New Existence Results on Group Divisible Designs and 3-GDDs

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DECLARATION

I, Zebene Girma Tefera with student ID number GSR/6462/10, hereby declare that this thesis is my own original work. The information derived from the literature has been duly acknowledged in the text and a list of references is provided. No part of this dissertation has been previously presented for assessment or completion of any degree or qualification at this or any other institution.

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Approval

This thesis has been conducted under our supervision and has granted our approval for submission.

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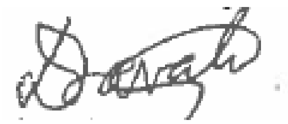

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This certifies that Zebene Girma Tefera's thesis, ***New Existence Results on Group Divisible Designs and 3-GDDs***, which was submitted to fulfill the requirements for the Doctor of Philosophy (in Mathematics) degree, complies with university regulations and is original and of a quality that is considered acceptable.

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List of symbols

Symbol	**Meaning**
$(\mathbb{V}, \mathbb{B})$	A combinatorial block design consisting of a set $\mathbb{V}$ and a collection $\mathbb{B}$ of subsets of $\mathbb{V}$
BIBD	Balanced Incomplete Block Design
BIBD(v, k, λ)	A BIBD on v points, with blocks each of size k where each pair of distinct elements occurs in λ blocks
PBDs	Pairwise Balanced Designs
GDD	Group Divisible Design
GDD$(n,\, m,\, k,\, \lambda_1,\, \lambda_2)$	A GDD with b blocks of size k on m groups of size n each where each pair of elements from the same group occurs in λ_1 blocks and each pair of elements from different groups occurs in λ_2 blocks.
MOLS	Mutually Orthogonal Latin Square
PBIBD	Partially Balanced Incomplete Block Design
STS	Steiner Triple System
SQS	Steiner Quadruple System
t-GDD	t-Group Divisible Design
3-GDD	3-Group Divisible Design
3-PBIBD	3-Partially Balanced Incomplete Block Design

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Abstract

Combinatorial design theory offers a framework for efficiently and systematically arranging objects with given criterion and studying these arrangement (combinatorial designs) and their applications is various fields. For example, combinatorial designs are used in statistical design theory, computer science, cryptography and coding theory. Two specific combinatorial designs are group divisible designs (GDDs) and t-GDDs.

The aim of our study is to find new existence results on GDDs and t-GDDs when $t = 3$. In particular, we gave some new constructions and necessary conditions for GDDs with two groups and block size four. An extension of the concepts of group divisible designs is called t-GDD. It is obtained by combining the concepts of group divisible designs and t-design. So for example, a 3-GDD(n, 2, k, λ_1, λ_2) is defined by combining the concepts of GDDs and a 3-design. 3-GDDs with 2 groups and block size 4 have been studied in the literature. We obtain new results on 3-GDDs with three groups and block size five, as well as with four groups and block size five.

In this work, certain required necessary conditions for these GDDs to exist are designed, several new constructions are given, and specific instances of non-existence are proved.

Chapter 1

Introduction

The question of whether or not it is possible to find a collection of subsets of a finite set that meet certain "balance" attributes is the focus of the area of combinatorial design theory [45]. Deciding on the existence of such collection is the fundamental question. The requirement for balance distinguishes the designs examined here from random collections of subsets. Both existence and non-existence findings are important in modern design theory. There are still a lot of unanswered questions regarding the existence of particular design types. For example, Steiner systems, projective geometries, affine geometries, graph designs, orthogonal Latin squares, orthogonal arrays, group divisible designs and t-designs.

The development of combinatorial design theory has strong ties to basic and higher mathematics. The study of combinatorial designs did not establish itself as a distinct academic discipline until the twentieth century, despite the fact that some of its most well-known accomplishments date back to the work of many scholars in the eighteenth and nineteenth centuries. In the 1920s, Fisher and his associates created the mathematical framework for experimental design, which gave rise to the discipline of combinatorial design theory. But Bose achieved a lot more [2–4]. Bose and his colleagues set the groundwork beginning in the 1930s, solidifying the field's emergence as a mathematical field of study by forging strong connections with group theory, finite geometry, number theory, and finite fields. The eighteenth and nineteenth centuries saw the development of many design types that are examined today in the context of mathematical puzzles and brain teasers. Because of its applications in the planning and evaluation of statistical studies, design theory as a mathematical field flourished in the twentieth century. Applications for combinatorial designs are numerous: they are employed in coding theory to create error-correcting codes, which are necessary for dependable data transfer; they are involved in the creation of safe encryption techniques in cryptography; designs aid in maximizing the efficiency of algorithms and data structures in computer science; and they support survey and experiment design in statistical designs [12, 19, 42, 48]. Furthermore, there are plenty of additional applications for designs, including networking, mathematical biology, lotteries, scheduling tournaments, and group testing [8, 22, 41].

A group divisible design, GDD(n, m, k, λ_1, λ_2), is an ordered triple (X, $\mathbb{G}$, $\mathbb{B}$) where X is a mn-set of symbols, $\mathbb{G}$ is a partition of X into m sets called groups of size n each, and $\mathbb{B}$ is a collection of k-subsets called blocks of X, such that each pair of symbols from the same group occurs in exactly λ_1 blocks and each pair of symbols from different groups occurs in exactly λ_2 blocks [1,5,9,24,28,30,31,34,35,38,39,45]. Any two symbols occurring together in the same group are called first associates, and pairs of symbols occurring in different groups are called second associates. It can be necessary to have different group sizes and in the occurrence of a specific member or pair of separate elements.
Since they can be used to create new designs (such as balanced incomplete designs, orthogonal arrays, transversal designs, etc.), group divisible designs (GDDs) have been investigated for their importance in coding and statistics [27, 43, 44]. Since the work of Bose and Shimamoto in 1952, who started categorizing such designs [4], the existence of such GDDs has been extensively studied. In order to answer some existence problems related to group divisible designs, a variety of mathematical studies have been conducted. The indices λ_1 and λ_2, as well as variations in the number of groups and block sizes, have been the main subjects of these studies. Zhu and Ge [49] addressed mixed GDDs with block size 4 and 3 groups; Hurd and Sarvate, for example, studied problems that involve 2 groups and block size 4 [20], 3 groups and block size 4, odd and even group divisible designs [21].

Given a set X consisting of v entries and positive integers k, of v elements, and given positive integers k, t ($t \leq k \leq v$), and λ, a requirement for the existence of a t-design, a broader term primarily represented by a $t-(v, k, \lambda)$ design, is known [14, 15, 17] as

$$\lambda.\frac{\binom{v-i}{t-i}}{\binom{k-i}{t-i}} = integer, i = 0, 1, 2, 3, ..., t-1. \tag{1.1}$$

Finding the values of k, t, and X for which this requirement is satisfied gives us the necessary conditions. Then the main problem is to prove that the design exists for those values.

Balanced Incomplete Block Designs (BIBD) are configurations where t = 2. It has been proved by Reiss [32] and Moore [26] that for BIBD with $k = 3$, $\lambda = 1$ (known also as the Steiner triple systems) condition (1.1) is sufficient. But less is known about configurations with $t > 2$.

GDDs and t-designs have been studied by many authors [6, 7, 15, 16], and the references therein. However, little is known about t-GDDs.

A 3-GDD was recently developed by combining the concepts of a t-design and a group divisible design, and some initial results were obtained [33, 36, 37]. The authors proved in [37] that the necessary conditions are sufficient for a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$ to exist, with the possible exception of $n \equiv 1, 3 \pmod 6), n \neq 3, 7, 13$ and $\lambda_1 > \lambda_2$. For $n \equiv 1, 7, 9 \pmod{12}$, the authors in [33] settled that the necessary conditions are sufficient for the existence of a 3-GDD(n, 2, 4, λ_1, λ_2).

In this dissertation, we have studied 3-GDDs with 4 groups and block size 5, 3-GDDs with 3 groups and block size 5, and Group Divisible designs with 2 groups and block size 4.

This thesis constitutes six chapters including Chapter 1, and the next 5 chapters are as follows.

In Chapter 2, we review various types of designs that are important to the objectives of our study by giving definitions and preliminary concepts. The designs introduced are Balanced Incomplete Block Designs (BIBDs), Pairwise Balanced Designs (PBDs), t-designs, group divisible designs (GDDs), and 3-GDDs. Furthermore, several combinatorial techniques for constructing designs, including difference sets, Latin squares, factorizations of graphs, and block complementation are introduced.

Chapter 3 is devoted to the study of 3-GDDs, which is defined by combining t-designs when t=3 and Group Divisible designs. It mainly considered the case where the design consists of three groups, each of size n, and a block size of five. We denote such GDD by 3-GDD(n, 3, 5, μ_1, μ_2). By including more than two groups, this work expands previous work [33, 37] that used block sizes of four and two groups. In this chapter, certain necessary conditions and their proofs for the existence of such designs are given, the proofs of some constructions especially when $\mu_2 = 0$ are presented and an infinite families of existences for the 3-GDD when $n = 3$ are given.

Chapter 4 continued to focus on the study of 3-GDDs. We explicitly considered the case when the required designs have four groups of size n each and block size five. Such a GDD is denoted by 3-GDD(n, 4, 5, μ_1, μ_2). This chapter establishes certain necessary conditions for the existence of such GDDs and provides many non-existence results.

In Chapter 5, we study group divisible designs with two groups of the same size and blocks of size four. Such GDD is denoted by GDD(n, 2, 4, λ_1, λ_2). This chapter provides additional necessary conditions for the existence of such designs, new constructions using some combinatorial tools, and indicate directions for future work.

Finally, in Chapter 6, we summarise the work we have done and outline some open problems to provide a direction for the future work.

1.0.1 Methods

Given a combinatorial design (hereafter referred to as a design) with essential parameters, we worked out what relationships and conditions those parameters had to have. We also decided for which values of the parameters the design could exist. To put it briefly, we started by obtaining the necessary conditions for the design's existence. Next, we proved the existence or non-existence of a design for those parameters for which the requirements are met. In other words, we demonstrated whether or not the necessary conditions are sufficient for the existence of the design.

Direct constructions and recursive constructions are the two main primary approaches used to construct designs [13]. Building a design with certain criteria is aided by the use of a direct method. Usually, it makes use of difference sets, congruences, or finite fields. Recursive design construction involves building larger designs from smaller ones. This dissertation uses both approaches.

The following papers, which are organized as chapters in this thesis, were written.

1. 3-Group Divisible Designs With Three Groups and Block Size Five
   Zebene Girma Tefera, Dinesh G. Sarvate, and Samuel Asefa Fufa
   Journal of Mathematics (Volume 2023), Article 1010091.

2. 3-GDDs With Four Groups and Block Size Five
   Zebene Girma Tefera and Samuel Asefa Fufa
   SINET: Ethiopian Journal of Science (Volume 46(2), 2023), Article.

3. Group Divisible Designs With Two Groups and Block Size Four
   Zebene Girma Tefera and Dinesh G. Sarvate
   Working Manuscript

Chapter 2

Definitions and Preliminaries

Most of the definitions, notation and preliminaries are from the well known text books, for example Stinson [45], Lindner and Rodger [24], R. Mathon and A. Rosa [25] and the Hand Book of Design Theory. Other results are from different research articles, for instance H. Hanani [14–17], D. G. Sarvate, W. Cowden, B. Wilbroad, et al [11, 18, 33, 37].

Definition 2.0.1. A pair (X, $\mathbb{B}$) where X is a set of objects known as points, and $\mathbb{B}$ is a collection (i.e., multiset) of nonempty subsets of X called blocks is called a combinatorial design, or just a design.

- A design is called simple if the design does not have repeated blocks.
- Recall $\mathbb{B}$ is a multiset of blocks not a set of blocks. Therefore a design can have repeating blocks.
- Sometimes the notation [ ] is used to list the elements of a multiset along with their multiplicities.
- A multiset is a set if and only if every element in it has multiplicity one. For instance, [a, b, c] = {a, b, c}, but $[a, b, e, b] \neq \{a, b, e, b\} = \{a, b, e\}$.

2.1 Balanced incomplete block designs

Definition 2.1.1. Given three natural numbers, v, k and λ, such that $v > k \geq 2$, a balanced incomplete block design, which is abbreviated as (v, k, λ)-BIBD, is a design (X, $\mathbb{B}$) that satisfies the following conditions:

1. $|X| = v$
2. Every block has exactly k points in it, and
3. Every pair of different points occurs in exactly λ blocks.
   The above condition 3 is called the "balance" property. Due to the fact that $k<v$, every block in a BIBD is a proper subset, consequently it is referred to as an incomplete block design. If $\lambda>1$, then repeated blocks might be contained in a BIBD.

Example 2.1.2. The blocks of a $(7, 3, 1)$-BIBD are
$\mathbb{B} = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 7\}, \{3, 5, 6\}\}$, where $X = \{1, 2, 3, 4, 5, 6, 7\}$.
Instead of writing blocks as $\{1, 2, 3\}$, we may write them as 123 to save space.

Example 2.1.3. The blocks of a $(10, 4, 2)$-BIBD are:
$\mathbb{B} = \{0123, 0145, 0246, 0378, 0579, 0689, 1278, 1369, 1479, 1568, 2359, 2489, 2567, 3458, 3467\}$
where $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Theorem 2.1.4. *Every point in a (v, k, λ)-BIBD occurs in precisely $r = \frac{\lambda(v-1)}{k-1}$ blocks.*

Proof. Let $(X, \mathbb{B})$ be a (v, k, λ)-BIBD. Assume $x \in X$, and let r_x be the number of blocks that contain x. Describe a collection I as
$I = \{(y, A) : y \in X, \, y \neq x, \, A \in \mathbb{B}, \, \{x, y\} \subseteq A\}$.
We will use two different approaches to compute $|I|$.
Initially, selecting $y \in X$ such that $y \neq x$ can be done in $v - 1$ ways. There are λ blocks A such that $\{x, y\} \subseteq A$ for every such y. Accordingly,

$$|I| = \lambda(v - 1) \tag{2.1}$$

Alternatively, there exist r_x means to select a block A so that x is a member of A. There are $k - 1$ ways to select $y \in A$, where $y \neq x$, for any choice A. Thus,

$$|I| = r_x(k - 1) \tag{2.2}$$

When Equations 2.1 and 2.2 are combined, we obtain $\lambda(v - 1) = r_x(k - 1)$.
Since $r_x = \frac{\lambda(v-1)}{k-1}$ is independent of x, the conclusion can be drawn. □

For the BIBD, r is known as the replication number.

Theorem 2.1.5. *A (v, k, λ)-BIBD has exactly.* $b = \frac{vr}{k} = \frac{\lambda(v^2-v)}{k^2-k}$ *blocks.*

Proof. Suppose $(X, \mathbb{B})$ is a (v, k, λ)-BIBD and $b = |\mathbb{B}|$. Let set I is given by
$I = \{(x, A)\colon\ x \in X,\ A \in \mathbb{B},\ x \in A\ \}$.
There are two distinct methods to calculate $|I|$.
First, choosing $x \in X$ can be done in v ways. There exist r blocks A with $x \in A$ for every such x. For this reason,

$$|I| = vr \tag{2.3}$$

$|I| = vr$.
Second, selecting a block $A \in \mathbb{B}$ can be done in b various ways. There exist k methods to select $x \in A$ for every choice of A. Thus,

$$|I| = bk \tag{2.4}$$

From Equations 2.3 and 2.4, we conclude that

$$bk = vr \tag{2.5}$$

□

Since b and r must both be integers, Theorem 2.1.4 and Theorem 2.1.5 lead us to the following corollary.

Corollary 2.1.6. $\lambda(v-1) \equiv 0 \pmod{k-1}$ *and* $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$ *if a-(v, k, λ)-BIBD exists.*

The above corollary helps us deduce when there exist no BIBDs with a given specific parameter sets.

Example 2.1.7. Given a parameter set $v = 8$, $k = 3$, and $\lambda = 1$, we deduce that there is no $(8, 3, 1)$-BIBD, since $\lambda(v-1) = 7 \not\equiv 0 \pmod 2$,

Example 2.1.8. Similarly given a parameter set $v = 19$, $k = 4$, and $\lambda = 1$ we deduce that there is no $(19, 4, 1)$-BIBD, since $\lambda v(v-1) = 342 \not\equiv 0 \pmod{12}$.

Occasionally, the notation (v, b, r, k, λ)-BIBD is used when we need to record the values of all five parameters.

2.2 Steiner Triple Systems(STS)

A $(v, 3, 1)$-BIBD is a Steiner triple system of order v, or STS(v). The most basic kind of "interesting" BIBDs are Steiner triple systems, since BIBDs with $k = 2$ are trivial to construct.

Example 2.2.1. Following are the triples of a STS(7), i.e., the blocks of a $(7, 3, 1)$-BIBD written vertically:

$$\begin{array}{ccccccc} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 4 & 6 & 4 & 5 & 4 & 5 \\ 3 & 5 & 7 & 6 & 7 & 7 & 6 \end{array}$$

where $X = \{1, 2, 3, 4, 5, 6, 7\}$.
Figure 2.1 shows an excellent diagrammatic illustration of this BIBD. The circle and the six lines in this diagram represent the BIBD's blocks.

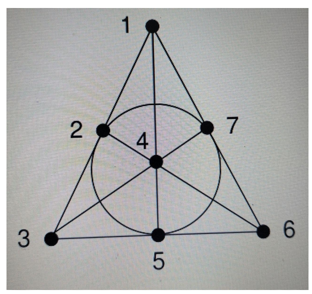

Figure 2.1: The Fano Plane: A (7, 3, 1)-BIBD

Example 2.2.2. If $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, the blocks of a $(9, 3, 1)$-BIBD are:

$$\begin{array}{cccccccccccc} 1 & 4 & 7 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\ 2 & 5 & 8 & 4 & 5 & 6 & 5 & 6 & 4 & 6 & 4 & 5 \\ 3 & 6 & 9 & 7 & 8 & 9 & 9 & 7 & 8 & 8 & 9 & 7 \end{array}$$

The next Lemma discusses the necessary and sufficient conditions for existence of an STS(v).

Lemma 2.2.3. *A STS(v) exists if and only if* $v \equiv 1, 3 \pmod 6$, $v \geq 7$.

Proof. We obtain $r = \frac{\lambda(v-1)}{k-1} = \frac{v-1}{2}$ since $k = 3$ and $\lambda = 1$.
Consequently, v is odd since $v = 2r + 1$. It is now possible to calculate $b = \frac{vr}{k} = \frac{v(v-1)}{6}$. It follows that $v(v-1) \equiv 0 \pmod 6$ since b is an integer. Only when $v \equiv 0, 1, 3, 4 \pmod 6$ is this congruence satisfied. On the other hand, $v \equiv 1, 3 \pmod 6$, since v is odd. □

2.3 Incidence Matrices

An incidence matrix is frequently a convenient way for representing a BIBD. Particularly for computer programs and certain applications Incidence matrices are necessary

Definition 2.3.1. Consider a design (X=$\{a_1, a_2..., a_v\}$, $\mathbb{B} = \{B_1, B_2, ..., B_b\}$). The $v \times b$ $0-1$ matrix denoted by $M = (m_{s,t})_{v\times b}$ is known as the incidence matrix of a (X, $\mathbb{B}$) design, given by

$$m_{s,t} = \begin{cases} 1, & \text{when } a_s \in B_t \\ 0, & \text{when } a_s \notin B_t \end{cases} \tag{2.6}$$

Remark 2.3.2. The following characteristics of the incidence matrix, M, of a (v, b, r, k, λ)-BIBD are satisfied:

1. There are exactly k "1"s in each column and r "1"s in each row of M.
2. Two different rows of M both contain "1"s in exactly λ columns.

