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Penalty and Barriers methods for constrained optimization problems

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Declaration

This project is my original work and has not been presented for a degree in any other University and all the sources of information used for the project have been fully acknowledged.

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Abstract

Constrained optimization problems are relatively more complex than unconstrained optimization problems. Some of these complexities are minimized by penalty and barrier methods. Penalty and barrier methods are approximating of constrained optimization problems by unconstrained optimization problems or sequence of unconstrained optimization problem to find the solution of a given constrained optimization problem. In penalty function method the constrained problem is replaced by unconstrained (sequence of unconstrained) problem by adding a term to the objective function that prescribes a high cost for violation of the constraints and in barrier method the problem is replaced by unconstrained (sequence of unconstrained) problem through adding a term that favors points in the interior of the feasible region over those near the boundary. Barrier requires that the interior of the feasible sets must be nonempty and therefore, they are used with problems having only inequality constraints (there is no interior for equality constraints). Even though, these methods are fundamental, they have their own series limitations to find its approximate solution to the constrained problem. In these methods we have to do with penalty parameter μ , and this certainly makes problem of unconstrained optimization of the penalized objective function. With those limitations, these methods are very fundamental to find best solutions of constrained optimization problems with some restrictions.

Keywords: Convex analysis, Unconstrained optimization, constrained optimization, penalty method, Barrier methods.

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Notations

- $N_\epsilon(x^*)$: ϵ -neighborhood of x^*
- $\nabla^2 f$: The second derivative of f which is called Hessian matrix
- $\|d\|$: The norm of a vector d
- $\frac{\partial f}{\partial x}$: Partial derivative of f with respect to x
- $H(x)$: Hessian matrix of a function at x
- ∇f : gradient of real valued function f
- R : set of real numbers
- R^n : n - dimensional space
- $\nabla^T f$: transpose of the gradient
- L : Lagrangian function
- Q_μ : auxiliary function for quadratic penalty method with penalty parameter μ
- $f \in C^1$: f is once continuously differentiable function
- $f \in C^2$: f is twice continuously differentiable function
- $L(\cdot, \lambda, \mu)$: Lagrangian function with Lagrange multipliers λ and μ
- f_μ : auxiliary function for penalty and barrier methods with parameter μ
- $\alpha(x)$: penalty function
- $B(x)$: barrier function

Chapter 1

Introduction

Optimization is a subject in applied mathematics, computational mathematics and operations research which has wide applications in science, engineering, business management, military and space technology. The subject is involved in optimal solution of problems which are defined mathematically, i.e., given a practical problem, the “best” solution to the problem can be found from lots of schemes by means of scientific methods and tools. It involves the study of optimality conditions of the problems, the construction of model problems, the determination of algorithmic method of solution, the establishment of convergence theory of the algorithms, and numerical experiments with typical problems and real life problems [10].

Optimization is the act of obtaining the best result under given circumstances. In design, construction, and maintenance of any engineering system, engineers have to take many technological and managerial decisions at several stages. The ultimate goal of all such decisions is either to minimize the effort required or to maximize the desired benefit. Since the effort required or the benefit desired in any practical situation can be expressed as a function of certain decision variables, optimization can be defined as the way of determining the conditions that yields the optimal value of a function [7].

The general form of optimization problem is [7]

$$\text{minimum } f(x)$$

subject to

$$x \in X \tag{1.1}$$

Where $x \in R^n$ is a decision variable, $f(x)$ is an objective function, $X \subseteq R^n$ a constraint set or feasible region.

Particularly, if the constraint set $X = R^n$, the optimization problem (1.1) is called unconstrained optimization problem, otherwise it is constrained optimization problem.

Moreover, the general mathematical optimization problem is [10]

$$\text{minimum } f(x)$$

subject to

$$\begin{aligned} h_i(x) &= 0, i \in I \\ g_j(x) &\leq 0, j \in J \end{aligned} \tag{1.2}$$

where I and J are the index set of equality constraint and inequality constraint, respectively, $h_i(x) = 0$, $g_j(x)$ are constraint function. When both objective and constraint functions are linear the problem is called linear programming. Otherwise, the problem is called nonlinear programming [10].

There are many techniques available for the solution of a constrained nonlinear programming problem. All the methods can be classified into two broad categories: direct methods and indirect methods. In the direct methods, the constraints are handled in an explicit manner, whereas in most of the indirect methods, the constrained problem is solved as a sequence of unconstrained minimization problems. From these indirect method, penalty and barrier method are some of the techniques used to solve constrained optimization problem through transforming to unconstrained optimization problem [3, 6, 2, 9].

Penalty and barrier methods are techniques for approximating constrained optimization problems by unconstrained problem. The approximation is accomplished in the case of penalty methods by adding to the objective function a term that prescribes a high cost for violation of the constraints, and in the case of barrier methods by adding a term that favors points interior to the feasible region over those near the boundary. Associated with these methods is a parameter μ that determines the severity of penalty or barriers and consequently the degree to which the unconstrained problem approximates the original constrained problems. There are two fundamental issues associated with these methods. The first has to do with how well the unconstrained problem approximates the constrained one. This is essential in examining whether, as the parameter μ is increased toward infinity, the solution of unconstrained problem converges to a solution of constrained problems. The other issue, most important from a practical viewpoint, is the question of

how to solve a given unconstrained problem when its objective function contains a penalty or barrier term. It turns out that as μ is increased to yield a good approximating problem, the corresponding structure of the resulting unconstrained problems becomes increasingly unfavorable thereby slowing the convergence rate of many algorithms that might be applied. To make this approach practical by overcoming its inherently slow convergence, finds it appropriate to bring into play nearly all aspects of optimization theory; including Lagrange multipliers and necessary conditions, etc [10, 2, 1, 4].

In this study, we deal with the solutions of constrained optimization using penalty and barrier methods. Constrained optimization problems are not that much simple to obtain its optimal solution and so one of the techniques used to simplify some of the difficulties is transforming the problem to unconstrained optimization problem. Therefore, in this project we apply the penalty and barrier method in the constrained optimization problem. This project is presented; some preliminary concepts in the first part and the two methods (penalty and barrier) in the second part.

Chapter 2

Preliminary concepts

2.1 Definiteness of Matrix

Definiteness of a matrix is important in searching best solution(s) of constrained or unconstrained optimization problem. Thus, to apply this fundamental concept we have to define formally as follows.

Definition 2.1.1. [4]

1. A symmetric $m \times m$ matrix B is called positive semi definite if $x^T Bx \geq 0$ for all $x \in R^m$.
2. A symmetric $m \times m$ matrix B is called positive definite if $x^T Bx > 0$ for all $x \in R^m$ and $x \neq 0$.
3. If B is a square matrix with order $m \times m$, then eigenvalues of B are scalars λ such that $Bx = \lambda x$, where $x \neq 0$, $x \in R^m$.
4. The eigenvalues of symmetric matrices are all real numbers, while non-symmetric matrices may have imaginary eigenvalues.

Based on the above definitions, different basic concepts are constructed. Some of these basic properties are the following [4].

1. If all the eigenvalues of a matrix B are positive (non-negative), then B is positive definite (semi-definite).
2. If B is positive definite, then B^{-1} is positive definite.
3. Let A be a symmetric with order $m \times m$ and Q be a positive semi-definite with order $m \times m$. Assume that $x^T A x > 0$ for all $x \neq 0$ satisfying $x^T Q x = 0$. Then there exists a scalar μ such that $A + \lambda Q$ is positive definite.

4. A matrix $A = B^T B$ is positive semi-definite, since $x^T B^T B x = |Bx|^2 \geq 0$.

2.2 Convex Analysis

In order to develop a theory directed toward characterizing global, rather than local, minimum points, it is necessary to introduce some sort of convexity assumptions. This results not only in a more potent, although more restrictive, theory but also provides an interesting geometric interpretation of the second-order sufficiency result.

Definition 2.2.1. [2] Let the set S be given. If for any $y_1, y_2 \in S$, $\lambda y_1 + (1 - \lambda)y_2 \in S$, $\forall \lambda \in [0, 1]$, then S is called convex. Geometrically, A set S in R^n is called convex if the line segment joining any two points of the set also belonging to the set.

A convex combination of a finite set of vectors $\{y_1, y_2, \dots, y_n\}$ in R^n is any vector x of the form

$$x = \sum_{i=1}^n \alpha_i y_i,$$

where $\sum_{i=1}^n \alpha_i = 1$, and $\alpha_i \geq 0$ for all $i = 1 : n$.

The convex hull of the set S containing $\{y_1, y_2, \dots, y_n\}$, denoted by $\text{conv}(S)$, is the set of all convex combinations of S . In other words, $x \in \text{conv}(S)$ if and only if x can be represented as a convex combination of $\{y_1, y_2, \dots, y_n\}$.

If non-negativity of the multipliers α_i for $i = 1, 2, \dots, n$ is dropped, then the combination is called an affine combination.

A cone is a non empty set C with the property that for all $x \in C$ we have $x \in C \Rightarrow \alpha x \in C$, for all $\alpha \geq 0$.

Note that cones are not necessarily convex. However, the cone generated by $\{y_1, y_2, \dots, y_n\}$ is the set of all vectors x of the form $x = \sum_{i=1}^n \alpha_i y_i$, where $\alpha_i \geq 0$ for all $i = 1 : n$ is a convex cone.

Definition 2.2.2. [10] Suppose that $S \subseteq R^n$ is a nonempty convex set and the function f is defined as $f : S \rightarrow R$. If for any $y_1, y_2 \in S$ and all $\lambda \in [0, 1]$,

$$f(\lambda y_1 + (1 - \lambda)y_2) \leq \lambda f(y_1) + (1 - \lambda)f(y_2), \quad (2.1)$$

then f is convex function on S . If the inequality in (2.1) is change to strict inequality for all $y_1 \neq y_2$, that is,

$$f(\lambda y_1 + (1 - \lambda)y_2) < \lambda f(y_1) + (1 - \lambda)f(y_2), \quad (2.2)$$

then f is called a strict convex function on S .

If there is a constant $b > 0$ such that for arbitrary $y_1, y_2 \in S$,

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(y_1) + (1 - \lambda)f(y_2) - \frac{1}{2}c\lambda\|y_1 - y_2\|^2, \quad (2.3)$$

then f is said to be a uniformly or strongly convex function on S [5].

If $-f$ is a convex (that is either strictly convex, or uniformly convex) function on S , then f is called a concave (either strictly concave, or uniformly concave) function.

Theorem 2.2.1. [10]

1. Suppose g_i is convex function on a convex set $S \subseteq R^m$ and $\theta_i \geq 0$ is a real number for each $i = 1 : m$. Then $\sum_{i=1}^m \theta_i g_i$ is also a convex function on S .
2. If h is convex and non-negative on a convex set S , then h^2 is also convex on S .
3. If g is a convex function on a convex set S , then the function

$$g^+(x) = \max\{g(x), 0\}$$

is also convex on S .

Proof. 1. Assume that $z, y \in S$, $\lambda \in [0, 1]$ and $g(x) = \sum_{i=1}^m \alpha_i g_i(x)$, then

$$\begin{aligned} g(\lambda z + (1 - \lambda)y) &= \sum_{i=1}^m \theta_i g_i(\lambda z + (1 - \lambda)y) \\ &= \theta_1 g_1(\lambda z + (1 - \lambda)y) + \theta_2 g_2(\lambda z + (1 - \lambda)y) + \cdots + \theta_m g_m(\lambda z + (1 - \lambda)y) \\ &\leq \theta_1 [\lambda g_1(z) + (1 - \lambda)g_1(y)] + \cdots + \theta_m [\lambda g_m(z) + (1 - \lambda)g_m(y)] \\ &= \lambda [\theta_1 g_1(z) + \cdots + \theta_m g_m(z)] + (1 - \lambda) [\theta_1 g_1(y) + \cdots + \theta_m g_m(y)] \\ &= \lambda g(z) + (1 - \lambda)g(y) \end{aligned}$$

Therefore, $g(x) = \sum_{i=1}^m \theta_i g_i(x)$ is convex function on S .

