

A SAGBI BASIS FOR SOME SUBALGEBRAS OF POLYNOMIAL RINGS

by

Dawit Solomon Tadesse

e-mail: devatasol@gmail.com

A Thesis

submitted to

the Department of Mathematics

Presented in Partial Fulfillment
of the Requirements for the Degree of
Doctor of Philosophy in Mathematics

Addis Ababa University



Addis Ababa, Ethiopia

March 13, 2019

©

by

Dawit Solomon Tadesse

March 13, 2019

All Rights Reserved

Department of Mathematics
College of Natural Sciences
Addis Ababa University

This is to certify that the thesis prepared by Dawit Solomon Tadesse entitled: “*A SAGBI BASIS FOR SOME SUBALGEBRAS OF POLYNOMIAL RINGS*” submitted in partial fulfilment of the requirements for the Degree of Doctor of Philosophy in Mathematics complies with the regulations of the University and meets the accepted standards with respect to originality and quality.

Signed by the Examining Committee:

_____	_____	_____
Examiner	Signature	Date
_____	_____	_____
Examiner	Signature	Date
_____	_____	_____
Advisor	Signature	Date
_____	_____	_____
Co-Advisor	Signature	Date

Chair of Department or Graduate Program Coordinator

Abstract

A SAGBI BASIS FOR SOME SUBALGEBRAS OF POLYNOMIAL RINGS

Dawit Solomon Tadesse

Addis Ababa University, 2018

The term “SAGBI” is an acronym for “Subalgebra Analogue to Gröbner Bases for Ideals”. There exist finitely generated subalgebras of $\mathbb{k}[x_1, x_2, \dots, x_n]$ which have no finite SAGBI basis with respect to any monomial order. There are also subalgebras which may or may not have finite SAGBI basis depending on the monomial order defined on $\mathbb{k}[x_1, x_2, \dots, x_n]$. We know that, there are uncountably many monomial orders defined on $\mathbb{k}[x_1, x_2, \dots, x_n]$ for $n \geq 2$. It is still an important open problem to classify subalgebras that have a finite SAGBI basis. In addition to this, two or more generators of a subalgebra may not form a SAGBI basis. Torstensson et al. provide sufficient and necessary conditions for two generators form a SAGBI basis, in the case of univariate polynomial ring. An other important open problem is to provide a sufficient and/or necessary condition when three or more generators form a SAGBI basis in the univariate polynomial ring as well as two or more generators form a SAGBI basis in a multivariate polynomial ring. This thesis provides sufficient conditions when two generators of a subalgebra form a SAGBI basis in a multivariate polynomial ring. We also investigate sufficient and necessary condition when generators with consecutive degrees form a SAGBI basis in the univariate polynomial ring. Finally we conjecture that subalgebras generated by two polynomials have a finite SAGBI basis with respect to any monomial order. We prove our conjecture in certain cases.

Dedication

For My Grandmothers Dinkinesh Desta and Etabeba Weldesemaet, with all my love and Gratitude.

Acknowledgements

First of all I would like to thank my Almighty God for being my ultimate source of wisdom, courage, and inspiration.

I would like to express my sincere appreciation to my supervisor Dr. Mohammed S. Tesemma, for teaching (introducing) me Gröbner basis, SAGBI basis, invariant theory and Lie algebra, for showing me much patience, energy, and friendship throughout my studies and for his continual help, guidance and encouragement.

I would like to extend my thank to my Co-supervisor Dr. Berhanu Bekele, for teaching me Commutative algebra and Gröbner bases, for all his encouragement and guidance throughout my stay, for his valuable comments and questions whenever I have presented my findings to our “Algebra seminar group”.

I would like to thank professor Haohao Wang for carefully reading the complete thesis and giving valuable comments.

I would like to thank professor U.M.Swamy, Dr. K.Venkateswarlu, Dr. Gebrelassie Baraki, Professor Omer Yayenie, Dr. Mengistu Goa, Dr. Semu Mitiku, and Dr. Tadesse Abdi, for giving me a wide range of courses.

I would like to thank all the staff members of Mathematics department at Addis Ababa university and University of Gondar for their cooperation and encouragement. I gratefully acknowledge the generous financial support of University of Gondar, Addis Ababa University and International Science Program (ISP) .

I would like to thank Dr. Litgebih Wendie, Dr. Dawit Chernet, Solomon Belete, Mulugeta Habte, Eyerusalem W\Yohannes, Girum Aklilu, Megibaru Esayiyas, Dr. Ketsela Hailu, Dr. Dereje Kifle and my other colleagues for many helpful conversations.

Finally I would like to thank my wife w\ro Meseret Nega and My daughters Mahidere Dawit and Zema Dawit, who sacrifices a lot and missing me for long time. My warmest thank goes to my mother Negedework, my father Solomon, my brothers Abinet, Yohannes, Sofonias, Efrem, Frew and Negash, My sisters Lemlem, Biruktwit, Birtukan, Betelhem, Hermela and Hiwot, my dear friend Endalcachew for their encouragement, appreciation and financial support.

Dawit Solomon Tadesse
Addis Ababa, Ethiopia.

Table of Contents

DEDICATION	iv
ACKNOWLEDGMENT	v
NOTATIONS	vii
INTRODUCTION	viii
1 Preliminaries	1
1.1 Polynomial \mathbb{k} -algebras	1
1.2 Monomial Orders and Initial Algebra	7
1.2.1 Monomial orders	8
1.2.2 Initial Algebra	11
2 Fundamentals of SAGBI bases	13
2.1 Definition and Examples of SAGBI basis	13
2.2 Subduction	20
2.3 SAGBI basis test	26
2.4 SAGBI basis and Integral Extension	30
3 SAGBI bases in a univariate polynomial ring	35
3.1 Existence of finite SAGBI basis	36
3.2 Two generators' Case	38
3.3 Generators with Consecutive degrees	39

4	SAGBI basis in a Multivariate Polynomial Ring	53
4.1	SAGBI basis for a \mathbb{k} subalgebra generated by two polynomials	53
4.2	Further on SAGBI bases for \mathbb{k} -subalgebras generated by two polynomials	60
	Future Work	69
	Bibliography	70
	A Counter example	75

NOTATIONS

Unless and otherwise stated, through out this thesis, We use the following notations.

\mathbb{k}	A field with characteristic zero.
\mathbb{N}	The set of all non-negative integers.
$\mathbb{N}_{\geq k}$	The set of all non-negative integers greater than or equal to k .
\mathbb{Q}	The field of rational numbers.
\mathbb{Q}_0^+	The set of nonnegative rational numbers.
$\mathbb{k}[\mathbf{x}]$	A polynomial ring over \mathbb{k} in n indeterminates x_1, x_2, \dots, x_n .
$\mathbf{x}^{\mathbf{a}}$	A monomial $x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$ where $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{N}^n$.
\mathcal{M}	The set of all monomials in $\mathbb{k}[\mathbf{x}]$.
Ω	The collection of all possible monomial orders on $\mathbb{k}[\mathbf{x}]$.
$ \mathbf{a} $	The total degree of a monomial $\mathbf{x}^{\mathbf{a}}$.
$\deg \mathbf{x}^{\mathbf{a}}$	The multi-degree of a monomial $\mathbf{x}^{\mathbf{a}}$.
\mathcal{G}_{mon}	The set of all \mathcal{G} – <i>power</i> products.
$\mathbb{k}[\mathcal{G}]$	The \mathbb{k} – <i>subalgebra</i> generated by \mathcal{G} .
$\langle \mathbf{a} \rangle$	An additive monoid generated by \mathbf{a} .
$\text{in}_{\succeq} f$	The initial (leading) monomial of f with respect to $\succeq \in \Omega$.
$\text{lt}_{\succeq} f$	The leading term of f with respect to $\succeq \in \Omega$.
$\text{lc}_{\succeq} f$	The leading Coefficient of f with respect to $\succeq \in \Omega$.
$\text{In}_{\succeq} \mathcal{R}$	The initial Algebra of a \mathbb{k} – <i>subalgebra</i> \mathcal{R} of $\mathbb{k}[\mathbf{x}]$ w.r.t $\succeq \in \Omega$.
$f \xrightarrow{B} g$	g is a first subductum of f over B .
$f \xrightarrow{B^*} g$	g is a final subductum of f over B .

INTRODUCTION

The notion of SAGBI basis was introduced at the end of 1980's by Robbiano and Sweedler [24], and independently by Kapur and Madelener [14]. The term **SAGBI** is an acronym for “**S**ubalgebra **A**nalogue to **G**röbner **B**ases for **I**deals”.

Robbiano and Sweedler (1990) realized that, like Gröbner bases, SAGBI bases generate the subalgebra. The Membership problem is also the same in both Gröbner theory and SAGBI theory. They have also developed the subduction algorithm and the SAGBI basis test by introducing the SAGBI analog of S - polynomials called T-polynomials [24].

Unfortunately, the SAGBI analog of Buchberger algorithm is discouraging due to the existence of subalgebras which have no finite SAGBI basis. Actually, a SAGBI basis for a subalgebra is not a carbon copy of a Gröbner basis for an ideal. Unlike ideals, subalgebras of a finitely generated algebra may not be finitely generated. For example, the subalgebra $\mathcal{R} = \mathbb{k}[x, xy, xy^2, \dots]$ of $\mathbb{k}[x, y]$ is not finitely generated, and hence there exists a subalgebra of $\mathbb{k}[x_1, x_2, \dots, x_n]$, for $n \geq 2$, which has no a finite SAGBI basis whatsoever. This is not surprising since there exists an infinitely generated submonoid of \mathbb{N}^n for $n \geq 2$, and hence the associated monoid algebra can not be finitely generated. There is also a subalgebra of $\mathbb{k}[x_1, x_2, \dots, x_n]$, $n \geq 2$, which has a finite SAGBI basis with one monomial order and has no finite SAGBI basis with another monomial order [24]. That is, a subalgebra may or may not have a finite SAGBI basis depending on the monomial order defined on $\mathbb{k}[x_1, x_2, \dots, x_n]$, and there are infinitely many monomial orders defined on $\mathbb{k}[x_1, x_2, \dots, x_n]$ for $n \geq 2$. It is still an important open problem to classify subalgebras that have a finite SAGBI basis.

In the rest of this introduction we will state some of the known results in SAGBI bases that interest our research and also make the statement of the dissertation.

Robbiano and Sweedler [24] also pointed out that SAGBI basis is not invariant under ring automorphism. That is, a subalgebra with finite SAGBI basis can be isomorphic to a subalgebra which has no finite SAGBI basis.

Francois Ollivier [23] conjectured that “If B is finitely generated integrally closed \mathbb{k} - subalgebra of $\mathbb{k}[x_1, x_2, \dots, x_n]$, then B has a finite SAGBI basis with any admissible monomial ordering”. He partially answered his conjecture for 2-dimensional

graded algebra.

Anna Torstensson ([28],[29]), as part of her PhD thesis, studied SAGBI bases for subalgebras in the univariate case. One of her results was a necessary and sufficient condition when two polynomials form a SAGBI basis.

Patrik Nordbeck [22] studied when SAGBI bases are compatible with composition of polynomials. J. Alem Kahn [16] further studied the Nordbeck Composition problem under different monomial orders.

Our work is mainly related to the work of Torstensson. In particular we will show the following in this thesis.

- i) Show that a certain family of pair of polynomials, where the subalgebra they generate have a finite SAGBI basis.
- ii) Show necessary and sufficient condition when generators with consecutive degrees form a SAGBI basis in the univariate polynomial ring.

This thesis is divided into four chapters. The first chapter provides all the necessary preliminaries which are useful in our discussions in the main text of the thesis. We made a discussion on the basics of \mathbb{k} – *subalgebras* of a \mathbb{k} – *algebra*, monomial orders, and initial algebra of a \mathbb{k} – *subalgebra* of a polynomial ring.

The second chapter deals with the fundamental concepts of SAGBI basis. We divided this chapter in to four sections. The first section of this chapter provides the definition and some important examples of a SAGBI basis. Proposition 2.1.5 yields some equivalent statements to the definition of SAGBI basis. The second section of this chapter explains the subduction process and some important properties of SAGBI basis. The third section of this chapter is concerned with the SAGBI basis test and the SAGBI basis construction. The last section of this chapter focuses on finiteness of SAGBI basis, we will see the application of E.Artin-Tate theorem to construct a sufficient condition when a subalgebra has a finite SAGBI basis.

The third and the fourth chapters present the main findings of the research. In the third chapter we discuss on the sufficient and necessary conditions when three

or more generators of a subalgebra form a SAGBI basis in the case of univariate polynomial ring.

In the fourth chapter we focus on the multivariate \mathbb{k} – *subalgebras*. We have sufficient condition for two generators form a SAGBI basis in the case of multivariate polynomial ring. In addition we give a conjecture, “any \mathbb{k} – *subalgebra* generated by two polynomials has a finite SAGBI basis”. Finally we present our future plan.

Chapter 1

Preliminaries

In this chapter, we assemble the basic and necessary background from commutative algebra and SAGBI bases theory. Some theorems in SAGBI bases theory that we refer frequently are also stated in this chapter. For further reading and proofs references are also provided. Examples will also be presented along the way. Through out this thesis, \mathbb{k} denotes a field of characteristic zero.

1.1 Polynomial \mathbb{k} -algebras

In this section, we briefly introduce the notion of \mathbb{k} -subalgebras and \mathbb{k} -affine subalgebras of a polynomial ring $\mathbb{k}[\mathbf{x}]$. Given a monomial $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$, the lattice point $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{N}^n$, where \mathbb{N} stands for the set of non-negative integers, is called the multidegree of $\mathbf{x}^{\mathbf{a}}$ and we denote it by $\deg(\mathbf{x}^{\mathbf{a}})$. The total degree of $\mathbf{x}^{\mathbf{a}}$, denoted by $|\mathbf{a}|$, is defined as $|\mathbf{a}| = a_1 + a_2 + \dots + a_n$.

Let A be a ring. An A -algebra is a ring B together with a homomorphism $f : A \rightarrow B$. If A is a field, then $\ker(f) = \{0\}$, as being an ideal of A . Hence f is injective. So, we can consider A as a subring

of B ($A \cong f(A)$). Therefore, over the field \mathbb{k} , we define algebra as follows.

Definition 1.1.1. A \mathbb{k} – algebra is a ring containing the field \mathbb{k} as a sub-ring [[9], p.2].

For example, our polynomial ring $\mathbb{k}[\mathbf{x}]$ is a \mathbb{k} – algebra.

Definition 1.1.2. A subset \mathcal{R} of a \mathbb{k} – algebra B is said to be a \mathbb{k} – subalgebra of B if it is a subring of B which contains \mathbb{k} .

Example 1.1.3. .

Let $\mathcal{R} := \{0, f : f \in \mathbb{k}[\mathbf{x}] \text{ and } \mathbf{x}^{\mathbf{a}} \in \text{supp}(f)^1 \text{ implies } |\mathbf{a}| \text{ is even} \}$. Then \mathcal{R} is a \mathbb{k} – subalgebra of $\mathbb{k}[\mathbf{x}]$.

Remark 1.1.4. Let \mathcal{G} be a nonempty subset of $\mathbb{k}[\mathbf{x}]$. The set

$$\mathcal{G}_{mon} := \left\{ \prod_{i=1}^t (g_i)^{a_i} : g_i \in \mathcal{G} \text{ and } 0 \neq t, a_i \in \mathbb{N} \right\}$$

is a multiplicative monoid generated by \mathcal{G} .

An element of \mathcal{G}_{mon} is called a \mathcal{G} – power product or \mathcal{G} – monomial.

Definition 1.1.5. Let \mathcal{G} be a nonempty subset of $\mathbb{k}[\mathbf{x}]$. The \mathbb{k} – subalgebra of $\mathbb{k}[\mathbf{x}]$ generated by \mathcal{G} is the smallest \mathbb{k} – subalgebra of $\mathbb{k}[\mathbf{x}]$ that contains \mathcal{G} . It is denoted by $\mathbb{k}[\mathcal{G}]$.

Explicitly, the \mathbb{k} – subalgebra generated by \mathcal{G} can be expressed as $\mathbb{k}[\mathcal{G}] := \{ \sum_{i=1}^r (k_i n_i) : k_i \in \mathbb{k}, n_i \in \mathcal{G}_{mon} \}$, a set of finite \mathbb{k} - linear combinations of \mathcal{G} – power products. In other words, $\mathbb{k}[\mathcal{G}]$ is a \mathbb{k} – subspace of $\mathbb{k}[\mathbf{x}]$ with basis \mathcal{G}_{mon} . A \mathbb{k} – subalgebra \mathcal{R} of $\mathbb{k}[\mathbf{x}]$ is said to be **finitely generated** (\mathbb{k} – affine) if there exists a finite subset \mathcal{G} of \mathcal{R} such that $\mathcal{R} = \mathbb{k}[\mathcal{G}]$.

¹supp(f) = $\{ \mathbf{x}^{\mathbf{a}} : \mathbf{x}^{\mathbf{a}} \text{ is a monomial in } f \text{ with nonzero coefficients} \}$.

Example 1.1.6. Consider example 1.1.3, where \mathcal{R} is the subalgebra generated by polynomials of even total degree. Then \mathcal{R} is a finitely generated \mathbb{k} -subalgebra of $\mathbb{k}[\mathbf{x}]$, generated by $\mathcal{G} = \{x_i x_j : 1 \leq i, j \leq n\}$.

Since $\mathcal{G} \subseteq \mathcal{R}$ and \mathcal{R} is a \mathbb{k} -subalgebra of $\mathbb{k}[\mathbf{x}]$, we have $\mathbb{k}[\mathcal{G}] \subseteq \mathcal{R}$. To show $\mathcal{R} \subseteq \mathbb{k}[\mathcal{G}]$, assume $0 \neq f \in \mathcal{R}$. If $x_1^{a_1} \dots x_n^{a_n} \in \text{supp}(f)$ then $a_1 + a_2 + \dots + a_n$ is even.

Case-1: If each a_i is even, then $x_1^{a_1} \dots x_n^{a_n} = (x_1^2)^{t_1} \dots (x_n^2)^{t_n} \in \mathcal{G}_{\text{mon}}$

case-2: If a_i is odd for some i , then there should be a_j which is also odd. In other words, the number of odd coordinates in (a_1, \dots, a_n) is even. Hence $x_i^{a_i} x_j^{a_j} = x_i^{2t_i} x_j^{2t_j} x_i x_j \in \mathcal{G}_{\text{mon}}$, which directly implies $x_1^{a_1} \dots x_n^{a_n} \in \mathcal{G}_{\text{mon}}$.

Both the above cases show that each monomial in f is contained in \mathcal{G}_{mon} and so, $f \in \mathbb{k}[\mathcal{G}]$. Hence $\mathcal{R} \subseteq \mathbb{k}[\mathcal{G}]$. Consequently, $\mathcal{R} = \mathbb{k}[\mathcal{G}]$. Moreover, we can simply observe that $\{x_i x_j : 1 \leq i, j \leq n\}$ is a finite set of cardinality $\frac{n^2+n}{2}$. \square

Remark 1.1.7. If $\mathcal{R} = \mathbb{k}[\mathcal{G}]$ is a \mathbb{k} -affine subalgebra of $\mathbb{k}[\mathbf{x}]$, then we can choose a set of finite generators from \mathcal{G} .

Note that, if $\langle f_1, f_2, \dots, f_r \rangle$ is an ideal generated by $\{f_1, f_2, \dots, f_r\}$ and $f_i \mid f_j$ for some $i \neq j$, then we can remove f_j from the generating set. For example, since $x^2 \mid x^3$ we have $\langle x^2, x^3 \rangle = \langle x^2 \rangle$. However, since x^3 can not be expressed as a power product of x^2 , $\mathbb{k}[x^2, x^3] \neq \mathbb{k}[x^2]$. But, if g_1 is a power product of g_2, \dots, g_m , where $g_i s'$ are in the generating set of the subalgebra, then we can remove g_1 from the generating set.

Example 1.1.8. Given a positive integer p , and $\mathcal{R} = \mathbb{k}[y^p, x, xy, xy^2, \dots]$.

For each $m \geq p$, by applying division algorithm, $m = qp + r$ for some $q, r \in \mathbb{N}$ with $r \leq p - 1$. So, $xy^m = (xy^r)y^{qp}$. Hence $\{y^p, x, xy, xy^2, \dots, xy^{p-1}\}$ is a finite subset of $\{y^p, x, xy, xy^2, xy^3, \dots\}$ which can generate \mathcal{R} .

Example 1.1.9. $\mathcal{R} = \mathbb{k}[x, xy, xy^2, xy^3, \dots]$ is a \mathbb{k} -subalgebra of $\mathbb{k}[x, y]$ which is not \mathbb{k} -affine.

To the contrary assume \mathcal{R} is \mathbb{k} -affine. By remark 1.1.7, we can take a finite generators from $\{x, xy, xy^2, xy^3, \dots\}$. Assume $A = \{x, xy, xy^2, \dots, xy^m\}$, for some $m \in \mathbb{N}$, is a set of finite generators of \mathcal{R} . Since xy^{m+1} is a monomial in $\mathcal{R} \setminus A$, it is a finite product of elements of A . But the exponent of x in xy^{m+1} is 1 and each element in A has x as a factor. So, xy^{m+1} must be one of the element's of A . This is a contradiction with $xy^{m+1} \notin A$. Therefore, \mathcal{R} is not \mathbb{k} -affine. \square

Proposition 1.1.10. A \mathbb{k} -Algebra \mathcal{R} is finitely generated if and only if $\mathcal{R} \cong \mathbb{k}[x_1, x_2, \dots, x_m]/I$ for some $m \in \mathbb{N}$ and an ideal I of $\mathbb{k}[x_1, x_2, \dots, x_m]$.

Proof. If $\mathcal{R} \cong \mathbb{k}[x_1, x_2, \dots, x_m]/I$, since $\mathbb{k}[x_1, \dots, x_m]/I$ is a finitely generated \mathbb{k} -algebra, generated by $x_1 + I, \dots, x_m + I$, we can say that \mathcal{R} is finitely generated. Conversely, let \mathcal{R} be a \mathbb{k} -algebra finitely generated by c_1, \dots, c_m . Then $\phi : \mathbb{k}[x_1, \dots, x_m] \rightarrow \mathcal{R}$, that send $x_j \mapsto c_j$, is a ring epimorphism. So by the fundamental theorem of homomorphism, $\mathcal{R} \cong \mathbb{k}[x_1, x_2, \dots, x_m]/I$ where $I = \ker \phi$. \square

Definition 1.1.11. An A -module M is said to be Noetherian if and only if every ascending chain of submodules of M is stationary.

From Module theory we know that, an A -module M is Noetherian if and only if every submodule of M is finitely generated. We have also that, every finitely generated Module over a Noetherian ring is Noetherian. Interested readers can read further on Noetherian Modules in [2].

Proposition 1.1.12. *Let $A \subseteq B \subseteq C$ be rings.*

- a) *If B is a finitely generated A – module, then it is also a finitely generated A – algebra.*
- b) *If B is a finitely generated A – algebra and C is a finitely generated B – algebra, then C is a finitely generated A – algebra.*

Proof.

a) Suppose B is generated by b_1, b_2, \dots, b_m as an A – module.

Therefore, $B = b_1A + b_2A + \dots + b_mA \subseteq A[b_1, b_2, \dots, b_m] \subseteq B$.

Hence $B = A[b_1, b_2, \dots, b_m]$.

b) Suppose $B = A[b_1, b_2, \dots, b_m]$ and $C = B[c_1, c_2, \dots, c_k]$ for some $b_i \in B$ and $c_j \in C$.

Claim: $C = A[b_1, b_2, \dots, b_m, c_1, c_2, \dots, c_k]$.

Since all the b_i 's and c_j 's are in C and $A \subseteq C$, we have

$$A[b_1, b_2, \dots, b_m, c_1, c_2, \dots, c_k] \subseteq C.$$

So, it is enough to show that $C \subseteq A[b_1, b_2, \dots, b_m, c_1, c_2, \dots, c_k]$.

$$\begin{aligned}
f \in C &\Rightarrow f = \sum_{i=1}^t g_i s_i \text{ where } g_i \in B \text{ and } s_i \in \{c_1, c_2, \dots, c_k\}_{\text{mon}}. \\
&\Rightarrow f = \sum_{i=1}^t \left(\sum_{j=1}^{r_i} a_{i_j} h_{i_j} \right) s_i \text{ where } a_{i_j} \in A \text{ and } h_{i_j} \in \{b_1, b_2, \dots, b_m\}_{\text{mon}}. \\
&\Rightarrow f = \sum_{i=1}^t \sum_{j=1}^{r_i} a_{i_j} (h_{i_j} s_i), \text{ and } h_{i_j} s_i \in \{b_1, b_2, \dots, b_m, c_1, c_2, \dots, c_k\}_{\text{mon}} \\
&\Rightarrow f \in A[b_1, b_2, \dots, b_m, c_1, c_2, \dots, c_k]
\end{aligned}$$

Hence proposition (b) holds. \square

From Hilbert basis theorem ², We know that any ideal of $\mathbb{k}[\mathbf{x}]$ is finitely generated. But example 1.1.9 shows that, unlike ideals \mathbb{k} – *subalgebras* may not be finitely generated. The following proposition gives a sufficient condition for a \mathbb{k} – *subalgebra* of \mathbb{k} – *affine algebra* is \mathbb{k} – *affine*.

Proposition 1.1.13. (*E. Artin-Tate Theorem*) *Let A and B be two \mathbb{k} – algebras such that $A \subseteq B$. Assume that B is a finitely generated \mathbb{k} – algebra and a finitely generated A – module. Then A is \mathbb{k} – affine.*

Proof. Let $\{f_1, f_2, \dots, f_m\}$ be generators of B as a \mathbb{k} -algebra, and $\{g_1, g_2, \dots, g_t\}$ be generators of B as an A -module. Since f_r and $g_i g_j$ are elements of B , we have the following expression.

$$f_r = \sum_{s=1}^t a_{rs} g_s, \text{ where } a_{rs} \in A \quad (1.1.1)$$

$$g_i g_j = \sum_{k=1}^t a_{ijk} g_k, \text{ where } a_{ijk} \in A \quad (1.1.2)$$

²Hilbert basis theorem :- Let \mathbf{R} be a noetherian ring. Then the polynomial ring $\mathbf{R}[\mathbf{x}]$ is again noetherian.

Let $A_0 = \mathbb{k}[a_{rs}, a_{ijk}]$. Clearly A_0 is finitely generated \mathbb{k} -subalgebra of A . Since \mathbb{k} is a field (and hence Noetherian ring), A_0 is Noetherian. Now let us proof that B is Noetherian A_0 -module.

Observe that each element of B is a polynomial in f_r with coefficients in \mathbb{k} . Substituting (1.1.1) and making repeated use of (1.1.2) shows that each element of B is a linear combination of the g_j with coefficients in A_0 . Hence B is finitely generated A_0 -module. So, since A_0 is Noetherian, B is Noetherian A_0 -module.

Since every submodules of a Noetherian module is finitely generated, A is finitely generated A_0 -module. Hence A is finitely generated A_0 -algebra.

Now we have, A is finitely generated A_0 -algebra and A_0 is finitely generated \mathbb{k} -algebra. Hence A is finitely generated \mathbb{k} -algebra. \square

1.2 Monomial Orders and Initial Algebra

Ordering monomials is a mandatory process in the study of “SAGBI” basis. In the case of the univariate polynomial ring, we can arrange monomials simply by comparing their degrees. But this is not the way of comparing monomials in the case of multivariate polynomial ring. So, we need to define an ordering on the set of monomials which satisfies certain properties.

In this section, we concisely describe the desirable properties of monomial orders, and define the initial algebra for a \mathbb{k} – *subalgebra*. Through out this thesis, we denote the set of monomials by \mathcal{M} , and the set of monomial orders defined on $\mathbb{k}[\mathbf{x}]$ by Ω .

1.2.1 Monomial orders

Let B be a nonempty set. A binary relation \succeq on B is said to be a **partial order** on B if it is reflexive ($x \succeq x, \forall x \in B$), anti-symmetric ($x \succeq y \succeq x \Rightarrow x = y, \forall x, y \in B$) and transitive ($x \succeq y \succeq z \Rightarrow x \succeq z, \forall x, y, z \in B$). A partial order on B is said to be a total order if any two elements in B are comparable, that is, $\forall x, y \in B$ either $x \succeq y$ or $y \succeq x$.

We write $x \succ y$ if $x \succeq y$ and $x \neq y$. We also write $x \preceq y$ if and only if $y \succeq x$.

Definition 1.2.1. A monomial order on $\mathbb{k}[\mathbf{x}]$ is a total order \succeq on \mathcal{M} which satisfies the following conditions:

- 1) $\mathbf{x}^{\mathbf{a}} \succeq 1$ for all $\mathbf{x}^{\mathbf{a}} \in \mathcal{M}$ (1 is the smallest element in \mathcal{M})
- 2) $\mathbf{x}^{\mathbf{a}} \succ \mathbf{x}^{\mathbf{b}}$ implies $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{c}} \succ \mathbf{x}^{\mathbf{b}}\mathbf{x}^{\mathbf{c}}$ for all $\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}}, \mathbf{x}^{\mathbf{c}} \in \mathcal{M}$ (\succeq is compatible with multiplication)

Example 1.2.2. In each of the following familiar examples of monomial orders, assume $x_1 \succ x_2 \succ \dots \succ x_n$.

1. We define the lexicographical order, \succeq_{lex} , on \mathcal{M} as follows:

For $\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}} \in \mathcal{M}$;

$\mathbf{x}^{\mathbf{a}} \succeq_{lex} \mathbf{x}^{\mathbf{b}} \Leftrightarrow$ the left-most nonzero coordinate of $\mathbf{a} - \mathbf{b}$ is positive .

2. We define the degree lexicographical order, \succeq_{deglex} , on \mathcal{M} as follows:

For $\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}} \in \mathcal{M}$;

$$\mathbf{x}^{\mathbf{a}} \succeq_{deglex} \mathbf{x}^{\mathbf{b}} \Leftrightarrow \begin{cases} |\mathbf{a}| > |\mathbf{b}| \text{ or} \\ |\mathbf{a}| = |\mathbf{b}| \text{ and } \mathbf{x}^{\mathbf{a}} \succeq_{lex} \mathbf{x}^{\mathbf{b}} \end{cases}$$

3. We define the degree reverse lexicographical order, \succeq_{degrev} , on \mathcal{M} as follows:

For $\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}} \in \mathcal{M}$;

$$\mathbf{x}^{\mathbf{a}} \succeq_{degrev} \mathbf{x}^{\mathbf{b}} \Leftrightarrow \begin{cases} |\mathbf{a}| > |\mathbf{b}| \text{ or} \\ |\mathbf{a}| = |\mathbf{b}| \text{ and the right-most nonzero coordinate} \\ \text{of } \mathbf{b} - \mathbf{a} \text{ is positive} \end{cases}$$

4. Fix $\mathbf{w} \in \mathbb{R}_{\geq 0}^n$. Define a weight order $\succeq_{\mathbf{w}}$ on \mathcal{M} as follows:

For $\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}} \in \mathcal{M}$;

$$\mathbf{x}^{\mathbf{a}} \succeq_{\mathbf{w}} \mathbf{x}^{\mathbf{b}} \Leftrightarrow \begin{cases} \mathbf{a} \cdot \mathbf{w} > \mathbf{b} \cdot \mathbf{w} \text{ or} \\ \mathbf{a} \cdot \mathbf{w} = \mathbf{b} \cdot \mathbf{w} \text{ and } \mathbf{x}^{\mathbf{a}} \succeq_{lex} \mathbf{x}^{\mathbf{b}} \end{cases}$$

5. In (iv) above, if the coordinates of \mathbf{w} are linearly independent over the field \mathbb{Q} , the monomial order induced by the weight order $\succeq_{\mathbf{w}}$ becomes simply, for $\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}} \in \mathcal{M}$,

$$\mathbf{x}^{\mathbf{a}} \succeq_{\mathbf{w}} \mathbf{x}^{\mathbf{b}} \Leftrightarrow \mathbf{a} \cdot \mathbf{w} \geq \mathbf{b} \cdot \mathbf{w}.$$

Proposition 1.2.3. *Let $\succeq \in \Omega$. Then the following hold:*

1. *If $\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}} \in \mathcal{M}$ with $\mathbf{x}^{\mathbf{a}} \mid \mathbf{x}^{\mathbf{b}}$, then $\mathbf{x}^{\mathbf{b}} \succeq \mathbf{x}^{\mathbf{a}}$*
2. *Any chain of the form $\mathbf{x}^{\beta_1} \succ \mathbf{x}^{\beta_2} \succ \mathbf{x}^{\beta_3} \succ \mathbf{x}^{\beta_4} \succ \dots$ is finite.*

By Applying Dickson's lemma ³, see [[6], p.72], the definition of monomial order is equivalent if we replace (1) by; \mathcal{M} is well ordered ⁴ under \succeq . That is, A monomial order is a well order that preserves multiplication.

