# GAMMA RAY PRODUCTION IN AN ACCRETING NEUTRON STARS

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Degree of Master of Science in Physics

By

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*To werkinesh*

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## Abstract

Inverse compton (IC) scattering of lower frequency photons by relativistic electrons was used as a method for the production of high-energy gamma-rays in an accreting neutron stars. A spectrum function is derived.Based on which we obtained two new results. First there is a maximum scattered photon energy for a given resonant scattering which depends on both the incident electron energy and the magnetic field, but is independent of the incident photon energy. It is also found that the common upper limit of the scattered photon energy is the highest scattered photon energy for any given incident photon and electron energy. Thus this may have physical implications for Gamma-ray production.

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# Introduction

$\mathbf{b}_1$Most of the known bright Gamma-ray sources in our galaxy are compact objects in binary systems. Among which pulsars are a well- established class of Gamma-ray sources .More over the extraction of gravitational potential energy from material which accretes on to a gravitating body is now known to be the principal source of power in several types of close binary systems; and is widely believed to provide the power supply in active galactic nuclei and quasars.

When gas is accreted by a compact star, it is heated so much that a large fraction of its gravitational energy is transformed into electromagnetic radiation Salpeter(1964), Zeldovich(1964),Zeldovich& Guseynov(1965)[1]. Inverse compton scattering has attracted more and more attention and a great deal of work has been done on this subject (Daugherty & Harding 1989;Dermer 1990;Zhang & Qiao 1997;Harding & Muslimov 1998).Daugherty & Harding (1989) studied the gamma ray generation by Monte Carlo simulation based on Herold's crossection (Herold 1979) of magnetic compton scatering in the electron rest frame(ERF)[2].

It is believed that inverse compton scattering in strong magnetic fields near the surfaces of neutron stars is one of the possible mechanisms responsible for pulsed Gamma-ray emissions. Theorotically it is believed that inverse compton scattering in strong magnetic field can produce Gamma-rays in the range between $1 - 100 Mev$. In this thesis we will be concerned on the production of these range of Gamma-rays on the surface of neutron star from inverse compton scattering in strong magnetic field enhanced by accretion. Thus we are aimed at the analytical study of Gamma-ray production based on the compton scattering crossection in the lab frame (LF) where we have found it helpful in astrophysics since a crossection in the LF on scattering of lower frequency photon by relativistic electron can be reduced to Herold's non-relativistic result. This will not be recovered from a relativistic transformation of Herold's crossection in ERF.

The first chapter of this thesis devoted to the discussion of the physics of neutron stars,in which the formation, structure, internal composition, and evolution of neutron stars is described. In chapter two the source and generating mechanisms of Gamma-rays through inverse compton scattering in strong magnetic field are briefly discussed. The calculation of the crossection of the required Compton scattering is fully dealt with in Chapter 3. The derivation of the spectrum function of scattered photon is given in Chapter 4.

# Chapter 1

# The Physics Of Neutron Star

## 1.1 Introduction

Neutron stars are some of the densest manifestation of massive objects in the universe. In the following sections we will describe the formation , structure, internal composition and life of neutron star. Observations such as mass transfer (accretion) in neutron stars is also included with the discussion of the limit on the steady accretion rate ,$\dot{M}(gs^{-1})$ , which is the Eddington limit,the reason for mass transfer as a Roche lobe overflow in our case, formation of accretion disc and accretion column.

## 1.2 Neutron Star

The term "neutron star" refers to a star with a mass of order of $1.5M_{\odot}$ [$M_{\odot} = 1$ solar mass], a radius of$\sim 12km$, and magnetic field $\leq 10^{13}Gauss$. The nucleonic component of neutron stars is mostly neutrons and some protons (and enough electrons and muons to neutralize the matter). Neutron stars are formed in the aftermath of the gravitational collapse of the core of a massive star [3].

A cloud in space collapses due to gravitation of the particles. In a normal star in hydrodynamic equilibrium the particles 'feel' gravity as well but do not collapse because gravity is balanced mostly by radiation pressure and gas pressure. The radiation pressure is produced as a result of high energy photons created due to nuclear fusion inside the star's nucleus. If the temperature is below the fusion temperature of an available element a little contraction can raise the temperature to the ignition level [4].

A star which has no fuel left for fusion in its core, collapses to a white dwarf , a neutron star or a black hole depending on its mass. In a white dwarf gravity is balanced by the degenerate-electron pressure. The average density of a white dwarf is of the order $10^7 gcm^{-3}$. For a massive predecessor ($M_* > 10M_\odot$) the degenerate electron pressure is not sufficient to prevent further collapse. Such a massive star will collapse to a neutron star in which degenerate-neutron pressure balances gravity. If the star is too massive to be supported by degenerate-neutron pressure it will totally collapse to become a black hole.

A neutron star has five major regions :The inner and the outer cores, the crust,the envelope, and the atmosphere, as shown in figure below (Figure 1.1).

The atmosphere and the envelope contain a negligible amount of mass, but the atmosphere plays an important role in shaping the emergent photon spectrum, and the envelope influences the transport and the release of thermal energy from the star's surface .

The crust extends from 1 to $2km$ below the surface. It contains nuclei which vary in atomic weight. The crustal density varies from less than about $\rho \leq 10^6 gcm^{-3}$ to $\rho \geq 2.4 \times 10^{14} gcm^{-3}$ near the core-crust interface which contains mostly neutron-rich nuclei. The crust contains less than 10% of the total moment of inertia and hence little mass. The central density of neutron star is about $10^{15}$-$10^{16} gcm^{-3}$. Thus a neutron star may be felt as one huge nucleus composed of $10^{60}$ nucleons.

The internal temperature of a neutron star soon after birth is $\sim 10^9 k$ but cools quickly to $\sim 10^7 k$ in a matter of $10^2$ - $10^3$ years as a result of neutrino emission. Cooling by photon emission will take over after the first $10^5$ years. Pulsar are rotating magnetized neutron stars with magnetic fields typically in the range $2 \times 10^{10} - 2 \times 10^{13}$ Gauss[5]. The so called millisecond pulsars have much weaker magnetic fields.

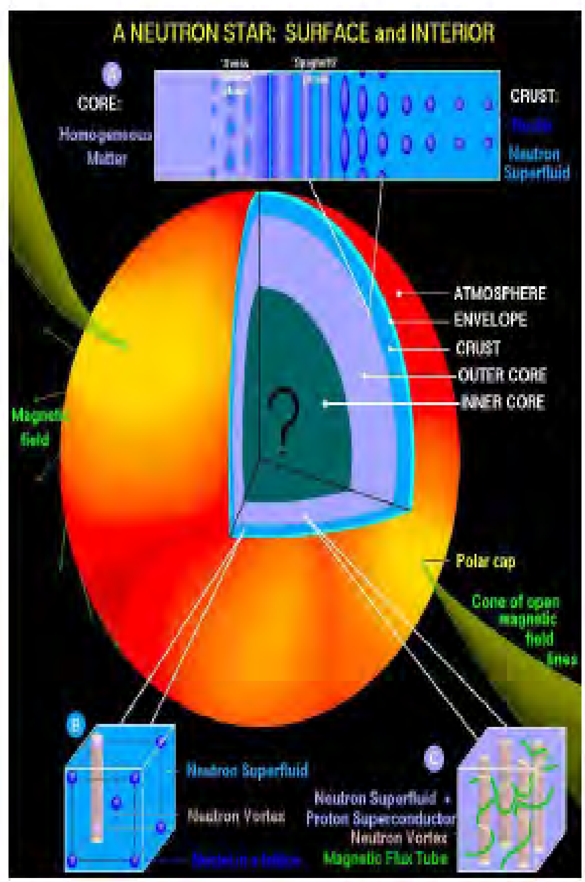


Figure 1.1: Neutron star structure.

### 1.2.1 Neutron Star Formation and Magnetic Fields

Neutron star is formed when a Core of a massive star burns into iron. Which is followed by iron core collapse. However as a result of conservation of angular momentum, there will be an increase in the number of rotation. Plasma density gradient is inherent to the degenerate system would lead to separated charges.

The spinning of this separated charges which comes as a result of plasma diffusion is a source for the magnetic moment. Thus an increase in the magnetic moment will increase the magnetic field. The magnetic fields generated by the spinning charges are **dipolar**[6].

$$B_{out} = \frac{- \mid Q \mid \omega}{3c}(\frac{R^2}{r^3})(3\cos\theta\mathbf{e}_r - \mathbf{k}) \qquad (1.2.1)$$

where $Q$ is the charge, $\omega$ is the angular frequency of the charge, c the speed of light, $\mathbf{e}_r$ and $\mathbf{k}$ are both unit vectors in the radial direction and along z-axis respectively.

When the neutron star density is reached, the energy from gravitational collapse has heated the matter to 10% of its rest mass ( $100 MeV or 10^{12}$ K per nucleon). Thus the temperature and lepton number gradients becomes so high that neutrinos are trapped inside the core which has implication of dynamo action resulting in an even larger B in a matter of 30 seconds [7].

Neutron star have rapid rotational rate initially about $1ms$. They spin down due to gravitational radiation, EM radiation, thus neutron stars life depends on the torque from these.

### 1.2.2 Magnetars

Magnetars are very strongly mangetized stars. The strong magnetic braking in these stars cause rapid spin down ( 10000 years). Thus they are easily observable only for a short period of time.Only about a dozen known so far. Starquakes and glitches produce gamma ray bursts (GRBs). GRBs are very poorly understood, where the strongest of them increase the conductivity of the Earths ionosphere from a distance of 10 kpc (roughly the distance to the galactic centre).They have biosphere killing potential. Short GRBs may be caused due to magnetar gamma flares. Magnetars are of various types some of these are soft gamma-ray repeaters (SGR) and Anomalous X-ray pulsars (AXP). Despit there similarity it is still not known that why SGRs burst but AXPs don't. Where the stronger the magnetic field is, the faster they spin down (motion to the right). Thus magnetars are observable only for $10^4$ years while normal pulsars are observable for $10^7$ years. Thus our galaxy may contain millions of dead magnetars [7].

## 1.3 Accretion In Neutron Star

In a system in which a compact object (neutron star) and a stellar companion orbit each other, mass will be transferred to (accreted by) the neutron star from its companion as shown in figure below.(Fig:1.2)

The overall process of accretion,such as the limit on the steady accretion rate, $\dot{M}(gs^{-1})$ , which is the Eddington limit, the reason for mass transfer as a Roche lobe overflow in our case, formation of accretion disc and accretion column in neutron star will be presented in the following section.

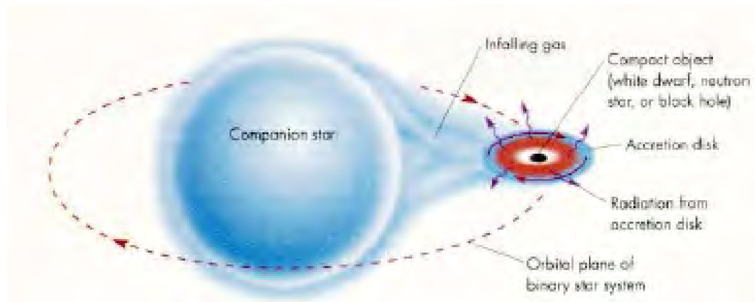


Figure 1.2: A binary system of a compact object and a companion star. Accreting gas forms a disk around the compact object. In the accretion disk particles transport their angular momentum outward by friction and approach the neutron star.

### 1.3.1 Accretion As A Source Of Power

The extraction of gravitational potential from material which accretes on to gravitating body is now known to be the principal source of power in several types of close binary systems. It is widely believed to provide the power supply in an active galactic nuclei and quasars.

This increasing recognition of the importance of accretion has accompanied the dramatic expansion of observational techniques in astronomy, in particular the exploitation of the full range of the electromagnetic spectrum from the radio to x-rays and gamma-rays. At the same time, the existence of compact objects has been placed beyond doubt by the discovery of pulsars and black holes.

Thus, the new role for gravity arises because accretion on to compact objects is a natural and powerful mechanisms for producing high-energy radiation.[8]

### 1.3.2 The Eddington Limit

Consider a steady spherically symmetrical accretion, assuming the accreting material as hydrogen exerts a force mostly on the free electrons by Thomson scattering. If S is the radiant energy flux (erg $s^{-1}cm^{-2}$) & $\sigma_T = 6.7 \times 10^{-25} cm^2$ is the Thomson cross-section, then in this limit the outward radial force on each electron equals the rate at which it absorbs momentum, $\sigma_T \frac{S}{c}$.The radiation pushes out electron-proton pairs against the total gravitational force $GM\frac{(m_p+m_e)}{r^2} \approx GM\frac{m_p}{r^2}$, ,r is a radial distance from the center.If the luminosity of the accreting source is $L(ergs^{-1})$, we have $S = \frac{L}{4\pi r^2}$, so the net inward force on an electron-proton pair is

$$(GMm_p - \frac{L\sigma_T}{4\pi c})\frac{1}{r^2} \tag{1.3.1}$$

This determines the luminosity for which this expression vanishes,the Eddington limit,

$$L_{Edd} = \frac{4\pi M m_p c}{\sigma_T} \approx 1.3 \times 10^{38} (\frac{M}{M_\odot}) ergs^{-1} \tag{1.3.2}$$

At greater luminosities the outward pressure of radiation would exceed the inward gravitational attraction and accretion would be halted. If all the luminosity of the source were derived from accretion this would switch of the source. If some, or all, of it were produced by other means, for example nuclear burning, then the outer layers of material would begin to be blown off and the source would not be steady. For stars with a given mass-luminosity relation this argument yields a maximum stable mass. In accretion powered objects the Eddington limit implies a limit on the steady accretion rate, $\dot{M}(gs^{-1})$. Since for a body of mass M and radius $R_\star$ the gravitational potential energy released by the accretion of a mass m on to its surface is

$$\triangle E_{acc} = \frac{GMm}{R_\star} \tag{1.3.3}$$

where G is the gravitational constant,A neutron star with a radius $R_* \sim 10Km$, mass $M \sim M_\odot$, then the yield $\triangle E_{acc}$ is about $10^{20}$ erg per accreted gram. Now if all of the kinetic energy of infalling matter is given up to radiation. The accretion luminosity is

$$L_{acc} = \frac{GM\dot{M}}{R_*} \tag{1.3.4}$$

Reexpressing Eq.(1.3.2) in terms of typical orders of magnitude writing the accretion rate as $\dot{M}$=$10^{16}\dot{M}_{16}gs^{-1}$ we have

$$L_{acc} = 1.3 \times 10^{36}\dot{M}_{16}(\frac{M}{M_\odot})(\frac{10km}{R_\star})ergs^{-1} \tag{1.3.5}$$

Obviously the ratios $(\frac{M}{M_\odot})$ and $(\frac{10km}{R_\star})$ are of the order of unity for neutron stars. Since $\dot{M}$ in a close binary systems involving neutron star is of order of $10^{16}gs^{-1}(\sim 1.5 \times 10^{-10}M_\odot yr^{-1})$, we have $\dot{M}_{16} \sim 1$, and luminosity $10^{36}ergs^{-1}$ represent value commonly found in such system. Further by comparison with $Eq.$(1.2.3) it can be seen for steady accretion $\dot{M}_{16}$is limited by $10^2$. Thus, accretion rate must be less than about $10^{18}gs^{-1}$ if the assumptions made in deriving the Eddington limit is valid [8].