2. Suppose $z, w \in S$ and $\lambda \in [0, 1]$. Then

$$\begin{aligned} h^2(\lambda z + (1 - \lambda)w) &= (\lambda h(z) + (1 - \lambda)h(w))^2 \\ &\leq [\lambda h(z) + (1 - \lambda)h(z)]^2 \\ &= \lambda h^2(z) + (1 - \lambda)h^2(w) - (1 - \lambda)\lambda(h(z) - h(w))^2 \\ &\leq \lambda h^2(z) + (1 - \lambda)h^2(w) \end{aligned}$$

Hence, f^2 is convex function.

3. Suppose $z, w \in S$ and $\lambda \in [0, 1]$. Then

$$\begin{aligned} g^+(\lambda z + (1 - \lambda)w) &= \max\{g(\lambda z + (1 - \lambda)w), 0\} \\ &\leq \max\{\lambda g(z) + (1 - \lambda)g(w), 0\}, \text{ since } g \text{ is convex} \\ &\leq \max\{\lambda g(z), 0\} + \{(1 - \lambda)g(w), 0\}, \text{ by maximum property} \\ &= \lambda \max\{g(z), 0\} + (1 - \lambda) \max\{g(w), 0\} = \lambda g^+(z) + (1 - \lambda)g^+(w) \end{aligned}$$

Hence, the given function g^+ is convex. □

For a differentiable and convex function, the behavior of differential convex functions can be expressed using the following theorem.

Theorem 2.2.2. [10] Let $S \subseteq R^n$ be a nonempty open convex set and let $f : S \rightarrow R$ be a differentiable function. Then f is convex if and only if

$$f(w) \geq f(z) + \nabla f(z)^T(w - z), \forall z, w \in S \quad (2.4)$$

Similarly, f is strictly convex on S if and only if

$$f(w) > f(z) + \nabla f(z)^T(w - z), \forall z, w \in S, w \neq z \quad (2.5)$$

Proof. (\Rightarrow): Let $f(z)$ be a convex function, then for all $\lambda \in (0, 1)$,

$$f(\lambda w + (1 - \lambda)z) \leq \lambda f(w) + (1 - \lambda)f(z).$$

Hence,

$$\frac{f(z + \lambda(w - z)) - f(z)}{\lambda} \leq f(w) - f(z).$$

Letting $\lambda \rightarrow 0$ yields

$$\nabla f(z)^T(w - z) \leq f(w) - f(z).$$

(\Leftarrow): Assume that (2.4) holds. Take any $z_1, z_2 \in S$ and assume $z = \lambda z_1 + (1 - \lambda)z_2$, $0 < \lambda < 1$. Then

$$f(z_1) \geq f(z) + \nabla f(z)^T(z_1 - z),$$

$$f(z_2) \geq f(z) + \nabla f(z)^T(z_2 - z).$$

Therefore,

$$\begin{aligned} \lambda f(z_1) + (1 - \lambda)f(z_2) &\geq f(z) + \nabla f(z)^T(\lambda z_1 + (1 - \lambda)z_2 - z) \\ &= f(\lambda z_1 + (1 - \lambda)z_2) \end{aligned}$$

which indicates that $f(z)$ is a convex function.

Using the definition of strictly convex, we have

$$f(z + \lambda(w - z)) - f(z) < \lambda(f(w) - f(z)).$$

Then, using (2.4) and the above expression, we have

$$\lambda \nabla f(z)^T(w - z) \leq f(z + \lambda(w - z)) - f(z) < \lambda(f(w) - f(z))$$

which is the required expression in (2.5). \square

Theorem 2.2.3. [10] Let $S \subseteq R^n$ be a nonempty open convex set and let $f : S \rightarrow R$ be twice continuously differentiable. Then

1. f is convex if and only if the second derivative $G(x) = \nabla^2 f(x)$ is positive semi-definite for all points in S .
2. f is strictly convex if its second derivative G is positive definite for all points in S .

Proof. 1. **Backward part:** Suppose that the matrix $G(x) = \nabla^2 f(x)$ is positive semi-definite at each point $x \in S$ and $x, \bar{x} \in S$. Using the mean-value theorem, we have

$$f(x) = f(\bar{x}) + \nabla f(\bar{x})^T(x - \bar{x}) + \frac{1}{2}(x - \bar{x})^T \nabla^2 f(\hat{x})(x - \bar{x}),$$

where $\hat{x} = \bar{x} + \theta(x - \bar{x})$, $\theta \in (0, 1)$. Since $\hat{x} \in S$, it follows from the assumption that

$$f(x) \geq f(\bar{x}) + \nabla f(\bar{x})^T(x - \bar{x}).$$

Hence f is a convex function by Theorem (2.2.2).

Forward: Suppose that f is a convex function and $\bar{x} \in S$. We want to prove that $q^T \nabla^2 f(\bar{x}) q \geq 0, \forall q \in R^n$. Since S is open, there is $\delta > 0$ such that $|\lambda| < \delta, \bar{x} + \lambda q \in S$. Using Theorem (2.2.2), we obtained

$$f(\bar{x} + \lambda q) \geq f(\bar{x}) + \lambda \nabla f(\bar{x})^T q. \quad (2.6)$$

Because of $f(x)$ is twice differentiable at \bar{x} , then

$$f(\bar{x} + \lambda q) = f(\bar{x}) + \lambda \nabla f(\bar{x})^T q + \frac{\lambda^2}{2} q^T \nabla^2 f(\bar{x}) q + O(\|\lambda q\|^2). \quad (2.7)$$

By Substituting (2.6) into (2.7), we get the result

$$\frac{1}{2} \lambda^2 q^T \nabla^2 f(\bar{x}) q + O(\|\lambda q\|^2) \geq 0.$$

Using the Division by λ^2 and setting $\lambda \rightarrow 0$, it follows the result

$$q^T \nabla^2 f(\bar{x}) q \geq 0$$

□

2.3 Unconstrained optimization problem

An unconstrained optimization is a problem of finding the minimum of a function $f(x)$ without any restriction on the vector x . Such problems arise in more real life applications. However, in this part we consider such problems since optimality conditions for constrained optimization become a logical extension of the conditions for unconstrained problems. Furthermore, one strategy for solving a constrained problem is to solve a sequence of unconstrained optimization problems. Methods for unconstrained problems are of more general value, though, since they form the foundation for methods used to solve constrained optimization problems.

Mathematically, unconstrained optimization is defined as [6];

$$\text{minimize } f(x)$$

subject to

$$x \in R^n \quad (2.8)$$

where no constraints are placed on the variables $x = (x_1, \dots, x_n)^T \in R^n$.

To investigate about solution of unconstrained optimization problem, first we have to define the following basic concepts.

Definition 2.3.1. [8] Consider the problem in (2.8) and let $x^* \in R^n$. Then

1. If $f(x^*) \leq f(x)$ for all $x \in R^n$, x^* is called a global or absolute minimum.
2. If there is an ϵ -neighborhood $N_\epsilon(x^*)$ around x^* such that $f(x^*) \leq f(x)$ for each $x \in N_\epsilon(x^*)$, x^* is called a local minimum or relative minimum.
3. If $f(x^*) < f(x)$ for all $x \in N_\epsilon(x^*)$, $x \neq x^*$ for some $\epsilon > 0$, x^* is called a strict local minimum.

Generally, it is possible for a function to have a local minimum but have no global minimum, to have neither global nor local minimum, to have both local and global minimum, to have multiple global minimum and other combinations.

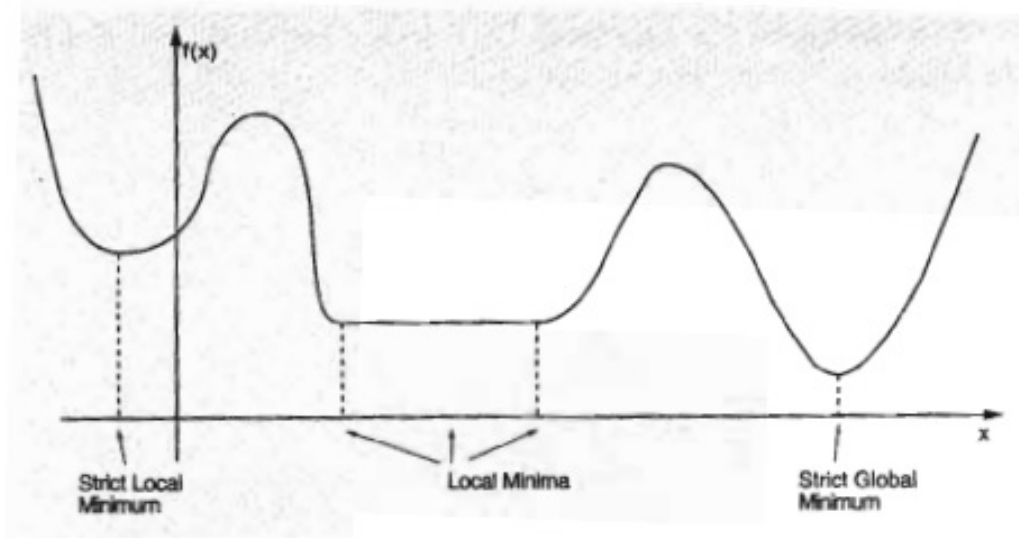


Figure 2.1: Unconstrained local and global minima in one dimension

Definition 2.3.2. [10] Let f be defined R^n to R and $x \in R^n$. A vector $d \in R^n$ is a descent direction of f at x if there is a number $\sigma > 0$ such that $f(x + rd) < f(x)$, for all $r \in (0, \sigma)$.

Theorem 2.3.1. [10] Suppose that the function defined from R^n to R is differentiable at $x \in R^n$. If there is a vector $d \in R^n$ such that $\nabla f(x)^T d < 0$, then d is said to be a descent direction of f at x .

Proof. By Taylor's series expansion,

$$f(x + rd) = f(x) + r\nabla f(x)^T d + O(r).$$

Since $\nabla f(x)^T d < 0$, there is $\delta > 0$ such that

$$f(x + rd) = f(x) + r\nabla f(x)^T d + O(r) < f(x), \forall r \in (0, \delta).$$

Thus, by definition d is a descent direction of f at x . \square

2.3.1 Necessary and sufficient optimality condition

To determine a possible characterization of local or global minimum of a function, the differentiability assumption is important, first-order and second necessary condition gives information about the optimal solution.

Theorem 2.3.2. [*First-order necessary condition*] [9] Suppose that $f : R^n \rightarrow R$ is differentiable at x^* . If there is a vector d with $\nabla f(x^*)^T d < 0$, there is a $\delta > 0$ such that $f(x^* + \lambda d) < f(x^*)$ for each $\lambda \in (0, \delta)$, so that d is a descent direction of f at x^* .

Proof. By the differentiability of f at x^* , we get the result

$$f(x^* + \lambda d) = f(x^*) + \lambda \nabla f(x^*)^T d + \lambda \|d\| \alpha(x^*; \lambda d)$$

, where $\alpha(x^*; \lambda d) \rightarrow 0$ as $\lambda \rightarrow 0$.

By arranging the terms and dividing with λ , $\lambda \neq 0$, we obtained

$$\frac{f(x^* + \lambda d) - f(x^*)}{\lambda} = \nabla f(x^*)^T d + \|d\| \alpha(x^*; \lambda d)$$

Since $\nabla f(x^*)^T d < 0$ and $\alpha(x^*; \lambda d) \rightarrow 0$ as $\lambda \rightarrow 0$, there exists $\delta > 0$ such that $\nabla f(x^*)^T d + \|d\| \alpha(x^*; \lambda d) < 0$ for all $\lambda \in (0, \delta)$. Consequently, we get $\frac{f(x^* + \lambda d) - f(x^*)}{\lambda} < 0$ and $f(x^* + \lambda d) < f(x^*)$. \square

Corollary 2.3.1. [9] If the function f defined from R^n to R is differentiable at x^* and x^* is a local minimum solution, then $\nabla f(x^*) = 0$.