Example 2.3.3. Consider the $(7, 3, 1)$-BIBD. The blocks of the design (written horizontally) are: $123, 145, 167, 246, 257, 347, 356$. The 7×7 matrix below represents the design's incidence matrix:

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Theorem 2.3.4. *Suppose $2 \leq k < v$ and M is a vxb $0-1$ matrix. M is the incidence matrix of a (v, b, r, k, λ)-BIBD if and only if $MM^T = \lambda J_v + (r-\lambda)I_v$ and $u_vM = ku_b$. where u_n indicates the vector of length n where each coordinate is a "1", J_n indicates the nxn matrix in which each entry is a "1", and I_n represents a nxn identity matrix.*

Proof. To begin, let $(X, \mathbb{B})$ represent a design in which $X = \{x_1, x_2..., x_v\}$ and $\mathbb{B} = \{A_1, A_2, ..., A_b\}$.
From MM^T, the (i, j)-entry is

$$\sum_{h=0}^{b} m_{i,h}m_{j,h} = \begin{cases} \lambda, & \text{if } i \neq j \\ r, & \text{if } i = j. \end{cases} \tag{2.7}$$

According to Remark 2.3.2, each main diagonal entry of MM^T's is r, and each entry on its off the diagonal is λ. As a result, $MM^T = \lambda J_v + (r-\lambda)I_v$. Additionally, the number of 1's in column i of M is equal to the i^{th} entry of u_vM. This equals k according to property 1 in Remark 2.3.2. Thus, $u_vM = ku_b$.
On the other hand, let's say that M is a vxb $0-1$ matrix so that $u_vM = ku_b$ and $MM^T = \lambda J_v + (r-\lambda)I_v$.
Assume that the design with an incidence matrix of M is $(X, \mathbb{B})$. It is evident that $|X| = v$ and $|\mathbb{B}| = b$. It is evident from the equation $u_vM = ku_b$ that each block in $\mathbb{B}$ has k points in it. Every point occurs in r blocks, and every pair of points occurs in exactly λ blocks, according to the equation $MM^T = \lambda J_v + (r-\lambda)I_v$. $(X, \mathbb{B})$ is therefore a (v, b, r, k, λ)-BIBD. □

If the property "There are exactly r 1's in each row of M " in Remark 2.3.2 is removed, the converse of Theorem 2.3.4 fails to hold.

Definition 2.3.5. A design $(X, \mathbb{B})$ is said to be pairwise balanced (abbreviated PBD) if each pair of different points appears in precisely λ blocks, where λ is a natural number. Additionally, if every point $x \in X$ appears in exactly r blocks $A \in \mathbb{B}$, where r is a natural number, then $(X, \mathbb{B})$ is a regular pairwise balanced design.

Blocks of size $|X|$, or entire blocks, are permissible in a PBD $(X, \mathbb{B})$. A $(X, \mathbb{B})$ pairwise balanced design is considered trivial if it only contains complete blocks. A $(X, \mathbb{B})$ design is considered proper pairwise balanced if it does not contain any complete blocks.

Theorem 2.3.6. *Consider a $v \times b$ $0-1$ matrix M. In a regular pairwise balanced design with v entries and b blocks, M is the incidence matrix if and only if there are natural number r and λ ensuring $MM^T = \lambda J_v + (r-\lambda)I_v$.*

For instance, consider the following 6×11 matrix:

$$M = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

M is the incidence matrix of the regular pairwise balanced design $(X, \mathbb{B})$, where X=$\{1,2,3,4,5,6\}$ and $\mathbb{B}$=$\{\{1,2,3\},\{4,5,6\},\{1,4\},\{1,5\},\{1,6\},\{2,4\},\{2,5\},\{2,6\},\{3,4\},\{3,5\},\{3,6\}\}$.
In this case, $\lambda = 1$, $v = 6$, $b = 11$, and $r = 4$. Due to the fact that there are two blocks that are size three and nine blocks that are size two, the design is not a BIBD. The equality $MM^T = \lambda J_v + (r-\lambda)I_v$ can be readily verified. But since $u_6M = (3,3,2,2,2,2,2,2,2,2,2)$, for every integer k, $u_6M \neq ku_b$.

2.4 New BIBDs from Old

We provide two elementary techniques in this section for creating new BIBDs from existing ones. One may refer to the initial construction as the "sum construction." The task is to form the collection of all the blocks in both designs given two BIBDs on an identical point set.

Theorem 2.4.1. *If a $(v,k,\lambda_1)$and (v,k,λ_2)-BIBD's exist, then a $(v,k,\lambda_1+\lambda_2)$-BIBD exists.*

Corollary 2.4.2. *Let s is an integers and $s \geq 1$. If a (v,k,λ)-BIBD exists, then a $(v,k,s\lambda)$-BIBD exists.*

"Block complementation" is the other alternative construction method. Assume that $(X,\mathbb{B})$ is a BIBD and $X \setminus A$ is used in place of each block $A \in \mathbb{B}$. The following theorem states that the outcome is also a BIBD.

Theorem 2.4.3. *(Block Complementation) Assume a (v, b, r, k, λ)-BIBD exists, with $k \leq v - 2$. Then, a $(v, b, b - r, v - k, b - 2r + \lambda)$-BIBD also exists.*

Proof. Let us assume that $(X, \mathbb{B})$ is a (v, b, r, k, λ)-BIBD. Our goal is to prove that $(X, \{X \setminus A : A \in \mathbb{B}\})$ is a BIBD. There are v points and b blocks in this design, which is evident because each block has $v - k \geq 2$ points and each point appears in $b - r$ blocks. That leaves us with the simple task of showing that each pair of points occurs in precisely $b - 2r + \lambda$ blocks.
Let $x \neq y$, where $x,\ y \in X$. Define

$$a_1 = |\{A \in \mathbb{B} : x, y \in A\}| \tag{2.8}$$

$$a_2 = |\{A \in \mathbb{B} : x \in A, y \notin A\}| \tag{2.9}$$

$$a_3 = |\{A \in \mathbb{B} : x \notin A, y \in A\}| \tag{2.10}$$

and

$$a_4 = |\{A \in \mathbb{B} : x \notin A, y \notin A\}| \tag{2.11}$$

Afterward, it is clear that

$$\begin{gathered} a_1 = \lambda, \\ a_1 + a_2 = r, \\ a_1 + a_3 = r, \text{ and} \\ a_1 + a_2 + a_3 + a_4 = b. \end{gathered}$$

It is simple to solve these four equations to get the desired result, $a_4 = b$ - $2r + \lambda$, □

Example 2.4.4. A $(7, 4, 2)$-BIBD is the complement of a $(7, 3, 1)$-BIBD, and a $(9, 6, 5)$-BIBD is the complement of a $(9, 3, 1)$-BIBD.

The blocks(written horizontally) of $(7, 3, 1)$-BIBD and $(7, 4, 2)$-BIBD

$(7, 3, 1)$-BIBD	$(7, 4, 2)$-BIBD
1 2 3	4 5 6 7
1 4 5	2 3 6 7
1 6 7	2 3 4 5
2 4 6	1 3 5 7
2 5 7	1 3 4 6
3 4 7	1 2 5 6
3 5 6	1 2 4 7

2.5 Symmetric, Residual and Derived BIBDs

Definition 2.5.1. A symmetric BIBD is one in which $b = v$ (or equivalently, $r = k$, or $\lambda(v-1) = k(k-1)$), occurs. (Note that the incidence matrix is not implied to be symmetric by this terminology.)

Definition 2.5.2. Suppose that $(\mathrm{X}, \mathbb{B})$ is a symmetric $(v,\ k,\ \lambda)$-BIBD, and let $A_0 \in \mathbb{B}$. Define a derived design
$\mathrm{Der}(\mathrm{X}, \mathbb{B}, A_0) = (A_0, \{A \cap A_0 : A \in \mathbb{B}, A \neq A_0\})$
and a residual design
$\mathrm{Res}(\mathrm{X}, \mathbb{B}, A_0) = (\mathrm{X} \backslash A_0, \{A \setminus A_0 : A \in \mathbb{B}, A \neq A_0\})$.

In other words, a derived design is obtained by first eliminating every point that is not in a specific block A_0. All of the points in A_0 are deleted to create the residual design.
Note $\mathrm{Der}(\mathrm{X}, \mathbb{B}, A_0)$ is a $\mathrm{BIBD}(k, v-1, k-1, \lambda, \lambda-1)$, while $\mathrm{Res}(\mathrm{X}, \mathbb{B}, A_0)$ is a $\mathrm{BIBD}(v-k, v-1, k, k-\lambda, \lambda)$.
If the block sizes are at least two and at most the number of points minus one, it is evident that the derived and residual designs are BIBDs as demonstrated below.

Theorem 2.5.3. *Suppose that $(X, \mathbb{B})$ is a symmetric (v, k, λ)-BIBD, and let $A_0 \in \mathbb{B}$. Then Der$(X, \mathbb{B}, A_0)$ is a $(k, v-1, k-1, \lambda, \lambda-1)$-BIBD provided that $\lambda \geq 2$.*
Furthermore, Res$(X, \mathbb{B}, A_0)$ is a $(v-k, v-1, k, k-\lambda, \lambda)$-BIBD provided that $k \geq \lambda + 2$.

Proof. $\mathrm{Der}(X, \mathbb{B}, A_0)$ is a BIBD with the stated parameters provided that $k > \lambda \geq 2$ (k is the number of points in the derived design, and the blocks have size λ). However, $k > \lambda$ in any symmetric BIBD because $\lambda(v-1) = k(k-1)$ and $v > k$, so this condition is superfluous.

$\mathrm{Res}(X, \mathbb{B}, A_0)$ is a BIBD with the stated parameters provided that $v - k > k - \lambda \geq 2$ (where $v - k$ is the number of points in the residual design, and the blocks have size $k - \lambda$). We now prove that $v - k > k - \lambda$ in a symmetric BIBD.
Suppose that $v \leq 2k - \lambda$; then we have $k(k-1) = \lambda(v-1) \leq \lambda(2k - \lambda - 1)$. This is equivalent to $(k-\lambda)(k-\lambda-1) \leq 0$. But k and λ are integers, so this last inequality holds if and only if $k = \lambda$ or $k = \lambda + 1$. We are assuming that $k \geq \lambda + 2$, so we have a contradiction. Therefore, there is no need for the condition $v - k > k - \lambda$. $\square$

Example 2.5.4. Since $2(11-1) = 5(5-1)$, a $(11, 5, 2)$-BIBD is symmetric. The two BIBDs constructed from a $(11, 5, 2)$-BIBD are: derived design which is a $(5, 2, 1)$-BIBD and residual design which is a $(6, 3, 2)$-BIBD.
The 11 blocks in a $(11, 5, 2)$-BIBD are:
{1, 3, 4, 5, 9}, { 2, 4, 5, 6, 10}, {0, 3, 5, 6, 7,}, {1, 4, 6, 7, 8}, {2, 5, 7, 8, 9}, {3, 6, 8, 9, 10}, {0, 4, 7, 9, 10}, {0, 1, 5, 8, 10}, {0, 1, 2, 6, 9}, {1, 2, 3, 7, 10}, {0, 2, 3, 4, 8}.
Let A_0 be $\{1, 3, 4, 5, 9\}$
By splitting each of the remaining 10 blocks of $(11, 5, 2)$-BIBD into two parts, we have $(5, 2, 1)$-BIBD on the set $\{1, 3, 4, 5, 9\}$ with blocks:
{ 4, 5}, {3, 5 }, {1, 4 }, {5, 9 }, {3, 9}, {4, 9 }, {1, 5}, {1, 9}, {1, 3}, {3, 4} and
a $(6, 3, 2)$-BIBD on the set $\{0, 2, 6, 7, 8, 10\}$ with blocks:
{2, 6, 9 }, {6, 7, 9}, {6, 7, 8 }, {2, 7, 8}, {6, 8, 10}, {0, 7, 10 }, {0, 8, 10 }, {0, 2, 6 }, {2, 7, 10 }, {0, 2, 8 }.

In the next section we are giving an example of a design which can be considered as a complete design. Though we will not use these designs in our thesis, they are used to construct BIBDs as well as other combinatorial designs effectively.

2.6 Latin Squares

Definition 2.6.1. A Latin square of order n with entries from an n-set X is an $n \times n$ array L in which every cell contains an element of X such that every row as well as every column of L is a permutation of X.
It is easy to construct a Latin square of any order $n \geq 1$. For example, we could take the first row to be

1	2	3	...	n

and then shift this row cyclically to the right by $1, 2, ..., n-1$ positions to construct the remaining $n-1$ rows.

Example 2.6.2. By cycling $\{1, 2, 3, 4, 5, 6, 7\}$, one can obtain a Latin square of order 7.

$L =$

1	2	3	4	5	6	7
7	1	2	3	4	5	6
6	7	1	2	3	4	5
5	6	7	1	2	3	4
4	5	6	7	1	2	3
3	4	5	6	7	1	2
2	3	4	5	6	7	1

The addition table of $\mathbb{Z}_n$ also provides a Latin square, therefore we can say that there is a Latin square of order n for each non-negative integer n.

Definition 2.6.3. Two Latin squares L_1 and L_2 are said to be orthogonal if for each $(x, y) \in \{1, 2, ..., n\}$ x $\{1, 2, ..., n\}$ there is exactly one ordered pair (i, j) such that cell (i, j) of L_1 contains the symbol x and cell (i, j) of L_2 contains the symbol y.

Example 2.6.4. $L_1 =$

1	3	2
3	2	1
2	1	3

and $L_2 =$

1	2	3
3	1	2
2	3	1

are orthogonal Latin squares of order 3.

Now we will introduce a powerful algebraic technique used to construct designs extensively. We will also use this technique in Chapter 3.

2.7 Difference Set

Definition 2.7.1. Assume that $(G, +)$ is a finite group of order v, where "0" represents the identity element. A (v, k, λ) difference set in $(G, +)$ is a subset D of G that satisfies the following properties: Let k and λ be positive integers such that $2 \leq k < v$.

1. Every element in $G \setminus \{0\}$ is contained exactly λ times in the multiset $[x - y : x, y \in D, x \neq y]$,
2. $|D| = k$

We shall not require that G be an Abelian group unless specifically mentioned. In numerous instances, $G = (Z_v, +)$, the group of integers modulo v.
Note if a (v, k, λ)-difference set exists, then $\lambda(v - 1) = k(k - 1)$.

Example 2.7.2. A $(21, 5, 1)$-difference set in $(Z_{21}, +)$:
Let $D = \{0, 1, 6, 8, 18\}$.
If we compute the differences (modulo 21) from the pairs of distinct elements in D, we obtain the following differences:
$0-1 = 20,\ 0-6 = 15,\ 0-8 = 13,\ 0-18 = 3,\ 1-0 = 1,\ 1-6 = 16,\ 1-8 = 14,\ 1-18 = 4$
$6-0 = 6,\ 6-1 = 5,\ 6-8 = 19,\ 6-18 = 9,\ 8-0 = 8,\ 8-1 = 7,\ 8-6 = 2,\ 8-18 = 11$
$18-0 = 18,\ 18-1 = 17,\ 18-6 = 12,\ 18-8 = 10$
Observe that as a difference between two elements in D, we obtained every element of $Z_{21} \setminus \{0\}$ precisely once. Therefore D is a $(21, 5, 1)$-difference set in $(Z_{21}, +)$.

Example 2.7.3. In $(Z_{15}, +)$, $D = \{0, 1, 2, 4, 5, 8, 10\}$ is a difference set of a $(15, 7, 3)$-BIBD.

Graph factorizations provide another important tools in the construction of designs.

2.8 1-factorization and 2-factorization of a complete graph

Definition 2.8.1. (i). A complete graph K_n is a graph on n vertices where each distinct pair of vertices is connected by an edge.

(ii). A 1-factor of a graph G is a set of pairwise disjoint edges which partition the vertex set of G.

(iii). A 1-factorization of a graph G is a partition of the edge set of G into 1-factors.

(iv). A 2-factor of a graph is a set of edges in which each vertex appears exactly twice.

(v). A 2-factorization of the complete graph K_n is a set of 2-factors that partitions the edges of the complete graph.

Example 2.8.2. K_4 has 3 1-factors:

1	3	1	2	1	2
2	4	3	4	4	3

Next section introduces the main designs we study in this project.

2.9 t-designs and GDDs

Definition 2.9.1. Given a set X of v elements, and given positive integers k, t ($t \leq k \leq v$), and λ, a t-(v, k, λ) design is a system of subsets of X, having k elements each, such that every subset of X having t elements is contained in exactly λ sets of the system.

Repeated blocks are permitted in a t-(v, k, λ)-design. Obviously, in a t-(v, k, λ) design, no block can be repeated if $\lambda = 1$. A simple t-design is one that has no repeating blocks. It is often true that creating simple t-(v, k, λ) designs is harder than creating non-simple ones when $\lambda > 1$.
A t-$\left(v, k, \lambda\binom{v-t}{k-t}\right)$-design is obtained if we take λ copies of each k-subset of a v-set, where $k < v$. Keep in mind that a 2-(v, k, λ)-design is equivalent to a (v, k, λ)-BIBD. For t-designs with $t > 2$, existence outcomes are known in a far less number than for the BIBDs.

Following example illustrate that the trivial t-design, the set of all k-subsets of a v-set gives us a t-(v, k, λ_t) design for $t = 1, \cdots, k-1, k$.
Where $v = 7$, $k = 4$, $t = 3$, $\lambda = 1$ and λ_t= $\lambda\binom{v-t}{k-t}$=4.

Example 2.9.2. A 3-$(7, 4, 4)$:

1 2 3 7	1 2 3 6	1 2 3 5	1 2 3 4	4 5 6 7
1 4 5 7	1 4 5 6	1 3 4 5	1 2 4 5	2 3 6 7
1 5 6 7	1 4 6 7	1 3 6 7	1 2 6 7	2 3 4 5
2 4 6 7	2 4 5 6	2 3 4 6	1 2 4 6	1 3 5 7
2 5 6 7	2 4 5 7	2 3 5 7	1 2 5 7	1 3 4 6
3 4 6 7	3 4 5 7	2 3 4 7	1 3 4 7	1 2 5 6
3 5 6 7	3 4 5 6	2 3 5 6	1 3 5 6	1 2 4 7

Definition 2.9.3. A Steiner Quadruple System (SQS(n)) is an ordered pair $(X, \mathbb{B})$ where X is a finite set of n symbols and $\mathbb{B}$ is a collection of 4-subsets of X called blocks (quadruples) with the property that every 3-subset of X is a subset of exactly one quadruple $\mathbb{B}$.
A SQS(n) is also denoted by 3-$(n, 4, 1)$.

A specific instance of a t-design is a SQS(n). The following simple 3-$(n, 4, \lambda)$ exist [23]:

1. 3-$(n, 4, 1)$ for $n \equiv 2, 4 \pmod 6$,

2. 3-$(n, 4, 2)$ for $n \equiv 2, 4, 5 \pmod 6$,

3. 3-$(n, 4, 3)$ for $n \geq 4$ and

4. 3-$(2^n + 1, 4, 6t)$ for any $n \geq 2$ and $1 \leq t \leq \frac{2^{n-1}-1}{3}$.

Example 2.9.4. A simple construction of SQS(8) = 3-$(8, 4, 1)$:
Inserting 8 to each block in $(7, 3, 1)$-BIBD in Example 2.4.4, and combining with the blocks of $(7, 4, 2)$-BIBD, we have $(8, 4, 3)$-BIBD, in fact it is a 3-design, a 3-$(8, 4, 1)$ which is also called a Steiner Quadruple System SQS(8).

1 2 3 8	4 5 6 7
1 4 5 8	2 3 6 7
1 6 7 8	2 3 4 5
2 4 6 8	1 3 5 7
2 5 7 8	1 3 4 6
3 4 7 8	1 2 5 6
3 5 6 8	1 2 4 7

Definition 2.9.5. A group divisible design, GDD$(n, m, k, \lambda_1, \lambda_2)$, is a collection of k-element subsets (called blocks) of an mn-set X, which satisfies the following properties:

i) The mn elements of X are partitioned into m subsets (called groups) of size n each.

ii) Points within the same group are called the first associates of each other and appear together in λ_1 blocks while any two points not in the same group are called the second associates and appear together in λ_2 blocks.

Definition 2.9.6. (a) A GDD$(n, m, 4, \lambda_1, \lambda_2)$ is said to be even if every block intersects each group exactly twice.

(b) A GDD$(n, m, 4, \lambda_1, \lambda_2)$ is said to be odd if every block intersects each group in exactly one or three points.

Example 2.9.7. The blocks of a GDD$(3, 2, 4, 3, 2)$ with two groups $G_1 = \{1, 2, 3\}$ and $G_2 = \{a, b, c\}$ are {1, 2, 3, a}, {1, 2, 3, b}, {1, 2, 3, c}, {1, a, b, c}, {2, a, b, c} and {3, a,

b, c}.
In Example 2.9.7, each block meets each group at precisely one point or three points, and hence this design is odd.