### 1.3.3 Roche Lobe Overflow

Roche lobe overflow happens when the star gets old enough to expand far out to fill up to $L_1 - 1^{st}$ lagrangian point. If the distance between the two star is sufficiently small, the Roche lobe can become smaller than the actual volume of one of the stars, causing its matter to be gravitationally captured by the other star via Accretion disc.[8]

Roche lobe overflow was first studied by Edouard Roche. The Roche approach is to consider the orbit of a test particle in the gravitational potential due to two massive bodies orbiting each other under the influence of their mutual gravitational attractions(which some times called restricted three-body problem). Thus, the two stars execute Kepler orbits about each other in a plane. The Roche problem assumes these orbits to be circular. Which is a good approximation for binary systems, since tidal effects tend to circularize originally eccentric orbits on time scale short compared to the time scale over which mass transfer occurs. Moreover assumption is made that the two stars are 'centrally condensed' in the sense that they can be considered as point masses for dynamical purposes. If we write the two masses as $M_1M_\odot$ and $M_2M_\odot$, where $0.1 < M_1$ , $M_2 < 100$ for typical stars, the binary separation , "a", can be expressed in terms of the fundamental observational quantity, the binary period, p, through Kepler's law, as

$$4\pi^2 a^3 = G(M_1 + M_2)M_\odot p^2 \tag{1.3.6}$$

For binary periods of the order of years ,days , or , hours ,"a" can generally be expressed in alternative forms:

$$a = \left\{ \begin{array}{l} 1.5 \times 10^{13} M_1^{\frac{1}{3}} (1+q)^{\frac{1}{3}} p_{yr}^{\frac{2}{3}} cm, \\ 2.9 \times 10^{11} M_1^{\frac{1}{3}} (1+q)^{\frac{1}{3}} p_{day}^{\frac{2}{3}} cm, \\ 3.5 \times 10^{10} M_1^{\frac{1}{3}} (1+q)^{\frac{1}{3}} p_{hr}^{\frac{2}{3}} cm, \end{array} \right\}. \tag{1.3.7}$$

where $p_{yr} = p$ in years etc, and $q = \frac{M_2}{M_1}$

$M_1M_\odot$ and $M_2M_\odot$ are masses of the primary star and the secondary star respectively. These are shown in the figure below (Fig:1.3) .

According to the Roche model the gravitational equipotential surfaces, makes contacts at $L_1$, the first lagrangian point where the gravitational forces of the companion

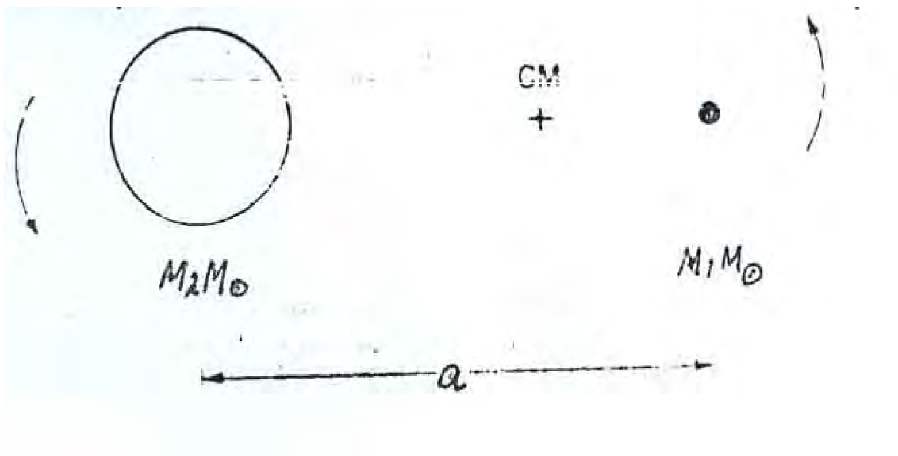


Figure 1.3: A binary system with a compact star of mass $M_1 M_\odot$ and a 'normal' star of mass $M_2 M_\odot$ orbiting their common center of mass with separation "$a$".

and the compact object and the centrifugal forces cancel. Each such points is associated with two lobes called Roche lobes as shown in the figure below (Fig:1.4). When, due to its evolutionary phase of expansion, the companion fills its Roche lobe, matter will stream through $L_1$ to the compact object. This process is called Roche lobe overflow.

### 1.3.4 Accretion Disc

The initial trajectory of matter issued from $L_1$ would be an elliptical orbit lying in the binary plane. The presence of the secondary causes the orbit to precess slowly. The stream will therefore intersect itself, resulting in dissipation of energy. However since the angular momentum is conserved, the gas will tend to the orbit of lowest energy for a given angular moment, i.e. a circular orbit.

In most cases the total mass of gas in the disk is so small that we can neglect

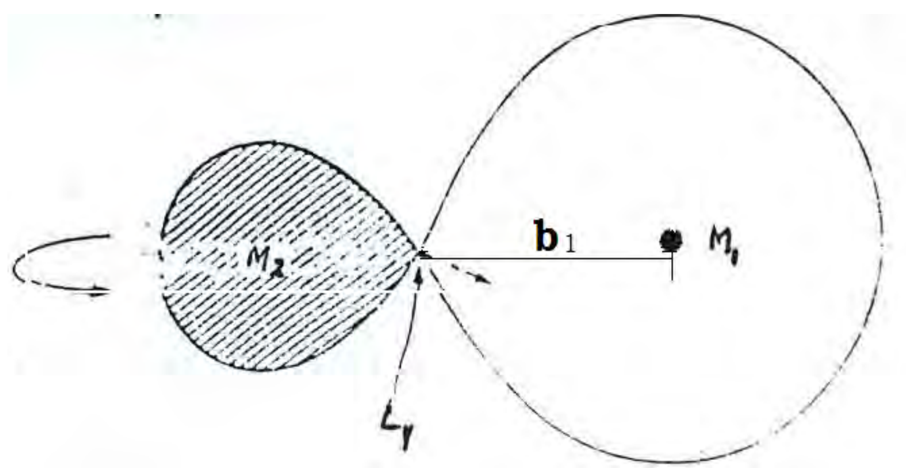


Figure 1.4: A binary system with the secondary star filling the roche lobe and transferring mass through $L_1$ into the lobe of the compact primary.

the self-gravity of the disk. The circular orbit is then Keplerian with angular velocity $\Omega_K(R) = (GM_1/R^3_{circ})^{1/2}$. The radius of this circular orbit is called the circularization radius $R_{circ}$. It is determined from the relation $(GM_1R_{circ})^{1/2} = b_1^2\omega$. Then $R_{circ}/a = (1+q)[0.5 - 0.227\log q]^4$.

Within the ring of radius $R_{circ}$, there will be dissipative processes, such as collisions, shocks, and viscous dissipation. These will convert some of the energy of the ordered bulk orbital motion into internal energy (heat). Eventually some of this energy is radiated and therefore lost from the gas. As a result the gas has to sink deeper into the gravitational potential of the primary, orbiting it more closely. This in turn will make it lose angular momentum. So most of the gas will spiral inwards towards the primary through a series of approximately circular orbits. The angular momentum is transferred outwards through the disk by viscous torques. The outer parts of the ring will gain angular momentum and will spiral outwards. The original ring of matter at $R = R_{circ}$ will spread to both smaller and larger radii by this process, to form an ***accretion disk***[9,10].

### 1.3.5 Accretion Column

Column accretion occurs when the accreting star possess a magnetic field strong enough to disrupt the inner regions of the disc channeling the accretion flow in such a way that nearly resembles free-fall on the magnetic polar caps.

# Chapter 2

# Gamma Ray Sources And Generating Mechanisms

## 2.1 Introduction

Pulsars are the brightest sources of gamma-rays. They emit radiation in the whole electromagnetic spectrum and they are a perfect target for multi wavelength campaigns[11]. Here we will discuss the general behavior of the seven well-studied gamma ray pulsars. In addition to this the main mechanisms of gamma-rays production(inverse compton(IC) scattering , Bremsstrahlung and $\pi^0$-decays ) will be presented below.

## 2.2 Gamma Ray Pulsars

There are now convincing evidences that pulsars are rotating, highly-magnetized neutron stars emitting pulsed radiation because of rotation. Pulsars are a well established class of gamma-ray sources. Young pulsars with high spin rates are the most powerful and the most likely to produce gamma rays. A small fraction of the rotational energy is converted into radiation and the bulk of radiation is emitted in gamma-rays.

Pulsars were first discovered in 1967. Today more than 1700 radio pulsars are known, while pulsars in other wavelengths are more difficult to detect, because of a combination of emission geometry and telescopes sensitivity. About 80 x-ray pulsars are known and only 7 gamma-ray pulsars. Eventhough there are only 7 confirmed gamma-ray pulsars, but we have good reason to think there are more out there [12].

The current knowledge about pulsars is still far to be complete, and the study and discovery of new gamma-ray pulsars will provide a perfect tool for probing these extreme intriguing sources.

The three brightest point sources in the gamma-ray sky appeared to be the Vela (PSR B0833-45), Crab (PSR B0531+21) and Geminga (PSR J0633+1746). Pulsation of Vela and Crab were detected by SAS-2 and extended by the COS B mission. A major step in understanding gamma-ray pulsars came with the launch of the Compton Gamma Ray Observatory (CGRO). CGRO carried four experiments (COMPTEL, OSSE, BATSE, EGRET) that increased to seven the number of currently known gamma-ray pulsars. EGRET was a pair conversion telescope that discovered the gamma-ray emission from PSR B1706-44,6 PSR B1055-52,7 PSR B1951+328 and detected the modulation of the gamma-rays from Geminga. The BATSE telescope, mainly devoted to Gamma Ray Bursts, detected the pulsar PSR B1509-58 that was observed also by COMPTEL up to 10 MeV. Some general facts can be derived by looking at the seven high-confidence gamma-ray pulsars in a multi wavelength context and by comparing them to the population of radio pulsars. The light curves look different at different energies, from which we can infer that the emission mechanisms are the result of a combination of geometry and energy band. For most of them the light curve has two peaks. Not all have been observed at EGRET energies, e.g. PSR

B1509-58 as been seen by COMPTEL up to 10 MeV.

The three brightest gamma-ray pulsars(Vela, Geminga and Crab) show a phase-dependent spectrum with no simple phase-energy pattern, while for the others a phase-resolved spectroscopic study is not possible because of the low statistics. Although there is a concentration of millisecond radio pulsars, there is not yet an high-confidence detection of a gamma-ray millisecond pulsar, with exception for a low-confidence detection of PSR J0218+4232.1

### 2.2.1 The Standard Model For Pulsars

In the so-called "Standard model" of pulsars the pulsed emission is explained as a consequence of the rapid rotation of a neutron star with high magnetic field. In 1969 Goldreich and Julian solved the equation of an aligned rotator in vacuum, showing that the environment around the pulsar must be filled by a plasma forming a magnetosphere corotating with the star. Since the magnetosphere is rotating at the same angular speed of the star, there is a critical radius, named radius of the light cylinder $R_c$, where the particles in the magnetosphere should corotate at the speed of light.

The field lines at this distance would eventually be swept out due to relativistic effects, creating some field lines that close at a distance larger than $R_c$ (See Fig:2.1). Particles can be extracted from the surface, since the electric fields are several orders of magnitude greater than the gravitational force. They are accelerated along the magnetic field lines. The particles traveling along the open field lines can reach Lorentz factors of about $\gamma \simeq 10^7$, and eventually emit radiation. The radiation losses are then not only due to magnetic dipole radiation but mainly due to magnetospheric emission. The last closed field lines define a region on the neutron star surface called

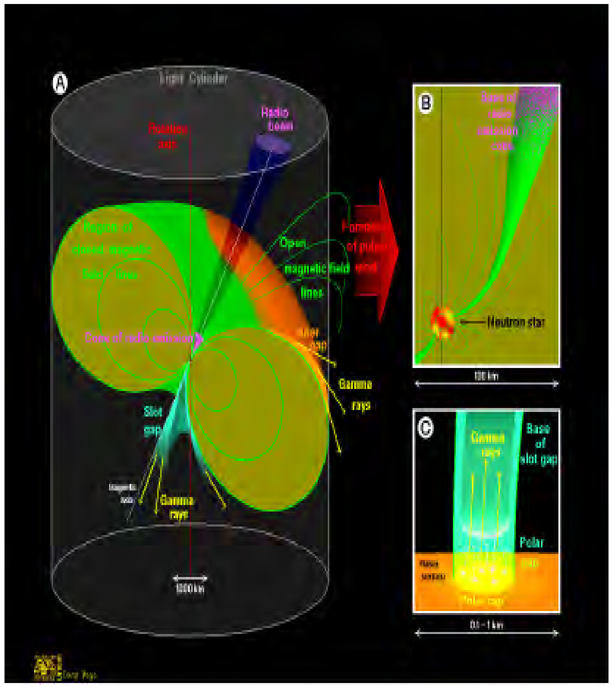


Figure 2.1: Pulsars Model

Polar Caps, located around the magnetic poles of the star. The bulk of the rotational energy is converted into pulsar wind, while a minor fraction is converted into radiation. According to the standard model the radio emission takes place above the magnetic poles and the rotation of the star produce a pulsed emission because the rotation axis and the magnetic axis are misaligned, as displayed in (Fig:2.1). The emission pattern sweep the line of sight of the observers as in a lighthouse. The gamma-ray emission takes place along the open field lines, where particles can be accelerated up to very high energies.