Proof. Suppose that $\nabla f(x^*) \neq 0$. Then, for $d = -\nabla f(x^*)$, we have $\nabla f(x^*)^T d = -\|\nabla f(x^*)\|^2$. By theorem (2.3.2), there is a $\delta > 0$ such that $f(x^* + \lambda d) < f(x^*)$ for all $\lambda \in (0, \delta)$. This is a contradiction to the assumption x^* is a local minimum. Therefore, $\nabla f(x^*) = 0$. \square

Theorem 2.3.3. [*Second-order necessary condition*][9] If f is defined from R^n to R and two times differentiable at x^* and x^* is local minimum solution, then $\nabla f(x^*) = 0$ and the matrix $G(x^*)$ is positive semi-definite matrix.

Proof. From the differentiability of f at x^* , for any direction d , we have

$$f(x^* + \lambda d) = f(x^*) + \lambda \nabla f(x^*)^T d + \frac{1}{2} \lambda^2 d^T G(x^*) d + \lambda^2 \|d\|^2 \alpha(x^*; \lambda d), \quad (2.9)$$

where $\alpha(x^*; \lambda d) \rightarrow 0$ as $\lambda \rightarrow 0$.

Since x^* is a local minimum, from corollary (2.3.3), we have $\nabla f(x^*) = 0$. Rearranging the terms in (2.9) and dividing by $\lambda^2 > 0$, we get

$$\frac{f(x^* + \lambda d) - f(x^*)}{\lambda^2} = \frac{1}{2} d^T G(x^*) d + \|d\|^2 \alpha(x^*; \lambda d), \quad (2.10)$$

Since x^* is a local minimum, $f(x^* + \lambda d) \geq f(x^*)$ for adequately small values of λ . From (2.10) it is clear that $\frac{1}{2} d^T G(x^*) d + \|d\|^2 \alpha(x^*; \lambda d) \geq 0$ for necessary small values of λ . Using the limit concept as $\lambda \rightarrow 0$, it provides that $d^T G(x^*) d \geq 0$ and hence since d is arbitrary, $H(x^*)$ is positive semi-definite. \square

Theorem 2.3.4. [*Second-order Sufficient optimality condition*] [9]

Suppose that f is defined from R^n to R and is twice continuously differentiable at x^* . If $\nabla f(x^*) = 0$ and the second derivative matrix $G(x^*)$ is positive definite, then x^* is a strictly local minimum.

Proof. From the given, f is two times differentiable at x^* , follows that, for each $x^* \in R^n$

$$f(x) = f(x^*) + \nabla f(x^*)^T (x - x^*) + \frac{1}{2} (x - x^*)^T G(x^*) (x - x^*) + \|x - x^*\|^2 \alpha(x^*; (x - x^*)), \quad (2.11)$$

where $\alpha(x^*; x - x^*) \rightarrow 0$ as $x \rightarrow x^*$.

Assume that, x^* is not strictly local minimum. Then there exists a sequence $\{x_k\}$ which converges to the local minimum x^* such that $f(x_k) \leq f(x^*)$, $x_k \neq x^*$ for all k . Using this sequence, and the assumptions $f(x_k) \leq f(x^*)$ and $\frac{x_k - x^*}{\|x_k - x^*\|}$ by d_k , (2.11) implies that

$$\frac{1}{2} (d_k)^T G(x^*) (d_k) + \alpha(x^*; (x - x^*)) \leq 0, \text{ for each } k \quad (2.12)$$

But $\|d_k\| = 1$ for each k and hence there exists an index set K such that $\{d_k\}_K$ converges to d , where $\|d\| = 1$. Considering this subsequence and the fact that $\alpha(x^*; x - x^*) \rightarrow 0$ as $k \in K$ approaches ∞ , (2.12) implies that $d^T G(x^*) d \leq 0$. This contradicts the assumption that $G(x^*)$ is positive definite since $\|d\| = 1$. Therefore, x^* is a strict local minimum. \square

Theorem 2.3.5. [10] Let $S \subset R^n$ be a nonempty convex set, $f : S \rightarrow R$ and $x^* \in S$ be a local minimizer for $\min_{x \in S} f(x)$. Then,

1. If f is convex, then x^* is also a global minimizer.
2. If f is strictly convex, then x^* is a unique global minimizer.

Proof. 1. Suppose that f is a convex and x^* is a local minimum. Then there is a δ -neighborhood $N(x^*, \delta)$ such that

$$f(x) \geq f(x^*), \forall x \in S \cap N(x^*, \delta) \quad (2.13)$$

Assume that x^* is not a global minimum. Then we can determine some values $\hat{x} \in S$ such that $f(\hat{x}) < f(x^*)$. Using the definition of convexity of f , we obtained for $\lambda \in]0, 1[$,

$$\begin{aligned} f(\lambda\hat{x} + (1-\lambda)x^*) &\leq \lambda f(\hat{x}) + (1-\lambda)f(x^*) \\ &< \lambda f(x^*) + (1-\lambda)f(x^*) = f(x^*) \end{aligned} \quad (2.14)$$

But for sufficiently small $\lambda > 0$, $\lambda\hat{x} + (1-\lambda)x^* \in S \cap B(x^*, \delta)$. Therefore, (2.14) contradicts to (2.13). Thus, x^* is a global minimum.

2. From part one, we have x^* is a global minimum due to the convexity concept. Therefore, we need to prove the uniqueness only. suppose that x^* is not unique global minimum, so that we can find $x \in S, x \neq x^*$, such that $f(x) = f(x^*)$. By strict convexity of f ,

$$f\left(\frac{x+x^*}{2}\right) < \frac{f(x)+f(x^*)}{2} = f(x^*) \quad (2.15)$$

Using the convexity of S , $\frac{x+x^*}{2} \in S$. Therefore, (2.15) contradicts the fact that x^* is an optimal solution.

□

Remark 1.

1. If $\nabla f(x^*) = 0$ and the matrix $G(x)$ is positive semi-definite for all x , then f is convex and consequently, x^* is a global minimum.
2. Let $f : R^n \rightarrow R$ be a differentiable convex function. Then x^* is a global minimum if and only if $\nabla f(x) = 0$.

2.3.2 Solution methods of unconstrained optimization

There are many approaches which are important to find optimal solution of unconstrained optimization problems. Some of these methods are gradient methods and Newton's methods. To express all these method clearly, consider the unconstrained optimization problem;

$$\text{minimum } f(x), x \in R^n \quad (2.16)$$

where $x = (x_1, x_2, \dots, x_n)$ and f is a real valued function on R^n .

In this optimization problem, the optimal solution can be obtained by solving $\nabla f(x) = 0$ analytically. However, since $\nabla f(x) = 0$ is a system of nonlinear equations for some complex functions f it is not simple to find its solution. Thus, such type of complexity is minimized by applying numerical techniques on $\nabla f(x) = 0$. Moreover, the numerical techniques are managed using the gradient and Newton methods.

Usually, the optimization method is an iterative one for finding the minimum of an optimization problem. The basic idea is that, given an initial point $x_0 \in R^n$, one generates an iterate sequence $\{x_k\}$ by means of some iterative rule, such that when $\{x_k\}$ is a finite sequence, the last point is the optimal solution of the optimization problem; when $\{x_k\}$ is infinite, it has a limit point which is the optimal solution of the optimization problem. A typical behavior of an algorithm which is regarded as acceptable is that the iterates x_k move steadily towards the neighborhood of a local minimizer x^* , and then rapidly converge to the point x^* . When a given convergence rule is satisfied, the iteration will be terminated.

Gradient method:

The idea of this method is to decrease the function value as much as possible in order to reach the minimum early by using negative its gradient as descent direction. This method is also known as steepest method.

Suppose that $f(x)$ is continuously differentiable near x_k , and the gradient $g_k = \nabla f(x) \neq 0$. From the Taylor expansion

$$f(x) = f(x_k) + (x - x_k)^t g_k + O(\|x - x_k\|) \quad (2.17)$$

If we set $x - x_k = \alpha d_k$, then the direction d_k satisfying $d_k^t g_k < 0$ is called a descent direction that is such that $f(x) < f(x_k)$. By fixing α , it follows that

the smaller value $d_k^t g_k$ (i.e., the larger value $|d_k^t g_k|$) is, the faster the function value decreases. By the Cauchy-Schwartz inequality

$$|d_k^t g_k| \leq \|d_k^t\| \|g_k\|, \quad (2.18)$$

we have that the value $d_k^t g_k$ is the smallest if and only if $d_k = -g_k$. Thus, $-g_k$ is the steepest descent direction.

Therefore, the general iteration form of steepest descent method is given by

$$x_{k+1} = x_k - \alpha_k g_k = x_k - \alpha_k \nabla f(x_k). \quad (2.19)$$

Based on the above concepts, the algorithm of steepest descent method is summarized as;

Step- 0: Let $0 \leq \epsilon \ll 1$ be the termination tolerance. Given an initial point $x_0 \in R^n$, Set $k = 0$.

Step- 1: If $\|g_k\| \leq \epsilon$, stop. Otherwise, let $d_k = -g_k$.

Step- 2: Find the step length factor α_k , such that

$$f(x_k + \alpha_k d_k) = \min_{\alpha \geq 0} f(x_k + \alpha d_k);$$

Step- 3: Compute $x_{k+1} = x_k + \alpha_k d_k = x_k - \alpha_k g_k$.

Step- 4: Set $k = k + 1$ and then return to Step-1.

The rate of global convergence of the steepest descent (gradient) method is described based on the following theorem.

Theorem 2.3.6. Let $f \in C^1$. Then each accumulation point of the iterative sequence x_k generated by the steepest descent Algorithm with exact line search is a stationary point.

Proof. Let \bar{x} be any accumulation point of x_k and K an infinite index set such that $\lim_{k \in K} x_k = \bar{x}$. Set $d_k = -\nabla f(x_k)$. Since $f \in C^1$, the sequence $\{d_k | k \in K\}$ is uniformly bounded and $\|d_k\| = \|\nabla f(x_k)\|$. Since $f(x_{k+1}) \leq f(x_k)$, $\forall k$ and $d_k^t \nabla f(x_k) \leq 0$, it follows that $\|\nabla f(\bar{x})\|^2 = 0$, i.e., $\nabla f(\bar{x}) = 0$. Therefore, \bar{x} is a stationary point. \square

Conjugate gradient method:

For many problems, the steepest descent method is not the actual “steepest”, but is very slow. Although the method usually works well in the early steps, as a stationary point is approached, it descends very slowly with zigzagging phenomena. Therefore, this class of methods can be viewed as a modification of the steepest descent method, where in order to avoid the zigzagging effect, at each iteration the direction is modified by a combination of the earlier directions:

$$d_k = -g_k + \beta_k d_{k-1} = -\nabla f(x_k) + \beta_k d_{k-1}. \quad (2.20)$$

These corrections ensure that d_1, \dots, d_n are so-called conjugate directions. This means that there exist a matrix B such that $d_i^t B d_j = 0$, for all $i \neq j$.

Having the direction d_k , the next iterate is calculated in the usual way;

$$x_{k+1} = x_k + \alpha_k d_k = x_k + \alpha_k (-\nabla f(x_k) + \beta_k d_{k-1}). \quad (2.21)$$

where the optimal step length $\alpha_k = \min_{\alpha \geq 0} f(x_k + \alpha d_k)$ or its approximation. The parameter β_k can be calculated using Fletcher and Reeves formula as:

$$\beta_k = \frac{\|f(x_k)\|^2}{\|f(x_{k-1})\|^2}.$$

Therefore, except the way of computing d_k the remaining concepts conjugate gradient method is the same as steepest descent method.

Newton’s method:

The basic idea of Newton’s method for unconstrained optimization is to iteratively use the quadratic approximation q_k to the objective function f at the current iterate x_k and to minimize the approximation q_k . This method is the most complex and also the fastest of the gradient methods. A problem of this method is, that it is equally attracted by all points where the gradient is zero, which can be minima, maxima and saddle points. So it is necessary that the function is locally convex (that means the Hessian matrix has to be positive definite) in order to guarantee that the computed direction is a descent direction.

