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ON THE SPECTRUM OF REGULAR
WEIGHTED STURM-LIOUVILLE PROBLEMS
IN THE SET OF INTEGERS

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I declare that this thesis "**On the Spectrum of Regular Weighted Sturm-Liouville Problems in the Set of Integers**" is an original work done by me under the supervision of Dr. Tadesse Abdi, Department of Mathematics, Addis Ababa University, Addis Ababa, Ethiopia.

Yonas Mesfin
January, 2021

*This work is dedicated to my father Mesfin Getachew
who has been there for me from day one.*

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Abstract

In this thesis, certain infinite subsets of the set of positive integers are investigated as possible spectra of regular weighted Sturm-Liouville eigenvalue problem with separated homogeneous boundary conditions. With the (conditional) exception of the set of square integers, it is shown that all the sets considered herein are not spectra of such a problem. We also state three necessary conditions in order for a set to be a spectrum of such eigenvalue problems. Concepts adopted from the area of study in Mathematical analysis known as Asymptotic analysis will figure prominently in the proofs of the main results.

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List of Symbols

$\sim$: Asymptotic equivalence between two functions or two sets¹.

δ : Asymptotic (Natural) density.

χ_A : Characteristic (Indicator) function of set A.

$O(\cdot)$: The order of magnitude function.

δ_B : Relative density with respect to base set B.

$\lfloor \cdot \rfloor$: The floor function.

$L[a, b]$: The space of integrable functions on $[a, b]$.

$L_2[a, b]$: The space of square integrable functions on $[a, b]$.

$\mathbb{C}$: The set of Complex numbers

$\mathbb{N}$: The set of Natural numbers.

$\mathbb{R}$: The set of Real numbers.

¹The intended category to be understood from the context

Introduction

The study of the spectrum of Sturm-Liouville problems on a compact interval, developed by Jacques Charles Francois Sturm (1803 - 1855) and Joseph Liouville (1809 - 1882) in the nineteenth century, is one that has a long and rich history. Subsequent investigations in this area have spanned numerous topics from operator theory to complex function theory [10]. Its applicability ranges from the study of unbounded self-adjoint operators to laying the mathematical foundations for the analysis of the Schrodinger equation and scattering matrix in quantum mechanics [13].

In [31], A. Zettl raised a problem which is restated below for ease of reference:

Problem: "Find a Sturm-Liouville problem whose spectrum is the primes."

A.B. Mingarelli, in [18]; proved that there does not exist a regular Sturm-Liouville problem with separated homogeneous boundary conditions whose spectrum is the set of prime numbers. In the process of addressing the aforementioned question A.B. Mingarelli posed the question "Does there

exist a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions with a spectrum composed of an infinite set of prime numbers if the parameter dependence is nonlinear?”. This led to further work on related topics as will be discussed in subsequent chapters.

In [31], A. Zettl also points out that given any finite set of distinct real numbers, a Sturm-Liouville problem can be found whose spectrum is precisely that set. It is this result that led us to pose the question of whether or not a similar result can be obtained for infinite sets of positive integers and to investigate the possibility of extracting a set of necessary conditions to be satisfied by a set which is the spectrum of a Sturm-Liouville problem.

The relevance of the concept of asymptotic density in relation to the problems addressed in this thesis arises from the possibility to compute the asymptotic density of the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions. This has been made possible due to the existence of an estimate for the n^{th} eigenvalue of such a boundary value problem. The following are the main questions posed and answered in this thesis (presented in Chapter 5):

1. Is there any infinite subset of the set of positive integers with a positive asymptotic density which is the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions?

2. Is there a subset $\{a_n\}$ of the set of positive integers with $\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$; which is the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions?
3. Is the set of square natural numbers the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions?
4. Is the set $F = \{m: m = r^t, t \geq 3, t \in \mathbb{N}, r = 1, 2, 3, \dots\}$ the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions?
5. Is there a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions whose spectrum contains arbitrarily long arithmetic progressions? In other words, for every natural number k , does there exist an arithmetic progression, of the eigenvalues of such a boundary value problem, with k terms?
6. Are there any necessary conditions in order for a subset of the set of positive integers to be a spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions?

Factors that have aided our efforts at addressing these inquiries include:

1. the fact that a regular weighted Sturm-Liouville problem, equipped with separated (homogeneous) boundary conditions possess the desirable prop-

erty that their eigenvalues are **real** and their spectra are **discrete**;

2. availability of asymptotic **estimate** for the n^{th} positive eigenvalue of such a problem which was given by Atkinson and Mingarelli (1987), namely

$$\lambda_n^+ \sim \frac{n^2 \pi^2}{(\int_a^b \sqrt{(\frac{r(x)}{p(x)})_+} dx)^2}$$

and

3. availability of other indispensable results and concepts in analytic number theory, asymptotic analysis, etc., including the concepts of asymptotic and relative densities along with the concepts of large and small sets.

The main objective of this thesis is to take certain infinite subsets of the set of positive integers and investigate them as possible spectra of regular weighted Sturm-Liouville eigenvalue problem with separated homogeneous boundary conditions. The overarching objective being the extraction of some necessary conditions against which we can check whether or not a given set of positive integers is the spectrum of such a boundary value problems. To this end, the content of this thesis is organised as follows:

In Chapter one, we provide an overview of the significance of second-order linear ODEs (ordinary differential equations) and define the general second-order linear ODE. Chapter one also consists of a discussion of the solvability of second-order linear ODEs and demonstrate the convertibility of every sec-

ond order linear ODE into a Sturm-Liouville equation.

In Chapter two, we consider some features of Sturm-Liouville problems and define Sturm-Liouville equations. Chapter two also contains a description of the different types of boundary conditions as well as the definition and properties of regular Sturm-Liouville problems.

In Chapter three, we discuss earlier works that linked Sturm-Liouville problems and the set of prime numbers; we give the adopted estimate for the n^{th} positive eigenvalue of regular weighted Sturm-Liouville eigenvalue problem with separated homogeneous boundary conditions as adopted from [3] a joint work by F.V. Atkinson and A.B. Mingarelli.

In Chapter four, we introduce the concepts of asymptotic density and the concepts of large sets and small sets.

In Chapter five, we present the main findings of this thesis. The preliminary lemmas and their proofs will be presented in section 5.1. We then state and provide the proofs of the theorems and corollaries in section 5.2.

Finally, we provide concluding remarks.

Chapter 1

Second-Order Linear Ordinary Differential Equations

1.1 Overview of Second-Order Linear ODEs

The theory of second-order linear ODEs is a theory with rich and far reaching consequences. This significance can be attributed to the fact that many results concerning second-order linear ODEs are generalizable to higher order linear ODEs. And this generalization can be achieved with no change in the underlying principles. The only catch being an increased complexity in the involved details. In addition, second-order linear ODEs are of great significance in both applied and theoretical Physics. Their relevance to Physics is in connection with the study of vibrations in Mechanics and

the development of the theory of electric circuits in Electricity.

