon(x)$ be the mollification of $u^\varepsilon(x)$

$\|u_\rho^\varepsilon - u^\varepsilon\|_{p,\Omega} \le \frac{\varepsilon}{2}$ for sufficiently small ρ

$u_\rho^\varepsilon \in c_0^\infty(\Omega)$ for sufficient small ρ

Hence, $\|u_\rho^\varepsilon - u\|_{p,\Omega} = \|u_\rho^\varepsilon - u^\varepsilon + u^\varepsilon - u\|_{p,\Omega}$

$\le \|u_\rho^\varepsilon - u^\varepsilon\|_{p,\Omega} + \|u^\varepsilon - u\|_{p,\Omega}$

$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for sufficiently small ρ

Note that $u_\rho^{(\varepsilon)} \in c_0^\infty(\Omega)$ if $\rho < dist\{\Omega', \partial\Omega\}$

1.3. Weak derivatives

1.3.1 Definition and properties of weak derivative

Definition 1:

Let $\Omega \subset \mathbb{R}^n$ be an open set, $\alpha \in \mathbb{N}_0^n, \alpha \neq 0$ and $f, g \in L_{1,loc}(\Omega)$ the function g is the weak (distributional) derivative of function f on order α on

$\Omega \left(g = D_w^\alpha f \right)$ if $\forall \varphi \in c_0^\infty(\Omega)$

$$\int_\Omega f D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega D_w^\alpha f \varphi dx = (-1)^\alpha \int_\Omega g \varphi dx$$, where $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$

Hence, the classical derivative $\partial^\alpha f$ is also the weak derivative of course,

$\partial^\alpha f$ may exist in the weak sense without existing in the classical sense.

Remark:

1. To define the weak derivative $\partial^\alpha f$, we don't need the existence of derivatives of the smaller order (like in the classical definition)
2. The weak derivative is defined as an element of $L_{1,loc}(\Omega)$, so we can change it on some set of measure zero.
3. Note that if a function $f \in L_{1,loc}(\Omega)$ has a weak derivative $D_w^\alpha f$ on Ω then automatically $D_w^\alpha f \in L_{1,loc}(\Omega)$

Example 1: For n = 1, $\Omega = \mathbb{R}$, $|x|_w' = \operatorname{sgn} x$

Proof: $\forall \varphi \in c_0^\infty(\mathbb{R})$

$$\int_{\infty}^{\infty} |x| \varphi'(x)\, dx = -\int_{-\infty}^{0} x\varphi'(x) + \int_0^\infty x\varphi'(x)\, dx$$

$$= -x\varphi(x)|_{-\infty}^{0} + x\varphi(x)|_0^\infty + \int_{-\infty}^{0} \varphi(x) dx - \int_0^\infty \varphi(x) dx$$

$$= \int_{-\infty}^{0} \varphi(x) dx - \int_0^\infty \varphi(x) dx \qquad (8)$$ and

$$(-1)\int_{-\infty}^{\infty} \operatorname{sgn} x \varphi(x) dx = (-1)\left[-\int_{-\infty}^{0} \varphi(x)\, dx + \int_0^\infty \varphi(x) \right]$$

$$= \int_{-\infty}^{0} \varphi(x)dx - \int_{0}^{\infty} \varphi(x)dx \qquad (9)$$

Hence from (8) and (9) we can see that $|x|_w^{'} = \operatorname{sgn} x$

Example 2: for n = 1, $\Omega = \mathbb{R}$ the weak derivative $(\operatorname{sgn} x)_x^{'}$ does not exist on $\mathbb{R}$

Proof:

Suppose $g \in L_{1,loc}(\mathbb{R})$ is a weak derivative then $\forall \varphi \in c_0^\infty(\mathbb{R})$

$$\int_{-\infty}^{\infty} \operatorname{sgn} x\varphi^{'} = (-1)\int_{-\infty}^{\infty} g\varphi(x)dx$$

$$\Rightarrow \qquad -\int_{-\infty}^{0} \varphi^{'}(x)\,dx + \int_{0}^{\infty} \varphi^{'}(x)\,dx = -\varphi(0) - \varphi(0) = -2\varphi(0)$$

$$\Rightarrow \qquad (-1)\int_{-\infty}^{\infty} g\varphi(x)dx = -2\varphi(0) \qquad (10)$$

Taking $\varphi(x) = x\psi(x)$ with arbitrary $\psi \in c_0^\infty(\mathbb{R})$ then from (10)

$$\int g(x)\varphi(x)\,dx = 0$$

Thus $g \sim 0$ this leads to a contradiction

Properties of $D_w^\alpha f$

1. **Lemma**: uniqueness of weak derivative

A weak α^{th} partial derivative of f, if it exists is uniquely defined up to a set of measure zero.

Proof:

Assume $g, h \in L_{1,loc}(\Omega)$ satisfying

$\Rightarrow \quad \int_\Omega fD^\alpha \varphi dx = (-1)^{|\alpha|}\int_\Omega g\varphi dx = (-1)^{|\alpha|}\int_\Omega h\varphi dx$ for all test function $f \in c_0^\infty(\Omega)$

$\Rightarrow \quad g - h = 0$ a.e

$\Rightarrow \quad g = h$ a.e

Example 3: Let n = 1 $,\Omega=(0,2)$ and $f(x)=\begin{cases} x, if\, 0<x\le 1 \\ 1, if\, 1\le x<2 \end{cases}$

Define $g(x)=\begin{cases} 1, if\, 0<x\le 1 \\ 0, if\, 1<x<2 \end{cases}$ show that $f^{'}=g$ in the weak sense

Solution:

Let $\varphi\in c_0^\infty(\Omega)$,then $\int_0^2 f\varphi^{'}dx=\int_0^1 x\varphi^{'}dx+\int_1^2\varphi^{'}dx$

$$=-\int_0^1\varphi dx+\varphi|_0^1+\varphi|_1^2$$

$$=-\int_0^2 g\varphi dx$$

Therefore, $f^{'}=g$ in the weak sense

Example 4: Let n = 1 $,\Omega=(0,2)$ and $f(x)=\begin{cases} x, if\, 0<x\le 1 \\ 2, if\, 1<x<2 \end{cases}$

Show that $f^{'}$ does not exist in the weak sense

Solution:

Suppose there exist $g\in L_{1,loc}(\Omega)$ satisfying

$$\int_0^2 f\varphi^{'}dx=-\int_0^2 g\varphi^{'}dx \text{ for all } \varphi\in c_0^\infty(\mathbb{R})$$

Now, $$-\int_0^2 g\varphi dx=\int_0^2 f\varphi^{'}dx=\int_0^1 x\varphi^{'}dx+2\int_1^2\varphi^{'}dx$$

$$=-\int_0^1\varphi dx+x\varphi|_0^1+2\varphi|_1^2$$

$$=-\int_0^1\varphi dx+\varphi(1)-2\varphi(1)$$

$$=-\int_0^1\varphi dx-\varphi(1)$$

$$\Rightarrow\varphi(1)=\int_0^2 g\varphi dx-\int_0^1\varphi dx \qquad (11)$$

Choose a sequence $\{\varphi_m\}_{m=1}^{\infty}$ of test functions satisfying $0 \le \varphi_m \le 1, \varphi_m(1), \varphi_m(x) \to 0$