³Dicksons lemma: Let I be a monomial ideal. Then each set of monomial generators of I contains a finite set which generates I .

⁴ \mathcal{M} is well ordered means every nonempty subset of \mathcal{M} has a least element.

Proof. (i) Suppose $\mathbf{x}^a \mid \mathbf{x}^b$. There exist $\mathbf{x}^c \in \mathcal{M}$ such that $\mathbf{x}^b = \mathbf{x}^a \mathbf{x}^c$. Since $\succeq \in \Omega$ and $\mathbf{x}^c \in \mathcal{M}$ we have $\mathbf{x}^c \succeq 1$. This directly implies that $\mathbf{x}^b = \mathbf{x}^a \mathbf{x}^c \succeq \mathbf{x}^a 1 = \mathbf{x}^a$.

(ii) Assume $\mathbf{x}^{\beta_1} \succ \mathbf{x}^{\beta_2} \succ \mathbf{x}^{\beta_3} \succ \mathbf{x}^{\beta_4} \succ \dots$, and consider the monomial ideal I generated by $\{\mathbf{x}^{\beta_1}, \mathbf{x}^{\beta_2}, \mathbf{x}^{\beta_3}, \dots\}$. By Dickson's lemma, there is an $m \in \mathbb{N}$ such that I is generated by $\{\mathbf{x}^{\beta_1}, \mathbf{x}^{\beta_2}, \mathbf{x}^{\beta_3}, \dots, \mathbf{x}^{\beta_m}\}$.

Claim:- For any $i \in \mathbb{N}$, $\mathbf{x}^{\beta_{m+i}} = \mathbf{x}^{\beta_m}$.

Since $\mathbf{x}^{\beta_{m+i}} \in I$, we have that

$$\mathbf{x}^{\beta_{m+i}} = \sum_{j=1}^m h_j \mathbf{x}^{\beta_j}, \text{ where } h_j \in \mathbb{k}[\mathbf{x}] \text{ for each } j \in \{1, 2, \dots, m\} \quad (1.2.1)$$

Observe that, each monomial in $h_j \mathbf{x}^{\beta_j}$ is divisible by \mathbf{x}^{β_j} . Thus every monomial of the right hand side of equation (1.2.1) is divisible by \mathbf{x}^{β_j} for some $j \in \{1, 2, \dots, m\}$. Moreover, $\mathbf{x}^{\beta_{m+i}}$ must appear as the power product of a monomial on the right side of equation (1.2.1). Therefore, $\mathbf{x}^{\beta_{m+i}}$ is divisible by \mathbf{x}^{β_j} for some $j \in \{1, 2, \dots, m\}$. Hence by (i) above, $\mathbf{x}^{\beta_{m+i}} \succeq \mathbf{x}^{\beta_j}$ for some $j \in \{1, 2, \dots, m\}$. Thus, for each $i \in \mathbb{N}$, $\mathbf{x}^{\beta_{m+i}} \succeq \mathbf{x}^{\beta_j} \succeq \mathbf{x}^{\beta_m} \succeq \mathbf{x}^{\beta_{m+i}}$. Hence the claim. \square

It is natural to ask, how many monomial orders can be defined on \mathcal{M} . The following proposition definitely answer this question.

Proposition 1.2.4 (The cardinality of Ω). *1. There is unique monomial order defined on a polynomial ring in one variable over \mathbb{k} . (For $n = 1$, Ω is a singleton set.)*

2. There are uncountably many distinct monomial orders on a polynomial ring $\mathbb{k}[x_1, x_2, \dots, x_n]$ with $n \geq 2$. (For $n \geq 2$, Ω is uncountable.)

Proof. (i) Since $1 \mid x \mid x^2 \mid x^3 \dots$, then by proposition 1.2.3(i) $1 \preceq x \preceq x^2 \preceq x^3 \preceq \dots$, and is the only ordering.

(ii) Let θ and β be distinct positive transcendental numbers and consider $\mathbf{w}_\theta = (1, \theta, \theta^2, \dots, \theta^{n-1})$ and $\mathbf{w}_\beta = (1, \beta, \beta^2, \dots, \beta^{n-1})$. Clearly Coordinates of \mathbf{w}_θ as well as \mathbf{w}_β are linearly independent over the field \mathbb{Q} . Hence, by example 1.2.2(V), both $\succeq_{\mathbf{w}_\theta}$ and $\succeq_{\mathbf{w}_\beta}$ are in Ω . We need to show $\succeq_{\mathbf{w}_\theta}$ and $\succeq_{\mathbf{w}_\beta}$ are different.

With out loss of generalities, assume $\theta > \beta$. Due to the denseness property of \mathbb{Q} , there exist $\frac{a}{b} \in \mathbb{Q}^+$ such that $\theta > \frac{a}{b} > \beta$. Let $\mathbf{a} = (a, 0, 0, \dots, 0)$ and $\mathbf{b} = (0, b, 0, \dots, 0)$. Clearly $\mathbf{b} \cdot \mathbf{w}_\theta > \mathbf{a} \cdot \mathbf{w}_\theta$, and $\mathbf{b} \cdot \mathbf{w}_\beta < \mathbf{a} \cdot \mathbf{w}_\beta$. Hence $\succeq_{\mathbf{w}_\theta}$ and $\succeq_{\mathbf{w}_\beta}$ are different. That is, distinct positive transcendental numbers can be associated with distinct monomial orders. Hence there are uncountably many monomial orders. \square

1.2.2 Initial Algebra

Definition 1.2.5. Let $\succeq \in \Omega$ and $f = \alpha_1 \mathbf{x}^{\beta_1} + \alpha_2 \mathbf{x}^{\beta_2} + \dots + \alpha_r \mathbf{x}^{\beta_r}$ be a polynomial in $\mathbb{k}[\mathbf{x}]$ with $\mathbf{x}^{\beta_1} \succ \mathbf{x}^{\beta_2} \succ \dots \succ \mathbf{x}^{\beta_r}$ and $\alpha_1 \neq 0$. Then

1. The initial (leading monomial) of f is defined as $\text{in}_\succeq(f) = \mathbf{x}^{\beta_1}$.
2. The leading coefficient of f is defined as $\text{lc}_\succeq(f) = \alpha_1$.
3. The leading term of f is defined as $\text{lt}_\succeq(f) = \alpha_1 \mathbf{x}^{\beta_1}$

Remark 1.2.6. Suppose $f_i \in \mathbb{k}[\mathbf{x}]$ for $i = 1, 2, \dots, m$ and given $\succeq \in \Omega$.

1. If $\sum_{i=1}^m (f_i)$ is not zero and f_h is chosen with maximal lead monomial among the nonzero f_i 's, then $\text{in}_\succeq(f_h) \succeq \text{in}_\succeq(\sum_{i=1}^m f_i)$.

2. If there is a unique f_h with maximal lead monomial among the nonzero f'_i s, then $\sum_{i=1}^m f_i$ is nonzero and $\text{in}_{\succeq}(\sum_{i=1}^m f_i) = \text{in}_{\succeq}(f_h)$.
3. If all f'_i s are nonzero, then $\text{in}_{\succeq} \prod_{i=1}^m f_i = \prod_{i=1}^m \text{in}_{\succeq}(f_i)$.

Definition 1.2.7. Let $\succeq \in \Omega$ and \mathcal{R} be a \mathbb{k} – subalgebra of $\mathbb{k}[\mathbf{x}]$. The initial algebra of \mathcal{R} with respect to \succeq , denoted by $\text{In}_{\succeq} \mathcal{R}$, is a \mathbb{k} – subalgebra generated by $\{\text{in}_{\succeq}(f) : f \in \mathcal{R} - \{0\}\}$.

One can raise the following basic questions about initial algebras.

- 1) Given $\succeq \in \Omega$ and a finitely generated \mathbb{k} – subalgebra \mathcal{R} of $\mathbb{k}[\mathbf{x}]$.
When does $\text{In}_{\succeq} \mathcal{R}$ finitely generated?
- 2) Let $\mathcal{R} = \mathbb{k}[f_1, f_2, \dots, f_m]$. When does $\text{In}_{\succeq} \mathcal{R} = \mathbb{k}[\text{in}_{\succeq} f_1, \text{in}_{\succeq} f_2, \dots, \text{in}_{\succeq} f_m]$?

In the next chapters we will try to answer the above two basic questions.

Chapter 2

Fundamentals of SAGBI bases

2.1 Definition and Examples of SAGBI basis

The Buchberger algorithm computes a “ better ” generating set for an ideal of a polynomial ring, called Gröbner Basis. The *initial ideal* of an ideal I with respect to a given monomial order \succeq , denoted by $In_{\succeq}I$, is the ideal generated by $\{\text{in}_{\succeq} f : f \in I \setminus \{0\}\}$. By Dickson’s lemma, there exist nonzero polynomials f_1, f_2, \dots, f_m in I such that $In_{\succeq}I = \langle \text{in}_{\succeq} f_1, \text{in}_{\succeq} f_2, \dots, \text{in}_{\succeq} f_m \rangle$. In this case, We call $\mathcal{G} = \{f_1, f_2, \dots, f_m\}$ a Gröbner basis for I with respect to \succeq .

A theory for subalgebra generators that is similar to Gröbner bases theory exists, called SAGBI bases, developed by Robbiano and Sweedler [24]. The term SAGBI stands for **S**ubalgebra **A**nalog to **G**röbner **B**ases for **I**deals. This section basically focus on the definition and some examples of SAGBI basis.

Definition 2.1.1. Let \mathcal{R} be a \mathbb{k} – subalgebra of $\mathbb{k}[\mathbf{x}]$ and $\succeq \in \Omega$. A nonempty subset \mathcal{S} of \mathcal{R} is said to be a SAGBI basis for \mathcal{R} with respect to \succeq if $In_{\succeq}\mathcal{R} = \mathbb{k}[\text{in}_{\succeq} f : f \in \mathcal{S} \setminus \{0\}]$.

Trivially, any \mathbb{k} – subalgebra \mathcal{R} is a SAGBI basis for itself. But we

are interested if we can determine a **finite** SAGBI basis for a given \mathbb{k} – subalgebra of $\mathbb{k}[\mathbf{x}]$ with respect to a given monomial order.

Example 2.1.2. Let \mathcal{R} be the subalgebra \mathbb{k} of $\mathbb{k}[\mathbf{x}]$. For a given monomial order \succeq , observe that $In_{\succeq}\mathcal{R} = \mathbb{k}[\text{in}_{\succeq} f : f \in \mathcal{R} \setminus \{0\}] = \mathbb{k}[1]$. Hence $\{1\}$ is a SAGBI basis for \mathcal{R} . Actually, every set of nonzero elements of \mathbb{k} can be a SAGBI basis for \mathbb{k} .

Example 2.1.3 ([24], p.67). Given $\succeq \in \Omega$. The set of elementary symmetric polynomials \mathcal{S} is a SAGBI basis with respect to \succeq for the subalgebra \mathcal{R} of symmetric polynomials.

Indeed; if f is symmetric polynomial and $\text{deg}(\text{in}_{\succeq} f) = (a_1, a_2, \dots, a_n)$, then $a_1 \geq a_2 \geq \dots \geq a_n$, that is, the coordinates of $\text{deg}(\text{in}_{\succeq} f)$ are in decreasing order. Thus, the initial of the elementary symmetric polynomials are $x_1, x_1x_2, x_1x_2x_3, \dots, x_1x_2x_3\dots x_n$. Since the coordinates of $\text{deg}(\text{in}_{\succeq} f)$ are in decreasing order, we can express $\text{in}_{\succeq} f$ as a power product of $x_1, x_1x_2, x_1x_2x_3, \dots, x_1x_2x_3\dots x_n$. Hence, $In_{\succeq}\mathcal{R} = \mathbb{k}[x_1, x_1x_2, x_1x_2x_3, \dots, x_1x_2x_3\dots x_n] = \mathbb{k}[\text{in}_{\succeq} g : g \in \mathcal{S}]$. Therefore, by definition 2.1.1, \mathcal{S} is a SAGBI basis for \mathcal{R} . \square

Lemma 2.1.4. *Let \mathcal{R} be a \mathbb{k} – subalgebra of $\mathbb{k}[\mathbf{x}]$ and $\succeq \in \Omega$. Then*

- a) $A = \{\text{in}_{\succeq} f : f \in \mathcal{R} \setminus \{0\}\}$ is a multiplicative monoid.
- b) $B = \{\text{deg}(\text{in}_{\succeq} f) : f \in \mathcal{R} \setminus \{0\}\}$ is an additive monoid.
- c) A mapping $\alpha : A \rightarrow B$ defined by $\alpha(\text{in}_{\succeq} f) = \text{deg}(\text{in}_{\succeq} f)$ is a monoid isomorphism.

Proof. a) Since $\text{in}_{\succeq} f \text{in}_{\succeq} g = \text{in}_{\succeq}(fg)$, and $fg \in \mathcal{R} \setminus \{0\}$ for any $f, g \in \mathcal{R} \setminus \{0\}$, A is a multiplicative monoid.

b) Since $\deg(\text{in}_{\succeq} f) + \deg(\text{in}_{\succeq} g) = \deg(\text{in}_{\succeq}(fg))$, and $fg \in \mathcal{R} \setminus \{0\}$ for any $f, g \in \mathcal{R} \setminus \{0\}$, B is an additive monoid.

c) Clearly α is a bijection from A to B . For any $f, g \in \mathcal{R} \setminus \{0\}$,

$$\begin{aligned}
\alpha(\text{in}_{\succeq} f \text{ in}_{\succeq} g) &= \alpha(\text{in}_{\succeq} fg) \\
&= \deg(\text{in}_{\succeq} fg) \\
&= \deg(\text{in}_{\succeq} f \text{ in}_{\succeq} g) \\
&= \deg(\text{in}_{\succeq} f) + \deg(\text{in}_{\succeq} g) \\
&= \alpha(\text{in}_{\succeq} f) + \alpha(\text{in}_{\succeq} g)
\end{aligned}$$

Hence α is a monoid isomorphism. \square

Theorem 2.1.5. *Let \mathcal{R} be a \mathbb{k} -subalgebra of $\mathbb{k}[\mathbf{x}]$, \mathcal{S} be a nonempty subset of \mathcal{R} and $\succeq \in \Omega$. Then the following statements are equivalent.*

- i) \mathcal{S} is a SAGBI basis for \mathcal{R} with respect to \succeq .*
- ii) $\{\text{in}_{\succeq} f : f \in \mathcal{S} \setminus \{0\}\}$ generates the multiplicative monoid $\{\text{in}_{\succeq} f : f \in \mathcal{R} \setminus \{0\}\}$.*
- iii) $\{\deg(\text{in}_{\succeq} f) : f \in \mathcal{S} \setminus \{0\}\}$ generates the additive monoid $\{\deg(\text{in}_{\succeq} f) : f \in \mathcal{R} \setminus \{0\}\}$.*

Proof. Define $A = \{\text{in}_{\succeq} f : f \in \mathcal{R} \setminus \{0\}\}$

$B = \{\deg(\text{in}_{\succeq} f) : f \in \mathcal{R} \setminus \{0\}\}$

$C =$ the multiplicative monoid generated by $\{\text{in}_{\succeq} g : g \in \mathcal{S} \setminus \{0\}\}$

$D =$ the additive monoid generated by $\{\deg(\text{in}_{\succeq} g) : g \in \mathcal{S} \setminus \{0\}\}$

$i) \Rightarrow iii)$

Suppose \mathcal{S} is a SAGBI basis for \mathcal{R} . For any $f \in \mathcal{R} \setminus \{0\}$ we have the

following.

$$\begin{aligned}
f \in \mathcal{R} \setminus \{0\} &\Rightarrow \text{in}_{\succeq} f \in \text{In}_{\succeq} \mathcal{R} \\
&\Rightarrow \text{in}_{\succeq} f \in \mathbb{k}[\text{in}_{\succeq} g : g \in \mathcal{S} \setminus \{0\}] \text{ (by hypothesis, (i))} \\
&\Rightarrow \text{in}_{\succeq} f = \sum_{i=1}^m \gamma_i \left(\prod_{j=1}^{t_i} (\text{in}_{\succeq} g_{j_i})^{\alpha_{j_i}} \right), \text{ where } g_{j_i} \in \mathcal{S} \setminus \{0\} \\
&\Rightarrow \text{in}_{\succeq} f = \prod_{j=1}^{t_i} (\text{in}_{\succeq} g_{j_i})^{\alpha_{j_i}}, \text{ for some } i \in \{1, 2, \dots, m\}.
\end{aligned}$$

If we consider α as in lemma 2.1.4 we have,

$$\begin{aligned}
\deg(\text{in}_{\succeq} f) &= \alpha(\text{in}_{\succeq} f) \\
&= \alpha\left(\prod_{j=1}^{t_i} (\text{in}_{\succeq} g_{j_i})^{\alpha_{j_i}}\right). \\
&= \sum_{j=1}^{t_i} \alpha(\text{in}_{\succeq} g_{j_i})^{\alpha_{j_i}} \text{ (by lemma 2.1.4).} \\
&= \sum_{j=1}^{t_i} \alpha_{j_i} \deg(\text{in}_{\succeq} g_{j_i}).
\end{aligned}$$

Hence $B \subseteq D$.

On the other hand, since $\{\deg(\text{in}_{\succeq} f) : f \in \mathcal{S} \setminus \{0\}\} \subseteq B$ and B is an additive monoid, we have $D \subseteq B$. Hence $B = D$.

iii) \Rightarrow ii)

Let $f \in \mathcal{R} \setminus \{0\}$. By hypothesis, (iii), $\deg(\text{in}_{\succeq} f) = \sum_{i=1}^t \gamma_i \deg(\text{in}_{\succeq} g_i)$,

where $g_i \in \mathcal{S} \setminus \{0\}$ and $\gamma_i \in \mathbb{N}$. Thus,

$$\begin{aligned}
\text{in}_{\succeq} f &= \alpha^{-1}(\text{deg}(\text{in}_{\succeq} f)) \\
&= \alpha^{-1}\left(\sum_{i=1}^t \gamma_i \text{deg}(\text{in}_{\succeq} g_i)\right) \\
&= \prod_{i=1}^t (\alpha^{-1}(\text{deg}(\text{in}_{\succeq} g_i)))^{\gamma_i} \quad (\text{ by lemma 2.1.4}). \\
&= \prod_{i=1}^t (\text{in}_{\succeq} g_i)^{\gamma_i}
\end{aligned}$$

Therefore, $\text{in}_{\succeq} f \in C$ and hence $A \subseteq C$. In addition to this, since $\{(\text{in}_{\succeq} g) : g \in \mathcal{S} \setminus \{0\}\} \subseteq A$ and A is a multiplicative monoid, we have $C \subseteq A$. Hence $A = C$.

ii) \Rightarrow i)

$$\begin{aligned}
f \in \text{In}_{\succeq} \mathcal{R} &\Rightarrow f = \sum_{i=1}^m a_i \prod_{j=1}^{t_i} (\text{in}_{\succeq} h_{j_i})^{\alpha_{j_i}}, \text{ where } h_{j_i} \in \mathcal{R} \setminus \{0\}. \\
&\Rightarrow f = \sum_{i=1}^m a_i \prod_{j=1}^{t_i} \left(\prod_{r=1}^{l_j} (\text{in}_{\succeq} g_{r_{j_i}})^{\beta_{r_{j_i}}} \right)^{\alpha_{j_i}}, \text{ where } g_{r_{j_i}} \in \mathcal{S} \setminus \{0\}, \text{ by (ii)}. \\
&\Rightarrow f \in \mathbb{k}[\{\text{in}_{\succeq} g : g \in \mathcal{S} \setminus \{0\}\}]
\end{aligned}$$

Therefore, $\text{In}_{\succeq} \mathcal{R} \subseteq \mathbb{k}[\{\text{in}_{\succeq} g : g \in \mathcal{S} \setminus \{0\}\}]$.

On the other hand, since $\mathcal{S} \subseteq \mathcal{R}$, $\mathbb{k}[\{\text{in}_{\succeq} g : g \in \mathcal{S} \setminus \{0\}\}] \subseteq \text{In}_{\succeq} \mathcal{R}$.

Hence $\text{In}_{\succeq} \mathcal{R} = \mathbb{k}[\{\text{in}_{\succeq} g : g \in \mathcal{S} \setminus \{0\}\}]$. \square

Proposition 2.1.6. *Let \mathcal{S} be a nonempty subset of $\mathbb{k}[\mathbf{x}]$ and $\succeq \in \Omega$.*

If the initials of any two distinct elements of \mathcal{S}_{mon} are distinct, then

\mathcal{S} is a SAGBI basis for $\mathbb{k}[\mathcal{S}]$.

Proof.

$$\begin{aligned}
h \in \mathbb{k}[\mathcal{S}] \setminus \{0\} &\Rightarrow h = \sum_{i=1}^r k_i m_i, \text{ for distinct } m_i \in \mathcal{S}_{\text{mon}} \text{ and } 0 \neq k_i \in \mathbb{k}. \\
&\Rightarrow \text{in}_{\succeq}(h) = \max_{\succeq} \{\text{in}_{\succeq} m_i : i \in \{1, 2, \dots, r\}\}
\end{aligned}$$

Hence by Theorem 2.1.5, \mathcal{S} is a SAGBI basis for $\mathbb{k}[\mathcal{S}]$ with respect to \succeq . \square

Example 2.1.7. Let $f = x + y$ and $g = xy$. Then $\mathcal{S} = \{f, g\}$ is a SAGBI basis for $\mathbb{k}[\mathcal{S}]$ with respect to any monomial order \succeq .

Indeed, for the monomial order with $x \succeq y$, consider arbitrary $m_1 \in \mathcal{S}_{mon}$, then we can find α_1 and β_1 in \mathbb{N} such that

$$\begin{aligned} m_1 &= f^{\alpha_1} g^{\beta_1} \\ &= (x + y)^{\alpha_1} (xy)^{\beta_1} \\ &= \sum_{i=0}^{\alpha_1} \binom{\alpha_1}{i} (x^{\alpha_1-i} y^i) (xy)^{\beta_1} \\ &= \sum_{i=0}^{\alpha_1} \binom{\alpha_1}{i} x^{\alpha_1+\beta_1-i} y^{i+\beta_1} \end{aligned}$$

Since $\binom{\alpha_1}{0} \neq 0$ and $x^{\alpha_1+\beta_1} y^{\beta_1} = x^{\alpha_1+\beta_1-i} x^i y^{\beta_1} \succeq x^{\alpha_1+\beta_1-i} y^{i+\beta_1}$ for each $i \geq 0$, (as $x \succeq y$), we have $\text{in}_{\succeq}(m_1) = x^{\alpha_1+\beta_1} y^{\beta_1}$.

Similarly for another $m_2 \in \mathcal{S}_{mon}$ we can find α_2 and β_2 such that $m_2 = f^{\alpha_2} g^{\beta_2}$ and $\text{in}_{\succeq}(m_2) = x^{\alpha_2+\beta_2} y^{\beta_2}$. Thus, $\text{in}_{\succeq}(m_1) = \text{in}_{\succeq}(m_2)$ implies that $m_1 = m_2$. In other words, the initials of any two distinct elements of \mathcal{S}_{mon} are distinct. Hence by proposition 2.1.6, \mathcal{S} is a SAGBI basis for $\mathbb{k}[\mathcal{S}]$ with respect to \succeq with $x \succeq y$. By symmetry it holds for the monomial order with $y \succeq x$ too.

Proposition 2.1.8 ([24], p.68). *A singleton set $\mathcal{S} = \{f\}$, where $0 \neq f \in \mathbb{k}[\mathbf{x}]$, is a SAGBI basis for $\mathbb{k}[\mathcal{S}]$ with respect to any $\succeq \in \Omega$.*

Proof. Suppose $m_1, m_2 \in \mathcal{S}_{mon}$. There exist $\alpha_1, \alpha_2 \in \mathbb{N}$ such that

$m_1 = f^{\alpha_1}$ and $m_2 = f^{\alpha_2}$.

$$\begin{aligned} \text{in}_{\succeq} m_1 = \text{in}_{\succeq} m_2 &\Rightarrow \text{in}_{\succeq} f^{\alpha_1} = \text{in}_{\succeq} f^{\alpha_2} \\ &\Rightarrow \alpha_1 = \alpha_2 \\ &\Rightarrow f^{\alpha_1} = f^{\alpha_2} \\ &\Rightarrow m_1 = m_2 \end{aligned}$$

Thus, distinct \mathcal{S} -monomials have distinct initials. Hence by proposition 2.1.6, $\mathcal{S} = \{f\}$ is a SAGBI basis for $\mathbb{k}[\mathcal{S}]$. \square

Corollary 2.1.9. *Let f and g be monic polynomials in $\mathbb{k}[\mathbf{x}]$. Then $\mathbb{k}[f] = \mathbb{k}[g]$ if and only if f and g differ only by their constant terms.*

Proof. If f and g differ only by their constant terms, trivially $\mathbb{k}[f] = \mathbb{k}[g]$.

Conversely, assume $\mathbb{k}[f] = \mathbb{k}[g]$.

Let $\deg(\text{in}_{\succeq} f) = \mathbf{a}$ and $\deg(\text{in}_{\succeq} g) = \mathbf{b}$. Since f is a SAGBI basis for $\mathbb{k}[f]$ and $g \in \mathbb{k}[f]$, we have $\mathbf{b} = t\mathbf{a}$ for some $t \in \mathbb{N}$. Again, since g is a SAGBI basis for $\mathbb{k}[g]$ and $f \in \mathbb{k}[g]$, we have $\mathbf{a} = r\mathbf{b}$ for some $r \in \mathbb{N}$. Hence $t = 1 = r$, that is, $\mathbf{a} = \mathbf{b}$. Since f and g are monic the leading term of f must cancel the leading term of g in $f - g$. That is, $\text{in}_{\succeq}(f - g) \prec \text{in}_{\succeq} f$. Hence $f - g \in \mathbb{k}$. \square

Example 2.1.10. $\mathbb{k}[x^2y + xy] \neq \mathbb{k}[x^2y + 2xy]$ and $\mathbb{k}[xy + x] \neq \mathbb{k}[xy + y]$.

We can also apply proposition 2.1.6 for monomial \mathbb{k} -subalgebras of $\mathbb{k}[\mathbf{x}]$. A \mathbb{k} -subalgebra is said to be **monomial** \mathbb{k} -subalgebra if it can be generated by monomials.

Example 2.1.11 ([24], p.68). Given $\succeq \in \Omega$. If \mathcal{S} is a non empty set of monomials, then \mathcal{S} is a SAGBI basis for $\mathbb{k}[\mathcal{S}]$ with respect to \succeq .

Observe that, distinct power products of monomials have distinct initials. Hence by proposition 2.1.6, \mathcal{S} is a SAGBI basis for $\mathbb{k}[\mathcal{S}]$.

In particular $\{x_1, x_2, \dots, x_n\}$ is a SAGBI basis for $\mathbb{k}[\mathbf{x}]$. □

Remark 2.1.12. A finitely generated monomial subalgebra has a finite SAGBI basis. In example 1.1.6 observe that, \mathcal{R} is generated by a finite set of monomials, and hence it has a finite SAGBI basis with respect to any monomial order \succeq .

2.2 Subduction

In this section, we present the process of \mathbb{k} -subalgebra reduction of an element of $\mathbb{k}[\mathbf{x}]$ over a subset B of $\mathbb{k}[\mathbf{x}]$. Using the language of Robbiano and Sweedler, instead of calling \mathbb{k} -subalgebra reduction we shall use the name subduction.

Let $B \subseteq \mathbb{k}[\mathbf{x}]$, $f \in \mathbb{k}[\mathbf{x}]$, and $\succeq \in \Omega$. If $f \neq 0$ and $\text{in}_{\succeq} f = \prod_{i=1}^{t_1} \text{in}_{\succeq} g_{i_1}^{\alpha_{i_1}}$ for some $g_{i_1} \in B$ and $\alpha_{i_1}, t_1 \in \mathbb{N}$, put $f_1 := f - \gamma_1 \prod_{i=1}^{t_1} g_{i_1}^{\alpha_{i_1}}$ where $\gamma_1 = \frac{lc(f)}{lc(\prod_{i=1}^{t_1} g_{i_1}^{\alpha_{i_1}})} \in \mathbb{k} \setminus \{0\}$. Hence we can express f as $f = \gamma_1 \prod_{i=1}^{t_1} g_{i_1}^{\alpha_{i_1}} + f_1$. In this case, f is subduced to f_1 in one step (or f_1 is a first subductum of f) over B . This relation of f and f_1 can be denoted by $f \xrightarrow{B} f_1$.

One can easily observe that, our construction of γ_1 forces to eliminate the leading term of f . So, we have either $f_1 = 0$ or $\text{in}_{\succeq} f_1 \succ \text{in}_{\succeq} f$.

If $f_1 \neq 0$ and $\text{in}_{\succeq} f_1 = \prod_{i=1}^{t_2} \text{in}_{\succeq} g_{i_2}^{\alpha_{i_2}}$ for some $g_{i_2} \in B$, and $\alpha_{i_2}, t_2 \in \mathbb{N}$, put $f_2 := f_1 - \gamma_2 \prod_{i=1}^{t_2} g_{i_2}^{\alpha_{i_2}}$ where $\gamma_2 = \frac{lc(f_1)}{lc(\prod_{i=1}^{t_2} g_{i_2}^{\alpha_{i_2}})} \in \mathbb{k} \setminus \{0\}$. Hence we

can re-express f as $f = \gamma_1 \prod_{i=1}^{t_1} g_{i_1}^{\alpha_{i_1}} + \gamma_2 \prod_{i=1}^{t_2} g_{i_2}^{\alpha_{i_2}} + f_2$. Observe that either $f_2 = 0$ or $\text{in}_{\succeq} f \succ \text{in}_{\succeq} f_1 \succ \text{in}_{\succeq} f_2$. Due to the well ordering property of \succeq , this process terminates at some f_r , and hence f can be written as

$$f = \gamma_1 \prod_{i=1}^{t_1} g_{i_1}^{\alpha_{i_1}} + \gamma_2 \prod_{i=1}^{t_2} g_{i_2}^{\alpha_{i_2}} + \dots + \gamma_r \prod_{i=1}^{t_r} g_{i_r}^{\alpha_{i_r}} + f_r$$

where $f_r = 0$ or $\text{in}_{\succeq} f_r \neq \text{in}_{\succeq} h$, for any $h \in B_{\text{mon}}$. In this case, we call f_r a *remainder* or a *final subductum* of f over B . We denote this inductive limit of subduction of f over B by $f \xrightarrow{B^*} f_r$.

Since we have the possibility to choose g_{ij} s' in many ways at each step, we may have so many different subductions and hence a final subductum of f over any subset B may not be unique. In the subduction process, it is important to observe that $\text{in}_{\succeq} f = \text{in}_{\succeq} \prod_{i=1}^{t_1} g_{i_1}^{\alpha_{i_1}} \succ \text{in}_{\succeq} \prod_{i=1}^{t_2} g_{i_2}^{\alpha_{i_2}} \succ \dots \succ \text{in}_{\succeq} \prod_{i=1}^{t_r} g_{i_r}^{\alpha_{i_r}} \succ \text{in}_{\succeq} f_r$, that is, at each stage we eliminate the leading term. We sum up the above subduction process by the following algorithm.