## 2.3 Gamma-ray generating mechanisms

The main mechanisms for gamma-ray production in a galactic scenario are inverse compton (IC) scattering (Inverse Compton scattering involves the scattering of low energy photons to high energies by ultra relativistic electrons so that the photons gain and the electrons lose energy. The process is called inverse because the electrons lose energy rather than the photons, the opposite of the standard Compton effect.) of lower frequency photons, relativistic Bremsstrahlung and $\pi^0$-decay from hadronic interactions. The common feature of all these mechanisms is that they require the presence of a population of relativistic particles (electron or positrons in the first to cases, protons or ions in the latter).[13-16]

Inverse compton scattering is expected to produce Gamma-rays with energies in the range $1 - 100Mev$; However those Gamma-rays with energies $> 100Mev$ are expected from $\pi^0$-decay and Bremsstrahlung. The dominant contribution at energies greater than 100Mev is expected to be from Gamma-rays produced in the decay of neutral pions ($\pi^0$) created in the collision of high energy protons and nuclei with protons and nuclei of the interstellar gas.

In this thesis we only consider IC scattering since our main concern is the production of Gamma-rays by IC scattering of lower frequency photons.

# Chapter 3

# COMPTON SCATTERING IN STRONG MAGNETIC FIELDS IN LABORATORY FRAME

## 3.1 Introduction

Since Herold's study of the Compton scattering in strong magnetic fields in the electron rest frame(ERF), his expression of cross-section has been widely used in astrophysical calculations. However, in actual calculations the cross-section in the laboratory frame (LF) is required. The difficulty of obtaining a LF version of Herold's cross-section in the ERF is due to the fact that the QED processes in an external magnetic are relativistic invariant only in the direction of the field and, therefore, no exact relativistic transformations are available to recover the full expression of cross-section in the LF from Herold,s expression. In this chapter, we aim at a derivation of an exact LF version of Herold's full expression of cross-section in the ERF [17].

## 3.2 Cross-section in laboratory frame

Taking into account the fact that the ground Landau level is not degenerate due to the spinor $u_{1,0}(x_1, k) = 0$, the initial and final states of an electron-photon scattering system can be represented by:

$$|i, t_i\rangle = c_0^+(p_i, t_i) a_{\lambda_i}^+(k_i, t_i)|0\rangle \tag{3.2.1}$$

$$|f, t_f\rangle = c_0^+(p_f, t_f) a_{\lambda_f}^+(k_f, t_f)|0\rangle \tag{3.2.2}$$

where, $c_0^+ = c_{2,0}^+$ is the electron creation operator, $a_\lambda^+$ is the photon creation operator and $p_{i(f)}$ denotes the incident(outgoing) electron momentum.We stress here that the operators in these two expressions are Heisenberg ones, not the free ones, meaning that interaction have been considered.

### 3.2.1 Derivation of the magnetic Feynman propagator

Consider the Dirac equation in the form,

$$[i\not{\partial} - e\not{A}(x) - m]\psi(x, t)] = 0 \tag{3.2.3}$$

in which $e > 0$ is assumed and the asymmetry gauge is taken,i.e.$A(x) = (0, Bx_1, 0)$.The solution of(3.2.3)with positive energy is:

$$\psi_{s,n}^+(x, t) = N u_{s,n}(x_1, p) e^{[-iE_n(p_3)t + ip_2x_2 + ip_3x_3]} \quad s = 1, 2 \tag{3.2.4}$$

where $E_n$indicates the Landau energy of an electron and $N$ is a normalization constant.

$$E_n(p_3) = (m^2 + p_3^2 + 2neB)^{1/2}, N = [\frac{E_n(p_3) + m}{2E_n(p_3)}]^{1/2} \tag{3.2.5}$$

The spinors $u_{1,n}(x_1,p)$,$u_{2,n}(x_1,p)$ are given by:

$$u_{1,n}(x_1,p) = \begin{bmatrix} I_{n-1}(x_1,p_2) \\ 0 \\ \frac{p_3}{E_n(p_3)+m} I_{n-1}(x_1,p_2) \\ \frac{i\sqrt{2neB}}{E_n(p_3)+m} I_n(x_1,p_2) \end{bmatrix} \tag{3.2.6}$$

$$u_{2,n}(x_1,p) = \begin{bmatrix} 0 \\ I_n(x_1,p_2) \\ \frac{-i\sqrt{2neB}}{E_n(p_3)+m} I_{n-1(x_1,p_2)} \\ \frac{-p_3}{E_n(p_3)+m} I_n(x_1,p_2) \end{bmatrix} \tag{3.2.7}$$

in which $I_n(x_1,p_2)$ is the harmonic oscillator wave function,

$$I_n(x_1,p_2) = (\lambda\sqrt{\Pi}2^n n!)^{-1/2} e^{[-1/2(\frac{x_1}{\lambda}+\lambda p_2)^2]} H_n(\frac{x_1}{\lambda}+\lambda p_2) \tag{3.2.8}$$

obeying the orthogonal condition

$$\int dx_1 I_n(x_1,p_2) I_m(x_1,p_2) = \delta_{nm} \tag{3.2.9}$$

where$\lambda^{-1} = \sqrt{eB}$.The solution of (2.1.5) with negative energy is

$$\psi^{(-)}_{s,n}(x) = N v_{s,n}(x_1,p) e^{[iE_n(p_3)t+ip_2x_2+ip_3x_3]} s = 1,2 \tag{3.2.10}$$

with the spinors $v_{1,n}(x_1,p)$ $v_{2,n}(x_1,p)$given by

$$v_{1,n}(x_1,p) = \begin{bmatrix} \frac{-p_3}{E_n(p_3)+m} I_{n-1}(x_1,p_2) \\ \frac{-i\sqrt{2neB}}{E_n(p_3+m} I_n(x_1,p_2) \\ I_{n-1}(x_1,p_2) \\ 0 \end{bmatrix} \tag{3.2.11}$$

$$v_{2,n}(x_1,p) = \begin{bmatrix} \frac{i\sqrt{2neB}}{E_n(p_3+m} I_{n-1}(x_1,p_2) \\ \frac{p_3}{E_n(p_3+m} I_n(x_1,p_2) \\ 0 \\ I_n(x_1,p_2) \end{bmatrix} \tag{3.2.12}$$

It is clear that the spinors $u_{s,n}(x_1,p)$,$v_{s,n}(x_1,p)$,s=1,2 form a complete and orthogonalized basis in the four-dimensional vector space. Therefore the Dirac field operators can be expanded as:

$$\psi(x)=\sum_{n=0}^{\infty}\sum_{s=1}^{2}\frac{1}{L}\sum_{p_2,p_3}[c_{s,n}(p,t)u_{s,n}(x_1,p)+d^{+}_{s,n}(p,t)v_{s,n}(x_1,p)]e^{(ip_2x_2+ip_3x_3)} \quad (3.2.13)$$

$$\psi^{+}(x)=\sum_{n=0}^{\infty}\sum_{s=1}^{2}\frac{1}{L}\sum_{p_2,p_3}[c^{+}_{s,n}(p,t)u^{+}_{s,n}(x_1,p)+d_{s,n}(p,t)v^{+}_{s,n}(x_1,p)]e^{(-ip_2x_2-ip_3x_3)} \quad (3.2.14)$$

We know that the Feynman propagator of an electron is given by:

$$S_F(x,y)=-i\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle=-i\langle 0|\psi(x)\psi^{+}(y)\gamma^0|0\rangle, \quad (3.2.15)$$

for $t_x>t_y$

$$\begin{aligned}\Rightarrow S_F(x,y) &= -i\langle 0|\sum_{n=0}^{\infty}\sum_{s=1}^{2}\frac{1}{L^2}\sum_{p_2,p_3}\{[c_{s,n}(p,t)u_{s,n}(x_1,p)+d^{+}s,n(p,t)v_{s,n}(x_1,p)]\\ &\times [c^{+}_{s,n}(p,t)u^{+}_{s,n}(y_1,p)+d_{s,n}(p,t)v^{+}_{s,n}(y_1,p)]\} \quad (3.2.16)\\ &\times e^{ip_2(x_2-y_2+ip_3(x_3-y_3)}\gamma^0|0\rangle\end{aligned}$$

$$\begin{aligned}\Rightarrow S_F(x,y) &= -i\langle 0|\sum_{n=0}^{\infty}\sum_{s=1}^{2}\frac{1}{L^2}\sum_{p_2,p_3}\{[c_{s,n}(p,t)u_{s,n}(x_1,p)c^{+}_{s,n}(p,t)u^{+}_{s,n}(y_1,p)]\\ &+ [c_{s,n}(p,t)u_{s,n}(x_1,p)d_{s,n}(p,t)v^{+}_{s,n}(y_1,p)]\\ &+ [d^{+}s,n(p,t)v_{s,n}(x_1,p)c^{+}_{s,n}(p,t)u^{+}_{s,n}(y_1,p)] \quad (3.2.17)\\ &+ [d^{+}s,n(p,t)v_{s,n}(x_1,p)d_{s,n}(p,t)v^{+}(y_1,p)]\}\times e^{ip_2(x_2-y_2+ip_3(x_3-y_3)}\gamma^0|0\rangle\end{aligned}$$

From the commutation relations between the annihilation and destruction operators,we have

$$c_{s,n}(p_1,t)c^{+}_{r,m}(p_2,t)+c^{+}_{r,m}(p_2,t)c_{s,n}(p,t)=\delta_{sr}\delta_{nm}\delta_{p_1p_2} \quad (3.2.18)$$

$$d_{s,n}(p_1,t)d^{+}_{r,m}(p_2,t) + d^{+}_{r,m}(p_2,t)d_{s,n}(p_1,t) = \delta_{sr}\delta_{nm}\delta_{p_1p_2} \tag{3.2.19}$$

Therefore,

$\langle 0|c_{s,n}(p_1,t)c^{+}_{r,m}(p_2,t)|0\rangle = \delta_{sr}\delta_{nm}\delta_{p_1p_2}$

and

$\langle 0|d_{s,n}(p_1,t)d^{+}_{r,m}(p_2,t)|0\rangle = \delta_{sr}\delta_{nm}\delta_{p_1p_2}$,

from the properties of raising and lowering operators.

Using these two identities in(2.1.19), we will have

$$\begin{aligned} S_F(x,y) &= -i\langle 0|\sum_{n=0}^{\infty}\sum_{s=1}^{2}\frac{1}{L^2}\sum_{p_2,p_3}\{[c_{s,n}(p,t)u_{s,n}(x_1,p)c^{+}_{s,n}(p,t)u^{+}_{s,n}(y_1,p)] \\ &\times\ e^{ip_2(x_2-y_2+ip_3(x_3-y_3)}\gamma^0|0\rangle \end{aligned} \tag{3.2.20}$$

$$\Rightarrow S_F(x,y) = \frac{-i}{L^2}\sum_{n=0}^{\infty}\sum_{s=1}^{2}\sum_{p_2,p_3} u_{s,n}(x_1,p)u^{+}_{s,n}(y_1,p)e^{ip_2(x_2-y_2+ip_3(x_3-y_3)}\gamma^0 \tag{3.2.21}$$

$$= \frac{-i}{L^2}\sum_{n=0}^{\infty}\sum_{p_2,p_3}\{u_{1,n}(x_1,p)u^{+}_{1,n}(y_1,p)+u_{2,n}(x_1,p)u^{+}_{2,n}(y_1,p)\}e^{\{}ip_2(x_2-y_2)+ip_3(x_3-y_3)\}\gamma^0\} \tag{3.2.22}$$

$$\begin{aligned} u_{1,n}(x_1,p)u^{+}_{1,n}(y_1,p) &= \begin{bmatrix} I_{n-1}(x_1,p_2) \\ 0 \\ \frac{p_3}{E_n(p_3)+m}I_{n-1}(x_1,p_2) \\ \frac{-i\sqrt{2neB}}{E_n(p_3)+m}I_n(x_1,p_2) \end{bmatrix} \\ &\times\ \begin{bmatrix} I_{n-1}(y_1,p_2) & 0 & \frac{p_3}{E_n(p_3)+m}I_{n-1}(y_1,p_2) & i\frac{\sqrt{2neB}}{E_n(p_3)+m}I_n(y_1,p_2) \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} I_{n-1}I_{n-1} & 0 & \frac{p_3}{E_n(p_3)+m}I_{n-1}I_{n-1} & \frac{i\sqrt{2neB}}{E_n(p_3)+m}I_{n-1}I_n \\ 0 & 0 & 0 & 0 \\ \frac{p_3}{E_n(p_3)+m}I_{n-1}I_{n-1} & 0 & \frac{p_3^2}{(E_n(p_3)+m)^2}I_{n-1}I_{n-1} & -ip_3\frac{\sqrt{2neB}}{(E_n(p_3)+m)^2}I_{n-1}I_{n-1} \\ \frac{i\sqrt{2neB}}{E_n(p_3)+m}I_nI_{n-1} & 0 & \frac{I\sqrt{2neB}}{(E_n(p_3)+m)^2}I_nI_{n-1} & \frac{2neB}{(E_n(p_3)+m)^2}I_nI_n \end{bmatrix} \tag{3.2.23}$$

$$u_{2,n}(x_1,p)u^{+}_{2,n}(y_1,p) = \begin{bmatrix} 0 \\ I_n(x_1,p_2) \\ \frac{-i\sqrt{2neB}}{E_n(p_3)+m} I_{n-1}(x_1,p_2) \\ \frac{-p_3}{E_n(p_3)+m} I_n(x_1,p_2) \end{bmatrix} \times \begin{bmatrix} 0 & I_n(y_1,p_2) & \frac{i\sqrt{2neB}}{E_n(p_3)+m} I_{n-1}(y_1,p_2) & \frac{-p_3}{E_n(p_3)+m} I_n(y_1,p_2) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I_nI_n & \frac{i\sqrt{2neB}}{E_n(p_3)+m} I_nI_{n-1} & \frac{-p_3}{E_n(p_3)+m} I_nI_n \\ 0 & \frac{-i\sqrt{2neB}}{E_n(p_3)+m} I_{n-1}I_n & \frac{2neB}{(E_n(p_3)+m)^2} I_{n-1}I_{n-1} & \frac{i\sqrt{2neB}}{(E_n(p_3)+m)^2} I_{n-1}I_n \\ 0 & \frac{-p_3}{E_n(p_3)+m} I_nI_n & \frac{-ip_3\sqrt{2neB}}{(E_n(p_3)+m)^2} I_nI_{n-1} & \frac{p_3^2}{(E_n(p_3)+m)^2} I_nI_n \end{bmatrix} \quad (3.2.24)$$