For all $x \ne 1$ replacing φ by φ_m in (11) and sending $m \to \infty$ we have

$$1 = \lim_{m\to\infty} \varphi_m(1) = \lim_{m\to\infty}\left[\int_0^2 g\varphi_m dx - \int_0^1 \varphi_m dx\right] = 0 \text{ a.e}$$

Which is a contradiction .Therefore, does not exist in the weak sense

2. Linearity:

If $u_1, u_2 \in L_{1,loc}(\Omega)$ and there exist weak derivatives $v_1 = D_w^{\alpha} u_1$,

$v_2 = D_w^{\alpha} u_2 \in L_{1,loc}(\Omega)$ then there exist $D_w^{\alpha}(c_1u_1 + c_2u_2)$ and

$D_w^{\alpha}(c_1u_1 + c_2u_2) = c_1 D_w^{\alpha} u_1 + c_2 D_w^{\alpha} u_2$, $c_1, c_2 \in \mathbb{C}$

Proof:

$$\int_{\Omega} (c_1u_1 + c_2u_2) D^{\alpha}\varphi(x)\,dx = c_1\int_{\Omega} u_1 D^{\alpha}\varphi(x)dx + \int_{\Omega} u_2 D^{\alpha}\varphi(x)dx$$

$$= (-1)^{|\alpha|} c_1 \int_{\Omega} v_1\varphi dx + (-1)^{|\alpha|} c_2 \int_{\Omega} v_2 \varphi dx$$

$$= (-1)^{|\alpha|} \int_{\Omega} \underbrace{(c_1v_1 + c_2v_2)}_{D^{\alpha}(c_1u_1 + c_2u_2)} dx$$

$$= D^{\alpha}(c_1u_1 + c_2u_2)$$

3. If $v = D_w^{\alpha} u$ in Ω , then $v = D_w^{\alpha} u$ in Ω' for any $\Omega' \subset \Omega$

4. Mollification of the weak derivative

"Derivative of mollification is equal to mollification of derivative" this is true in any bounded strictly interior domain $\Omega' \subset\subset \Omega$ Suppose that $u, v \in L_{1,loc}(\Omega)$ and $v = \partial^{\alpha} u$ then

$$v_{\rho}(x) = \partial^{\alpha} u_{\rho}(x) \text{ if } \rho < dist\{x, \partial\Omega\} \qquad (12)$$

The function u_{ρ} and v_{ρ} are smooth; the derivative $\partial^{\alpha} u_{\rho}$ in (12) is understood in the classical sense.

Proof: Let $\rho < dist\{x, \partial\Omega\}$ we have

$$u_\rho(x) = \int_\Omega \omega_\rho(x-y)u(y)\,dy$$

Then $\partial^\alpha u_\rho(x) = \int_\Omega \partial_x^\alpha \omega_\rho(x-y)u(y)\,dy$

Note that $\partial_x^\alpha \omega_\rho(x-y) = (-1)^{|\alpha|}\partial_y^\alpha \omega_\rho(x-y)$

Hence $\partial_x^\alpha u_\rho(x) = (-1)^{|\alpha|}\int_\Omega \partial_y^\alpha \omega_\rho(x-y)u(y)\,dy$

Since $\rho < dist\{x, \partial\Omega\}$, then for $\eta(y) : \omega_\rho(x-y)$

we have $\eta(y) \in c_0^\infty(\Omega)$

By definition of the weak derivative $\partial^\alpha u = v$, we obtain

$$\partial^\alpha u_\rho(x) = \int_\Omega \omega_\rho(x-y)v(y)\,dy = v_\rho(x)$$

5. Suppose that $u \in L_{1,loc}(\Omega)$ and there exist the weak derivative

$\partial^\alpha u \in L_p(\Omega)$, $1 \le p < \infty$ then $\left\|\partial^\alpha u_\rho - \partial^\alpha u\right\|_{p,\Omega'} \to 0$ as $\rho \to 0$

for any bounded strictly domain $\Omega' \subset\subset \Omega$

Proof:

This follows from property (4) of mollification and property (4) of weak

derivatives $\partial^\alpha u = v \in L_p(\Omega)$; $\partial^\alpha u_\rho = v_\rho$ in Ω' (for sufficiently small ρ);

$$\left\|v_\rho - v\right\|_{p,\Omega'} \to 0 \text{ as } \rho \to 0$$

Remark:

If we extended $u(x)$ by zero on $\mathbb{R}^n \setminus \Omega$, then in general the weak derivative $\partial^\alpha u$ in $\mathbb{R}^n$ does not exist. Hence, we have convergence $\partial^\alpha u_\rho \xrightarrow{\rho \to 0} \partial^\alpha u$ in $L_p(\Omega')$ only for bounded strictly interior domain Ω'

Theorem 2: If $D^\alpha u = v$ and $D^\beta v = w$ in the weak sense then $D^{\alpha+\beta} u = w$ in the weak sense

Proof:

Let $\psi \in C_0^\infty(\Omega)$ and $\phi = D^\beta \psi$. Then

$$\int_\Omega u D^{\alpha+\beta}\psi = (-1)^{|\alpha|}\int_\Omega \phi v dx$$

$$= (-1)^{|\alpha|}\int_\Omega v D^\beta \psi dx$$

$$= (-1)^{|\alpha|+|\beta|}\int_\Omega \psi w dx$$

Another definition of the weak derivative

Definition 2:

Suppose that $u, v \in L_{1,loc}(\Omega)$ and there exists a sequence $u_m \in C^1(\Omega)$, $m \in \mathbb{N}$,

Such that $u_m \xrightarrow{m\to\infty} u$ and $\partial^\alpha u_m \xrightarrow{m\to\infty} v$ in $L_{1,loc}(\Omega)$. Here α is a multi-index and $|\alpha| = l$ .then v is called the weak derivative of u in $\Omega : \partial^\alpha u = v.$

Remark: definition 1 $\Leftrightarrow$ definition 2

Proof:

1) Definition 1 $\Leftarrow$ Definition 2.

Since $u_m \in C^1(\Omega)$, then

$$\int_\Omega u_m \partial^\alpha \varphi dx = (-1)^{|\alpha|}\int_\Omega \partial^\alpha u_m \varphi dx \ , \ \forall \varphi \in C_0^\infty(\Omega). \tag{13}$$

For φ fixed , the left-hand side of (13) tends to $\int_\Omega u\partial^\alpha \varphi dx$ as $m \to \infty$:

$$\left|\int_\Omega (u_m - u)\partial^\alpha \varphi dx\right| \le \max\left|\partial^\alpha \varphi\right| \int_{\sup p}|u_m - u|\, dx \xrightarrow{m\to\infty} 0.$$

Similarly, the right- hand side of (13) tends to $(-1)^{|\alpha|}\int_\Omega v\varphi dx.$

Consequently,

$$\int_\Omega u\partial^\alpha \varphi dx = (-1)^{|\alpha|}\int_\Omega v\varphi dx, \qquad \forall \varphi \in C_0^\infty(\Omega)$$

It means that $v = \partial^\alpha u$ in the sense of Definition 1.