Let f be twice continuously differentiable real valued function, $x_k \in R^n$ and the Hessian $\nabla^2 f(x_k)$ is positive definite. Then, the quadratic approximation of f at the current point x_k is;

$$f(x_k + s) \approx q_k(s) = f(x_k) + \nabla f(x_k)^t s + \frac{1}{2} s^t \nabla^2 f(x_k) s, \quad (2.22)$$

where $s = x - x_k$. Minimizing q_k yields

$$x_{k+1} = x_k - [\nabla^2 f(x_k)]^{-1} \nabla f(x_k), \quad (2.23)$$

which is called Newton's formula. By setting $H_k = \nabla^2 f(x_k)$ and $g_k = \nabla f(x_k)$, we write expression (2.23) as

$$x_{k+1} = x_k - H_k^{-1} g_k, \quad (2.24)$$

where $s_k = x_{k+1} - x_k = -H_k^{-1} g_k$ is a Newton's direction.

If H_k is positive definite, then $g_k^t s_k = -g_k^t H_k^{-1} g_k < 0$ and so the Newton's direction is a descent direction. Therefore, the Newton's algorithm is stated as follows;

Step- 0: Let $0 \leq \epsilon \ll 1$ be the termination tolerance. Given an initial point $x_0 \in R^n$, Set $k = 0$.

Step- 1: If $\|g_k\| \leq \epsilon$, stop. Otherwise, let $H_k = \nabla^2 f(x_k)$;

Step- 2: Compute $s_k = -H_k^{-1} g_k$;

Step- 3: Compute $x_{k+1} = x_k + s_k$;

Step- 4: Set $k = k + 1$ and then go to Step-1.

2.4 Constrained optimization problem

A general formulation for constrained optimization problem is [10]

$$\text{minimum } f(x)$$

subject to

$$\begin{aligned} h_i(x) &= 0, i = 1, 2, \dots, n \\ g_j(x) &\leq 0, j = 1, 2, 3, \dots, m \end{aligned} \quad (2.25)$$

where m, n are positive integers and all the involved functions are real valued.

If $m = 0 = n$, the problem (2.25) is an unconstrained optimization problem; if $m = 0$, the problem is called an equality constrained optimization problem; if all the constraints $h_i, g_j, (i = 1 : n, j = 1 : m)$ are linear functions, the problem (2.25) is called a linearly constrained optimization problem. A linearly constrained optimization problem with quadratic objective function $f(x)$ is said to be a quadratic programming problem.

Definition 2.4.1. [10] The set of all points

$$X = \{x | h_i(x) = 0, g_j(x) \leq 0, i = 1 : n, j = 1 : m\} \quad (2.26)$$

is called feasible set or feasible region of the problem (2.25).

Definition 2.4.2. [10] Let $I = \{1, 2, \dots, n\}$, $J = \{1, 2, \dots, m\}$ and $I(x) = \{j : g_j(x) = 0, j \in J\}$. For any $x \in R^n$, the set

$$A(x) = I \cup I(x) \quad (2.27)$$

is an index set of active constraints at x and $g_j(x)$ ($j \in A(x)$) is an active constraint at x , $g_j(x)$ ($j \notin A(x)$) is inactive constraint at x .

Assume that $A(x^*)$ is an index set of the active constraints of problem (2.25) at x^* , then, from the observation about inactive constraints, it is enough for us to solve the constrained optimization problem

$$\min f(x)$$

subject to

$$g_i(x) = 0, i \in A(x) \quad (2.28)$$

In general, it is easier to solve equality constraint problem (2.28) than the original problem (2.25).

To state the optimality condition of constrained optimization problem, first we have to introduce the Lagrangian function as [9];

$$L(x, \lambda) = f(x) + \sum_{i=1}^n \lambda_i h_i(x) + \sum_{j=1}^m \mu_j g_j(x), \quad (2.29)$$

where, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in R^n$ and $\mu = (\mu_1, \mu_2, \dots, \mu_m) \in R^m$ are Lagrange multiplier vectors of h and g respectively.

2.4.1 Optimality Conditions

Definition 2.4.3. [10] Let $x^* \in X$, $d \in R^n$, $d \neq 0$. If there exists $\delta > 0$ such that $x^* + td \in X, \forall t \in [0, \delta]$, then d is said to be a feasible direction of X at x^* . The set of all feasible directions of X at x^* is

$$FD(x^*, X) = \{d | x^* + td \in X, \forall t \in [0, \delta]\}. \quad (2.30)$$

Definition 2.4.4. [10] Let $x^* \in X$ and $d \in R^n$. If

$$\begin{aligned} d^T \nabla h_i(x^*) &= 0, i \in I, \\ d^T \nabla g_j(x^*) &\geq 0, j \in I(x^*), \end{aligned}$$

then d is said to be a linearized feasible direction of X at x^* . The set of all linearized feasible directions of X at x^* is

$$[10]LFD(x^*, X) = \{d \mid d^T \nabla h_i(x^*) = 0, i \in I, d^T \nabla g_j(x^*) \geq 0, j \in I(x^*)\}. \quad (2.31)$$

Definition 2.4.5. [4](*Regular point*) Let x^* be a vector such that $h(x^*) = 0, g(x^*) \leq 0$ and for some $\epsilon > 0$, $h \in C^1$ and $g \in C^1$ on $S(x^*; \epsilon)$. Then x^* is regular point if the gradients $\nabla h_1(x^*), \nabla h_2(x^*), \dots, \nabla h_n(x^*)$ and $\nabla g_j(x^*)$, for $j \in A(x^*)$, are linearly independent.

Theorem 2.4.1. [4][*Karush-Kun-Tucker(KKT) necessary condition*] Let x^* be a local minimizer of problem (2.25) and assume that $f, h, g \in C^1$ on $S(x^*; \epsilon)$ for some $\epsilon > 0$ and x^* is a regular point. Then there exist unique vectors $\lambda^* \in R^n, \mu^* \in R^m$ such that

$$\nabla_x L(x^*, \lambda^*, \mu^*) = 0 \quad (2.32)$$

$$\mu_j^* \geq 0, \mu_j^* g_j(x^*) = 0, \forall i = 1, 2, \dots, m. \quad (2.33)$$

The conditions $\mu_j g_j(x^*) = 0$, for all $i = 1 : m$, are complementarity conditions; they imply that either constraint $g_j(x^*)$ is active or $\mu_j = 0$, or possibly both. In particular, the Lagrange multipliers corresponding to inactive inequality constraints are zero.

Moreover, if $f, h, g \in C^2$ on $S(x^*; \epsilon)$, then for all $z \in R^n$ satisfying $\nabla h(x^*)^T z = 0$ and $\nabla g_j(x^*)^T z = 0, i \in A(x^*)$, the

$$z^T \nabla_{xx} L(x^*, \lambda^*, \mu^*) \geq 0.$$

Theorem 2.4.2. [2][*KKT Sufficient Condition*] Let $x^* \in R^n$ such that $h(x^*) = 0, g(x^*) \leq 0$ and for some $\epsilon > 0$, $f, h, g \in C^2$ on $S(x^*; \epsilon)$. Assume that there exist vectors $\lambda \in R^n$ and $\mu \in R^m$ such that

$$\nabla_x L(x^*, \lambda^*, \mu^*) = 0 \quad (2.34)$$

$$\mu_j \geq 0, \mu_j g_j(x^*) = 0, \forall j = 1 : m \quad (2.35)$$

and for every $z \neq 0$ satisfying $\nabla h(x^*)^T z = 0, \nabla g_j(x^*) \leq 0, \forall j \in A(x^*)$ and $\nabla g_j(x^*)^T z = 0, \forall j \in A(x^*)$ with $\mu_j > 0$, we have

$$z^T \nabla_{xx} L(x^*, \lambda^*, \mu^*) > 0. \quad (2.36)$$

Then x^* is a strict local minimizer for the original constrained problem (1.1).

Theorem 2.4.3. [4] Assume that f and g_1, \dots, g_m are convex and continuously differentiable functions on R^n . Let $x^* \in R^n$ and $\mu^* \in R^m$ satisfy

$$\nabla f(x^*) + \nabla g(x^*)\mu^* = 0,$$

$$g(x^*) \leq 0, \mu^* \geq 0, g_j(x^*)\mu_j^* = 0, j = 1 : m.$$

Then x^* is a global minimizer.

Chapter 3

Penalty and barrier methods

3.1 Introduction

The constrained optimization problems can be solve using different methods are based on minimizing the Lagrangian function while attempting to attain and maintain feasibility. When inequality constraints are present, these methods generalize the simplex method. They solve a sequence of sub-problems with a changing active set (or basis) until a solution to the original constrained problem is found.

There are some major disadvantages to this approach. First, as the number of constraints increases, the number of potential sub-problems increases exponentially. While the hope is that the algorithm will consider only a small proportion of these sub-problems, there is no known method to guarantee that this indeed will be the case. Second, the idea of keeping the constraints satisfied exactly, although easily achieved in the case of linear constraints, is much more difficult to accomplish in the case of nonlinear constraints, and in some cases may not be desirable.

Therefore, some of these difficulties can be remove based on penalization methods. These methods solve a constrained optimization problem by solving a sequence of unconstrained optimization problems. The hope is that in the limit, the solutions of the unconstrained problems will converge to the solution of the constrained problem. The unconstrained problems involve an auxiliary function that incorporates the objective function or the Lagrangian function, together with “penalty” terms that measure violations of the constraints (various techniques differ in the way the auxiliary function is defined). The auxiliary function also includes one or more parameters

that determine the relative importance of the constraints in the auxiliary function. By changing these parameters appropriately, a sequence of problems is generated where the effect of the constraints becomes increasingly pronounced. In contrast to active-set methods, the auxiliary function takes into account all constraints, even when inequalities are present, and thus the combinatorial difficulties of guessing a correct active set are avoided. Further, since penalization techniques do not attempt to keep the constraints satisfied exactly, they can be more suitable for handling nonlinear constraints.

Although penalization methods ameliorate some of the difficulties associated with feasible-point methods, they introduce difficulties of their own. In particular, straightforward applications of penalization can give rise to ill-conditioning. Sophisticated modern interpretations of this idea, however, resolve some of these issues and can be effective methods for challenging optimization problems.

3.2 Penalty method

Penalty method transform a constrained problem into a single unconstrained problem or into a sequence of unconstrained problems. The constraints are placed into the objective function via a penalty parameter in a way that penalizes any violation of the constraints.

3.2.1 Formulation of Penalty problems

Consider the constrained optimization problem

$$\text{minimize } h(x)$$

subject to

$$\begin{aligned} r_i(y) &= 0, i = 1, 2, \dots, n \\ k_j(y) &\leq 0, j = 1, 2, 3, \dots, m \end{aligned} \tag{3.1}$$

where m, n are positive integers, y is independent variable and all the functions involved in the problem are real valued.

The general corresponding penalty function must be a positive penalty for infeasible points and there is no penalty function for feasible points. For the constraints like $k_j(y) \leq 0, j = 1, 2, 3, \dots, m$ and $r_i(y) = 0, i = 1, 2, \dots, n$, an

appropriate penalty function β is defined by

$$\beta(y) = \sum_{j=1}^m \phi[k_j(x)] + \sum_{i=1}^n \psi[r_i(x)] \quad (3.2)$$

where ϕ and ψ are continuous functions that satisfy the properties:

$$\begin{aligned} \phi(z) = \phi(k_j) = 0 & \quad \text{if} \quad z \leq 0 \quad \text{and} \quad \phi(z) > 0 \quad \text{if} \quad z > 0 \\ \psi(z) = \psi(r_i) = 0 & \quad \text{if} \quad z = 0 \quad \text{and} \quad \psi(z) > 0 \quad \text{if} \quad z \neq 0 \end{aligned} \quad (3.3)$$

The fundamental functions ϕ and ψ can be expressed as follows;

$$\phi(z) = [\max\{0, z\}]^p \quad \text{and} \quad \psi(z) = |z|^p$$

where p is a positive integer. Hence, the penalty function β is usually defined as

$$\beta(y) = \sum_{j=1}^m [\text{maximum}\{0, k_j(y)\}]^p + \sum_{i=1}^n |r_i(y)|^p \quad (3.4)$$

Therefore, the function $h(y) + \lambda\beta(x)$ is the corresponding auxiliary function.