The ideas behind GDDs and t-designs can be generalized in a variety of ways. In [37] GDDs and t-designs are generalized for GDDs with two groups, block size k and $t = 3$ as in the next section.

2.10 A new concept: 3-GDDs

Definition 2.10.1. A 3-GDD(n, 2, k, λ_1, λ_2) is a set X of $2n$ elements partitioned into two parts of size n, called groups, together with a collection of k-subsets of X, called blocks, such that

(i) every 3-subset of each group occurs in λ_1 blocks and

(ii) every 3-subset where two elements are from one group and one element from the other group occurs in λ_2 blocks.

According to [37], the necessary conditions for the existence of 3-GDD(n, 2, 4, λ_1, λ_2) are:

$$(n-1)(n-2)\lambda_1 \equiv 0 \pmod{6}. \tag{2.12}$$

From the requirement that the number of blocks b is an integer, we have

$$n(n-1)(n-2)\lambda_1 + 3n^2(n-1)\lambda_2 \equiv 0 \pmod{12}. \tag{2.13}$$

Example 2.10.2. If $G_1 = \{a, b, c, d\}$ and $G_2 = \{1, 2, 3, 4\}$, where $X = G_1 \cup G_2$, then a 3-GDD(4, 2, 4, 0, 1) exists and its blocks are vertically written as:

$$\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\
2 & 2 & 3 & 3 & 4 & 4 & 3 & 3 & 4 & 4 & 4 & 4 \\
a & c & a & b & a & b & a & b & a & b & a & c \\
b & d & d & c & c & d & c & d & d & c & b & d
\end{array}$$

Lemma 2.10.3. *A necessary condition for the existence of a 3-GDD(n, 2, k, λ_1, λ_2) for odd n and k even is that λ_1 and λ_2 must be of the same parity.*

Theorem 2.10.4. *For every positive integer* n*, a* 3-*GDD(*n*, 2, 4, 0, 2) exists and for every even number* n*, there exists a* 3-*GDD(*n*, 2, 4, 0, 1).*

Theorem 2.10.5. *Necessary conditions are sufficient for the existence of a* 3-*GDD*(n, 2, 4, λ_1, λ_2) *for*

1. $n \equiv 0, 2, 4, 5 \pmod 6$ *and* $n = 7$.
2. $n \equiv 1 \pmod 6$.
3. $n \equiv 3 \pmod 6$ *and integers* $t,\ s,\ m \geq 0,$ *where* $\lambda_1 = 3t + 3s,\ \lambda_2 = t + 3s + 2m.$
4. $n \equiv 3 \pmod 6$ *except when* $\lambda_1 \equiv 9 \pmod{12}$ *and* $n \equiv 3 \pmod{12}$.

The next chapter studies and gives new results on the existence of 3-GDDs with three groups and block size five.

Chapter 3

3-GDDs With Three Groups and Block Size Five

This chapter focus on the study of 3-GDDs and most of the results from this chapter are published in our paper [47]. We mainly consider the case when the required design has three groups of size n and the block size 5. Such a GDD is denoted by a 3-GDD(n, 3, 5, μ_1, μ_2). Some necessary conditions for the existence of such designs together with their proofs are given, the proofs of some constructions especially when $\mu_2 = 0$ are presented and existence of infinite families of 3-GDDs when $n = 3$ is given.

3.1 Preliminary results

A necessary condition for the existence of a t-(v, k, λ) design is given below [15].

$$\lambda.\frac{\binom{v-h}{t-h}}{\binom{k-h}{t-h}} = integer, h = 0, 1, 2, 3, ..., t-1.$$

The problem of determining the values of k, t and X (where X is a v-set of points) for which this condition is also sufficient is not yet solved completely, and even for $t = 3$ not much is known about these configurations.

Example 3.1.1. A 3-$(8, 4, 1)$ design on the set $X = \{1, 2, ..., 8\}$ is given by the blocks: $\{1, 2, 5, 6\}$, $\{3, 4, 7, 8\}$, $\{1, 3, 5, 7\}$, $\{2, 4, 6, 8\}$, $\{1, 4, 5, 8\}$, $\{2, 3, 6, 7\}$, $\{1, 2, 3, 4\}$, $\{5, 6, 7, 8\}$, $\{1, 2, 7, 8\}$, $\{3, 4, 5, 6\}$, $\{1, 3, 6, 8\}$, $\{2, 4, 5, 7\}$, $\{1, 4, 6, 7\}$, $\{2, 3, 5, 8\}$.

The concepts of a GDD and a 3-design can be merged to define a 3-GDD in many ways. We earlier introduced a 3-GDD given by [47], here is another way these concepts can be merged as done in [47] to define a 3′-GDD as follows:

Definition 3.1.2. A 3′-GDD(n, m, k, Λ_1, Λ_2), for $m > 2$, is a set X of mn symbols partitioned into m parts of size n called groups together with a collection of k-subsets of X called blocks, such that

(i) every 3-subset of each group occurs in Λ_1 blocks and

(ii) every mixed 3-subset, meaning either two symbols are from one group and one symbol from the other group or all three symbols are from different groups, occurs in Λ_2 blocks.

Definition 3.1.2 coincides with the definition given in [33, 37] where a 3-subset of X has only two choices, all symbols from one of the two groups or one symbol from a group and two symbols from the other group. Such GDDs are denoted by a 3-GDD(n, 2, 4, λ_1, λ_2) instead of 3′-GDD.

- The necessary conditions are sufficient for the existence of a 3-GDD(n, 2, 4, λ_1, λ_2) for $n \equiv 1, 7, 9 \pmod{12}$ [33].
- Except possibly when $n \equiv 1, 3 \pmod 6$, $n \neq 3$, 7, 13, the necessary conditions are sufficient for the existence of a 3-GDD(n, 2, 4, λ_1, λ_2) and $\lambda_1 > \lambda_2$ [37].

Lemma 3.1.3. *[33] If a 3-(2n, 4, λ_2) and a 3-(2n, 4, $\lambda_1 - \lambda_2$) exists, then a 3-GDD(n, 2, 4, λ_1, λ_2) exists.*

Example 3.1.4. Where $X = \{1, 2, 3, a, b, c\}$, $G_1 = \{1, 2, 3\}$ and $G_2 = \{a, b, c\}$, the blocks of a 3-GDD$(3, 2, 4, 3, 1)$ are: $\{1, 2, 3, a\}$, $\{1, 2, 3, b\}$, $\{1, 2, 3, c\}$, $\{a, b, c, 1\}$, $\{a, b, c, 2\}$, $\{a, b, c, 3\}$.

Following necessary conditions(Table 3.1, where the values of λ_1 and λ_2 are given modulo 6) and the existence results of a 3-GDD$(n, 2, 4, \lambda_1, \lambda_2)$ are given in [33, 37]

Table 3.1: Congruence Restrictions for 3-GDD(n, 2, 4, λ_1, λ_2)

$\lambda_1 \backslash \lambda_2$	0	1	2	3	4	5
0	all n	n even	all n	n even	all n	n even
1	$2,4$ (mod 6)	$1,2,4,5$ (mod 6)	$2,4$ (mod 6)	$1,2,4,5$ (mod 6)	$2,4$ (mod 6)	$1,2,4,5$ (mod 6)
2	$1,2,4,5$ (mod 6)	$2,4$ (mod 6)	$1,2,4,5$ (mod 6)	$2,4$ (mod 6)	$1,2,4,5$ (mod 6)	$2,4$ (mod 6)
3	n even	all n	n even	all n	n even	all n
4	$1,2,4,5$ (mod 6)	$2,4$ (mod 6)	$1,2,4,5$ (mod 6)	$2,4$ (mod 6)	$1,2,4,5$ (mod 6)	$2,4$ (mod 6)
5	$2,4$ (mod 6)	$1,2,4,5$ (mod 6)	$2,4$ (mod 6)	$1,2,4,5$ (mod 6)	$2,4$ (mod 6)	$1,2,4,5$ (mod 6)

One may relax condition (ii) in Definition 3.1.2 to define a 3-PBIBD as follows:

Definition 3.1.5. A 3-PBIBD (Partially Balanced Incomplete Block Design), 3-PBIBD(n, m, k, θ_1, θ_2, θ_3) is a collection of k-subsets of an mn set X, where X is partitioned in to m groups of order n such that:

(i) Every triple formed from symbols of only a single group occurs in θ_1 blocks.

(ii) Every triple formed from symbols of only two groups occurs in θ_2 blocks.

(iii) Every triple formed from symbols of all three groups occurs in θ_3 blocks.

Definition 3.1.6. A 3-GDD(n, m, k, μ_1, μ_2) is a pair (X, $\mathbb{B}$), where X is a set of mn elements partitioned into m n-subsets (groups) and $\mathbb{B}$ is a collection of k-subsets (blocks) of X such that

(i) every triple occurs in exactly μ_1 blocks if it contains elements from at most 2 groups,

(ii) it occurs in exactly μ_2 blocks if it has all three elements from different groups.

Definition 3.1.6 is the subject matter of this chapter and we explicitly consider the case in which $m = 3$ and $k = 5$. Such GDD is denoted by 3-GDD(n, 3, 5, μ_1, μ_2).

Remark 3.1.7. A 3-GDD(n, 3, 5, μ_1, μ_2) is a 3-PBIBD(n, 3, 5, θ_1, θ_2, θ_3), where θ_1=θ_2, denoted by μ_1, while θ_3 is denoted by μ_2.

When $\mu_1 = \mu_2$, a 3-GDD(n, 3, 5, μ_1, μ_2) is actually a 3-($3n$, 5, μ_1). It follows that a 3-GDD(n, 3, 5, μ_1, μ_2) exists if and only if a 3-($3n$, 5, μ_1) exists.

The next three sections of this chapter discuss our research findings.

3.2 Necessary conditions

In this section, assuming a 3-GDD(n, 3, 5, μ_1, μ_2) exists, we obtain some necessary conditions for the existence of a 3-GDD(n, 3, 5, μ_1, μ_2).
Let λ_1 denote the number of blocks containing a first associate pair, i.e. pairs of the form $\{x_1, x_2\}$, where x_1 and x_2 are from the same group, λ_2 denote the number of blocks containing a second associate pair, i.e. pairs of the form $\{x, y\}$ where x and y are from different groups, r and b respectively denote the replication number and the number of blocks in the 3-GDD.

Theorem 3.2.1. *Given a 3-GDD*$(n, 3, 5, \mu_1, \mu_2)$*,*

$$r = \frac{(n-1)(7n-2)\mu_1 + 2n^2\mu_2}{12} \tag{3.1}$$

$$b = \frac{n}{20}((n-1)(7n-2)\mu_1 + 2n^2\mu_2) \tag{3.2}$$

$$\lambda_1 = \frac{(3n-2)}{3}\mu_1 \tag{3.3}$$

$$\lambda_2 = \frac{2(n-1)\mu_1 + n\mu_2}{3} \tag{3.4}$$

Proof. The above results can be proved as follows:
(3.1). We count the number of triples containing a fixed element x in the design in two ways. First, given an element x, it appears in $\binom{n-1}{2}$ triples of the type $(3, 0)$ and in $2n(n-1)+2\binom{n}{2}$ triples of the type $(2, 1)$ (There are $2n(n-1)$ triples of the type $(2, 1)$ where x and an element from the same group as x appear with an element of another group, and there are $2\binom{n}{2}$ triples of type $(2, 1)$ where x appears with two elements from another group). So, in sum, x is in $(\binom{n-1}{2}+ 3n(n-1))\mu_1$ triples of the form $(3, 0)$ or $(2, 1)$ in the design.
Similarly, there are n^2 triples of the type $(1, 1, 1)$ containing x, and are repeated μ_2 times.

All together, there are $(\binom{n-1}{2}+3n(n-1))\mu_1+n^2\mu_2$ triples containing x.
Second, in every block containing x, there are 6 triples containing x and x occurs in r blocks, which means there are $6r$ triples containing x in the design.
Hence, $6r=(\binom{n-1}{2}+3n(n-1))\mu_1+n^2\mu_2$, and $r=\frac{(n-1)(7n-2)\mu_1+2n^2\mu_2}{12}$
(3.2). Again counting in two ways, in a design with block size 5 and b blocks, there are $b\binom{5}{3}=10b$ triples. On the other hand, there must be exactly $3\binom{n}{3}\mu_1$ triples of type $(3,0)$, $3n^2(n-1)\mu_1$ triples of type $(2,1)$ and $n^3\mu_2$ triples of type $(1,1,1)$ in the design.
Therefore, $10b=3\binom{n}{3}\mu_1+3n^2(n-1)\mu_1+n^3\mu_2$ gives equation (3.2).
(3.3). Let (x_1,x_2) be a first associate pair and occurs in λ_1 blocks. The triples containing (x_1,x_2) can only be of type $(3,0)$ or $(2,1)$. There are (n-2) triples of the type $(3,0)$ and $2n$ triples of the type $(2,1)$ containing (x_1,x_2). Now, since these triples occur μ_1 times and since the block containing a first associate pair, say (x_1,x_2) contains three triples containing (x_1,x_2), we have : $\lambda_1=\frac{(3n-2)}{3}\mu_1$.
(3.4). Let (x,y) be a second associate pair. This pair occurs in triples of the type $(2,1)$ and $(1,1,1)$ only. It occurs in $2(n-1)\mu_1$ triples of type $(2,1)$ and $n\mu_2$ triples of type $(1,1,1)$. Each of the λ_2 blocks containing the pair (x,y) has three triples containing (x,y).
Hence, $3\lambda_2=2(n-1)\mu_1+n\mu_2$ and $\lambda_2=\frac{2(n-1)\mu_1+n\mu_2}{3}$. ☐

As a consequence of Theorem 3.2.1, we have the following Corollaries:

Corollary 3.2.2. *For $\mu_1 \not\equiv 0 \pmod 3$, a 3-GDD($n$, 3, 5, μ_1, μ_2) does not exist.*

Proof. From equation (3.3) in Theorem 3.2.1, λ_1 is not an integer when $\mu_1 \not\equiv 0 \pmod 3$. ☐

Corollary 3.2.3. *In a 3-GDD(n, 3, 5, μ_1, μ_2), $\lambda_1 \neq 0$.*

Proof. As the number of groups is less than the block size, $\mu_1 \neq 0$. Again from (3.3) in Theorem 3.2.1, λ_1 is a multiple of μ_1, and hence $\lambda_1 \neq 0$ ☐

From the original necessary conditions in Theorem 3.2.1, when $\mu_1=\mu_2=\lambda$, we get 3-$(3n,5,\lambda)$, and from (3.3) and (3.4), we must have $\lambda \equiv 0 \pmod 3$. In addition, from (3.1) and (3.2) in Theorem 3.2.1, the following result follows:

Corollary 3.2.4. *The necessary conditions for the existence of 3-$(3n,5,\lambda)$ are satisfied only under the following cases:*

- *If $n\equiv 2, 7, 10, 14, 15 \pmod{20}$, then $\lambda\equiv 0 \pmod 3$.*

- *If* $n\equiv 0$, 4, 5, 9, 12, 17 $\pmod{20}$, *then* $\lambda \equiv 0 \pmod 6$.

- *If* $n\equiv 3$, 6, 11, 18 $\pmod{20}$, *then* $\lambda \equiv 0 \pmod{15}$.

- *If* $n\equiv 1$, 8, 13, 16 $\pmod{20}$, *then* $\lambda \equiv 0 \pmod{30}$.

Theorem 3.2.5. *In a* 3-*GDD*$(n,\ 3,\ 5,\ \mu_1,\ \mu_2)$, $\mu_2 < 3\mu_1$.

Proof. In a 3-GDD$(n,\ 3,\ 5,\ \mu_1,\ \mu_2)$, triples of the type $(1,1,1)$ occur in the blocks of type $(3,1,1)$ and $(2,2,1)$ only. In each of these blocks which can have a $(1,1,1)$ triples, the number of $(2,1)$ triples are more than the number of $(1,1,1)$ triples.
Hence, $3n^2(n-1)\mu_1 > n^3\mu_2$, implies $\mu_2 < \frac{3(n-1)\mu_2}{n} < 3\mu_1$. □

Theorem 3.2.6. *For* $n \le 5$, *a* 3-*GDD*$(n,\ 3,\ 5,\ \mu_1,\ 0)$ *doesn't exist.*

Proof. As $\mu_2 = 0$, blocks are of configuration $(5,0)$, $(4,1)$ and $(3,2)$ only. The maximum number of $(2,1)$ triples occur in $(3,2)$ configuration blocks and ratio is 9:1 (in each block of configuration $(3,2)$, there are nine $(2,1)$ and one $(3,0)$ triples). Counting the number of $(2,1)$ triples $(3n^2(n-1)\mu_1)$ and the number of $(3,0)$ triples $(\frac{n(n-1)(n-2)}{2})$, their ratio must be less than or equal to 9.
Thus $\frac{3n^2(n-1)\mu_1}{\frac{n(n-1)(n-2)}{2}} \le 9$
$\Rightarrow \frac{6n}{n-2} \le 9$, which gives $n \ge 6$. □

From (3.3) and (3.4) in Theorem 3.2.1, the necessary conditions are satisfied under the following conditions.