The general **Second-Order Linear ODE** is given by:

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = r(x) \quad (1.1)$$

with the assumption that $a_2 \neq 0$ on the given domain.

1.2 Solvability

In general a second-order ODE can be solved using ordinary methods (if initial conditions at some point are specified) and the Picard-Lindelof theorem ensures that the ODE has a unique solution in a neighbourhood of the point where the initial conditions have been specified. But if in place of specifying initial values at a single point, it is desired to specify values at two different points (boundary values), e.g. $y(a) = 0$ and $y(b) = 1$, the problem turns out to be much more difficult. In such cases, in general, equation 1.1 cannot be solved explicitly in terms of known elementary functions and hence at times requires solutions involving infinite series. This is where the significance of the study of Sturm-Liouville equations, discussed in the next section, comes into play. This is due to the property that every second-order linear ODE can be transformed into an equation of the Sturm-Liouville form. And (self-adjoint) Sturm-Liouville problems have an infinite number of eigenvalues, and the corresponding eigenfunctions form a complete and orthogonal set which makes orthogonal expansions possible. Which in turn makes it easier

to solve for the expansion coefficients of the eigenfunctions of a second-order ODE written in Sturm-Liouville form.

1.3 Conversion to Sturm-Liouville Form

The procedure to convert a second-order linear ODE to the Sturm-Liouville form is to use an integrating factor $\mu(x)$ such that Equation (1.1) multiplied by

$$\mu(x) = \frac{1}{a_2(x)} e^{\int \frac{a_1(x)}{a_2(x)} dx}$$

would be converted to the Sturm-Liouville form:

$$\frac{d}{dx} \left[p(x) \frac{dy}{dx} \right] + q(x)y = G(x)$$

where $p(x) = e^{\int \frac{a_1(x)}{a_2(x)} dx}$, $q(x) = p(x) \frac{a_0(x)}{a_2(x)}$ and $G(x) = p(x) \frac{r(x)}{a_2(x)}$.¹

Example

For instance, consider the following second-order linear homogeneous ODE with variable coefficients:

$$x^2 y'' + xy' + 2y = 0$$

¹For a proof of the fact that any linear second-order ODE can be converted into a Sturm-Liouville form refer to [15].

to recast it into its SL form we need only multiply it with

$$\mu(x) = \frac{1}{x^2} e^{\int \frac{x}{x^2} dx} = \frac{1}{x}$$

we get

$$xy'' + y' + \frac{2}{x}y = 0$$

by (trivial) inspection we can then get to the required SL form given by:

$$(xy')' + \frac{2}{x}y = 0.$$

Chapter 2

Sturm-Liouville Problems

2.1 Features of Sturm-Liouville Problems

Sturm-Liouville theory studies the existence and asymptotic behavior of the eigenvalues, the corresponding qualitative theory of the eigenfunctions and their completeness in the function space. As an example, in quantum mechanics, the one-dimensional (time-independent) Schrodinger equation is a Sturm-Liouville problem.

In a series of papers in 1836 and 1837 C. F. Sturm and J. Liouville set forth many properties of the class of boundary value problems associated with their names.¹

¹C. F. Sturm is also famous for a theorem on the number of real zeros of a polynomial, and in addition, did extensive work in physics and mechanics.

J. Liouville on the other hand was the founder, and for 39 years the editor of an influential mathematical journal. One of his most important results was the proof (in 1844) of the existence of transcendental numbers.

Sturm-Liouville theory is important in applied mathematics, particularly in the cases where mathematical models of physical phenomena lead to the evaluation of separable linear partial differential equations.

In pure mathematics, the importance of Sturm-Liouville equation lies in:

1. oscillation theory, which was initiated by Jacques Charles Francois Sturm in his investigations of Sturm-Liouville problems in 1836. Based on those investigations, he showed that the n^{th} eigenfunction of a Sturm-Liouville problem has precisely $n - 1$ roots;
2. the convertibility of any linear second order ODE into a Sturm-Liouville equation which has the desirable property that its set of eigenfunctions is an orthogonal set and
3. the fact that it commonly arises in relation to the theoretical study of separable linear Partial Differential Equations.

2.2 Sturm-Liouville Equations

Classical **Sturm-Liouville Equation**, named after J. C. F. Sturm and J. Liouville, who studied them in the mid-1800s, is a real second-order linear differential equation of the form:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = -\lambda r(x)y \quad (2.1)$$

where $p, q, r: [a, b] \rightarrow \mathbb{R}$, and λ is generally a complex parameter; and $\frac{1}{p}$, $q, r \in L[a, b]$.

In the language of operator theory, a Sturm-Liouville operator is an operator given by:

$$L[y] = \frac{d}{dx} \left[p(x) \frac{dy}{dx} \right] + q(x)y$$

Hence, a Sturm-Liouville equation can thus be written as:

$$L[y] + \lambda r(x)y = 0$$

In the cases where $p(x)$, $q(x)$ and $r(x)$ are continuous on the finite closed interval $[a, b]$; and $p(x)$ has continuous first derivative, y is called a solution if it is continuously differentiable on (a, b) and satisfies Equation (2.1) at every point in (a, b) . In addition, the unknown function y is required to satisfy the given boundary conditions. The function $r(x)$, is called the weighting or density function.

The solutions of Equation (2.1), with appropriate boundary conditions, are called eigenfunctions. The solutions of this equation satisfy many amenable mathematical properties under appropriate boundary conditions.

Finding λ for which there exists a non-trivial solution comprises a significant portion in solving any Sturm-Liouville problem. Such values of λ , whenever they exist, are called eigenvalues of the problem. The corresponding non-trivial solutions are eigenfunctions associated to each λ .

2.3 Boundary Conditions

Boundary conditions for a solution y of a Sturm-Liouville differential equation on interval $[a, b]$ are classified as follows:

1. **Mixed Boundary Conditions** are Boundary conditions of the form

$$c_1 y(a) + d_1 y'(a) = \alpha$$

$$c_2 y(b) + d_2 y'(b) = \beta$$

where, $c_1, d_1, c_2, d_2, \alpha$ and β are constants. Mixed boundary conditions are also called mixed Dirichlet-Neumann boundary conditions.

In the cases where $\alpha = \beta = 0$ the boundary conditions are said to be homogeneous. Special cases are:

1.1. **Dirichlet Boundary Conditions:** $d_1 = d_2 = 0$.

1.2. **Neumann Boundary Conditions:** $c_1 = c_2 = 0$.