2) Definition 1 $\Rightarrow$ Definition 2.

Let $u.v \in L_{1,loc}(\Omega)$, and let $v = \partial^\alpha u$ in the sense of Definition 1.

We want to find a sequence $u_m \in C^\infty(\Omega)$ such that $u_m \xrightarrow{m\to\infty} u$ and

$$\partial^\alpha u_m \xrightarrow{m\to\infty} v \text{ in } L_{1,loc}(\Omega).$$

Let $\{\Omega'_m\}, m\in\mathbb{N},$ be a sequence of bounded domains such that

$$\Omega'_m \subset\subset \Omega, \quad \Omega'_m \subset \Omega'_{m+1} \text{ and } \bigcup_{m\in\mathbb{N}} \Omega'_m = \Omega$$

We put $\quad u^{(m)}(x) = \begin{cases} u(x), if x\in\Omega'_m \\ 0, otherwise \end{cases}$

Then $u^{(m)} \in L_1(\Omega)$. Consider the mollification of $u^{(m)} : u_\rho^{(m)} \in C^\infty(\Omega)$ as $m\to\infty$

We put $\quad u_m(x) = u_{\rho_m}^{(m)}(x), \quad x\in\Omega.$

Then $u_m \in C^\infty(\Omega)$ and $u_m \xrightarrow{m\to\infty} u$ in $L_{1,loc}(\Omega)$.

By property (4) of mollification.

Next , by property (5) we have $\partial^\alpha u_m \xrightarrow{m\to\infty} v$ in $L_{1,loc}(\Omega)$

Thus , $v=\partial^\alpha u$ in the sense of Definition 2.

Theorem 3:

Let $u_m \in L_{1,lock}(\Omega)$ and $u_m \xrightarrow{m\to\infty} u$ in $L_{1,loc}(\Omega)$. suppose that there exist weak derivatives $\partial^\alpha u_m \in L_{1,loc}(\Omega)$. and $\partial^\alpha u_m \xrightarrow{m\to\infty} v$ in $L_{1,loc}(\Omega)$. then $v=\partial^\alpha u$ In other words, the operator ∂^α is closed.

Proof:

By Definition 1, for $\partial^\alpha u_m$ we have

$$\int_\Omega u_m \partial^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega \partial^\alpha u_m \varphi dx, \qquad \forall \varphi \in c_0^\infty(\Omega)$$

$$\downarrow m\to\infty \qquad\qquad \downarrow m\to\infty$$

$$\int_\Omega u \partial^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega v \varphi dx, \qquad \forall \varphi \in c_0^\infty(\Omega)$$

$\Rightarrow v=\partial^\alpha u$ in the sense of definition 1.

1.4. Definition of sobolev space based on L_p space

1. 4.1 Definition of $W_p^l(\Omega)\ (1\le p<\infty), l\in\mathbb{N}_0$

Definition:

Suppose $1\le p\le\infty$ and $l\in\mathbb{N}_0, \Omega$ be a non- empty open set.

The sobolev space $W_p^l(\Omega)$ of order l based on $L_p(\Omega)$ is defined by

$W_p^l(\Omega)=\{u\in L_p(\Omega):\partial^\alpha u\in L_p(\Omega),\ for|\alpha|\le l\}$

Remark:

For $p=2, l\ge 0, W_2^l(\Omega)=H^l(\Omega)$ is a Hilbert space with respect to the inner product.

Definition:

If $f\in W_p^l(\Omega)$ we define its norm to be

$$\|f\|_{W_p^l(\Omega)}=\begin{cases}\left(\sum_{|\alpha|\le l}\int_\Omega |D^\alpha f|^p\right)^{\frac{1}{p}}, if 1\le p<\infty\\ \sum_{|\alpha|\le l} ess\sup|D^\alpha f|, if p=\infty\end{cases}$$

Definitions:

a) Let $\{f_m\}_{m=1}^\infty, f\in W_p^l(\Omega)$. We say f_m converges to f in $W_p^l(\Omega)$ written $f_m\to f$ in $W_p^l(\Omega)$, Provided $\lim_{m\to\infty}\|f_m\to f\|_{W_p^l(\Omega)}=0$

b) We write $f_m\to f$ in $W_p^l(\Omega)$ to mean $f_m\to f$ in $W_p^l(v)$ for each $v\subset\subset\Omega$

Remark:

1. The norm $\sum_{|\alpha|\le l}\|\partial^\alpha u\|_{p,\Omega}$ is equivalent to the standard norm.
2. $W_p^0(\Omega)=L_p(\Omega)$

Proposition

$W_p^l(\Omega)$ is complete.

In other words, $W_p^l(\Omega)$ is a Banach space

Proof:

Let $\{u_m\}$ be a fundamental sequence in $W_p^l(\Omega)$. It is equivalent to the fact that all sequences $\{\partial^\alpha u_m\}$ for $|\alpha| \le l$ are fundamental sequences in $L_p(\Omega)$.

Since $L_p(\Omega)$ is complete, there exist fundamental $u, v_\alpha \in L_p(\Omega)$ such that

$$u_m \xrightarrow{L_p(\Omega)} u, \quad \partial^\alpha u_m \xrightarrow{L_p(\Omega)} v_\alpha \text{ as } m \to \infty$$

Then $u_m \to u$, $\partial^\alpha u_m \to v_\alpha$ in $L_{1,loc}(\Omega)$ by theorem 3 , $v_\alpha = \partial^\alpha u$

Hence, $u_m \xrightarrow{W_p^l(\Omega)} u$ as $m \to \infty$

If $p = 2$, then space $W_p^l(\Omega)$ is a Hilbert space with the inner product

$$(u,v)_{W_p^l(\Omega)} = \int_\Omega \sum_{|\alpha| \le l} \partial^\alpha u(x) \overline{\partial^\alpha v(x)} dx$$

For $W_p^l(\Omega)$ another notation $H'(\Omega)$ is often used: $W_p^l(\Omega) = H'(\Omega)$ Using the properties of weak derivative, we can show that the class $W_p^l(\Omega)$ is invariant with respect Smooth (c^1 - class $W_p^l(\Omega)$) change of variables.

1.4.2 Definition of $\overset{\circ}{W}{}_p^l(\Omega)$

Definition:

The closure of $c_0^\infty(\Omega)$ in the norm of $W_p^l(\Omega)$ is denoted by $\overset{\circ}{W}{}_p^l(\Omega)$.