Lemma 3.2.7. *In regards to*
(i). λ_1, *for every* n, $\mu_1 \equiv 0 \pmod 3$ *and* μ_2 *is free.*
(ii). λ_2, *when*
- $n \equiv 0 \pmod 3$, $\mu_1 \equiv 0 \pmod 3$ *and* μ_2 *is free or*
- $n \equiv 1 \pmod 3$, μ_1 *is free and* $\mu_2 \equiv 0 \pmod 3$ *or*
- $n \equiv 2 \pmod 3$, $\mu_1 + \mu_2 \equiv 0 \pmod 3$

As the values of b and r must also be integers, Table 3.2 is a table of congruence restrictions for a 3-GDDs with 3 groups and block size 5 (all values are considered to be in terms of $\pmod{60}$ unless otherwise stated)

Table 3.2: Congruence restrictions for 3-GDD(n, 3, 5, μ_1, μ_2)

	r	r	b	b
n	μ_1	μ_2	μ_1	μ_2
0	0 (mod 6)	All	All	All
1, 41	All	0 (mod 6)	All	0 (mod 10)
2, 14, 22, 34, 42, 54	All	0 (mod 3)	$\Yup_1$	$\Yup_1$
3	$*_1$	$*_1$	$\Yup_2$	$\Yup_2$
4, 32, 44, 52	$*_2$	$*_2$	$\Yup_1$	$\Yup_1$
5, 25	All	0 (mod 6)	All	0 (mod 2)
6	0 (mod 3)	All	All	0 (mod 5)
7, 19, 47, 59	$*_3$	$*_3$	$\Yup_3$	$\Yup_3$
8	$*_2$	$*_2$	$\Yup_8$	$\Yup_8$
9, 57	$*_4$	$*_4$	$\Yup_4$	$\Yup_4$
10, 50	All	0 (mod 3)	All	All
11, 31	$*_3$	$*_3$	$\Yup_5$	$\Yup_5$
12, 24	0 (mod 6)	All	$\Yup_1$	$\Yup_1$
13, 53	All	0 (mod 6)	$\Yup_6$	$\Yup_6$
15	$*_1$	$*_1$	$\Yup_7$	$\Yup_7$
16, 56	$*_2$	$*_2$	All	0 (mod 5)
17, 29, 37, 49	All	0 (mod 6)	$\Yup_4$	$\Yup_4$
18	0 (mod 3)	All	$\Yup_8$	$\Yup_8$
20, 40	$*_2$	$*_2$	All	All
21	$*_4$	$*_4$	All	0 (mod 10)
23, 43	$*_3$	$*_3$	$\Yup_2$	$\Yup_2$
26, 46	All	0 (mod 3)	All	0 (mod 5)
27, 39	$*_1$	$*_1$	$\Yup_3$	$\Yup_3$
28	$*_2$	$*_2$	$\Yup_7$	$\Yup_7$
30	0 (mod 3)	All	All	All
33	$*_4$	$*_4$	$\Yup_6$	$\Yup_6$
35, 55	$*_3$	$*_3$	$\Yup_7$	$\Yup_7$
36	0 (mod 6)	All	All	0 (mod 5)
38, 58	All	0 (mod 3)	$\Yup_8$	$\Yup_8$
45	$*_4$	$*_4$	All	0 (mod 2)
48	0 (mod 6)	All	$\Yup_7$	$\Yup_7$
51	$*_1$	$*_1$	$\Yup_5$	$\Yup_5$

Where,

$\Yup_1$ denotes the set of all ordered pairs (μ_1, μ_2) such that

$$4\mu_1 + \mu_2 \equiv 0 \pmod 5,$$

$$\mu_1 + 4\mu_2 \equiv 0 \pmod 5,$$

$$2\mu_1 + 3\mu_2 \equiv 0 \pmod 5,$$
$$3\mu_1 + 2\mu_2 \equiv 0 \pmod 5$$

$\Yup_2$ denotes the set of all ordered pairs (μ_1, μ_2) such that
$$\mu_1 + \mu_2 \equiv 0 \pmod{10}.$$

$\Yup_3$ denotes the set of all ordered pairs (μ_1, μ_2) such that
$$9\mu_1 + \mu_2 \equiv 0 \pmod{10}.$$
$$\mu_1 + 9\mu_2 \equiv 0 \pmod{10},$$
$$3\mu_1 + 7\mu_2 \equiv 0 \pmod{10}$$

$\Yup_4$ denotes the set of all ordered pairs (μ_1, μ_2) such that
$$4\mu_1 + \mu_2 \equiv 0 \pmod{10},$$
$$2\mu_1 + 3\mu_2 \equiv 0 \pmod{10},$$
$$6\mu_1 + 9\mu_2 \equiv 0 \pmod{10},$$
$$8\mu_1 + 7\mu_2 \equiv 0 \pmod{10}$$

$\Yup_5$ denotes the set of all ordered pairs (μ_1, μ_2) such that
$$5\mu_1 + \mu_2 \equiv 0 \pmod{10},$$
$$5\mu_1 + 3\mu_2 \equiv 0 \pmod{10},$$
$$5\mu_1 + 9\mu_2 \equiv 0 \pmod{10},$$
$$5\mu_1 + 7\mu_2 \equiv 0 \pmod{10}$$

$\Yup_6$ denotes the set of all ordered pairs (μ_1, μ_2) such that
$$8\mu_1 + 3\mu_2 \equiv 0 \pmod{10},$$
$$2\mu_1 + 7\mu_2 \equiv 0 \pmod{10},$$
$$6\mu_1 + \mu_2 \equiv 0 \pmod{10},$$
$$4\mu_1 + 9\mu_2 \equiv 0 \pmod{10}$$

$\Yup_7$ denotes the set of all ordered pairs (μ_1, μ_2) such that
$$\mu_1 + \mu_2 \equiv 0 \pmod 2$$

$\Yup_8$ denotes the set of all ordered pairs (μ_1, μ_2) such that
$$\mu_1 + \mu_2 \equiv 0 \pmod 5$$

$*_1$ denotes the set of all ordered pairs (μ_1, μ_2) such that
$$\mu_1 + 3\mu_2 \equiv 0 \pmod 6,$$
$$5\mu_1 + 3\mu_2 \equiv 0 \pmod 6$$

$*_2$ denotes the set of all ordered pairs (μ_1, μ_2) such that

$$3\mu_1 + 4\mu_2 \equiv 0 \pmod 6,$$
$$3\mu_1 + 2\mu_2 \equiv 0 \pmod 6$$

$*_3$ denotes the set of all ordered pairs (μ_1, μ_2) such that

$$3\mu_1 + \mu_2 \equiv 0 \pmod 6,$$
$$3\mu_1 + 5\mu_2 \equiv 0 \pmod 6$$

$*_4$ denotes the set of all ordered pairs (μ_1, μ_2) such that

$$4\mu_1 + 3\mu_2 \equiv 0 \pmod 6,$$
$$2\mu_1 + 3\mu_2 \equiv 0 \pmod 6,$$

b, r, λ_1 and λ_2 must be positive integers for a design to exist. From Lemma 3.2.7 and Table 3.2, we have a more compact necessary conditions for different values of n as follows:

Lemma 3.2.8. *For*

(*i*). $n \equiv 0 \pmod{10}$, *μ_1 and μ_2 are free in regards to the number of blocks b.*

(*ii*). $n \equiv 0 \pmod 3$ *and* $\mu_1 \equiv 0 \pmod 3$, *λ_1 & λ_2 are integers for any chosen μ_2.*

(*iii*). $n \equiv 1, 2 \pmod 3$, $\mu_1 \equiv 0 \pmod 3$ *and* $\mu_2 \equiv 0 \pmod 3$, *λ_1 & λ_2 are integers.*

(*iv*). $n \equiv 1, 5 \pmod{12}$ *and* $\mu_2 \equiv 0 \pmod 6$ *or* $n \equiv 2, 10 \pmod{12}$ *and* $\mu_2 \equiv 0 \pmod 3$, *r is an integer for any chosen μ_1.*

(*v*). $n \equiv 1, 2, 4, 7, 10 \pmod{12}$ *and* $n \equiv 5, 17, 41, 53 \pmod{60}$, *$\mu_2$ is multiple of* 3.

(*vi*). $n \equiv 6 \pmod{10}$ *and* $n \equiv 1 \pmod{20}$, *μ_2 is multiple of* 5.

3.3 Some constructions when $\mu_2 = 0$

In this section, the proofs of some constructions when $\mu_2 = 0$ are given.

Theorem 3.3.1. (*a*). *All blocks of a design are of type* $(3, 2)$ *constitute a* 3-*PBIBD*(n, 3, 5, $n(n-1)$, $\frac{3(n-1)(n-2)}{2}$, 0)*, where the number of blocks of such design is* $6\binom{n}{3}\binom{n}{2}$.

(*b*). *When all blocks are of type* $(5, 0)$*, there exisits a* 3-*PBIBD*(n, 3, 5, $\frac{(n-3)(n-4)}{2}$, 0, 0).

Proof.

(a). When all blocks are of type $(3,2)$, the number of blocks of such design is $6\binom{n}{3}\binom{n}{2}$, where $3\binom{n}{3}$ gives the number of 3-subsets of 3 groups and $2\binom{n}{2}$ gives the number of 2-subsets for each choice of a 3-subset. This also tells the number of blocks containing every $(3,0)$ triple is $2\binom{n}{2}=n(n-1)$.

Similarly, let $\{a,b,x\}$ be $(2,1)$ triple, where the pair $\{a,b\}$ is from say G_1 and the element x is from G_2. There are $(n-2)(n-1)$ triples containing $\{a,b,x\}$ (one element each from G_1 and G_2) or there are $\binom{n-1}{2}$ triples containing $\{a,b,x\}$ (3-subsets of G_2), and hence in total there are $\frac{3(n-1)(n-2)}{2}$ blocks of size five containing $\{a,b,x\}$.

(b). When all blocks are of type $(5,0)$, the number of blocks containing every $(3,0)$ triple is $\binom{n-3}{2}=\frac{(n-3)(n-4)}{2}$. □

Example 3.3.2. A 3-GDD(6, 3, 5, 30, 0) exists.
In Theorem 3.3.1 (a), if $n=6$, then, $\theta_1=\theta_2=n(n-1)=\frac{3(n-1)(n-2)}{2}=30$.

Blocks are of type $(4,1)$ if there are four elements from a single group and one element from other group.

Theorem 3.3.3. *If all the blocks are of type $(4,1)$, then a 3-GDD(n, 3, 5, μ_1, 0) does not exist.*

Proof. when all blocks are of type $(4,1)$, a fixed $(3,0)$ triple occurs in $2n(n-3)$ and a fixed $(2,1)$ triple occurs in $\binom{n-2}{2}$ blocks.
Hence, $2n(n-3)=\binom{n-2}{2}$, which is true only for $3n=-2$, which is impossible. □

In Theorem 3.2.6, a 3-GDD(n, 3, 5, μ_1, 0) does not exist when $n \leq 5$. But, when $n \geq 6$, a 3-GDD(n, 3, 5, μ_1, 0) exists for some μ_1.

Theorem 3.3.4. *For $n \geq 6$, a 3-GDD(n, 3, 5, $\frac{3(n-1)(n-2)(n-3)(n-4)}{4}$, 0) exists.*

Proof. When $n \geq 6$, taking $\binom{n-3}{2}=\frac{(n-3)(n-4)}{2}$ copies of the blocks of Theorem 3.3.1(a) together with $\frac{3(n-1)(n-2)}{2}-n(n-1)=\frac{(n-1)(n-6)}{2}$ copies of the blocks of Theorem 3.3.1(b) give the blocks of 3-GDD(n, 3, 5, $\frac{3(n-1)(n-2)(n-3)(n-4)}{4}$, 0). □

Example 3.3.5. By Theorem 3.3.4 above, the following 3-GDDs exist:
3-GDD(6, 3, 5, 90, 0), 3-GDD(7, 3, 5, 270, 0), 3-GDD(8, 3, 5, 630, 0),
3-GDD(9, 3, 5, 1260, 0), 3-GDD(10, 3, 5, 2268, 0), 3-GDD(11, 3, 5, 3780, 0),
3-GDD(12, 3, 5, 5940, 0), etc.

Theorem 3.3.6. *There exists a* 3-*GDD*(n, 3, 5, $\frac{(n-1)(n-2)(n-3)(n-4)}{4}$, 0) *for* $n \geq 6$ *and* $n \not\equiv 2 \pmod 3$.

Proof. If $n \not\equiv 2 \pmod 3$ and $n \geq 6$, then both $\frac{(n-1)(n-6)}{2}$ and $\frac{(n-3)(n-4)}{2}$ are divisible by 3. Joining $\frac{(n-3)(n-4)}{6}$ copies of the blocks of Theorem 3.3.1(a) together with $\frac{(n-1)(n-6)}{6}$ copies of the blocks of Theorem 3.3.1(b) give the blocks of 3-GDD(n, 3, 5, $\frac{(n-1)(n-2)(n-3)(n-4)}{4}$, 0). □

Example 3.3.7. By Theorem 3.3.6 above, the following 3-GDDs exist:
3-GDD(6, 3, 5, 30, 0), 3-GDD(7, 3, 5, 90, 0), 3-GDD(9, 3, 5, 420, 0),
3-GDD(10, 3, 5, 756, 0), 3-GDD(12, 3, 5, 1980, 0), 3-GDD(13, 3, 5, 2970, 0),
3-GDD(15, 3, 5, 6006, 0), etc.

3.4 3-GDD(3, 3, 5, μ_1, μ_2)

In this section, we give some families of existences along with several examples of the 3-GDD when $n = 3$. In addition, some relationship between 3′-GDD(n, m, k, Λ_1, Λ_2)(Definition 3.1.2) and 3-PBIBD(n, m, k, θ_1, θ_2, θ_3) (Definition 3.1.5) is designed.
We begin with the following three examples (Example 3.4.1, 3.4.2, 3.4.3) of a 3-PBIBD(3, 3, 5, θ_1, θ_2, θ_3).

Example 3.4.1. A 3-PBIBD(3, 3, 5, 0, 3, 4) with $G_1 = \{1, 2, 3\}$, $G_2 = \{a, b, c\}$, $G_3 = \{x, y, z\}$ exists and the blocks(written vertically) of this design are:

$$\begin{array}{ccccccccc|ccccccccc|ccccccccc}
1&1&1&1&1&1&2&2&2&3&1&2&1&2&3&2&3&1&1&1&2&1&1&2&1&1&2\\
2&2&2&3&3&3&3&3&3&a&a&a&a&a&a&b&b&b&2&3&3&2&3&3&2&3&3\\
a&a&b&a&a&b&a&a&b&b&b&b&c&c&c&c&c&c&a&b&c&b&c&a&c&a&b\\
b&c&c&b&c&c&b&c&c&x&x&y&x&x&y&x&x&y&x&x&x&x&x&x&y&y&y\\
y&z&x&z&x&y&x&y&z&y&z&z&y&z&z&y&z&z&y&y&y&z&z&z&z&z&z
\end{array}$$

Example 3.4.2. A 3-PBIBD(3, 3, 5, 6, 3, 0) exists and the blocks of this design are obtained by union of every G_i with every two element set of G_j for $i \neq j$, where i, j=1, 2, 3.

$$\begin{array}{cccccc|cccccc|cccccc}
1&1&1&1&1&1&a&a&a&a&a&a&x&x&x&x&x&x\\
2&2&2&2&2&2&b&b&b&b&b&b&y&y&y&y&y&y\\
3&3&3&3&3&3&c&c&c&c&c&c&z&z&z&z&z&z\\
a&a&b&x&x&y&1&1&2&x&x&y&1&1&2&a&a&b\\
b&c&c&y&z&z&2&3&3&y&z&z&2&3&3&b&c&c
\end{array}$$

Example 3.4.3. 3-PBIBD(3, 3, 5, 9, 3, 3) exists. Its blocks are obtained by union of every group with each singleton set of both remaining groups.

$$
\begin{array}{ccccccccc|ccccccccc|ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & a & a & a & a & a & a & a & a & a & x & x & x & x & x & x & x & x & x \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & b & b & b & b & b & b & b & b & b & y & y & y & y & y & y & y & y & y \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & c & c & c & c & c & c & c & c & c & z & z & z & z & z & z & z & z & z \\
a & a & a & b & b & b & c & c & c & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\
x & y & z & x & y & z & x & y & z & x & y & z & x & y & z & x & y & z & a & b & c & a & b & c & a & b & c
\end{array}
$$

Remark 3.4.4. When $n = 3$, from the original necessary conditions in Theorem 3.2.1; r, b, λ_1 and λ_2 respectively are integers if

$$\mu_1 \equiv 0 \pmod 3, \ *_1 : \mu_1 + 3\mu_2 \equiv 0 \pmod 6 \text{ and } ⅄_2 : \ \mu_1 + \mu_2 \equiv 0 \pmod{10}.$$

$*_1$ and $⅄_2$ denote the set of all ordered pairs (μ_1, μ_2) such that $\mu_1 + 3\mu_2 \equiv 0 \pmod 6$ and $\mu_1 + \mu_2 \equiv 0 \pmod{10}$, respectively.

Now, to determine all the possible values of μ_1 and μ_2 satisfying the necessary conditions The values of μ_1 and μ_2 satisfying $\mu_1 \equiv 0 \pmod 3$ and $*_1 : \mu_1 + 3\mu_2 \equiv 0 \pmod 6$ are given in Table 3.3 as follows:

Table 3.3: $\mu_1 \equiv 0 \pmod 3$ and $\mu_1 + 3\mu_2 \equiv 0 \pmod 6$.

μ_1	μ_2
0 (mod 6)	0 (mod 2)
3 (mod 6)	1 (mod 2)

The values of μ_1 and μ_2 satisfying $\mu_1 \equiv 0 \pmod 3$ and $⅄_2$: $\mu_1 + \mu_2 \equiv 0 \pmod{10}$ are given in Table 3.4:

Table 3.4: $\mu_1 \equiv 0 \pmod 3$ and $\mu_1 + \mu_2 \equiv 0 \pmod{10}$.

μ_1	μ_2	μ_1	μ_2
0 (mod 30)	0 (mod 10)	15 (mod 30)	5 (mod 10)
3 (mod 30)	7 (mod 10)	18 (mod 30)	2 (mod 10)
6 (mod 30)	4 (mod 10)	21 (mod 30)	9 (mod 10)
9 (mod 30)	1 (mod 10)	24 (mod 30)	6 (mod 10)
12 (mod 30)	8 (mod 10)	27 (mod 30)	3 (mod 10)

From Table 3.3 and Table 3.4, taking all the possible combinations of μ_1 and μ_2, we have a more refined necessary conditions as follows:

- $\mu_1 \equiv 3 \pmod{30}$ and $\mu_2 \equiv 7 \pmod{10}$ or
- $\mu_1 \equiv 9 \pmod{30}$ and $\mu_2 \equiv 1 \pmod{10}$ or
- $\mu_1 \equiv 15 \pmod{30}$ and $\mu_2 \equiv 5 \pmod{10}$ or
- $\mu_1 \equiv 21 \pmod{30}$ and $\mu_2 \equiv 9 \pmod{10}$ or
- $\mu_1 \equiv 27 \pmod{30}$ and $\mu_2 \equiv 3 \pmod{10}$ or
- $\mu_1 \equiv 0 \pmod{30}$ and $\mu_2 \equiv 0 \pmod{10}$ or
- $\mu_1 \equiv 6 \pmod{30}$ and $\mu_2 \equiv 4 \pmod{10}$ or
- $\mu_1 \equiv 12 \pmod{30}$ and $\mu_2 \equiv 8 \pmod{10}$ or
- $\mu_1 \equiv 18 \pmod{30}$ and $\mu_2 \equiv 2 \pmod{10}$ or
- $\mu_1 \equiv 24 \pmod{30}$ and $\mu_2 \equiv 6 \pmod{10}$.

Example 3.4.5. 3-GDD(3, 3, 5, 9, 11) exists and its blocks are obtained by combining the blocks of Example 3.4.3 with two copies of the blocks of Example 3.4.1.

1	1	1	1	1	1	1	1	1	a	a	a	a	a	a	a	a	a	x	x	x	x	x	x	x	x	x
2	2	2	2	2	2	2	2	2	b	b	b	b	b	b	b	b	b	y	y	y	y	y	y	y	y	y
3	3	3	3	3	3	3	3	3	c	c	c	c	c	c	c	c	c	z	z	z	z	z	z	z	z	z
a	a	a	b	b	b	c	c	c	1	1	1	2	2	2	3	3	3	1	1	1	2	2	2	3	3	3
x	y	z	x	y	z	x	y	z	x	y	z	x	y	z	x	y	z	a	b	c	a	b	c	a	b	c
1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	2	2	2	2	2	2	3	3	3
a	a	a	a	b	b	a	a	a	a	b	b	a	a	a	a	b	b	a	a	b	b	c	c	a	a	b
b	b	c	c	c	c	b	b	c	c	c	c	b	b	c	c	c	c	x	x	x	y	x	y	x	y	x
x	y	x	z	y	z	y	z	x	y	x	z	x	z	y	z	x	y	y	z	y	z	z	z	y	z	z
1	1	1	2	2	2	2	2	2	1	1	1	2	2	2	3	3	3	1	1	1	2	2	2	3	3	3
3	3	3	3	3	3	3	3	3	a	a	b	a	a	b	a	a	b	a	a	b	a	a	b	a	a	b
b	c	c	a	a	b	b	c	c	b	c	c	b	c	c	b	c	c	b	c	c	b	c	c	b	c	c
y	x	x	x	y	x	x	x	y	x	y	x	y	x	x	x	x	y	x	y	x	y	x	x	x	x	y
z	y	z	z	z	y	z	y	z	z	z	y	z	y	z	y	z	z	z	z	y	z	y	z	y	z	z

Example 3.4.6. 3-GDD(3, 3, 5, 6, 4) exists and its blocks are obtained by joining blocks of Example 3.4.1 and Example 3.4.2.

1	1	1	1	1	1	2	2	2	3	1	2	1	2	3	2	3	1	1	1	2	1	1	2	1	1	2
2	2	2	3	3	3	3	3	3	a	a	a	a	a	a	b	b	b	2	3	3	2	3	3	2	3	3
a	a	b	a	a	b	a	a	b	b	b	b	c	c	c	c	c	c	a	b	c	b	c	a	c	a	b
b	c	c	b	c	c	b	c	c	x	x	y	x	x	y	x	x	y	x	x	x	x	x	x	y	y	y
y	z	x	z	x	y	x	y	z	y	z	z	y	z	z	y	z	z	y	y	y	z	z	z	z	z	z

1	1	1	1	1	1	a	a	a	a	a	a	x	x	x	x	x	x
2	2	2	2	2	2	b	b	b	b	b	b	y	y	y	y	y	y
3	3	3	3	3	3	c	c	c	c	c	c	z	z	z	z	z	z
a	a	b	x	x	y	1	1	2	x	x	y	1	1	2	a	a	b
b	c	c	y	z	z	2	3	3	y	z	z	2	3	3	b	c	c

For $t \geq 1$, taking t copies of Example 3.4.6 we have 3-GDD$(3, 3, 5, 6t, 4t)$.