In particular, for $k_j(y) = 0, \forall j = 1, 2, \dots, m$ in (3.1), we have an optimization problem with equality constraints and its penalty function becomes $\beta(y) = \sum_{i=1}^n \psi[r_i(y)]$. Besides, if $r_i(y) = 0$ for all $i = 1, 2, \dots, n$, the optimization problem in (3.1) becomes a problem with inequalities restriction and its corresponding penalty function is $\beta(y) = \sum_{j=1}^m \phi[k_j(y)]$.

Consider the following primal and penalty problems;

Primal problem [9]:

$$\text{minimize} \quad h(y)$$

subject to

$$\begin{aligned} r_i(y) &= 0, i = 1, 2, \dots, n \\ k_j(y) &\leq 0, j = 1, 2, 3, \dots, m \\ y &\in Y \end{aligned} \quad (3.5)$$

where $h, r_i(y), i = 1, 2, \dots, n$ and $k_j(y), j = 1, 2, 3, \dots, m$ are continuous functions defined on R^n , and Y is a not empty set in R^n which represents simple constraints that could easily be handled explicitly such as lower and upper bounds on the variables.

Penalty problem [9]:

Let β be a continuous function of the form (3.2) satisfying the properties stated in (3.3).

The basic penalty function technique attempts to find

$$\sup \Theta(\lambda)$$

subject to $\lambda \geq 0$ where $\Theta(\lambda) = \inf\{h(y) + \lambda\beta(y) : y \in Y\}$.

Then based on the above information, fundamental theorems related with penalty method stated as follow.

Lemma 3.2.1. [9] Assume that $h, k_1, \dots, k_m, r_1, \dots, r_n$ are continuous in R^n , and let Y be a nonempty set in R^n . Suppose that β is continuous in R^n given by (3.2), and for any λ , there exists an $y_\lambda \in Y$ such that $\Theta(\lambda) = h(y_\lambda) + \lambda\beta(y_\lambda)$. Then, the following statements are true:

1. $\inf\{h(y) : y \in Y, k(y) \leq 0, r(y) = 0\} \geq \sup_{\lambda \geq 0} \Theta(\lambda)$, where $\Theta(\lambda) = \inf\{h(y) + \lambda\beta(y) : y \in Y\}$ and k is a function with components k_1, k_2, \dots, k_m , and r is a function with components r_1, r_2, \dots, r_n .
2. $h(y_\lambda)$ is decreasing function of $\lambda \geq 0$, $\Theta(\lambda)$ is a non-decreasing function of λ and $\beta(y_\lambda)$ is a non-increasing function of λ .

Proof. 1. Assuming $y \in Y$ with $k(y) \leq 0$ and $r(y) = 0$, and $\beta(y) = 0$ with $\lambda \geq 0$, we have

$$h(y) = h(y) + \lambda\beta(y) \geq \inf\{h(x) + \lambda\beta(x) : x \in Y\} = \Theta(\lambda).$$

This implies that $h(y) \geq \Theta(\lambda)$. Since $h(y) \geq \inf\{h(y) : y \in Y, k(y) \leq 0, r(y) = 0\}$ and $\Theta(\lambda) \leq \sup \Theta_{\lambda \geq 0}(\lambda)$, $\inf\{h(y) : y \in Y, k(y) \leq 0, r(y) = 0\} \geq \sup \Theta_{\lambda \geq 0}(\lambda)$. Therefore, the required inequality holds true.

2. By assuming $\gamma < \lambda$ and basic definition of $\Theta(\gamma)$, $\Theta(\mu)$, the following two results are obtained:

$$h(y_\lambda) + \gamma\beta(x_\lambda) \geq h(y_{\gamma}) + \gamma\beta(y_\gamma) \quad (3.6)$$

$$h(y_\gamma) + \lambda\beta(y_\gamma) \geq h(y_\lambda) + \lambda\beta(y_\lambda) \quad (3.7)$$

If we add these two expressions and arranging suitably, we have

$$(\lambda - \gamma)[\beta(y_\gamma) - \beta(y_\lambda)] \geq 0$$

Because of $\lambda > \gamma$, we obtained $\beta(y_\gamma) \geq \beta(y_\lambda)$. It yields from (1.1) that $h(y_\lambda) \geq h(y_\gamma)$ for $\gamma \geq 0$. If we add and subtract $\lambda\beta(y_\lambda)$ to the left-hand side of (3.6), then

$$h(y_\lambda) + \lambda\beta(y_\lambda) + (\gamma - \lambda)\beta(y_\lambda) \geq \Theta(\gamma)$$

Since $\lambda \geq \gamma$ and $\beta(x_\lambda) \geq 0$, the above expressions yields that $\Theta(\lambda) \geq \Theta(\gamma)$. Therefore, $\Theta(\lambda)$ and $h(y_\lambda)$ are non-decreasing functions, $\beta(y_\lambda)$ is non-increasing function. \square

Theorem 3.2.1. [2] Consider the constrained optimization problem:

$$\begin{aligned} & \text{minimize } h(y) \\ & k_j(y) \leq 0, j = 1 : m \\ & r_i(y) = 0, i = 1 : n \\ & y \in Y \end{aligned} \tag{3.8}$$

where $h, k_1, \dots, k_m, r_1, \dots, r_n$ are continuous functions on R^n and Y is not empty set in R^n . Assume that the optimization has a feasible solution and let β be a continuous function given by (3.2). Besides, assume that for any λ there is a solution $y_\lambda \in Y$ to the problem for minimizing $h(y) + \lambda\beta(y)$ subject to $y \in Y$ and that $\{y_\lambda\}$ is contained in a compact subset of Y . Then

$$\inf\{h(y) : y \in Y, k(y) \leq 0, r(y) = 0\} = \sup_{\lambda \geq 0} \Theta(\lambda) = \lim_{\lambda \rightarrow \infty} \Theta(\lambda)$$

, where $\Theta(\lambda) = \inf\{h(y) + \lambda\beta(y) : y \in Y\} = h(y_\lambda) + \lambda\beta(y_\lambda)$. Furthermore, the limit y^* of any convergent sub-sequence of $\{y_\lambda\}$ is an optimal solution to the original constrained optimization problem and $\lambda\beta(y_\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$.

Proof. Using the second part of Lemma (3.2.1), i.e., $\Theta(\mu)$ is non-decreasing, $\sup_{\lambda \geq 0} \Theta(\lambda) = \lim_{\lambda \rightarrow \infty} \Theta(\lambda)$. Firstly we have prove that $\beta(y_\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$. Let x be a feasible and $\epsilon > 0$ and y_1 be an optimal solution to the problem to minimizing $h(y) + \lambda\beta(y)$ subject to $y \in Y$ for $\lambda = 1$. If $\lambda \geq \frac{1}{\epsilon}|h(x) - h(y_1)| + 2$, then using second Part of Lemma (3.2.1), we obtained $h(y_\lambda) \geq h(y_1) + \epsilon$.

Secondly, prove that $\beta(y_\lambda) \leq \epsilon$. Show this by the concept of contradiction as, assume that $\beta(y_\lambda) \geq \epsilon$. By first part of Lemma (3.2.1), we have a result

$$\begin{aligned} \inf\{h(y) : k(y) \leq 0, r(y) = 0, y \in Y\} & \geq \Theta(\lambda) = h(y_\lambda) + \lambda\beta(y_\lambda) \\ & \geq h(y_1) + \lambda\beta(y_\lambda) > h(y_1) + |h(x) - h(y_1)| + 2\epsilon > h(x) \end{aligned}$$

This expression is impossible in view of the feasibility of x . Thus, $\beta(y_\lambda) \leq \epsilon$ for all $\lambda \geq \frac{1}{\epsilon}|h(x) - h(y_1)| + 2$. Since $\epsilon > 0$ is arbitrary, $\beta(y_\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$. Then

$$\sup_{\lambda \geq 0} \Theta(\lambda) \geq \Theta(\lambda_k) = h(y_{\lambda_k}) + \lambda_k \beta(y_{\lambda_k}) \geq h(y_{\lambda_k})$$

. Due to the continuity of h and $y_{\lambda_k} \rightarrow y^*$, the above expression becomes

$$\sup_{\lambda \geq 0} \Theta(\lambda) \geq h(y^*) \quad (3.9)$$

Since $\beta(y_\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$, $\beta(y^*) = 0$; that is, y^* is a feasible solution to the original problem. From (3.9) and first Part of Lemma (3.2.1), it follows that y^* is an optimal solution to the original problem and that $\sup_{\lambda \geq 0} \Theta(\lambda) = h(y^*)$. Note that $\lambda\beta(y_\lambda) = \Theta(\lambda) - h(y_\lambda)$. As $\lambda \rightarrow \infty$, $\Theta(\lambda)$ and $h(y_\lambda)$ both approaches to $h(y^*)$, and hence $\lambda\beta(y_\lambda)$ approaches zero. \square

Corollary 3.2.1. If $\beta(y_\lambda) = 0$ for some λ , then y_λ is an optimal solution to the problem.

Proof. If $\beta(y_\lambda) = 0$, then y_λ is a feasible solution to the problem. Additionally, since

$$\inf\{h(y) : k(y) \leq 0, r(y) = 0, y \in Y\} \geq \Theta(\lambda) = h(y_\lambda) + \lambda\beta(y_\lambda) = h(y_\lambda),$$

it follows that y_λ is an optimal solution. \square

Remark 2.

1. The significance assumption that y_μ is contained in a compact subset of Y holds true if X is compact.
2. Without the assumption that y_λ is contained in a compact subset in Y , it is possible that the optimal objective values of the primal and penalty problems are not equal.

Generally, the optimal solution of the original primal problem can be determined through solving the transformed sequence of unconstrained optimization of the form:

$$\text{minimize } h(y) + \lambda\beta(y)$$

subject to $y \in Y$ for an increasing sequence of penalty parameters. The optimal points $\{y_\lambda\}$ are generally infeasible, but as the penalty parameter λ is made large, the points generated approach an optimal solution from outside the feasible region.

If the value of $p = 2$ in equation (3.4), then the penalty function is called quadratic penalty function and written as

$$\begin{aligned}\beta(y) &= \sum_{j=1}^m [\text{maximum}\{0, k_j(y)\}]^2 + \sum_{i=1}^n |r_i(y)|^2 \\ &= \sum_{j=1}^m [\text{maximum}\{0, k_j(y)\}]^2 + \sum_{i=1}^n (r_i(y))^2\end{aligned}\quad (3.10)$$

The quadratic penalty method adds a multiple of the square of the violation of each constraint to the objective. As the constraints get squared the new objective function retains the properties of smoothness and differentiability. Because of its simplicity and intuitive appeal, this approach is used often in practice, although it has some important disadvantages [10].