So, $\overset{\circ}{W}{}_p^l(\Omega)$ is a subspace in the space $W_p^l(\Omega)$

Proposition:

Let $u \in \overset{\circ}{W}{}_p^l(\Omega)$, and let

$$\tilde{u}(x) = \begin{cases} u(x), x \in \Omega \\ 0, x \in \mathbb{R}^n \setminus \Omega \end{cases}$$

Then $\tilde{u} \in W_p^l(\Omega)$ for any Ω_1 such that $\Omega \subset \Omega_1$ .in particular $\tilde{u} \in W_p^l(\Omega)$

Proof:

By definition of $\overset{\circ}{W}{}_p^l(\Omega)$, there exist a sequence $u_m \in c_0^\infty(\Omega)$

Such that $u_m \xrightarrow{W_p^l(\Omega)} u$ as $m \to \infty$,we get

$$\tilde{u}_m(x) = \begin{cases} u_m(x), x \in \Omega \\ o, otherwise \end{cases}$$

Then $\tilde{u}_m \in c_0^\infty(\Omega_1)$ and $\tilde{u}_m \xrightarrow{W_p^l(\Omega)} \tilde{u}$ as $m \to \infty$

(Since $\|\tilde{u}_m - \tilde{u}\|_{W_p^l(\Omega_1)} = \|u_m - u\|_{W_p^l(\Omega)}$)

Hence, $\tilde{u} \in \overset{\circ}{W}{}_p^l(\Omega_1)$

Theorem 4:

Let $u \in \overset{\circ}{W}{}_p^l(\Omega)$ and let

$$\tilde{u}(x) = \begin{cases} u(x), x \in \Omega \\ 0, x \in \Re^n \setminus \Omega \end{cases}$$

There for mollifications $u_\rho(x)$ we have $u_\rho \xrightarrow{\rho \to 0} u$ in $W_p^l(\Omega)$

Proof:

We have already proved that $\tilde{u} \in W_p^l(\Re^n)$. then $\partial^\alpha \tilde{u} \in L_p(\Re^n)$, $|\alpha| \le l$ By property (4) and (5) of $\partial^\alpha u$ (mollification of the weak derivative) $\partial^\alpha \tilde{u}_\rho \xrightarrow{\rho \to 0} \partial^\alpha \tilde{u}$ in $L_p(\Omega)$, $|\alpha| \le l$ It means that $\tilde{u}_\rho \xrightarrow{\rho \to 0} \tilde{u}$ in $W_p^l(\Omega)$. But by definition of mollification, $\tilde{u} = u$ in Ω, and $\tilde{u}_\rho = u_\rho$.

So, $u_\rho \xrightarrow{\rho\to 0} u$ in $W_p^l(\Omega)$

1.4.3 Integration by parts

Proposition:

Let $u \in W_p^l(\Omega)$ and $v \in \overset{\circ}{W}{}_p^l(\Omega)$, where $\frac{1}{p}+\frac{1}{p'}=1$. Then

$$\int_\Omega \partial^\alpha u v dx = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha v dx \quad , \ |\alpha| \le l \qquad (14)$$

Proof:

Let $v_m \in c_0^\infty(\Omega)$ and $v_m \to v$ as $m \to \infty$ in $\overset{\circ}{W}{}_p^l(\Omega)$.

by definition of the weak derivative $\partial^\alpha u$, we have

$$\int_\Omega \partial^\alpha u v_m dx = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha v_m dx \qquad (15)$$

Let us show that

$$\int_\Omega \partial^\alpha u v_m dx \xrightarrow{m\to\infty} \int_\Omega \partial^\alpha u v dx,$$
$$\int_\Omega u \partial^\alpha v_m dx \xrightarrow{m\to\infty} \int_\Omega u \partial^\alpha v dx.$$

We have

$$\left| \int_\Omega \partial^\alpha u (v_m - v) dx \right| \le \left(\int_\Omega |\partial^\alpha u|^p \right)^{\frac{1}{p}} \left(\int_\Omega |v_m - v|^{p'} \right)^{\frac{1}{p'}}$$
$$\le \|u\|_{W_p^l(\Omega)} \|v_m - v\|_{W_p^l(\Omega)}$$

$$\to 0 \text{ as } m \to \infty$$

$$\left| \int_\Omega u \left(\partial^\alpha v_m - \partial^\alpha v \right) dx \right| \le \left(\int_\Omega |u|^p dx \right)^{\frac{1}{p}} \left(\int_\Omega |\partial^\alpha v_m - \partial^\Omega v|^{p'} dx \right)^{\frac{1}{p'}}$$
$$\le \|u\|_{W_p^l(\Omega)} \|v_m - v\|_{W_p^l(\Omega)}$$
$$\to 0 \text{ as } m \to \infty$$

Tending to the limit in (15) as $m \to \infty$, we obtain (14)

PART-II

2. Application to elliptic partial differential equation

2.1 Embedding

In the application of sobolev space on partial differential equations, it is important to have inequalities between functions in sobolev space. An application of the concept of continuous embedding gives us some of these inequalities. In general embedding theorems gives relations between different functional spaces.

Definition:

Let x and y be two Banach spaces.We then say that x is continuously embedded in y and write $x \subset y$. If $x \subset y$ and there exists a constant c such that $\|x\|_y \le c\|x\|_x$, $\forall x \in x$ We define the embedding operator $I : x \to y$, which takes $x \in X$ into the same element x consider as an element of y. The fact that $x \subset y$ is equivalent to the fact that the embedding operator $I : x \to y$ is continuous linear operator.

If $\|x\|_y \le c\|x\|_x \quad \forall x \in x$,then $\|I\| \le c$

Definition:

If X is dense subspace o y we say that I is a continuous dense injection. If $x \subset y$ and the embedding operator $I : x \to y$ is compact operator, and then we say that X is compactly embedded into y .the compactness of operator I is equivalent to the fact that any bounded set in X is a compact set in y .

Example:

$H^1(\Omega)$ is continuously embedded in $L_2(\Omega)$ .Hence

$\|\varphi\|_{L_2(\Omega)} \le c\|\varphi\|_{H^1(\Omega)} \quad \forall \varphi \in H^1(\Omega)$, where $c \le 1$.

This is equivalent to $\|\varphi\|_{L_2(\Omega)} \le c\left(\|\varphi\|_{L_2(\Omega)} + \|\nabla\varphi\|_{L_2(\Omega)}\right)$ for some constant k.

2.2 The trace theorem

For the application of sobolev space to boundary value problems, it is necessary to restrict functions in sobolev spaces to the boundary of the domain in order to satisfy the assigned boundary conditions. On the other hand, how smooth do our boundary data have to be so that functions in $H^s(\Omega)$ can take such values? In this section we will introduce the trace operator which restricts a function defined on the domain $\overline{\Omega}$ to the boundary of Ω, denoted by $\partial\Omega$.

We write $x \in \mathbb{R}^n$ as $x = (x', x_n), x' = (x_1, \ldots x_n) \in \mathbb{R}^{n-1}$. consider the traces of functions on the hyper-plane $x_n = 0$. we define the trace operator

$$\gamma_0 : C_0^\infty(\Re^n) \to C_0^\infty(\Re^{n-1}) \ , \quad (\gamma_0 u)(x') = u(x', 0)$$

2.3 extension theorem

Let $k \in \mathbb{N}_0, s > k + \frac{1}{2}$

Denote $H^{\langle s-\frac{1}{2}\rangle}(\Re^{n-1}) = H^{s-\frac{1}{2}}(\Re^{n-1}) \times H^{s-\frac{3}{2}}(\Re^{n-1}) \times \ldots \times H^{s-k-\frac{1}{2}}(\Re^{n-1})$.