Theorem 3.4.7. *There exists a* 3-*GDD*$(3,\ 3,\ 5,\ 9q+3p,\ 11q-3p)$ *for* $p \leq q$.

Proof. Blocks of the design are obtained by taking p copies of 3-GDD(3, 3, 5, 12, 8) together with $q-p$ copies of Example 3.4.5. □

Corollary 3.4.8. *There exists a* 3-*GDD*$(3,\ 3,\ 5,\ 9q+3p+6,\ 11q-3p+4)$ *for* $p \leq q$.

Proof. Taking blocks of 3-GDD$(3,\ 3,\ 5,\ 9q+3p,\ 11q-3p)$ together with blocks of Example 3.4.6 give 3-GDD$(3,\ 3,\ 5,\ 9q+3p+6,\ 11q-3p+4)$. □

Corollary 3.4.9. *For all* $p \geq 1$, 3-$(9,\ 5,\ 30p)$ *exists.*

Proof. Blocks of such design are obtained by taking p-copies of 3-GDD(3, 3, 5, 12, 8) together with $2p$-copies of Example 3.4.5. □

Corollary 3.4.10. *There exists a* 3-$(9,\ 5,\ 30p+15)$ *for all* $p \geq 0$.

Proof. For $p \geq 0$, blocks of 3-$(9,\ 5,\ 30p+15)$ are obtained by joining p-copies of blocks of 3-GDD(3, 3, 5, 12, 8), $2p+1$-copies of blocks of Example 3.4.5 and blocks of Example 3.4.6
. □

Example 3.4.11. There exists a 3-$(9, 5, 15)$ and its blocks are obtained by combining the blocks of 3-GDD$(3, 3, 5, 9, 11)$ and 3-GDD$(3, 3, 5, 6, 4)$.

1	1	1	1	1	1	1	1	1	a	a	a	a	a	a	a	a	a	x	x	x	x	x	x	x	x	x
2	2	2	2	2	2	2	2	2	b	b	b	b	b	b	b	b	b	y	y	y	y	y	y	y	y	y
3	3	3	3	3	3	3	3	3	c	c	c	c	c	c	c	c	c	z	z	z	z	z	z	z	z	z
a	a	a	b	b	b	c	c	c	1	1	1	2	2	2	3	3	3	1	1	1	2	2	2	3	3	3
x	y	z	x	y	z	x	y	z	x	y	z	x	y	z	x	y	z	a	b	c	a	b	c	a	b	c
1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	2	2	2	2	2	2	3	3	3
a	a	a	a	b	b	a	a	a	a	b	b	a	a	a	a	b	b	a	a	b	b	c	c	a	a	b
b	b	c	c	c	c	b	b	c	c	c	c	b	b	c	c	c	c	x	x	x	y	x	y	x	y	x
x	y	x	z	y	z	y	z	x	y	x	z	x	z	y	z	x	y	y	z	y	z	z	z	y	z	z
1	1	1	2	2	2	2	2	2	1	1	1	2	2	2	3	3	3	1	1	1	2	2	2	3	3	3
3	3	3	3	3	3	3	3	3	a	a	b	a	a	b	a	a	b	a	a	b	a	a	b	a	a	b
b	c	c	a	a	b	b	c	c	b	c	c	b	c	c	b	c	c	b	c	c	b	c	c	b	c	c
y	x	x	x	y	x	x	x	y	x	y	x	y	x	x	x	x	y	x	y	x	y	x	x	x	x	y
z	y	z	z	z	y	z	y	z	z	z	y	z	y	z	y	z	z	z	z	y	z	y	z	y	z	z
1	1	1	1	1	1	2	2	2	3	1	2	1	2	3	2	3	1	1	1	2	1	1	2	1	1	2
2	2	2	3	3	3	3	3	3	a	a	a	a	a	a	b	b	b	2	3	3	2	3	3	2	3	3
a	a	b	a	a	b	a	a	b	b	b	b	c	c	c	c	c	c	a	b	c	b	c	a	c	a	b
b	c	c	b	c	c	b	c	c	x	x	y	x	x	y	x	x	y	x	x	x	x	x	x	y	y	y
y	z	x	z	x	y	x	y	z	y	z	z	y	z	z	y	z	z	y	y	y	z	z	z	z	z	z

1	1	1	1	1	1	a	a	a	a	a	a	x	x	x	x	x	x
2	2	2	2	2	2	b	b	b	b	b	b	y	y	y	y	y	y
3	3	3	3	3	3	c	c	c	c	c	c	z	z	z	z	z	z
a	a	b	x	x	y	1	1	2	x	x	y	1	1	2	a	a	b
b	c	c	y	z	z	2	3	3	y	z	z	2	3	3	b	c	c

The next Theorem discusses the situation in which the existence of a $3'$-GDD(Definition 3.1.2) guarantees the existence of 3-PBIBD(Definition 3.1.5).

Theorem 3.4.12. *(Block Complementation)*
Let $k \leq mn - 3$. If a $3'$-GDD(n, m, k, Λ_1, Λ_2) exists, then a 3-PBIBD(n, m, $nm - k$, θ_1, θ_2, θ_3) also exists, where
$\theta_1 = b - 3r + 3\lambda_1 - \Lambda_1$, $\theta_2 = b - 3r + \lambda_1 + 2\lambda_2 - \Lambda_2$, $\theta_3 = b - 3r + 3\lambda_2 - \Lambda_2$. *$r$, λ_1 & λ_2*

respectively are the replication number, number of first associate pairs and number of second associate pairs in the $3'$-GDD.

Proof. Suppose a $3'$-GDD(n, m, k, Λ_1, Λ_2) exists.
Let (X, $\mathbb{B}$) is a 3-GDD(n, m, k, Λ_1, Λ_2), where X is a mn set of points partitioned in to m groups of size n each, and $\mathbb{B}$ is a collection of k-subsets(blocks) of X.
To Show (X,$\{X\backslash\beta$: $\beta\in\mathbb{B}\}$) is a 3-PBIBD(n, m, $mn-k$, θ_1, θ_2, θ_3).
This design has mn points partitioned in to m-parts(groups) of size n, and every block contains $mn-k\geq 3$ symbols.
To show every triple of type:

(i). $(3,0)$ occurs in $\theta_1{=}b-3r+3\lambda_1-\Lambda_1$ blocks.

(ii). $(2,1)$ occurs in $\theta_2{=}b-3r+\lambda_1+2\lambda_1-\Lambda_2$ blocks.

(iii). $(1,1,1)$ occurs in $\theta_3{=}b-3r+3\lambda_2-\Lambda_2$ blocks.

(i). To show every triple of type $(3,0)$ occurs in $\theta_1{=}b-3r+3\lambda_1-\Lambda_1$ blocks.
Let (x,y,z) is triple of type $(3,0)$ in (X, $\mathbb{B}$). There are $3r-3\lambda_1+\Lambda_1$ blocks containing x or y or z(containing at least one of them), which means $b-3r+3\lambda_1-\Lambda_1$ blocks in (X, $\mathbb{B}$) do not contain any one of the three elements.
Therefore, in (X, $\{X\backslash\beta$:$\beta\in\mathbb{B}\}$), (x,y,z) occurs in $\theta_1{=}b-3r+3\lambda_1-\Lambda_1$ blocks.

(ii). To show every triple of type $(2,1)$ occurs in $\theta_2{=}b-3r+\lambda_1+2\lambda_2-\Lambda_2$ blocks.
Let (x,y,z) is triple of type $(2,1)$ in (X, $\mathbb{B}$). There are $3r-\lambda_1-2\lambda_2+\Lambda_2$ blocks containing x or y or z(each block containing (x,y,z) contains one first and two second associate pairs).
Thus, in (X, $\{X\backslash\beta$:$\beta\in\mathbb{B}\}$), there are $\theta_2{=}b-3r+\lambda_1+2\lambda_1-\Lambda_2$ blocks containing x, y and z.

(iii). Similarily, if (x,y,z) is triple of type $(1,1,1)$ in (X, $\mathbb{B}$), there are $3r-3\lambda_2+\Lambda_2$ blocks containing x or y or z(and each block containing (x,y,z) contains three second associate pairs), and as a result there are $\theta_3{=}b-3r+3\lambda_2-\Lambda_2$ blocks not containing the triple.
Thus, in (X, $\{X\backslash\beta$:$\beta\in\mathbb{B}\}$), there are $\theta_3{=}b-3r+3\lambda_2-\Lambda_2$ blocks containing x, y and z. □

Corollary 3.4.13. *(a). If a $3'$-GDD(3, 3, 4, Λ_1, Λ_2) exists then 3-GDD(3, 3, 5, $\frac{9\Lambda_2+\Lambda_1}{4}$, $\frac{11\Lambda_2-\Lambda_1}{4}$) exists.*

(b). If a 3-GDD(3, 3, 5, μ_1, μ_2) exists, then $3'$-GDD(3, 3, 4, $\frac{11\mu_1-9\mu_2}{5}$, $\frac{\mu_1+\mu_2}{5}$) exists.

Proof.

(a). Suppose $3'$-GDD(3, 3, 4, Λ_1, Λ_2) exists.
By Theorem 3.4.12, when $m = 3$ and $k = 4$, a 3-PBIBD(3, 3, 5, θ_1, θ_2, θ_3) exists, and $n = 3$ yields θ_1=θ_2=$\frac{9\Lambda_2+\Lambda_1}{4}$ and θ_3=$\frac{11\Lambda_2-\Lambda_1}{4}$.

(b). If a 3-GDD(3, 3, 5, μ_1, μ_2) exists, then taking the complement of its blocks gives a 3-PBIBD(3, 3, 4, β_1, β_2, β_3), where $\beta_1 = \frac{11\mu_1-9\mu_2}{5}$, β_2=β_3=$\frac{\mu_1+\mu_2}{5}$. $\square$

As a result of Corollary 3.4.13(b), we have the following examples of a $3'$-GDDs when $k = 4$.

Example 3.4.14. A $3'$-GDD(3, 3, 4, 6, 2) exists and its blocks are obtained by taking the complement of the blocks of Example 3.4.6.

Example 3.4.15. A 3-(9, 4, 6) exists. Its blocks are obtained by combining the complements of the blocks of Example 3.4.5 and Example 3.4.6.

1	1	1	1	1	1	1	1	1	a	a	a	a	a	a	a	a	a	x	x	x	x	x	x	x	x	x
2	2	2	2	2	2	2	2	2	b	b	b	b	b	b	b	b	b	y	y	y	y	y	y	y	y	y
3	3	3	3	3	3	3	3	3	c	c	c	c	c	c	c	c	c	z	z	z	z	z	z	z	z	z
a	a	a	b	b	b	c	c	c	1	1	1	2	2	2	3	3	3	1	1	1	2	2	2	3	3	3
x	y	z	x	y	z	x	y	z	x	y	z	x	y	z	x	y	z	a	b	c	a	b	c	a	b	c
1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	2	2	2	2	2	2	3	3	3
a	a	a	a	b	b	a	a	a	a	b	b	a	a	a	a	b	b	a	a	b	b	c	c	a	a	b
b	b	c	c	c	c	b	b	c	c	c	c	b	b	c	c	c	c	x	x	x	y	x	y	x	y	x
x	y	x	z	y	z	y	z	x	y	x	z	x	z	y	z	x	y	y	z	y	z	z	z	y	z	z
1	1	1	2	2	2	2	2	2	1	1	1	2	2	2	3	3	3	1	1	1	2	2	2	3	3	3
3	3	3	3	3	3	3	3	3	a	a	b	a	a	b	a	a	b	a	a	b	a	a	b	a	a	b
b	c	c	a	a	b	b	c	c	b	c	c	b	c	c	b	c	c	b	c	c	b	c	c	b	c	c
y	x	x	x	y	x	x	x	y	x	y	x	y	x	x	x	x	y	x	y	x	y	x	x	x	x	y
z	y	z	z	z	y	z	y	z	z	z	y	z	y	z	y	z	z	z	z	y	z	y	z	y	z	z

1	1	1	1	1	1	2	2	2	3	1	2	1	2	3	2	3	1	1	1	2	1	1	2	1	1	2
2	2	2	3	3	3	3	3	3	a	a	a	a	a	a	b	b	b	2	3	3	2	3	3	2	3	3
a	a	b	a	a	b	a	a	b	b	b	b	c	c	c	c	c	c	a	b	c	b	c	a	c	a	b
b	c	c	b	c	c	b	c	c	x	x	y	x	x	y	x	x	y	x	x	x	x	x	x	y	y	y
y	z	x	z	x	y	x	y	z	y	z	z	y	z	z	y	z	z	y	y	y	z	z	z	z	z	z

1	1	1	1	1	1	a	a	a	a	a	a	x	x	x	x	x	x
2	2	2	2	2	2	b	b	b	b	b	b	y	y	y	y	y	y
3	3	3	3	3	3	c	c	c	c	c	c	z	z	z	z	z	z
a	a	b	x	x	y	1	1	2	x	x	y	1	1	2	a	a	b
b	c	c	y	z	z	2	3	3	y	z	z	2	3	3	b	c	c

In general, taking the complements of the the blocks of Corollary 3.4.10, the following result follows.

Corollary 3.4.16. *For all* $p \geq 0$*, a* 3-(9, 4, $12p + 6$) *exists.*

Chapter 4

3-GDDs With Four Groups and Block Size Five

This chapter continues to focus on the definition of 3-GDDs and we explicitly consider the case when the required designs have four groups of size n each and block size five. Throughout this chapter, such a GDD is denoted by 3-GDD(n, 4, 5, μ_1, μ_2).
This chapter is organized as follows: Section 4.1 presents some well-known definitions and concepts that will be used in the succeeding section. Section 4.2 is the result section and it presents and discusses the finding of our work published in [46].

4.1 Preliminary results

In this section, we review some important definitions and concepts that will be used later to obtain the main results.

Definition 4.1.1. [37] A 3-GDD$(n, 2, k, \lambda_1, \lambda_2)$ is a set X of $2n$ elements partitioned into two parts of size n, called groups, together with a collection of k-subsets of X, called blocks, such that

(i) every 3-subset of each group occurs in λ_1 blocks and

(ii) every 3-subset where two elements are from one group and one element from the other group occurs in λ_2 blocks.

Example 4.1.2. A 3-GDD$(3, 2, 4, 3, 1)$: Let $X = \{1, 2, 3, a, b, c\}$, G_1 =$\{1, 2, 3\}$ and $G_2 = \{a, b, c\}$.
Then $\mathbb{B} = \{\{1, 2, 3, a\}, \{1, 2, 3, b\}, \{1, 2, 3, c\}, \{a, b, c, 1\}, \{a, b, c, 2\}, \{a, b, c, 3\}\}$ gives the required blocks of the GDD.

The above definition can be easily generalized to the following definition.

Definition 4.1.3. A 3-GDD(n, m, k, μ_1, μ_2) is a pair $(X, \mathbb{B})$ where X is a set of mn elements partitioned into m n-subsets (groups) and $\mathbb{B}$ is a collection of k-subsets (blocks) of X such that

(i) every triple occurs exactly μ_1 times if it contains elements from at most 2 groups,

(ii) it occurs exactly μ_2 times if it has all three elements from different groups.

We extrapolate the above definition to a new type of a t-design.

Definition 4.1.4. A 3-PBIBD (3 Partially Balanced Incomplete Block Design) denoted by 3-PBIBD$(n, m, k, \Lambda_1, \Lambda_2, \Lambda_3)$, is a 3-design where

(i) Every triple formed from elements of only a single group occurs in Λ_1 blocks of the 3-design.

(ii) Every triple formed from elements of only two groups occurs in Λ_2 blocks of the 3-design.

(iii) Every triple formed from elements of all three groups occurs in Λ_3 blocks of the 3-design.

A 3-GDD is a 3-PBIBD$(n, 4, 5, \Lambda_1, \Lambda_2, \Lambda_3)$ where $\Lambda_1 = \Lambda_2$, denoted by μ_1, while Λ_3 is denoted by μ_2.
Definition 4.1.3, is the subject matter of this chapter and we explicitly consider the case in which $m = 4$ and $k = 5$. Throughout this chapter, such a GDD is denoted by 3-GDD$(n, 4, 5, \mu_1, \mu_2)$.

Remark 4.1.5. A 3-GDD$(n, 4, 5, \mu_1, \mu_2)$ where $\mu_1 = \mu_2$ is the same as a 3-$(4n, 5, \mu_1)$. Hence a 3-GDD$(n, 4, 5, \mu_1, \mu_1)$ exists if and only if a 3-$(4n, 5, \mu_1)$ exists.

4.2 New Results

Assuming a 3-GDD(n, 4, 5, μ_1, μ_2) exists, we count the number of blocks containing any given element(called the replication number r), the number of blocks, say λ_1, containing a given first associate pair, the number of blocks, say λ_2, containing a given second associate pair and the required number of blocks, say b, for the design.

4.2.1 Necessary Conditions

Theorem 4.2.1. *In 3-PBIBD(n, m, k, Λ_1, Λ_2, Λ_3) with $k = 5$ and $m = 4$:*

$$r = \frac{(n-1)(n-2)\Lambda_1 + 9n(n-1)\Lambda_2 + 6n^2\Lambda_3}{12} \tag{4.1}$$

$$b = \frac{n}{15}((n-1)(n-2)\Lambda_1 + 9n(n-1)\Lambda_2 + 6n^2\Lambda_3) \tag{4.2}$$

$$\lambda_1 = \frac{(n-2)\Lambda_1 + 3n\Lambda_2}{3} \tag{4.3}$$

$$\lambda_2 = \frac{2(n-1)\Lambda_2 + 2n\Lambda_3}{3} \tag{4.4}$$

Proof. **(i).** We count the number of triples containing a fixed element x in the design in two ways.

First way: Given an element x, first it appears in $\binom{n-1}{2}$ triples of the type $(3,0)$. Second it appears in a total of $3n(n-1)$+$3\binom{n}{2}$ triples of the type $(2,1)$ because there are $3n(n-1)$ triples of the type $(2,1)$ where x and an element from the same group as x appear with an element of another group and there are $3\binom{n}{2}$ triples of type $(2,1)$ where x appears with two elements from another group. Third, there are $3n^2$ triples of the type $(1,1,1)$ containing x, and these triples are repeated Λ_1, Λ_2 and Λ_3 times respectively.

Hence there are $\binom{n-1}{2}\Lambda_1$+ $(3n(n-1)$+$3\binom{n}{2})\Lambda_2$ +$3n^2\Lambda_3$ triples containing x.

Second way: In every block containing x, there are 6 triples containing x and x occurs in r blocks, which means there are $6r$ triples containing x.

Hence, equating the two expressions thus obtained we have

$$6r=\binom{n-1}{2}\Lambda_1+ (3n(n-1)+3\binom{n}{2})\Lambda_2 +3n^2\Lambda_3$$

and

$$r=\tfrac{(n-1)(n-2)\Lambda_1+9n(n-1)\Lambda_2+6n^2\Lambda_3}{12}.$$

(ii) We count the number of triples in the design in two ways.
First way: There are 4 groups and the block size 5, and hence the number of triples in the design are $4\binom{n}{3}\Lambda_1$+$6n^2(n-1)\Lambda_2$+$4n^3\Lambda_3$.
Second way: There are b blocks in the design, hence the number of triples are $\binom{5}{3}b$.
Equating we get :

$$\binom{5}{3}b= 4\binom{n}{3}\Lambda_1+6n^2(n-1)\Lambda_2+4n^3\Lambda_3.$$

Hence we have:

$$b=\tfrac{n}{15}((n-1)(n-2)\Lambda_1+9n(n-1)\Lambda_2+6n^2\Lambda_3).$$

(iii). Now we count the number of blocks containing a given pair of points, say (x_1, x_2), from the same group in two ways.

(a). There are $(n-2)$ triples of the type$(3,0)$ and $3n$ triples of the type $(2,1)$ containing (x_1, x_2). In sum, there are $(n-2)\Lambda_1$ + $3n\Lambda_2$ triples containing (x_1, x_2).