3.2.2 Optimality of Penalty problem using KKT Lagrange Multipliers

Within certain conditions, we can use the solutions to the sequence of penalty problems to recover the KKT Lagrange multipliers associated with the constraints at optimality. For simplicity, suppose that $Y = R^n$ and consider the primal optimization problem;

$$\text{minimize } h(y)$$

subject to

$$\begin{aligned}r_i(y) &= 0, i = 1, 2, \dots, n \\ k_j(y) &\leq 0, j = 1, 2, \dots, m\end{aligned}\quad (3.11)$$

Suppose that the penalty function β is given by (3.2), ϕ and ψ are continuously differentiable with $\phi'(z) \geq 0$ for all z and $\phi'(z) = 0$ for $z \leq 0$. Assuming that the conditions of Theorem (3.2.1) hold true, since y_λ solves the problem to minimizing $h(y) + \lambda\beta(y)$, the gradient of the objective function of this corresponding penalty problem must vanish at y_λ . This gives

$$\nabla h(y_\lambda) + \sum_{j=1}^m \lambda \phi'[k_j(y_\lambda)] \nabla k_j(y_\lambda) + \sum_{i=1}^n \lambda \psi'[r_i(y_\lambda)] \nabla r_i(y_\lambda) = 0, \text{ for all } \lambda \quad (3.12)$$

Without loss of generality, assume that the generated sequence $\{y_\lambda\}$ converges to an accumulation point y^* . Denote that $I = \{j : k_j(y^*) = 0\}$ to be the set of inequality constraints that are binding at y^* . Since $k_j(y) < 0$

for all $j \notin I$ by Theorem (3.2.1), we have $k_j(y_\lambda) < 0$ for λ sufficiently large, yielding $\lambda\phi'[k_j(y_\lambda)] = 0$. Hence, we can write the foregoing identity (3.12) as;

$$\nabla h(y_\lambda) + \sum_{j \in I} (w_\lambda)_j \nabla k_j(y_\lambda) + \sum_{i=1}^n (v_\lambda)_i \nabla r_i(y_\lambda) = 0 \quad (3.13)$$

for all λ large enough, where w_λ and v_λ are vectors having components

$$(w_\lambda)_j = \lambda\phi'[k_j(y_\lambda)] \geq 0, (v_\lambda)_i = \lambda\psi'[r_i(y_\lambda)] \quad (3.14)$$

for all $j \in I$ and for all $i = 1, 2, \dots, n$. Then, if y^* is a regular solution, there exist unique Lagrangian multipliers $w_j \geq 0, j \in I$ and $v_i, i = 1, \dots, n$ such that

$$\nabla h(y^*) + \sum_{j \in I} (w_j)^* \nabla k_j(y^*) + \sum_{i=1}^n (v_i)^* \nabla r_i(y^*) = 0 \quad (3.15)$$

Since k, r, ϕ and ψ are all continuously differentiable and $\{y_\lambda\} \rightarrow y^*$ which is a regular point, we must have that $(w_\lambda)_j \rightarrow w_j^*$ for all $j \in I$ and $(v_\lambda)_i \rightarrow v_i^*$ for all $i = 1, \dots, n$. Hence, for sufficiently large values of λ , the multipliers can be used to estimate the KKT Lagrange multipliers at optimality.

In particular, if β is the quadratic penalty function given by $\beta(y) = \sum_{j=1}^m [\text{maximum}\{0, k_j(y)\}]^2 + \sum_{i=1}^n [r_i(y)]^2$, then $\phi(z) = [\text{maximum}\{0, z\}]^2$, $\phi'(z) = 2[\text{maximum}\{0, z\}]$, $\psi(z) = z^2$ and $\psi'(z) = 2z$. Hence, expression (3.14) becomes

$$(w_\lambda)_j = 2\lambda\text{maximum}\{0, k_j(y_\lambda)\}, (v_\lambda)_i = 2\lambda r_i(y_\lambda) \quad (3.16)$$

for all $j \in I$ and for all $i = 1, 2, \dots, n$.

Example 3.2.1. [5] Consider the problem *minimize* $h(y) = y_1^2 + 2y_2^2$ subject to $k(y) = 1 - y_1 - y_2 \leq 0; y_1, y_2 \in R$.

Solution: The given problem is a constrained optimization with inequality constraint only. Then, its quadratic penalty function is

$$\beta(y) = [\text{maximum}\{0, 1 - y_1 - y_2\}]^2.$$

Thus, $\beta(y) = 0$ for $k(y) = 0$ and $\beta(y) = (1 - y_1 - y_2)^2$ for $k(y) < 0$.

Then the unconstrained problem is

$$h_{\lambda_k}(y) = h(y) + \lambda_k \alpha(y) = y_1^2 + 2y_2^2 + \mu_k \beta(y).$$

If $\beta(y) = 0$, then $h_{\lambda_k}(y) = y_1^2 + 2y_2^2$ and the optimal solution to minimize $h_{\lambda_k}(y) = y_1^2 + 2y_2^2 + \lambda_k\beta(y)$ using KKT condition is $y^* = (y_1, y_2) = (0, 0)$ and this is infeasible. Therefore,

$$h_{\lambda_k}(y) = y_1^2 + 2y_2^2 + \mu_k\beta(y) = y_1^2 + 2y_2^2 + \lambda_k(1 - y_1 - y_2)^2.$$

Now, by using the necessary condition for optimality (i.e., $\nabla h_{\lambda_k}(y) = 0$) we have the following:

$$\begin{aligned}\frac{\partial h_{\lambda_k}}{\partial y_1} &= 2y_1 - 2\lambda_k(1 - y_1 - y_2) = 0 \\ \frac{\partial h_{\lambda_k}}{\partial y_2} &= 4y_2 - 2\lambda_k(1 - y_1 - y_2) = 0\end{aligned}$$

This implies that

$$\begin{aligned}\frac{y_1}{\lambda_k} &= 1 - y_1 - y_2 \\ \frac{2y_2}{\lambda_k} &= 1 - y_1 - y_2\end{aligned}$$

By solving these system of equations, we get $x_1 = 2x_2$ and $y_{\lambda_k} = (\frac{2\lambda_k}{2+3\lambda_k}, \frac{\lambda_k}{2+3\lambda_k})$. As $\lambda_k \rightarrow \infty$, $y_{\lambda_k} \rightarrow y^* = (\frac{2}{3}, \frac{1}{3})$ and we have

$$w_{\lambda_k} = 2\lambda_k \text{maximum}\{0, k_j(y_{\lambda_k})\} = 2\lambda_k(1 - y_1 - y_2) = 2\lambda_k(1 - \frac{2\lambda_k}{2+3\lambda_k} - \frac{\lambda_k}{2+3\lambda_k}) = \frac{4\lambda_k}{2+3\lambda_k}$$

Thus, as $\lambda_k \rightarrow \infty$, $w_{\lambda_k} \rightarrow w^* = \frac{4}{3}$ and the optimal solution of original problem is $y^* = (\frac{2}{3}, \frac{1}{3})$.

Example 3.2.2. [9] Consider the constrained optimization problem

$$\text{minimize } h(y_1, y_2) = -y_1y_2$$

subject to

$$k(y_1, y_2) = y_1 + 2y_2 - 4 = 0$$

Then a sequence of unconstrained minimization problems

$$\text{minimize } h_{\lambda_k}(y) = -y_1y_2 + \lambda_k(y_1 + 2y_2 - 4)^2$$

is solved for increasing values of the penalty parameter λ_k . The necessary conditions for optimality of the unconstrained problem are

$$\begin{aligned}-y_2 + 2\lambda_k(y_1 + y_2 - 4) &= 0 \\ -y_1 + 4\lambda_k(y_1 + y_2 - 4) &= 0\end{aligned}$$

For $\lambda_k > \frac{1}{4}$ this yields the solution $y_{1k} = \frac{8\lambda_k}{4\lambda_k - 1}$, $y_{2k} = \frac{4\lambda_k}{4\lambda_k - 1}$ which is a local as well as a global minimizer. As $\lambda_k \rightarrow \infty$ for $\lambda_k > \frac{1}{4}$, we obtain the optimal solution of the original problem $y^* = (2, 1)$ and the Lagrange multiplier is $v_{\lambda_k} = -\lambda_k k'(y_{\lambda_k}) = \frac{-4\lambda_k}{4\lambda_k - 1} \rightarrow -1 = v$.

3.2.3 Drawback of Penalty method

- Difficulties Related to Ill-Conditioning

We now examine the nature of the ill-conditioning in the Hessian matrix $\nabla^2 h_{\lambda_k}(y)$. An understanding of the properties of this matrix and the similar Hessian's that arise in other penalty and barrier methods, is essential in choosing effective algorithms for the minimization problem at each iteration [4].

Since the penalty method is based on the solution of the problem of the form

$$\text{minimize } h_{\lambda_k}(y)$$

subject to

$$y \in R^n \tag{3.17}$$

it is natural to inquire about the degree of difficulty in solving such problems. When $h, r, k \in C^n$, the degree of difficulty for solving problem (3.17) depends on the the eigenvalue structure of the Hessian matrix $\nabla^2 h_{\lambda_k}(y)$ [4].

The condition number, defined as

$$\frac{\text{maximum eignevalue of } \nabla^2 f_\mu}{\text{minimum eignevalue of } \nabla^2 f_\mu}$$

can get arbitrarily large. As a consequence of all this, the numerical solution of the Newton equation $\nabla^2 f_\mu d = -\nabla f_\mu$ is very susceptible to rounding error and the resulting search directions can be inaccurate and ineffective. Similar difficulties can occur if we attempt to minimize $f_\mu(x)$ using quasi-Newton or conjugate gradient methods [5].

To see this, let us consider the Hessian of the quadratic penalized objective function Q_{μ_k} ,

$$\nabla^2 Q_{\mu_k} = [\nabla^2 f(x) + \sum_{i=1}^n \mu_k h_k \nabla^2 h_i(x)] + \mu_k \sum_{i=1}^n \nabla h_i \nabla^T h_i.$$

When x is close to the minimizer of $Q_{\mu_k}(x)$, then $\mu_k h_i(x)$ approaches to the corresponding Lagrangian multiplier, λ^* , of the original constrained problem and we have the following estimation:

$$\nabla^2 Q_{\mu_k}(x) \approx \nabla^2 L(x, \lambda^*) + \mu_k \sum_{i=1}^n \nabla h_i \nabla^T h_i.$$

where $\nabla^2 L(x, \lambda^*)$ is Hessian of the Lagrangian function with multiplier λ^* to the given constrained problem.

But $\nabla^2 L(x, \lambda^*)$ is independent of μ_k and the matrix $\mu_k \sum_{i=1}^n \nabla h_i \nabla^T h_i$ with nonzero eigenvalues of order μ_k is depend on μ_k . Therefore, the matrix

$$\nabla^2 L(x, \lambda^*) + \mu_k \sum_{i=1}^n \nabla h_i \nabla^T h_i.$$

has some of its eigenvalues approaching a constant, while others are of order μ_k . Since μ_k is approaching ∞ , the increasing ill-conditioning of $\nabla^2 Q_{\mu_k}(x)$ is apparent.

Example 3.2.3. Consider the Hessian matrix of the Auxiliary function $f_\mu(x) = x_1^2 + 2x_2^2 + \mu(1 - x_1 - x_2)^2$,

$$H = \begin{pmatrix} 2 + 2\mu & 2\mu \\ 2\mu & 4 + 2\mu \end{pmatrix}$$

Suppose we want to find its eigenvalues by solving

$$|H - \lambda I| = \lambda^2 - (6 + 4\mu)\lambda + (8 + 12\mu) = 0.$$

This yields $\lambda_1 = (3 + 2\mu) - \sqrt{4\mu^2 + 1}$ and $\lambda_2 = (3 + 2\mu) + \sqrt{4\mu^2 + 1}$
Taking the ratio of largest and smallest eigenvalue yields

$$\frac{(3 + 2\mu) + \sqrt{4\mu^2 + 1}}{(3 + 2\mu) - \sqrt{4\mu^2 + 1}}$$

It is clear that as $\mu \rightarrow \infty$, the limit of the preceding ratio (the condition number) also goes to ∞ . This indicates that as the iterations proceed and we start to increase the value of μ , the Hessian of unconstrained function that we are minimizing becomes increasingly ill-conditioned. This is a common situation and is especially problematic if we are using a method for the unconstrained optimization that requires the use of Hessian.

The steepest descent is also out of the question as a possible solution method, since the rate of convergence of methods of steepest descent applied to a functional is determined by the ratio of smallest to the largest eigenvalues of the Hessian of that function. It follows in particular that the steepest descent method applied to f_{μ_k} converges slowly for large μ . Even Newton's method can encounter significant difficulties if μ_k is very high, and the starting point

for minimizing $f_{\mu_k}(x)$ is not near a solution [10].