There exists a linear continuous operator

$$p : H^{\langle s-\frac{1}{2}\rangle}(\Re^{n-1}) \to H^s(\Re^n),$$

Such that, if $\varphi = (\varphi_0, \varphi_1, \ldots \varphi_k) \in H^{\langle s-\frac{1}{2}\rangle}(\Re^{n-1}), u = p_\varphi \in H^s(\Re^n)$

Then $\varphi_j = \gamma_j u,\ j = 0, 1, \ldots, k.$ we have

$$\|u\|^2_{H^s(\Re^n)} \le c\|\varphi\|^2_{H^{\langle s-\frac{1}{2}\rangle}(\Re^{n-1})} = c\sum_{j=0}^{k}\|\varphi_j\|^2_{H^{s-j-\frac{1}{2}}(\Re^n)}$$

2. 4 Spaces $H^s(\Omega)$

Let $\Omega \subset \Re^n$ be a domain. There are different ways of definition of the sobolev spaces $H^s(\Omega)$

Approach I

Definition 1

$H^s(\Omega)$ is the class of restrictions to Ω of functions in $H^s(\Re^n)$:

$$u \in H^s(\Omega) \Leftrightarrow \exists v \in H^s(\mathbb{R}^n), \quad v \backslash \Omega .$$

Approach II

Case $s \ge 0$

Definition 2

$H^s(\Omega)$ is the set of functions in $L_2(\Omega)$, such that their weak derivatives up to order $k = [s]$ also belong to $L_2(\Omega)$, and the following norm is finite: $\|u\|_{H^s} < \infty$,

$$\|u\|_{H^s}^2 \underset{=}{def} \begin{cases} \sum_{|\alpha| \le s} \int_\Omega |\partial^\alpha u|^2 \, dx, if s = [s] \\ \sum_{|\alpha| \le k} \int_\Omega |\partial^\alpha u|^2 \, dx + \sum_{|\alpha| = k} \int_\Omega \int_\Omega \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)| dxdy}{|x-y|^{n+2\{s\}}}, \\ if s \ne [s] = k, \{s\} = s - k \end{cases} \qquad (1)$$

Comments:

1. If $\Omega \subset \Re^n$ is a bounded domain of Lipschitz class, then both definitions giving the same spaces: Def 1 $\Leftrightarrow$ Def 2. If $H^s(\Omega), s \ge 0$,is the sobolev space in the sense of Def 2,there exists a linear continuous extension operator $\prod : H^s(\Omega) \to H^s(\Re^n)$

2. The spaces $W_p^s(\Omega)$ with fractional $s \ge 0$ and $p \ne 2$ can be defined by analogy with Def 2 (With “2” replaced by “p”).

3 The embedding theorems can be generalized for spaces of fraction order.

Next, the space $\overset{\circ}{H}{}^s(\Omega)$ is defined.

Definition 3

$\overset{\circ}{H}{}^s(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm (1).

Definition 4 :

Let $s>0$ .then, by definition, $H^{-s}(\Omega)=\left(\overset{\circ}{H}{}^{s}(\Omega)\right)^{*}$,i.e. , $H^{-s}(\Omega)$ is the space of linear

continuous functional on $\overset{\circ}{H}{}^{s}(\Omega)$ with the norm

$$\|u\|_{H^{-s}}=\sup_{0\neq\varphi\in\overset{0}{H}{}^{s}(\Omega)}\frac{|<u,\varphi>|}{\|\varphi\|_{H^{s}(\Omega)}}$$

By analogy with $H^{s}(\mathbb{R}^{n})$ and $H^{-s}(\mathbb{R}^{n})$, for $u\in H^{-s}(\Omega),\varphi\in\overset{\circ}{H}{}^{s}(\Omega)$

We denote

$$\langle u,\varphi\rangle=\int_{\Omega}u(x)\varphi(x)\,dx$$

Comments

1. $\overset{\circ}{H}{}^{s}(\Omega)=H^{s}(\Omega)$ for $s<\frac{1}{2}$

2. Let $u\in\overset{\circ}{H}{}^{s}(\Omega)$, $p_0u(x)=\begin{cases}u(x),x\in\Omega\\0,x\in\Re^{n}\setminus\Omega\end{cases}$ Then $p_0:\overset{\circ}{H}{}^{s}(\Omega)\to H^{s}(\Re^{n})$

is continuous, if $s\neq m+\frac{1}{2},m\in\mathbb{N}_0$.

3. Let $\Omega\subset\Re^{n}$ be a bounded domain with Lipschitz boundary.

Then $H^{-s}(\Omega)$ coincides with the Space of restrictions to

Ω of distributions$\in H^{-s}(\Re^{n})$, if $s\neq m+\frac{1}{2},m\in\mathbb{N}_0$

4. $H^{s}(\Omega)$ is invariant with respect to diffeomorphisms of class $C^{1},l\geq|s|,l\in\mathbb{N}$.

Friedrichs inequality

If Ω is a bounded domain in $\Re^n$, then for any function $u \in \overset{\circ}{W}{}_p^l(\Omega)$

We have

$$\|u\|_{p,\Omega} \le (diam\Omega)^l - u -_{p,\Omega} \qquad (2)$$

Here $$-u-_{p,l,\Omega} = \left(\sum_{|\alpha = l|} \|\partial^\alpha u\|_{p,\Omega}^p \right)^{\frac{1}{p}} \qquad (3)$$

Proof:

Since $C_0^\infty(\Omega)$ is dense in $\overset{\circ}{W}{}_p^l(\Omega)$, it suffices to prove (2) for $u \in C_0^\infty(\Omega)$

1) So, let $u \in C_0^\infty(\Omega)$. Let Q be a cube with the edge $d = diam\Omega$, such that $\Omega \subset Q$. We extend $u(x)$ by zero to $Q \setminus \Omega$.we can choose the coordinate system so that $Q = \{x : 0 < x_j < d,\ j = 1, \dots, n\}$ Obviously, $u(x) = \int_0^{x_n} \frac{\partial u}{\partial x_n}(x', x_n)\, dy, x \in Q$.

Here $x = (x_1, \dots x_{n-1}, x_n) = (x', x_n)$ Then, by the Holder inequality,

$$|u(x)|^p \le \left(\int_0^{x_n} \left| \frac{\partial u}{\partial x_n}(x', y) \right|^p dy \right) \underbrace{\left(\int_0^{x_n} 1 dy \right)^{p/p'}}_{\le d^{p/p'}} \le d^{p/p'} \int_0^d \left| \frac{\partial u(x', x_n)}{\partial x_n} \right|^p dx_n$$

Here $\frac{1}{p} + \frac{1}{p'} = 1$

We integrate both sides of this inequality.

$$\int_\Omega |u|^p\, dx = \int_Q |u|^p\, dx \le d^{p/p'} \left(\int_0^d dx_n \right) \left(\int_Q \left| \frac{\partial u}{\partial x_n} \right|^p dx \right)$$

$$\overset{\frac{p}{p'}+1=p}{=} d^p \int_\Omega \left|\frac{\partial u}{\partial x_n}\right|^p dx$$

We have provided that

$$\|u\|_{p,\Omega} \leq (diam\Omega)\left(\int_\Omega \left|\frac{\partial u}{\partial x_n}\right|^p dx\right)^{\frac{1}{p}} \leq (diam\Omega) - u -_{p,1,\Omega} \qquad (4)$$