(b). In a design with block size 5, the block containing (x_1, x_2) contains 3 triples containing (x_1, x_2). Since (x_1, x_2) occurs in λ_1 blocks, there are $3\lambda_1$ such triples.
Since (a) and (b) counts the same,

$$3\lambda_1=(n-2)\Lambda_1 + 3n\Lambda_2$$

and

$$\lambda_1=\tfrac{(n-2)\Lambda_1+3n\Lambda_2}{3}.$$

(iv). Again counting the number of blocks containing a given pair of points from different groups, say (x, y) in two ways:
First, the pair occurs in triples of the type $(2,1)$ and (1, 1, 1) only. It occurs in $2(n-1)\Lambda_2$ triples of type $(2,1)$ or $2n\Lambda_3$ triples of type (1, 1, 1).
Hence, there are $2(n-1)\Lambda_2 + 2n\Lambda_3$ triples containing the pair (x, y).
Second, each of the λ_2 blocks containing the pair (x, y) has three triples containing (x, y). Hence, there are $3\lambda_2$ triples containing (x, y).
Since both ways count the same number of specific blocks, the result follows. □

Example 4.2.2. A 3-PBIBD$(2, 4, 5, 0, 4, 7)$ exists.

If $G_1 = \{1, 2\}$, $G_2 = \{a, b\}$, $G_3 = \{x, y\}$ and G_4=$\{w, z\}$, then the blocks are constructed by combining both elements of a group with every-three subsets(taken from distinct groups) of the union of the remaining three groups.

1	1	1	1	1	1	1	1	a	a	a	a	a	a	a	a
2	2	2	2	2	2	2	2	b	b	b	b	b	b	b	b
a	a	a	a	b	b	b	b	1	1	1	1	2	2	2	2
x	x	y	y	x	x	y	y	x	x	y	y	x	x	y	y
w	z	w	z	w	z	w	z	w	z	w	z	w	z	w	z
x	x	x	x	x	x	x	x	w	w	w	w	w	w	w	w
y	y	y	y	y	y	y	y	z	z	z	z	z	z	z	z
1	1	1	1	2	2	2	2	1	1	1	1	2	2	2	2
a	a	b	b	a	a	b	b	a	a	b	b	a	a	b	b
w	z	w	z	w	z	w	z	x	y	x	y	x	y	x	y

Example 4.2.3. 3-PBIBD(3, 4, 5, 36, 9, 3) exists.

If $G_1 = \{1, 2, 3\}$, $G_2 = \{a, b, c\}$, $G_3 = \{x, y, z\}$ and $G_4 = \{w, s, t\}$, then the blocks of the design are constructed by joining G_i for $i = 1, 2, 3, 4$ with every two subsets of H=$\bigcup_{j=1, j\neq i}^{4} G_j$.

Example 4.2.4. A 3-PBIBD$(4, 4, 5, 12, 1, 0)$ exists.

With $G_1 = \{1, 2, 3, 4\}$, $G_2 = \{a, b, c, d\}$, $G_3 = \{x, y, z, w\}$ and $G_4 = \{r, s, t, u\}$, the blocks of this design are obtained by joining all four elements of a group with every element of the union of the remaining three groups.

1	1	1	1	1	1	1	1	1	1	1	1	a	a	a	a	a	a	a	a	a	a	a	a
2	2	2	2	2	2	2	2	2	2	2	2	b	b	b	b	b	b	b	b	b	b	b	b
3	3	3	3	3	3	3	3	3	3	3	3	c	c	c	c	c	c	c	c	c	c	c	c
4	4	4	4	4	4	4	4	4	4	4	4	d	d	d	d	d	d	d	d	d	d	d	d
a	b	c	d	x	y	z	w	r	s	t	u	1	2	3	4	x	y	z	w	r	s	t	u
x	x	x	x	x	x	x	x	x	x	x	x	r	r	r	r	r	r	r	r	r	r	r	r
y	y	y	y	y	y	y	y	y	y	y	y	s	s	s	s	s	s	s	s	s	s	s	s
z	z	z	z	z	z	z	z	z	z	z	z	t	t	t	t	t	t	t	t	t	t	t	t
w	w	w	w	w	w	w	w	w	w	w	w	u	u	u	u	u	u	u	u	u	u	u	u
1	2	3	4	a	b	c	d	r	s	t	u	1	2	3	4	a	b	c	d	x	y	z	w

In a 3-PBIBD(n, 4, 5, Λ_1, Λ_2, Λ_3), when $\Lambda_1 = \Lambda_2$ is denoted by μ_1 and Λ_3 is denoted by μ_2, from Theorem 4.2.1, the following result follows;

Corollary 4.2.5. *In 3-GDD(n, 4, 5, μ_1, μ_2), the replication number, r, number of blocks, b, number of blocks containing a first associate pair, λ_1, and number of blocks containing a second associate pair, λ_2, are calculated as follows:*

$$r = \frac{(n-1)(5n-1)\mu_1 + 3n^2\mu_2}{6} \tag{4.5}$$

$$b = \frac{2n}{15}((n-1)(5n-1)\mu_1 + 3n^2\mu_2) \tag{4.6}$$

$$\lambda_1 = \frac{(4n-2)}{3}\mu_1 \tag{4.7}$$

$$\lambda_2 = \frac{2(n-1)\mu_1 + 2n\mu_2}{3} \tag{4.8}$$

Corollary 4.2.6. *In a 3-GDD(n, 4, 5, μ_1, μ_2), $\mu_1 \neq 0$ and $\lambda_1 \neq 0$.*

Proof. As the number of groups is less than the block size, $\mu_1 \neq 0$. Again from (4.7), λ_1 is a multiple of μ_1, and hence $\lambda_1 \neq 0$ □

From Equations 4.5 and 4.6, the necessary conditions are satisfied under the following conditions:

Lemma 4.2.7. *For*

i. $n \equiv 0 \pmod 5$, *μ_1 and μ_2 are free in regards to the number of blocks b.*

ii. $n \equiv 0 \pmod 6$ *and* $\mu_1 \equiv 0 \pmod 6$, *r is an integer for any chosen μ_2.*

iii. $n \equiv 1 \pmod 6$ *and even μ_2, r is an integer for any chosen μ_1.*

iv. $n \equiv 4 \pmod 6$ *and even μ_1, r is an integer for any chosen μ_2.*

From Equations 4.7 and 4.8, a more compact necessary condition is given as follows:

Lemma 4.2.8. *In regards to:*

i. *λ_1, for $n \equiv 2 \pmod 3$, μ_1 and μ_2 are free and for $n \not\equiv 2 \pmod 3$, $\mu_1 \equiv 0 \pmod 3$ and μ_2 is free.*

ii. λ_2, *for* $n \equiv 0 \pmod 3$, $\mu_1 \equiv 0 \pmod 3$ *and* μ_2 *is free, when* $n \equiv 1 \pmod 3$, μ_1 *is free and* $\mu_2 \equiv 0 \pmod 3$, *and for* $n \equiv 2 \pmod 3$, $\mu_1 + 2\mu_2 \equiv 0 \pmod 3$.

• From Corollary 4.2.5, as the values of b and r must be integers, we have table congruence restrictions (all values are considered to be in terms of $\pmod{30}$ unless otherwise stated) in Table 4.1.

Table 4.1: Congruence restrictions for 3-GDD(n, 4, 5, μ_1, μ_2)

	r		b	
n	μ_1	μ_2	μ_1	μ_2
0	$0 \pmod 6$	All	All	All
1,11	All	$0 \pmod 2$	All	$0 \pmod 5$
2, 7, 17	$0 \pmod 2$	All	$*_1$	$*_1$
3, 9	$*_2$	$*_2$	$*_3$	$*_3$
4, 8, 14, 28	$0 \pmod 2$	All	$*_3$	$*_3$
5, 25	All	$0 \pmod 2$	All	All
6	$0 \pmod 6$	All	All	$0 \pmod 5$
10, 20	$0 \pmod 2$	All	All	All
12	$0 \pmod 6$	All	$*_1$	$*_1$
13, 19, 23, 29	All	$0 \pmod 2$	$*_3$	$*_3$
15	$*_2$	$*_2$	All	All
16, 26	$0 \pmod 2$	All	All	$0 \pmod 5$
18, 24	$0 \pmod 6$	All	$*_3$	$*_3$
21	$*_2$	$*_2$	All	$0 \pmod 5$
22	$0 \pmod 2$	All	$*_1$	$*_1$
27	$*_2$	$*_2$	$*_1$	$*_1$

where,

$*$ denotes the set of all ordered pairs (μ_1, μ_2) such that
$$\mu_1 + 2\mu_2 \equiv 0 \pmod 3$$
$*_1$ denotes the set of all ordered pairs (μ_1, μ_2) such that
$$\mu_1 + 3\mu_2 \equiv 0 \pmod 5$$

$*_2$ denotes the set of all ordered pairs (μ_1, μ_2) such that
$$4\mu_1 + 3\mu_2 \equiv 0 \pmod 6$$
$*_3$ denotes the set of all ordered pairs (μ_1, μ_2) such that
$$\mu_1 + 4\mu_2 \equiv 0 \pmod 5$$

Remark 4.2.9. From Table 4.1 and Lemma 4.2.8, we have more compact necessary conditions for different values of n as follows:

• $n \equiv 0 \pmod{30}$, for all μ_2 and $\mu_1 \equiv 0 \pmod 6$

• $n \equiv 1 \pmod{30}$, $\mu_1 \equiv 0 \pmod 3$ and $\mu_2 \equiv 0 \pmod{30}$.

• $n \equiv 4, 28 \pmod{30}$, $\mu_1 \equiv 0 \pmod{30}$ and $\mu_2 \equiv 0 \pmod{15}$ or, $\mu_1 \equiv 6 \pmod{30}$ and $\mu_2 \equiv 6 \pmod{15}$ or, $\mu_1 \equiv 12 \pmod{30}$ and $\mu_2 \equiv 12 \pmod{15}$ or,$\mu_1 \equiv 18 \pmod{30}$ and $\mu_2 \equiv 3 \pmod{15}$ or, $\mu_1 \equiv 24 \pmod{30}$ and $\mu_2 \equiv 9 \pmod{15}$.

• $n \equiv 5 \pmod{30}$, $\mu_1 \equiv 0 \pmod 3$ and $\mu_2 \equiv 0 \pmod 6$ or, $\mu_1 \equiv 2 \pmod 3$ and $\mu_2 \equiv 2 \pmod 6$ or,$\mu_1 \equiv 1 \pmod 3$ and $\mu_2 \equiv 4 \pmod 6$.

• $n \equiv 6 \pmod{30}$, $\mu_1 \equiv 0 \pmod 6$ and $\mu_2 \equiv 0 \pmod 5$.

• $n \equiv 7 \pmod{30}$, $\mu_1 \equiv 0 \pmod{15}$ and $\mu_2 \equiv 0 \pmod{30}$ or, $\mu_1 \equiv 3 \pmod{15}$ and $\mu_2 \equiv 24 \pmod{30}$ or, $\mu_1 \equiv 6 \pmod{15}$ and $\mu_2 \equiv 18 \pmod{30}$ or, $\mu_1 \equiv 9 \pmod{15}$ and $\mu_2 \equiv 12 \pmod{30}$ or, $\mu_1 \equiv 12 \pmod{15}$ and $\mu_2 \equiv 6 \pmod{30}$.

• $n \equiv 10 \pmod{30}$, $\mu_1 \equiv 0 \pmod 6$ and $\mu_2 \equiv 0 \pmod 3$.

• $n \equiv 11 \pmod{30}$, $\mu_1 \equiv 0 \pmod 3$ and $\mu_2 \equiv 0 \pmod{30}$ or, $\mu_1 \equiv 1 \pmod 3$ and $\mu_2 \equiv 10 \pmod{30}$ or, $\mu_1 \equiv 2 \pmod 3$ and $\mu_2 \equiv 20 \pmod{30}$.

• $n \equiv 12 \pmod{30}$, $\mu_1 \equiv 0 \pmod{30}$ and $\mu_2 \equiv 0 \pmod 5$ or, $\mu_1 \equiv 6 \pmod{30}$ and $\mu_2 \equiv 3 \pmod 5$ or, $\mu_1 \equiv 12 \pmod{30}$ and $\mu_2 \equiv 1 \pmod 5$ or, $\mu_1 \equiv 18 \pmod{30}$ and $\mu_2 \equiv 4 \pmod 5$ or, $\mu_1 \equiv 24 \pmod{30}$ and $\mu_2 \equiv 2 \pmod 5$.

• $n \equiv 13, 19 \pmod{30}$, $\mu_1 \equiv 0 \pmod{15}$ and $\mu_2 \equiv 0 \pmod{30}$ or, $\mu_1 \equiv 3 \pmod{15}$ and $\mu_2 \equiv 18 \pmod{30}$ or, $\mu_1 \equiv 6 \pmod{15}$ and $\mu_2 \equiv 6 \pmod{30}$ or, $\mu_1 \equiv 9 \pmod{15}$ and $\mu_2 \equiv 24 \pmod{30}$ or, $\mu_1 \equiv 12 \pmod{15}$ and $\mu_2 \equiv 12 \pmod{30}$.

• $n \equiv 15 \pmod{30}$, $\mu_1 \equiv 0 \pmod 3$ and $\mu_2 \equiv 0 \pmod 2$.

• $n \equiv 16 \pmod{30}$, $\mu_1 \equiv 0 \pmod 6$ and $\mu_2 \equiv 0 \pmod{15}$.

• $n \equiv 18, 24 \pmod{30}$, $\mu_1 \equiv 0 \pmod{30}$ and $\mu_2 \equiv 0 \pmod 5$ or, $\mu_1 \equiv 6 \pmod{30}$ and $\mu_2 \equiv 1 \pmod 5$ or, $\mu_1 \equiv 12 \pmod{30}$ and $\mu_2 \equiv 2 \pmod 5$ or, $\mu_1 \equiv 18 \pmod{30}$ and $\mu_2 \equiv 3 \pmod 5$ or, $\mu_1 \equiv 24 \pmod{30}$ and $\mu_2 \equiv 4 \pmod 5$.

• $n \equiv 20 \pmod{30}$, $\mu_1 \equiv 0 \pmod 6$ and $\mu_2 \equiv 0 \pmod 3$ or, $\mu_1 \equiv 2 \pmod 6$ and $\mu_2 \equiv 2$

(mod 3) or, $\mu_1 \equiv 4 \pmod 6$ and $\mu_2 \equiv 1 \pmod 3$.

• $n \equiv 21 \pmod{30}$, $\mu_1 \equiv 0 \pmod 3$ and $\mu_2 \equiv 0 \pmod{10}$.

• $n \equiv 22 \pmod{30}$, $\mu_1 \equiv 0 \pmod{30}$ and $\mu_2 \equiv 0 \pmod{15}$, $\mu_1 \equiv 6 \pmod{30}$ and $\mu_2 \equiv 3 \pmod{15}$, $\mu_1 \equiv 12 \pmod{30}$ and $\mu_2 \equiv 6 \pmod{15}$, $\mu_1 \equiv 18 \pmod{30}$ and $\mu_2 \equiv 9 \pmod{15}$, $\mu_1 \equiv 24 \pmod{30}$ and $\mu_2 \equiv 12 \pmod{15}$.

• $n \equiv 25 \pmod{30}$, $\mu_1 \equiv 0 \pmod 3$ and $\mu_2 \equiv 0 \pmod 6$.

• $n \equiv 26 \pmod{30}$, $\mu_1 \equiv 0 \pmod 6$ and $\mu_2 \equiv 0 \pmod{15}$ or, $\mu_1 \equiv 2 \pmod 6$ and $\mu_2 \equiv 5 \pmod{15}$ or, $\mu_1 \equiv 4 \pmod 6$ and $\mu_2 \equiv 10 \pmod{15}$.

⊛ When $n \equiv 2 \pmod{30}$, $\mu_1 \equiv 0 \pmod 2 \cap *_1 \cap *$, and $\mu_2 \equiv *_1 \cap *$, when $*_1$ and $*$ are as given above, and

$\cap$ denotes intersection where we are abusing the notation not just here but in what follows. When we use congruence relations like $\mu_1 \equiv 0 \pmod 2$ and $*_1$ with $\cap$, we mean it represents the intersection of the set of all ordered pairs (μ_1, μ_2) where μ_2 is any number but μ_1 is only even (i.e. satisfies the used congruence $\mu_1 \equiv 0 \pmod 2$) with the set of ordered pairs (μ_1, μ_2) such that $\mu_1 + 3\mu_2 \equiv 0 \pmod 5$.

Taking all the possible combinations, the original necessary conditions in Corollary 4.2.5 are satisfied for:

• $\mu_1 \equiv 0 \pmod{30}$ and $\mu_2 \equiv 0 \pmod{15}$ or, $\mu_1 \equiv 2 \pmod{30}$ and $\mu_2 \equiv 11 \pmod{15}$ or, $\mu_1 \equiv 4 \pmod{30}$ and $\mu_2 \equiv 7 \pmod{15}$ or, $\mu_1 \equiv 6 \pmod{30}$ and $\mu_2 \equiv 3 \pmod{15}$ or, $\mu_1 \equiv 8 \pmod{30}$ and $\mu_2 \equiv 14 \pmod{15}$ or, $\mu_1 \equiv 10 \pmod{30}$ and $\mu_2 \equiv 10 \pmod{15}$ or, $\mu_1 \equiv 12 \pmod{30}$ and $\mu_2 \equiv 6 \pmod{15}$ or, $\mu_1 \equiv 14 \pmod{30}$ and $\mu_2 \equiv 2 \pmod{15}$ or, $\mu_1 \equiv 16 \pmod{30}$ and $\mu_2 \equiv 13 \pmod{15}$ or, $\mu_1 \equiv 18 \pmod{30}$ and $\mu_2 \equiv 9 \pmod{15}$ or, $\mu_1 \equiv 20 \pmod{30}$ and $\mu_2 \equiv 5 \pmod{15}$ or, $\mu_1 \equiv 22 \pmod{30}$ and $\mu_2 \equiv 1 \pmod{15}$ or, $\mu_1 \equiv 24 \pmod{30}$ and $\mu_2 \equiv 12 \pmod{15}$ or, $\mu_1 \equiv 26 \pmod{30}$ and $\mu_2 \equiv 8 \pmod{15}$ or, $\mu_1 \equiv 28 \pmod{30}$ and $\mu_2 \equiv 4 \pmod{15}$.

• When $n \equiv 3, 9 \pmod{30}$, $\mu_1 \equiv 0 \pmod 3 \cap *_2 \cap *_3$, and $\mu_2 \equiv *_2 \cap *_3$.

• When $n \equiv 8, 14 \pmod{30}$, $\mu_1 \equiv 0 \pmod 2 \cap *_3 \cap *$, and $\mu_2 \equiv *_3 \cap *$.

• When $n \equiv 17 \pmod{30}$, $\mu_1 \equiv *_1 \cap *$, and $\mu_2 \equiv 0 \pmod 2 \cap *_1 \cap *$.

• When $n \equiv 23, 29 \pmod{30}$, $\mu_1 \equiv *_3 \cap *$, and $\mu_2 \equiv 0 \pmod 2 \cap *_3 \cap *$.

• When $n \equiv 27 \pmod{30}$, $\mu_1 \equiv 0 \pmod 3 \cap *_2 \cap *_1$, and $\mu_2 \equiv *_2 \cap *_1$.

Where $\cap$ denotes intersection.

⊛ To show how the possible values μ_1 and μ_2 in Remark 4.2.9 are determined, let us take the following two cases.

i. When $n \equiv 3, 9 \pmod{30}$, $\mu_1 \equiv 0 \pmod{3} \cap *_2 \cap *_3$, and $\mu_2 \equiv *_2 \cap *_3$.

ii. When $n \equiv 4, 28 \pmod{30}$

i. When $n \equiv 3, 9 \pmod{30}$, the necessary conditions in Lemma 4.2.8 and Table 4.1 are satisfied for $\mu_1 \equiv 0 \pmod{3} \cap *_2 \cap *_3$, and $\mu_2 \equiv *_2 \cap *_3$.

Where $\mu_1 \equiv 0 \pmod{3} \cap *_2 \cap *_3$ and $\mu_2 \equiv *_2 \cap *_3$, represent the intersection of the set of all ordered pairs (μ_1, μ_2) where μ_2 is any number but μ_1 satisfies the congruence $\mu_1 \equiv 0 \pmod{3}$ with the set of ordered pairs (μ_1, μ_2) such that $*_2$: $4\mu_1 + 3\mu_2 \equiv 0 \pmod{6}$ and the set of ordered pairs (μ_1, μ_2) such that $*_3$: $\mu_1 + 4\mu_2 \equiv 0 \pmod{5}$.