The Subduction Algorithm [[24], p. 64]

INPUT : $B \subseteq \mathbb{k}[\mathbf{x}]$, $f \in \mathbb{k}[\mathbf{x}]$ and $\succeq \in \Omega$

OUTPUT : A final subductum f_r of f over B

INITIALIZE : $f_0 = f$

WHILE : $f_i \neq 0$ and $\text{in}_{\succeq} f_i = \prod_{j=1}^r (\text{in}_{\succeq} g_{j_i})^{\alpha_{j_i}}$ for some $g_{j_i} \in B$ and $\alpha_{j_i} \in \mathbb{N}$

Do: $f_{i+1} = f_i - \gamma_i \prod_{j=1}^r (g_{j_i})^{\alpha_{j_i}}$ where $\gamma_i \in \mathbb{k}$ is chosen so that f_i and $\gamma_i \prod_{j=1}^r (g_{j_i})^{\alpha_{j_i}}$ have the same leading coefficient.

Return: f_{i+1}

Remark 2.2.1. ([24], Remark .8, p. 65) The following are important points in the subduction algorithm

1. The subduction algorithm always terminates.
2. The WHILE loop stops at either $f_i = 0$ or $\text{in}_{\succeq} f_i \neq \text{in}_{\succeq} h$ for all $h \in B_{\text{mon}}$.
3. If \mathcal{R} is a subalgebra of $\mathbb{k}[\mathbf{x}]$ and $B \subseteq \mathcal{R}$, then $f \in \mathcal{R}$ iff $f_1 \in \mathcal{R}$ (and hence each $f_i \in \mathcal{R}$).
4. If B is a SAGBI basis for the \mathbb{k} – subalgebra \mathcal{R} and $f \in \mathcal{R}$, then $f \xrightarrow{B^*} 0$.

Theorem 2.2.2. ([24], Proposition 1.16, p. 67) Let \mathcal{R} be a \mathbb{k} – subalgebra of a polynomial ring $\mathbb{k}[\mathbf{x}]$ and $\succeq \in \Omega$. If \mathcal{S} is a SAGBI basis for \mathcal{R} with respect to \succeq then $\mathcal{R} = \mathbb{k}[\mathcal{S}]$.

Proof. Suppose \mathcal{S} is a SAGBI basis for \mathcal{R} with respect to \succeq .

Claim: $\mathcal{R} = \mathbb{k}[\mathcal{S}]$

Since $\mathcal{S} \subseteq \mathcal{R}$ and \mathcal{R} is a \mathbb{k} – subalgebra of $\mathbb{k}[\mathbf{x}]$ we have $\mathbb{k}[\mathcal{S}] \subseteq \mathcal{R}$.

On the other hand, let $0 \neq f \in \mathcal{R}$.

The subduction of f over \mathcal{S} yields

$$f = \gamma_1 \prod_{i=1}^{t_1} g_{i_1}^{\alpha_{i_1}} + \gamma_2 \prod_{i=1}^{t_2} g_{i_2}^{\alpha_{i_2}} + \dots + \gamma_r \prod_{i=1}^{t_r} g_{i_r}^{\alpha_{i_r}} + f_r,$$

where f_r is a final subductem of f over \mathcal{S} . Since \mathcal{S} is a SAGBI basis for \mathcal{R} , by Remark 2.2.1 (iv), $f_r = 0$. Hence $f \in \mathbb{k}[\mathcal{S}]$.

Thus, $\mathcal{R} = \mathbb{k}[\mathcal{S}]$. □

We say that \mathcal{S} is a SAGBI basis (with out reference to a subalgebra) to mean \mathcal{S} is a SAGBI basis for the subalgebra it generates.

Corollary 2.2.3. *Let \mathcal{R} and \mathcal{T} be \mathbb{k} -subalgebras of $\mathbb{k}[\mathbf{x}]$ with $\mathcal{R} \subseteq \mathcal{T}$. If $In_{\succeq}\mathcal{R} = In_{\succeq}\mathcal{T}$ for some $\succeq \in \Omega$, then $\mathcal{R} = \mathcal{T}$.*

Proof. $In_{\succeq}\mathcal{R} = In_{\succeq}\mathcal{T}$ and $\mathcal{R} \subseteq \mathcal{T}$, implies that \mathcal{R} is a SAGBI basis for \mathcal{T} . Hence $\mathcal{T} = \mathbb{k}[\mathcal{R}] = \mathcal{R}$. \square

The above corollary fails if we remove the condition $\mathcal{R} \subseteq \mathcal{T}$. For example, if $\mathcal{R} = \mathbb{k}[x^2y + xy]$ and $\mathcal{T} = \mathbb{k}[x^2y + 2xy]$, then $In_{\succeq}\mathcal{R} = In_{\succeq}\mathcal{T}$ for any $\succeq \in \Omega$. But by corollary 2.1.9, $\mathcal{R} \neq \mathcal{T}$.

Remark 2.2.4. • The membership problem of \mathbb{k} -subalgebra is solved in Remark 2.2.1(iv). That is, for $f \in \mathbb{k}[\mathbf{x}]$; $f \in \mathcal{R}$ if and only if the subduction of f over a SAGBI basis \mathcal{S} terminates at zero.

- The converse of Theorem 2.2.2 does not hold. For example, consider $\mathcal{S} = \{x^5, x^6, x^7 + x^4\}$, \mathcal{S} is not a SAGBI basis for $\mathbb{k}[\mathcal{S}]$. Because $x^9 = x^5(x^7 + x^4) - (x^6)^2 \in \mathbb{k}[\mathcal{S}]$ but $9 = \deg x^9$ can not be expressed as an \mathbb{N} -linear combination of 5, 6, and 7.

Theorem 2.2.2 assures that, if a subalgebra is not finitely generated, then it does not have finite SAGBI basis. The converse of this statement is not true. That is, there exist a finitely generated \mathbb{k} -subalgebra of $\mathbb{k}[\mathbf{x}]$ that does not have finite SAGBI basis. Robianno and Sweedler [24] give the following example to show the existence of finitely generated \mathbb{k} -subalgebra that does not have finite SAGBI basis with respect to any monomial ordering.

Example 2.2.5. ([24], Example 1.20, p.68) Let \mathcal{R} be a subalgebra of $\mathbb{k}[x, y]$ (finitely) generated by $f = x + y, g = xy$, and $h = xy^2$. Then

\mathcal{R} does not have finite SAGBI basis with respect to any monomial order.

First of all let us identify a useful family of elements in \mathcal{R} and a useful family of elements which are not in \mathcal{R} .

IN \mathcal{R} : $g = xy, h = xy^2, h_m = xy^m = (x + y)xy^{m-1} - (xy)xy^{m-2}$, for $m \in N_{\geq 3}$. This implies that $xy^m \in \mathcal{R}$ for all $m \in N_{\geq 1}$.

NOT IN \mathcal{R} : For $j \geq 1$, \mathcal{R} contains no element having y^j as a homogeneous component.

To the contrary, assume there is an element $t \in \mathcal{R}$ with y^j as a homogeneous component. Since \mathcal{R} is graded algebra, $y^j \in \mathcal{R}$. Since \mathcal{R} is generated by f, g and h , every element of \mathcal{R} is a function of f, g and h .

Assume $r(x + y, xy, xy^2) = y^j$.

For $x = 0$, $r(y, 0, 0) = y^j$

For $y = 0$, $r(x, 0, 0) = 0$. This is not acceptable.

Hence \mathcal{R} does not contain an element having y^j as a homogeneous component.

For any monomial order \succeq on $\mathbb{k}[x, y]$, either $x \succeq y$ or $y \succeq x$

For $x \succeq y$

By IN \mathcal{R} , $\mathcal{S} = \{x + y, xy, xy^2, \dots\} \subseteq \mathcal{R}$. We claim to show that \mathcal{S} is a SAGBI basis for \mathcal{R} and \mathcal{R} has no finite SAGBI basis.

For $j \geq 1$, \mathcal{R} contains no elements having y^j as a leading monomial.

To the contrary, suppose q is a polynomial with leading monomial y^j .

In the monomial ordering y^j is the smallest monomial of degree j .

Hence the degree j component of q must be a non-zero scalar times y^j . By NOT IN \mathcal{R} , $q \notin \mathcal{R}$. Hence the initial of any non constant

element $p \in \mathcal{R}$ is of the form $x^i y^j$ with $i \geq 1$.

$$\begin{aligned} \text{in}_{\succeq} p &= x^{i-1} x y^j \\ &= \text{in}_{\succeq} (x + y)^{i-1} \text{in}_{\succeq} (x y^j) \end{aligned}$$

Hence, by Theorem 2.1.5, \mathcal{S} is a SAGBI basis for \mathcal{R} .

Suppose \mathcal{R} has a finite SAGBI basis T . Choose m large enough so that no element of T has leading monomial $x y^m$. Since T is a SAGBI basis for \mathcal{R} , $x y^m$ is a power product of initials of elements of T . Since x occurs to the first power, it follows that T has an element with initial of the form $x y^i$ with $0 \leq i$ and an element with initial y^j with $1 \leq j$. This contradicts NOT IN \mathcal{R} . Hence \mathcal{R} has no finite SAGBI basis.

For $y \succeq x$

Since $(x + y)xy = x^2 y + x y^2$, $\{x + y, xy, x^2 y\}$ can also generate \mathcal{R} . Hence with x and y interchanged, the same reasoning as above shows that $\{x + y, xy, x^2 y, x^3 y, \dots\}$ is a SAGBI basis for \mathcal{R} and \mathcal{R} has no finite SAGBI basis.

Remark 2.2.6. Example 2.1.7 shows that $\{x + y, xy\}$ form a SAGBI basis. Example 2.1.11 shows that $\{xy, xy^2\}$ form a SAGBI basis. In similar fashion with example 2.1.7, we can easily show that $\{x + y, xy^2\}$ can form a SAGBI basis. So, every pair of elements in $\{x + y, xy, xy^2\}$ can generate a \mathbb{k} -subalgebra with finite SAGBI basis. Can we conclude that every \mathbb{k} -subalgebra generated by two polynomials has a finite SAGBI basis? We will answer this question in chapter 4.

2.3 SAGBI basis test

Consider a finite subset $\mathcal{S} = \{f_1, f_2, \dots, f_m\}$ of $\mathbb{k}[\mathbf{x}]$. Since the elements in $\mathbb{k}[\mathcal{S}]$ are \mathbb{k} -linear combinations of \mathcal{S} -power products and the initials of each \mathcal{S} -power products is clearly a product of initials of elements in \mathcal{S} , one might think that the condition in the definition on SAGBI basis should always be satisfied. However, the leading terms of the summands in the \mathbb{k} -linear combinations of \mathcal{S} -power products might cancel and the initial of the \mathbb{k} -linear combinations of \mathcal{S} -power products need not be a product of initials of elements in \mathcal{S} . In this case \mathcal{S} fails to be a SAGBI basis. In this section, we introduce *critical pairs*, which are the most important pairs of polynomials in the SAGBI test theorem.

Definition 2.3.1. Let $\emptyset \neq \mathcal{S} \subseteq \mathbb{k}[\mathbf{x}]$ and \mathcal{R} be the \mathbb{k} -subalgebra of $\mathbb{k}[\mathbf{x}]$ generated by \mathcal{S} . An element $f \in \mathcal{R} \setminus \{0\}$ is said to have a lower representation in \mathcal{S} if $f = \sum_{i=1}^r \gamma_i m_i$ for some $\gamma_i \in \mathbb{k} \setminus \{0\}$ and $m_i \in \mathcal{S}_{mon}$ with $\text{in}_{\succeq} f \succeq \max_{\succeq} \{\text{in}_{\succeq}(m_1), \text{in}_{\succeq}(m_2), \dots, \text{in}_{\succeq}(m_r)\}$.

Proposition 2.3.2 ([24], Proposition 2.3, p.70). *Let $\emptyset \neq \mathcal{S} \subseteq \mathbb{k}[\mathbf{x}]$ and \mathcal{R} be the \mathbb{k} -subalgebra of $\mathbb{k}[\mathbf{x}]$ generated by \mathcal{S} . The following conditions are equivalent.*

- (1) \mathcal{S} is a SAGBI basis
- (2) Each element $f \in \mathcal{R} \setminus \{0\}$ has a lower representation in \mathcal{S} .

Definition 2.3.3. Let $\emptyset \neq \mathcal{S} \subseteq \mathbb{k}[\mathbf{x}]$ and $\succeq \in \Omega$. A pair $(p, q) \in \mathcal{S}_{mon} \times \mathcal{S}_{mon}$ is said to be an \mathcal{S} -critical pair if $\text{in}_{\succeq} p = \text{in}_{\succeq} q$.

An \mathcal{S} -critical pair (p, q) is said to be trivial if p and q have a non-constant common factor in \mathcal{S} .

For an \mathcal{S} -critical pair (p, q) , there exists $0 \neq \gamma \in \mathbb{k}$ such that $lt_{\succeq} p = \gamma lt_{\succeq} q$. For such $0 \neq \gamma \in \mathbb{k}$ we call the difference $p - \gamma q$ a T -polynomial of (p, q) and denoted by $T(p, q)$.¹

Remark 2.3.4. The following are simple consequences of the definition of \mathcal{S} -critical pairs and T -polynomials.

- (1) (p, q) is an \mathcal{S} -critical pair if and only if (q, p) is an \mathcal{S} -critical pair.
- (2) For any \mathcal{S} -critical pair (p, q) , $\text{in}_{\succeq} T(p, q) \prec \text{in}_{\succeq} p = \text{in}_{\succeq} q$
- (3) $T(p, q) = p - \gamma q$ if and only if $T(q, p) = q - \frac{1}{\gamma} p = -\frac{1}{\gamma} T(p, q)$
- (4) $T(p, q) = 0$ if and only if $p = \gamma q$ for some $0 \neq \gamma \in \mathbb{k}$.
- (5) If (p, q) and (u, v) are \mathcal{S} -critical pairs and $m \in \mathbb{N}$ then (pu, qv) , (pv, qu) and (p^m, q^m) are \mathcal{S} -critical pairs.

Note that, there are generally an infinite number of critical pairs, but the set of critical pairs may be finitely generated and it is time saving to use a generating set for the SAGBI basis test.

Definition 2.3.5. Let $\emptyset \neq \mathcal{S} \subseteq \mathbb{k}[\mathbf{x}]$ and C be the set of all \mathcal{S} -critical pairs. A non empty subset A of C is said to be a generating set for C if for all $(u, v) \in C$ there exist $(u_1, v_1), (u_2, v_2), \dots, (u_t, v_t)$ with either $(u_i, v_i) \in A$ or $(v_i, u_i) \in A$ for each $i \in \{1, 2, 3, \dots, t\}$ such that $u = \prod_{i=1}^t u_i$ and $v = \prod_{j=1}^t v_j$.

¹In Buchberger theory the name S -polynomial is used to refer *Syzygy*. The SAGBI analogue for S -polynomial is T -polynomial. We used the name T -polynomial since Robbiano and Sweedler used “tête-à-tête”.

Theorem 2.3.6 ([24], Theorem 2.8, p. 71). *[SAGBI Test]*

Let $\emptyset \neq \mathcal{S} \subseteq \mathbb{k}[\mathbf{x}]$, $\succeq \in \Omega$, C be the set of all \mathcal{S} -critical pairs and A be a generating set for C . Then the following statements are equivalent.

- (1) \mathcal{S} is a SAGBI basis with respect to \succeq .
- (2) Each non-zero T – polynomial of an element in A has a lower representation in \mathcal{S} .
- (3) Each non-zero T – polynomial of an element in A has at least one subduction over \mathcal{S} which terminates at 0.
- (4) Every subduction of each non-zero T – polynomial of an element in C over \mathcal{S} terminates at 0.

Remark 2.3.7. Proposition 2.1.6 can be a consequence of the SAGBI test theorem. Indeed, if any two \mathcal{S} – power products have the same initial, then the two \mathcal{S} – power products are equal. Hence the corresponding T -polynomial should be zero. Hence by SAGBI test theorem \mathcal{S} is a SAGBI basis.

“As in classical Buchberger theory, the SAGBI test theorem is the key to construct SAGBI basis. If a set \mathcal{G} is not a SAGBI basis, there will be at least one $(p, q) \in A$ where $T(p, q)$ does not subduce to 0. These final subductums of elements not equal to zero are what to add to the set \mathcal{G} to get closer to SAGBI basis” [24]. In the following section, we will see how can we construct a SAGBI basis from a subset \mathcal{G} of $\mathbb{k}[\mathbf{x}]$.

SAGBI basis construction

One begins with a generating set for a \mathbb{k} -subalgebra, and wishes to find a SAGBI basis for the \mathbb{k} -subalgebra. The construction starts with a set \mathcal{G} and leads to a SAGBI basis for the \mathbb{k} -subalgebra \mathcal{R} generated by \mathcal{G} . If \mathcal{G} is finite and \mathcal{R} has a finite SAGBI basis, our SAGBI basis construction really terminates in a finite number of steps.

SAGBI basis construction “algorithm” [[24], p. 77]

Let \mathcal{G} be a finite generator of a \mathbb{k} -subalgebra \mathcal{R} of the polynomial ring $\mathbb{k}[\mathbf{x}]$.

INITIALIZE : $\mathcal{G}_0 = \mathcal{G}$;

Let A_j be a generating set for \mathcal{G}_j -critical pairs. For each $(p, q) \in A_j$, let $r(p, q)$ be a final subductum of $T(p, q)$ over \mathcal{G}_j , and $F_j = \{r(p, q) : r(p, q) \neq 0\}$.

WHILE $F_j \neq \emptyset$

DO $\mathcal{G}_{j+1} = F_j \cup \mathcal{G}_j$

FINALIZE : Set $\mathcal{G}_\infty = \cup_j \mathcal{G}_j$.

Remark 2.3.8. Let $\mathcal{G}_j s'$ and $F_j s'$ be given as in the above “algorithm”. Then the following hold.

- $\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \dots$
- All the \mathcal{G}_j , and hence \mathcal{G}_∞ are contained in \mathcal{R} .
- If $\mathcal{G}_{j+1} = \mathcal{G}_j$ (and hence $F_{j+1} = F_j = \emptyset$), then $\mathcal{G}_j = \mathcal{G}_{j+1} = \mathcal{G}_{j+2} = \dots$ and $\mathcal{G}_j = \mathcal{G}_\infty$ and hence the above SAGBI basis construction algorithm stop at a finite stage.

Theorem 2.3.9 ([24], Theorem 3.5, p. 77). (*SAGBI construction theorem*)

Let \mathcal{G} be a finite generator of a \mathbb{k} -subalgebra \mathcal{R} of the polynomial ring $\mathbb{k}[\mathbf{x}]$.

- \mathcal{G}_∞ is a SAGBI basis for \mathcal{R}
- \mathcal{G}_j is a SAGBI basis for \mathcal{R} if $\mathcal{G}_j = \mathcal{G}_{j+1}$

2.4 SAGBI basis and Integral Extension

In this section we will see definitions and some properties of integral element, integral closure and integrally closed algebra. In addition to this we will see the relationship between integrally closed algebras and algebras with finite SAGBI basis.

Definition 2.4.1. Let $A \subseteq B$ be an extension of commutative rings with the same 1. An element s in B is said to be an integral element over A if there is a monic polynomial $f \in A[x]$ with $f(s) = 0$. That is, if there are $m \in \mathbb{N}_{\geq 1}$ and $a_1, a_2, \dots, a_m \in A$ with $s^m + a_1 s^{m-1} + \dots + a_{m-1} s + a_m = 0$. We say that B is integral over A if every element of B is integral element over A .

Note that if A is a field, then integral and algebraic are the same.

Proposition 2.4.2 ([2], Proposition 5.1, pp.59-60). *Let $A \subseteq B$ be an extension of commutative rings and $s \in B$. Then the following statements are equivalent;*

- (1) s is integral element over A .
- (2) $A[s]$ is finitely generated A -module.

(3) $A[s]$ is contained in some subring $C \subseteq B$ so that C is finitely generated as A -module.

(4) There is a faithful $A[s]$ -module M that is finitely generated as A -module.

Corollary 2.4.3 ([2], Corollary 5.2, p.60). *Let $A \subseteq B$ be an extension of commutative rings. $s_1, s_2, \dots, s_n \in B$ are integral elements over A if and only if $A[s_1, s_2, \dots, s_n]$ is finitely generated A -module.*

Corollary 2.4.4 ([2], Corollary 5.3, p.60). *Let $A \subseteq B$ be an extension of commutative rings.*

$$\bar{A} := \{x \in B : x \text{ is integral element over } A\}$$

is a subring of B which contains A . This subring is called the integral closure of A in B .

Corollary 2.4.5 ([15], Theorem 8.4, p.105). *If B is generated by integral elements over A as an A -module, then B is integral over A .*

Remark 2.4.6. Corollary 2.4.3 and 2.4.5 shows that any finitely generated A -algebra that is also integral over A is finitely generated A -module.

Corollary 2.4.7 ([2], Corollary 5.4, pp.60-61). *Let $R \subseteq S \subseteq T$ be commutative rings with S is integral over R and T is integral over S . Then T is integral over R .*

Corollary 2.4.8 ([24], Proposition 4.7, p.82). *Let \mathcal{R} be a \mathbb{k} -subalgebra of $\mathbb{k}[\mathbf{x}]$ and $\succeq \in \Omega$. If $\mathbb{k}[\mathbf{x}]$ is integral over $In_{\succeq} \mathcal{R}$ then \mathcal{R} has a finite SAGBI basis.*

Proof. Clearly, $\mathbb{k}[\mathbf{x}]$ is finitely generated as a \mathbb{k} - algebra. Moreover, Since $\mathbb{k}[\mathbf{x}]$ is integral over $In_{\succeq}\mathcal{R}$, $\mathbb{k}[\mathbf{x}]$ is finitely generated as $In_{\succeq}\mathcal{R}$ -module. Therefore, By proposition 1.1.13(E.Artin-Tate Theorem), $In_{\succeq}\mathcal{R}$ is finitely generated as a \mathbb{k} -Algebra. Hence by definition of SAGBI basis, \mathcal{R} has a finite SAGBI basis. \square

Theorem 2.4.9 ([24], Proposition 4.9, p. 82). *Let \mathcal{R} be a \mathbb{k} – subalgebra of a polynomial algebra $\mathbb{k}[\mathbf{x}] = \mathbb{k}[x_1, x_2, \dots, x_n]$ and $\succeq \in \Omega$. Then $\mathbb{k}[\mathbf{x}]$ is integral over $In_{\succeq}\mathcal{R}$ if and only if for each $i = 1, 2, \dots, n$, $\exists f_i \in \mathcal{R}$ such that $in_{\succeq} f_i = x_i^{d_i}$ for some $d_i \in \mathbb{N}_{>0}$.*

Proof. Suppose $\mathbb{k}[\mathbf{x}]$ is integral over $In_{\succeq}\mathcal{R}$. For each $i = 1, 2, \dots, n$, x_i satisfies

$$x_i^m + g_1 x_i^{m-1} + \dots + g_{m-1} x_i + g_m = 0$$

for some $g_i \in In_{\succeq}\mathcal{R}$. To cancel out x_i^m , at least one g_t must have a nonzero term with monomial of the form $x_i^{d_i}$. Since $In_{\succeq}\mathcal{R}$ is a monomial algebra, each of the individual terms of the g_i 's lie in $In_{\succeq}\mathcal{R}$. Thus $x_i^{d_i} \in In_{\succeq}\mathcal{R}$. Therefore, $\exists f_i \in \mathcal{R}$ such that $in_{\succeq} f_i = x_i^{d_i}$.

Conversely, assume $x_1^{d_1}, x_2^{d_2}, \dots, x_n^{d_n} \in In_{\succeq}\mathcal{R}$ for some $d_1, d_2, \dots, d_n \in \mathbb{N}_{>0}$. Since each x_i satisfies a monic polynomial equation $x^{d_i} - x_i^{d_i} = 0$ in $In_{\succeq}\mathcal{R}[x]$, we have each x_i is integral over $In_{\succeq}\mathcal{R}$. Therefore, $\mathbb{k}[\mathbf{x}]$ is integral over $In_{\succeq}\mathcal{R}$. \square

Corollary 2.4.10. *Let \mathcal{R} be a \mathbb{k} – subalgebra of $\mathbb{k}[x_1, x_2, \dots, x_n]$ and $\succeq \in \Omega$. If there exist $f_1, f_2, \dots, f_n \in \mathcal{R}$ and positive integer d_1, d_2, \dots, d_n such that $in_{\succeq} f_i = x_i^{d_i}$, then \mathcal{R} has a finite SAGBI basis.*

Proof. By Theorem 2.4.9, $\mathbb{k}[x_1, x_2, \dots, x_n]$ is integral over $in_{\succeq} \mathcal{R}$. By corollary 2.4.8, \mathcal{R} has finite SAGBI basis. \square

The following is one of the discouraging example, given by Robbiano and Sweedler [24], where finiteness of SAGBI basis depend on the monomial order.

Proposition 2.4.11 ([24], Example 4.11, pp. 83-84). [*Robbiano and Sweedler Example*]

Let \mathcal{R} be the \mathbb{k} -subalgebra of $\mathbb{k}[x_1, x_2]$ (finitely) generated by $x_1, x_1x_2 - x_2^2, x_1x_2^2$. Then \mathcal{R} has a finite SAGBI basis for a monomial order with $x_2 \succ x_1$ but has no finite SAGBI basis for a monomial order with $x_1 \succ x_2$.

Proof. For the monomial order with $x_2 \succ x_1$, there exist $f_1 = x_1$ and $f_2 = -x_2^2 + x_1x_2$ in \mathcal{R} with $\text{in}_{\succeq} f_1 = x_1$ and $\text{in}_{\succeq} f_2 = x_2^2$. Hence by Corollary 2.4.10, \mathcal{R} has finite SAGBI basis.

Let us consider a monomial order \succeq with $x_1 \succ x_2$.

For any $m \in \mathbb{N}$, set

$$g_{2m} = x_1x_2^{2m}$$

and

$$g_{2m+1} = (m+1)x_1x_2^{2m+1} - x_2^{2m+2}$$

Observe that $g_0 = x_1, g_1 = x_1x_2 - x_2^2$ and $g_2 = x_1x_2^2$ are generators for \mathcal{R} .

Moreover, for any $m \geq 1$, one can verify that

$$g_{2m+1} = mg_0g_{2m} - g_1g_{2m-1}$$

and

$$g_{2m+2} = \frac{1}{2m}[2g_0g_{2m+1} - (2m+2)g_1g_{2m}].$$

Thus for any $m \in \mathbb{N}$, g_{2m} & g_{2m+1} are members of \mathcal{R} . Since $x_1 \succ x_2$, we have $\text{in}_{\succeq} g_{2m+1} = x_1x_2^{2m+1}$. It follows that $\{x_1, x_1x_2, x_1x_2^2, x_1x_2^3, \dots\} \subseteq$

$\text{in}_{\succeq} \mathcal{R}$. Hence $\mathbb{k}[x_1, x_1x_2, x_1x_2^2, x_1x_2^3, \dots] \subseteq \text{in}_{\succeq} \mathcal{R}$.

Notice that in degree n , $\mathbb{k}[x_1, x_1x_2, x_1x_2^2, x_1x_2^3, \dots]$ has \mathbb{k} basis consisting of the monomials

$$x_1^n, x_1^{n-1}x_2, x_1^{n-2}x_2^2, \dots, x_1x_2^{n-1}.$$

Thus $\mathbb{k}[x_1, x_1x_2, x_1x_2^2, x_1x_2^3, \dots]$ and $\text{in}_{\succeq} \mathcal{R}$ have the same Hilbert-Poincaré series. Hence $\mathbb{k}[x_1, x_1x_2, x_1x_2^2, x_1x_2^3, \dots] = \text{in}_{\succeq} \mathcal{R}$. Thus $\text{in}_{\succeq} \mathcal{R}$ is not finitely generated and \mathcal{R} does not have a finite SAGBI basis. \square

Finally let us see whether the property of having a finite SAGBI basis is preserved by an isomorphism. That is, Assume $\alpha : \mathcal{R} \rightarrow \mathcal{S}$ is an isomorphism between \mathbb{k} -subalgebras of $\mathbb{k}[\mathbf{x}]$, and \mathcal{R} has a finite SAGBI basis with respect to a given monomial ordering \succeq . Does \mathcal{S} have a finite SAGBI basis with respect to \succeq ? Let us see the following example.

Example 2.4.12 ([24], Example 4.13, p. 84). ‘let \mathcal{R} be the \mathbb{k} -subalgebra of $\mathbb{k}[x, y]$ generated by $x+y, xy, xy^2$. From example 2.2.5, we know that \mathcal{R} does not have finite SAGBI basis with respect to any monomial ordering. Consider an automorphism α on $\mathbb{k}[x, y]$ which sends x to $x+y$ and y to $-y$. Observe that α is its own inverse and α maps $x+y$ to x and xy to $-xy - y^2$. Thus $x, xy + y^2 \in \alpha(\mathcal{R})$. If $\mathbb{k}[x, y]$ has a monomial ordering with $y \succeq x$, then it follows that $x, y^2 \in \text{In}_{\succeq}(\alpha(\mathcal{R}))$. By corollary 2.4.10, $\alpha(\mathcal{R})$ has a finite SAGBI basis. This shows that the property of having a finite SAGBI basis is not invariant under isomorphism.

Chapter 3

SAGBI bases in a univariate polynomial ring

The existence of finite SAGBI basis for a \mathbb{k} – *subalgebra* of a univariate polynomial ring $\mathbb{k}[x]$ is guaranteed. The main problem in the case of univariate polynomial ring is that whether the generators of a \mathbb{k} – *subalgebra* themselves can form a SAGBI basis. In this chapter, we present necessary and/or sufficient conditions when two or more polynomials form a SAGBI basis.

Let us start our discussion by the following motivational example of Torstensson et al. ([29], p.1087).

Let $f = x^3 + a_2x^2 + a_1x + a_0$ and $g = x^2 + b_1x + b_0$. Is it possible to find a monomial x in $\mathbb{k}[f, g]$?

For some convenient choice $\alpha, \beta, \gamma, \phi$ in \mathbb{k} , we get a polynomial $h(x) = (f^2 - g^3) - \alpha fg - \beta g^2 - \gamma f - \phi g$, which has probably degree 1. However, Torstensson [28] shows that $\mathbb{k}[f, g]$ does not contain a polynomial of degree 1. In 1970, N. S. Mendelsohn [19] proved that a subsemigroup of \mathbb{N} generated by two relatively prime numbers m and k contains all integers greater than or equal to $(m - 1)(k - 1)$.

So, If we are given a positive integer $q \geq 2$, then we can find a polynomial $h \in \mathbb{k}[f, g]$ with $\deg h = q$.

3.1 Existence of finite SAGBI basis

We know that the only ordering on a univariate polynomial ring is the degree ordering. This ordering is defined naturally as $x^i \succeq x^j$ if and only if $i \geq j$, where \geq is the usual greater than or equal to ordering on the set \mathbb{N} . Consequently, we can express any polynomial f in $\mathbb{k}[x]$ in the form of $f = a_mx^m + a_{m-1}x^{m-1} + \dots + a_1x + a_0$ for some $m \in \mathbb{N}$, where $a_m, \dots, a_0 \in \mathbb{k}$ and $a_m \neq 0$. In this case:

- the degree of f , $\deg(f) = m$
- the initial of f , $\text{in}(f) = x^m$
- the leading coefficient of f , $lc(f) = a_m$
- the leading term of f , $lt(f) = a_mx^m$.

Theorem 3.1.1 ([24], Corollary 4.8, p.82). *Every \mathbb{k} – subalgebra of $\mathbb{k}[x]$ is finitely generated.*

Proof. Given a \mathbb{k} – subalgebra \mathcal{R} of $\mathbb{k}[x]$.