This is inequality (2) for $l = 1$

2) in order to prove (2) with $l > 1$, we integrate (4)

$$\int_\Omega \left|\frac{\partial u}{\partial x_n}\right|^p dx \leq d^p \int_\Omega \left|\frac{\partial^2 u}{\partial x_n{}^2}\right|^p dx$$

$$\Rightarrow \qquad \int_\Omega |u|^p \, dx \leq d^{l_p} \int_\Omega \left|\frac{\partial^l u}{\partial x_n^l}\right|^p dx \leq d^{l_p} - u - p_{p,l,\Omega}$$

Riesz Theorem:

Let H be a Hilibert space and $l(u), u \in H$ be a continuous linear functional on H

Then there exists such element $v \in H$ that $l(u) = (u,v)_H$. this element v is unique and $\|l\| = \|v\|_H$

1) Let $N = Kerl = \{z \in H : l(z) = 0\}$. Then N is a closed subspace in H. indeed, if $z_j \in N$ and $z_j \xrightarrow{j\to\infty} z$ in H, then $l(z_j) \xrightarrow{j\to\infty} l(z)$. Since $l(z_j) = 0$,

It follows that $l(z) = 0, i.e., z \in N$.

2) If $N = H,$ then $l(u) = 0, \forall u \in H$.in this case $v = 0$ if $N \neq H,$ then $N^\perp \neq \{0\}$ (where $N^\perp$ is the orthogonal complement of N). So, there exists $v_0 \in N^\perp, v_0 \neq 0$. then, $l(v_0) \neq 0$

3) For $\forall u \in H$ consider $u - \dfrac{l(u)}{l(v_0)} v_0 \in N$.

(indeed, $l\left(u-\frac{l(u)}{l(v_0)}v_0\right)=l(u)-\frac{l(u)}{l(v_0)}l(v_0)=0$.)

Since $v_0 \in N^{\perp}$,we have

$$\left(u-\frac{l(u)}{l(v_0)}v_0\right)=0 \Rightarrow (u,v_0)=l(u)\frac{\|v_0\|^2}{l(v_0)}$$

Denote $v=\frac{\overline{l(v_0)}}{\|v_0\|^2}v_0$. then $l(u)=(u,v)$

4. Uniqueness

If $(u,v)=(u,\tilde{v}), \forall u \in H$, then $v-\tilde{v} \perp H \Rightarrow v-\tilde{v}=0$.

5. The norm of l.

$$\|l\| = \sup_{0\neq\varphi\in H}\frac{|l(u)|}{\|u\|_H} = \sup_{0\neq\varphi\in H}\frac{|(u,v)|}{\|u\|_H} = \|v\|_H$$

Indeed,

$\frac{|(u,v)|}{\|u\|_H} \le \|v\|_H$ for $\forall 0\neq u\in H$, and for $u=v$ we have $\frac{|(u,v)|}{\|u\|_H} \le \|v\|_H$

Let $l(u)$ be a continuous linear functional on $H^s(\Re^n)$.

It means that $l: H^s \to \mathbb{C}$,

a) $l(c_1u_1+c_2u_2)=c_1l(u_1)+c_2l(u_2), \forall u_1,u_2 \in H^s, \forall c_1,c_2 \in \mathbb{C}$,

b) $|l(u)| \le c\|u\|_{H^s}, \forall u \in H^s(\Re^n)$.

Then norm $\|l\|$ of a function l is defined by the formula

$$\|l\| = \sup_{0\neq u\in H^s}\frac{|l(u)|}{\|u\|_{H^s}}$$

2.5 Dirichlet problem for the poisson equation

Let $\Omega \subset \Re^n$ be a bounded domain. Consider the classical Dirichlet problem:

$$\begin{cases} -\Delta u = F, x \in \Omega \\ u \setminus \partial\Omega = g \end{cases} \qquad (5)$$

If $\Phi(x)$ is arbitrary function in Ω such that $\Phi \setminus \partial\Omega = g$,then the function

$v(x) = u(x) - \Phi(x)$ is solution of the problem

$$\begin{cases} -\Delta v = F, x \in \Omega \\ v \setminus \partial\Omega = 0 \end{cases} \qquad (6)$$

Where $f(x) = F(x) + \Delta\Phi(x)$. First , we'll study problem (6) with homogeneous boundary condition. In the classical setting of problem (6), the boundary is sufficiently smooth, $f \in C(\overline{\Omega})$and solution $v \in C^2(\overline{\Omega})$. Now we want to define weak solution of problem (6) under wide conditions on $\partial\Omega$ and f .Let us formally multiply equation $-\Delta v = f$ by the test function $\varphi \in C_0^\infty$ and integrate over Ω.

Then $v(x)$ satisfies the integral identity

$$\int_\Omega \nabla v \overline{\nabla \varphi} dx = \int_\Omega f \overline{\varphi} dx \qquad , \forall \varphi \in C_0^\infty \qquad (7)$$

The left-hand side is well-defined for any $v \in \overset{0}{H}1_{(\Omega)} = \overset{\circ}{\boldsymbol{W}}{}_{\boldsymbol{2}}^{\boldsymbol{1}}\boldsymbol{(\Omega)}$, $\boldsymbol{\varphi} \in \overset{0}{H}1_{(\Omega)}$

And the right- hand side is well-defined for $f \in H^{-1}(\Omega)$, $\varphi \in \overset{0}{H}1_{(\Omega)}$

(since $H^{-1}(\Omega)$ is the dual space to $\overset{0}{H}1_{(\Omega)}$with respect to L_2-duality). The boundary condition $v \setminus \partial\Omega = 0$ we understand in the sense that $v \in \overset{0}{H}1_{(\Omega)}$ .then we can consider arbitrary domains.

Definition

Let $\Omega \subset \Re^n$ be arbitrary bounded domain. A function $v \in \overset{0}{H}1_{(\Omega)}$is called a weak solution of the Dirichlet problem (2) with $f \in H^{-1}(\Omega)$, if v satisfies the identity (7) for any $\varphi \in \overset{0}{H}1_{(\Omega)}$

Theorem 1:

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then, for any $f \in H^{-1}(\Omega)$, there exist unique (weak) Solution $v \in \overset{0}{H}1_{(\Omega)}$of the Dirichlet problem (6) we have , $\|v\|_{H^1(\Omega)} \le C\|f\|_{H^{-1}(\Omega)}$.

Proof:

1. The form

$$[v,\varphi] := \int_\Omega \nabla v \overline{\nabla\varphi} dx, \;\; v,\varphi \in \overset{0}{H}1_{(\Omega)}$$

defines an inner product in the space $\overset{0}{H}1_{(\Omega)}$. The corresponding norm $[v,v]^{1/2}$

is equivalent to the standard norm $\|v\|_{H^1(\Omega)} = \left(\int_\Omega \left(|v|^2 + |\nabla v|^2\right) dx\right)^{\frac{1}{2}}$. this follows

from the Friedrichs inequality:

$$\int_\Omega |v|^2 dx \le C_\Omega \int_v |\nabla v|^2 dx, \;\; \forall v \in \overset{0}{H}1_{(\Omega)}$$

(here it is important that Ω is bounded).