Summing (2.1.26)and (2.1.28),we will obtain:

$$\begin{bmatrix} I_{n-1}I_{n-1} & 0 & \frac{p_3}{E_n(p_3)+m} I_{n-1}I_{n-1} & \frac{i\sqrt{2neB}}{E_n(p_3)+m} I_{n-1}I_n \\ 0 & I_nI_n & \frac{i\sqrt{2neB}}{E_n(p_3)+m} I_nI_{n-1} & \frac{-p_3}{E_n(p_3)+m} I_nI_n \\ \frac{p_3}{E_n(p_3)+m} I_{n-1}I_{n-1} & \frac{-i\sqrt{2neB}}{E_n(p_3)+m} I_{n-1}I_n & \frac{(p_3^2+2neB)}{(E_n(p_3)+m)^2} I_{n-1}I_{n-1} & 0 \\ \frac{i\sqrt{2neB}}{E_n(p_3)+m} I_nI_{n-1} & \frac{-p_3}{E_n(p_3)+m} I_nI_n & 0 & \frac{(p_3^2+2neB)}{(E_n(p_3)+m)^2} I_nI_n \end{bmatrix} \quad (3.2.25)$$

We know that

$E_n(p_3) = (m^2+p_3^2+2neB)^{\frac{1}{2}} \Rightarrow p_3^2+2neB = E_n^2(p_3)-m^2 = (E_n(p_3)-m)(E_n(p_3)+m)$

After inserting this relation into (2.1.29) and multiplying by $\gamma^0$, the propagator will have the form:

$$S_F(x,y) = \frac{1}{L^2}\sum_{p_2,p_3}\int \frac{d\omega}{2\Pi}\sum_{n=0}^{\infty} \frac{S_n(x_1,y_1,p)}{(\omega+i\epsilon)(E_n+m)} \times e^{[-i\omega(t_x-t_y)+ip_2(x_2-y_2)+ip_3(x_3-y_3)]} \quad (3.2.26)$$

where,

$$S_n(x_1,y_1,p)=\begin{bmatrix}(E_n+m)I_{n-1}I_{n-1} & 0 & -p_3I_{n-1}I_{n-1} & i\sqrt{2neB}I_{n-1}I_n\\ 0 & (E_n(p_3)+m) & -i\sqrt{2neB}I_nI_{n-1} & p_3I_nI_n\\ p_3I_nI_n & -i\sqrt{2neB}I_{n-1}I_n & -(E_n-m)I_{n-1}I_{n-1} & 0\\ i\sqrt{2neB}I_nI_{n-1} & -p_3I_nI_n & 0 & -(E_n-m)I_nI_n\end{bmatrix} \tag{3.2.27}$$

### 3.2.2 Calculation of the scattering amplitude

$$\psi^+(x)=\sum_{n=0}^{\infty}\sum_{s=1}^{2}\frac{1}{L}\sum_{p_2,p_3}[c^+_{s,n}(p,t)u^+_{s,n}(x_1,p)+d_{s,n}(p,t)v^+_{s,n}(x_1,p)]e^{-ip_2x_2-ip_3x_3} \tag{3.2.28}$$

$$\begin{aligned}\langle 0|Tc_0(p_f,t_f)\bar{\psi}(r_1,t)|0\rangle &= \langle 0|c_0(p_f,t_f)\bar{\psi}(r_1,t)|0\rangle, t_f\to\infty\\ &= \langle 0|c_0(p_f,t_f)\psi^+(r_1,t)\gamma^0|0\rangle\\ &= \sum_{n=0}^{\infty}\sum_{s=1}^{2}\frac{1}{L}\sum_{p_2,p_3}[\langle 0|c_0(p_f,t_f)c^+_{s,n}(p,t)u^+_{s,n}(x_1,p)\gamma^0|0\rangle\\ &+ \langle 0|c_0(p_f,t_f)d_{s,n}(p,t)v^+_{s,n}(x_1,p)\gamma^0|0\rangle]\times e^{-i(p_2x_2+p_3x_3)}\\ &= \sum_{n=0}^{\infty}\sum_{s=1}^{2}\frac{1}{L}\sum_{p_2,p_3}[\langle 0|c_0(p_f,t_f)c^+_{s,n}(p,t)u^+_{s,n}(x_1,p)\gamma^0|0\rangle]\times e^{-i(p_2x_2+p_3x_3)}\\ &= \frac{1}{L}\sum_{p_2,p_3}[\langle 0|c_0(p_f,t_f)c^+_{2,0}(p,t)u^+_{2,0}(x_1,p)\gamma^0|0\rangle]\times e^{-i(p_2x_2+p_3x_3)}\\ &= \frac{1}{L}\sum_{p_2,p_3}\bar{u_0}(x_1,p_f)e^{-i(p_2x_2+p_3x_3)}\end{aligned} \tag{3.2.29}$$

The incident (outgoing) electron momentum can be expressed as $p_{i(f)}=(0,a_{i(f)},p_{i(f)})$, meaning the incident (outgoing) electron momentum along the direction of the magnetic field (taken as the Z direction) is $p_{i(f)}$ and the center of the Landau orbit is$-\lambda^2a_{i(f)}$

$$\Rightarrow \langle 0|Tc_0(p_f,t_f)\bar{\psi}(r_1,t)|0\rangle=\frac{1}{L}\bar{u_0}(x_1,p_f)e^{(-iE_ft_f-ip_fz_1-ia_fy_1+iE_ft_1)} \tag{3.2.30}$$

similarly,

$$\langle 0|Ta_{\lambda_f}(k_f,t_f)A_\mu(\vec{r_1},t)|0\rangle = \frac{1}{\sqrt{v}}\frac{1}{\sqrt{2\omega_f}}e_\mu^{(\lambda_f)}e^{(-i\omega_f t_f + i\omega_f t_1 - i\vec{k_f}.\vec{r_1})} \tag{3.2.31}$$

but,$i\vec{k_f}.\vec{r_1} = ik_{fx}x_1 + ik_{fy}y_1 + ik_{fz}z_1$,

where,$k_{fz} = k_f cos\theta_f$

$$\langle 0|Ta_{\lambda_f}(k_f,t_f)A_\mu\vec{r_1},t)|0\rangle = \frac{1}{\sqrt{v}}\frac{1}{\sqrt{2\omega_f}}e_\mu^{(\lambda_f)}e^{(-i\omega_f t_f + i\omega_f t_1 - ik_{fx}x_1 - ik_{fy}y_1 - ik_f cos\theta_f z_1)} \tag{3.2.32}$$

$$\langle 0|T\psi(\vec{r_2},t)c_0^+(p_i,t_i)|0\rangle = \frac{1}{L}u_0(x_2,p_i)e^{(iE_i t_i + ip_i z_2 + ia_i y_2 - iE_i t_2)} \tag{3.2.33}$$

$$\langle 0|Ta_{\lambda_i}^+(k_i,t_i)A_\mu(\vec{r_1},t)|0\rangle = \frac{1}{\sqrt{V}}\frac{1}{\sqrt{2\omega_i}}e_\mu^{(\lambda_i)}e^{(i\omega_i t_i - i\omega_i t_2 + i\vec{k_i}.\vec{r_2})} \tag{3.2.34}$$

$$\langle 0|Ta_{\lambda_i}^+(k_i,t_i)A_\mu(\vec{r_1},t)|0\rangle = \frac{1}{\sqrt{V}}\frac{1}{\sqrt{2\omega_i}}e_\mu^{(\lambda_i)}e^{(i\omega_i t_i - i\omega_i t_2 + ik_{ix}x_2 + ik_{iy}y_2 + ik_i cos\theta_i z_2)} \tag{3.2.35}$$

The scattering matrix can be generally expressed by:

$$S_{fi} = \lim_{t_i\to-\infty,t_f\to\infty}\langle 0|T[c_0(p_f,t_f)a_{\lambda_f}(k_f,t_f)c_0^+(p_i,t_i)a_{\lambda_i}^+(k_i,t_i)]|0\rangle \tag{3.2.36}$$

Introducing the incoming interaction picture and making perturbation expansions, one then obtains under the Born approximation:

$$\begin{aligned} S_{fi} &= \lim_{t_i\to-\infty,t_f\to\infty} e^2\int d^4x_1 d^4x_2 \langle 0|Tc_0(p_f,t_f)\bar{\psi}(x_1)|0\rangle \\ &\times \gamma_\mu\langle 0|T\psi(x_1)\bar{\psi}(x_2)|0\rangle\gamma_\nu \\ &\times [\langle 0|Ta_{\lambda_f}(k_f,t_f)A_\mu(x_1)|0\rangle\langle 0|Ta_{\lambda_i}^+(k_i,t_i)A_{\nu(x_2)}|0\rangle \\ &+ \langle 0|Ta_{\lambda_f}(k_f,t_f)A_\nu(x_2)|0\rangle\langle 0|Ta_{\lambda_i}^+(k_i,t_i)A_\mu(x_1)|0\rangle] \\ &\times \langle 0|T\psi(x_2)c_0^+(p_i,t_i)|0\rangle \end{aligned} \tag{3.2.37}$$

$$
\begin{aligned}
S_{fi} &= \lim_{t_i \to -\infty, t_f \to \infty} e^2 \int d^4x_1 d^4x_2 \langle 0|Tc_0(p_f, t_f)\bar{\psi}(x_1)|0\rangle \\
&\times \gamma_\mu \langle 0|T\psi(x_1)\bar{\psi}(x_2)|0\rangle \gamma_\nu \\
&\times \langle 0|Ta_{\lambda_f}(k_f, t_f)A_\mu(x_1)|0\rangle \langle 0|Ta^+_{\lambda_i}(k_i, t_i)A_{\nu(x_2)}|0\rangle \\
&\times \langle 0|T\psi(x_2)c_0^+(p_i, t_i)|0\rangle \\
&+ \lim_{t_i \to -\infty, t_f \to \infty} e^2 \int d^4x_1 d^4x_2 \langle 0|Tc_0(p_f, t_f)\bar{\psi}(x_1)|0\rangle \\
&\times \gamma_\mu \langle 0|T\psi(x_1)\bar{\psi}(x_2)|0\rangle \gamma_\nu \\
&\times \langle 0|Ta_{\lambda_f}(k_f, t_f)A_\nu(x_2)|0\rangle \langle 0|Ta^+_{\lambda_i}(k_i, t_i)A_\mu(x_1)|0\rangle] \\
&\times \langle 0|T\psi(x_2)c_0^+(p_i, t_i)|0\rangle
\end{aligned}
$$

for this two expressions the respective feyman diagram will be as follows

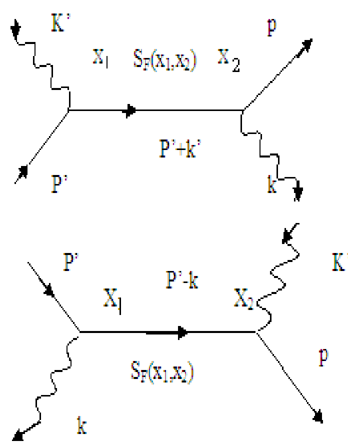


Figure 3.1: feyman diagram

$$
\begin{aligned}
S_{fi} &= \lim_{t_i \to -\infty, t_f \to \infty} e^2 \int d^4x_1 d^4x_2 \frac{1}{L} \overline{u_0}(x_1, p_f) e^{-i(E_f t_f - E_f t_1) - i(p_f z_1 + a_f y_1)} \\
&\times \gamma_\mu \langle 0 | T\psi(x_1)\overline{\psi}(x_2)|0\rangle \gamma_\nu \\
&\times [\frac{1}{\sqrt{v}} \frac{1}{\sqrt{2\omega_f}} e_\mu^{(\lambda_f)} e^{(-i\omega_f t_f + i\omega_f t_1 - ik_{fx}x_1 - ik_{fy}y_1 - ik_f cos\theta_f z_1)} \\
&\times \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2\omega_i}} e_\mu^{(\lambda_i)} e^{(i\omega_i t_i - i\omega_i t_2 + ik_{ix}x_2 + ik_{iy}y_2 + ik_i cos\theta_i z_2)} \\
&+ \frac{1}{\sqrt{v}} \frac{1}{\sqrt{2\omega_f}} e_\nu^{(\lambda_f)} e^{-i(\omega_f t_f - \omega_f t_2) - ik_{fx}x_2 - ik_{fy}y_2 - ik_f cos\theta_f z_2} \\
&\times \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2\omega_i}} e_\mu^{(\lambda_i)} e^{i(\omega_i t_i - i\omega_i t_1) + ik_{ix}x_1 + ik_{iy}y_1 + ik_i cos\theta_i z_1}] \\
&\times \frac{1}{L} u_0(x_2, p_i) e^{i(E_i t_i - E_i t_2) + i(p_i z_2 + a_i y_2)} \qquad (3.2.38)
\end{aligned}
$$

$$
\begin{aligned}
S_{fi} &= \lim_{t_i \to -\infty, t_f \to \infty} \frac{1}{L^2 V} \frac{1}{\sqrt{4\omega_i \omega_f}} \exp(2) \int d^4x_1 d^4x_2 \bar{u}_0(x_1, p_f) \gamma_\mu \\
&\times \frac{1}{L^2} \sum_{q_y, q_z} \int \frac{dq_0}{2\pi} \sum_{n=0}^{\infty} \frac{S_n(x_1, x_2, q)}{q_0^2 - E_n^2(q_z) + i\epsilon} \\
&\times \exp[iq_0(t_2 - t_1) + iq_y(y_1 - y_2) + iq_z(z_1 - z_2)]\gamma_\nu \\
&\times \{\big[\exp[-i(E_f t_f - E_f t_1) - i(p_f z_1 + a_f y_1)] e_\mu^{(\lambda_f)} \\
&\times \exp[-i(\omega_f t_f - \omega_f t_1) - ik_{fx}x_1 - ik_{fy}y_1 - ik_f \cos\theta_f z_1] \\
&\times e_\nu^{(\lambda_i)} \exp[i(\omega_i t_i - \omega_i t_2) + ik_{ix}x_2 + ik_{iy}y_2 + ik_i \cos\theta_i z_2]\big] \\
&+ \big[\exp[-i(E_f t_f - E_f t_1) - i(p_f z_1 + a_f y_1)] e_\nu^{(\lambda_f)} \\
&\times \exp[-i(\omega_f t_f - \omega_f t_2) - ik_{fx}x_2 - ik_{fy}y_2 - ik_f \cos\theta_f z_2] \\
&\times e_\mu^{(\lambda_i)} \exp[i(\omega_i t_i - \omega_i t_1) + ik_{ix}x_1 + ik_{iy}y_1 + ik_i \cos\theta_i z_1]\big]\} \\
&\times u_0(x_2, p_i) \exp[i(E_i t_i - E_i t_2) + i(p_i z_2 + a_i y_2)] \qquad (3.2.39)
\end{aligned}
$$