Put $B = \{\deg f : f \in \mathcal{R} \setminus \{0\}\}$. Clearly B is an additive subsemi-group of \mathbb{N} . By William Y. Sit and Man-Keung Siu [26], B is finitely generated. Let $\mathcal{S} = \{f_1, f_2, \dots, f_m\} \subseteq \mathcal{R}$ such that $\{\deg(f_1), \deg(f_2), \dots, \deg(f_m)\}$ generates B . By Theorem 2.1.5, \mathcal{S} is a (finite) SAGBI basis for \mathcal{R} . Thus, by Theorem 2.2.2, $\mathcal{R} = \mathbb{k}[\mathcal{S}]$. □

Unlike \mathbb{k} – subalgebras of a multivariate polynomial algebra, each \mathbb{k} – subalgebra of a univariate polynomial algebra $\mathbb{k}[x]$ has a finite

SAGBI basis. So, in univariate polynomial algebra, our focus is not to determine finite SAGBI basis. Our main target here is to determine sufficient conditions when a given set of generators form a SAGBI basis. For example, consider a subset $\mathcal{G} = \{x^2 + x, x^2\}$ of $\mathbb{k}[x]$. Observe that $x = x^2 + x - x^2 \in \mathbb{k}[x^2 + x, x^2]$, the initial of both polynomials in \mathcal{G} is x^2 and x can not be expressed as a power product of x^2 . Thus, \mathcal{G} is not a SAGBI basis for $\mathbb{k}[x^2 + x, x^2]$. This example is actually given by Robbiano and Sweedler ([24], Example 1.18, p.68) to show that generators of a \mathbb{k} -subalgebra may not be a SAGBI basis. The following simple result also notifies that there are infinitely many sets of generators which can not form a SAGBI basis.

Proposition 3.1.2. *Let $f, g \in \mathbb{k}[x]$ be monic polynomials with the same degree. Then $\{f, g\}$ is a SAGBI basis if and only if f and g differ only by their constant terms.*

Proof. Assume that $\{f, g\}$ is a SAGBI basis. Since f and g are monic polynomials with the same degree, the degree of $f - g$ is strictly less than the degree of f . This fact implies that $f - g \in \mathbb{k}$.

Conversely, assume f and g differ only by their constant terms. By Corollary 2.1.9, we have that $\mathbb{k}[f] = \mathbb{k}[g]$. Hence $\{f\}$ is a SAGBI basis for $\mathbb{k}[f, g]$. Thus, $\{f, g\}$ is a SAGBI basis.

□

3.2 Two generators' Case

We should understand the main results of Torstensson et al. ([28],[29]). They focus on the sufficient (and necessary) conditions when two generators form a SAGBI basis in a univariate polynomial algebra over a field \mathbb{k} . We briefly state the main results of Torstensson as follow.

Theorem 3.2.1. ([28], Theorem 10, p.570) *If $f, g \in \mathbb{k}[x]$ are of relatively prime degrees, then $\{f, g\}$ is a SAGBI basis.*

Corollary 3.2.2. ([28], Corollary 11, p.570) *If $f, g \in \mathbb{k}[x]$ are of degree at least two that are relatively prime then $\mathbb{k}[f, g] \neq \mathbb{k}[x]$.*

Theorem 3.2.3. ([28], Theorem 12, p.571) *Let $f, g \in \mathbb{k}[x]$ be such that $\deg f \mid \deg g$. Then $\{f, g\}$ is a SAGBI basis if and only if g is a polynomial in f .*

Theorem 3.2.4. ([28], Theorem 14, p.571) *Let $f, g \in \mathbb{k}[x]$ and $d = \gcd(\deg f, \deg g)$. Then $\{f, g\}$ is a SAGBI basis if and only if there is a polynomial $h \in \mathbb{k}[x]$ of degree d and polynomials F and G such that $f = F \circ h$ and $g = G \circ h$.*

Theorem 3.2.4 does not hold for three or more generators. The following example is given by Torestenson et al.[28] to show the above Theorem does not hold for three generators.

Example 3.2.5. ([29], p.572) Let $f = x^2 - x, g = x^3$, and $r = x^5$. Here $\gcd(\deg f, \deg g, \deg r) = 1$, observe that the three polynomials have also pair wise relatively prime degrees. Now $h = x$, and polynomials $F = f, G = g$, and $R = r$ can satisfy the sufficient condition

for the converse. But $\{f, g, r\}$ can not be a SAGBI basis.

For, $x = fg + f^2 + 2g - r - f \in \mathbb{k}[f, g, r]$ and $\deg(x) = 1$ can not be expressed as an \mathbb{N} -linear combination of $\deg f = 2, \deg g = 3$, and $\deg r = 5$.

Theorem 3.2.6. ([29], Theorem 24, p.1100) *Let f and g be polynomials in $\mathbb{k}[x]$ of degree m and k respectively. Let $m' = \frac{m}{\gcd(k, m)}$, $k' = \frac{k}{\gcd(k, m)}$ and $f^{k'} - g^{m'} \xrightarrow{\{f, g\}^*} h$ with $\deg h = l$. If $\gcd(l, \gcd(m, k)) = 1$, then $\{f, g, h\}$ is a SAGBI basis.*

Remark 3.2.7. In Theorem 3.2.6, the condition $f^{k'} - g^{m'} \xrightarrow{\{f, g\}^*} h$ is very important. If h is not a final subductum of $f^{k'} - g^{m'}$ over $\{f, g\}$, then the theorem would no longer be valid. Look at the following counter example:

Let $f = x^6 + 3x^3, g = x^4 + 2x$ and $h = x^3 + x$.

Clearly $\gcd(\deg h, \gcd(\deg f, \deg g)) = \gcd(3, \gcd(6, 4)) = 1$. But $\{f, g, h\}$ can not be a SAGBI basis. This is because $-x = f^2 - g^3 + 3f - h \in \mathbb{k}[f, g, h]$ and $\deg(-x) = 1$ is not an \mathbb{N} -linear combination of 3, 4 and 6. Observe that h can satisfy all the conditions of the theorem except $f^{q'} - g^{p'} \xrightarrow{\{f, g\}^*} h$.

3.3 Generators with Consecutive degrees

If f and g are non-constant univariate polynomials with consecutive degrees, then by Theorem 3.2.1, $\{f, g\}$ is a SAGBI basis. But we have examples of three or more generators with consecutive degrees that can not form a SAGBI basis. In this section we present necessary and sufficient conditions when three generators with consecutive

degrees form a SAGBI basis.

Theorem 3.3.1. *Let f_1, f_2, \dots, f_m be non-constant polynomials in a univariate polynomial ring $\mathbb{k}[x]$ with degrees $m, m+1, \dots, 2m-1$ respectively. Then $\{f_1, f_2, \dots, f_m\}$ is a SAGBI basis if and only if the subalgebra $\mathbb{k}[f_1, f_2, \dots, f_m]$ does not contain a non-constant polynomial h with $\deg h < m$.*

Proof. Suppose $\{f_1, f_2, \dots, f_m\}$ is a SAGBI basis. For each non-constant polynomial $h \in \mathbb{k}[f_1, f_2, \dots, f_m]$, $\deg h$ is an \mathbb{N} -linear combination of $\deg f_1, \deg f_2, \dots, \deg f_m$. Since $\deg f_i \geq m$ for each $i \in \{1, 2, \dots, m\}$, and at least one of the coefficient's of the linear combination is nonzero, we have that $\deg h \geq m$. Hence $\mathbb{k}[f_1, f_2, \dots, f_m]$ does not contain a non-constant polynomial h with $\deg h < m$.

Conversely, assume $\mathbb{k}[f_1, f_2, \dots, f_m]$ does not contain a non-constant polynomial h with $\deg h < m$. We want to show that $\{f_1, f_2, \dots, f_m\}$ is a SAGBI basis.

Let h be a non-constant polynomial in $\mathbb{k}[f_1, f_2, \dots, f_m]$. By assumption, $\deg h \geq m$.

$$\begin{aligned} \deg h \geq m &\Rightarrow \deg h = km + i \text{ for some } k \geq 1 \text{ and } i \in \{0, 1, 2, \dots, m-1\} \\ &\Rightarrow \deg h = (k-1)m + m + i = (k-1)\deg f_1 + \deg f_{i+1} \end{aligned}$$

Hence by Theorem 2.1.5, $\{f_1, f_2, \dots, f_m\}$ is a SAGBI basis. \square

Observe that, for each $q \geq m$, there exists $h \in \mathbb{k}[f_1, f_2, \dots, f_m]$ such that $\deg h = q$. In other words, each positive integer $q \geq m$ can be expressed as an \mathbb{N} -linear combination of $m, m+1, \dots, 2m-1$. However, if $q < m$ there may or may not exist a polynomial $t \in$

$\mathbb{k}[f_1, f_2, \dots, f_m]$ such that $\deg t = q$. The existence of such t depends on whether $\{f_1, f_2, \dots, f_m\}$ is a SAGBI basis or not.

Example 3.3.2. Let $f_1 = x^3, f_2 = x^4 + x, f_3 = x^5$. Since $x^2 = f_2^2 - f_1 f_3 - 2f_3 \in \mathbb{k}[f_1, f_2, f_3]$ and $\deg x^2 = 2 < 3 = \deg f_1$, we have $\{f_1, f_2, f_3\}$ is not a SAGBI basis.

In Theorem 3.3.1, the degree of f_1 is equal to the number of generators f_1, f_2, \dots, f_m . The following counter example explains that the two statements in Theorem 3.3.1 are not equivalent if we remove f_m from the generating set.

Example 3.3.3. Let $f(x) = x^4 + x^3, g(x) = x^5 + x^4$ and $h(x) = x^6$. Then $\{f, g, h\}$ can not form a SAGBI basis even though $\mathbb{k}[f, g, h]$ does not contain linear polynomials, quadratic polynomials and cubic polynomials.

Indeed, $x^7 = f^2 + g^2 - f(g + h) - h \in \mathbb{k}[f, g, h]$, but we can not express $\deg(x^7) = 7$ as an \mathbb{N} -linear combination of 4, 5, and 6. Hence, by Theorem 2.1.5, $\{f, g, h\}$ can not form a SAGBI basis.

Moreover, using singular, we found that $\{x^4 + x^3, x^5 + x^4, x^6, x^7\}$ is a SAGBI basis for $\mathbb{k}[f, g, h]$. Hence, by Theorem 3.3.1, $\mathbb{k}[f, g, h] = \mathbb{k}[f, g, h, x^7]$ does not contain linear polynomials, quadratic polynomials and cubic polynomials.

Example 3.3.4. Let f_1, f_2, f_3 be polynomials in $\mathbb{k}[x]$ with degrees 4, 5 and 6 respectively. Then $\{f_1, f_2, f_3\}$ is a SAGBI basis if and only if $\mathbb{k}[f_1, f_2, f_3]$ does not contain polynomials of degree 1, 2, 3, and 7.

Suppose $\{f_1, f_2, f_3\}$ is a SAGBI basis. Since 1, 2, 3, and 7 can not be expressed as \mathbb{N} -linear combination of 4, 5, and 6 we can say

that $\mathbb{k}[f_1, f_2, f_3]$ does not contain polynomials of degree 1, 2, 3, and 7.

Conversely, assume that $\mathbb{k}[f_1, f_2, f_3]$ does not contain polynomials of degree 1, 2, 3, and 7. Let h be a non-constant polynomial in $\mathbb{k}[f_1, f_2, f_3]$. If $\deg h$ is either 4, 5, or 6, we can express $\deg h = \deg f_i$, for some $i \in \{1, 2, 3\}$. So, assume $\deg h \geq 8$.

$$\deg h \geq 8 \Rightarrow \deg h = 4k + r \text{ where } k \geq 2 \text{ and } r \in \{0, 1, 2, 3\}.$$

case-1 : if $r = 0$

$$\begin{aligned} \deg h &= 4k \\ &= k \deg f_1 \end{aligned}$$

case-2 : if $r = 1$

$$\begin{aligned} \deg h &= 4k + 1 \\ &= 4(k - 1) + 5 \\ &= (k - 1) \deg f_1 + \deg f_2 \end{aligned}$$

case-3 : if $r = 2$

$$\begin{aligned} \deg h &= 4k + 2 \\ &= 4(k - 1) + 6 \\ &= (k - 1) \deg f_1 + \deg f_3 \end{aligned}$$

case-4 : if $r = 3$

$$\begin{aligned} \deg h &= 4k + 3 \\ &= 4(k - 2) + 11 \\ &= (k - 2) \deg f_1 + \deg f_2 + \deg f_3 \end{aligned}$$

In all cases $\deg h$ can be expressed as an \mathbb{N} – *linear* combination of $\deg f_1, \deg f_2$ and $\deg f_3$. Hence, by Theorem 2.1.5, $\{f_1, f_2, f_3\}$ is a SAGBI basis. \square

In the explanation of the above example observe that the subsemi-group $\langle 4, 5, 6 \rangle$ of \mathbb{N} contains all integers greater than or equal to 8.

Lemma 3.3.5. *Let f_1, f_2, \dots, f_t be polynomials in $\mathbb{k}[x]$ with $\deg f_i = m + i - 1$, for some positive integer $m > t$. If $\{f_1, f_2, \dots, f_t\}$ is a SAGBI basis then the \mathbb{k} – subalgebra $\mathbb{k}[f_1, f_2, \dots, f_t]$ does not contain polynomials of degree $1, 2, 3, \dots, m - 1, m + t, m + t + 1, \dots, 2m - 1$.*

Proof. Suppose $\{f_1, f_2, \dots, f_t\}$ is a SAGBI basis. Since all $1, 2, \dots, m - 1$ are less than $\deg f_1 = m$, we can not express $1, 2, \dots, m - 1$ as \mathbb{N} – *linear* combination of $\deg f_1, \deg f_2, \dots, \deg f_t$. Hence $\mathbb{k}[f_1, \dots, f_t]$ does not contain polynomials of degree $1, 2, \dots, m - 1$.

Suppose $h \in \mathbb{k}[f_1, \dots, f_t]$ with $m + t \leq \deg h \leq 2m - 1$. Since $\{f_1, f_2, \dots, f_t\}$ is a SAGBI basis, by Theorem 2.1.5, we have $\deg h = \alpha_1 \deg f_1 + \alpha_2 \deg f_2 + \dots + \alpha_t \deg f_t$ for some $\alpha_1, \alpha_2, \dots, \alpha_t \in \mathbb{N}$. Since $\deg h > \deg f_t = m + t - 1$, at least two of α_i and α_j should be nonzero (or at least one α_i should be greater than or equal to 2). Hence,

$$\begin{aligned}
 \deg h &= \alpha_1 \deg f_1 + \alpha_2 \deg f_2 + \dots + \alpha_t \deg f_t. \\
 &\geq \alpha_i \deg f_i + \alpha_j \deg f_j. \\
 &= \alpha_i(m + i - 1) + \alpha_j(m + j - 1). \\
 &= \alpha_i m + \alpha_j m + \alpha_i(i - 1) + \alpha_j(j - 1). \\
 &\geq 2m.
 \end{aligned}$$

This contradicts with $\deg h \leq 2m - 1$. Therefore, $\mathbb{k}[f_1, \dots, f_t]$ does not contain polynomials of degree $m + t, \dots, 2m - 1$.

Hence $\{f_1, f_2, \dots, f_t\}$ is a SAGBI basis implies that $\mathbb{k}[f_1, f_2, \dots, f_t]$ does not contain polynomials of degree $1, 2, 3, \dots, m - 1, m + t, m + t + 1, \dots, 2m - 1$ \square

Theorem 3.3.6. *Given an even integer $m \geq 4$. Let $f_1, f_2, \dots, f_{\frac{m}{2}+1}$ be polynomials in $\mathbb{k}[x]$ with $\deg f_i = m + i - 1$. Then $\{f_1, f_2, \dots, f_{\frac{m}{2}+1}\}$ is a SAGBI basis if and only if $\mathbb{k}[f_1, f_2, \dots, f_{\frac{m}{2}+1}]$ does not contain polynomials of degree $1, 2, \dots, m - 1, m + \frac{m}{2} + 1, m + \frac{m}{2} + 2, \dots, 2m - 1$.*

Proof. Since $m > \frac{m}{2} + 1$ for any $m \geq 4$, by Lemma 3.3.5, $\{f_1, f_2, \dots, f_{\frac{m}{2}+1}\}$ is a SAGBI basis implies that $\mathbb{k}[f_1, f_2, \dots, f_{\frac{m}{2}+1}]$ does not contain polynomials of degree $1, 2, \dots, m - 1, m + \frac{m}{2} + 1, m + \frac{m}{2} + 2, \dots, 2m - 1$. Conversely, suppose $\mathbb{k}[f_1, \dots, f_{\frac{m}{2}+1}]$ does not contain polynomials of degree $1, 2, \dots, m - 1, m + \frac{m}{2} + 1, m + \frac{m}{2} + 2, \dots, 2m - 1$. We want to show that $\{f_1, \dots, f_{\frac{m}{2}+1}\}$ is a SAGBI basis.

Let $h \in \mathbb{k}[f_1, \dots, f_{\frac{m}{2}+1}] \setminus \mathbb{k}$. Then either $m \leq \deg h \leq m + \frac{m}{2}$ or $\deg h \geq 2m$.

If $m \leq \deg h \leq m + \frac{m}{2}$, then $\deg h = \deg f_i$ for some $i \in \{1, 2, \dots, \frac{m}{2} + 1\}$. Assume $\deg h \geq 2m$. By division algorithm, $\deg h = qm + r$ where $q \geq 2$ and $r \in \{0, 1, \dots, m - 1\}$.

Case 1:- If $0 \leq r \leq \frac{m}{2}$,

$$\begin{aligned} \deg h &= qm + r \\ &= (q - 1)m + m + r \\ &= (q - 1) \deg f_1 + \deg f_{r+1} \end{aligned}$$

Case 2:- If $\frac{m}{2} < r \leq m - 1$, then r can be expressed as $r = \frac{m}{2} + i - 1$

for some $i \in \{2, \dots, \frac{m}{2}\}$. So,

$$\begin{aligned}
\deg h &= qm + r \\
&= (q - 2)m + 2m + \frac{m}{2} + i - 1, \text{ for some } i \in \{2, \dots, \frac{m}{2}\} \\
&= (q - 2)m + (m + \frac{m}{2} - 1) + (m + i), \text{ for some } i \in \{2, \dots, \frac{m}{2}\} \\
&= (q - 2) \deg f_1 + \deg f_{\frac{m}{2}} + \deg f_{i+1}, \text{ for some } i \in \{2, \dots, \frac{m}{2}\}
\end{aligned}$$

In both cases we can express $\deg h$ as \mathbb{N} - linear combination of $\deg f_1, \deg f_2, \dots, \deg f_{\frac{m}{2}+1}$. Hence, by Theorem 2.1.5, $\{f_1, \dots, f_{\frac{m}{2}+1}\}$ is a SAGBI basis. \square

Theorem 3.3.7. *Given an odd integer $m \geq 3$. Let $f_1, f_2, \dots, f_{\frac{m-1}{2}+1}$ be polynomials in $\mathbb{k}[x]$ with $\deg f_i = m+i-1$. Then $\{f_1, f_2, \dots, f_{\frac{m-1}{2}+1}\}$ is a SAGBI basis if and only if $\mathbb{k}[f_1, f_2, \dots, f_{\frac{m-1}{2}+1}]$ does not contain polynomials of degree $1, 2, \dots, m-1, m + \frac{m-1}{2} + 1, m + \frac{m-1}{2} + 2, \dots, 2m-1$.*

Proof. Since $m > \frac{m-1}{2} + 1$ for any $m \geq 3$, by Lemma 3.3.5, if $\{f_1, f_2, \dots, f_{\frac{m-1}{2}+1}\}$ is a SAGBI basis then $\mathbb{k}[f_1, f_2, \dots, f_{\frac{m-1}{2}+1}]$ does not contain polynomials of degree $1, 2, \dots, m-1, m + \frac{m-1}{2} + 1, \dots, 2m-1$. Conversely, suppose $\mathbb{k}[f_1, \dots, f_{\frac{m-1}{2}+1}]$ does not contain polynomials of degree $1, 2, \dots, m-1, m + \frac{m-1}{2} + 1, m + \frac{m-1}{2} + 2, \dots, 2m-1$. We want to show that $\{f_1, \dots, f_{\frac{m-1}{2}+1}\}$ is a SAGBI basis.

Let $h \in \mathbb{k}[f_1, \dots, f_{\frac{m-1}{2}+1}] \setminus \mathbb{k}$. Then either $m \leq \deg h \leq m + \frac{m-1}{2}$ or $\deg h \geq 2m$

If $m \leq \deg h \leq m + \frac{m-1}{2}$, then $\deg h = \deg f_i$, for some $i \in \{1, 2, \dots, \frac{m-1}{2} + 1\}$. Assume $\deg h \geq 2m$. By division algorithm, $\deg h = qm + r$, where $q \geq 2$, and $r \in \{0, 1, \dots, m-1\}$.

Case 1:- If $0 \leq r \leq \frac{m-1}{2}$,

$$\begin{aligned} \deg h &= qm + r \\ &= (q-1)m + m + r \\ &= (q-1) \deg f_1 + \deg f_{r+1} \end{aligned}$$

Case 2:- If $\frac{m-1}{2} < r \leq m-1$, then r can be expressed as $r = \frac{m-1}{2} + i - 1$ for some $i \in \{2, \dots, \frac{m-1}{2} + 1\}$. So,

$$\begin{aligned} \deg h &= qm + r \\ &= (q-2)m + 2m + \frac{m-1}{2} + i - 1, \text{ for } i \in \{2, \dots, \frac{m-1}{2} + 1\} \\ &= (q-2)m + (m + \frac{m-1}{2}) + (m + i - 1), \text{ for } i \in \{2, \dots, \frac{m-1}{2} + 1\} \\ &= (q-2) \deg f_1 + \deg f_{\frac{m-1}{2}+1} + \deg f_i, \text{ for } i \in \{2, \dots, \frac{m-1}{2} + 1\} \end{aligned}$$

In both cases we can express $\deg h$ as \mathbb{N} -linear combination of $\deg f_1, \deg f_2, \dots, \deg f_{\frac{m-1}{2}+1}$. Hence, by Theorem 2.1.5, $\{f_1, \dots, f_{\frac{m-1}{2}+1}\}$ is a SAGBI basis. \square

Corollary 3.3.8. *Let f and g be polynomials in $\mathbb{k}[x]$ with $\deg f = m$ and $\deg g = m+1$ for some $m \in \mathbb{N}_{\geq 2}$. Then $\mathbb{k}[f, g]$ does not contain polynomials of degree $1, 2, \dots, m-1, m+2, m+3, \dots, 2m-1$.*

Proof. Since any two consecutive positive integers are relatively prime, by Theorem 3.2.1 $\{f, g\}$ is a SAGBI basis. Thus, by Lemma 3.3.5, $\mathbb{k}[f, g]$ does not contain polynomials of degree $1, 2, \dots, m-1, m+2, m+3, \dots, 2m-1$. \square

More specifically we have the following;

- If $\deg f = 2$ and $\deg g = 3$, then $\mathbb{k}[f, g]$ does not contain linear polynomials. But, by Theorem 3.3.1, the subalgebra $\mathbb{k}[f, g]$ contains a polynomial of degree d , for any $d \geq 2$.

- If $\deg f = 3$ and $\deg g = 4$, then $\mathbb{k}[f, g]$ does not contain polynomials of degree 1, 2 and 5. But, by Theorem 3.3.7, the subalgebra $\mathbb{k}[f, g]$ contains a polynomial of degree d , for any $d \geq 6$.
- If $\deg f = 4$ and $\deg g = 5$, then $\mathbb{k}[f, g]$ does not contain polynomials of degree 1, 2, 3, 6 and 7. Moreover, by the Theorem of N.S.Mendelsohn [19], we know that $\mathbb{k}[f, g]$ does not contain a polynomial of degree 11 but for any $d \in \mathbb{N}_{\geq 12}$ there is a polynomial $h \in \mathbb{k}[f, g]$ such that $\deg h = d$.

The number of generators in Theorem 3.3.6 and Theorem 3.3.7 depends on the degree of f_1 . The following two Theorems give sufficient and necessary conditions when three generators with consecutive degree form a SAGBI basis.

Theorem 3.3.9. *Let $m \geq 4$ be an even integer and \mathcal{H} be an additive monoid generated by $\{m, m+1, m+2\}$. Then $q \in \mathcal{H}$ if and only if $q \notin \cup_{i=0}^{\frac{m}{2}-1} \{t \in \mathbb{N} : i(m+2) < t < (i+1)m\}$.*

Proof. Assume $q \in \mathcal{H}$.

$$\begin{aligned} q \in \mathcal{H} &\Rightarrow q = \alpha_1 m + \alpha_2(m+1) + \alpha_3(m+2) \text{ for some } \alpha_1, \alpha_2, \alpha_3 \in \mathbb{N} \\ &\Rightarrow q = (\alpha_1 + \alpha_2 + \alpha_3)m + (\alpha_2 + 2\alpha_3) \text{ for some } \alpha_1, \alpha_2, \alpha_3 \in \mathbb{N} \end{aligned}$$

Claim $q \notin \cup_{i=0}^{\frac{m}{2}-1} \{t \in \mathbb{N} : i(m+2) < t < (i+1)m\}$.

To the contrary assume $i(m+2) < q < (i+1)m$, for some $i \in \{0, 1, 2, \dots, \frac{m}{2} - 1\}$. Hence, $q = (\alpha_1 + \alpha_2 + \alpha_3)m + (\alpha_2 + 2\alpha_3)$ is greater than $i(m+2) = im + 2i$. Thus, either $\alpha_1 + \alpha_2 + \alpha_3 > i$ or $\alpha_1 + \alpha_2 + \alpha_3 \leq i$ and $\alpha_2 + 2\alpha_3 > 2i$.

Case-1:- If $\alpha_1 + \alpha_2 + \alpha_3 > i$

$$\begin{aligned}\alpha_1 + \alpha_2 + \alpha_3 > i &\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 \geq i + 1 \\ &\Rightarrow q \geq (\alpha_1 + \alpha_2 + \alpha_3)m \geq (i + 1)m\end{aligned}$$

Which contradicts with $q < (i + 1)m$

Case-2:- If $\alpha_1 + \alpha_2 + \alpha_3 \leq i$ and $\alpha_2 + 2\alpha_3 > 2i$.

$$\begin{aligned}\alpha_2 + 2\alpha_3 > 2i &\Rightarrow \alpha_2 + 2\alpha_3 > 2(\alpha_1 + \alpha_2 + \alpha_3) \\ &\Rightarrow 0 > 2\alpha_1 + \alpha_2\end{aligned}$$

Which contradicts with $\alpha_1, \alpha_2 \in \mathbb{N}$

Hence $q \notin \cup_{i=0}^{\frac{m}{2}-1} \{t \in \mathbb{N} : i(m + 2) < t < (i + 1)m\}$.

Conversely, assume $q \notin \cup_{i=0}^{\frac{m}{2}-1} \{t \in \mathbb{N} : i(m + 2) < t < (i + 1)m\}$.

Claim $q \in \mathcal{H}$

$$q \notin \cup_{i=0}^{\frac{m}{2}-1} \{t \in \mathbb{N} : i(m + 2) < t < (i + 1)m\}$$

$$\Rightarrow q \notin (i(m + 2), (i + 1)m), \text{ for all } i = 0, 1, \dots, \frac{m}{2} - 1$$

$$\Rightarrow q \in [(i + 1)m, (i + 1)(m + 2)], \text{ for some } i = 0, 1, \dots, \frac{m}{2} - 2$$

$$\text{or } q \in [(\frac{m}{2})m, \infty)$$

Case-1:- If $q \in [(i + 1)m, (i + 1)(m + 2)]$, for some $i = 0, 1, \dots, \frac{m}{2} - 2$,

then we have $(i + 1)m \leq q \leq (i + 1)m + 2(i + 1)$. So, $q = (i + 1)m + r$, where $0 \leq r \leq 2(i + 1)$, Now,

Case-1.1:- If r is even

$\frac{r-2}{2}$ is an integer which is less than or equal to i , that is, $i - \frac{r-2}{2} \in \mathbb{N}$,

and

$$\begin{aligned}q &= (i + 1)m + r \\ &= (i - \frac{r-2}{2})m + \frac{r}{2}(m + 2)\end{aligned}$$

Hence $q \in \mathcal{H}$

Case-1.2:- If r is odd

$\frac{r-1}{2}$ is a non-negative integer which is less than or equal to i , that is,
 $i - \frac{r-1}{2} \in \mathbb{N}$, and

$$\begin{aligned} q &= (i+1)m + r \\ &= \left(i - \frac{r-1}{2}\right)m + \frac{r-1}{2}(m+2) + (m+1) \end{aligned}$$

Hence $q \in \mathcal{H}$.

Case-2:- If $q \in [(\frac{m}{2})m, \infty)$

$(\frac{m}{2})m \leq q \Rightarrow q = pm + r$, where $p \geq \frac{m}{2}$ and $r \in \{0, 1, \dots, m-1\}$

Case-2.1:- If r is even

$\frac{r}{2}$ is a non-negative integer which is less than or equal to p , that is,
 $p - \frac{r}{2} \in \mathbb{N}$, and

$$\begin{aligned} q &= pm + r \\ &= \left(p - \frac{r}{2}\right)m + \frac{r}{2}(m+2) \end{aligned}$$

Hence $q \in \mathcal{H}$

Case-2.2:- If r is odd

$\frac{r+1}{2}$ is a non-negative integer which is between 1 and p , that is,
 $p - \frac{r+1}{2}$, and $\frac{r+1}{2} - 1 \in \mathbb{N}$, and

$$\begin{aligned} q &= pm + r \\ &= \left(p - \frac{r+1}{2}\right)m + \left(\frac{r+1}{2} - 1\right)(m+2) + (m+1) \end{aligned}$$

Hence $q \in \mathcal{H}$.

In all possible cases $q \in \mathcal{H}$. Hence the Theorem Holds. \square

Corollary 3.3.10. *Given an even integer $m \geq 4$. Let $f_1, f_2, f_3 \in \mathbb{k}[x]$ with $\deg f_1 = m$, $\deg f_2 = m+1$ and $\deg f_3 = m+2$. Then*

$\{f_1, f_2, f_3\}$ is a SAGBI basis if and only if $\mathbb{k}[f_1, f_2, f_3]$ does not contain a polynomial h with $\deg h \in \cup_{i=0}^{\frac{m}{2}-1} \{t : i(m+2) < t < (i+1)m\}$.

Theorem 3.3.11. *Let $m \geq 5$ be an odd integer and \mathcal{H} be an additive monoid generated by $\{m, m+1, m+2\}$. Then $q \in \mathcal{H}$ if and only if $q \notin \cup_{i=0}^{\frac{m-1}{2}-1} \{t \in \mathbb{N} : i(m+2) < t < (i+1)m\}$.*

Proof. Assume $q \in \mathcal{H}$.

$$\begin{aligned} q \in \mathcal{H} &\Rightarrow q = \alpha_1 m + \alpha_2(m+1) + \alpha_3(m+2), \text{ for some } \alpha_1, \alpha_2, \alpha_3 \in \mathbb{N} \\ &\Rightarrow q = (\alpha_1 + \alpha_2 + \alpha_3)m + (\alpha_2 + 2\alpha_3), \text{ for some } \alpha_1, \alpha_2, \alpha_3 \in \mathbb{N} \end{aligned}$$

Claim $q \notin \cup_{i=0}^{\frac{m-1}{2}-1} \{t \in \mathbb{N} : i(m+2) < t < (i+1)m\}$.