1.3. **Periodic Boundary Conditions** are Boundary conditions of the form:

$$y(a) = y(b)$$

$$y'(a) = y'(b)$$

1.4. Separated Homogeneous Boundary Conditions:

$$c_1y(a) + d_1y'(a) = 0, c_1^2 + d_1^2 > 0 \quad (2.2)$$

$$c_2y(b) + d_2y'(b) = 0, c_2^2 + d_2^2 > 0 \quad (2.3)$$

In applications, each boundary condition may be used to put in place certain restricting conditions on the phenomenon under study. For instance, in the case of the heat equation, Dirichlet boundary conditions correspond to maintaining a fixed temperature at the ends of the rod.

On the other hand, the Neumann boundary conditions would correspond to no heat flow across the ends, or insulating conditions, as there would be no temperature gradient at those points. The more general boundary conditions allow for partially insulated boundaries.

And in the case of the periodic boundary condition, consider the heated rod that has been bent to form a circle. Then the two end points are physically the same. It would thus be expected that the temperature and the temperature gradient should agree at those points. For this case we write $u(a) = u(b)$ and $u'(a) = u'(b)$. Boundary value problems using these conditions have to be handled differently than the above homogeneous conditions.

Among the previously defined boundary conditions, we will exclusively consider Separated homogeneous boundary conditions in all subsequent discussions.

2.4 Regular Sturm-Liouville Problems

Definition 1. A Sturm-Liouville problem (given by Equation (2.1) along with appropriate boundary conditions) is said to be **Regular** if $p(x) > 0$, $r(x) > 0$, and $p(x)$, $p'(x)$, $q(x)$ and $r(x)$ are all continuous functions over the finite interval $[a, b]$.

Properties of Regular Sturm-Liouville Problems

Regular Weighted Sturm-Liouville Eigenvalue Problems have the following properties²:

1. Eigenvalues of the regular weighted Sturm-Liouville Eigenvalue Problems are real.
2. Eigenfunctions of a regular Sturm-Liouville Problem corresponding to the distinct eigenvalues are orthogonal w.r.t. the weight function $r(x)$ on $[a, b]$. In other words, if the eigenfunctions u and v correspond to the distinct eigenvalues λ_1 and λ_2 then

$$\int_a^b u(x)v(x)r(x)dx = 0$$

3. The regular Sturm-Liouville problem has simple³ eigenvalues. Thus an eigenfunction that corresponds to an eigenvalue is unique up to a constant multiple.
4. The regular Sturm-Liouville operator L is self-adjoint⁴: i.e., whenever u ,

²The statements of the listed properties are taken from [16].

³An eigenvalue λ_n of L is called simple if its algebraic multiplicity $m_l(\lambda_n) = 1$.

⁴A densely defined linear operator L on a Hilbert space is self-adjoint if it equals its

$v \in C^2(a, b) \cap C^1[a, b]$ and satisfy the regular Sturm-Liouville problem and its boundary conditions then

$$\int_a^b (vL[u] - uL[v])dx = 0$$

5. A self-adjoint regular weighted Sturm-Liouville eigenvalue problem has an infinite number of real eigenvalues λ_n that are simple and can be ordered in such a way that $\lambda_1 < \lambda_2 < \dots < \lambda_n \dots$, with $\lim_{n \rightarrow \infty} \lambda_n = \infty$

adjoint. That is to say, L is self-adjoint if

- i. the domain of L coincides with the domain of the adjoint, and
- ii. the operator L agrees with its adjoint on this common domain.

Chapter 3

Sturm-Liouville Problems and the Set of Prime Numbers

In [31], A. Zettl inextricably linked Sturm-Liouville Theory and Number Theory by posing the question, Does there exist a Sturm-Liouville problem whose spectrum consists of infinitely many prime numbers?. The first work addressing this question was done by A.B. Mingarelli in [18] as will be discussed in this chapter. The availability of results providing estimates for the n^{th} eigenvalue in the spectrum of particular classes of Sturm-Liouville problems have made it possible for A. B. Mingarelli (2011) and R. Amirov and I. Adalar (2017) to work on and answer A. Zettl's question for the considered classes of Sturm-Liouville problems.

3.1 Estimate for the n^{th} Positive Eigenvalue

A.B. Mingarelli (in [18]) used the following estimate for the n^{th} positive eigenvalue of the Sturm-Liouville eigenvalue problem with separated homogeneous boundary conditions (given in Equations (2.1) and (2.2) respectively) under the assumptions that $p(x)$ has a finite number of turning points and that $(\frac{r(x)}{p(x)})_+$ is not a.e. equal to zero. Where $(\cdot)_+$ denotes the positive part of the function in question. Then, by a direct application of the Atkinson-Mingarelli theorem¹, the positive eigenvalues, λ_n^+ , of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions satisfy:

$$\lambda_n^+ \sim \frac{n^2 \pi^2}{(\int_a^b \sqrt{(\frac{r(x)}{p(x)})_+} dx)^2}$$

3.2 Outline of Earlier Works

In [18], A.B. Mingarelli answered the question of whether there exists a Sturm-Liouville problem whose spectrum is the set of prime numbers posed by A. Zettl in [31]. That is, he showed the non-existence of a regular Sturm-Liouville problem whose spectrum² is the set of prime numbers. In doing so, A.B. Mingarelli used results from Analytic Number Theory, specifically

¹See [3]

²A spectrum of an eigenvalue problem is the set of all eigenvalues of the problem.

a result providing an asymptotically equal value to the n^{th} prime number.

The Prime Number Theorem yields the following asymptotic equality: ³

$$p_n \sim n \log(n)$$

where p_n denotes the n^{th} prime number.

A.B. Mingarelli also used results on asymptotic estimates for the n^{th} positive eigenvalue of a regular Sturm-Liouville eigenvalue problem given in the previous section, namely:

$$\lambda_n^+ \sim \frac{n^2 \pi^2}{(\int_a^b \sqrt{(\frac{r(x)}{p(x)})_+} dx)^2}$$

$$\text{Now, } \lim_{n \rightarrow \infty} \frac{p_n}{\lambda_n^+} \sim \lim_{n \rightarrow \infty} \frac{K^2 n \log(n)}{n^2 \pi^2} = \lim_{n \rightarrow \infty} \frac{K^2 \log(n)}{n \pi^2} = \frac{K^2}{\pi^2} \lim_{n \rightarrow \infty} \frac{\log(n)}{n}$$

$$\text{where } K = \int_a^b \sqrt{(\frac{r(x)}{p(x)})_+} dx.$$

But, $\frac{\log(n)}{n} \rightarrow 0$ as $n \rightarrow \infty$ implies that $\lim_{n \rightarrow \infty} \frac{p_n}{\lambda_n^+}$ tends to 0 as $n \rightarrow \infty$.