$$\Rightarrow S_{fi} = \frac{(2\pi)^5}{L^2 V} \quad * \quad \frac{e^2}{2\sqrt{\omega_i \omega_f}} \sum_{n=0}^{\infty} \frac{1}{L^2} \sum_{q_y, q_z} \int d^4x_1 d^4x_2 \int dq_0 \bar{u_0}(x_1, p_f)$$
$$\times \quad [\hat{e_f} \frac{S_n(x_1, x_2, q)}{(E_i + \omega_i)^2 - E_n^2(q_z) + i\epsilon}$$
$$\times \quad \hat{e_i} \delta(\omega_f + E_f - q_0) \delta(q_0 - \omega_i - E_i)$$
$$\times \quad \delta(k_{iy} + a_i - q_y) \delta(q_y - k_{fy} - a_f) \delta(q_z - k_f \cos\theta_f - p_f)$$
$$\times \quad \delta(k_i \cos\theta_i + p_i - q_z) \exp(-ik_{fx}x_1 + ik_{ix}x_2) + \hat{e_i} \frac{S_n(x_1, x_2, q)}{(E_i - \omega_f)^2 - E_n^2(q_z) + i\epsilon}$$
$$\times \quad \hat{e_f} \delta(\omega_f - E_i + q_0) \delta(E_f - \omega_i - q_0)$$
$$\times \quad \delta(k_{iy} - a_f + q_y) \delta(a_i - k_{fy} - q_y) \delta(q_z + k_i \cos\theta_i - p_f)$$
$$\times \quad \delta(p_i - k_f \cos\theta_f - q_z) \exp(ik_{ix}x_1 - ik_{fx}x_2)] u_0(x_2, p_i)$$
$$\times \quad \lim_{t_i \to -\infty, t_f \to \infty} \exp[-i(E_f + \omega_f)t_f + i(E_i - \omega_i)t_i] \tag{3.2.40}$$

after substituting all the expressions given from(2.1.33)to(2.1.39) into(2.1.40), where$\hat{e_f} = e_\mu^{(\lambda_f)} \gamma_\mu$,$\hat{e_i} = e_\mu^{(\lambda_i)} \gamma_\mu$.

The phase factor at the end of(2.1.42)can be ignored, since the cross-section$\sim |S_{fi}|^2$. The spinors $\bar{u_0}(x_1, p_f)$,$u_0(x_2, p_i)$ can be expressed as:

$$\bar{u_0}(x_1, p_f) = [\frac{E_f + m}{2E_f}]^{\frac{1}{2}} I_0(x_1, a_f) \bar{u_f}(p_f) \tag{3.2.41}$$

$$u_0^T(x_2, p_i) = [\frac{E_i + m}{2E_i}]^{\frac{1}{2}} I_0(x_2, a_i) u_i^T(p_i), \tag{3.2.42}$$

where

$$\bar{u_f}(p_f) = \begin{pmatrix} 0 & 1 & 0 & \frac{p_f}{E_f + m} \end{pmatrix} \tag{3.2.43}$$

$$u_i^T(p_i) = \begin{pmatrix} 0 & 1 & 0 & \frac{-p_i}{E_i + m} \end{pmatrix} \tag{3.2.44}$$

$$
\begin{aligned}
S_{fi} &= \frac{(2\Pi)^5}{L^2V}\frac{\exp(2)}{\sqrt{4\omega_i\omega_f}}\sum_{n=0}^{\infty}\frac{1}{L^2}\sum_{q_y,q_z}[\frac{(E_f+m)(E_i+m)}{4E_iE_f}]^{\frac{1}{2}} \\
&\times \int d^4x_1d^4x_2\int dq_0 I_0(x_1,a_f)\bar{u}_f(p_f)I_0(x_2,a_i)u_i(p_i) \\
&\times \{\hat{e_f}\frac{S_n(x_1,x_2,q)}{(E_i+\omega_i)^2-E_n^2(q_z)+i\epsilon}\hat{e_i}\delta(\omega_f+E_f-q_0)\delta(q_0-\omega_i-E_i) \\
&\times \delta(k_{iy}+a_i-q_y)\delta(q_y-k_{fy}-a_f)\delta(q_z-k_f\cos\theta_f-p_f) \\
&\times \delta(k_i\cos\theta_i+p_i-q_z)\exp(-ik_{fx}x_1+ik_{ix}x_2) \\
&+ \hat{e_i}\frac{S_n(x_1,x_2,q)}{(E_i+\omega_i)^2-E_n^2(q_z)+i\epsilon} \\
&\times \hat{e_f}\delta(\omega_f-E_i+q_0)\delta(E_f-\omega_i-q_0) \\
&\times \delta(k_{iy}-a_f+q_y)\delta(a_i-k_{fy}-q_y)\delta(q_z+k_i\cos\theta_i-p_f) \\
&\times \delta(p_i+k_f\cos\theta_f-q_z)\exp(ik_{ix}x_1-ik_{fx}x_2)\} \qquad (3.2.45)
\end{aligned}
$$

Let $I_1=\int d^4x_1d^4x_2\exp(-ik_{fx}x_1)I_0(x_1,a_f)S_n(x_1,x_2,q)\exp(ik_{ix}x_2)I_0(x_2,a_i)$,and

$I_2=\int d^4x_1d^4x_2\exp(+ik_{ix}x_1)I_0(x_1,a_f)S_n(x_1,x_2,q)\exp(-ik_{fx}x_2)I_0(x_2,a_i)$

With the help of the integral:

$\int_{-\infty}^{\infty}d\xi H_n(\xi)\exp[-(\xi-\alpha)^2]=\sqrt{\Pi}(2\alpha)^n$,these two integrals yield values,

$$
\begin{aligned}
I_1 &= \frac{(\lambda^2k_f^+k_i^-)^{(n-1)}}{2^nn!}A_n\exp[-\frac{\lambda^2}{4}(k_i^2\sin^2\theta_i+k_f^2\sin^2\theta_f)] \\
&\times \exp[i\lambda^2(a_fk_{fx}+\frac{1}{2}k_{fx}k_{fy})-i\lambda^2(a_ik_{ix}+\frac{1}{2}k_{ix}k_{iy})], \qquad (3.2.46)
\end{aligned}
$$

where $k_i^{\pm}=k_{ix}\pm ik_{iy}$ and $k_f^{\pm}=k_{fx}\pm ik_{fy}$ and $A_n$ is defined by

$$
A_n=\begin{pmatrix}
2n(q_0+m) & 0 & -2nq_z & -2nk_i^- \\
0 & (q_0+m)\lambda^2k_f^+k_i^- & -2nk_f^+ & q_z\lambda^2k_f^+k_i^- \\
2nq_z & -2nk_i^- & -2n(q_0-m) & 0 \\
2nk_f^+ & -q_z\lambda^2k_f^+k_i^- & 0 & -(q_0-m)\lambda^2k_f^+k_i^-
\end{pmatrix} \qquad (3.2.47)
$$

Similarly,

$$\begin{aligned} I_2 &= \frac{(\lambda^2 k_f^+ k_i^-)^{(n-1)}}{2^n n!} B_n \exp[-\frac{\lambda^2}{4}(k_i^2 \sin^2\theta_i + k_f^2 \sin^2\theta_f)] \\ &\times \exp[-i\lambda^2(a_f k_{ix} - \frac{1}{2}k_{ix}k_{iy}) + i\lambda^2(a_i k_{fx} + \frac{1}{2}k_{fx}k_{fy})], \end{aligned} \quad (3.2.48)$$

with the matrix $B_n$ given by

$$B_n = \begin{pmatrix} 2n(q_0+m) & 0 & -2nq_z & 2nk_f^- \\ 0 & (q_0+m)\lambda^2 k_f^- k_i^+ & 2nk_i^+ & q_z\lambda^2 k_f^- k_i^+ \\ 2nq_z & -2nk_f^- & -2n(q_0-m) & 0 \\ -2nk_i^+ & -q_z\lambda^2 k_f^- k_i^+ & 0 & -(q_0-m)\lambda^2 k_f^- k_i^+ \end{pmatrix} \quad (3.2.49)$$

$$\begin{aligned} S_{fi} &= \frac{(2\Pi)^5}{L^2 V}\frac{\exp(2)}{\sqrt{4\omega_i\omega_f}}\sum_{n=0}^{\infty}\frac{1}{L^2}\sum_{q_y,q_z}[\frac{(E_i+m)(E_f+m)}{4E_iE_f}]^{\frac{1}{2}} \\ &\times \frac{(\lambda^2 k_f^+ k_i^-)^{(n-1)}}{2^n n!}\exp[-\frac{\lambda^2}{4}(k_i^2\sin^2\theta_i + k_f^2\sin^2\theta_f)] \\ &\times \{\frac{\bar{u_f}(p_f)\hat{e_f}A_n\hat{e_i}u_i(p_i)}{(E_i+\omega_i)^2 - E_n^2(q_z) + i\epsilon} \\ &\times \exp[i\lambda^2(a_f k_{fx} + \frac{1}{2}k_{fx}k_{fy}) - i\lambda^2(a_i k_{ix} + \frac{1}{2}k_{ix}k_{iy})] \\ &\times \int dq_0 \delta(\omega_f + E_f - q_0)\delta(q_0 - \omega_i - E_i)\delta(k_{iy} + a_i - q_y) \\ &\times \delta(q_y - k_{fy} - a_f)\delta(q_z - k_f\cos\theta_f - p_f)\delta(k_i\cos\theta_i - p_i - q_z) \\ &+ \frac{\bar{u_f}(p_f)\hat{e_i}B_n\hat{e_f}u_i(p_i)}{(E_i-\omega_f)^2 - E_n^2(q_z) + i\epsilon} \\ &\times \exp[-i\lambda^2(a_f k_{ix} - \frac{1}{2}k_{ix}k_{iy}) + i\lambda^2(a_i k_{fx} + \frac{1}{2}k_{fx}k_{fy})] \\ &\times \int dq_0 \delta(\omega_f - E_i + q_0)\delta(E_f - \omega_i - q_0)\delta(k_{iy} - a_f + q_y) \\ &\times \delta(a_i - k_{fy} - q_y)\delta(p_i - k_f\cos\theta_f - q_z)\delta(q_z + k_i\cos\theta_i - p_f)\} \end{aligned} \quad (3.2.50)$$

But,

$$\sum_{q_y,q_z}\int dq_0\delta(\omega_f+E_f-q_0)\delta(q_0-\omega_i-E_i)\delta(k_{iy}+a_i-q_y)\delta(q_y-k_{fy}-a_f)\delta(q_z-k_f\cos\theta_f-p_f)$$

$$*\delta(k_i\cos\theta_i - p_i - q_z) = \delta(\omega_f + E_f - \omega_i - E_i)\delta(k_{iy} + a_i - k_{fy} - a_f)\delta(k_i\cos\theta_i + p_i - k_f\cos\theta_f - p_f)$$

Similarly,

$$\sum_{q_y,q_z}\int dq_0\delta(\omega_f - E_i + q_0)\delta(E_f - \omega_i - q_0)\delta(k_{iy} - a_f + q_y)\times\delta(a_i - k_{fy} - q_y)\delta(p_i - k_f\cos\theta_f - q_z)$$

$$*\delta(k_i\cos\theta_i + q_z - p_f) = \delta(\omega_f + E_f - \omega_i - E_i)\delta(k_{iy} + a_i - k_{fy} - a_f)\delta(k_i\cos\theta_i + p_i - k_f\cos\theta_f - p_f)$$

$$\begin{aligned} \Rightarrow S_{fi} &= \frac{(2\Pi)^3}{L^2V}\frac{\exp(2)}{\sqrt{4\omega_i\omega_f}}\sum_{n=0}^{\infty}\frac{1}{L^2}[\frac{(E_i+m)(E_f+m)}{4E_iE_f}]^{\frac{1}{2}} \\ &\times \frac{(\lambda^2k_f^+k_i^-)^{(n-1)}}{2^nn!}\exp[-\frac{\lambda^2}{4}(k_i^2\sin^2\theta_i + k_f^2\sin^2\theta_f)] \\ &\times \{\frac{P_1}{(E_i+\omega_i)^2 - E_n^2(q_z) + i\epsilon} \\ &\times \exp[i\lambda^2(a_fk_{fx} + \frac{1}{2}k_{fx}k_{fy}) - i\lambda^2(a_ik_{ix} + \frac{1}{2}k_{ix}k_{iy})] \\ &+ \frac{P_2}{(E_i-\omega_f)^2 - E_n^2(q_z) + i\epsilon} \\ &\times \exp[-i\lambda^2(a_fk_{ix} - \frac{1}{2}k_{ix}k_{iy}) + i\lambda^2(a_ik_{fx} + \frac{1}{2}k_{fx}k_{fy})]\} \\ &\times \delta(\omega_f + E_f - \omega_i - E_i)\delta(k_{iy} + a_i - k_{fy} - a_f) \\ &\times \delta(k_i\cos\theta_i + p_i - k_f\cos\theta_f - p_f), \end{aligned} \tag{3.2.51}$$

where $P_1 = \bar{u_f}(p_f)\hat{e_f}A_n\hat{e_i}u_i(p_i)$ and$P_2 = \bar{u_f}(p_f)\hat{e_i}B_n\hat{e_f}u_i(p_i)$, where$\hat{e_f} = e_\mu^{\lambda_f}\gamma_\mu$ ,$\hat{e_i} = e_\mu^{\lambda_i}\gamma_\mu$