To the contrary, assume $i(m+2) < q < (i+1)m$, for some $i \in \{0, 1, 2, \dots, \frac{m-1}{2} - 1\}$. Hence, $q = (\alpha_1 + \alpha_2 + \alpha_3)m + (\alpha_2 + 2\alpha_3)$ is greater than $i(m+2) = im + 2i$. Thus, either $\alpha_1 + \alpha_2 + \alpha_3 > i$ or $\alpha_1 + \alpha_2 + \alpha_3 \leq i$ and $\alpha_2 + 2\alpha_3 > 2i$.

Case-1:- If $\alpha_1 + \alpha_2 + \alpha_3 > i$

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 > i &\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 \geq i + 1 \\ &\Rightarrow q \geq (\alpha_1 + \alpha_2 + \alpha_3)m \geq (i+1)m \end{aligned}$$

Which contradicts with $q < (i+1)m$

Case-2:- If $\alpha_1 + \alpha_2 + \alpha_3 \leq i$ and $\alpha_2 + 2\alpha_3 > 2i$.

$$\begin{aligned} \alpha_2 + 2\alpha_3 > 2i &\Rightarrow \alpha_2 + 2\alpha_3 > 2(\alpha_1 + \alpha_2 + \alpha_3) \\ &\Rightarrow 0 > 2\alpha_1 + \alpha_2 \end{aligned}$$

Which contradicts with $\alpha_1, \alpha_2 \in \mathbb{N}$

Hence $q \notin \cup_{i=0}^{\frac{m-i}{2}-1} \{t \in \mathbb{N} : i(m+2) < t < (i+1)m\}$.

Conversely, assume $q \notin \cup_{i=0}^{\frac{m-1}{2}-1} \{t \in \mathbb{N} : i(m+2) < t < (i+1)m\}$.

Claim $q \in \mathcal{H}$

$$q \notin \cup_{i=0}^{\frac{m-1}{2}-1} \{t \in \mathbb{N} : i(m+2) < t < (i+1)m\}$$

$$\Rightarrow q \notin (i(m+2), (i+1)m),$$

$$\text{for all } i = 0, 1, \dots, \frac{m-1}{2} - 1$$

$$\Rightarrow q \in [(i+1)m, (i+1)(m+2)],$$

$$\text{for some } i = 0, 1, \dots, \frac{m-1}{2} - 2 \text{ or } q \in [\frac{m-1}{2}m, \infty)$$

Case-1:- If $q \in [(i+1)m, (i+1)(m+2)]$ for some $i = 0, 1, \dots, \frac{m-1}{2} - 2$, then we have $(i+1)m \leq q \leq (i+1)m + 2(i+1)$ for some $i = 0, 1, \dots, \frac{m-1}{2} - 2$. So, $q = (i+1)m + r$, where $0 \leq r \leq 2(i+1)$. Now,

Case-1.1:- If r is even

$\frac{r-2}{2}$ is an integer which is less than i , that is, $i - \frac{r-2}{2} \in \mathbb{N}$ and

$$\begin{aligned} q &= (i+1)m + r \\ &= (i - \frac{r-2}{2})m + \frac{r}{2}(m+2) \end{aligned}$$

Hence $q \in \mathcal{H}$

Case-1.2:- If r is odd

$\frac{r-1}{2}$ is a non-negative integer which is less than or equal to i , that is, $i - \frac{r-1}{2} \in \mathbb{N}$ and

$$\begin{aligned} q &= (i+1)m + r \\ &= (i - \frac{r-1}{2})m + \frac{r-1}{2}(m+2) + (m+1) \end{aligned}$$

Hence $q \in \mathcal{H}$.

Case-2:- If $q \in [\frac{m-1}{2}m, \infty)$

$\frac{m-1}{2}m \leq q \Rightarrow q = pm + r$, where $p \geq \frac{m-1}{2}$ and $r \in \{0, 1, \dots, m-1\}$

Case-2.1:- If r is even

$\frac{r}{2}$ is a non-negative integer which is less than or equal to p , that is,
 $p - \frac{r}{2} \in \mathbb{N}$ and

$$\begin{aligned} q &= pm + r \\ &= \left(p - \frac{r}{2}\right)m + \frac{r}{2}(m + 2) \end{aligned}$$

Hence $q \in \mathcal{H}$

Case-2.2:- If r is odd

$\frac{r+1}{2}$ is a non-negative integer which is between 1 and p , that is,
 $p - \frac{r+1}{2}$, and $\frac{r+1}{2} - 1 \in \mathbb{N}$ and

$$\begin{aligned} q &= pm + r \\ &= \left(p - \frac{r+1}{2}\right)m + \left(\frac{r+1}{2} - 1\right)(m + 2) + (m + 1) \end{aligned}$$

Hence $q \in \mathcal{H}$.

In all possible cases $q \in \mathcal{H}$. Hence the Theorem Holds □

Corollary 3.3.12. *Given an odd integer $m \geq 5$. Let $f_1, f_2, f_3 \in \mathbb{k}[x]$ with $\deg f_1 = m$, $\deg f_2 = m+1$ and $\deg f_3 = m+2$. Then $\{f_1, f_2, f_3\}$ is a SAGBI basis if and only if $\mathbb{k}[f_1, f_2, f_3]$ does not contain a polynomial h with $\deg h \in \cup_{i=0}^{\frac{m-1}{2}-1} \{t : i(m+2) < t < (i+1)m\}$.*

Chapter 4

SAGBI basis in a Multivariate Polynomial Ring

In this chapter we present sufficient and necessary conditions when \mathbb{k} – *subalgebras* have finite SAGBI basis with respect to a given monomial order. We support our results by examples. Finally we conjecture that any \mathbb{k} – *subalgebra* generated by two polynomials has a finite SAGBI basis with respect to a given monomial order.

4.1 SAGBI basis for a \mathbb{k} subalgebra generated by two polynomials

Some times the \mathbb{k} – *subalgebra* $\mathbb{k}[f, g]$ may be contained in the \mathbb{k} – *subalgebra* generated by a monomial θ while $\theta \notin \mathbb{k}[f, g]$. In this section we proved that such \mathbb{k} – *subalgebra* has a finite SAGBI basis. We also present a sufficient (and necessary) condition when $\{f, g\}$ is a SAGBI basis.

Lemma 4.1.1. *Let L be a one dimensional subspace of \mathbb{R}^m . Then the set of lattice points $S = L \cap \mathbb{N}^m$ is a monoid generated by one*

element.

Proof. Since L and \mathbb{N}^m are submonoids of $(\mathbb{R}^m, +)$, and the intersection of two submonoids of a monoid is again a submonoid, we can say that S is a monoid. So, we need only to show that S is generated by one element as a monoid.

Let $\mathbf{a} = (a_1, a_2, \dots, a_m)$ be a nonzero vector in S of minimal norm (under the standard Euclidean norm).

Claim:- $S = \langle \mathbf{a} \rangle$

Since $\mathbf{a} \in S$, we have $\langle \mathbf{a} \rangle \subseteq S$. On the other hand, for arbitrary $\mathbf{v} = (v_1, v_2, \dots, v_m) \in S$, there is some $\lambda \in \mathbb{Q}_0^+$ such that $\mathbf{v} = \lambda \mathbf{a}$. Take a nonzero component a_{t_0} of \mathbf{a} . By division algorithm, $\exists! q_{t_0}, r_{t_0} \in \mathbb{Z}$ such that $v_{t_0} = q_{t_0} a_{t_0} + r_{t_0}$ where $0 \leq r_{t_0} < a_{t_0}$. Thus, since $\lambda a_{t_0} = v_{t_0} = q_{t_0} a_{t_0} + r_{t_0}$, we have $(\lambda - q_{t_0}) a_{t_0} = r_{t_0}$, and from the fact that $0 \leq r_{t_0} < a_{t_0}$, we have $0 \leq \lambda - q_{t_0} < 1$.

Now let $r_i = v_i - q_{t_0} a_i$ for each $i \in \{1, 2, 3, \dots, m\}$. We claim that $0 \leq r_i < a_i$ for each i . Indeed, $r_i = v_i - q_{t_0} a_i = \lambda a_i - q_{t_0} a_i = (\lambda - q_{t_0}) a_i$ and hence $0 \leq r_i < a_i$.

To summarize, we have shown that $v_i = q_{t_0} a_i + r_i$ where $0 \leq r_i < a_i$, and hence $\mathbf{v} = q_{t_0} \mathbf{a} + \mathbf{r}$ where $\mathbf{r} = (r_1, r_2, \dots, r_m)$. It follows that $\mathbf{r} = \mathbf{v} - q_{t_0} \mathbf{a} \in S$. This directly implies that $\mathbf{r} = \mathbf{0}$ (as \mathbf{a} is a vector of minimal norm and the norm of \mathbf{r} is smaller than the norm of \mathbf{a}). Hence $\mathbf{v} = q_{t_0} \mathbf{a} \in \langle \mathbf{a} \rangle$. Therefore, $S = \langle \mathbf{a} \rangle$ □

Theorem 4.1.2. *Let $f, g \in \mathbb{k}[\mathbf{x}]$. If the multi-degrees of each monomial in f and g all lie on a line through the origin, then $\mathbb{k}[f, g]$ has a finite SAGBI basis with respect to a given $\succeq \in \Omega$.*

Proof. Let L be the set of points on a line passing through the origin

and one of the (nonzero) multi-degrees of the monomials in f or g . Clearly L is a one dimensional subspace of \mathbb{R}^n . Hence, by lemma 4.1.1, $S = L \cap \mathbb{N}^n$ is generated by one element as a monoid, say $S = \langle \mathbf{u} \rangle$. Since the multi-degrees of each monomial in f and g are members of S , we can express each monomial in f and g in the form of $(\mathbf{x}^{\mathbf{u}})^m$ for some $m \in \mathbb{N}$. If we replace $\mathbf{x}^{\mathbf{u}}$ by a generic variable y , the polynomials f and g can be seen as a polynomial in one variable y . By Theorem 3.1.1, $\mathbb{k}[f, g]$ has a finite SAGBI basis. \square

Remark 4.1.3. The above theorem hold for arbitrary finite set of generators.

Theorem 4.1.2 does not imply that f and g themselves form a SAGBI basis. Actually we can construct an example which satisfies all the sufficient condition of Theorem 4.1.2, but $\{f, g\}$ does not form a SAGBI basis.

Example 4.1.4. Let $f = x^6y^{12} + xy^2$ and $g = x^3y^6$. The multi-degree of each monomial in f and g lie on the line passing through $(0, 0)$ and $(1, 2)$. Hence, Theorem 4.1.2 assures that, $\mathbb{k}[f, g]$ has finite SAGBI basis with respect to any monomial ordering. But $\{f, g\}$ itself is not a SAGBI basis with respect to a given $\succeq \in \Omega$. This is because, $xy^2 = f - g^2 \in \mathbb{k}[f, g]$ can not be expressed as a power product of $\text{in}_{\succeq} f = x^6y^{12}$ and $\text{in}_{\succeq} g = x^3y^6$ (observe that $xy^2 \nmid x^6y^{12}$).

If we consider f and g that satisfy the sufficient condition of Theorem 4.1.2, and a component of multi degree of $\text{in}_{\succeq} f$ and corresponding component of multi-degree of $\text{in}_{\succeq} g$ are relatively prime, then $\{f, g\}$ can form a SAGBI basis with respect to \succeq . We can explain this fact

using the following example.

Example 4.1.5. Let $f = 2x^{15}y^{30} + \sqrt{3}x^{12}y^{24} + \pi x^4y^8$ and $g = \sqrt{2}x^4y^8 + 3x^3y^6 + xy^2$. Then with respect to a given $\succeq \in \Omega$, $\{f, g\}$ is a SAGBI basis.

In the above example note that all the multi-degrees of each monomial in both f and g lie on the line passing through the origin and $(1, 2)$. Observe also that the first component, 15, of $\deg(\text{in}_{\succeq} f)$ and the first component, 4, of $\deg(\text{in}_{\succeq} g)$ are relatively prime.

We can rewrite f and g as $f = 2(xy^2)^{15} + \sqrt{3}(xy^2)^{12} + \pi(xy^2)^4$ and $g = \sqrt{2}(xy^2)^4 + 3(xy^2)^3 + xy^2$. Now replace xy^2 by a generic variable t and obtain $f = 2t^{15} + \sqrt{3}t^{12} + \pi t^4$ and $g = \sqrt{2}t^4 + 3t^3 + t$. Since the $\deg f(t) = 15$ and $\deg g(t) = 4$ are relatively prime, by Theorem 3.2.1, $\{f, g\}$ itself is a SAGBI basis.

Moreover, there are also examples of f and g satisfying the sufficient condition of Theorem 4.1.2, and all components of multi-degrees of $\text{in}_{\succeq} f$ are not relatively prime with corresponding components of multi-degrees of $\text{in}_{\succeq} g$, but $\{f, g\}$ form a SAGBI basis.

Example 4.1.6. Let $f = x^6y^{10} + x^3y^5$ and $g = x^{21}y^{35} + x^{12}y^{20}$. Then $\{f, g\}$ is a SAGBI basis with respect to any $\succeq \in \Omega$.

Here the multi-degrees of each monomial in both f and g lie on the line passing through the origin and $(3, 5)$. Observe also that both components of $\text{in}_{\succeq} f$ are not relatively prime with corresponding components of $\text{in}_{\succeq} g$. However, we can rewrite f and g as, $f = (x^3y^5)^2 + x^3y^5$ and $g = (x^3y^5)^7 + (x^3y^5)^4$. Now replace x^3y^5 by a generic variable t and obtain $f(t) = t^2 + t$ and $g(t) = t^7 + t^4$. Since

the $\deg f(t) = 2$ and $\deg g(t) = 7$ are relatively prime, by Theorem 3.2.1, $\{f, g\}$ itself is a SAGBI basis. \square

Corollary 4.1.7. *Let f and g be polynomials which satisfy the sufficient condition of Theorem 4.1.2, and $\mathbf{a} = (a_1, a_2, \dots, a_n)$ and $\mathbf{b} = (b_1, b_2, \dots, b_n)$ be the multi-degrees of $\text{in}_{\succeq} f$ and $\text{in}_{\succeq} g$ respectively. If $d_i := \gcd(a_i, b_i)$ and $\gcd(d_i, d_j) = 1$ for some $i, j \in \{1, 2, \dots, n\}$, then $\{f, g\}$ is a SAGBI basis.*

Proof. Since f and g satisfy the sufficient condition of Theorem 4.1.2, each monomial of f and g can be expressed as $(\mathbf{x}^{\mathbf{u}})^q$ for some $q \in \mathbb{N}$ and $\mathbf{u} = (u_1, u_2, \dots, u_n) \in L \cup \mathbb{N}^n$. So $\mathbf{a} = m\mathbf{u}$ and $\mathbf{b} = k\mathbf{u}$ for some $m, k \in \mathbb{N}$. Put $t = \gcd(m, k)$. Since $\mathbf{a} = m\mathbf{u}$ and $\mathbf{b} = k\mathbf{u}$, we have $a_i = mu_i$, $b_i = ku_i$, $a_j = mu_j$ and $b_j = ku_j$. Therefore, $d_i = \gcd(a_i, b_i) = \gcd(mu_i, ku_i) = \gcd(m, k)u_i = tu_i$, and similarly $d_j = tu_j$. Since d_i and d_j are relatively prime, we have $t = 1$, and hence m and k are relatively prime. Moreover, from compatibility property of monomial order we have $\mathbf{x}^{q\mathbf{u}} \succeq \mathbf{x}^{p\mathbf{u}}$ if and only if $q \geq p$. Hence after replacing $\mathbf{x}^{\mathbf{u}}$ by a generic variable y we obtain $\deg(f) = m$ and $\deg(g) = k$. Hence, by Theorem 3.2.1, $\{f, g\}$ is a SAGBI basis. \square

Observe that, if f and g satisfy the sufficient condition of Theorem 4.1.2, then there exists a monomial $\theta \in \mathbb{k}[x_1, \dots, x_n]$ and $F, G \in \mathbb{k}[y]$ such that $f = F \circ \theta$ and $g = G \circ \theta$. This observation can be easily seen in the proof of Theorem 4.1.2 where $\theta = \mathbf{x}^{\mathbf{u}}$.

The following theorem is an easy consequence of the above observation and the main result of Torstensson ([28], Theorem15, P.572).

Theorem 4.1.8. *Let $f, g \in \mathbb{k}[x_1, x_2, \dots, x_n]$ and $F, G \in \mathbb{k}[t]$ such that $f = F \circ \theta$ and $g = G \circ \theta$ for some nonconstant monomial θ in x_1, x_2, \dots, x_n . Then the following are equivalent:*

- 1) $\{f, g\}$ is a SAGBI basis in $\mathbb{k}[x_1, x_2, \dots, x_n]$.
- 2) $\{F, G\}$ is a SAGBI basis in $\mathbb{k}[t]$.
- 3) There exists a polynomial $h \in \mathbb{k}[t]$ of degree $d := \gcd(\deg F, \deg G)$ and polynomials $F^*, G^* \in \mathbb{k}[t]$ such that $F = F^* \circ h$ and $G = G^* \circ h$.

Proof. The proof of 2) \Leftrightarrow 3) is given by Torstensson in [[28], P.572]. So, It is enough to proof 1) \Leftrightarrow 2). Suppose $\theta \in \mathcal{M} \setminus \{1\}$. For a given $\succeq \in \Omega$, $\deg F = m$ and $\deg G = k$ if and only if $\text{in}_{\succeq} f = \theta^m$ and $\text{in}_{\succeq} g = \theta^k$.

Suppose $\{f, g\}$ is a SAGBI basis in $\mathbb{k}[x_1, x_2, \dots, x_n]$.

Claim:- $\{F, G\}$ is a SAGBI basis in $\mathbb{k}[t]$.

(F^p, G^q) is an $\{F, G\}$ - critical pair

$$\Rightarrow \text{in } F^p = \text{in } G^q$$

$$\Rightarrow t^{mp} = t^{kq}$$

$$\Rightarrow mp = kq$$

$$\Rightarrow \theta^{mp} = \theta^{kq}$$

$$\Rightarrow (f^p, g^q) \text{ is an } \{f, g\} \text{ - critical pair}$$

$$\Rightarrow T(f^p, g^q) \text{ has a lower representation in } \{f, g\}$$

(by SAGBI Test theorem).

$$\begin{aligned} \Rightarrow f^p - cg^q &= \sum_{i=1}^s \gamma_i f^{a_i} g^{b_i}, \text{ where } a_i m + b_i k < pm = qk \\ &\text{for each } i \in \{1, 2, \dots, s\} \\ \Rightarrow F^p - cG^q &= \sum_{i=1}^s \gamma_i F^{a_i} G^{b_i}, \text{ where } a_i m + b_i k < pm = qk \\ &\text{for each } i \in \{1, 2, \dots, s\}. \\ &\text{(replace } \theta \text{ by the generic variable } t) \end{aligned}$$

Hence by SAGBI Test theorem, $\{F, G\}$ is a SAGBI basis in $\mathbb{k}[t]$

Conversely, assume $\{F, G\}$ is a SAGBI basis in $\mathbb{k}[t]$.

Claim:- $\{f, g\}$ is a SAGBI basis in $\mathbb{k}[x_1, x_2, \dots, x_n]$ with respect to a given $\succeq \in \Omega$.

(f^p, g^q) is an $\{f, g\}$ - critical pair

$$\begin{aligned} \Rightarrow \text{in}_{\succeq} f^p &= \text{in}_{\succeq} g^q \\ \Rightarrow \theta^{mp} &= \theta^{kq} \\ \Rightarrow mp &= kq \\ \Rightarrow t^{mp} &= t^{kq} \\ \Rightarrow (F^p, G^q) &\text{ is an } \{F, G\} \text{ - critical pair} \\ \Rightarrow T(F^p, G^q) &\text{ has a lower representation in } \{F, G\} \\ &\text{(by SAGBI Test theorem).} \\ \Rightarrow F^p - cG^q &= \sum_{i=1}^s \gamma_i F^{a_i} G^{b_i} \text{ where } a_i m + b_i k < pm = qk \\ &\text{for each } i \in \{1, 2, \dots, s\} \\ \Rightarrow (F^p - cG^q) \circ \theta &= \left(\sum_{i=1}^s \gamma_i F^{a_i} G^{b_i} \right) \circ \theta \\ \Rightarrow (f^p - cg^q) &= \left(\sum_{i=1}^s \gamma_i f^{a_i} g^{b_i} \right) \text{ where } a_i m + b_i k < pm = qk \\ &\text{for each } i \in \{1, 2, \dots, s\}. \end{aligned}$$

Hence by SAGBI Test theorem, $\{f, g\}$ is a SAGBI basis in $\mathbb{k}[x_1, x_2, \dots, x_n]$ with respect to $\succeq \in \Omega$. □

4.2 Further on SAGBI bases for \mathbb{k} -subalgebras generated by two polynomials

In this section we present some sufficient conditions when two generators in multivariate polynomial ring form a SAGBI basis. In addition to this we have seen some sufficient conditions when a \mathbb{k} -subalgebra of a multivariate polynomial ring generated by two polynomials have a finite SAGBI basis. Finally we state a conjecture on the existence of finite SAGBI basis for \mathbb{k} -subalgebra generated by two polynomials.

Theorem 4.2.1 ([18], Proposition 6.6.11, p.485). *Let $\succeq \in \Omega$ and $f_1, f_2, \dots, f_m \in \mathbb{k}[\mathbf{x}]$. If $\deg(\text{in}_{\succeq} f_i)s'$, for $i \in \{1, 2, \dots, m\}$, are linearly independent vectors over the field \mathbb{Q} , then $\{f_1, f_2, \dots, f_m\}$ is a SAGBI basis .*

Proof. Let $\deg(\text{in}_{\succeq} f_i) = \mathbf{u}_i$ and assume $\mathbf{u}_i s'$ are linearly independent vectors over \mathbb{Q} , for $i \in \{1, 2, \dots, m\}$.

$(\prod_{i=1}^m f_i^{a_i}, \prod_{i=1}^m f_i^{b_i})$ is an $\{f_1, f_2, \dots, f_m\}$ – critical pairs

$$\Rightarrow \text{in}_{\succeq}(\prod_{i=1}^m f_i^{a_i}) = \text{in}_{\succeq}(\prod_{i=1}^m f_i^{b_i}).$$

$$\Rightarrow \sum_{i=1}^m a_i \mathbf{u}_i = \sum_{i=1}^m b_i \mathbf{u}_i.$$

$$\Rightarrow \sum_{i=1}^m (a_i - b_i) \mathbf{u}_i = 0.$$

$$\Rightarrow a_i - b_i = 0, \text{ for } i \in \{1, 2, \dots, m\} \text{ (as } \mathbf{u}_i \text{ are linearly independent over } \mathbb{Q}\text{)}.$$

$$\Rightarrow a_i = b_i \text{ for each } i \in \{1, 2, \dots, m\}.$$

$$\Rightarrow \prod_{i=1}^m f_i^{a_i} = \prod_{i=1}^m f_i^{b_i}.$$

$$\Rightarrow T(\prod_{i=1}^m f_i^{a_i}, \prod_{i=1}^m f_i^{b_i}) = 0.$$

Hence, by SAGBI Test theorem, $\{f_1, f_2, \dots, f_m\}$ is a SAGBI basis. \square

In the case of two generators, we can explicitly restate the above Theorem as follows.

Corollary 4.2.2. *Given $\succeq \in \Omega$, and $f, g \in \mathbb{k}[\mathbf{x}]$. Let $\mathbf{a} = (a_1, a_2, \dots, a_n)$ and $\mathbf{b} = (b_1, b_2, \dots, b_n)$ be multi-degrees of $\text{in}_{\succeq} f$ and $\text{in}_{\succeq} g$ respectively. If $a_i b_j \neq a_j b_i$ for some $i, j \in \{1, 2, \dots, n\}$, then $\{f, g\}$ is a SAGBI basis. In particular if $a_i = 0$ and $b_i \neq 0$ for some $i \in \{1, 2, \dots, n\}$, then $\{f, g\}$ is a SAGBI basis.*

Proof. Suppose $a_i b_j \neq a_j b_i$ for some $i, j \in \{1, 2, \dots, n\}$. We can simply observe that both \mathbf{a} and \mathbf{b} are non-zero.

Claim:- $\{\mathbf{a}, \mathbf{b}\}$ is linearly independent over \mathbb{Q} .

To the contrary assume $\{\mathbf{a}, \mathbf{b}\}$ is linearly dependent over \mathbb{Q} . This directly implies that $\mathbf{a} = t\mathbf{b}$ for some $t \in \mathbb{Q}$. Hence

$$a_i b_j = (tb_i) b_j = b_i (tb_j) = b_i a_j = a_j b_i.$$

Which is a contradiction with our assumption $a_i b_j \neq a_j b_i$. Hence $\{\mathbf{a}, \mathbf{b}\}$ is linearly independent over \mathbb{Q} . Hence by Theorem 4.2.1 $\{f, g\}$ is a SAGBI basis. \square

Example 4.2.3. Let $f = x^3 y^2 + x^3 y + x^2 y^4$ and $g = x^5 y^4 + xy$. Then $\{f, g\}$ is a SAGBI basis with respect to lexicographical ordering.

This is because $\deg(\text{in}_{>_{lex}} f) = (3, 2)$, $\deg(\text{in}_{>_{lex}} g) = (5, 4)$ and $3 \cdot 4 \neq 5 \cdot 2$.

Example 4.2.4. Any two elementary symmetric polynomials can form a SAGBI basis with respect to any $\succeq \in \Omega$. This is because we can find $a_i = 0$ and $b_i \neq 0$ for some i .

Corollary 4.2.5. Let $\succeq \in \Omega$ and $f, g \in \mathbb{k}[\mathbf{x}]$. If $\text{in}_{\succeq} f = x_i^{d_i}$ for some $d_i \in \mathbb{N} - \{0\}$ and $\text{in}_{\succeq} g = x_j t$ with $i \neq j$ where t is any monomial in $\mathbb{k}[\mathbf{x}]$, then $\{f, g\}$ form a SAGBI basis.

Proof. One can easily observe that $a_j = 0$ and $b_j \neq 0$. Hence by corollary 4.2.2 $\{f, g\}$ is a SAGBI basis. \square

Example 4.2.6. Consider $f = x + y$ and $g = xy$. Observe that f and g satisfies the sufficient condition of Corollary 4.2.5 for every monomial ordering. So, $\{f, g\}$ is a SAGBI basis with respect to any monomial ordering.

Theorem 4.2.7. Given $\succeq \in \Omega$ and $f, g \in \mathbb{k}[\mathbf{x}]$ with $\deg(\text{in}_{\succeq} f) = \frac{k}{m} \deg(\text{in}_{\succeq} g)$ where $\gcd(k, m) = 1$. If $f^m - g^k \xrightarrow{\{f, g\}^*} 0$, then $\{f, g\}$ is a SAGBI basis.

Proof. Assume f and g are monic polynomials in $\mathbb{k}[\mathbf{x}]$. Let $\deg(\text{in}_{\succeq} f) = \mathbf{u}$ and $\deg(\text{in}_{\succeq} g) = \mathbf{v}$. Assume $\mathbf{u} = \frac{k}{m} \mathbf{v}$ where k and m are relatively

prime numbers. Since $f^m - g^k \xrightarrow{\{f,g\}^*} 0$, we can express $f^m - g^k$ as

$$f^m - g^k = \sum_{i=1}^r f^{\alpha_i} g^{\beta_i} \text{ with } m\mathbf{u} = k\mathbf{v} \succ \alpha_i\mathbf{u} + \beta_i\mathbf{v} \text{ for each } i.$$

Let $T(f^a g^b, f^c g^d)$ be arbitrary nonzero T -polynomial in f and g .

Thus, $a\mathbf{u} + b\mathbf{v} = c\mathbf{u} + d\mathbf{v}$

$\Rightarrow (a - c)\mathbf{u} = (d - b)\mathbf{v}$ (With out loss of generality, assume $a > c$)

$\Rightarrow \mathbf{u} = \frac{d-b}{a-c}\mathbf{v}$

$\Rightarrow d - b = tk$ and $a - c = tm$ for some $t \in \mathbb{N}$

Now we have

$$\begin{aligned} T(f^a g^b, f^c g^d) &= f^a g^b - f^c g^d \\ &= f^c f^{a-c} g^b - f^c g^b g^{d-b} \\ &= f^c g^b (f^{a-c} - g^{d-b}) \\ &= f^c g^b (f^{tm} - g^{tk}) \\ &= f^c g^b ((f^m)^t - (g^k)^t) \\ &= f^c g^b (f^m - g^k) \left(\sum_{j=0}^{t-1} f^{m(t-1-j)} g^{kj} \right) \\ &= f^c g^b \left(\sum_{i=1}^r f^{\alpha_i} g^{\beta_i} \right) \left(\sum_{j=0}^{t-1} f^{m(t-1-j)} g^{kj} \right) \\ &= \sum_{i=1}^r \sum_{j=0}^{t-1} f^{\alpha_i + c + m(t-j-1)} g^{\beta_i + b + kj} \end{aligned}$$

Moreover; for each i and j ,

$$\begin{aligned}
& \deg(\text{in}_{\succeq} f^{\alpha_i+c+m(t-j-1)} g^{\beta_i+b+kj}) \\
&= (\alpha_i + c + m(t - j - 1))\mathbf{u} + (\beta_i + b + kj)\mathbf{v} \\
&= (\alpha_i\mathbf{u} + \beta_i\mathbf{v}) + c\mathbf{u} + b\mathbf{v} + mt\mathbf{u} - mj\mathbf{u} - m\mathbf{u} \\
&+ kj\mathbf{v} \\
&\prec m\mathbf{u} + c\mathbf{u} + b\mathbf{v} + mt\mathbf{u} - mj\left(\frac{k}{m}\mathbf{v}\right) - m\mathbf{u} + kj\mathbf{v} \\
&= c\mathbf{u} + b\mathbf{v} + mt\mathbf{u} \\
&= c\mathbf{u} + b\mathbf{v} + (a - c)\mathbf{u} \quad (\text{as } mt = a - c) \\
&= a\mathbf{u} + b\mathbf{v}
\end{aligned}$$

Therefore, every T-polynomial in f and g has a lower representation. Hence by SAGBI test theorem, Theorem 2.3.6, $\{f, g\}$ is a SAGBI basis. \square

Theorem 4.2.8. *Given $\succeq \in \Omega$ and $f, g \in \mathbb{k}[\mathbf{x}]$ with $\deg(\text{in}_{\succeq} f) = \frac{k}{m} \deg(\text{in}_{\succeq} g)$ where $\gcd(k, m) = 1$. If $f^m - g^k \xrightarrow{\{f, g\}^*} h$, and $\deg(\text{in}_{\succeq} h)$ and $\deg(\text{in}_{\succeq} f)$ are linearly independent over \mathbb{Q} , then $\{f, g, h\}$ is a SAGBI basis with respect to \succeq .*

Proof. Fix a monomial order \succeq on $\mathbb{k}[x_1, \dots, x_n]$, and $\text{in}_{\succeq}(f) = \mathbf{x}^{\mathbf{u}}$, $\text{in}_{\succeq}(g) = \mathbf{x}^{\mathbf{v}}$ and $\text{in}_{\succeq} h = \mathbf{x}^{\mathbf{w}}$ for some $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{N}^n$.

It is given that $\mathbf{u} = \frac{k}{m}\mathbf{v}$ where $k, m \in \mathbb{N}$ with $\gcd(k, m) = 1$, and h be the final subductum of $f^m - g^k$ over $\{f, g\}$. So, we can express $f^m - g^k$ as

$$f^m - g^k = \sum_{i=1}^r f^{a_i} g^{b_i} + h \text{ with } m\mathbf{u} = k\mathbf{v} \succ a_i\mathbf{u} + b_i\mathbf{v} \text{ for each } i.$$

With out loss of generality, assume f, g and h are monic polynomials.

Suppose \mathbf{w} and \mathbf{u} are linearly independent over \mathbb{Q} .

Claim: $\{f, g, h\}$ is a SAGBI basis for $\mathbb{k}[f, g]$. Equivalently, each nonzero T -polynomial in f, g and h has lower representation in f, g and h .