The possible values of μ_1, which is multiple of 3, and the corresponding values of μ_2 satisfying $*_2$: $4\mu_1 + 3\mu_2 \equiv 0 \pmod{6}$ are given in Table 4.2.

Table 4.2: $\mu_1 \equiv 0 \pmod{3}$ and $4\mu_1 + 3\mu_2 \equiv 0 \pmod{6}$

μ_1	μ_2
$0 \pmod{3}$	$0 \pmod{2}$

Again, the values of μ_1 and μ_2 satisfying $*_3$:$\mu_1 + 4\mu_2 \equiv 0 \pmod{5}$, when $\mu_1 \equiv 0 \pmod{3}$ are as given in the Table 4.3.

Table 4.3: $\mu_1 \equiv 0 \pmod{3}$ and $\mu_1 + 4\mu_2 \equiv 0 \pmod{5}$.

μ_1	μ_2
$0 \pmod{15}$	$0 \pmod{5}$
$3 \pmod{15}$	$3 \pmod{5}$
$6 \pmod{15}$	$1 \pmod{5}$
$9 \pmod{15}$	$4 \pmod{5}$
$12 \pmod{15}$	$2 \pmod{5}$

Finally, taking all possible combinations of μ_1 and μ_2 in Table 4.2 and Table 4.3, we get the possible values of μ_1 and μ_2 satisfying $\mu_1 \equiv 0 \pmod{3}$, $*_2$:$4\mu_1 + 3\mu_2 \equiv 0 \pmod{6}$ and $*_3$:$\mu_1 + 4\mu_2 \equiv 0 \pmod{5}$ as shown in the Table 4.4.

Table 4.4: $\mu_1 \equiv 0 \pmod 3$, $4\mu_1 + 3\mu_2 \equiv 0 \pmod 6$ and $\mu_1 + 4\mu_2 \equiv 0 \pmod 5$

μ_1	μ_2
$0 \pmod{15}$	$0 \pmod{10}$
$3 \pmod{15}$	$8 \pmod{10}$
$6 \pmod{15}$	$6 \pmod{10}$
$9 \pmod{15}$	$4 \pmod{10}$
$12 \pmod{15}$	$2 \pmod{10}$

ii. When $n \equiv 4, 28 \pmod{30}$, from Lemma 4.2.8 and Table 4.1, the replication number (r), number of blocks (b), λ_1 which denotes the number of blocks containing a given first associate pair, and λ_2 which denotes the number of blocks containing a given second associate pair are integers when

$$\mu_1 \equiv 0 \pmod 6,\ \mu_2 \equiv 0 \pmod 3 \text{ and } *_3{:}\mu_1 + 4\mu_2 \equiv 0 \pmod 5$$

Now all possible values of (μ_1, μ_2) satisfying $\mu_1 \equiv 0 \pmod 6$ and $*_3{:}\mu_1 + 4\mu_2 \equiv 0 \pmod 5$ as shown in Table 4.5.

Table 4.5: $\mu_1 \equiv 0 \pmod 6$ and $\mu_1 + 4\mu_2 \equiv 0 \pmod 5$.

μ_1	μ_2
$0 \pmod{30}$	$0 \pmod 5$
$6 \pmod{30}$	$1 \pmod 5$
$12 \pmod{30}$	$2 \pmod 5$
$18 \pmod{30}$	$3 \pmod 5$
$24 \pmod{30}$	$4 \pmod 5$

But $\mu_2 \equiv 0 \pmod 3$, when $n \equiv 4, 28 \pmod{30}$. Hence, Table 4.6 summarises the possible values of μ_1 and μ_2 satisfying the original necessary conditions in Corollary 4.2.5.

Table 4.6: $n \equiv 4, 28 \pmod{30}$.

μ_1	μ_2
$0 \pmod{30}$	$0 \pmod{15}$
$6 \pmod{30}$	$6 \pmod{15}$
$12 \pmod{30}$	$12 \pmod{15}$
$18 \pmod{30}$	$3 \pmod{15}$
$24 \pmod{30}$	$9 \pmod{15}$

Theorem 4.2.10. *3-GDD(n, 4, 5, μ_1, 0) doesn't exist.*

Proof. If a 3-GDD(n, 4, 5, μ_1, 0) exists, then blocks of the design are of type $(5, 0)$, $(4, 1)$ or $(3, 2)$ only.
Let x, y and z denotes the number of blocks of type $(5, 0)$, $(4, 1)$ and $(3, 2)$ respectively

(i). Note that for $n = 3$, both x and y are zeros and for $n = 4$, x is zero.

• When $n = 3$, only allowed blocks are of type $(3, 2)$ and such blocks give 9 $(2, 1)$ triples and 1 $(3, 0)$ triple.

So in a design with z blocks, there are z triples of type $(3, 0)$ and $9z$ triples of type $(2, 1)$, but z=$4\mu_1$ (number of triples of type $(3, 0)$) and $9z$=$108\mu_1$(number of triples of type $(2, 1)$). From the two equations, we get $4\mu_1 = 12\mu_1$, which is impossible.

• When $n = 4$, number of triples of type $(3, 0)$ is

$$4y + z = 16\mu_1 \tag{4.9}$$

and number of triples of type $(2, 1)$ is given by

$$6y + 9z = 288\mu_1 \tag{4.10}$$

From Equations 4.9 and 4.10, simultaneously solving for z gives

$$5z = 176\mu_1 \tag{4.11}$$

From Equation 4.11, $z > 35\mu_1$ and hence Equation 4.9 is impossible.

(ii). For $n \geq 5$, the number of triples of type $(3, 0)$ and $(2, 1)$ are respectively given by:

$$10x + 4y + z = \frac{2n(n-1)(n-2)\mu_1}{3} \tag{4.12}$$

and

$$6y + 9z = 6n^2(n-1)\mu_1 \tag{4.13}$$

Solving for z from Equation 4.13 and substituting this value in to Equation 4.12 gives:

$$30x + 10y + 2n^2(n-1)\mu_1 = 2n(n-1)(n-2)\mu_1 \tag{4.14}$$

Since $2n^2(n-1)\mu_1 > 2n(n-1)(n-2)\mu_1$, Equation 4.14 is impossible.

□

Theorem 4.2.11. *Given 3-GDD(n, 4, 5, μ_1, μ_2), $\mu_1 \geq \frac{2n\mu_2}{7(n-1)}$*

Proof. In a 3-GDD(n, 4, 5, μ_1, μ_2), the blocks which have a $(1,1,1)$ are of the form $(2,1,1,1)$, $(3,1,1)$, and $(2,2,1)$. In each of these types of blocks, the number of $(2,1)$ triples are 3, 6 and 6 and $(1,1,1)$ triples are 7, 3 and 4 respectively. So except in the blocks of type $(2,1,1,1)$, the number of $(2,1)$ triples are more in other two types of blocks.
Suppose there are x blocks of type $(3,1,1)$, y blocks of type $(2,2,1)$ and z blocks of type $(2,1,1,1)$.
The the number of $(2,1)$ triples is

$$6x + 6y + 3z = 6n^2(n-1)\mu_1 \tag{4.15}$$

The the number of $(1,1,1)$ triples is

$$3x + 4y + 7z = 4n^3\mu_2 \tag{4.16}$$

From Equation 4.15 $3x + 3y + 3z \leq 6x + 6y + 3z = 6n^2(n-1)\mu_1$, which gives

$$x + y + z \leq 2n^2(n-1)\mu_1 \tag{4.17}$$

From Equation 4.16 $4n^3\mu_2 = 3x + 4y + 7z \leq 7x + 7y + 7z$, and this gives

$$\frac{4n^3\mu_2}{7} \leq x + y + z \tag{4.18}$$

Now, from Equations 4.17 and 4.18, we have

$$\frac{4n^3\mu_2}{7} \leq x + y + z \leq 2n^2(n-1)\mu_1 \tag{4.19}$$

This gives

$$\mu_1 \geq \tfrac{2n\mu_2}{7(n-1)}.$$

□

4.2.2 Some Non-existence results

Remark 4.2.12. When μ_1, $\mu_2 \neq 0$, blocks of the 3-GDD$(n, 4, 5, \mu_1, \mu_2)$ are of type $(5, 0)$, $(4, 1)$, $(3, 2)$, $(3, 1, 1)$, $(2, 2, 1)$ or $(2, 1, 1, 1)$.

If u, v, x, y, z and w denote the number of blocks of type $(5, 0)$, $(4, 1)$, $(3, 2)$, $(3, 1, 1)$, $(2, 2, 1)$ and $(2, 1, 1, 1)$ respectively, then

$$10u + 4v + x + y = \frac{2n(n-1)(n-2)}{3}\mu_1, \tag{4.20}$$

$$6v + 9x + 6y + 6z + 3w = 6n^2(n-1)\mu_1, \tag{4.21}$$

and

$$3y + 4z + 7w = 4n^3\mu_2 \tag{4.22}$$

where Equations 4.20, 4.21 and 4.22 denote the number of triples of type (3, 0), (2, 1) and (1, 1, 1) respectively.

Theorem 4.2.13. *A 3-GDD$(n, 4, 5, \mu_1, \mu_2)$ does not exist if:*

(i). $n = 3$ and $\mu_2 < \frac{4}{9}\mu_1$.

(ii). $n = 4$ and $\mu_2 < \frac{3}{8}\mu_1$.

(iii). $n \geq 5$ and $\mu_1 < \frac{2n}{7(n-1)}\mu_2$.

Let u, v, x, y, z and w denotes the number of blocks of type $(5, 0)$, $(4, 1)$, $(3, 2)$, $(3, 1, 1)$, $(2, 2, 1)$ and $(2, 1, 1, 1)$ respectively.

Proof. **(i)**. When $n = 3$, we have $u = v = 0$.

Multiplying Equation 4.21 by $\frac{2}{3}$ and subtracting Equation 4.20 gives,

$$5x + 3y + 4z + 2w = 68\mu_1. \tag{4.23}$$

Subtracting Equation 4.23 from Equation 4.22 gives

$$5w = 108\mu_2 - 68\mu_1 + 5x. \tag{4.24}$$

From Equation 4.20, $x \leq 4\mu_1$ and Equation 4.24 becomes

$$5w \leq 108\mu_2 - 68\mu_1 + 20\mu_1 = 108\mu_2 - 48\mu_1$$

and

$$\text{Since } w \text{ can not be negative, } 108\mu_2 - 48\mu_1 \geq 0.$$

Which implies

$$\mu_2 \geq \tfrac{4}{9}\mu_1.$$

Therefore, a 3-GDD$(3, 4, 5, \mu_1, \mu_2)$ does not exist when $\mu_2 < \frac{4}{9}\mu_1$. $\square$

Proof. **(ii)**. When $n = 4$, we have $u = 0$.
Similarly, multiplying Equation 4.21 by $\frac{2}{3}$ and subtracting Equation 4.20 gives

$$5x + 3y + 4z + 2w = 176\mu_1. \tag{4.25}$$

Subtracting Equation 4.25 from Equation 4.22, we have

$$5w = 256\mu_2 - 176\mu_1 + 5x. \tag{4.26}$$

From Equation 4.20, $x \leq 16\mu_1$ and $5x \leq 80\mu_1$, we have

$$5w \leq 256\mu_2 - 176\mu_1 + 80\mu_1 \text{ and } 5w \leq 256\mu_2 - 96\mu_1$$

As w can not be negative,

$$256\mu_2 - 96\mu_1 \geq 0 \text{ implies } \mu_2 \geq \tfrac{3}{8}\mu_1.$$

Therefore, a 3-GDD$(4, 4, 5, \mu_1, \mu_2)$ does not exist when $\mu_2 < \frac{3}{8}\mu_1$. $\square$

Proof. **(iii).** From Equations 4.20, 4.21 and 4.22, solving for x in terms of the remaining, we have

$$5x = 4n^2(n-1)\mu_1 - \frac{2n(n-1)(n-2)}{3}\mu_1 - 4n^3\mu_2 + 10u + 5w \tag{4.27}$$

From Equation 4.20 and Equation 4.21, $10u \leq 4\binom{n}{3}\mu_1$ and $w \leq 2n^2(n-1)\mu_1 \Rightarrow$
$5w \leq 10n^2(n-1)\mu_1$ and hence

$$5x \leq 4n^2(n-1)\mu_1 - 4\binom{n}{3}\mu_1 \text{ - } 4n^3\mu_2 + 4\binom{n}{3}\mu_1 + 10n^2(n-1)\mu_1$$

and

$$5x \leq 14n^2(n-1)\mu_1 - 4n^3\mu_2.$$

Now as x can not be a negative number

$$14n^2(n-1)\mu_1 - 4n^3\mu_2 \geq 0$$

which implies

$$\mu_1 \geq \tfrac{2n}{7(n-1)}\mu_2.$$

Hence, a 3-GDD$(n, 4, 5, \mu_1, \mu_2)$ does not exist when $\mu_1 < \frac{2n}{7(n-1)}\mu_2$ and $n \geq 5$. $\square$

Remark 4.2.14. When $n = 3$, the necessary conditions for 3-GDD$(n, 4, 5, \mu_1, \mu_2)$ in Remark 4.2.9 and as given in the Table 4.4 are satisfied only when:
⊛ $\mu_1 \equiv 0 \pmod{15}$ and $\mu_2 \equiv 0 \pmod{10}$ or,
⊛ $\mu_1 \equiv 3 \pmod{15}$ and $\mu_2 \equiv 8 \pmod{10}$ or,
⊛ $\mu_1 \equiv 6 \pmod{15}$ and $\mu_2 \equiv 6 \pmod{10}$ or,
⊛ $\mu_1 \equiv 9 \pmod{15}$ and $\mu_2 \equiv 4 \pmod{10}$ or,
⊛ $\mu_1 \equiv 12 \pmod{15}$ and $\mu_2 \equiv 2 \pmod{10}$.

Even though the original necessary conditions are satisfied, there are designs that do not exist. From Theorem 4.2.13 (i) and Remark 4.2.14, we have the following:

Corollary 4.2.15. *For non-negative integers s and t:*

i. *If $t > \frac{3s}{2}$, then* 3-*GDD*$(3, 4, 5, 15t, 10s)$ *and* 3-*GDD*$(3, 4, 5, 15t + 9, 10s + 4)$ *do not exist.*

ii. 3-*GDD*$(3, 4, 5, 15t + 3, 10s + 8)$ *does not exist if $t > \frac{3s+2}{2}$.*

iii. 3-*GDD*$(3, 4, 5, 15t + 6, 10s + 6)$ *does not exist if $t > \frac{3s+1}{2}$.*

iv. 3-*GDD*$(3, 4, 5, 15t + 12, 10s + 2)$ *does not exist if $t > \frac{3s-1}{2}$*

Even when the necessary conditions are satisfied, here below are lists of some of these designs which do not exist for $n = 3$:

3-GDD$(3, 4, 5, 15t, 10)$ for $t \geq 2$, 3-GDD$(3, 4, 5, 15t, 20)$ for $t \geq 4$, 3-GDD$(3, 4, 5, 15t, 30)$ for $t \geq 5$, 3-GDD$(3, 4, 5, 15t + 9, 4)$ for $t \geq 1$, 3-GDD$(3, 4, 5, 15t + 9, 14)$ for $t \geq 2$, 3-GDD$(3, 4, 5, 15t + 3, 8)$ for $t \geq 2$, 3-GDD$(3, 4, 5, 15t + 3, 18)$ for $t \geq 3$, 3-GDD$(3, 4, 5, 15t + 12, 2)$ for $t \geq 0$, 3-GDD$(3, 4, 5, 15t + 12, 12)$ for $t \geq 2$, 3-GDD$(3, 4, 5, 15t + 6, 6)$ for $t \geq 1$, 3-GDD$(3, 4, 5, 15t + 6, 16)$ for $t \geq 3$, etc.

Remark 4.2.16. For $n = 4$, the necessary conditions for 3-GDD$(n,\ 4,\ 5,\ \mu_1,\ \mu_2)$ are satisfied only when:
$\mu_1 \equiv 0 \pmod{30}$ and $\mu_2 \equiv 0 \pmod{15}$, or $\mu_1 \equiv 6 \pmod{30}$ and $\mu_2 \equiv 6 \pmod{15}$, or $\mu_1 \equiv 12 \pmod{30}$ and $\mu_2 \equiv 12 \pmod{15}$, or $\mu_1 \equiv 18 \pmod{30}$ and $\mu_2 \equiv 3 \pmod{15}$, or $\mu_1 \equiv 24 \pmod{30}$ and $\mu_2 \equiv 9 \pmod{15}$.

When $n = 4$, from Theorem 4.2.13 (*ii*) and Remark 4.2.16, the following result follows:

Corollary 4.2.17. *For non-negative integers s and t:*

i. If $t > \frac{4s}{3}$, *then* 3-*GDD*$(4, 4, 5, 30t, 15s)$ & 3-*GDD*$(4, 4, 5, 30t + 24, 15s + 9)$ *do not exist.*

ii. 3-*GDD*$(4, 4, 5, 30t + 6, 15s + 6)$ *does not exist if* $t > \frac{4s+1}{3}$.

iii. 3-*GDD*$(4, 4, 5, 30t + 12, 15s + 12)$ *does not exist if* $t > \frac{4s+2}{3}$.

iv. 3-*GDD*$(4, 4, 5, 30t + 18, 15s + 3)$ *does not exist if* $t > \frac{4s-1}{3}$.

The next chapter studies and gives new results on the existence of GDDs with two groups and block size four.

Chapter 5

Group Divisible Designs With Two Groups and Block size Four

Now we turn our attention to a certain case of GDDs where $t = 2$ and where the existence problem is still open. Together [9] and [10] settle the existence problem for GDD with $k = 3$. For $k = 4$ and $m > 3$, the existence problem when $\lambda_1 = 0$ is settled by Hanani [5]. But $\lambda_1 \neq 0$ has not been settled. When the number of groups is less than the block size, the problem to prove the existence is specially hard.

5.1 Preliminary results

Very little is known about the existence of these designs and the known results from [21] are summerized below when the number of groups is 2.

Lemma 5.1.1. *Suppose GDD*$(n, 2, 4, \lambda_1, \lambda_2)$ *has replication number* r*, and suppose* b *is the number of blocks. Let* s*,* t *and* u *denote the number of blocks with* $4, 3$ *and* 2 *points, respectively from the same group. The necessary conditions for the existence of the GDD include the following:*
(a) $\lambda_1(n-1) + n\lambda_2 \equiv 0 \pmod 3$ *and* $n(\lambda_1(n-1) + n\lambda_2) \equiv 0 \pmod 6$.
(b) $\lambda_2 \leq \frac{2\lambda_1(n-1)}{n}$
(c) $6s + 3t + 2u = 2\lambda_1\binom{n}{2} = n(n-1)\lambda_1,$ *and* $3t + 4u = n^2\lambda_2,$ *and* $s + t + u = b$

Lemma 5.1.2. (a) *For even GDDs,* $n\lambda_2 = 2\lambda_1(n-1)$*. Moreover, if* n *is even, even GDDs* *exist with* $\lambda_1 = \frac{n}{2}$ *and* $\lambda_2 = n-1$. *For n odd, even GDDs exist with* $\lambda_1 = n$ *and* $\lambda_2 = 2(n-1)$. (b) *For any Odd GDD,* $\lambda_1 = tn$ *and* $\lambda_2 = t(n-1)$ *for some* t. *The minimum possible* t *is* $t = 1$ *for* $n \equiv 0, 1 \pmod 3$ *and* $t = 3$ *for* $n \equiv 2 \pmod 3$.

Theorem 5.1.3. *A* $GDD(10+12j, 2, 4, (5+6j)t + 2u + 3v, (9+12j)t + 3v)$ *exists for all* $j, t, u, v \geq 0$.