2. the right-hand side of (7) is

$$l_f(\varphi) = \int_\Omega f\overline{\varphi} dx$$

$l_f(\varphi)$ is antilinear continuous functional on $\varphi \in \overset{0}{H}1_{(\Omega)}$:

$$|l_f(\varphi)| \le \|f\|_{H^{-1}(\Omega)} \|\varphi\|_{H^1(\Omega)}.$$

We rewrite (3) in the following form:

$$[v,\varphi] = l_f(\varphi). \tag{8}$$

By the Riesz theorem, for antilinear continuous functional l_f on $\overset{0}{H}1_{(\Omega)}$ there

exists unique function $v \in \overset{0}{H}1_{(\Omega)}$ such that $l_f(\varphi) = [v,\varphi]$, and the norm of

this functional is equal to the norm of v.(now we consider $\overset{0}{H}1_{(\Omega)}$ as the Hilibert space

with the inner product $[.,.]$.) then , by the Riesz theorem ,

$$\|l_f\| = \sup_{0\neq\varphi\in \overset{0}{H}^1(\Omega)} \frac{|l_f(\varphi)|}{[\varphi,\varphi]^{\frac{1}{2}}} = [v,v]^{\frac{1}{2}} \tag{9}$$

Thus, v is the unique solution of (8) $\Leftrightarrow$ (7).since, by definition of the class $H^{-1}(\Omega)$,

$$\|f\|_{H^{-1}(\Omega)} = \sup_{0\neq\varphi\in \overset{0}{H}^1(\Omega)} \frac{|l_f(\varphi)|}{\|\varphi\|_{H^1(\Omega)}}$$

and $\|\varphi\|_{H^1(\Omega)} \asymp [\varphi,\varphi]^{\frac{1}{2}}$, it follows form (9) that

$$\|v\|_{H^1(\Omega)} \leq C\|f\|_{H^{-1}(\Omega)}$$

Theorem 2:

Let $\Omega\subset\Re^n$ be a bounded domain of class C^1. Let $F\in H^{-1}(\Omega)$, $g\in H^{\frac{1}{2}}(\partial\Omega)$.

Then there exists unique weak solution $u\in H^1(\Omega)$ of Problem (5). We have

$$\|u\|_{H^1(\Omega)} \leq C\left(\|F\|_{H^{-1}(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\partial\Omega)}\right) \tag{10}$$

Proof:

1. by extension theorem, for $g\in H^{\frac{1}{2}}(\partial\Omega)$,there exists extension $G = p_\Omega g\in H^1(\Omega)$

Such that $\gamma_0 G = g$ and

$$\|G\|_{H^1(\Omega)} \leq C_1\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \tag{11}$$

If $u\in H^1(\Omega)$ and $\gamma_0 u = g$. Then $v = u - G\in H^1(\Omega)$ and $\gamma_0 v = 0$.

This is equivalent to the fact that $v\in \overset{\circ}{H}^1(\Omega)$. Function v is solution

of the problem

$$\begin{cases} -\Delta v = f \\ v\setminus\partial\Omega = 0 \end{cases} \tag{12}$$

Where $f = F + \Delta G$. From $G \in H^1(\Omega)$ it follows that $\Delta G \in H^{-1}(\Omega)$ and

$\|\Delta G\|_{H^{-1}(\Omega)} \le C_2 \|G\|_{H^1(\Omega)}$. Then $f \in H^{-1}(\Omega)$ and

$\|\Delta G\|_{H^{-1}(\Omega)} \le \|F\|_{H^{-1}(\Omega)} + C_2 \|G\|_{H^1(\Omega)} \le \|F\|_{H^{-1}(\Omega)} + C_1 C_2 \|g\|_{H^{\frac{1}{2}}(\partial\Omega)}$.

By theorem 1, there exists unique solution $v \in \overset{\circ}{H}{}^1(\Omega)$ of the problem (12), and

$\|v\|_{H^1(\Omega)} \le C_3 \|f\|_{H^{-1}}$. then $u = v + G$ is unique solution of the problem (5), and

$$\begin{aligned} \|u\|_{H^1} &\le \|v\|_{H^1(\Omega)} + \|G\|_{H^1} \\ &\le c_3 \|f\|_{H^{-1}(\Omega)} + c_1 \|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\le c_3 \|f\|_{H^{-1}(\Omega)} + (c_1 c_2 c_3 + c_1) \|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \end{aligned}$$

2.6 Dirichlet problem with spectral parameter

Now we consider the problem

$$\left.\begin{aligned} -\Delta u = \lambda u + f(x), x \in \Omega \\ u \setminus \partial\Omega = 0 \end{aligned}\right\} \qquad (13)$$

with spectral parameter λ. Here Ω is bounded.

Definition:

Let $\Omega \subset \mathbb{R}^n$ be arbitrary bounded domain. Let $f \in H^{-1}(\Omega)$. A function

$u \in \overset{\circ}{H}{}^1(\Omega)$ satisfying identity

$$\int_\Omega \nabla u \overline{\nabla \varphi} dx = \lambda \int_\Omega u \overline{\varphi} dx + \int_\Omega f(x) \overline{\varphi} dx, \quad \forall \varphi \in \overset{\circ}{H}{}^1(\Omega) \qquad (14)$$

is called a weak solution of problem (13).

As before, we denote $[u, \varphi] = \int_\Omega \nabla u \overline{\nabla \varphi} dx$. this is inner product in $\overset{\circ}{H}{}^1(\Omega)$.

The form $\int_\Omega u \overline{\varphi} dx$, $u, \varphi \in \overset{\circ}{H}{}^1(\Omega)$ is continuous sesquilinear form in $\overset{\circ}{H}{}^1(\Omega)$.

By the Riesz theorem for such form it can be represented as $[Au, \varphi]$, where

A is linear continuous operator in $\overset{\circ}{H}{}^1(\Omega)$.

Obviously, $\int_{\Omega} u\bar{\varphi}dx = \overline{\left(\int_{\Omega} \varphi\bar{u}dx\right)}$, so $[Au,\varphi] = \overline{[A\varphi,u]} = [u, A\varphi]$, $\forall u,\varphi \in \overset{\circ}{H}{}^1(\Omega)$.

It follows that $A = A^*$.

Next, $[Au,u] = \int_{\Omega} |u|^2 \, dx > 0$ if $u \neq 0$.so, $A > 0$.

Lemma:

The operator A is compact operator in $\overset{\circ}{H}{}^1(\Omega)$.

Proof:

This follows from the embedding theorem: $\overset{\circ}{H}{}^1(\Omega)$ is compactly embedded in $L_2(\Omega)$.

We'll use the following property of compact operators: T is a compact operator in the Hilbert space H, if and only if for any sequence $\{u_k\}$ which converges weakly in H, the sequence $\{Tu_k\}$ converges strongly in H.