By the definition of " $\sim$ ", $p_n \sim \lambda_n^+ \Leftrightarrow \lim_{n \rightarrow \infty} \frac{p_n}{\lambda_n^+}$ tends to 1 as $n \rightarrow \infty$.

Hence, $p_n \approx \lambda_n^+$. Which proves that the set of primes is not the spectrum of such a problem.

In fact, the above argument, adopted from [18]; proves more than the aforementioned assertion. Specifically, it also proves that any infinite subsequence of distinct prime numbers is not the spectrum of a regular weighted Sturm-

³See [24]

liouville problem with separated homogeneous boundary conditions.

In [18], A. B. Mingarelli, also conjectured that there may exist a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions with a spectrum composed of an infinite set of prime numbers if the parameter dependence in Equation (2.1) (or boundary conditions (2.2)) is nonlinear.

In a subsequent joint work, in response to the above conjecture in [18], R. Amirov and I. Adalar, showed in [1]; that there is no function $q(x) \in L_2(0, 1)$ which is the potential of a Sturm-Liouville problem with Dirichlet boundary condition whose spectrum is the set of prime numbers. R. Amirov and I. Adalar based their proof on the previously mentioned asymptotic distribution of prime numbers, namely

$$p_n \sim \text{li}(n) = \int_0^n \frac{dt}{\ln t}$$

and the following asymptotic estimate for the eigenvalues for $N(\lambda) = \lambda$

$$\pi \lambda_n = n\pi + \frac{\int_0^1 q(x)}{2n\pi} + O(n^{-\frac{1}{2}})^4.$$

Using these asymptotic estimates, R. Amirov and I. Adalar, proved the following two theorems:

⁴O(f), the "Big O" notation, is defined as the set of all functions g such that there exists a positive constant C for which $f(x) \leq C g(x)$.

1. If $N(\lambda) = \frac{\lambda}{\ln \lambda}$ then there is no function $q \in L_2[0, 1]$ such that the spectrum of Equation (2.1) (with Dirichlet boundary conditions) is the set of prime numbers.
2. If $N(\lambda) = \text{li}(\lambda)$ then there is no function $q \in L_2[0, 1]$ such that the spectrum of Equation (2.1) (with Dirichlet boundary conditions) is the set of prime numbers.

Although the consideration of regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions whose parameter dependence is non-linear is a potentially fruitful endeavor, in this thesis; we will only consider those problems whose parametric dependence is linear as those investigated by A. B. Mingarelli in [18]. The novelty of the problems considered in this thesis being that they are specifically posed in order to make it possible for us to extract a set of necessary conditions based on which we can conclude whether or not a set is a spectrum of a regular weighted Sturm-Liouville eigenvalue problem with separated homogeneous boundary conditions.

Chapter 4

Asymptotic Density and Large/Small Sets

To motivate the discussions in the subsequent chapters of this thesis, we will begin by providing definitions, basic properties and examples of relevant concepts¹. That is, upcoming discussions will focus on concepts deemed (by us) to be instrumental in stating and proving the main findings of this work.

4.1 Preliminaries

To avoid any ambiguities, unless explicitly stated otherwise; the term "set" should subsequently be understood to refer to an infinite set.

¹In this work, the relevance of a concept is gaged based on the degree to which it is applicable in stating and proving the main results.

Definition 2. Let $\mathbb{N}$ denote the set of natural numbers (positive integers), and let $2^{\mathbb{N}}$ denote the set of subsets (power set) of $\mathbb{N}$. For $A \in 2^{\mathbb{N}}$, define the **Counting Function** $A(n)$ (which counts the number of elements of set A that are less than or equal to the natural number n) as:

$$A(n) = \sum_{i=1}^n \chi_A(i)$$

Definition 3. The **Lower Asymptotic Density** $\underline{\delta}$ of a sequence A is expressed by the nonnegative real number $\underline{\delta}$ defined by the formula

$$\underline{\delta}(A) = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \chi_A(i) = \liminf_{n \rightarrow \infty} \sum_{a \in A, 1 \leq a \leq n} \frac{1}{n},$$

Definition 4. The **Upper Asymptotic Density** $\bar{\delta}$ of a sequence A is expressed by the real number

$$\bar{\delta}(A) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \chi_A(i) = \limsup_{n \rightarrow \infty} \sum_{a \in A, 1 \leq a \leq n} \frac{1}{n},$$

4.2 Asymptotic Density

Definition 5. The **Asymptotic Density** $\delta(A)$ is the value for which both the lower and upper asymptotic densities exist and are equal. That is, assuming that both the lower and upper asymptotic densities exist,

$$\delta(A) = \underline{\delta}(A) = \bar{\delta}(A)$$

$$\Rightarrow \delta(A) = \lim_{n \rightarrow \infty} \frac{A(n)}{n}$$

Properties of Asymptotic Density

The following are the basic properties of asymptotic density, as stated in [28]:

1. For all $A \subseteq \mathbb{N}$, $0 \leq \delta(A) \leq 1$;
2. $\delta(\mathbb{N}) = 1$;
3. If $A \sim B$, then $\delta(A) = \delta(B)$
4. If $A \cap B = \emptyset$, then $\delta(A) + \delta(B) \leq \delta(A \cup B)$;
5. For all A, B ; $\delta(A) + \delta(B) \leq 1 + \delta(A \cap B)$

If a function $\delta : 2^{\mathbb{N}} \rightarrow \mathbb{R}$ is an Asymptotic Density, then it will satisfy the previously given set of (five) properties. [28]

The following properties of asymptotic density are adopted from [22].

- 6, The asymptotic density of any finite set of integers is equal to zero.
7. Let $A = \{a_n\}$ be an infinite sequence of positive integers. If $\lim_{n \rightarrow \infty} \frac{n}{a_n}$

exists, then so does $\delta(A) = \lim_{n \rightarrow \infty} \frac{A(n)}{n}$ and they are equal.

8. Let A and B be sequences of positive integers possessing asymptotic density. If $A \cup B = C$ and $A \cap B = \emptyset$, then C also possesses asymptotic density and

$$\delta(C) = \delta(A) + \delta(B)$$

The following property is a corollary to property 8.

9. If $\delta(A)$ exists, then $\delta(A^c)$ exists and $\delta(A) = 1 - \delta(A^c)$.

10. Let A be a sequence of positive integers with asymptotic density $\delta(A)$. If A is altered by including or deleting a set of integers possessing zero asymptotic density, then the resulting sequence possesses asymptotic density $\delta(A)$.