Indeed, let $h_1 = f^{\alpha_1}g^{\alpha_2}h^{\alpha_3} - f^{\beta_1}g^{\beta_2}h^{\beta_3}$ be a T -polynomial for some $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathbb{N}$. It follows that:

$$\alpha_1\mathbf{u} + \alpha_2\mathbf{v} + \alpha_3\mathbf{w} = \beta_1\mathbf{u} + \beta_2\mathbf{v} + \beta_3\mathbf{w}$$

$$\Rightarrow (\alpha_1 - \beta_1)\mathbf{u} + (\alpha_2 - \beta_2)\mathbf{v} + (\alpha_3 - \beta_3)\mathbf{w} = 0$$

$$\Rightarrow (\alpha_1 - \beta_1)\mathbf{u} + (\alpha_2 - \beta_2)\frac{m}{k}\mathbf{u} + (\alpha_3 - \beta_3)\mathbf{w} = 0$$

$$\Rightarrow [k(\alpha_1 - \beta_1) + m(\alpha_2 - \beta_2)]\mathbf{u} + k(\alpha_3 - \beta_3)\mathbf{w} = 0$$

$\Rightarrow k(\alpha_1 - \beta_1) + m(\alpha_2 - \beta_2) = 0$ and $(\alpha_3 - \beta_3) = 0$ (as \mathbf{u} and \mathbf{w} are linearly independent)

$\Rightarrow k(\alpha_1 - \beta_1) = m(\beta_2 - \alpha_2)$ and $\alpha_3 = \beta_3$, but $\alpha_1 \neq \beta_1$ and $\alpha_2 \neq \beta_2$, other wise $h_1 = 0$.

With out loss of generality, assume $\alpha_1 > \beta_1$

$$\Rightarrow \frac{k}{m} = \frac{\beta_2 - \alpha_2}{\alpha_1 - \beta_1}$$

$\Rightarrow \alpha_1 - \beta_1 = tm$ and $\beta_2 - \alpha_2 = tk$ for some $t \in \mathbb{N}$ (as $(k, m) = 1$).

Therefore;

$$\begin{aligned}
h_1 &= f^{\alpha_1} g^{\alpha_2} h^{\alpha_3} - f^{\beta_1} g^{\beta_2} h^{\beta_3} \\
&= h^{\alpha_3} f^{\beta_1} g^{\alpha_2} (f^{\alpha_1 - \beta_1} - g^{\beta_2 - \alpha_2}) \\
&= h^{\alpha_3} f^{\beta_1} g^{\alpha_2} (f^{tm} - g^{tk}) \\
&= h^{\alpha_3} f^{\beta_1} g^{\alpha_2} [(f^m)^t - (g^k)^t] \\
&= h^{\alpha_3} f^{\beta_1} g^{\alpha_2} (f^m - g^k) [f^{m(t-1)} + f^{m(t-2)} g^k + \dots + f^m g^{k(t-2)} + g^{k(t-1)}] \\
&= h^{\alpha_3} f^{\beta_1} g^{\alpha_2} \left(\sum_{i=1}^r f^{a_i} g^{b_i} + h \right) \left(\sum_{j=0}^{t-1} f^{m(t-1-j)} g^{kj} \right). \\
&= \sum_{i=1}^r \sum_{j=0}^{t-1} f^{a_i + \beta_1 + m(t-1-j)} g^{b_i + \alpha_2 + kj} h^{\alpha_3} + \sum_{j=0}^{t-1} f^{m(t-1-j) + \beta_1} g^{kj + \alpha_2} h^{\alpha_3 + 1}
\end{aligned}$$

Now for each i and j

$$\begin{aligned}
&\deg(\text{in}_{\succeq} f^{a_i + \beta_1 + m(t-1-j)} g^{b_i + \alpha_2 + kj} h^{\alpha_3}) \\
&= (a_i + \beta_1 + m(t-1-j))\mathbf{u} + (b_i + \alpha_2 + kj)\mathbf{v} + \alpha_3\mathbf{w} \\
&= (a_i\mathbf{u} + b_i\mathbf{v}) + \beta_1\mathbf{u} + \alpha_2\mathbf{v} + \alpha_3\mathbf{w} + mt\mathbf{u} - m\mathbf{u} - mj\mathbf{u} + kj\mathbf{v} \\
&\prec m\mathbf{u} + \beta_1\mathbf{u} + \alpha_2\mathbf{v} + \alpha_3\mathbf{w} + (\alpha_1 - \beta_1)\mathbf{u} - m\mathbf{u} - mj\left(\frac{k}{m}\mathbf{v}\right) + kj\mathbf{v} \\
&= \alpha_1\mathbf{u} + \alpha_2\mathbf{v} + \alpha_3\mathbf{w}
\end{aligned}$$

Moreover for each j ,

$$\begin{aligned}
&\deg(\text{in}_{\succeq} f^{m(t-1-j) + \beta_1} g^{kj + \alpha_2} h^{\alpha_3 + 1}) \\
&= (m(t-1-j) + \beta_1)\mathbf{u} + (kj + \alpha_2)\mathbf{v} + (\alpha_3 + 1)\mathbf{w} \\
&= mt\mathbf{u} - m\mathbf{u} - mj\mathbf{u} + \beta_1\mathbf{u} + kj\mathbf{v} + \alpha_2\mathbf{v} + \alpha_3\mathbf{w} + \mathbf{w} \\
&\prec mt\mathbf{u} - m\mathbf{u} - mj\left(\frac{k}{m}\mathbf{v}\right) + \beta_1\mathbf{u} + kj\mathbf{v} + \alpha_2\mathbf{v} + \alpha_3\mathbf{w} + m\mathbf{u} \text{ (as } \mathbf{w} \prec m\mathbf{u} \text{)} \\
&= (\alpha_1 - \beta_1)\mathbf{u} + \beta_1\mathbf{u} + \alpha_2\mathbf{v} + \alpha_3\mathbf{w} \text{ (as } mt = \alpha_1 - \beta_1 \text{)} \\
&= \alpha_1\mathbf{u} + \alpha_2\mathbf{v} + \alpha_3\mathbf{w}
\end{aligned}$$

Thus, each T-polynomial has lower representation in f , g and h . Therefore, by Theorem 2.3.6, $\{f, g, h\}$ is a SAGBI basis for $\mathbb{k}[f, g]$.

□

Our aim was to prove that any \mathbb{k} – *subalgebra* of $\mathbb{k}[\mathbf{x}]$ generated by two polynomials f and g has a finite SAGBI basis with respect to a given monomial ordering $\succeq \in \Omega$. If $\deg(\text{in}_{\succeq} f)$ and $\deg(\text{in}_{\succeq} g)$ are *linearly independent vectors* over the field \mathbb{Q} , then, by Theorem 4.2.1, $\{f, g\}$ is a SAGBI basis for $\mathbb{k}[f, g]$. Otherwise $\deg(\text{in}_{\succeq} f) = \frac{k}{m} \deg(\text{in}_{\succeq} g)$ where $\gcd(k, m) = 1$. In this case, compute the final subductum of $f^m - g^k$ over $\{f, g\}$ with respect to \succeq , say h_0 . If $h_0 = 0$, by Theorem 4.2.7 $\{f, g\}$ is a SAGBI basis for $\mathbb{k}[f, g]$. So assume $h_0 \neq 0$. If $\deg(\text{in}_{\succeq} h_0)$ and $\deg(\text{in}_{\succeq} f)$ are *linearly independent vectors* over the field \mathbb{Q} , then by Theorem 4.2.8 $\{f, g, h_0\}$ is a SAGBI basis for $\mathbb{k}[f, g]$. Otherwise $\deg(\text{in}_{\succeq} f) = \frac{r}{t} \deg(\text{in}_{\succeq} h_0)$, for som $r, t \in \mathbb{N}$ where $\gcd(r, t) = 1$. In this case, compute the final subductum of $f^t - h_0^r$ over $\{f, g, h_0\}$ with respect to \succeq , say h_1 . We hope that, if $\deg(\text{in}_{\succeq} h_1)$ and $\deg(\text{in}_{\succeq} f)$ are *linearly independent vectors* over the field \mathbb{Q} , then $\{f, g, h_0, h_1\}$ is a SAGBI basis for $\mathbb{k}[f, g]$. Unfortunately, *by applying singular*, we obtain a counter example which we put it in appendix A.

In our counter example observe that, $\deg(\text{in}_{\succeq_{lex}} f_1) = \frac{4}{3} \deg(\text{in}_{\succeq_{lex}} f_2)$ and f_3 is the final subductum of $f_1^3 - f_2^4$. Moreover $\deg(\text{in}_{\succeq_{lex}} f_1) = \frac{16}{39} \deg(\text{in}_{\succeq_{lex}} f_3)$ and f_4 is the final subductum of $f_1^{39} - f_3^{16}$. Here $\deg(\text{in}_{\succeq_{lex}} f_1)$ and $\deg(\text{in}_{\succeq_{lex}} f_4)$ are *linearly independent* over the field \mathbb{Q} . But $\{f_1, f_2, f_3, f_4\}$ can not be a SAGBI basis for $\mathbb{k}[f_1, f_2]$. This is because there is $f_5 \in \mathbb{k}[f_1, f_2]$ such that the $\deg(\text{in}_{\succeq_{lex}} f_5)$

can not be expressed as an \mathbb{N} – *linear* combination of $\deg(\text{in}_{\succeq_{lex}} f_1)$, $\deg(\text{in}_{\succeq_{lex}} f_2)$, $\deg(\text{in}_{\succeq_{lex}} f_3)$ and $\deg(\text{in}_{\succeq_{lex}} f_4)$.

Conjecture 4.2.9. A \mathbb{k} – *subalgebra* of $\mathbb{k}[\mathbf{x}]$, generated by any two polynomials $f, g \in \mathbb{k}[\mathbf{x}]$ has a finite SAGBI basis with respect to a given monomial order \succeq .

Future Work

Classifying a \mathbb{k} – *subalgebra* with finite SAGBI basis is still an open problem. So we need more good sufficient and \ or necessary conditions for a \mathbb{k} – *subalgebra* has a finite SAGBI basis in the case of multivariate polynomial ring. It is also an open problem to set good sufficient and \ or necessary conditions for three generators form a SAGBI basis in the case of univariate polynomial ring.

Our conjecture, “ *A \mathbb{k} – subalgebra generated by two polynomials has a finite SAGBI basis with respect to any given monomial order*”, is also an other problem which needs to be proved or disproved.

There is also a conjecture given by F. Ollivier on his paper “ Canonical bases and Tame automorphism ”. He conjectured that Every finitely generated integrally closed \mathbb{k} – *subalgebra* has a finite SAGBI basis. It is our future plane to prove or disprove F. Ollivier conjecture.

Bibliography

- [1] Adams, W.W., Lousta.unau, P. *An introduction to Grobner bases*, Graduate studies in mathematics, American mathematical society, volume 3, 1994.
- [2] Atiyah, M.F., Macdonald, I.G. *Introduction to commutative algebra*, Addison - Wesley publishing Company, 1969.
- [3] Becker, T., Weispfenning, V., Kredel, H. *Gröbner Bases, A Computational Approach to Commutative Algebra*, Graduate Texts in Mathematics, Springer, NewYork., 1993.
- [4] Bravo, A. *Some Facts About Canonical Subalgebra Bases*, Trends in Commutative Algebra, MSRI Publications, Volume 51, 2004, pp. 247-254.
- [5] Campbell, E.H.E.A., Wehlau, D.L. *Invariant theory and algebraic transformation Groups VIII, Modular invariant theory*, Encyclopaedia of mathematical sciences, Volume 139, Springer Heidelberg Dordrecht London New York, 2011, pp. 83 - 97.
- [6] Cox, D., Little, J., O'Shea, D. *Ideals, Varieties, and Algorithms, An Introduction to Computational Algebraic Geometry and Commutative Algebra*, Third edition, Springer-Verlag, New York, 2007.

- [7] Cox, D., Little, J., O'Shea, D. *Using Algebraic Geometry*, Second Edition, Springer-Verlag, New York, 2005.
- [8] Dummit, D.S., Foote, R.M. *Abstract Algebra*, third edition, John Wiley and Sons, 2004, pp. 691 - 693.
- [9] Ene, V., Herzog, J. *Gröbner Bases in Commutative Algebra*, American mathematical society, Graduate studies in mathematics Volume 130, 2012.
- [10] Gale, D. *Subalgebras of an algebra with a single generator are finitely generated*, American Mathematical Society, Volume 8 (1957), 929930.
- [11] Gathmann, A. *Commutative Algebra*, Class Notes TU Kaiserslautern, 2013/14.
- [12] Gopalakrishnan, N.S. *commutative Algebra*, University of Poons, Oxonian press, New Delhi, 1984.
- [13] Greuel G-M., Pfister, G. *A Singular Introduction to Commutative Algebra*, second edition, Springer, Berlin, 2008.
- [14] Kapur, D., Madlener, K. *A completion procedure for computing a canonical basis for a k -subalgebra* in Computers and mathematics (Cambridge MA 1989), Springer, New York, 1989, pp. 1-11.
- [15] Kemper, G. *A course in commutative algebra*, Graduate text in mathematics, Volume 256, Springer, Berlin, 2011.

- [16] Khan, J.A. *Further on the composition of SAGBI bases*, International Electronic Journal of Algebra, Volume 20 (2016) 100-110.
- [17] Kreuzer, M., Robbiano, L. *Computational Commutative Algebra 1*, Springer-Verlag Berlin Heidelberg, New York, 2000.
- [18] Kreuzer, M., Robbiano, L. *Computational Commutative Algebra 2*, Springer-Verlag Berlin Heidelberg, New York, 2005.
- [19] Mendelsohn, N.S. *A linear diophantine equation with applications to non-negative matrices*, Annals of the New York academy of sciences, v.175, 1970.
- [20] Nordbeck, P. *Canonical bases for subalgebras of factor algebras*, Computer Science Journal of Moldova, vol.7, 1999, pp. 63-77.
- [21] Nordbeck, P. *Canonical subalgebra bases in non-commutative polynomial rings*, In Proceedings of ISSAC 98, ACM Press, New York, (1998), pp. 140-146.
- [22] Nordbeck, P. *SAGBI bases under composition*, Journal of Symbolic Computatin, 2002, pp. 67-76.
- [23] Ollivier, F. *"Canonical Bases: Relations with standard bases, finiteness conditions and application to tame automorphisms*, Effective methods in algebraic geometry, Volume 94, Progr. math. Birkhäuser Boston, Boston, MA, 1991, pp.379-400.
- [24] Robbiano, L., Sweedler, M. *Subalgebra bases, Commutative Algebra*, in Lecture Notes in Mathematics, Volume 1430, Springer, Berlin, 1990, pp. 61 - 87.

- [25] Robbiano, L. *Term orderings on the polynomial ring*, Proceedings of the EUROCAL 85, in Lecture Notes in Computer Science 204, Springer , 1985, pp. 513 - 517.
- [26] Sit, W.Y., Siu, M.-K. *On the Subsemigroups of N* , Mathematical Association of America, Mathematics Magazine, Volume 48, 1975, pp 225-227.
- [27] Sturmfels, B. *Gröbner Bases and Convex Polytopes*, American Mathematical Society, University Lecture Series, Volume 8 (1995), pp. 99 - 103
- [28] Torstensson, A. *Canonical Bases for Subalgebras on Two Generators in the Univariate Polynomial Ring*, Beiträge zur Algebra und Geometrie, Contributions to Algebra and Geometry, Volume 43 (2002), No. 2, pp. 565-577.
- [29] Torstensson, A., Ufnarovski, V., Öfverbeck, H. *Using resultants for SAGBI basis verification in the univariate polynomial ring*, Journal of Symbolic Computation, Volume 40 (2005), pp. 1087-1105.

Appendix A

Counter example

Let $f_1 = x^{32}y^{16} + x^4y^2$ and $f_2 = x^{24}y^{12} + x^6y^3 + xy$. Then $\{f_1, f_2, f_3, f_4, f_5\}$ is a SAGBI basis for $\mathbb{k}[f_1, f_2]$ with respect to \succeq_{lex} , where

$$f_1 = x^{32}y^{16} + x^4y^2$$

$$f_2 = x^{24}y^{12} + x^6y^3 + xy$$

$$f_3 = x^{78}y^{39} + x^{73}y^{37} - \frac{3}{4}x^{68}y^{34} + \frac{3}{2}x^{60}y^{30} + 3x^{55}y^{28} + \frac{3}{2}x^{50}y^{26} + x^{42}y^{21} - \frac{3}{4}x^{40}y^{20} + 3x^{37}y^{19} + 3x^{32}y^{17} + x^{27}y^{15} + \frac{1}{4}x^{24}y^{12} + x^{19}y^{10} + \frac{3}{2}x^{14}y^8 - \frac{1}{4}x^{12}y^6 + x^9y^6 + \frac{1}{4}x^4y^4$$

$$f_4 = x^{307}y^{154} + \frac{3}{2}x^{302}y^{152} - \frac{3}{4}x^{302}y^{151} + x^{297}y^{150} - \frac{9}{4}x^{297}y^{149} + \frac{5}{4}x^{294}y^{147} + \frac{1}{4}x^{292}y^{148} - \frac{9}{4}x^{292}y^{147} + \frac{27}{32}x^{292}y^{146} + \frac{29}{4}x^{289}y^{145} - \frac{3}{4}x^{287}y^{145} + \frac{27}{16}x^{287}y^{144} + 15x^{284}y^{143} - \frac{45}{8}x^{284}y^{142} + \frac{27}{32}x^{282}y^{142} - \frac{27}{64}x^{282}y^{141} + 15x^{279}y^{141} - \frac{27}{2}x^{279}y^{140} - \frac{27}{64}x^{277}y^{139} + \frac{35}{8}x^{276}y^{138} + \frac{15}{2}x^{274}y^{139} - \frac{81}{4}x^{274}y^{138} + \frac{57}{32}x^{274}y^{137} + \frac{81}{1024}x^{272}y^{136} + \frac{105}{4}x^{271}y^{136} + \frac{3}{2}x^{269}y^{137} - \frac{27}{2}x^{269}y^{136} + \frac{171}{32}x^{269}y^{135} + \frac{525}{8}x^{266}y^{134} - \frac{153}{16}x^{266}y^{133} - \frac{27}{8}x^{264}y^{134} + \frac{171}{32}x^{264}y^{133} + \frac{135}{128}x^{264}y^{132} + \frac{175}{2}x^{261}y^{132} - \frac{621}{16}x^{261}y^{131} + \frac{57}{32}x^{259}y^{131} + \frac{135}{64}x^{259}y^{130} + \frac{65}{8}x^{258}y^{129} + \frac{525}{8}x^{256}y^{130} - \frac{585}{8}x^{256}y^{129} - \frac{1125}{128}x^{256}y^{128} + \frac{135}{128}x^{254}y^{128} - \frac{81}{64}x^{254}y^{127} + \frac{455}{8}x^{253}y^{127} + \frac{105}{4}x^{251}y^{128} - \frac{585}{8}x^{251}y^{127} + \frac{27}{32}x^{251}y^{126} - \frac{81}{64}x^{249}y^{125} + \frac{1365}{8}x^{248}y^{125} - \frac{315}{32}x^{248}y^{124} + \frac{35}{8}x^{246}y^{126} - \frac{585}{16}x^{246}y^{125} + \frac{81}{64}x^{246}y^{124} + \frac{281}{64}x^{246}y^{123} + \frac{81}{256}x^{244}y^{122} + \frac{2275}{8}x^{243}y^{123} - \frac{945}{16}x^{243}y^{122} - \frac{117}{16}x^{241}y^{123} + \frac{27}{32}x^{241}y^{122} + \frac{843}{64}x^{241}y^{121} + \frac{681}{64}x^{240}y^{120} + \frac{2275}{8}x^{238}y^{121} - \frac{4725}{32}x^{238}y^{120} -$$

$$\begin{aligned}
& \frac{855}{64}x^{238}y^{119} + \frac{27}{128}x^{236}y^{120} + \frac{843}{64}x^{236}y^{119} - \frac{63}{128}x^{236}y^{118} + \frac{681}{8}x^{235}y^{118} + \\
& \frac{1365}{8}x^{233}y^{119} - \frac{1575}{8}x^{233}y^{118} - \frac{1971}{64}x^{233}y^{117} + \frac{281}{64}x^{231}y^{117} - \frac{63}{64}x^{231}y^{116} + \\
& \frac{4767}{16}x^{230}y^{116} - 9x^{230}y^{115} + \frac{455}{8}x^{228}y^{117} - \frac{4725}{32}x^{228}y^{116} - \frac{1395}{32}x^{228}y^{115} - \\
& \frac{3855}{256}x^{228}y^{114} - \frac{63}{128}x^{226}y^{114} - \frac{27}{16}x^{226}y^{113} + \frac{4767}{8}x^{225}y^{114} - 63x^{225}y^{113} + \\
& \frac{65}{8}x^{223}y^{115} - \frac{945}{16}x^{223}y^{114} - \frac{1395}{32}x^{223}y^{113} + \frac{1521}{64}x^{223}y^{112} + \frac{165}{16}x^{222}y^{111} - \\
& \frac{27}{16}x^{221}y^{111} + \frac{23835}{32}x^{220}y^{112} - 189x^{220}y^{111} - \frac{1071}{128}x^{220}y^{110} - \frac{315}{32}x^{218}y^{112} - \\
& \frac{1395}{64}x^{218}y^{111} + \frac{4563}{128}x^{218}y^{110} + \frac{189}{64}x^{218}y^{109} + \frac{1485}{16}x^{217}y^{109} + \frac{297}{512}x^{216}y^{108} + \\
& \frac{4767}{8}x^{215}y^{110} - 315x^{215}y^{109} - \frac{3213}{64}x^{215}y^{108} - \frac{279}{64}x^{213}y^{109} + \frac{1521}{64}x^{213}y^{108} + \\
& \frac{567}{64}x^{213}y^{107} + \frac{1485}{4}x^{212}y^{107} - \frac{369}{64}x^{212}y^{106} + \frac{4767}{16}x^{210}y^{108} - 315x^{210}y^{107} - \\
& \frac{16065}{128}x^{210}y^{106} - \frac{561}{32}x^{210}y^{105} + \frac{1521}{256}x^{208}y^{106} + \frac{567}{64}x^{208}y^{105} - \frac{63}{32}x^{208}y^{104} + \\
& \frac{3465}{4}x^{207}y^{105} - \frac{369}{8}x^{207}y^{104} + \frac{681}{8}x^{205}y^{106} - 189x^{205}y^{105} - \frac{5355}{32}x^{205}y^{104} - \\
& \frac{117}{32}x^{205}y^{103} + \frac{121}{16}x^{204}y^{102} + \frac{189}{64}x^{203}y^{103} - \frac{63}{16}x^{203}y^{102} + \frac{10395}{8}x^{202}y^{103} - \\
& \frac{2583}{16}x^{202}y^{102} - \frac{549}{64}x^{202}y^{101} + \frac{681}{64}x^{200}y^{104} - 63x^{200}y^{103} - \frac{16065}{128}x^{200}y^{102} + \\
& \frac{555}{16}x^{200}y^{101} - \frac{6615}{256}x^{200}y^{100} + \frac{605}{8}x^{199}y^{100} - \frac{63}{32}x^{198}y^{100} - \frac{81}{64}x^{198}y^{99} + \frac{10395}{8}x^{197}y^{101} - \\
& \frac{2583}{8}x^{197}y^{100} - \frac{3843}{64}x^{197}y^{99} - 9x^{195}y^{101} - \frac{3213}{64}x^{195}y^{100} + \frac{555}{16}x^{195}y^{99} + \frac{1449}{64}x^{195}y^{98} + \\
& \frac{5445}{16}x^{194}y^{98} - \frac{165}{64}x^{194}y^{97} - \frac{81}{64}x^{193}y^{97} + \frac{3465}{4}x^{192}y^{99} - \frac{12915}{32}x^{192}y^{98} - \frac{11529}{64}x^{192}y^{97} - \\
& \frac{21}{64}x^{192}y^{96} - \frac{1071}{128}x^{190}y^{98} + \frac{555}{32}x^{190}y^{97} + \frac{4347}{128}x^{190}y^{96} + \frac{1815}{2}x^{189}y^{96} - \frac{1485}{64}x^{189}y^{95} + \\
& \frac{81}{128}x^{188}y^{94} + \frac{1485}{4}x^{187}y^{97} - \frac{2583}{8}x^{187}y^{96} - \frac{19215}{64}x^{187}y^{95} - \frac{63}{32}x^{187}y^{94} + \frac{135}{32}x^{186}y^{93} + \\
& \frac{111}{32}x^{185}y^{95} + \frac{1449}{64}x^{185}y^{94} + \frac{12705}{8}x^{184}y^{94} - \frac{1485}{16}x^{184}y^{93} - \frac{2925}{512}x^{184}y^{92} + \\
& \frac{1485}{16}x^{182}y^{95} - \frac{2583}{16}x^{182}y^{94} - \frac{19215}{64}x^{182}y^{93} - \frac{315}{64}x^{182}y^{92} - \frac{1701}{64}x^{182}y^{91} + \\
& \frac{1485}{32}x^{181}y^{91} + \frac{1449}{256}x^{180}y^{92} - \frac{231}{128}x^{180}y^{90} + \frac{7623}{4}x^{179}y^{92} - \frac{3465}{16}x^{179}y^{91} - \frac{2925}{64}x^{179}y^{90} + \\
& \frac{165}{16}x^{177}y^{93} - \frac{369}{8}x^{177}y^{92} - \frac{11529}{64}x^{177}y^{91} - \frac{105}{16}x^{177}y^{90} - \frac{441}{64}x^{177}y^{89} + \frac{7425}{32}x^{176}y^{89} - \\
& \frac{99}{128}x^{176}y^{88} - \frac{231}{64}x^{175}y^{88} + \frac{12705}{8}x^{174}y^{90} - \frac{10395}{32}x^{174}y^{89} - \frac{20475}{128}x^{174}y^{88} - \\
& \frac{69}{32}x^{174}y^{87} - \frac{369}{64}x^{172}y^{90} - \frac{3843}{64}x^{172}y^{89} - \frac{315}{64}x^{172}y^{88} + \frac{1575}{32}x^{172}y^{87} - \frac{945}{32}x^{172}y^{86} + \\
& \frac{22275}{32}x^{171}y^{87} - \frac{495}{64}x^{171}y^{86} - \frac{231}{128}x^{170}y^{86} - \frac{9}{16}x^{170}y^{85} + \frac{1815}{2}x^{169}y^{88} - \frac{10395}{32}x^{169}y^{87} - \\
& \frac{20475}{64}x^{169}y^{86} - \frac{483}{32}x^{169}y^{85} + \frac{227}{128}x^{168}y^{84} - \frac{549}{64}x^{167}y^{87} - \frac{63}{32}x^{167}y^{86} + \frac{1575}{32}x^{167}y^{85} +
\end{aligned}$$

$$\begin{aligned}
& \frac{63}{8}x^{167}y^{84} + \frac{22275}{16}x^{166}y^{85} - \frac{4455}{128}x^{166}y^{84} - \frac{165}{64}x^{166}y^{83} - \frac{9}{16}x^{165}y^{83} + \frac{5445}{16}x^{164}y^{86} - \\
& \frac{3465}{16}x^{164}y^{85} - \frac{102375}{256}x^{164}y^{84} - \frac{1449}{32}x^{164}y^{83} + \frac{315}{128}x^{164}y^{82} + \frac{681}{32}x^{163}y^{82} - \\
& \frac{21}{64}x^{162}y^{84} + \frac{1575}{64}x^{162}y^{83} + \frac{189}{16}x^{162}y^{82} - \frac{63}{64}x^{162}y^{81} + \frac{31185}{16}x^{161}y^{83} - \frac{1485}{16}x^{161}y^{82} - \\
& \frac{1485}{64}x^{161}y^{81} + \frac{459}{1024}x^{160}y^{80} + \frac{605}{8}x^{159}y^{84} - \frac{1485}{16x}y^{83} - \frac{20475}{64}x^{159}y^{82} - \frac{2415}{32}x^{159}y^{81} + \\
& \frac{945}{64}x^{159}y^{80} + \frac{7491}{64}x^{158}y^{80} - \frac{9}{64}x^{158}y^{79} + \frac{315}{64}x^{157}y^{81} + \frac{63}{8}x^{157}y^{80} - \frac{189}{64}x^{157}y^{79} + \\
& \frac{31185}{16}x^{156}y^{81} - \frac{10395}{64}x^{156}y^{80} - \frac{1485}{16}x^{156}y^{79} - \frac{465}{256}x^{156}y^{78} + \frac{121}{16}x^{154}y^{82} - \\
& \frac{1485}{64}x^{154}y^{81} - \frac{20475}{128}x^{154}y^{80} - \frac{2415}{32}x^{154}y^{79} + \frac{4725}{128}x^{154}y^{78} - \frac{945}{32}x^{154}y^{77} + \\
& \frac{12485}{32}x^{153}y^{78} - \frac{99}{64}x^{153}y^{77} + \frac{63}{32}x^{152}y^{78} - \frac{189}{64}x^{152}y^{77} - \frac{27}{32}x^{152}y^{76} + \frac{22275}{16}x^{151}y^{79} - \\
& \frac{6237}{32}x^{151}y^{78} - \frac{3465}{16}x^{151}y^{77} - \frac{465}{32}x^{151}y^{76} + \frac{35}{64}x^{150}y^{75} - \frac{165}{64}x^{149}y^{79} - \frac{2925}{64}x^{149}y^{78} - \\
& \frac{1449}{32}x^{149}y^{77} + \frac{1575}{32}x^{149}y^{76} - \frac{693}{32}x^{149}y^{75} + \frac{112365}{128}x^{148}y^{76} - \frac{495}{64}x^{148}y^{75} - \\
& \frac{99}{128}x^{148}y^{74} - \frac{63}{64}x^{147}y^{75} - \frac{27}{16}x^{147}y^{74} + \frac{22275}{32}x^{146}y^{77} - \frac{10395}{64}x^{146}y^{76} - \frac{10395}{32}x^{146}y^{75} - \\
& \frac{3255}{64}x^{146}y^{74} + \frac{45}{64}x^{146}y^{73} + \frac{455}{64}x^{145}y^{73} - \frac{2925}{512}x^{144}y^{76} - \frac{483}{32}x^{144}y^{75} + \frac{4725}{128}x^{144}y^{74} + \\
& \frac{315}{16}x^{144}y^{73} - \frac{5355}{256}x^{144}y^{72} + \frac{22473}{16}x^{143}y^{74} - \frac{1485}{64}x^{143}y^{73} - \frac{495}{64}x^{143}y^{72} - \frac{27}{32}x^{142}y^{72} - \\
& \frac{9}{64}x^{142}y^{71} + \frac{7425}{32}x^{141}y^{75} - \frac{1485}{16}x^{141}y^{74} - \frac{10395}{32}x^{141}y^{73} - \frac{3255}{32}x^{141}y^{72} + \frac{315}{64}x^{141}y^{71} + \\
& \frac{1365}{32}x^{140}y^{71} - \frac{3}{256}x^{140}y^{70} - \frac{69}{32}x^{139}y^{73} + \frac{945}{64}x^{139}y^{72} + \frac{315}{16}x^{139}y^{71} + \frac{21}{64}x^{139}y^{70} + \\
& \frac{52437}{32}x^{138}y^{72} - \frac{1485}{32}x^{138}y^{71} - \frac{4455}{128}x^{138}y^{70} - \frac{55}{64}x^{138}y^{69} - \frac{9}{64}x^{137}y^{69} + \frac{1485}{32}x^{136}y^{73} - \\
& \frac{4455}{128}x^{136}y^{72} - \frac{3465}{16}x^{136}y^{71} - \frac{16275}{128}x^{136}y^{70} + \frac{945}{64}x^{136}y^{69} + \frac{63}{64}x^{136}y^{68} + \frac{5005}{32}x^{135}y^{69} - \\
& \frac{9}{64}x^{135}y^{68} + \frac{315}{128}x^{134}y^{70} + \frac{315}{32}x^{134}y^{69} + \frac{63}{128}x^{134}y^{68} - \frac{9}{16}x^{134}y^{67} + \frac{22473}{16}x^{133}y^{70} - \\
& \frac{2079}{32}x^{133}y^{69} - \frac{1485}{16}x^{133}y^{68} - \frac{495}{64}x^{133}y^{67} + \frac{21}{64}x^{132}y^{66} + \frac{135}{32}x^{131}y^{71} - \frac{495}{64}x^{131}y^{70} - \\
& \frac{1485}{16}x^{131}y^{69} - \frac{3255}{32}x^{131}y^{68} + \frac{1575}{64}x^{131}y^{67} + \frac{189}{32}x^{131}y^{66} + \frac{25025}{64}x^{130}y^{67} - \\
& \frac{99}{128}x^{130}y^{66} - \frac{9}{64}x^{130}y^{65} + \frac{63}{32}x^{129}y^{67} + \frac{21}{64}x^{129}y^{66} - \frac{27}{16}x^{129}y^{65} + \frac{112365}{128}x^{128}y^{68} - \\
& \frac{2079}{32}x^{128}y^{67} - \frac{10395}{64}x^{128}y^{66} - \frac{495}{16}x^{128}y^{65} + \frac{45}{512}x^{128}y^{64} + \frac{105}{64}x^{127}y^{64} - \frac{99}{128}x^{126}y^{68} - \\
& \frac{1485}{64}x^{126}y^{67} - \frac{3255}{64}x^{126}y^{66} + \frac{1575}{64}x^{126}y^{65} + \frac{945}{64}x^{126}y^{64} - \frac{1323}{64}x^{126}y^{63} + \frac{45045}{64}x^{125}y^{65} - \\
& \frac{165}{64}x^{125}y^{64} - \frac{99}{64}x^{125}y^{63} + \frac{21}{256}x^{124}y^{64} - \frac{27}{16}x^{124}y^{63} - \frac{27}{128}x^{124}y^{62} + \frac{12485}{32}x^{123}y^{66} - \\
& \frac{1485}{32}x^{123}y^{65} - \frac{6237}{32}x^{123}y^{64} - \frac{1155}{16}x^{123}y^{63} + \frac{45}{64}x^{123}y^{62} + \frac{1365}{128}x^{122}y^{62} - \frac{165}{64}x^{121}y^{65} - \\
& \frac{465}{32}x^{121}y^{64} + \frac{945}{64}x^{121}y^{63} + \frac{315}{16}x^{121}y^{62} - \frac{1239}{64}x^{121}y^{61} + \frac{15015}{16}x^{120}y^{63} - \frac{1485}{256}x^{120}y^{62} -
\end{aligned}$$