Let $\{u_k\}$ be a weak convergent sequence in $\overset{\circ}{H}{}^1(\Omega)$. Since the embedding operator $J: \overset{0}{H}1_{(\Omega)} \hookrightarrow L_2(\Omega)$ is compact, $\{u_k\}$ converges strongly in $L_2(\Omega)$.
We want to check that $\{Au_k\}$ converges strongly in $L_2(\Omega)$ .since $\{u_k\}$ weakly converges in $\overset{\circ}{H}{}^1(\Omega)$, it follows that $\|u_k\|_{H^1(\Omega)}$ is uniformly bounded. A is a continuous operator; then also $\|Au_k\|_{H^1(\Omega)}$ is uniformly bounded. We have

$$\left[A(u_k - u_l), A(u_k - u_l)\right] = \int_{\Omega} (u_k - u_l)\overline{(Au_k - Au_l)}dx$$

$$\leq \underbrace{\|u_k - u_l\|_{L_2(\Omega)}}_{\to 0} \underbrace{\|Au_k - Au_l\|_{L_2(\Omega)}}_{\leq c}$$

$$\to 0 \text{ as } k,l \to \infty$$

$\{Au_k\}$ converges strongly in $\overset{\circ}{H}{}^1(\Omega)$. It follows that A is compact operator.

As before, the functional $l_f(\varphi) = \int_\Omega f\overline{\varphi}dx$ (where $f \in H^{-1}(\Omega)$) is continuous antilinear functional on $\varphi \in \overset{\circ}{H}{}^1(\Omega)$. By the Riesz theorem, there exists unique element $v \in \overset{\circ}{H}{}^1(\Omega)$ such that $\int_\Omega f\overline{\varphi}dx = [v,\varphi], \forall \varphi \in \overset{\circ}{H}{}^1(\Omega)$, and

$\|f\|_{H^1(\Omega)} \asymp \|v\|_{H^{-1}(\Omega)}$.

Now, we can rewrite identity (14) in the form

$$[u,\varphi] = \lambda[Au,\varphi] + [v,\varphi] \ , \quad \forall \varphi \in \overset{\circ}{H}{}^1(\Omega) \tag{15}$$

Which is equivalent to the equation

$$u - \lambda Au = v \ , \tag{16}$$

Where $v \in \overset{\circ}{H}{}^1(\Omega)$ is given , and we are looking for solution $u \in \overset{\circ}{H}{}^1(\Omega)$. Thus, we reduced the problem (13) to the abstract equation (16) with compact operator

A in the Hilbert space $\overset{\circ}{H}{}^1(\Omega)$.

We analyzes equation (16) , using the properties of compact operators.

The case $v = 0$ (which corresponds to $f = 0$)

$$\left.\begin{array}{l} -\Delta u = \lambda u, x \in \Omega \\ u \backslash \partial\Omega = 0 \end{array}\right\} \tag{17}$$

$\Leftrightarrow \quad u - \lambda Au = 0$

$\Leftrightarrow \quad Au = \mu u$ (where $\mu = \dfrac{1}{\lambda}$)

It is know that the spectrum of a compact operator is discrete: it consists of eigenvalues $\mu_j, j \in \mathbb{N},$ that may accumulate only to point $\mu = 0$; each

eigenvalues is of finite multiplicity (i.e., $\dim\ker\left(A - \mu_j I\right) < \infty$). In our case

$A = A^* > 0,$ then all eigenvalues μ_j are positive: $\mu_j > 0$.we enumerate eigenvalues in non-increasing order counting multiplicties $\mu_1 \geq \mu_2 \geq \ldots$

Then each eigenvalues corresponds to one eigenfunction $u_j : Au_j = \mu_j u_j, j \in \mathbb{N}$.

Eigenfunction $\{u_j\}$ are linearly independent. We have: $\mu_j \to 0$ as $j \to \infty$.

Then for the eigenvalues $\lambda_j = \dfrac{1}{\mu_j}$ of the Dirichlet problem (17) we have the following properties: $0 < \lambda_1 \le \lambda_2 \ldots, \lambda_j \to \infty$ as $j \to \infty$.

Thus , we have the following theorem.

Theorem 3:

The spectrum of the Dirichlet problem (17) is discrete. There exists non - trivial solution only if $\lambda = \lambda_j, j \in \mathbb{N}$. All eigenvalues are positive and have finite multiplicities. The only accumulation point is infinity: $\lambda_j \to \infty$ as $j \to \infty$.

The case $v \neq 0\,(f \neq 0)$

$$\left.\begin{array}{l} -\Delta u = \lambda u + f, x \in \Omega \\ u \setminus \partial\Omega = 0 \end{array}\right\} \qquad (18)$$

$$\Leftrightarrow \quad u - \lambda A u = v$$

For compact operator A it is known that , if $\lambda \neq \lambda_j \left(= \dfrac{1}{\mu_j} \right), \forall j \in \mathbb{N}$, then operator $(I - \lambda A)^{-1}$ is bounded. We can find unique solution

$$u = (I - \lambda A)^{-1} v,$$

and

$$\|u\|_{H^1(\Omega)} \le \underbrace{\left\|(I - \lambda A)^{-1}\right\|}_{=c_\lambda} \|v\|_{H^1(\Omega)}.$$

Since $\|v\|_{H^1(\Omega)} \asymp \|f\|_{H^{-1}(\Omega)}$, we arrive at the following theorem.

Theorem 4:

If $\lambda \notin \{\lambda_j\}_{j \in \mathbb{N}}$ (λ is not eigenvalue), then for any $f \in H^{-1}(\Omega)$ there exists Unique (weak) solution $u \in \overset{\circ}{H}{}^1(\Omega)$ of the problem (18), and

$$\|u\|_{H^1(\Omega)} \le c_\lambda \|f\|_{H^{-1}(\Omega)}.$$

Now, suppose that $\lambda = \lambda_j$, and $v \neq 0\left(f \neq 0\right)$. then, solution of the equation $u - \lambda_j Au = v$ exists, if v satisfies the solvability condition: $v \perp \ker\left(I - \lambda_j A\right)$.

It means that v is orthogonal (with respect to the inner product $[.,.]$) in $\overset{\circ}{H}{}^1\left(\Omega\right)$ to all eigenfunctions $\varphi_j^{(k)}, k = 1,\dots,p,$ corresponding to the eigenvalue λ_j (here p is the multiplicity of λ_j). Since $[v,\varphi] = \int_\Omega f\left(x\right)\overline{\varphi\left(x\right)}dx,$ this

Solvability condition is equivalent to:

$$\int_\Omega f\left(x\right)\overline{\varphi_j^{(k)}}dx = 0, \qquad k = 1,\dots,p. \tag{19}$$

The solution $u\left(x\right)$ is not unique, but is defined up to summand $\sum_{j=1}^{p} c_j \varphi_j^{(k)}$ with arbitrary constants c_j.

Theorem 5:

If $\lambda = \lambda_j$ is eigenvalue of the Dirichlet problem, and $\varphi_j^{(k)}, k = 1,\dots,p,$ are corresponding (linearly independent) eigenfunction, then problem (18) has solution for any $f \in H^{-1}\left(\Omega\right)$, which satisfies the solvability conditions (19). Solution is not unique and is represented as

$$u = u_0 + \sum_{j=1}^{p} c_j \varphi_j^{(k)},$$

Where u_0 is a fixed solution, and c_j are arbitrary constants.

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