Examples

1. The asymptotic density of the set of prime numbers in the set of natural numbers is:

$$\delta(P) = \lim_{n \rightarrow \infty} \frac{\pi(n)}{n} \sim \lim_{n \rightarrow \infty} \frac{\frac{n}{\ln(n)}}{n}$$

This implies,

$$\delta(P) = \lim_{n \rightarrow \infty} \frac{1}{\ln(n)}$$

which tends to 0 as $n \rightarrow \infty$.

2. The asymptotic density of the set of even positive integers in the set of natural numbers is²:

$$\delta(E) = \lim_{n \rightarrow \infty} \frac{\lfloor \frac{n}{2} \rfloor}{n} = \frac{1}{2}$$

Therefore, $\delta(E) = \frac{1}{2}$.

3. Since the set of natural numbers (denoted by $\mathbb{N}$) is the union of the set of even natural numbers (denoted by E) and the set of odd natural numbers (denoted by O) i.e., $\mathbb{N} = E \cup O$, with $E \cap O = \emptyset$ and since by property 2 of asymptotic density $\delta(\mathbb{N}) = 1$: by property 8 of asymptotic density we have:

$$\delta(O) = \delta(\mathbb{N}) - \delta(E) = 1 - \frac{1}{2} = \frac{1}{2}$$

Definition 6. A set H with $\delta(H) > 0$ is known as a set with positive (asymptotic) density in the set of natural numbers.

Hence, the set of even natural numbers as well as the set of odd natural

²In this proof, the following identities can be adopted:

- i. $\lfloor \frac{n}{2} \rfloor = \frac{n}{2}$ for n an even integer. And,
- ii. $\lfloor \frac{n}{2} \rfloor = \lceil \frac{n+1}{2} \rceil - 1 = \frac{n-1}{2}$ for n an odd integer.

In both cases $\lim_{n \rightarrow \infty} \frac{\lfloor \frac{n}{2} \rfloor}{n} = \frac{1}{2}$

Alternatively, consider the following inequality:

$$\frac{n-1}{2} \leq \lfloor \frac{n}{2} \rfloor \leq \frac{n+1}{2}.$$

Then, the desired limit can be obtained by applying the Squeeze Theorem.

numbers are sets with positive (asymptotic) density in the set of natural numbers.

4.3 Relative (Asymptotic) density

The concept of relative density is a generalization of the concept of asymptotic density. In other words, asymptotic density is a special case of the concept of relative density; where the set of natural numbers is taken to be the base set.³

In addition to considering the density of a set in relation to the set of natural numbers, one can also look at the density of a set relative to some base set⁴. This type of density is known as relative density. For ease of reference, "relative density" should be taken to mean "relative asymptotic density".

The **Relative Density** of a set A with respect to a base set B , denoted by $\delta_B(A)$, is given by:

$$\delta_B(A) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{\chi_A(i)}{\chi_B(i)} = \lim_{n \rightarrow \infty} \frac{A(n)}{B(n)}$$

where $\chi_A(i)$ is the characteristic function of the set $A \subseteq \mathbb{N}$ and $\chi_B(i)$ is the characteristic function of the set $B \subseteq \mathbb{N}$ and $A(n)$ and $B(n)$ represent the number of elements in set A and set B respectively, $B(n) \neq 0$. It can be

³See [28]

⁴Any subset of the set of Natural numbers can be taken to be a base. It's also possible to adopt a base differing from the natural sequences, for instance bases in algebraic number fields. But, due to the scope of the current work, we will not delve into an investigation of these concepts.

inferred from the restriction that the Empty set cannot be a base set.

The following are three basic properties of relative density:

1. For all $A \subseteq B$, $0 \leq \delta_B(A) \leq 1$;
2. If $\delta_B(A)$ exists and $C = B - A$, then $\delta_B(C)$ exists and
$$\delta_B(A) = 1 - \delta_B(C).$$
3. $[\delta_B(A) = 1 \wedge C = B - A] \Rightarrow \delta_B(C) = 0.$

4.4 Large Sets and Small Sets

Definition 7. A *Large Set* is a set of positive integers, $L = \{a_0, a_1, a_2, \dots\}$ with the property that the infinite sum of the reciprocals of the members of the set L diverges. [30]

That is, a set of positive integers is said to be a Large set if $\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$.

Definition 8. A *Small Set* is a set of positive integers, $S = \{b_0, b_1, b_2, \dots\}$ with the property that the infinite sum of the reciprocals of the members of the set L converges to a finite value. [30]

Examples

1. If $A = \{am + b\}$ is a set of natural numbers in an arithmetic progression, with a and b constant natural numbers and $m = 0, 1, 2, 3, \dots$; then A is a large set. [30]
2. The set containing the n^{th} powers of natural numbers is a small set; for all $n \geq 2$, $n \in \mathbb{N}$. [30]
3. The set of prime numbers is a large set. [11]

Properties

The following are some of the properties of small/large sets:⁵

1. Any subset of a small set is small.
2. The union of finitely many small sets is small. This is so since the sum of two convergent series is a convergent series.
3. The complement of any small set is a large set.

⁵Source [30]

Chapter 5

Results

5.1 Preliminary Lemmas

The lemmas we state and prove in this section play a foundational role in the proofs of the results stated in the subsequent section.

Lemma 1. *The set $S = \{\lambda_n^+\}$ of the positive eigenvalues of a regular weighted Sturm-Liouville problem, with separated (homogeneous) boundary conditions is a small set with in the set of natural numbers. That is, $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^+}$ converges to a finite value.*

Proof. Let $S = \{\lambda_n^+\}$ be the set of positive eigenvalues of a regular weighted Sturm-Liouville problem, with separated homogeneous boundary conditions.

From [3], $\lambda_n^+ \sim \frac{n^2\pi^2}{(\int_a^b \sqrt{(\frac{r(x)}{p(x)})_+} dx)^2}$.

By the definition of asymptotic equality¹,

this implies that; $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^+}$ and $\sum_{n=1}^{\infty} \frac{1}{\frac{n^2\pi^2}{(\int_a^b \sqrt{(\frac{r(x)}{p(x)})_+} dx)^2}}$ diverge (or converge)

at the same rate [29].

But, applying the solution to the Basel Problem², we have,

$$\sum_{n=1}^{\infty} \frac{1}{\frac{n^2\pi^2}{K^2}} = \sum_{n=1}^{\infty} \frac{K^2}{n^2\pi^2} = \frac{K^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \left(\frac{K^2}{\pi^2}\right) \left(\frac{\pi^2}{6}\right) = \frac{K^2}{6}$$

$$\text{where } K = \int_a^b \sqrt{\left(\frac{r(x)}{p(x)}\right)_+} dx.$$

But, $\frac{K^2}{6} \in \mathbb{R}$ as $K \in \mathbb{R}$ since $\frac{1}{p}$, q , r , $\in L(a, b)$ by assumption.