$$\begin{aligned}
& \frac{495}{64}x^{120}y^{61} - \frac{33}{128}x^{120}y^{60} - \frac{9}{16}x^{119}y^{61} - \frac{27}{64}x^{119}y^{60} + \frac{7491}{64}x^{118}y^{64} - \frac{1485}{64}x^{118}y^{63} - \\
& \frac{10395}{64}x^{118}y^{62} - \frac{3465}{32}x^{118}y^{61} + \frac{315}{128}x^{118}y^{60} + \frac{9}{32}x^{118}y^{59} + \frac{1365}{32}x^{117}y^{60} - \frac{465}{256}x^{116}y^{62} + \\
& \frac{315}{64}x^{116}y^{61} + \frac{945}{64}x^{116}y^{60} + \frac{105}{32}x^{116}y^{59} - \frac{585}{64}x^{116}y^{58} + \frac{15015}{16}x^{115}y^{61} - \frac{297}{32}x^{115}y^{60} - \\
& \frac{1485}{64}x^{115}y^{59} - \frac{165}{64}x^{115}y^{58} - \frac{27}{128}x^{114}y^{58} + \frac{681}{32}x^{113}y^{62} - \frac{495}{64}x^{113}y^{61} - \frac{1485}{16}x^{113}y^{60} - \\
& \frac{3465}{32}x^{113}y^{59} + \frac{315}{64}x^{113}y^{58} + \frac{63}{32}x^{113}y^{57} + \frac{15015}{128}x^{112}y^{58} - \frac{3}{256}x^{112}y^{56} + \frac{45}{64}x^{111}y^{59} + \\
& \frac{189}{32}x^{111}y^{58} + \frac{105}{32}x^{111}y^{57} - \frac{9}{16}x^{111}y^{56} + \frac{45045}{64}x^{110}y^{59} - \frac{693}{64}x^{110}y^{58} - \frac{1485}{32}x^{110}y^{57} - \\
& \frac{1485}{128}x^{110}y^{56} + \frac{7}{32}x^{109}y^{55} + \frac{227}{128}x^{108}y^{60} - \frac{99}{64}x^{108}y^{59} - \frac{4455}{128}x^{108}y^{58} - \frac{1155}{16}x^{108}y^{57} + \\
& \frac{1575}{256}x^{108}y^{56} + \frac{189}{32}x^{108}y^{55} + \frac{21}{128}x^{108}y^{54} + \frac{15015}{64}x^{107}y^{56} - \frac{9}{64}x^{107}y^{54} + \frac{63}{64}x^{106}y^{56} + \\
& \frac{105}{64}x^{106}y^{55} - \frac{27}{32}x^{106}y^{54} - \frac{9}{64}x^{106}y^{53} + \frac{25025}{64}x^{105}y^{57} - \frac{297}{32}x^{105}y^{56} - \frac{2079}{32}x^{105}y^{55} - \\
& \frac{495}{16}x^{105}y^{54} + \frac{105}{64}x^{104}y^{53} + \frac{33}{512}x^{104}y^{52} - \frac{9}{64}x^{103}y^{57} - \frac{495}{64}x^{103}y^{56} - \frac{495}{16}x^{103}y^{55} + \\
& \frac{315}{64}x^{103}y^{54} + \frac{315}{32}x^{103}y^{53} + \frac{63}{64}x^{103}y^{52} + \frac{45045}{128}x^{102}y^{54} - \frac{99}{128}x^{102}y^{52} - \frac{3}{64}x^{102}y^{51} + \\
& \frac{21}{64}x^{101}y^{53} - \frac{9}{16}x^{101}y^{52} - \frac{27}{64}x^{101}y^{51} + \frac{5005}{32}x^{100}y^{55} - \frac{1485}{256}x^{100}y^{54} - \frac{2079}{32}x^{100}y^{53} - \\
& \frac{3465}{64}x^{100}y^{52} + \frac{9}{256}x^{100}y^{50} + \frac{455}{64}x^{99}y^{51} - \frac{99}{128}x^{98}y^{54} - \frac{495}{64}x^{98}y^{53} + \frac{315}{128}x^{98}y^{52} + \\
& \frac{315}{32}x^{98}y^{51} + \frac{315}{128}x^{98}y^{50} - 9x^{98}y^{49} + \frac{6435}{16}x^{97}y^{52} - \frac{165}{64}x^{97}y^{50} - \frac{33}{64}x^{97}y^{49} - \\
& \frac{9}{64}x^{96}y^{50} - 2764x^{96}y^{49} - \frac{23}{1024}x^{96}y^{48} + \frac{1365}{32}x^{95}y^{53} - \frac{165}{64}x^{95}y^{52} - \frac{1485}{32}x^{95}y^{51} - \\
& \frac{2079}{32}x^{95}y^{50} + \frac{9}{32}x^{95}y^{48} + \frac{1365}{64}x^{94}y^{49} - \frac{55}{64}x^{93}y^{51} + \frac{45}{64}x^{93}y^{50} + \frac{189}{32}x^{93}y^{49} + \\
& \frac{105}{32}x^{93}y^{48} - 9x^{93}y^{47} + \frac{45045}{128}x^{92}y^{50} - \frac{1485}{256}x^{92}y^{48} - \frac{165}{64}x^{92}y^{47} - \frac{9}{64}x^{91}y^{47} - \\
& \frac{1}{32}x^{91}y^{46} + \frac{455}{64}x^{90}y^{51} - \frac{99}{128}x^{90}y^{50} - \frac{1485}{64}x^{90}y^{49} - \frac{3465}{64}x^{90}y^{48} + \frac{63}{64}x^{90}y^{46} + \\
& \frac{3}{64}x^{90}y^{45} + \frac{3003}{64}x^{89}y^{47} + \frac{45}{512}x^{88}y^{48} + \frac{63}{32}x^{88}y^{47} + \frac{315}{128}x^{88}y^{46} - \frac{585}{256}x^{88}y^{44} + \\
& \frac{15015}{64}x^{87}y^{48} - \frac{297}{32}x^{87}y^{46} - \frac{495}{64}x^{87}y^{45} + \frac{3}{32}x^{86}y^{44} + \frac{35}{64}x^{85}y^{49} - \frac{9}{64}x^{85}y^{48} - \\
& \frac{495}{64}x^{85}y^{47} - \frac{495}{16}x^{85}y^{46} + \frac{63}{32}x^{85}y^{44} + \frac{21}{64}x^{85}y^{43} + \frac{5005}{64}x^{84}y^{45} - \frac{1}{256}x^{84}y^{42} + \\
& \frac{9}{32}x^{83}y^{45} + \frac{63}{64}x^{83}y^{44} - \frac{9}{64}x^{83}y^{42} + \frac{15015}{128}x^{82}y^{46} - \frac{693}{64}x^{82}y^{44} - \frac{495}{32}x^{82}y^{43} + \\
& \frac{35}{64}x^{81}y^{42} - \frac{3}{256}x^{80}y^{46} - \frac{99}{64}x^{80}y^{45} - \frac{1485}{128}x^{80}y^{44} + \frac{315}{128}x^{80}y^{42} + \frac{63}{64}x^{80}y^{41} + \\
& \frac{6435}{64}x^{79}y^{43} - \frac{3}{64}x^{79}y^{40} + \frac{21}{128}x^{78}y^{42} - \frac{27}{128}x^{78}y^{40} - \frac{1}{64}x^{78}y^{39} + \frac{1365}{32}x^{77}y^{44} - \\
& \frac{297}{32}x^{77}y^{42} - \frac{693}{32}x^{77}y^{41} + \frac{455}{256}x^{76}y^{40} + \frac{3}{256}x^{76}y^{38} - \frac{9}{64}x^{75}y^{43} - \frac{165}{64}x^{75}y^{42} + \\
& \frac{63}{32}x^{75}y^{40} + \frac{105}{64}x^{75}y^{39} + \frac{6435}{64}x^{74}y^{41} - \frac{33}{128}x^{74}y^{38} - \frac{9}{64}x^{73}y^{38} - \frac{3}{64}x^{73}y^{37} +
\end{aligned}$$

$$\begin{aligned}
& \frac{1365}{128}x^{72}y^{42} - \frac{1485}{256}x^{72}y^{40} - \frac{693}{32}x^{72}y^{39} + \frac{3}{512}x^{72}y^{36} + \frac{273}{64}x^{71}y^{38} - \frac{33}{128}x^{70}y^{40} + \\
& \frac{63}{64}x^{70}y^{38} + \frac{105}{64}x^{70}y^{37} - \frac{9}{4}x^{70}y^{35} + \frac{5005}{64}x^{69}y^{39} - \frac{55}{64}x^{69}y^{36} - \frac{9}{256}x^{68}y^{36} - \\
& \frac{3}{64}x^{68}y^{35} + \frac{105}{64}x^{67}y^{40} - \frac{165}{64}x^{67}y^{38} - \frac{495}{32}x^{67}y^{37} + \frac{3}{64}x^{67}y^{34} + \frac{1001}{128}x^{66}y^{36} + \\
& \frac{9}{32}x^{65}y^{36} + \frac{63}{64}x^{65}y^{35} - \frac{9}{4}x^{65}y^{33} + \frac{3003}{64}x^{64}y^{37} - \frac{495}{256}x^{64}y^{34} - \frac{1}{64}x^{63}y^{33} + \\
& \frac{15}{128}x^{62}y^{38} - \frac{99}{128}x^{62}y^{36} - \frac{495}{64}x^{62}y^{35} + \frac{21}{128}x^{62}y^{32} + \frac{715}{64}x^{61}y^{34} + \frac{9}{256}x^{60}y^{34} + \\
& \frac{21}{64}x^{60}y^{33} - \frac{65}{256}x^{60}y^{30} + \frac{1365}{64}x^{59}y^{35} - \frac{99}{32}x^{59}y^{32} - \frac{9}{64}x^{57}y^{34} - \frac{165}{64}x^{57}y^{33} + \\
& \frac{21}{64}x^{57}y^{30} + \frac{6435}{512}x^{56}y^{32} + \frac{3}{64}x^{55}y^{31} - \frac{1}{64}x^{55}y^{28} + \frac{455}{64}x^{54}y^{33} - \frac{231}{64}x^{54}y^{30} - \\
& \frac{3}{256}x^{52}y^{32} - \frac{33}{64}x^{52}y^{31} + \frac{105}{256}x^{52}y^{28} + \frac{715}{64}x^{51}y^{30} - \frac{3}{128}x^{50}y^{26} + \frac{105}{64}x^{49}y^{31} - \\
& \frac{99}{32}x^{49}y^{28} + \frac{1}{1024}x^{48}y^{24} - \frac{3}{64}x^{47}y^{29} + \frac{21}{64}x^{47}y^{26} + \frac{1001}{128}x^{46}y^{28} - \frac{1}{64}x^{45}y^{24} + \\
& \frac{15}{64}x^{44}y^{29} - \frac{495}{256}x^{44}y^{26} + \frac{21}{128}x^{42}y^{24} - \frac{1}{4}x^{42}y^{21} + \frac{273}{64}x^{41}y^{26} - \frac{1}{256}x^{40}y^{22} + \\
& \frac{1}{64}x^{39}y^{27} - \frac{55}{64}x^{39}y^{24} + \frac{3}{64}x^{37}y^{22} - \frac{1}{4}x^{37}y^{19} + \frac{455}{256}x^{36}y^{24} - \frac{33}{128}x^{34}y^{22} + \\
& \frac{3}{512}x^{32}y^{20} + \frac{35}{64}x^{31}y^{22} - \frac{3}{64}x^{29}y^{20} + \frac{15}{128}x^{26}y^{20} - \frac{1}{256}x^{24}y^{18} + \frac{1}{64}x^{21}y^{18} + \\
& \frac{1}{1024}x^{16}y^{16} \\
f_5 = & x^{379}y^{190} + \frac{3}{2}x^{374}y^{188} - \frac{3}{4}x^{374}y^{187} + x^{369}y^{186} - \frac{9}{4}x^{369}y^{185} + \frac{9}{4}x^{366}y^{183} + \\
& \frac{1}{4}x^{364}y^{184} - \frac{9}{4}x^{364}y^{183} + \frac{27}{32}x^{364}y^{182} + \frac{45}{4}x^{361}y^{181} - \frac{3}{4}x^{359}y^{181} + \frac{27}{16}x^{359}y^{180} + \\
& \frac{45}{2}x^{356}y^{179} - \frac{69}{8}x^{356}y^{178} + \frac{27}{32}x^{354}y^{178} - \frac{27}{64}x^{354}y^{177} + \frac{45}{2}x^{351}y^{177} - \frac{45}{2}x^{351}y^{176} - \\
& \frac{27}{64}x^{349}y^{175} + \frac{77}{8}x^{348}y^{174} + \frac{45}{4}x^{346}y^{175} - \frac{135}{4}x^{346}y^{174} + \frac{69}{16}x^{346}y^{173} + \frac{81}{1024}x^{344}y^{172} + \\
& \frac{231}{4}x^{343}y^{172} + \frac{9}{4}x^{341}y^{173} - \frac{45}{2}x^{341}y^{172} + \frac{207}{16}x^{341}y^{171} + \frac{1155}{8}x^{338}y^{170} - \frac{315}{16}x^{338}y^{169} - \\
& \frac{45}{8}x^{336}y^{170} + \frac{207}{16}x^{336}y^{169} - \frac{27}{128}x^{336}y^{168} + \frac{385}{2}x^{333}y^{168} - \frac{1575}{16}x^{333}y^{167} + \\
& \frac{69}{16}x^{331}y^{167} - \frac{27}{64}x^{331}y^{166} + 26x^{330}y^{165} + \frac{1155}{8}x^{328}y^{166} - \frac{1575}{8}x^{328}y^{165} - \\
& \frac{1077}{128}x^{328}y^{164} - \frac{27}{128}x^{326}y^{164} - \frac{1053}{1024}x^{326}y^{163} + 182x^{325}y^{163} + \frac{231}{4}x^{323}y^{164} - \\
& \frac{1575}{8}x^{323}y^{163} + \frac{1035}{32}x^{323}y^{162} - \frac{1053}{1024}x^{321}y^{161} + 546x^{320}y^{161} - \frac{1365}{32}x^{320}y^{160} + \\
& \frac{77}{8}x^{318}y^{162} - \frac{1575}{16}x^{318}y^{161} + \frac{3105}{64}x^{318}y^{160} + \frac{805}{128}x^{318}y^{159} + \frac{81}{256}x^{316}y^{158} + \\
& 910x^{315}y^{159} - \frac{4095}{16}x^{315}y^{158} - \frac{315}{16}x^{313}y^{159} + \frac{1035}{32}x^{313}y^{158} + \frac{2415}{128}x^{313}y^{157} + \\
& \frac{3177}{64}x^{312}y^{156} + 910x^{310}y^{157} - \frac{20475}{32}x^{310}y^{156} + \frac{315}{128}x^{310}y^{155} + \frac{1035}{128}x^{308}y^{156} + \\
& \frac{2415}{128}x^{308}y^{155} - \frac{4149}{1024}x^{308}y^{154} + \frac{3177}{8}x^{307}y^{154} + 546x^{305}y^{155} - \frac{6825}{8}x^{305}y^{154} +
\end{aligned}$$

$$\begin{aligned}
& \frac{1575}{128}x^{305}y^{153} + \frac{805}{128}x^{303}y^{153} - \frac{4149}{512}x^{303}y^{152} + \frac{22239}{16}x^{302}y^{152} - \frac{2043}{32}x^{302}y^{151} + \\
& 182x^{300}y^{153} - \frac{20475}{32}x^{300}y^{152} + \frac{1575}{64}x^{300}y^{151} - \frac{8485}{256}x^{300}y^{150} - \frac{4149}{1024}x^{298}y^{150} - \\
& \frac{189}{256}x^{298}y^{149} + \frac{22239}{8}x^{297}y^{150} - \frac{14301}{32}x^{297}y^{149} + 26x^{295}y^{151} - \frac{4095}{16}x^{295}y^{150} + \\
& \frac{1575}{64}x^{295}y^{149} + \frac{5595}{64}x^{295}y^{148} + \frac{4543}{64}x^{294}y^{147} - \frac{189}{256}x^{293}y^{147} + \frac{111195}{32}x^{292}y^{148} - \\
& \frac{42903}{32}x^{292}y^{147} - \frac{609}{32}x^{292}y^{146} - \frac{1365}{32}x^{290}y^{148} + \frac{1575}{128}x^{290}y^{147} + \frac{16785}{128}x^{290}y^{146} - \\
& \frac{2295}{1024}x^{290}y^{145} + \frac{40887}{64}x^{289}y^{145} + \frac{297}{512}x^{288}y^{144} + \frac{22239}{8}x^{287}y^{146} - \frac{71505}{32}x^{287}y^{145} - \\
& \frac{1827}{16}x^{287}y^{144} + \frac{315}{128}x^{285}y^{145} + \frac{5595}{64}x^{285}y^{144} - \frac{6885}{1024}x^{285}y^{143} + \frac{40887}{16}x^{284}y^{143} - \\
& \frac{4455}{64}x^{284}y^{142} + \frac{22239}{16}x^{282}y^{144} - \frac{71505}{32}x^{282}y^{143} - \frac{9135}{32}x^{282}y^{142} + \frac{9093}{256}x^{282}y^{141} + \\
& \frac{5595}{256}x^{280}y^{142} - \frac{6885}{1024}x^{280}y^{141} - \frac{1557}{256}x^{280}y^{140} + \frac{95403}{16}x^{279}y^{141} - \frac{4455}{8}x^{279}y^{140} + \\
& \frac{3177}{8}x^{277}y^{142} - \frac{42903}{32}x^{277}y^{141} - \frac{3045}{8}x^{277}y^{140} + \frac{45465}{256}x^{277}y^{139} + \frac{5027}{64}x^{276}y^{138} - \\
& \frac{2295}{1024}x^{275}y^{139} - \frac{1557}{128}x^{275}y^{138} + \frac{286209}{32}x^{274}y^{139} - \frac{31185}{16}x^{274}y^{138} - \frac{2979}{64}x^{274}y^{137} + \\
& \frac{3177}{64}x^{272}y^{140} - \frac{14301}{32}x^{272}y^{139} - \frac{9135}{32}x^{272}y^{138} + \frac{45465}{128}x^{272}y^{137} - \frac{28665}{256}x^{272}y^{136} + \\
& \frac{25135}{32}x^{271}y^{136} - \frac{1557}{256}x^{270}y^{136} + \frac{243}{512}x^{270}y^{135} + \frac{286209}{32}x^{269}y^{137} - \frac{31185}{8}x^{269}y^{136} - \\
& \frac{20853}{64}x^{269}y^{135} - \frac{2043}{32}x^{267}y^{137} - \frac{1827}{16}x^{267}y^{136} + \frac{45465}{128}x^{267}y^{135} + \frac{3015}{64}x^{267}y^{134} + \\
& \frac{226215}{64}x^{266}y^{134} - \frac{1815}{32}x^{266}y^{133} + \frac{243}{512}x^{265}y^{133} + \frac{95403}{16}x^{264}y^{135} - \frac{155925}{32}x^{264}y^{134} - \\
& \frac{62559}{64}x^{264}y^{133} + \frac{8267}{256}x^{264}y^{132} - \frac{609}{32}x^{262}y^{134} + \frac{45465}{256}x^{262}y^{133} + \frac{9045}{128}x^{262}y^{132} - \\
& \frac{2727}{256}x^{262}y^{131} + \frac{75405}{8}x^{261}y^{132} - \frac{16335}{32}x^{261}y^{131} + \frac{81}{128}x^{260}y^{130} + \frac{40887}{16}x^{259}y^{133} - \\
& \frac{31185}{8}x^{259}y^{132} - \frac{104265}{64}x^{259}y^{131} + \frac{24801}{128}x^{259}y^{130} + \frac{4383}{64}x^{258}y^{129} + \frac{9093}{256}x^{257}y^{131} + \\
& \frac{3015}{64}x^{257}y^{130} - \frac{8181}{256}x^{257}y^{129} + \frac{527835}{32}x^{256}y^{130} - \frac{16335}{8}x^{256}y^{129} - \frac{31185}{512}x^{256}y^{128} + \\
& \frac{40887}{64}x^{254}y^{131} - \frac{31185}{16}x^{254}y^{130} - \frac{104265}{64}x^{254}y^{129} + \frac{124005}{256}x^{254}y^{128} + \frac{7749}{256}x^{254}y^{127} + \\
& \frac{48213}{64}x^{253}y^{127} + \frac{3015}{256}x^{252}y^{128} - \frac{8181}{256}x^{252}y^{127} - \frac{1977}{512}x^{252}y^{126} + \frac{316701}{16}x^{251}y^{128} - \\
& \frac{38115}{8}x^{251}y^{127} - \frac{31185}{64}x^{251}y^{126} + \frac{4543}{64}x^{249}y^{129} - \frac{4455}{8}x^{249}y^{128} - \frac{62559}{64}x^{249}y^{127} + \\
& \frac{41335}{64}x^{249}y^{126} + \frac{38745}{256}x^{249}y^{125} + \frac{241065}{64}x^{248}y^{125} - \frac{4455}{128}x^{248}y^{124} - \frac{2727}{256}x^{247}y^{125} - \\
& \frac{1977}{256}x^{247}y^{124} + \frac{527835}{32}x^{246}y^{126} - \frac{114345}{16}x^{246}y^{125} - \frac{218295}{128}x^{246}y^{124} + \frac{3381}{256}x^{246}y^{123} - \\
& \frac{4455}{64}x^{244}y^{126} - \frac{20853}{64}x^{244}y^{125} + \frac{124005}{256}x^{244}y^{124} + \frac{38745}{128}x^{244}y^{123} - \frac{1629}{8}x^{244}y^{122} + \\
& \frac{723195}{64}x^{243}y^{123} - \frac{22275}{64}x^{243}y^{122} - \frac{1977}{512}x^{242}y^{122} + \frac{171}{128}x^{242}y^{121} + \frac{75405}{8}x^{241}y^{124} -
\end{aligned}$$

$$\begin{aligned}
& \frac{114345}{16}x^{241}y^{123} - \frac{218295}{64}x^{241}y^{122} + \frac{23667}{256}x^{241}y^{121} + \frac{6071}{128}x^{240}y^{120} - \frac{2979}{64}x^{239}y^{123} + \\
& \frac{24801}{128}x^{239}y^{122} + \frac{38745}{128}x^{239}y^{121} - \frac{45}{2}x^{239}y^{120} + \frac{723195}{32}x^{238}y^{121} - \frac{200475}{128}x^{238}y^{120} - \\
& \frac{27555}{512}x^{238}y^{119} + \frac{171}{128}x^{237}y^{119} + \frac{226215}{64}x^{236}y^{122} - \frac{38115}{8}x^{236}y^{121} - \frac{1091475}{256}x^{236}y^{120} + \\
& \frac{71001}{256}x^{236}y^{119} + \frac{9513}{256}x^{236}y^{118} + \frac{18213}{32}x^{235}y^{118} + \frac{8267}{256}x^{234}y^{120} + \frac{38745}{256}x^{234}y^{119} - \\
& \frac{135}{4}x^{234}y^{118} - \frac{4923}{512}x^{234}y^{117} + \frac{1012473}{32}x^{233}y^{119} - \frac{66825}{16}x^{233}y^{118} - \frac{247995}{512}x^{233}y^{117} + \\
& \frac{459}{1024}x^{232}y^{116} + \frac{25135}{32}x^{231}y^{120} - \frac{16335}{8}x^{231}y^{119} - \frac{218295}{64}x^{231}y^{118} + \frac{118335}{256}x^{231}y^{117} + \\
& \frac{28539}{128}x^{231}y^{116} + \frac{200343}{64}x^{230}y^{116} - \frac{2043}{128}x^{230}y^{115} + \frac{7749}{256}x^{229}y^{117} - \frac{45}{2}x^{229}y^{116} - \\
& \frac{14769}{512}x^{229}y^{115} + \frac{1012473}{32}x^{228}y^{117} - \frac{467775}{64}x^{228}y^{116} - \frac{247995}{128}x^{228}y^{115} - \frac{1485}{256}x^{228}y^{114} + \\
& \frac{5027}{64}x^{226}y^{118} - \frac{16335}{32}x^{226}y^{117} - \frac{218295}{128}x^{226}y^{116} + \frac{118335}{256}x^{226}y^{115} + \frac{142695}{256}x^{226}y^{114} + \\
& \frac{189}{32}x^{226}y^{113} + \frac{333905}{32}x^{225}y^{114} - \frac{22473}{128}x^{225}y^{113} - \frac{45}{8}x^{224}y^{114} - \frac{14769}{512}x^{224}y^{113} - \\
& \frac{81}{128}x^{224}y^{112} + \frac{723195}{32}x^{223}y^{115} - \frac{280665}{32}x^{223}y^{114} - \frac{578655}{128}x^{223}y^{113} - \frac{1485}{32}x^{223}y^{112} + \\
& \frac{3339}{128}x^{222}y^{111} - \frac{1815}{32}x^{221}y^{115} - \frac{31185}{64}x^{221}y^{114} + \frac{71001}{256}x^{221}y^{113} + \frac{47565}{64}x^{221}y^{112} + \\
& \frac{945}{32}x^{221}y^{111} + \frac{3005145}{128}x^{220}y^{112} - \frac{112365}{128}x^{220}y^{111} - \frac{17523}{512}x^{220}y^{110} - \frac{4923}{512}x^{219}y^{111} - \\
& \frac{81}{64}x^{219}y^{110} + \frac{723195}{64}x^{218}y^{113} - \frac{467775}{64}x^{218}y^{112} - \frac{1735965}{256}x^{218}y^{111} - \frac{10395}{64}x^{218}y^{110} + \\
& \frac{7299}{256}x^{218}y^{109} + \frac{43407}{128}x^{217}y^{109} - \frac{31185}{512}x^{216}y^{112} + \frac{23667}{256}x^{216}y^{111} + \frac{142695}{256}x^{216}y^{110} + \\
& \frac{945}{16}x^{216}y^{109} - \frac{61581}{256}x^{216}y^{108} + \frac{601029}{16}x^{215}y^{110} - \frac{337095}{128}x^{215}y^{109} - \frac{87615}{256}x^{215}y^{108} - \\
& \frac{81}{128}x^{214}y^{108} + \frac{1233}{1024}x^{214}y^{107} + \frac{241065}{64}x^{213}y^{111} - \frac{66825}{16}x^{213}y^{110} - \frac{1735965}{256}x^{213}y^{109} - \\
& \frac{10395}{32}x^{213}y^{108} + \frac{51093}{256}x^{213}y^{107} + \frac{130221}{64}x^{212}y^{107} - \frac{1365}{256}x^{212}y^{106} + \frac{3381}{256}x^{211}y^{109} + \\
& \frac{28539}{128}x^{211}y^{108} + \frac{945}{16}x^{211}y^{107} - \frac{2445}{64}x^{211}y^{106} + \frac{1402401}{32}x^{210}y^{108} - \frac{337095}{64}x^{210}y^{107} - \\
& \frac{788535}{512}x^{210}y^{106} - \frac{3355}{256}x^{210}y^{105} + \frac{1233}{1024}x^{209}y^{105} + \frac{48213}{64}x^{208}y^{109} - \frac{200475}{128}x^{208}y^{108} - \\
& \frac{578655}{128}x^{208}y^{107} - \frac{51975}{128}x^{208}y^{106} + \frac{153279}{256}x^{208}y^{105} + \frac{819}{64}x^{208}y^{104} + \frac{477477}{64}x^{207}y^{105} - \\
& \frac{4095}{64}x^{207}y^{104} + \frac{9513}{256}x^{206}y^{106} + \frac{945}{32}x^{206}y^{105} - \frac{7335}{128}x^{206}y^{104} - \frac{531}{128}x^{206}y^{103} + \\
& \frac{601029}{16}x^{205}y^{106} - \frac{471933}{64}x^{205}y^{105} - \frac{262845}{64}x^{205}y^{104} - \frac{30195}{256}x^{205}y^{103} + \frac{1473}{128}x^{204}y^{102} + \\
& \frac{4383}{64}x^{203}y^{107} - \frac{22275}{64}x^{203}y^{106} - \frac{247995}{128}x^{203}y^{105} - \frac{10395}{32}x^{203}y^{104} + \frac{255465}{256}x^{203}y^{103} + \\
& \frac{2457}{32}x^{203}y^{102} + \frac{2387385}{128}x^{202}y^{103} - \frac{45045}{128}x^{202}y^{102} - \frac{8145}{512}x^{202}y^{101} + \frac{189}{32}x^{201}y^{103} - \\
& \frac{2445}{64}x^{201}y^{102} - \frac{1593}{128}x^{201}y^{101} + \frac{3005145}{128}x^{200}y^{104} - \frac{471933}{64}x^{200}y^{103} - \frac{1839915}{256}x^{200}y^{102} -
\end{aligned}$$