These two scalar matrix products (i.e.$P_1 and P_2$) for all combinations of $\gamma_1, \gamma_2, \gamma_3$ will have the form:

$$\begin{aligned} P_1 &= 2n[\frac{(q_0+m)P_iP_f}{(E_i+m)(E_f+m)} + (q_0-m)]e_f^+e_i^- \\ &+ 2n[\frac{P_f}{E_f+m} + \frac{P_i}{E_i+m}](k_ie_f^+e_{iz} + k_f^+e_{fz}e_i^- + q_ze_f^+e_i^-) \\ &+ \lambda^2k_f^+k_i^-[\frac{(q_0+m)P_iP_f}{(E_i+m)(E_f+m)} + (q_0-m) + \frac{q_zP_f}{E_f+m} + \frac{q_zP_i}{E_i+m}]e_{fz}e_{iz} \end{aligned} \tag{3.2.52}$$

$$\begin{aligned} P_2 &= 2n[\frac{(q_0+m)P_iP_f}{(E_i+m)(E_f+m)}+(q_0-m)]e_i^+e_f^- \\ &- 2n[\frac{P_f}{E_f+m}+\frac{P_i}{E_i+m}](k_i^+e_f^-e_{iz}+k_f^-e_i^+e_{fz}+q_ze_i^+e_f^-) \\ &+ \lambda^2k_f^-k_i^+[\frac{(q_0+m)P_iP_f}{(E_i+m)(E_f+m)}+(q_0-m)+\frac{q_zP_f}{E_f+m}+\frac{q_zP_i}{E_i+m}]e_{fz}e_{iz} \end{aligned} \quad (3.2.53)$$

Evaluating $\exp[i\lambda^2(a_fk_{fx}+\frac{1}{2}k_{fx}k_{fy})-i\lambda^2(a_ik_{ix}+\frac{1}{2}k_{ix}k_{iy})]$ and $\exp[-i\lambda^2(a_fk_{ix}-\frac{1}{2}k_{ix}k_{iy})+i\lambda^2(a_ik_{fx}-\frac{1}{2}k_{fx}k_{fy})]$

at the value $a_f = k_{iy}+a_i-k_{fy}$:

$$\begin{aligned} &\exp[i\lambda^2(a_fk_{fx}+\frac{1}{2}k_{fx}k_{fy})-i\lambda^2(a_ik_{ix}+\frac{1}{2}k_{ix}k_{iy})] \\ &= \exp[i\lambda^2(k_{iy}k_{fx}+a_ik_{fx}-k_{fx}k_{fy}+\frac{1}{2}k_{fx}k_{fy})-i\lambda^2(a_ik_{ix}+\frac{1}{2}k_{ix}k_{iy})] \\ &= \exp[i\lambda^2(k_{iy}k_{fx}+a_ik_{fx}-\frac{1}{2}k_{fx}k_{fy})-i\lambda^2(a_ik_{ix}+\frac{1}{2}k_{ix}k_{iy})] \\ &= \exp(i\lambda^2(k_{iy}k_{fx}))\exp[i\lambda^2(-\frac{1}{2}k_{fx}k_{fy}-\frac{1}{2}k_{ix}k_{iy})+i\lambda^2(a_ik_{fx}-a_ik_{ix})] \\ &= \exp(i\lambda^2(k_{iy}k_{fx}))\exp[-\frac{i\lambda^2}{2}(k_{ix}k_{iy}+k_{fx}k_{fy})-i\lambda^2a_i(k_{ix}-k_{fx})] \end{aligned}$$

Similarly

$$\begin{aligned} &\exp[-i\lambda^2(a_fk_{ix}-\frac{1}{2}k_{ix}k_{iy})+i\lambda^2(a_ik_{fx}-\frac{1}{2}k_{fx}k_{fy})] \\ &= \exp(i\lambda^2(k_{ix}k_{fy}))\exp[-\frac{i\lambda^2}{2}(k_{ix}k_{iy}+k_{fx}k_{fy})-i\lambda^2a_i(k_{ix}-k_{fx})] \end{aligned}$$

and substituting it back into the original equation together with $P_1$ and $P_2$, the full expression for the $S_{fi}$ will be:

$$\begin{aligned} S_{fi} &= \frac{(2\Pi)^3}{L^2V}\frac{\exp(2)}{\sqrt{4\omega_i\omega_f}}[\frac{(E_i+m)(E_f+m)}{4E_iE_f}]^{\frac{1}{2}}\exp[-\frac{\lambda^2}{4}(\omega_i^2\sin^2\theta_i+\omega_f^2\sin^2\theta_f)] \\ &\times \exp[-i\lambda^2a_i(k_{ix}-k_{fx})-i\frac{\lambda^2}{2}(k_{ix}k_{iy}+k_{fx}k_{fy})]X \\ &\times \delta(E_i+\omega_i-E_f-\omega_f)\delta(p_i+k_i\cos\theta_i-p_f-k_f\cos\theta_f) \\ &\times \delta(a_i-k_{iy}-a_f-k_{fy}) \end{aligned} \quad (3.2.54)$$

where $\omega_i$ and $\omega_f$ are frequencies of the incident and scattered photons ,$E_i$ and $E_f$ represent the energies of the incident and scattered electrons, respectively,$\theta_i(\theta_f)$ denotes the angle between the incoming(outgoing)photon and the magnetic field. $X$ is given by the following expression as follows:

$$
\begin{aligned}
X &= \sum_{n=0}^{\infty} \frac{1}{n!}[(\frac{\lambda^2 k_i^- k_f^+}{2})^n \exp(i\lambda^2 k_{iy} k_{fx})(\frac{X_1}{(E_i+\omega_i)^2 - E_{i,n+1}^2} + \frac{X_2}{(E_i+\omega_i)^2 - E_{i,n}^2}) \\
&+ (\frac{\lambda^2 k_i^- k_f^+}{2})^n \exp(i\lambda^2 k_{ix} k_{fy})(\frac{X_1'}{(E_i-\omega_f)^2 - E_{f,n+1}^2} + \frac{X_2'}{(E_i-\omega_f)^2 - E_{f,n}^2})]
\end{aligned}
$$

in which $k_i^{\pm} = k_{ix} \pm ik_{iy}, k_f^{\pm} = k_{fx} \pm ik_{fy}$,

$$E_{i,n}^2 = m^2 + (p_i + \omega_i \cos\theta_i)^2 + 2neB \tag{3.2.55}$$

$$E_{f,n}^2 = m^2 + (p_i - \omega_f \cos\theta_f)^2 + 2neB \tag{3.2.56}$$

and $X_i, X_i', i = 1, 2$,are given by

$$
\begin{aligned}
X_1 &= [\frac{(E_i+\omega_i+m)p_i(p_i+\omega_i\cos\theta_i - \omega_f\cos\theta_f)}{(E_i+m)(E_f+m)} + (E_i+\omega_i-m)]e_i^- e_f^+ \\
&+ (\frac{p_i}{E_i+m} + \frac{p_i+\omega_i\cos\theta_i-\omega_f\cos\theta_f}{E_f+m}) \\
&\times [k_f^+ e_{fz} e_i^- + k_i^- e_{iz} e_f^+ - (p_i+\omega_i\cos\theta_i) e_f^+ e_i^-]
\end{aligned} \tag{3.2.57}
$$

$$
\begin{aligned}
X_2 &= [\frac{(E_i+\omega_i+m)p_i(p_i+\omega_i\cos\theta_i - \omega_f\cos\theta_f)}{(E_i+m)(E_f+m)} \\
&+ (\frac{p_i}{E_i+m} + \frac{p_i+\omega_i\cos\theta_i-\omega_f\cos\theta_f}{E_f+m}) \\
&\times (p_i+\omega_i\cos\theta_i) + (E_i+\omega_i-m)]e_{fz} e_{iz}
\end{aligned} \tag{3.2.58}
$$

$$
\begin{aligned}
X_1' &= [\frac{(E_i-\omega_f+m)p_i(p_i+\omega_i\cos\theta_i-\omega_f\cos\theta_f)}{(E_i+m)(E_f+m)}+(E_i-\omega_f-m)]e_i^+e_f^- \\
&+ -(\frac{p_i}{E_i+m}+\frac{p_i+\omega_i\cos\theta_i-\omega_f\cos\theta_f}{E_f+m}) \\
&\times [k_i^+e_{iz}e_f^-+k_f^-e_{fz}e_+^++(p_i-\omega_f\cos\theta_f)e_i^+e_f^-] \qquad (3.2.59)
\end{aligned}
$$

$$
\begin{aligned}
X_2' &= [\frac{(E_i-\omega_f+m)p_i(p_i+\omega_i\cos\theta_i-\omega_f\cos\theta_f)}{(E_i+m)(E_f+m)}+(E_i-\omega_f-m) \\
&+ (\frac{p_i}{E_i+m}+\frac{p_i+\omega_i\cos\theta_i-\omega_f\cos\theta_f}{E_f+m})(p_i-\omega_f\cos\theta_f)]e_{iz}e_{fz} \qquad (3.2.60)
\end{aligned}
$$

where $e_i$ and $e_f$ are polarizations of the incident and scattered photons and $e_i^{\pm}=e_{ix}\pm ie_{iy}$,$e_f^{\pm}=e_{fx}\pm\pm ie_{fy}$.The conservation of energy and momentum along the z direction,i.e.

$E_f=E_i+\omega_i-\omega_f$ and $p_f+k_f\cos\theta_f=p_i+k_i\cos\theta_i$ ,leads to

$$
\begin{aligned}
\omega_f &= \frac{1}{\sin^2\theta_f}\{E_i-p_i\cos\theta_f+\omega_i(1-\cos\theta_i\cos\theta_f)-[(E_i-p_i\cos\theta_f)^2 \\
&+ 2\omega_i(E_i\cos\theta_f-p_i)(\cos\theta_f-\cos\theta_i)+\omega_i^2(\cos\theta_f-\cos\theta_i)^2]^{\frac{1}{2}}\} \qquad (3.2.61)
\end{aligned}
$$

$$
\begin{aligned}
|S_{fi}|^2 &= \frac{(2\Pi)^6}{L^4V^2}\frac{e^4}{4\omega_i\omega_f}[\frac{(E_i+m)(E_f+m)}{4E_iE_f}]\exp[-\frac{\lambda^2}{2}(\omega_i^2\sin^2\theta_i+\omega_f^2\sin^2\theta_f)] \\
&\times \exp[-i2\lambda^2a_i(k_{ix}-k_{fx})-i\lambda^2(k_{ix}k_{iy}+k_{fx}k_{fy})]|X|^2 \\
&\times \delta^2(E_i+\omega_i-E_f-\omega_f)\delta^2(p_i+k_i\cos\theta_i-p_f-k_f\cos\theta_f) \\
&\times \delta^2(a_i-k_{iy}-a_f-k_{fy}) \qquad (3.2.62)
\end{aligned}
$$

In the expressions for $X_1$,$X_1'$,$X_2$,and $X_2'$putting $p_i+\omega_i\cos\theta_i-\omega_f\cos\theta_f=p_f$,we get the following.

$$X_1=\{[A-B(p_i+\omega_i\cos\theta_i)e_f^+e_i^-+B(k_f^+e_{fz}e_i^-+k_i^-e_{iz}e_f^+)\}(E_f+m)^{-1}(E_i+m)^{-1},$$

$$X_2 = \{[A + B(p_i + \omega_i \cos\theta_i)e_{fz}e_{iz}\}(E_f + m)^{-1}(E_i + m)^{-1},$$

$$X_1' = \{[A' - B(p_i - \omega_f \cos\theta_f)]e_f^- e_i^+ - B(k_f^- e_{fz}e_i^+ + k_i^+ e_{iz}e_f^-)\}(E_f + m)^{-1}(E_i + m)^{-1} and$$

$$X_2' = \{[A' + B(p_i - \omega_f \cos\theta_f)]e_{fz}e_{iz}\}(E_f + m)^{-1}(E_i + m)^{-1}$$

In the above two expressions the coefficients $A, A', B$ are defined by

$$A = (E_i + \omega_i + m)p_i p_f + (E_i + \omega_i - m)(E_i + m)(E_f + m) \tag{3.2.63}$$

$$A' = (E_i - \omega_f + m)p_i p_f + (E_i - \omega_f - m)(E_i + m)(E_f + m) \tag{3.2.64}$$

$$B = p_i(E_f + m) + p_f(E_i + m) \tag{3.2.65}$$

respectively.

Substituting these results back into the expression for $X$:

$$\begin{aligned} X &= \sum_{n=0}^{\infty} \frac{1}{n!}\Big[(\frac{\lambda^2 k_i^- k_f^+}{2})^n \exp(i\lambda^2 k_{iy}k_{fx}) \\ &\times (\frac{[A - B(p_i + \omega_i \cos\theta_i)]e_f^+ e_i^- + B(k_f^+ e_{fz}e_i^- + k_i^- e_{iz}e_f^+)(E_f + m)^{-1}(E_i + m)^{-1}}{(E_i + \omega_i)^2 - E_{i,n+1}^2} \\ &+ \frac{[A + B(p_i + \omega_i \cos\theta_i)]e_{fz}e_{iz}(E_f + m)^{-1}(E_i + m)^{-1}}{(E_i + \omega_i)^2 - E_{i,n}^2}) \\ &+ (\frac{\lambda^2 k_i^+ k_f^-}{2})^n \exp(i\lambda^2 k_{ix}k_{fy}) \\ &\times (\frac{[A' - B(p_i - \omega_f \cos\theta_f)]e_f^- e_i^+ - B(k_f^- e_{fz}e_i^+ + k_i^+ e_{iz}e_f^-)(E_f + m)^{-1}(E_i + m)^{-1}}{(E_i - \omega_f)^2 - E_{f,n+1}^2} \\ &+ \frac{[A' + B(p_i - \omega_f \cos\theta_f)]e_{fz}e_{iz}(E_f + m)^{-1}(E_i + m)^{-1}}{(E_i - \omega_f)^2 - E_{f,n}^2})\Big] \end{aligned} \tag{3.2.66}$$

$$
\begin{aligned}
X &= \sum_{n=0}^{\infty}\frac{1}{n!}\Big[\big(\frac{\lambda^2\omega_i\omega_f\sin\theta_i\sin\theta_f}{2}e^{-i(\phi_i-\phi_f)}\big)^n e^{i\lambda^2\omega_i\omega_f\sin\theta_i\sin\theta_f\cos\phi_f\sin\phi_i} \\
&\times \big(\frac{[A-B(p_i+\omega_i\cos\theta_i)]e_f^+e_i^- + B(k_f^+e_{fz}e_i^- + k_i^-e_{iz}e_f^+)(E_f+m)^{-1}(E_i+m)^{-1}}{(E_i+\omega_i)^2-E_{i,n+1}^2} \\
&+ \frac{[A+B(p_i+\omega_i\cos\theta_i)]e_{fz}e_{iz}(E_f+m)^{-1}(E_i+m)^{-1}}{(E_i+\omega_i)^2-E_{i,n}^2}\big) \\
&+ \big(\frac{\lambda^2\omega_i\omega_f\sin\theta_i\sin\theta_f}{2}e^{i(\phi_i-\phi_f)}\big)^n e^{i\lambda^2\omega_i\omega_f\sin\theta_i\sin\theta_f\sin\phi_f\cos\phi_i} \\
&\times \big(\frac{[A'-B(p_i-\omega_f\cos\theta_f)]e_f^-e_i^+ - B(k_f^-e_{fz}e_i^+ + k_i^+e_{iz}e_f^-)(E_f+m)^{-1}(E_i+m)^{-1}}{(E_i-\omega_f)^2-E_{f,n+1}^2} \\
&+ \frac{[A'+B(p_i-\omega_f\cos\theta_f)]e_{fz}e_{iz}(E_f+m)^{-1}(E_i+m)^{-1}}{(E_i-\omega_f)^2-E_{f,n}^2}\big)\Big] \qquad (3.2.67)
\end{aligned}
$$