In general, the poor conditioning of this system will lead to significant errors in the computed value of d , regardless of the computational technique used to solve $\nabla^2 f_\mu d = -\nabla f_\mu$.

Note that, the ill-conditioning associated with the unconstrained minimization problem is a basic characteristic feature of penalty methods and represents the overriding factor in determining the manner in which these methods are operated.

- **Difficulties Related to Small Value of the Penalty parameter**

On the other hand, if we start the iteration from a small value of the penalty parameter, then the minimizer of the penalized objective function may diverge away from the feasible region.

Example 3.2.4. Consider the problem

$$\min f(x) = -5x_1^2 + x_2^2$$

subject to

$$h(x) = x_1 - 1 = 0$$

Then the penalized objective function

$$Q_\mu(x) = -5x_1^2 + x_2^2 + \mu(x_1 - 1)^2$$

is unbounded for $\mu < 10$.

To see this let us compute the minimizer using first order necessary conditions of optimality. Thus,

$$\nabla^2 Q_\mu = (-10x_1 + \mu(x_1 - 1), 2x_2) = (0, 0)$$

So that $x_\mu = (\frac{\mu}{\mu-10}, 0)$ and as $\mu \rightarrow 10$ from the left side, x is unbounded and we can not approach to the right solution $x^* = (1, 0)$. But for $\mu > 10$, as $\mu \rightarrow \infty$, we can approach to that solution.

3.3 Barrier Function Method

Barrier function methods are used to convert constrained optimization into an unconstrained optimization or a sequence of unconstrained optimizations.

These functions set a barrier against leaving the feasible region. If the optimal solution occurs at the boundary of the feasible region, the procedure moves from the interior to the boundary [9].

Barrier methods transform the original constrained problem into an unconstrained problem or a sequence of unconstrained problems; however the barrier (or called interior penalty) functions prevent the current solution from leaving the feasible region. These require that the interior of the feasible sets must be nonempty and therefore, they are used with problems having only inequality constraints, because there is no interior for equality constraint. If the optimal solution occurs at the boundary of feasible region, the procedure moves from the interior to the boundary. Thus, once the unconstrained minimization is started from any feasible point, the subsequent points generated will always lie within the feasible region, since the constraint boundaries act as barriers during the minimization process. This is the reason why the interior penalty method is also known as "barrier" method [5].

3.3.1 Formulation of Barrier Problems

Consider the Primal Problem;

$$\text{minimize } h(y)$$

subject to

$$k(y) \leq 0, y \in Y \quad (3.18)$$

where k is a vector function whose components are k_1, \dots, k_m . Here h, k_1, \dots, k_m are continuous functions defined on R^n and Y is a nonempty set in R^n .

If equality constraints are present, they are accommodated within the set Y . Alternatively, in the case of affine equality constraints, we can possibly eliminate them after solving for some variables in terms of the others, thereby reducing the dimension of the problem. The reason why this treatment is necessary is that barrier function methods require the set $\{y : k(y) < 0\}$ to be nonempty, which would obviously not be possible if the equality constraints $r(y) = 0$ were accommodated within the set of inequalities as $r(y) \leq 0$ and $r(y) \geq 0$.

Definition 3.3.1. (*Barrier Problem*) Find;

$$\inf \beta(\lambda)$$

subject to

$$\lambda > 0, \quad (3.19)$$

where $\beta(\lambda) = \inf\{h(y) + \lambda B(y) : k(y) < 0, y \in Y\}$. Here $B(y)$ is a barrier function that is non-negative and continuous over the region $\{y : k(y) < 0\}$ and approaches to ∞ as the boundary of the region $\{y : k(y) \leq 0\}$ is approached from the interior.

Definition 3.3.2. The barrier function B is defined by

$$B(y) = \sum_{i=1}^m \phi[k_i(y)] \quad (3.20)$$

where ϕ is a continuous function on $\{z : z = k_i(y) < 0\}$ and hold

$$\phi(z) \geq 0 \text{ if } z < 0 \text{ and } \lim_{z \rightarrow 0^-} \phi(z) = \infty. \quad (3.21)$$

The two fundamental barrier functions are;

1. $B(y) = -\sum_{i=1}^m \frac{1}{k_i(y)}$ for $k_i(y) < 0$ which is called Inverse function.
2. $B(y) = -\sum_{i=1}^m \ln \min\{1, -k_i(y)\}$ for $k_i(y) < 0$ which is known as Logarithmic function.

Because of the term $\min\{1, -k_i(y)\}$, the second barrier function is not differentiable. Since the property in (3.21) for ϕ is hold only in a neighborhood of $z = 0$, the popular barrier function is defined as,

$$B(y) = -\sum_{i=1}^m \ln[-k_i(y)] \quad (3.22)$$

which is also convergent.

In the above two cases, $\lim_{k_i(y) \rightarrow 0^-} B(y) \rightarrow \infty$. Thus, the auxiliary function is defined as

$$h_{\lambda_k}(y) = h(y) + \lambda B(y), \quad (3.23)$$

Where λ is a positive constant.

Ideally, if $k_i(y) < 0$ we have $B(y) = 0$ and if $k_i(y) \rightarrow 0$ we have $B(y) \rightarrow \infty$. So that we never leave the region $\{y | k_i(y) \leq 0\}$. But, $B(y)$ is not continuous for $k_i(y) = 0$. This causes serious computational problems during the unconstrained optimization problem. Hence, the function B is replaced by non-negative and continuous on $\{y | k_i(y) < 0\}$ which approaches infinity as the boundary is approached from the interior. Moreover, the barrier function at infeasible points is not necessarily defined.

Therefore, barrier problem is expressed as:

$$\text{minimize } h(y) + \lambda B(y)$$

subject to

$$\begin{aligned} k_i(y) &< 0, i = 1 : m \\ y &\in R^n \end{aligned} \tag{3.24}$$

From this we observe that the barrier problem itself is a constrained problem, and indeed the constraint is some what more complicated than in the original problem. The advantage of this problem is that it can be solved by using an unconstrained search technique. To find the solution one starts at an initial interior point and then searches from that point using steepest descent or some other iterative descent method applicable to unconstrained problems. Thus, although the barrier problem is from a formal viewpoint a constrained problem, from a computational view point it is unconstrained.

Lemma 3.3.1. [9] Let h, k_1, \dots, k_m be continuous functions on R^n and let Y be a nonempty closed set in R^n . Suppose that the set $\{y \in Y : k(y) < 0\}$ is not empty and that B is a barrier function of the form (3.21) which is continuous on $\{y : k(y) < 0\}$. Moreover, suppose that for any given $\lambda > 0$, if $\{x_k\}$ in Y satisfies $k(y_k) < 0$ and $h(y_k) + \lambda B(y_k) \rightarrow \beta(\lambda)$, then $\{y_k\}$ has convergent sub-sequence. Then,

1. For any λ , there is a $y_\lambda \in Y$ with $k(y_\lambda) < 0$ such that $\beta(\lambda) = h(y_\lambda) + \lambda B(y_\lambda) = \inf\{h(y) + \mu B(y) : y \in Y, k(y) < 0\}$.
2. $\inf\{h(y) : y \in Y, k(y) \leq 0\} \leq \inf\{\beta(\lambda) : \lambda > 0\}$.
3. For $\lambda > 0$, $h(y_\lambda)$ and $\beta(\lambda)$ are increasing functions of λ , and $B(y_\lambda)$ is a decreasing function of λ .

Proof. 1. Let $\lambda > 0$. Using the definition of β , there is a sequence $\{y_k\}$ in Y with $k(x_k) < 0$ such that $h(y_k) + \lambda B(y_k) \rightarrow \beta(\lambda)$. Based on the given assumption, $\{y_k\}$ has a sub-sequence $\{y_{\lambda_k}\}$ which converge to $y_\lambda \in Y$. Since k is continuous function, $k(y_\lambda) \leq 0$ and this shows that $k(y_\lambda) < 0$. If it is not true, then $k_i(y_\lambda) = 0$ for some i and since the barrier function B satisfied (3.3.2), $B(y_k) \rightarrow \infty$. Hence, $\beta(\lambda) = \infty$, which is not possible, since $\{y : y \in Y, k(y) < 0\}$ is assumed non-empty. Thus, $\beta(\lambda) = h(y_\lambda) + \lambda B(y_\lambda)$, for $y_\lambda \in Y$ and $k(y_\lambda) < 0$. Therefore, the required part is satisfied.

2. Since $B(y) \geq 0$ for $k(y) < 0$, for $\lambda \geq 0$, we get

$$\begin{aligned}\beta(\lambda) &= \inf\{h(y) + \lambda B(y) : y \in Y, k(y) < 0\} \\ &\geq \inf\{h(y) : y \in Y, k(y) < 0\} \\ &\geq \inf\{h(y) : y \in Y, k(y) \leq 0\}.\end{aligned}$$

Since the above expression is true for all $\lambda \geq 0$, $\inf\{h(y) : y \in Y, k(y) \leq 0\} \leq \inf\{\beta(\lambda) : \lambda > 0\}$.

3. Let $\lambda > \gamma > 0$. If $B(y) \geq 0$ and $k(y) < 0$, then $h(y) + \lambda B(y) \geq h(y) + \gamma B(y) \forall y \in Y$ with $k(y) < 0$. Therefore, $\beta(\lambda) \geq \beta(\gamma)$. Using part one, there is y_λ and y_γ such that

$$h(y_\lambda) + \lambda B(y_\lambda) \leq h(y_\gamma) + \lambda B(y_\gamma) \quad (3.25)$$

$$h(y_\gamma) + \gamma B(y_\gamma) \leq h(y_\lambda) + \gamma B(y_\lambda) \quad (3.26)$$

By adding and arranging expressions (3.25) and (3.26), we obtained $(\lambda - \gamma)[B(y_\lambda) - B(y_\gamma)] \leq 0$. Since $\lambda - \gamma > 0$, $B(y_\lambda) \leq B(y_\gamma)$. By substituting this in (3.26), it yields that $h(y_\gamma) \leq h(y_\lambda)$. Hence, the required part is true. \square

Theorem 3.3.1. [9] Suppose that $h : R^n \rightarrow R$ and $k : R^n \rightarrow R^m$ are continuous functions and Y is nonempty closed set in R^n . Assume that the set $\{y \in Y : k(y) < 0\}$ is nonempty. Moreover, suppose that the problem to find minimum of $h(y)$ subject to $k(y) \leq 0, y \in Y$ has an optimum solution y^* that satisfied the conditions, for a neighborhood D around y^* , there is a $y \in y \cap D$ such that $k(y) < 0$. Then

$$\text{minimum}\{h(y) : k(y) \leq 0, y \in Y\} = \lim_{\lambda \rightarrow 0^+} \beta(\lambda) = \inf_{\lambda > 0} \beta(\lambda)$$

Assuming $\beta(\lambda) = h(y_\lambda) + \lambda B(y_\lambda)$, for $y_\lambda \in Y$ and $k(y_\lambda) < 0$, the subsequence $\{y_k\}$ converges to the optimal solution of the primal optimization and as $\lambda \rightarrow 0^+$, $\lambda B(y_\lambda) \rightarrow 0$.

Proof. Assume that y^* is an optimum solution of the original optimization that satisfy all the given properties and $\epsilon > 0$. Using the definition of continuity of h and the given information, there exists a $\hat{y} \in Y$ with $k(\hat{y}) < 0$ such that $h(y^*) + \epsilon > h(\hat{y})$. Then, for $\lambda > 0$

$$h(y^*) + \epsilon + \lambda B(\hat{y}) > h(\hat{y}) + \lambda B(\hat{y}) \geq \beta(\lambda).$$

Using the concept of limit as $\lambda \rightarrow 0^+$, it yields $h(y^*) + \epsilon \geq \lim_{\lambda \rightarrow 0^+} \beta(\lambda)$ for all $\epsilon > 0$. Consequently, $h(y^*) \geq \lim_{\lambda \rightarrow 0^+} \beta(\lambda)$. By the above Lemma (3.3.1), $h(y^*) = \lim_{\lambda \rightarrow 0^+} \beta(\lambda)$.