5.1.1 Necessary and Sufficient Conditions

In a GDD$(n, 2, 4, \lambda_1, \lambda_2)$, each point of X appears in r of the b blocks, and r is called the replication number. For GDDs with block size 4 and two groups, the replication number r and the number of blocks b are given by:

$$r = \frac{\lambda_1(n-1) + \lambda_2 n}{3} \tag{5.1}$$

and

$$b = \frac{\lambda_1 n(n-1) + \lambda_2 n^2}{6} \tag{5.2}$$

These are obtained by noting that

$$r(4-1) = \lambda_1(n-1) + \lambda_2 n$$

and

$$b\tbinom{4}{2} = \lambda_1(\tbinom{n}{2} + \tbinom{n}{2}) + \lambda_2 n^2$$

As r and b must be integers, one can obtain the necessary conditions as given in [21] and reproduced in Table 5.1.

Table 5.1: Congruence Restrictions for GDD$(n, 2, 4, \lambda_1, \lambda_2)$

$\lambda_2 \backslash \lambda_1$	$0 \pmod 3$	$1 \pmod 3$	$2 \pmod 3$
$0 \pmod 6$	all n	$n \equiv 1 \pmod 3$	$n \equiv 1 \pmod 3$
$1 \pmod 6$	$n \equiv 0 \pmod 6$	$n \equiv 2 \pmod 6$	Impossible
$2 \pmod 6$	$n \equiv 0 \pmod 3$	Impossible	$n \equiv 2 \pmod 3$
$3 \pmod 6$	n even	$n \equiv 4 \pmod 6$	$n \equiv 4 \pmod 6$
$4 \pmod 6$	$n \equiv 0 \pmod 3$	$n \equiv 2 \pmod 3$	Impossible
$5 \pmod 6$	$n \equiv 0 \pmod 6$	Impossible	$n \equiv 2 \pmod 6$

For example, from Table 5.1, one can conclude that when $\lambda_1 \equiv 0 \pmod 3$ and $\lambda_2 \equiv 0 \pmod 6$, the necessary conditions are satisfied, that is both b and r are expressed as integers, for all n, whereas with $\lambda_1 \equiv 2 \pmod 3$ and $\lambda_2 \equiv 4 \pmod 6$ does not exist for any n.

Remark 5.1.4. In a GDD$(n, 2, 4, \lambda_1, \lambda_2)$, a block is of type $(4, 0)$ if it contains all the four elements from the same group. Similarly a block containing two elements each from a group is of type $(2, 2)$, where a block with three elements from one group and one from the other group is of type $(3, 1)$.
In a GDD$(n, 2, 4, \lambda_1, \lambda_2)$, a block of type $(2, 2)$ or $(3, 1)$ can cover at most three first and four second associate pairs.

Lemma 5.1.5. *[21]:*

(i) $b \geq \frac{\lambda_2 n^2}{4}$

(ii)$\lambda_2 < 2\lambda_1$

Proof.

(i) $\lambda_2 n^2$ counts the second associate pairs.
In GDD$(n, 2, 4, \lambda_1, \lambda_2)$ a block can cover at most 4 second associate pairs.
Therefore, $4b \geq \lambda_2 n^2$. i.e, $b \geq \frac{\lambda_2 n^2}{4}$.

(ii) $\lambda_2 n^2$ is the number of second associate pairs and there are at least two first associate pairs, in every block which has second associate pairs
The minimum number of blocks required for second associate pairs is $\frac{\lambda_2 n^2}{4}$.
Hence the number of first associate pairs in the blocks are $\frac{2\lambda_2 n^2}{4} = \frac{\lambda_2 n^2}{2}$.
Hence, $2\lambda_1 n(n-1) \geq \lambda_2 n^2 \Rightarrow \lambda_1 n(n-1) \geq \frac{\lambda_2 n^2}{2}$.
we must have $\lambda_2 \leq \frac{2\lambda_1(n-1)}{n} < 2\lambda_1$. □

Remark 5.1.6. $\lambda_1 = \lambda_2 = \lambda$
If $\lambda_1 = \lambda_2 = \lambda$, then we have a BIBD$(2n, 4, \lambda)$.
It is known that the necessary conditions are satisfied for the existence of a BIBD$(n, 4, \lambda)$; hence the same necessary conditions are sufficient for the existence of GDD$(n, 2, 4; \lambda_1, \lambda_2)$ when $\lambda_1 = \lambda_2$.

Remark 5.1.7. $\lambda_1 = 0$ is impossible

The blocks of a GDD$(n, 2, 4, 0, \lambda_2)$, must contains only second associate pairs.

If a block contains second associate pairs, then it contains either one element from one group and three from the other or two from each group (hence it contains 3 first associate pairs and 3 second associate pairs or 2 first associate pairs and 4 second associate pairs, respectively). Hence $\lambda_1 = 0$ is not possible.

Remark 5.1.8. $\lambda_2 = 0$

In this case, all blocks contain elements from the same group.

This means all blocks are of the form $(4, 0)$.

In this case we have a BIBD$(n, 4, \lambda_1)$ on each group and the necessary conditions are sufficient, where the necessary conditions are the same as the necessary conditions for the existence of BIBD$(n, 4, \lambda)'s$.

5.2 Results

In this section, we present our results.

In view of the remarks above, we let $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. Using Lemma 5.1.5, $\lambda_2 < 2\lambda_1$, we have the following results for various values of λ_1.

5.2.1 $\lambda_1 = 1$

We have GDD$(n, 2, 4, 1, \lambda_2)$ with $b = \frac{n(n-1)+\lambda_2 n^2}{6}$ and $r = \frac{(n-1)+\lambda_2 n}{3}$.

When $\lambda_1 = 1$, λ_2 must be 1.

As we have a BIBD$(2n, 4, 1)$ and the necessary condition for the existence of BIBD(n, 4, 1) is $n \equiv 2 \pmod 6$.

Hence the necessary condition for the existence of a GDD$(n, 2, 4, 1, 1)$ are satisfied if $n \equiv 2 \pmod 6$.

5.2.2 $\lambda_1 = 2$

λ_2 must be 0, 1, 2 or 3. But we need to construct GDD $(n, 2, 4, 2, 3)$, where $r = \frac{2(n-1)+3\lambda_2 n}{3}$ and $b = \frac{n(n-1)}{3} + \frac{n^2}{2}$.

The necessary condition for GDD$(n, 2, 4, 2, 3)$ to exist is $n \equiv 4 \pmod 6$.

Example 5.2.1. For $n = 4$, $G_1 = \{1, 2, 3, 4\}$ and $G_2 = \{a, b, c, d\}$, the blocks of a GDD(4, 2, 4, 2, 3) can be vertically written as;

$$\begin{array}{cccccccccccc}
1 & 1 & 3 & 3 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 \\
2 & 2 & 4 & 4 & 3 & 3 & 4 & 4 & 4 & 4 & 3 & 3 \\
a & c & a & c & a & b & a & b & a & b & a & b \\
b & d & b & d & c & d & c & d & d & c & d & c
\end{array}$$

5.2.3 $\lambda_2 = 1 + \lambda_1$

Hence, $r = \frac{\lambda_1(n-1)+(\lambda_1+1)n}{3} = \frac{n+\lambda_1(2n-1)}{3}$ and
$b = \frac{\lambda_1 n(n-1)+(\lambda_1+1)n^2}{6} = \frac{n(\lambda_1(2n-1)+n))}{6}$.

• $n \equiv 0 \pmod 6$
b is an integer and $r = \frac{6t+\lambda_1(12t-1)}{3}$, for integer t implies $\lambda_1 \equiv 0 \pmod 3$.

• $n \equiv 1 \pmod 6$
$b = \frac{(6t+1)(12t\lambda_1+\lambda_1+6t+1)))}{6}$.
Hence $\lambda_1 \equiv 5 \pmod 6$.
Similarly, r is integer when $\lambda_1 \equiv 5 \pmod 6$.

• $n \equiv 2 \pmod 6$ In this case, $b = \frac{(3t+1)(\lambda_1(12t+3)+(6t+2)))}{3}$.
Here 3 divides λ_1 if and only if 3 divides 2. But 3 can not divide 2.
Thus, $n \equiv 2 \pmod 6$ is not possible when $\lambda_2 = \lambda_1 + 1$.

• $n \equiv 3 \pmod 6$
$b = \frac{(2t+1)(\lambda_1(12t+5)+(6t+3)))}{2}$, b is integer if λ_1 is odd.
Hence $\lambda_1 \equiv 1 \pmod 2$.

• $n \equiv 4 \pmod 6$ we have $r = \frac{6t+4+\lambda_1(12t+7)}{3}$.
Now r is integer when 3 divides $7\lambda_1 + 4$, which means 3 divides $\lambda_1 + 1$.
Thus, $\lambda_1 \equiv 2 \pmod 3$.

• $n \equiv 5 \pmod 6$
Now, $b = \frac{(6t+5)(\lambda_1(12t+9)+(6t+5)))}{6}$.
b exists when 6 divides $9\lambda_1 + 5$, which means when 6 divides $3\lambda_1 + 5$, but this is not possible.

The above explanations can be summarized as shown in the Table 5.2.

Table 5.2: Congruence Restrictions for $GDD(n, 2, 4; \lambda_1, \lambda_1 + 1)$

n	λ_1	$\lambda_2 = \lambda_1 + 1$
$n \equiv 0 \pmod 6$	$\lambda_1 \equiv 0 \pmod 3$	$\lambda_2 \equiv 1 \pmod 3$
$n \equiv 1 \pmod 6$	$\lambda_1 \equiv 5 \pmod 6$	$\lambda_2 \equiv 0 \pmod 6$
$n \equiv 2, 5 \pmod 6$	Impossible	Impossible
$n \equiv 3 \pmod 6$	$\lambda_1 \equiv 1 \pmod 2$	Even
$n \equiv 4 \pmod 6$	$\lambda_1 \equiv 2 \pmod 3$	$\lambda_2 \equiv 0 \pmod 3$

Example 5.2.2. Consider the GDD$(6, 2, 4, 3, 4)$.
Here $n \equiv 0 \pmod 6$, $\lambda_1 \equiv 0 \pmod 3$ and $\lambda_1 \equiv 1 \pmod 3$. From Table 5.2, the necessary conditions for the existence of this design are satisfied. In fact, this design exists, and its blocks are constructed as shown below:
Using difference set $\{0, 1, 4, 6\}$, the 13 blocks of BIBD$(13, 4, 1)$, with replication number 4 are given as:

0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	0
4	5	6	7	8	9	10	11	12	0	1	2	3
6	7	8	9	10	11	12	0	1	2	3	4	5

Now using the blocks containing a fixed element (say 0), we obtain two groups
$G_1 = \{1, 4, 6, 7, 8, 11\}$ and $G_2 = \{2, 3, 5, 9, 10, 12\}$ for the required GDD.
Three copies of the blocks of BIBD$(13, 4, 1)$ not containing zero and the following blocks together give the GDD$(6, 2, 4, 3, 4)$ on G_1 and G_2.

12	2	7	1
1	7	3	2
4	8	5	9
6	11	12	10

3	9	8	4
1	7	3	2
4	8	5	9
6	11	12	10

5	10	11	6
1	7	3	2
4	8	5	9
6	11	12	10

The next example is an incomplete one, which needs more tasks to determine its existence or non-existence.

Example 5.2.3. Consider a GDD$(10, 2, 4, 2, 3)$

If the design GDD$(10, 2, 4, 2, 3)$ exists, then it has $b = 80$ (number of blocks), $r = 16$(replication number), 180 first associate pairs where each first associate pair is repeated in $\lambda_1 = 2$ blocks, and 300 second associate pairs, where each second associate pair is repeated in $\lambda_2 = 3$ blocks. In a design with 2 groups with block size 4, the blocks of the design are of the type $(4, 0)$, $(3, 1)$ or $(2, 2)$ only.

If s, t and u denote the number of blocks of type $(4, 0)$, $(3, 1)$ and $(2, 2)$ respectively, then we get the following systems of linear equations.

$$6s + 3t + 2u = 2\lambda_1 \binom{n}{2} = n(n-1)\lambda_1 = 10 \times 9 \times 2 = 180 \tag{5.3}$$

$$3t + 4u = n^2\lambda_2 = 10^2 \times 3 = 300 \tag{5.4}$$

$$s + t + u = b = 80 \tag{5.5}$$

Where Equations 5.3, 5.4 and 5.5 denote the number of first associate pairs, number of second associate pairs and number of blocks in the design respectively.

Solving for the three unknowns s, t and u, from the three equations 5.3, 5.4 and 5.5, we get

$$u = 60 + 3s \tag{5.6}$$

The possible values of s, t and u for a design to exist are given as shown(horizontally) in the Table 5.3

Table 5.3: Table of possible values for s, t and u:

s	t	u
0	20	60
1	16	63
2	12	66
3	8	69
4	4	72
5	0	75

If $G_1 = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $G_2 = \{a, b, c, d, e, f, g, h, i, j\}$ and we go with the option(first row) $s = 0$, $t = 20$, $u = 60$ in the Table 5.3.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7
8	9	i	g	i	j	j	j	h	f	g	d	h	f	i	h
b	a	c	e	b	c	e	a	e	d	b	a	f	c	d	g

1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2
2	3	4	5	6	7	8	9	2	3	4	5	6	7	3	4
9	j	j	j	g	h	f	h	i	f	b	h	i	i	j	j
a	d	b	g	c	d	e	e	c	a	a	c	e	g	d	i

2	2	2	2	2	2	2	2	2	2	3	3	3	3	3	3
5	6	7	8	3	4	5	6	7	8	4	6	5	7	8	9
j	h	g	g	e	i	f	h	c	e	8	9	g	j	h	i
h	f	f	d	b	e	d	b	a	c	c	b	e	f	g	h

3	3	3	3	4	4	4	4	4	4	4	4	5	5	5	5
4	5	6	7	5	6	7	8	9	5	6	7	6	6	8	7
e	h	i	f	9	j	f	i	g	g	h	h	7	8	i	j
a	b	g	c	d	f	b	d	c	a	d	c	a	e	f	e

5	6	6	7	7	8	j	j	i	g	g	f	e	e	d	c
9	7	9	8	9	9	i	h	h	f	d	e	d	d	c	b
i	8	j	9	i	j	b	a	a	b	a	a	c	b	b	a
f	d	g	b	a	c	9	8	8	8	9	6	9	7	5	6

In the above construction:

- $b = 80$, $r = 16$, 180 first associate pairs and each first associate pair is repeated in $\lambda_1 = 2$ blocks(as expected).
- But out of the expected 300 second associate pairs, only seven fail to satisfy $\lambda_2 = 3$.Those second associate pairs are :$(5, a)$, $(5, f)$, $(7, f)$, $(7, i)$, $(9, a)$, $(9, f)$ and $(9, i)$. The number of blocks in which these second associate pairs occured are given in the Table 5.4.

Table 5.4: Second associate pairs failing to satisfy $\lambda_2 = 3$:

pairs	Number of blocks in which the pair occurs
$(5, a)$	2
$(5, f)$	4
$(7, f)$	4
$(7, i)$	2
$(9, a)$	4
$(9, f)$	1
$(9, i)$	4

Note that one of the first critical designs for n = 10 + 12j is GDD(n, 2, 4, 2, 3), whose existence would immediately give us many missing designs. The existence of GDD(10, 2, 4, 2, 3) gives us many designs, including all those (small) missing designs for n = 10 [20].

Chapter 6

Conclusions and Future Work

6.1 Conclusions

Both GDDs and t-designs have been studied and have important applications in the field of Combinatorics. These two concepts have been combined to define a t-GDD which was the subject of the present work. This work has raised more challenging existence problems.

In this thesis, we studied a special type of t-GDDs when $t = 3$, with three or four groups and block size five.

In a 3-GDD with three groups and block size five denoted by 3-GDD(n, 3, 5, μ_1, μ_2), we have established some necessary conditions for its existence (Theorem 3.2.1, 3.2.5), proved the non-existence of a 3-GDD(n, 3, 5, μ_1, 0) when $n \leq 5$ (Theorem 3.2.6), and when $n \geq 6$, we have given several existence results for 3-GDD(n, 3, 5, μ_1, 0) (Theorem 3.3.4, 3.3.6). Furthermore, the relationship between $3'$-GDD(n, m, k, Λ_1, Λ_2)(Definition 3.1.2) and 3-PBIBD is obtained in Theorem 3.4.12. When $n = 3$, Corollary 3.4.13 proves that the existence of a $3'$-GDD(n, 3, 4, Λ_1, Λ_2) guarantees the existence of 3-GDD(3, 3, 5, $\frac{9\Lambda_2+\Lambda_1}{4}$, $\frac{11\Lambda_2-\Lambda_1}{4}$), while the existence of 3-GDD(3, 3, 5, μ_1, μ_2) shows the existence of $3'$-GDD(3, 3, 4, $\frac{11\mu_1-9\mu_2}{5}$, $\frac{\mu_1+\mu_2}{5}$). Lastly, we constructed many families of 3-GDDs when $n = 3$ in Theorem 3.4.7, Corollary 3.4.8, Corollary 3.4.9 and Corrollary 3.4.10.

For a 3-GDD with four groups and block size five which is denoted by 3-GDD(n, 4, 5, μ_1, μ_2), some necessary conditions for the existence of such designs are developed in Corollary 4.2.5. In Theorem 4.2.10, the non-existence of a 3-GDD(n, 4, 5, μ_1, μ_2) when $\mu_2 = 0$ is proved, and in Theorem 4.2.13 a more compact conditions for the non- existence of a

3-GDD(n, 4, 5, μ_1, μ_2) are established. At the end, when $n = 3$ and $m = 4$, several specific instances of non-existence are given in Corollary 4.2.15 and Corollary 4.2.17.

6.2 Future Work

This thesis offers many possibilities for further studies and applications. Some of them are listed below.

Problem 1. Prove the existence or the non-existence of these missing designs in a 3-GDD(n, 4, 5, μ_1, μ_2), when $n = 3$ and $n = 4$:

For instance, from Corollary 4.2.15, the existence or the non-existence of the following designs are not determined:

- 3-GDD$(3, 4, 5, 15t, 10s)$, when $t \leq \frac{3s}{2}$, $s > 0$ and $t > 0$.
- 3-GDD$(3, 4, 5, 15t + 9, 10s + 4)$, when $t \leq \frac{3s}{2}$
- 3-GDD$(3, 4, 5, 15t + 3, 10s + 8)$, when $t \leq \frac{3s+2}{2}$
- 3-GDD$(3, 4, 5, 15t + 6, 10s + 6)$, if $t \leq \frac{3s+1}{2}$.
- 3-GDD$(3, 4, 5, 15t + 12, 10s + 2)$, if $t \leq \frac{3s-1}{2}$

Similarly, when $n = 4$, there are missing designs in Corollary 4.2.17, whose existence are not determined:

- 3-GDD$(4, 4, 5, 30t, 15s)$ if $t \leq \frac{4s}{3}$, $s > 0$ and $t > 0$.
- 3-GDD$(4, 4, 5, 30t + 24, 15s + 9)$, when $t \leq \frac{4s}{3}$.
- 3-GDD$(4, 4, 5, 30t + 6, 15s + 6)$, when $t \leq \frac{4s+1}{3}$
- 3-GDD$(4, 4, 5, 30t + 6, 15s + 6)$ if $t \leq \frac{4s+1}{3}$.
- 3-GDD$(4, 4, 5, 30t + 12, 15s + 12)$, when $t \leq \frac{4s+2}{3}$.
- 3-GDD$(4, 4, 5, 30t + 18, 15s + 3)$ if $t \leq \frac{4s-1}{3}$.

Problem 2. Complete the construction of blocks of GDD$(10, 2, 4, 2, 3)$ in Example 5.2.3 or prove the design does not exist. This is an important problem as the existence of this design would immediately give us many missing designs for a GDD(n, 2, 4, 2, 3) when $n = 10 + 12j$ [20].

Problem 3. Study the relationships, if any, that exist between 3-GDDs and other combinatorial structures such as orthogonal arrays, latin squares, and transversal designs.

We are also thankful to a referee of one of our published papers for suggesting following problems for further exploration:

Problem 4. Discuss the existence of a general t-GDD instead of for only $t = 3$. How the concept of resolvability can be used for these designs?

Problem 5. To increase the importance of such designs, define the overall efficiency for a 3-GDDs.

Problem 6. A GDD with parameters: $v = mn$, b, r, k, $\mu_1 = 0$, $\mu_2 = 1$ is applicable in the construction of regular low-density parity-check (LDPC) codes free of 4-cycles [40]. Is a 3-GDD under certain conditions also applicable in LDPC codes?

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