Hence, $\sum_{n=1}^{\infty} \frac{1}{\frac{n^2\pi^2}{(\int_a^b \sqrt{(\frac{r(x)}{p(x)})_+} dx)^2}}$ is a convergent series.

And hence, $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^+}$ also converges [29].

¹Two functions f and g are said to be asymptotically equal (as $x \rightarrow x_0$) if and only if $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 1$

²The Basel problem is a problem in mathematical analysis with relevance to number theory, first posed by Pietro Mengoli in 1650 and solved by Leonhard Euler in 1734. The Basel problem asks for the precise summation of the reciprocals of the squares of the natural numbers, i.e. the precise sum of the infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}. \text{ Source [5].}$$

Therefore, by the definition of small sets³, $S = \{\lambda_n^+\}$ of the positive eigenvalues of a regular weighted Sturm-Liouville problem, with separated homogeneous boundary conditions is a small set.

□

Lemma 2. *The asymptotic density of the set of positive eigenvalues of a regular weighted Sturm-Liouville problem, with separated homogeneous boundary conditions; in the set of natural numbers is 0. That is, if $S = \{\lambda_n^+\}$ is the set of positive eigenvalues of a regular weighted Sturm-Liouville problem, with separated homogeneous boundary conditions, then $\delta(S) = 0$.*

Proof. From Lemma 1, S is a small set. As the set of Prime numbers is a large set in the set of natural numbers, this implies that S is sparser than the set of prime numbers in the set of natural numbers.

Thus, $0 \leq \delta(S) \leq \delta(P) = 0$

Therefore, $\delta(S) = 0$.

□

Lemma 3. *If G and H are two sets with $G \sim H$, then $\delta_H(G) = 1$.*

Proof. By the definition of asymptotic equality (of sets),

$G \sim H \Rightarrow (\exists M \in \mathbb{N})$ such that:

$(\forall n \geq M \text{ and } \exists k \in \mathbb{N}) |G(M) - H(M)| = k.$

³Definition 2

$$\Rightarrow (\forall n \geq M) \quad (G(n) = H(n) \pm k)$$

Hence,

$$\lim_{n \rightarrow \infty} \frac{G(n)}{H(n)} = \lim_{n \rightarrow \infty} \frac{H(n) \pm k}{H(n)} = 1.$$

Thus, by the definition of relative density;

$$\lim_{n \rightarrow \infty} \frac{G(n)}{H(n)} = 1 \Leftrightarrow \delta_H(G) = 1.$$

□

Lemma 4. *Let $S = \{\lambda_n^+\}$ be the set of positive eigenvalues of a regular weighted Sturm-Liouville problem, with separated (homogeneous) boundary conditions and let $R = \left\{ \frac{n^2 \pi^2}{\left(\int_a^b \sqrt{\left(\frac{r(x)}{p(x)} \right)_+ dx} \right)^2} \right\}$.*

Then, $S(n) \sim \lfloor \frac{n^{\frac{1}{2}} K}{\pi} \rfloor$, where $K = \int_a^b \sqrt{\left(\frac{r(x)}{p(x)} \right)_+ dx}$.

Proof. From [3] we have, $S \sim R$.

By Lemma 3, $S \sim R \Rightarrow \delta_R(S) = 1$.

Now, from the definition of relative density,

$$\delta_R(S) = 1 \Rightarrow \lim_{n \rightarrow \infty} \frac{S(n)}{R(n)} = 1.$$

By the definition of asymptotic equality⁴, $\lim_{n \rightarrow \infty} \frac{S(n)}{R(n)} = 1 \Rightarrow S(n) \sim R(n)$.

Now, $R = \left\{ \frac{n^2 \pi^2}{\left(\int_a^b \sqrt{\left(\frac{r(x)}{p(x)} \right)_+} dx \right)^2} \right\}$ implies that

$R(m) = \text{Number of elements in } \{T: T \leq m, \text{ for } m \in \mathbb{N}\}$, where $T = \frac{n^2 \pi^2}{K^2}$

Now, $\left[\frac{n^2 \pi^2}{K^2} \leq m \right]$ and $\left[\frac{n^2 \pi^2}{K^2} = m \Leftrightarrow \frac{n^2 \pi^2}{K^2} \text{ is a positive integer} \right]$ implies that

$$n^2 \leq \frac{mK^2}{\pi^2} \Rightarrow n \leq \frac{m^{\frac{1}{2}}K}{\pi} \Rightarrow n = \left\lfloor \frac{m^{\frac{1}{2}}K}{\pi} \right\rfloor$$

Hence, for a natural number n , $R(n) = \left\lfloor \frac{n^{\frac{1}{2}}K}{\pi} \right\rfloor$.

Now, $[S(n) \sim R(n) \text{ and } R(n) = \left\lfloor \frac{n^{\frac{1}{2}}K}{\pi} \right\rfloor] \Rightarrow S(n) \sim \left\lfloor \frac{n^{\frac{1}{2}}K}{\pi} \right\rfloor$. □

⁴The definition of asymptotic equality is applicable here as $S(n)$ and $R(n)$ are both positive integers for any $n \in \mathbb{N}$

5.2 Main Results

The following theorems are the main findings of this thesis.

Theorem 1. *Any subset of the set of positive integers with asymptotic density strictly greater than zero is not the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions.*

Proof. Let $A \subseteq \mathbb{N}$ with $\delta(A) > 0$.

Now, by Lemma 2, $\delta(S) = 0$.

Hence, $\delta(A) > 0 \Rightarrow \delta(A) > \delta(S) \Rightarrow \delta(A) \neq \delta(S)$.

By the contrapositive of the property 3 of asymptotic density,

$\delta(A) \neq \delta(S) \Rightarrow A \approx S$.

And, by the contrapositive of the implication, $A \approx S \Rightarrow A \neq S$.

Therefore, set A is not the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions whenever $\delta(A) > 0$. □

Corollary 1. *A set of all the terms of an arithmetic sequence $\{am + b\}$ (where a and b are positive natural numbers (constants) and $m = 0, 1, 2, 3, \dots$) is not the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions.*

Proof. Let $A = \{am + b\}$. Now, $\lim_{n \rightarrow \infty} \frac{n}{an + b} = \frac{1}{a}$ (a proof is given in [22]) and since $a > 0$ by assumption, $\frac{n}{an + b}$ exists. Then, by property 6 of asymptotic density, $\delta(A)$ exists and $\delta(A) = \frac{1}{a}$. And, as $a > 0$ by assumption, $\frac{1}{a} > 0$.

Hence, $\delta(A) > 0$.

By Theorem 1, $\delta(A) > 0$ implies that A is not the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions. $\square$

The following is an alternative proof of Corollary 1.