$$\begin{aligned}
& \frac{30195}{64}x^{200}y^{101} + \frac{7425}{512}x^{200}y^{100} + \frac{5061}{32}x^{199}y^{100} - \frac{4455}{128}x^{198}y^{104} - \frac{247995}{512}x^{198}y^{103} - \\
& \frac{10395}{64}x^{198}y^{102} + \frac{255465}{256}x^{198}y^{101} + \frac{12285}{64}x^{198}y^{100} - \frac{1071}{256}x^{198}y^{99} + \frac{4297293}{128}x^{197}y^{101} - \\
& \frac{75075}{64}x^{197}y^{100} - \frac{89595}{512}x^{197}y^{99} - \frac{2445}{256}x^{196}y^{100} - \frac{1593}{128}x^{196}y^{99} + \frac{729}{1024}x^{196}y^{98} + \\
& \frac{333905}{32}x^{195}y^{102} - \frac{337095}{64}x^{195}y^{101} - \frac{1103949}{128}x^{195}y^{100} - \frac{70455}{64}x^{195}y^{99} + \frac{7425}{64}x^{195}y^{98} + \\
& \frac{65793}{64}x^{194}y^{98} - \frac{315}{256}x^{194}y^{97} - \frac{27555}{512}x^{193}y^{101} - \frac{1485}{32}x^{193}y^{100} + \frac{153279}{256}x^{193}y^{99} + \\
& \frac{4095}{16}x^{193}y^{98} - \frac{5355}{256}x^{193}y^{97} + \frac{1432431}{32}x^{192}y^{99} - \frac{675675}{256}x^{192}y^{98} - \frac{447975}{512}x^{192}y^{97} - \\
& \frac{2673}{256}x^{192}y^{96} - \frac{531}{128}x^{191}y^{97} + \frac{729}{512}x^{191}y^{96} + \frac{200343}{64}x^{190}y^{100} - \frac{337095}{128}x^{190}y^{99} - \\
& \frac{1839915}{256}x^{190}y^{98} - \frac{211365}{128}x^{190}y^{97} + \frac{51975}{128}x^{190}y^{96} + \frac{711}{64}x^{190}y^{95} + \frac{65793}{16}x^{189}y^{96} - \\
& \frac{4095}{256}x^{189}y^{95} - \frac{1485}{256}x^{188}y^{98} + \frac{51093}{256}x^{188}y^{97} + \frac{12285}{64}x^{188}y^{96} - \frac{5355}{128}x^{188}y^{95} - \\
& \frac{12987}{64}x^{188}y^{94} + \frac{1432431}{32}x^{187}y^{97} - \frac{135135}{32}x^{187}y^{96} - \frac{1343925}{512}x^{187}y^{95} - \frac{13365}{128}x^{187}y^{94} + \\
& \frac{729}{1024}x^{186}y^{94} + \frac{563}{128}x^{186}y^{93} + \frac{18213}{32}x^{185}y^{98} - \frac{112365}{128}x^{185}y^{97} - \frac{262845}{64}x^{185}y^{96} - \\
& \frac{211365}{128}x^{185}y^{95} + \frac{51975}{64}x^{185}y^{94} + \frac{4977}{64}x^{185}y^{93} + \frac{723723}{64}x^{184}y^{94} - \frac{12285}{128}x^{184}y^{93} - \\
& \frac{1365}{256}x^{184}y^{92} + \frac{7299}{256}x^{183}y^{95} + \frac{2457}{32}x^{183}y^{94} - \frac{5355}{128}x^{183}y^{93} - \frac{315}{16}x^{183}y^{92} + \frac{4297293}{128}x^{182}y^{95} - \\
& \frac{315315}{64}x^{182}y^{94} - \frac{1343925}{256}x^{182}y^{93} - \frac{120285}{256}x^{182}y^{92} + \frac{2475}{512}x^{182}y^{91} + \frac{7339}{128}x^{181}y^{91} + \\
& \frac{6071}{128}x^{180}y^{96} - \frac{22473}{128}x^{180}y^{95} - \frac{788535}{512}x^{180}y^{94} - \frac{70455}{64}x^{180}y^{93} + \frac{259875}{256}x^{180}y^{92} + \\
& \frac{14931}{64}x^{180}y^{91} + \frac{105}{256}x^{180}y^{90} + \frac{723723}{32}x^{179}y^{92} - \frac{45045}{128}x^{179}y^{91} - \frac{4095}{64}x^{179}y^{90} + \\
& \frac{819}{64}x^{178}y^{92} - \frac{5355}{256}x^{178}y^{91} - \frac{945}{32}x^{178}y^{90} - \frac{765}{1024}x^{178}y^{89} + \frac{2387385}{128}x^{177}y^{93} - \\
& \frac{135135}{32}x^{177}y^{92} - \frac{1881495}{256}x^{177}y^{91} - \frac{40095}{32}x^{177}y^{90} + \frac{22275}{512}x^{177}y^{89} + \frac{12705}{32}x^{176}y^{89} - \\
& \frac{57}{512}x^{176}y^{88} - \frac{2043}{128}x^{175}y^{93} - \frac{87615}{256}x^{175}y^{92} - \frac{30195}{64}x^{175}y^{91} + \frac{51975}{64}x^{175}y^{90} + \\
& \frac{24885}{64}x^{175}y^{89} + \frac{315}{128}x^{175}y^{88} + \frac{2171169}{64}x^{174}y^{90} - \frac{225225}{256}x^{174}y^{89} - \frac{45045}{128}x^{174}y^{88} - \\
& \frac{1335}{256}x^{174}y^{87} - \frac{1071}{256}x^{173}y^{89} - \frac{315}{6}x^{173}y^{88} - \frac{2295}{1024}x^{173}y^{87} + \frac{477477}{64}x^{172}y^{91} - \\
& \frac{675675}{256}x^{172}y^{90} - \frac{1881495}{256}x^{172}y^{89} - \frac{280665}{128}x^{172}y^{88} + \frac{22275}{128}x^{172}y^{87} + \frac{1485}{256}x^{172}y^{86} + \\
& \frac{55055}{32}x^{171}y^{87} - \frac{315}{128}x^{171}y^{86} - \frac{17523}{512}x^{170}y^{90} - \frac{30195}{256}x^{170}y^{89} + \frac{51975}{128}x^{170}y^{88} + \\
& \frac{24885}{64}x^{170}y^{87} + \frac{1575}{256}x^{170}y^{86} - \frac{189}{64}x^{170}y^{85} + \frac{310167}{8}x^{169}y^{88} - \frac{405405}{256}x^{169}y^{87} - \\
& \frac{75075}{64}x^{169}y^{86} - \frac{14685}{256}x^{169}y^{85} - \frac{315}{64}x^{168}y^{86} - \frac{2295}{1024}x^{168}y^{85} + \frac{1545}{1024}x^{168}y^{84} + \\
& \frac{130221}{64}x^{167}y^{89} - \frac{75075}{64}x^{167}y^{88} - \frac{1343925}{256}x^{167}y^{87} - \frac{168399}{64}x^{167}y^{86} + \frac{51975}{128}x^{167}y^{85} +
\end{aligned}$$

$$\begin{aligned}
& \frac{1485}{32}x^{167}y^{84} + \frac{165165}{32}x^{166}y^{85} - \frac{4095}{256}x^{166}y^{84} - \frac{315}{256}x^{166}y^{83} - \frac{3355}{256}x^{165}y^{87} + \\
& \frac{7425}{64}x^{165}y^{86} + \frac{14931}{64}x^{165}y^{85} + \frac{525}{64}x^{165}y^{84} - \frac{945}{64}x^{165}y^{83} + \frac{2171169}{64}x^{164}y^{86} - \\
& \frac{135135}{64}x^{164}y^{85} - \frac{675675}{256}x^{164}y^{84} - \frac{73425}{256}x^{164}y^{83} + \frac{495}{512}x^{164}y^{82} - \frac{765}{1024}x^{163}y^{83} + \\
& \frac{1041}{64}x^{163}y^{82} + \frac{43407}{128}x^{162}y^{87} - \frac{45045}{128}x^{162}y^{86} - \frac{1343925}{512}x^{162}y^{85} - \frac{280665}{128}x^{162}y^{84} + \\
& \frac{155925}{256}x^{162}y^{83} + \frac{10395}{64}x^{162}y^{82} + \frac{411}{256}x^{162}y^{81} + \frac{363363}{32}x^{161}y^{83} - \frac{4095}{64}x^{161}y^{82} - \\
& \frac{4095}{256}x^{161}y^{81} + \frac{7425}{512}x^{160}y^{84} + \frac{4977}{64}x^{160}y^{83} + \frac{1575}{256}x^{160}y^{82} - \frac{945}{32}x^{160}y^{81} - \frac{31995}{256}x^{160}y^{80} + \\
& \frac{723723}{32}x^{159}y^{84} - \frac{135135}{64}x^{159}y^{83} - \frac{135135}{32}x^{159}y^{82} - \frac{220275}{256}x^{159}y^{81} + \frac{2475}{256}x^{159}y^{80} + \\
& \frac{14607}{128}x^{158}y^{80} + \frac{93}{512}x^{158}y^{79} + \frac{3339}{128}x^{157}y^{85} - \frac{4095}{64}x^{157}y^{84} - \frac{447975}{512}x^{157}y^{83} - \\
& \frac{40095}{32}x^{157}y^{82} + \frac{155925}{256}x^{157}y^{81} + \frac{10395}{32}x^{157}y^{80} + \frac{2877}{256}x^{157}y^{79} + \frac{605605}{32}x^{156}y^{81} - \\
& \frac{45045}{256}x^{156}y^{80} - \frac{12285}{128}x^{156}y^{79} - \frac{455}{256}x^{156}y^{78} + \frac{711}{64}x^{155}y^{81} + \frac{315}{128}x^{155}y^{80} - \frac{945}{32}x^{155}y^{79} - \\
& \frac{315}{64}x^{155}y^{78} + \frac{723723}{64}x^{154}y^{82} - \frac{405405}{256}x^{154}y^{81} - \frac{315315}{64}x^{154}y^{80} - \frac{220275}{128}x^{154}y^{79} + \\
& \frac{22275}{512}x^{154}y^{78} + \frac{495}{256}x^{154}y^{77} + \frac{33915}{64}x^{153}y^{78} + \frac{9}{512}x^{153}y^{77} - \frac{1365}{256}x^{152}y^{82} - \\
& \frac{89595}{512}x^{152}y^{81} - \frac{120285}{256}x^{152}y^{80} + \frac{51975}{128}x^{152}y^{79} + \frac{51975}{128}x^{152}y^{78} + \frac{8631}{256}x^{152}y^{77} - \\
& \frac{63}{64}x^{152}y^{76} + \frac{778635}{32}x^{151}y^{79} - \frac{45045}{128}x^{151}y^{78} - \frac{45045}{128}x^{151}y^{77} - \frac{1365}{64}x^{151}y^{76} + \\
& \frac{105}{256}x^{150}y^{78} - \frac{945}{64}x^{150}y^{77} - \frac{945}{128}x^{150}y^{76} + \frac{251}{1024}x^{150}y^{75} + \frac{65793}{16}x^{149}y^{80} - \frac{225225}{256}x^{149}y^{79} - \\
& \frac{135135}{32}x^{149}y^{78} - \frac{308385}{128}x^{149}y^{77} + \frac{7425}{64}x^{149}y^{76} + \frac{4455}{256}x^{149}y^{75} + \frac{440895}{256}x^{148}y^{76} - \\
& \frac{315}{256}x^{148}y^{75} - \frac{21}{128}x^{148}y^{74} - \frac{8145}{512}x^{147}y^{79} - \frac{13365}{128}x^{147}y^{78} + \frac{22275}{128}x^{147}y^{77} + \\
& \frac{10395}{32}x^{147}y^{76} + \frac{14385}{256}x^{147}y^{75} - \frac{189}{32}x^{147}y^{74} + \frac{778635}{32}x^{146}y^{77} - \frac{135135}{256}x^{146}y^{76} - \\
& \frac{225225}{256}x^{146}y^{75} - \frac{15015}{128}x^{146}y^{74} + \frac{45}{512}x^{146}y^{73} - \frac{189}{64}x^{145}y^{75} - \frac{315}{64}x^{145}y^{74} + \\
& \frac{3147}{1024}x^{145}y^{73} + \frac{65793}{64}x^{144}y^{78} - \frac{45045}{128}x^{144}y^{77} - \frac{675675}{256}x^{144}y^{76} - \frac{308385}{128}x^{144}y^{75} + \\
& \frac{51975}{256}x^{144}y^{74} + \frac{4455}{64}x^{144}y^{73} + \frac{495}{512}x^{144}y^{72} + \frac{264537}{64}x^{143}y^{74} - \frac{1365}{256}x^{143}y^{73} - \\
& \frac{315}{128}x^{143}y^{72} - \frac{2673}{256}x^{142}y^{76} + \frac{22275}{512}x^{142}y^{75} + \frac{10395}{64}x^{142}y^{74} + \frac{14385}{256}x^{142}y^{73} - \\
& \frac{945}{64}x^{142}y^{72} - \frac{189}{256}x^{142}y^{71} + \frac{605605}{32}x^{141}y^{75} - \frac{19305}{32}x^{141}y^{74} - \frac{405405}{256}x^{141}y^{73} - \\
& \frac{25025}{64}x^{141}y^{72} + \frac{495}{512}x^{141}y^{71} - \frac{315}{256}x^{140}y^{72} + \frac{2937}{128}x^{140}y^{71} + \frac{99}{512}x^{140}y^{70} + \frac{5061}{32}x^{139}y^{76} - \\
& \frac{12285}{128}x^{139}y^{75} - \frac{75075}{64}x^{139}y^{74} - \frac{220275}{128}x^{139}y^{73} + \frac{31185}{128}x^{139}y^{72} + \frac{10395}{64}x^{139}y^{71} + \\
& \frac{495}{64}x^{139}y^{70} + \frac{969969}{128}x^{138}y^{72} - \frac{4095}{256}x^{138}y^{71} - \frac{4095}{256}x^{138}y^{70} - \frac{105}{256}x^{138}y^{69} +
\end{aligned}$$

$$\begin{aligned}
& \frac{2475}{512}x^{137}y^{73} + \frac{1485}{32}x^{137}y^{72} + \frac{8631}{256}x^{137}y^{71} - \frac{315}{16}x^{137}y^{70} - \frac{945}{256}x^{137}y^{69} + \frac{363363}{32}x^{136}y^{73} - \\
& \frac{135135}{256}x^{136}y^{72} - \frac{135135}{64}x^{136}y^{71} - \frac{225225}{256}x^{136}y^{70} + \frac{2475}{512}x^{136}y^{69} + \frac{99}{256}x^{136}y^{68} + \\
& \frac{14545}{128}x^{135}y^{69} + \frac{99}{256}x^{135}y^{68} + \frac{723}{64}x^{134}y^{74} - \frac{4095}{256}x^{134}y^{73} - \frac{45045}{128}x^{134}y^{72} - \\
& \frac{220275}{256}x^{134}y^{71} + \frac{51975}{256}x^{134}y^{70} + \frac{31185}{128}x^{134}y^{69} + \frac{3465}{128}x^{134}y^{68} - \frac{9}{64}x^{134}y^{67} + \\
& \frac{692835}{64}x^{133}y^{70} - \frac{9009}{256}x^{133}y^{69} - \frac{4095}{64}x^{133}y^{68} - \frac{1365}{256}x^{133}y^{67} + \frac{1485}{256}x^{132}y^{70} + \\
& \frac{2877}{256}x^{132}y^{69} - \frac{945}{64}x^{132}y^{68} - \frac{945}{128}x^{132}y^{67} - \frac{56441}{1024}x^{132}y^{66} + \frac{165165}{32}x^{131}y^{71} - \\
& \frac{45045}{128}x^{131}y^{70} - \frac{135135}{64}x^{131}y^{69} - \frac{45045}{32}x^{131}y^{68} + \frac{7425}{512}x^{131}y^{67} + \frac{495}{128}x^{131}y^{66} + \\
& \frac{101745}{256}x^{130}y^{67} + \frac{99}{512}x^{130}y^{66} + \frac{3}{128}x^{130}y^{65} - \frac{315}{256}x^{129}y^{71} - \frac{4095}{64}x^{129}y^{70} - \frac{73425}{256}x^{129}y^{69} + \\
& \frac{7425}{64}x^{129}y^{68} + \frac{31185}{128}x^{129}y^{67} + \frac{3465}{64}x^{129}y^{66} - \frac{63}{64}x^{129}y^{65} + \frac{6235515}{512}x^{128}y^{68} - \\
& \frac{15015}{256}x^{128}y^{67} - \frac{45045}{256}x^{128}y^{66} - \frac{4095}{128}x^{128}y^{65} + \frac{411}{256}x^{127}y^{67} - \frac{189}{32}x^{127}y^{66} - \\
& \frac{945}{128}x^{127}y^{65} - \frac{109}{512}x^{127}y^{64} + \frac{55055}{32}x^{126}y^{69} - \frac{45045}{256}x^{126}y^{68} - \frac{405405}{256}x^{126}y^{67} - \\
& \frac{105105}{64}x^{126}y^{66} + \frac{7425}{256}x^{126}y^{65} + \frac{4455}{256}x^{126}y^{64} + \frac{165}{512}x^{126}y^{63} + \frac{264537}{256}x^{125}y^{65} - \\
& \frac{9}{64}x^{125}y^{63} - \frac{1365}{256}x^{124}y^{68} - \frac{14685}{256}x^{124}y^{67} + \frac{22275}{512}x^{124}y^{66} + \frac{10395}{64}x^{124}y^{65} + \\
& \frac{17325}{256}x^{124}y^{64} - \frac{189}{64}x^{124}y^{63} - \frac{63}{256}x^{124}y^{62} + \frac{692835}{64}x^{123}y^{66} - \frac{19305}{256}x^{123}y^{65} - \\
& \frac{45045}{128}x^{123}y^{64} - \frac{15015}{128}x^{123}y^{63} - \frac{63}{64}x^{122}y^{64} - \frac{945}{256}x^{122}y^{63} + \frac{2067}{1024}x^{122}y^{62} + \frac{33}{512}x^{122}y^{61} + \\
& \frac{12705}{32}x^{121}y^{67} - \frac{4095}{64}x^{121}y^{66} - \frac{225225}{256}x^{121}y^{65} - \frac{45045}{32}x^{121}y^{64} + \frac{10395}{256}x^{121}y^{63} + \\
& \frac{1485}{32}x^{121}y^{62} + \frac{1485}{512}x^{121}y^{61} + \frac{264537}{128}x^{120}y^{63} - \frac{315}{256}x^{120}y^{61} - \frac{59}{1024}x^{120}y^{60} - \\
& \frac{1335}{256}x^{119}y^{65} + \frac{2475}{256}x^{119}y^{64} + \frac{4455}{64}x^{119}y^{63} + \frac{3465}{64}x^{119}y^{62} - \frac{315}{64}x^{119}y^{61} - \frac{189}{128}x^{119}y^{60} + \\
& \frac{969969}{128}x^{118}y^{64} - \frac{19305}{256}x^{118}y^{63} - \frac{135135}{256}x^{118}y^{62} - \frac{75075}{256}x^{118}y^{61} + \frac{9}{256}x^{118}y^{59} - \\
& \frac{189}{256}x^{117}y^{61} + \frac{467}{32}x^{117}y^{60} + \frac{99}{512}x^{117}y^{59} + \frac{1815}{32}x^{116}y^{65} - \frac{4095}{256}x^{116}y^{64} - \frac{45045}{128}x^{116}y^{63} - \\
& \frac{225225}{256}x^{116}y^{62} + \frac{10395}{256}x^{116}y^{61} + \frac{10395}{128}x^{116}y^{60} + \frac{1485}{128}x^{116}y^{59} + \frac{415701}{128}x^{115}y^{61} - \\
& \frac{1365}{256}x^{115}y^{59} - \frac{105}{128}x^{115}y^{58} + \frac{495}{512}x^{114}y^{62} + \frac{4455}{256}x^{114}y^{61} + \frac{3465}{128}x^{114}y^{60} - \frac{315}{64}x^{114}y^{59} - \\
& \frac{945}{256}x^{114}y^{58} - \frac{83}{1024}x^{114}y^{57} + \frac{264537}{64}x^{113}y^{62} - \frac{15015}{256}x^{113}y^{61} - \frac{19305}{32}x^{113}y^{60} - \\
& \frac{135135}{256}x^{113}y^{59} + \frac{99}{256}x^{113}y^{57} + \frac{3625}{64}x^{112}y^{58} + \frac{99}{512}x^{112}y^{57} + \frac{9}{256}x^{112}y^{56} + \frac{121}{32}x^{111}y^{63} - \\
& \frac{315}{128}x^{111}y^{62} - \frac{12285}{128}x^{111}y^{61} - \frac{25025}{64}x^{111}y^{60} + \frac{7425}{256}x^{111}y^{59} + \frac{6237}{64}x^{111}y^{58} + \\
& \frac{3465}{128}x^{111}y^{57} + \frac{2078505}{512}x^{110}y^{59} - \frac{4095}{256}x^{110}y^{57} - \frac{1365}{256}x^{110}y^{56} + \frac{495}{256}x^{109}y^{59} +
\end{aligned}$$

$$\begin{aligned}
& \frac{495}{64}x^{109}y^{58} - \frac{189}{64}x^{109}y^{57} - \frac{315}{64}x^{109}y^{56} - \frac{401}{1024}x^{109}y^{55} + \frac{440895}{256}x^{108}y^{60} - \frac{9009}{256}x^{108}y^{59} - \\
& \frac{135135}{256}x^{108}y^{58} - \frac{45045}{64}x^{108}y^{57} + \frac{495}{256}x^{108}y^{55} + \frac{33}{512}x^{108}y^{54} + \frac{20349}{128}x^{107}y^{56} + \\
& \frac{33}{512}x^{107}y^{55} + \frac{9}{128}x^{107}y^{54} - \frac{45}{256}x^{106}y^{60} - \frac{4095}{256}x^{106}y^{59} - \frac{15015}{128}x^{106}y^{58} + \frac{7425}{512}x^{106}y^{57} + \\
& \frac{10395}{128}x^{106}y^{56} + \frac{10395}{256}x^{106}y^{55} - \frac{9}{256}x^{106}y^{53} + \frac{2078505}{512}x^{105}y^{57} - \frac{9009}{256}x^{105}y^{55} - \\
& \frac{1365}{64}x^{105}y^{54} + \frac{495}{512}x^{104}y^{56} - \frac{63}{64}x^{104}y^{55} - \frac{945}{256}x^{104}y^{54} - \frac{669}{1024}x^{104}y^{53} - \frac{33}{2}x^{104}y^{52} + \\
& \frac{33915}{64}x^{103}y^{58} - \frac{4095}{256}x^{103}y^{57} - \frac{45045}{128}x^{103}y^{56} - \frac{45045}{64}x^{103}y^{55} + \frac{1485}{256}x^{103}y^{53} + \\
& \frac{165}{256}x^{103}y^{52} + \frac{88179}{256}x^{102}y^{54} + \frac{9}{256}x^{102}y^{52} - \frac{1}{1024}x^{102}y^{51} - \frac{315}{256}x^{101}y^{57} - \frac{1365}{64}x^{101}y^{56} + \\
& \frac{2475}{512}x^{101}y^{55} + \frac{1485}{32}x^{101}y^{54} + \frac{10395}{256}x^{101}y^{53} - \frac{63}{256}x^{101}y^{51} + \frac{415701}{128}x^{100}y^{55} - \\
& \frac{15015}{256}x^{100}y^{53} - \frac{15015}{256}x^{100}y^{52} - \frac{9}{64}x^{99}y^{53} - 189/128x^{99}y^{52} + \frac{129}{1024}x^{99}y^{51} + \\
& \frac{14535}{128}x^{98}y^{56} - \frac{1365}{256}x^{98}y^{55} - \frac{45045}{256}x^{98}y^{54} - \frac{135135}{256}x^{98}y^{53} + \frac{1485}{128}x^{98}y^{51} + \frac{1485}{512}x^{98}y^{50} + \\
& \frac{37791}{64}x^{97}y^{52} - \frac{57}{1024}x^{97}y^{49} - \frac{455}{256}x^{96}y^{54} + \frac{495}{512}x^{96}y^{53} + \frac{4455}{256}x^{96}y^{52} + \frac{3465}{128}x^{96}y^{51} - \\
& \frac{189}{256}x^{96}y^{49} - \frac{7}{256}x^{96}y^{48} + \frac{264537}{128}x^{95}y^{53} - \frac{19305}{256}x^{95}y^{51} - \frac{15015}{128}x^{95}y^{50} - \frac{63}{256}x^{94}y^{50} + \\
& \frac{27}{8}x^{94}y^{49} + \frac{3}{256}x^{94}y^{47} + \frac{969}{64}x^{93}y^{54} - \frac{315}{256}x^{93}y^{53} - \frac{4095}{64}x^{93}y^{52} - \frac{75075}{256}x^{93}y^{51} + \\
& \frac{2079}{128}x^{93}y^{49} + \frac{495}{64}x^{93}y^{48} + \frac{415701}{512}x^{92}y^{50} - \frac{105}{256}x^{92}y^{47} + \frac{45}{512}x^{91}y^{51} + \frac{495}{128}x^{91}y^{50} + \\
& \frac{1485}{128}x^{91}y^{49} - \frac{315}{256}x^{91}y^{47} - \frac{21}{128}x^{91}y^{46} + \frac{264537}{256}x^{90}y^{51} - \frac{19305}{256}x^{90}y^{49} - \frac{45045}{256}x^{90}y^{48} + \\
& \frac{3}{512}x^{90}y^{45} + \frac{1443}{128}x^{89}y^{47} + \frac{9}{256}x^{89}y^{45} + \frac{969}{1024}x^{88}y^{52} - \frac{45}{256}x^{88}y^{51} - \frac{4095}{256}x^{88}y^{50} - \\
& \frac{15015}{128}x^{88}y^{49} + \frac{2079}{128}x^{88}y^{47} + \frac{3465}{256}x^{88}y^{46} + \frac{230945}{256}x^{87}y^{48} - \frac{455}{256}x^{87}y^{45} + \frac{99}{256}x^{86}y^{48} + \\
& \frac{1485}{512}x^{86}y^{47} - \frac{315}{256}x^{86}y^{45} - \frac{105}{256}x^{86}y^{44} + \frac{101745}{256}x^{85}y^{49} - \frac{15015}{256}x^{85}y^{47} - \frac{6435}{32}x^{85}y^{46} + \\
& \frac{33}{512}x^{85}y^{43} + \frac{6783}{256}x^{84}y^{45} + \frac{9}{256}x^{84}y^{43} + \frac{3}{1024}x^{84}y^{42} - \frac{3}{256}x^{83}y^{49} - \frac{315}{128}x^{83}y^{48} - \\
& \frac{4095}{128}x^{83}y^{47} + \frac{1485}{128}x^{83}y^{45} + \frac{2079}{128}x^{83}y^{44} + \frac{415701}{512}x^{82}y^{46} - \frac{1365}{256}x^{82}y^{43} + \frac{165}{512}x^{81}y^{45} - \\
& \frac{189}{256}x^{81}y^{43} - \frac{35}{64}x^{81}y^{42} + \frac{14535}{128}x^{80}y^{47} - \frac{9009}{256}x^{80}y^{45} - \frac{45045}{256}x^{80}y^{44} + \frac{165}{512}x^{80}y^{41} + \\
& \frac{12597}{256}x^{79}y^{43} + \frac{3}{256}x^{79}y^{41} + \frac{3}{512}x^{79}y^{40} - \frac{45}{256}x^{78}y^{46} - \frac{1365}{256}x^{78}y^{45} + \frac{1485}{256}x^{78}y^{43} + \\
& \frac{3465}{256}x^{78}y^{42} - \frac{1}{256}x^{78}y^{39} + \frac{37791}{64}x^{77}y^{44} - \frac{3003}{256}x^{77}y^{41} - \frac{63}{256}x^{76}y^{41} - \frac{105}{256}x^{76}y^{40} - \\
& 3x^{76}y^{38} + \frac{2907}{128}x^{75}y^{45} - \frac{4095}{256}x^{75}y^{43} - \frac{15015}{128}x^{75}y^{42} + \frac{495}{512}x^{75}y^{39} + \frac{37791}{512}x^{74}y^{41} + \\
& \frac{3}{1024}x^{74}y^{38} - \frac{105}{256}x^{73}y^{43} + \frac{495}{256}x^{73}y^{41} + \frac{495}{64}x^{73}y^{40} - \frac{7}{256}x^{73}y^{37} + \frac{88179}{256}x^{72}y^{42} - \\
& \frac{5005}{256}x^{72}y^{39} - \frac{9}{256}x^{71}y^{39} - \frac{21}{128}x^{71}y^{38} + \frac{2907}{1024}x^{70}y^{43} - \frac{1365}{256}x^{70}y^{41} - \frac{15015}{256}x^{70}y^{40} +
\end{aligned}$$

$$\begin{aligned}
& \frac{495}{256}x^{70}y^{37} + \frac{46189}{512}x^{69}y^{39} + \frac{99}{256}x^{68}y^{39} + \frac{1485}{512}x^{68}y^{38} - \frac{21}{256}x^{68}y^{35} + \frac{20349}{128}x^{67}y^{40} - \\
& \frac{6435}{256}x^{67}y^{37} - \frac{7}{256}x^{66}y^{36} + \frac{1}{1024}x^{66}y^{33} + \frac{171}{1024}x^{65}y^{41} - \frac{315}{256}x^{65}y^{39} - \frac{1365}{64}x^{65}y^{38} + \\
& \frac{693}{256}x^{65}y^{35} + \frac{46189}{512}x^{64}y^{37} + \frac{9}{256}x^{63}y^{37} + \frac{165}{256}x^{63}y^{36} - \frac{35}{256}x^{63}y^{33} + \frac{14535}{256}x^{62}y^{38} - \\
& \frac{6435}{256}x^{62}y^{35} + \frac{3}{1024}x^{61}y^{31} - \frac{45}{256}x^{60}y^{37} - \frac{1365}{256}x^{60}y^{36} + \frac{693}{256}x^{60}y^{33} + \frac{37791}{512}x^{59}y^{35} + \\
& \frac{33}{512}x^{58}y^{34} - \frac{35}{256}x^{58}y^{31} + \frac{969}{64}x^{57}y^{36} - \frac{5005}{256}x^{57}y^{33} + \frac{3}{1024}x^{56}y^{29} - \frac{3}{256}x^{55}y^{35} - \\
& \frac{105}{128}x^{55}y^{34} + \frac{495}{256}x^{55}y^{31} + \frac{12597}{256}x^{54}y^{33} - \frac{21}{256}x^{53}y^{29} + \frac{2907}{1024}x^{52}y^{34} - \frac{3003}{256}x^{52}y^{31} + \\
& \frac{1}{1024}x^{51}y^{27} - \frac{15}{256}x^{50}y^{32} + \frac{495}{512}x^{50}y^{29} + \frac{6783}{256}x^{49}y^{31} - \frac{7}{256}x^{48}y^{27} - \frac{1}{4}x^{48}y^{24} + \\
& \frac{171}{512}x^{47}y^{32} - \frac{1365}{256}x^{47}y^{29} + \frac{165}{512}x^{45}y^{27} + \frac{2907}{256}x^{44}y^{29} - \frac{1}{256}x^{43}y^{25} + \frac{19}{1024}x^{42}y^{30} - \\
& \frac{455}{256}x^{42}y^{27} + \frac{33}{512}x^{40}y^{25} + \frac{969}{256}x^{39}y^{27} - \frac{105}{256}x^{37}y^{25} + \frac{3}{512}x^{35}y^{23} + \frac{969}{1024}x^{34}y^{25} - \\
& \frac{15}{256}x^{32}y^{23} + \frac{171}{1024}x^{29}y^{23} - \frac{1}{256}x^{27}y^{21} + \frac{19}{1024}x^{24}y^{21} + \frac{1}{1024}x^{19}y^{19}.
\end{aligned}$$

DECLARATION

I, the undersigned, declare that the thesis is my original work, has not been presented for degrees in any other university and all sources of material used for the thesis have been duly acknowledged.

Name: Dawit Solomon Tadesse

Signature: _____

Place: College of Natural Sciences, Addis Ababa University

Date: March 13, 2019

This thesis has been submitted for examination with my approval as a university advisor.

Mohammed S. Tesemma (Advisor)

Brhanu Bekele (Co-Advisor)