For $B(y_\lambda) \geq 0$, y_λ is feasible to the given optimization problem and $\lambda \rightarrow 0^+$,

$$\beta(\lambda) = h(y_\lambda) + \lambda B(y_\lambda) \geq h(y_\lambda) \geq h(y^*).$$

Now using the concept of limit $\lambda \rightarrow 0^+$ and $h(y^*) = \lim_{\lambda \rightarrow 0^+} \beta(\mu)$, it yields $h(y_\lambda)$ and $h(y_\lambda) + \lambda B(y_\lambda)$ goes to $h(y^*)$. So that, as $\lambda \rightarrow 0^+$, $\lambda B(y_\lambda) \rightarrow 0$. In addition to this, if $\{y_\lambda\}$ has a sub-sequence which converges to y' , then $h(y') = h(y^*)$. Due to the feasibility of y_λ to the original problem for all λ , y' is feasible and optimum solution. \square

3.3.2 Optimality of Barrier problem using KKT Lagrange Multipliers

Under the conditions of regularity, the barrier method provides a sequence of Lagrange multipliers that converge to an optimum set of Lagrange multipliers. To observe this, consider the problem of finding a minimum $h(y)$ subject to $k_i(y) \leq 0$ for $i = 1 : m$ and $y \in Y = R^n$. Then, the barrier function problem becomes;

$$\text{minimum}_y \{h(y) + \lambda \sum_{i=1}^m \phi[k_i(y)] : k_i(y) < 0\}, \quad (3.27)$$

where ϕ holds true the conditions in (3.21).

To make our explanation simple, assume that h, k and ϕ are continuously differentiable functions, and the conditions of Lemma (3.3.1) and Theorem (3.3.1) are satisfied. In addition to this, suppose the optimal solution y^* which is a limit point of the sequence $\{y_\lambda\}$ is a regular point. That is, $\{y_\lambda\} \rightarrow y^*$. Then, if $I = \{i : k_i(y^*) = 0\}$ is the index set of active constraints at y^* , then there is a unique set of Lagrange multipliers w_i^* , $i = 1 : m$ such that

$$\nabla h(y^*) + \sum_{i=1}^m w_i^* \nabla k_i(y^*) = 0, \quad w_i^* \geq 0, \quad \text{for } i = 1 : m, \quad w_i^* = 0, \quad i \in I \quad (3.28)$$

Since y_λ is a solution of problem (3.3.2) with $k(y_\lambda) < 0$, for each $\lambda > 0$,

$$\nabla h(y_\lambda) + \sum_{i=1}^m (w_\lambda)_i \nabla k_i(y_\lambda) = 0 \quad (3.29)$$

where $(w_\lambda)_i \equiv \lambda \phi' [k_i(y_\lambda)]$, for each $i = 1 : m$. If $\lambda \rightarrow 0^+$, then $\{y_\lambda\} \rightarrow y^*$ and so that $(w_\lambda)_i \rightarrow 0$ for each $i \notin I$. Besides, due to the regularity of y^* and differentiability of the functions h, k, ϕ , we obtain $(w_\lambda)_i \rightarrow w_i^*$ for each $i \in I$. Thus, w_λ gives the set of Lagrange multipliers that approaches the optimal set of Lagrange multipliers w^* as $\lambda \rightarrow 0^+$.

Example 3.3.1.

$$\text{minimize } h(y) = y$$

subject to

$$1 - y \leq 0$$

Solution: One Barrier function of the given optimization is

$$B(y) = \frac{-1}{1-y}, y \neq 1.$$

The corresponding auxiliary function of the optimization is $h_\lambda(y) = h(y) + \lambda B(y) = y + [\frac{\lambda}{y-1}]$. From figure (3.1), simply we see that as λ goes to zero,

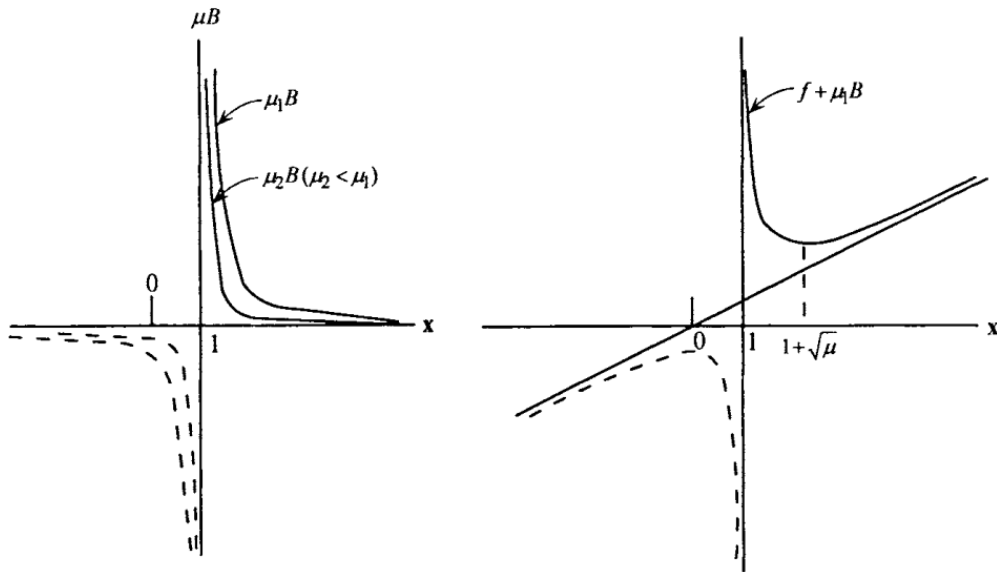


Figure 3.1: Barrier and auxiliary functions

λB approaches to a function with zero value on $y > 1$ and infinity value for $y = 1$. The function represented by dashes in Figure 3.1 correspond to the region $\{y : k(y) > 0\}$ and does not influence the way of finding optimal solution.

Finally, for arbitrary $\lambda > 0$, the barrier optimization is minimizing $h(y_\lambda) =$

$y + \frac{\lambda}{y-1}$ for the values $y > 1$. Thus, $h'_\lambda(y) = 1 - \frac{\lambda}{(y-1)^2}$ and the solution of $h'_\lambda(y) = 0$ gives us $y_\lambda = 1 + \sqrt{\lambda}$. Additionally, the barrier problem becomes $h_\lambda(\lambda) = 1 + 2\sqrt{\lambda}$ and as $\lambda \rightarrow 0^+$, $y_\lambda \rightarrow y^* = 1$, $h_\lambda \rightarrow h(y^*) = 1$. Therefore, optimal solution of the original problem is $y^* = 1$.

Example 3.3.2. minimize $h(y) = y_1^2 + 2y_2^2$ subject to $k(y) = 1 - y_1 - y_2 \leq 0$, $y \in R^2$

Solution: From the problem, its corresponding barrier function is defined as

$$B(y) = -\ln[-k(y)] = -\ln[y_1 + y_2 - 1]$$

The unconstrained optimization problem is

$$\text{minimize } h_\lambda(y) = y_1^2 + 2y_2^2 - \lambda \ln[y_1 + y_2 - 1].$$

Using the concept of necessary conditions ($\nabla f_\mu(x) = 0$), we get;

$$\frac{\partial h_\lambda}{\partial y_1} = 2y_1 - \frac{\lambda}{y_1 + y_2 - 1} = 0, \quad \frac{\partial h_\lambda}{\partial y_2} = 4y_2 - \frac{\lambda}{y_1 + y_2 - 1} = 0.$$

By solving y_1 and y_2 in terms λ , we obtain

$$y_\lambda = \left(\frac{1 \pm \sqrt{1 + 3\lambda}}{3}, \frac{1 \pm \sqrt{1 + 3\lambda}}{6} \right).$$

Since the negative signs leads to infeasibility, we have

$$y_\lambda = \left(\frac{1 + \sqrt{1 + 3\lambda}}{3}, \frac{1 + \sqrt{1 + 3\lambda}}{6} \right).$$

Therefore, as $\lambda \rightarrow 0^+$, $y_\lambda \rightarrow y^* = \left(\frac{2}{3}, \frac{1}{3}\right)$ and optimal solution of the original problem is $y^* = \left(\frac{2}{3}, \frac{1}{3}\right)$.

3.3.3 Computational difficulties of Barrier method

In solving constrained nonlinear programming problems using Barrier methods, several computational complexities are faced. One of these complexity is due to the behavior of barrier function B ; for small values of the parameter λ , ill-conditioning and round-off errors may face while minimizing $h(y) + \lambda B(y)$ for each $y \in Y$, that is as the boundary of the feasible set $\{y : k(y) \leq 0\}$ is approached. In fact, as the boundary approaches, and process uses discrete steps, a step leading outside the region $\{y : k(y) \leq 0\}$ may show a decrease in the value of $h(y) + \lambda B(y)$, an incorrect result. Thus, the constraint $k(y)$

must be to guarantee for the movement outside the feasible set.

To understand the potential ill-conditioning effect formally, we can examine the eigen values of the Hessian matrix for the objective function in (3.28) at the optimal solution y_λ as $\lambda \rightarrow 0^+$. Using (3.29) and the assumptions h, k, ϕ are twice continuously differentiable, the Hessian matrix is given by

$$[\nabla^2 h(y_\lambda) + \sum_{i=1}^m (w_\lambda)_i \nabla^2 k_i(y_\lambda)] + \lambda \sum_{i=1}^m \phi''[k_i(y_\mu)] \nabla k_i(y_\lambda) \nabla k_i(y_\lambda)^T \quad (3.30)$$

As $\lambda \rightarrow 0^+$, we get $\{y_\lambda\} \rightarrow y^*$ (for the convergent sub-sequence) and letting that y^* is regular, we obtained the optimal set of Lagrange multipliers $w_\lambda \rightarrow w^*$. So that, the expression in (3.30) goes to $\nabla^2 h_\lambda(y^*)$. Consequently, the Hessian matrix $\nabla^2 h_\lambda(y^*)$ increasingly ill-conditioned as the solution approaches. This ill-condition structure occurs even if the constrained optimization problem is in well condition.

Remark 3. From the two methods, penalty and barrier, the penalty methods considered preferable. The primary reasons are;

1. Barrier methods cannot deal with equality constraints without some modifications to the basic approach.
2. Barrier methods demand a feasible starting point and finding such a point often presents formidable difficulties in and of itself.
3. Barrier methods require that the search never leave the feasible region. This significantly increases the computational effort associated with the line search segment of the algorithm.

Conclusion

The basic idea, in penalty methods is to eliminate some or all of the constraints and add penalty term to the objective function which prescribes a high cost to infeasible points. The parameter associated with penalty method determines the severity of the penalty and as a consequence the extent to which the resulting unconstrained problem approximates the original constrained problem. This methods are not used in cases where feasibility must be maintained, for instance, if the objective function is undefined or ill-conditioned outside the feasible region. Moreover, this method is also called interior penalty method.

Besides to this, another basic method, barrier method, which is also called interior penalty method, in which that an optimal solution requires that a constraint be active (i.e.,tight) so that this optimal solutions lies on the boundary between feasibility and infeasibility. Knowing this, a penalty is applied to feasible solutions when the constraint is not active, so-called "interior solutions".

However, penalty and barrier methods suffer from some computational difficulties , in the absence of alternative software especially for no-derivative problems they are still recommended. Although penalty and barrier methods met with great initial success, their slow rates of convergence due to ill-conditioning of the associated Hessian led researchers to pursue other approaches.

Of the two methods, the penalty method considered preferable. The primary reasons are barrier penalty methods cannot deal with equality constraints without cumbersome modifications to the basic approach, demand a feasible starting point. Finding such a point often presents formidable difficulties in and of itself and require that the search in penalty method never leave the feasible region. This significantly increases the computational effort associated with constrained optimization problem.

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