Proof. Let $A = \{am + b\}$. Then, by results from Example 1 in section ,

$$A(n) = \lfloor \frac{n-b}{a} \rfloor + 1$$

Hence,

$$\delta(A) = \lim_{n \rightarrow \infty} \frac{\lfloor \frac{n-b}{a} \rfloor + 1}{n}$$

Now, consider the following inequality:

$$\frac{n-b-1}{a} \leq \lfloor \frac{n-b}{a} \rfloor \leq \frac{n-b+1}{a}$$

This implies that,

$$\lim_{n \rightarrow \infty} \frac{\frac{n-b-1}{a}}{n} \leq \lim_{n \rightarrow \infty} \frac{\lfloor \frac{n-b}{a} \rfloor + 1}{n} = \lim_{n \rightarrow \infty} \frac{\lfloor \frac{n-b}{a} \rfloor}{n} \leq \lim_{n \rightarrow \infty} \frac{\frac{n-b+1}{a}}{n}$$

But,

$$\lim_{n \rightarrow \infty} \frac{\frac{n-b-1}{a}}{n} = \lim_{n \rightarrow \infty} \left(\frac{1}{a} - \frac{b}{an} - \frac{1}{an} \right) = \frac{1}{a}$$

and

$$\lim_{n \rightarrow \infty} \frac{\frac{n-b+1}{a}}{n} = \lim_{n \rightarrow \infty} \left(\frac{1}{a} - \frac{b}{an} + \frac{1}{an} \right) = \frac{1}{a}$$

Hence, by the Squeeze Theorem,

$$\delta(A) = \lim_{n \rightarrow \infty} \frac{\lfloor \frac{n-b}{a} \rfloor + 1}{n} = \frac{1}{a}$$

And, as $a > 0$ by assumption, $\frac{1}{a} > 0$.

Hence, $\delta(A) > 0$.

By Theorem 1, $\delta(A) > 0$ implies that A is not the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions. $\square$

As a consequence, the following two results follow.

Corollary 2. *The set of even natural numbers is not the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions.*

Proof. Let E be the set of even positive integers. The set of even positive integers can be defined as:

$$E = \{k : [k = 2m](m = 1, 2, 3, \dots)\}$$

Thus, $E = \{2m\}$ is an arithmetic sequence or a special case of $A = \{am + b\}$ with $a = 2$ and $b = 0$. Then by Corollary 1 E is not the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions. $\square$

Corollary 3. *The set of odd natural numbers is not the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions.*

Proof. Let O be the set of odd positive integers. The set of odd positive integers is given by:

$$O = \{d : [d = 2m + 1](m = 0, 1, 2, 3, \dots)\}$$

Thus, $O = \{2m + 1\}$ is an arithmetic sequence or a special case of $A = \{am + b\}$ with $a = 2$ and $b = 1$. Then, by Corollary 1, O is not the spectrum

of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions.) $\square$

Theorem 2. *Any subset $\{a_n\}$ of the set of positive integers which is a large set⁵ (a set with $\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$) is not the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions.*

Proof. Let a_n denote the n^{th} term of the given sequence. Then, by assumption,

$$\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$$

Now, by Lemma 1, the set $\{\lambda_n^+\}$ of the positive eigenvalues of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions is a small set. But, by assumption, the set $\{a_n\}$ is a large set.

Therefore, any subset $\{a_n\}$ of the set of positive integers with $\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$; is not the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions. $\square$

⁵Refer to section 2.2 for more on large sets.

Corollary 4. *The set of prime numbers is not the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions.*

Proof. Based on the proof in [11], the sum of the reciprocals of the primes is a divergent series (i.e., the set of prime numbers is a large set).

Therefore, by Theorem 2, the set of prime numbers is not the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions. $\square$

Corollary 5. *An infinite subset, $J = \{a_n\}$, of the set of prime numbers with $\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$; is not the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions.*

Proof. Given $J = \{a_n\}$ and $\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$. This implies that J is a large set.

Therefore, based on the assertion and proof of Theorem 2, the set of prime numbers is not the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions. $\square$

Theorem 3. *If D is the set of square natural numbers; i.e., if $D = \{1, 4, 9, 16, 25, \dots\}$, then D is the spectrum of a regular weighted Sturm-Liouville problem with*

separated homogeneous boundary conditions implies that $K = \pi$,
where $K = \int_a^b \sqrt{\left(\frac{r(x)}{p(x)}\right)_+} dx$.

Proof. Let the set of square numbers, denoted by D , be the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions. And let $S = \{\lambda_n^+\}$ is the set of positive eigenvalues of a regular weighted Sturm-Liouville problem, with separated homogeneous boundary conditions.

Also, let $R = \left\{ \frac{n^2 \pi^2}{\left(\int_a^b \sqrt{\left(\frac{r(x)}{p(x)}\right)_+} dx \right)^2} \right\}$. Now, $D = S \Rightarrow D \sim S$.

And, $[D \sim S \wedge S \sim R] \Rightarrow D \sim R$; as the asymptotic density of sets is an equivalence relation (Transitivity).

Based on Lemma 3, this implies that $\delta_R(D) = 1$

Observe that,⁶

$$\begin{aligned} \delta_R(D) &= \lim_{n \rightarrow \infty} \frac{D(n)}{R(n)} \\ &= \lim_{n \rightarrow \infty} \frac{\lfloor n^{\frac{1}{2}} \rfloor}{\lfloor \frac{n^{\frac{1}{2}} K}{\pi} \rfloor} \\ &= \frac{\pi}{K} \end{aligned}$$

⁶From Lemma 4, $R(n) = \lfloor \frac{n^{\frac{1}{2}} K}{\pi} \rfloor$.

In evaluating the resulting limit involving the floor function, consider the inequality;

$$\lim_{n \rightarrow \infty} \frac{n^{\frac{1}{2}} - 1}{\frac{n^{\frac{1}{2}} K + 1}{\pi}} \leq \lim_{n \rightarrow \infty} \frac{\lfloor n^{\frac{1}{2}} \rfloor}{\lfloor \frac{n^{\frac{1}{2}} K}{\pi} \rfloor} \leq \lim_{n \rightarrow \infty} \frac{n^{\frac{1}{2}} + 1}{\frac{n^{\frac{1}{2}} K - 1}{\pi}}$$

And then apply the squeeze theorem for limits to evaluate $\delta_R(D)$, as both the given lower and upper bounds for $\delta_R(D)$ converge to $\frac{\pi}{K}$ as n tends to ∞ .

Hence, $\delta_R(D) = \frac{\pi}{K}$.

Now, $\delta_R(D) = \frac{\pi}{K} = 1 \Leftrightarrow K = \pi$

Therefore, D is the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions implies that $K = \pi$. $\square$

But, whether the converse (of Theorem 3) holds (for the case $K = \pi$); is still, to our knowledge, an open question.