Letting

$$
\begin{aligned}
Y_1 &= \sum_{n=0}^{\infty}\frac{1}{n!}\big(\frac{\lambda^2\omega_i\omega_f\sin\theta_i\sin\theta_f}{2}e^{-i(\phi_i-\phi_f)}\big)^n \exp(i\lambda^2\omega_i\omega_f\sin\theta_i\sin\theta_f\cos\phi_f\sin\phi_i) \\
&\times \Big[\frac{[A-B(p_i+\omega_i\cos\theta_i)]e_f^+e_i^- + B(k_f^+e_{fz}e_i^- + k_i^-e_{iz}e_f^+)}{(E_i+\omega_i)^2-E_{i,n+1}^2} \\
&+ \frac{[A+B(p_i+\omega_i\cos\theta_i)]e_{fz}e_{iz}}{(E_i+\omega_i)^2-E_{i,n}^2}\Big]
\end{aligned}
$$

$$
\begin{aligned}
Y_2 &= \sum_{n=0}^{\infty}\frac{1}{n!}\big(\frac{\lambda^2\omega_i\omega_f\sin\theta_i\sin\theta_f}{2}e^{i(\phi_i-\phi_f)}\big)^n \exp(i\lambda^2\omega_i\omega_f\sin\theta_i\sin\theta_f\sin\phi_f\cos\phi_i) \\
&\times \Big[\frac{[A'-B(p_i-\omega_f\cos\theta_f)]e_f^-e_i^+ - B(k_f^-e_{fz}e_i^+ + k_i^+e_{iz}e_f^-)}{(E_i-\omega_f)^2-E_{f,n+1}^2} \\
&+ \frac{[A'+B(p_i-\omega_f\cos\theta_f)]e_{fz}e_{iz}}{(E_i-\omega_f)^2-E_{f,n}^2}\Big]
\end{aligned}
$$

$\Rightarrow Y = (Y_1 + Y_2) = X(E_f+m)(E_i+m)$

$\Rightarrow X = \frac{Y}{(E_f+m)(E_i+m)}$

Substituting $|X|^2 = \frac{|Y|^2}{(E_f+m)^2(E_i+m)^2}$ in $|S_{fi}|^2$ :

$$\begin{aligned}
|S_{fi}|^2 &= \frac{(2\Pi)^6}{L^4V^2}\frac{e^4}{4\omega_i\omega_f}[\frac{(E_i+m)(E_f+m)}{4E_iE_f}]\exp[-\frac{\lambda^2}{2}(\omega_i^2\sin^2\theta_i+\omega_f^2\sin^2\theta_f)] \\
&\times \exp[-i2\lambda^2 a_i(k_{ix}-k_{fx})-i\lambda^2(k_{ix}k_{iy}+k_{fx}k_{fy})]\frac{|Y|^2}{(E_f+m)^2(E_i+m)^2} \\
&\times \delta^2(E_i+\omega_i-E_f-\omega_f)\delta^2(p_i+k_i\cos\theta_i-p_f-k_f\cos\theta_f) \\
&\times \delta^2(a_i-k_{iy}-a_f-k_{fy}) \qquad (3.2.68)
\end{aligned}$$

Taking the final states of the scattered photon and electron to be $\frac{Vd^3\omega_f}{(2\Pi)^3}$and$\frac{Vd^3p_f}{(2\Pi)^3}$ and $\frac{Vd^3\omega_f}{(2\Pi)^3} = \frac{V\omega_f^2 d\Omega_f}{(2\Pi)^3}$,and also using $e^4 = 16\Pi^2m^2r_0^2$

$$\begin{aligned}
\frac{|S_{fi}|^2}{T} &= \frac{(2\Pi)^6}{L^4V^2}\frac{16\Pi^2m^2r_0^2}{4\omega_i\omega_f}[\frac{1}{4E_iE_f}]\exp[-\frac{\lambda^2}{2}(\omega_i^2\sin^2\theta_i+\omega_f^2\sin^2\theta_f)] \\
&\times \exp[-i2\lambda^2 a_i(k_{ix}-k_{fx})-i\lambda^2(k_{ix}k_{iy}+k_{fx}k_{fy})] \\
&\times \frac{|Y|^2}{(E_f+m)(E_i+m)}\frac{T}{2\Pi}\frac{L}{2\Pi}\frac{1}{T}\delta^2(a_i-k_{iy}-a_f-k_{fy}) \qquad (3.2.69)
\end{aligned}$$

where we have used

$\delta^2(E_i+\omega_i-E_f-\omega_f) = \frac{T}{2\Pi}\delta(E_i+\omega_i-E_f-\omega_f) = \frac{T}{2\Pi}$,

$\delta^2(p_i+k_i\cos\theta_i-p_f-k_f\cos\theta_f) = \frac{L}{2\Pi}\delta(p_i+k_i\cos\theta_i-p_f-k_f\cos\theta_f) = \frac{L}{2\Pi}$ Note the above expression reduces because, from conservation of energy and conservation of momentum along z-axis the term inside the delta function is zero.

Dividing this result by the relative incident flux density, $\frac{(E_i-p_i\cos\theta_i)}{VE_i}$:

$$\begin{aligned}
\frac{|S_{fi}|^2}{T}\frac{VE_i}{(E_i-p_i\cos\theta_i)} &= \frac{(2\Pi)^6}{L^3V^2}\frac{m^2r_0^2}{4\omega_i\omega_f}[\frac{1}{E_iE_f}]\exp[-\frac{\lambda^2}{2}(\omega_i^2\sin^2\theta_i+\omega_f^2\sin^2\theta_f)] \\
&\times \exp[-i2\lambda^2 a_i(k_{ix}-k_{fx})-i\lambda^2(k_{ix}k_{iy}+k_{fx}k_{fy})] \\
&\times \frac{|Y|^2}{(E_f+m)(E_i+m)}\frac{VE_i}{(E_i-p_i\cos\theta_i)} \\
&\times \delta^2(a_i-k_{iy}-a_f-k_{fy}) \qquad (3.2.70)
\end{aligned}$$

Summing over the final states of the scattered photon and electron:

$$
\begin{aligned}
d\sigma &= \frac{(2\Pi)^6}{V^2}\frac{m^2 r_0^2}{4\omega_i\omega_f}[\frac{1}{E_f}]\exp[-\frac{\lambda^2}{2}(\omega_i^2\sin^2\theta_i+\omega_f^2\sin^2\theta_f)] \\
&\times \exp[-i2\lambda^2 a_i(k_{ix}-k_{fx})-i\lambda^2(k_{ix}k_{iy}+k_{fx}k_{fy})] \\
&\times \frac{|Y|^2}{(E_f+m)(E_i+m)}\frac{1}{(E_i-p_i\cos\theta_i)}\frac{V\omega_f^2 d\Omega_f}{(2\Pi)^3}\frac{Vd^3p_f}{(2\Pi)^3} \\
&\times \delta^2(a_i-k_{iy}-a_f-k_{fy}) \qquad (3.2.71)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\sigma}{d\Omega_f} &= \frac{r_0^2}{4}\frac{\omega_f}{\omega_i}\frac{m^2}{(E_f+m)(E_i+m)(E_i-p_i\cos\theta_i} \\
&\times \exp[-\frac{\lambda^2}{2}(\omega_i^2\sin^2\theta_i+\omega_f^2\sin^2\theta_f)]|Y|^2 \\
&\times \{\frac{1}{E_f}\exp[-i2\lambda^2 a_i(k_{ix}-k_{fx})-i\lambda^2(k_{ix}k_{iy}+k_{fx}k_{fy})] \\
&\times \delta^2(a_i-k_{iy}-a_f-k_{fy})d^3p_f\} \qquad (3.2.72)
\end{aligned}
$$

But,

$$
\begin{aligned}
\frac{1}{[E_f-(p_i+\omega_i\cos\theta_i-\omega_f\cos\theta_f)\cos\theta_f]} &= \{\frac{1}{E_f}\exp[-i2\lambda^2 a_i(k_{ix}-k_{fx})-i\lambda^2(k_{ix}k_{iy}+k_{fx}k_{fy})] \\
&\times \delta^2(a_i-k_{iy}-a_f-k_{fy})d^3p_f\}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{d\sigma}{d\Omega_f} &= \frac{r_0^2}{4}\frac{\omega_f}{\omega_i}\frac{m^2}{(E_i+m)(E_f+m)} \\
&\times \frac{\exp[\frac{-\lambda^2}{2}(\omega_i^2\sin^2\theta_i+\omega_f^2\sin^2\theta_f)]|Y|^2}{(E_i-p_i\cos\theta_i)[E_f-(p_i+\omega_i\cos\theta_i-\omega_f\cos\theta_f)\cos\theta_f]}, \qquad (3.2.73)
\end{aligned}
$$

in which $r_0$is the classical electron radius

The above result is performed by letting:

$\vec{k_i} = \omega_i(\sin\theta_i \cos\phi_i, \sin\theta_i \sin\phi_i, \cos\theta_i)$, and $\vec{k_i} = \omega_f(\sin\theta_f \cos\phi_f, \sin\theta_f \sin\phi_f, \cos\theta_f)$

Therefore,

$$(\frac{\lambda^2}{2}k_i^- k_f^+)^n = [\frac{\lambda^2}{2}\omega_i\omega_f \sin\theta_i \sin\theta_f e^{-i(\phi_i-\phi_f)}]^n$$

and

$$(\frac{\lambda^2}{2}k_i^+ k_f^-)^n = [\frac{\lambda^2}{2}\omega_i\omega_f \sin\theta_i \sin\theta_f e^{i(\phi_i-\phi_f)}]^n$$

using $k_i^- k_f^+ = (k_{ix} - ik_{iy})(k_{fx} + ik_{fy})$

### 3.2.3 Cross-section in the Electron rest frame

In the ERF,$p_i = 0$,(3.2.74) is reduced to Herold's expression,as expected. To simplify calculations,we choose the coordinate system with $\phi_i = 0$.Denoting $\phi_f = \phi$, from above section we have

$\vec{k_i} = \omega_i(\sin\theta_i \cos\phi_i, \sin\theta_i \sin\phi_i, \cos\theta_i)$, and $\vec{k_i} = \omega_f(\sin\theta_f \cos\phi_f, \sin\theta_f \sin\phi_f, \cos\theta_f)$

which will reduce to

$$k_i = \omega_i(\sin\theta_i, 0, \cos\theta_i), k_f = \omega_f(\sin\theta_f \cos\phi, \sin\theta_f \sin\phi, \cos\phi) \qquad (3.2.74)$$

Taking into account the fact that photons have only two transversal polarizations, the polarizations of incident and scattered photons can be chosen as

$$e_i^{(1)} = (-\cos\theta_i, 0, \sin\theta_i); e_i^{(2)} = (0, -1, 0) \qquad (3.2.75)$$

$$e_f^{(1)} = (-\cos\theta_f \cos\phi, -\cos\theta_f \sin\phi, \sin\theta_f); e_f^{(2)} = (\sin\phi, -\cos\phi, 0). \qquad (3.2.76)$$

The above choice is not unique but is convenient for calculations.Now we define reduced quantities

$$\Delta_i = \frac{\omega_i}{m}; \Delta_f = \frac{\omega_f}{m}; \Delta_0 = \frac{\omega_0}{m} \tag{3.2.77}$$

where$\omega_0 = \frac{eB}{m}$ is the cyclotron frequency.We denote the reduced Doppler frequencies by

$$\Delta_{ir} = \gamma(1 - \beta\cos\theta_i)\Delta_i; \Delta_{fr} = \gamma(1 - \beta\cos\theta_f)\Delta_f \tag{3.2.78}$$

Averaging over the polarizations of incident photons and summing over those of scattered photons, the total differential cross section will be:

$$\frac{d\sigma}{d\Omega_f} = \frac{1}{2} \sum_{e_i^{(1)}, e_f^{(1)}, e_i^{(2)}, e_f^{(2)}} \frac{r_0^2 \omega_f}{4\omega_i} \frac{m^2}{E_i + m} \frac{e^{\frac{-\lambda^2}{2}(\omega_i^2 \sin^2\theta_i + \omega_f^2 \sin^2\theta_f)|Y|^2}}{(E_f + m)(E_i - P_i\cos\theta_i)[E_f - (P_i + \omega_i\cos\theta_i - \omega_f\cos\theta_f)\cos\theta_f]} \tag{3.2.79}$$

where $Y = Y_1 + Y_2$

Note that:$\omega_f = \Delta_f m$,$\omega_i = \Delta_i m$ and $E_i + m = m(\frac{E_i}{m} + 1) = m(\gamma + 1)$and in similar way$E_f + m = E_i + \omega_i - \omega_f + m = m(\frac{E_i}{m} + \frac{\omega_i}{m} - \frac{\omega_f}{m} + 1 = m(1 + \gamma + \Delta_i - \Delta_f)$ and $\omega_i^2 = \Delta_i^2 m^2 = \lambda^2\omega_i^2 = \lambda^2\Delta_i^2 m^2 = \frac{B_c}{B}\Delta_i^2$ Similarly:

$\lambda^2\omega_f^2 = \lambda^2\Delta_f^2 m^2 = \frac{B_c}{B}\Delta_f^2$ and $E_i - P_i\cos\theta_i = E_i(1 - \frac{P_i}{E_i}\cos\theta_i = E_i(1 - \beta\cos\theta_i) = m\gamma(1 - \beta\cos\theta_i)$

$$E_f - (P_i + \omega_i\cos\theta_i - \omega_f\cos\theta_f)\cos\theta_f = E_f - P_i\cos\theta_f + \omega_i\cos\theta_i\cos\theta_f + \omega_f\cos\theta_f^2 \tag{3.2.80}$$