Theorem 4. *The set $F = \{m: m = r^t, t \geq 3, t \in \mathbb{N}, r = 1, 2, 3, \dots\}$ is not the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions.*

Proof. Given $S = \{\lambda_n^+\}$ is the set of positive eigenvalues of a regular weighted Sturm-Liouville problem, with separated homogeneous boundary conditions;

Let $R = \left\{ \frac{n^2 \pi^2}{\left(\int_a^b \sqrt{\left(\frac{r(x)}{p(x)} \right)_+} dx \right)^2} \right\}$.

And let $F = \{m : m = r^t, t \geq 3, t \in \mathbb{N}, r = 1, 2, 3, \dots\}$.

From the asymptotic estimate in [3], $\lambda_n^+ \sim \frac{n^2 \pi^2}{\left(\int_a^b \sqrt{\left(\frac{r(x)}{p(x)} \right)_+} dx \right)^2}$.

Now⁷, $\delta_R(F) = \lim_{n \rightarrow \infty} \frac{\lfloor n^{\frac{1}{t}} \rfloor}{\lfloor \frac{n^{\frac{1}{2}} K}{\pi} \rfloor}$, where $K = \int_a^b \sqrt{\left(\frac{r(x)}{p(x)} \right)_+} dx$.

In evaluating the above limit involving the floor function, consider the inequality;

$$\lim_{n \rightarrow \infty} \frac{n^{\frac{1}{t}} - 1}{\frac{n^{\frac{1}{2}} K + 1}{\pi}} \leq \lim_{n \rightarrow \infty} \frac{\lfloor n^{\frac{1}{t}} \rfloor}{\lfloor \frac{n^{\frac{1}{2}} K}{\pi} \rfloor} \leq \lim_{n \rightarrow \infty} \frac{n^{\frac{1}{t}} + 1}{\frac{n^{\frac{1}{2}} K - 1}{\pi}}$$

And then apply the squeeze theorem to evaluate $\delta_R(F)$, as both the given lower and upper bounds for $\delta_R(F)$ converge to 0 as n tends to ∞ .

Hence, $\delta_R(F) = 0$. Now, by the contrapositive of Lemma 3,

$$\delta_R(F) \neq 1 \Rightarrow F \not\sim R$$

And, as the asymptotic density of sets is an equivalence relation,

⁷From Lemma 4, $R(n) = \lfloor \frac{n^{\frac{1}{2}} K}{\pi} \rfloor$

$$F \approx R \Rightarrow F \approx S \Rightarrow F \neq S.$$

Therefore, the set $F = \{m : m = r^t, t \geq 3, t \in \mathbb{N}, r = 1, 2, 3, \dots\}$ is not the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions. $\square$

Theorem 5. *Provided that the converse of Erdos's conjecture on arithmetic progressions holds⁸, the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions doesn't contain arbitrarily long arithmetic progressions.*

Proof. Let $S = \{\lambda_n^+\}$ be the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions. From Lemma 1, S is a small set. If the converse of Erdos's conjecture (If set $A \subseteq \mathbb{N}$ contains arbitrarily long arithmetic progressions, then the sum of the reciprocals of the elements of a set A diverges.) holds, then its contrapositive (If the sum of the reciprocals of the elements of a set $A \subseteq \mathbb{N}$ converges, then set A doesn't contain arbitrarily long arithmetic progressions.) implies that the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions doesn't contain arbitrarily long arithmetic progressions. $\square$

⁸Erdos's conjecture (the Erdos - Turan conjecture) on arithmetic progressions, conjectured by Hungarian mathematician Paul Erdos, states that if the sum of the reciprocals of the elements of a set A of natural integers diverges, then A contains arbitrarily long arithmetic progressions. That is, if $\sum_{a_n \in A} \frac{1}{n} = \infty$ then set A contains arbitrarily long arithmetic progressions. This conjecture is still open.

Based on Lemma 3, Theorem 1 and Theorem 2, it will subsequently be shown that the three conditions, stated in the following theorem, are necessary conditions in order for a subset of the set of positive integers to be a spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions.

Theorem 6. *A set A which is a subset of the set of positive integers is a spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions implies that:*

6.1. $\delta(A) = 0$.⁹

Proof. Let $\delta(A) \neq 0$. Then, based on the definition of asymptotic density, we have,

$\delta(A) > 0$. By Theorem 1, this implies that A is not the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions.

Therefore, set A is the spectrum of such an eigenvalue problem implies that $\delta(A) = 0$.

□

6.2. $\delta_S(A) = 1$.¹⁰ Where $S = \{\lambda_n^+\}$ is the spectrum of a regular weighted

⁹i.e., the asymptotic (natural) density of set A is zero.

¹⁰i.e., the relative asymptotic density of set A with respect to base set S is one.

Sturm-Liouville problem with separated homogeneous boundary conditions.

Proof. Let $\delta_S(A) \neq 1$. By the contrapositive of Lemma 3, this implies that $A \approx S$ which further implies that $A \neq S$.

Therefore, set A is the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions implies that $\delta_S(A) = 1$. □

6.3. Set A is a small set.

Proof. Let A be a large set. Then, by Theorem 2, this implies that A is not the spectrum of a regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions.

Therefore, set A is the spectrum of such an eigenvalue problem implies that A is a small set. □

Whether or not the above three conditions are sufficient conditions as well is, to our knowledge, still an open question.

Conclusion

The spectrum of regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions has the property of being a discrete set. It is this quality, in addition to the fact that A.B. Mingarelli's work and later on R. Amirov's and I. Adalar's work proving that the set of Prime numbers are not a spectrum of regular weighted Sturm-Liouville problem with separated homogeneous boundary conditions, that beg the question on the existence of certain infinite subsets of the set of positive integers may be such a spectrum. Accordingly, in this thesis we have investigated this question for certain subsets of the set of positive integers and come up with some necessary conditions.

It should be noted that it's not only the main results but also the approaches adopted in proving those results that we consider to be of importance. The limitations of the techniques and concepts adopted in this thesis to prove the main results are that they only admit the spectrum of a Sturm-Liouville problem with a proven asymptotic estimate for the n^{th} eigenvalue. In spite of this shortcoming, the aforementioned set of techniques and concepts can

still prove to be valuable in tackling similar problems.

We conclude this work with the hope that another work will follow investigating the spectrum of a more general Sturm-Liouville eigenvalue problem in a future work. It's also the authors' wish that this work will inspire others to expand on it; or better yet, explore and embark upon totally new approaches of investigating the main problems in this thesis in particular and problems in the spectral analysis of Sturm-Liouville problems in general.

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