Using the usual trigonometric relation$\cos^2\theta_f = 1 - \sin^2\theta_f$and $E_f = E_i - \omega_i - \omega_f$

$$E_i - \omega_i - \omega_f - P_i\cos\theta_f + \omega_i\cos\theta_i\cos\theta_f + \omega_f\cos^2\theta_f \tag{3.2.81}$$

$$E_i + \omega_i(1 - \cos\theta_i\cos\theta_f) - P_i\cos\theta_f + \omega_f(\cos^2\theta_f - 1) \tag{3.2.82}$$

$$E_i - P_i \cos\theta_f + \omega_i(1 - \cos\theta_i \cos\theta_f) - \omega_f \sin^2\theta_f \tag{3.2.83}$$

This gives $= E_i(1 - \beta\cos\theta_f) + m\Delta_i(1 - \cos\theta_i\cos\theta_f) - m\Delta_f \sin^2\theta_f$

and this will reduce to:$m[\gamma(1 - \beta\cos\theta_f) + \Delta_i(1 - \cos\theta_i\cos\theta_f) - \Delta_f\sin^2\theta_f]$

Using all this back in the expression$\frac{d\sigma}{d\Omega_f}$gives

$$\begin{aligned}\frac{d\sigma}{d\Omega_f} &= \frac{1}{2}\sum_{e_i^{(1)},e_f^{(1)},e_i^{(2)},e_f^{(2)}} \frac{r_0^2}{4}\frac{\Delta_f}{m^2\Delta_{ir}}\frac{e^{\frac{-B_c}{2B}(\Delta_f^2\sin^2\theta_f+\Delta_i^2\sin^2\theta_i)}|Y|^2}{(\gamma+1)(1+\gamma+\Delta_i-\Delta_f)} \\ &\times \frac{1}{[\gamma(1-\beta\cos\theta_f)+\Delta_i(1-\cos\theta_i\cos\theta_f)-\Delta_f\sin^2\theta_f]}\end{aligned} \tag{3.2.84}$$

Where$\Delta_{ir} = \gamma(1 - \beta\cos\theta_i)\Delta_i$ Now

$$\begin{aligned}\frac{d\sigma}{d\Omega_f} &= \frac{r_0^2}{8}\frac{\Delta_f}{\Delta_{ir}(1+\gamma)(1+\gamma+\Delta_i-\Delta_f)} \times \frac{e^{\frac{-B_c}{2B}(\Delta_f^2\sin^2\theta_f+Delta_i^2\sin^2\theta_i)}}{[\gamma(1-\beta\cos\theta_f)+\Delta_i(1-\cos\theta_i\cos\theta_f)-\Delta_f\sin^2\theta_f]} \\ &\times \sum_{e_i^{(1)},e_f^{(1)},e_i^{(2)},e_f^{(2)}}\frac{|Y|^2}{m^2}\end{aligned}$$

$\sum_{e_i^{(1)},e_f^{(1)},e_i^{(2)},e_f^{(2)}}\frac{|Y|^2}{m^2}$ where$Y = Y_1 + Y_2$ is given by:

$$\frac{1}{m^2}|Y(1i \to 1f)|^2 + \frac{1}{m^2}|Y(1i \to 2f)|^2 + \frac{1}{m^2}|Y(2i \to 1f)|^2 + \frac{1}{m^2}|Y(2i \to 2f)|^2$$

Where $\lambda_i \to \lambda_f, \lambda_{i(f)} = 1_{i(f)}, 2_{i(f)}$represents the scattering from the polarization $\lambda_i$ to $\lambda_f$

$$|Y(1_i \to 1_f)| = |Y_1(1_1 \to 1_f) + Y_2(1_i \to 1_f)|$$

$\phi_i = 0, \phi_f = \phi$ and

$$\frac{\lambda^2 \omega_i \omega_f \sin\theta_i \sin\theta_f}{2} = \frac{B_c}{2B} \Delta_i \Delta_f \sin\theta_i \sin\theta_f = \xi$$

and $e^{-i(\phi_i - \phi_f)} = e^{i\phi}$ and $e^{[i\lambda^2 \omega_i \omega_f \sin\theta_i \sin\theta_f \cos\phi_f \sin\phi_i]} = e^0 = 1$

$$\frac{1}{(E_i + \omega_i)^2 - E_{i,n+1}^2} = \frac{1}{E_i^2 + \omega_i^2 + 2E_i\omega_i - m^2 - P_i^2 - \omega_i^2 \cos^2\theta_i - 2P_i\omega_i \cos\theta_i - 2(n+1)eB}, \tag{3.2.85}$$

using $E_{i,n+1}^2 = m^2 + (P_i + \omega_i \cos\theta_i)^2 + 2(n+1)eB$

$$\frac{1}{E_i^2 + 2E_i\omega_i - m^2 - P_i^2 - 2P_i\omega_i \cos\theta_i - 2(n+1)eB + \omega_i^2 \sin^2\theta_i} \tag{3.2.86}$$

$$= \frac{1}{2E_i\omega_i - 2P_i\omega_i \cos\theta_i - 2neB + \omega_i^2 \sin^2\theta_i} \tag{3.2.87}$$

$$= \frac{1}{2m\gamma m\Delta_i - 2m\gamma\beta m\Delta_i \cos\theta_i - 2nm^2\Delta_0 + m^2\Delta_i^2 \sin^2\theta_i} \tag{3.2.88}$$

$$= \frac{1}{m^2[2(\Delta_{ir} - n\Delta_0) + \Delta_i^2 \sin^2\theta_i]} \tag{3.2.89}$$

$$= \frac{1}{m^2} S_{i,n+1} \tag{3.2.90}$$

Similarly in $Y_2$ :

$$\frac{1}{(E_i - \omega_f)^2 - E_{f,n+1}^2} = \frac{1}{m^2[2(\Delta_{fr} + n\Delta_0) - \Delta_f^2 \sin^2\theta_f]} \tag{3.2.91}$$

$= \frac{1}{m^2} S_{f,n+1}$ and $\frac{\lambda^2 \omega_i \omega_f \sin\theta_i \sin\theta_f}{2} = \frac{B_c}{2B} \Delta_i \Delta_f \sin\theta_i \sin\theta_f = \xi$ and

$e^{i(\phi_i - \phi_f)} = e^{-i\phi}, e^{[i\lambda^2 \omega_i \omega_f \sin\theta_i \sin\theta_f \sin\phi_f \cos\phi_i]} = e^{i\frac{B_c}{B}\Delta_i \Delta_f \sin\theta_i \sin\theta_f \sin\phi} = e^{i2\xi \sin\phi} = e^{i\eta}$

where $\eta = 2\xi \sin\phi$

Therefore, $Y = Y_1 + Y_2$ will be:

$$\begin{aligned}
|Y| &= [A - B(P_i + \omega_i \cos\theta_i)e_f^+ e_i^- + B(K_f^+ e_{fz} e_i^- + K_i^- e_{iz} e_f^+)] \sum_{n=0}^{\infty} \frac{1}{n!}\xi^n \frac{1}{m^2} S_{i,n+1} e^{in\phi} \\
&+ [A + B(P_i + \omega_i \cos\theta_i)] e_{fz} e_{iz} \sum_{n=0}^{\infty} \frac{1}{n!}\xi^n \frac{1}{m^2} S_{i,n} e^{in\phi} \\
&+ [(A' - B(P_i - \omega_f \cos\theta_f))e_f^- e_i^+ - B(K_f^- e_{fz} e_i^+ + K_i^+ e_{iz} e_f^+)] \sum_{n=0}^{\infty} \frac{1}{n!}\xi^n \frac{1}{m^2} S_{f,n+1} e^{-in\phi+i\eta} \\
&+ [(A' + B(P_i - \omega_f \cos\theta_f))e_{fz} e_{iz}] \sum_{n=0}^{\infty} \frac{1}{n!}\xi^n \frac{1}{m^2} S_{f,n} e^{-in\phi+i\eta} \qquad (3.2.92)
\end{aligned}$$

To calculate $|Y(1_i \to 1_f)|$, we use the polarization components: $e_i^{(1)} = (\cos\theta_i, 0, \sin\theta_i)$ , $e_f^{(1)} = (-\cos\theta_f \cos\phi, -\cos\theta_f \sin\phi, \sin\theta_f)$ and

$K_i = \omega_i(\sin\theta_i, 0, \cos\theta_i), K_f = \omega_f(\sin\theta_f \cos\phi, \sin\theta_f \sin\phi, \cos\phi)$

Therefore:

$$e_f^- e_i^+ = e_{fx} e_{ix} + e_{fy} e_{iy} - i e_{fy} e_{ix} + i e_{fx} e_{iy} = \cos\theta_i \cos\theta_f \cos\phi - i \cos\theta_i \cos\theta_f \sin\phi$$

$$e_f^+ e_i^- = \cos\theta_i \cos\theta_f \cos\phi + i \cos\theta_i \cos\theta_f \sin\phi$$

$$\begin{aligned}
K_f^+ e_{fz} e_i^- + K_i^- e_{iz} e_f^+ &= (K_{fx} e_{ix} + K_{fy} e_{iy} + i K_{fy} e_{ix} - i K_{fx} e_{iy}) e_{fz} \\
&+ (K_{ix} e_{fx} + K_{iy} e_{fy} + i K_{ix} e_{fy} - i K_{iy} e_{fx}) e_{iz} \qquad (3.2.93)
\end{aligned}$$

$$\begin{aligned}
K_f^+ e_{fz} e_i^- + K_i^- e_{iz} e_f^+ &= (-\omega_f \sin\theta_f \cos\phi \cos\theta_i - i\omega_f \sin\theta_f \sin\phi \cos\theta_i) \sin\theta_f \\
&+ (-\omega_i \sin\theta_i \cos\phi \cos\theta_f - i\omega_i \sin\theta_i \sin\phi \cos\theta_f) \sin\theta_i
\end{aligned}$$

$e_{fz} e_{iz} = \sin\theta_i \sin\theta_f$

and

$$\begin{aligned}
K_f^- e_{fz} e_i^+ + K_i^+ e_{iz} e_f^- &= (-\omega_f \sin\theta_f \cos\phi \cos\theta_i + i\omega_f \sin\theta_f \sin\phi \cos\theta_i) \sin\theta_f \\
&+ (-\omega_i \sin\theta_i \cos\phi \cos\theta_f + i\omega_i \sin\theta_i \sin\phi \cos\theta_f) \sin\theta_i
\end{aligned}$$

The first term in the square bracket of (3.2.92),

$$[A - B(P_i + \omega_i \cos\theta_i)]e_f^+ e_i^- + B(K_f^+ e_{fz} e_i^- + K_i^- e_{iz} e_f^+)$$

$$\begin{aligned}
= &\ [(E_i + \omega_i + m)P_i P_f + (E_i + \omega_i - m)(E_i + m)(E_f + m)] \\
- &\ [\big(P_i(E_f + m) + P_f(E_i + m)\big)(P_i + \omega_i \cos\theta_i)][\cos\theta_i \cos\theta_f \cos\phi + i\cos\theta_i \cos\theta_f \sin\phi] \\
+ &\ [P_i(E_f + m) + P_f(E_i + m)][-\omega_f \sin^2\theta_f \cos\phi \cos\theta_i - i\omega_f \sin^2\theta_f \sin\phi \cos\theta_i \\
- &\ \omega_i \sin^2\theta_i \cos\phi \cos\theta_f - i\omega_i \sin^2\theta_i \sin\phi \cos\theta_f] \qquad (3.2.94)
\end{aligned}$$

$$\begin{aligned}
= &\ e^{i\phi} m^3 [[(\beta\gamma(1 + \gamma + \Delta_i)(\beta\gamma + \Delta_i \cos\theta_i - \Delta_f \cos\theta_f) + (\gamma - 1 + \Delta_f)(1 + \gamma)(1 + \Delta_i - \Delta_f)) \\
- &\ (\beta\gamma(1 + \gamma + \Delta_i - \Delta_f) + (\beta\gamma + \Delta_i \cos\theta_i - \Delta_f \cos\theta_f)(1 + \gamma))(\beta\gamma + \Delta_i \cos\theta_i)] \cos\theta_f \\
- &\ (\beta\gamma(1 + \gamma + \Delta_i - \Delta_f) + (\beta\gamma + \Delta_i \cos\theta_i - \Delta_f \cos\theta_i)(1 + \gamma)\Delta_f \sin^2\theta_f) \cos\theta_i \\
- &\ (\beta\gamma(1 + \gamma + \Delta_i - \Delta_f) + (\beta\gamma + \Delta_i \cos\theta_i - \Delta_f \cos\theta_f)(1 + \gamma)\Delta_i \sin^2\theta_i) \cos\theta_f] \qquad (3.2.95)
\end{aligned}$$

The second square bracket, $[(A + B(P_i + \omega_i \cos\theta_i))e_{fz} e_{iz})]$

$$\begin{aligned}
= &\ [(E_i + \omega_i + m)P_i P_f + (E_i + \omega_i - m)(E_i + m)(E_f + m)] \\
+ &\ [\big(P_i(E_f + m) + P_f(E_i + m)\big)(P_i + \omega_i \cos\theta_i)](\sin\theta_i \sin\theta_f \theta_i)]
\end{aligned}$$

$= m^3[a + b(\beta\gamma + \Delta_i \cos\theta_i)] \sin\theta_i \sin\theta_f$

The third square bracket, $[(A' - B(P_i - \omega_f \cos\theta_f))e_f^- e_i^+ - B(K_f^- e_{fz} e_i^+ + K_i^+ e_{iz} e_f^+)]$

$$\begin{aligned}
= &\ [[(E_i - \omega_f + m)P_i P_f + (E_i - \omega_f - m)(E_i + m)(E_f + m)] \\
- &\ (P_i(E_i + m) + P_f(E_i + m))(P_i - \omega_f \cos\theta_f)(\cos\theta_i \cos\theta_f \cos\phi - i\cos\theta_i \cos\theta_f \sin\phi) \\
- &\ (P_i(E_f + m) + P_f(E_i + m))(-\omega_f \sin^2\theta_f \cos\phi \cos\theta_i + i\omega_f \sin^2\theta_f \sin\phi \cos\theta_i \\
- &\ \omega_i \sin^2\theta_i \cos\phi \cos\theta_f + i\omega_i \sin^2\theta_i \sin\phi \cos\theta_f)] \qquad (3.2.96)
\end{aligned}$$

$$
\begin{aligned}
&= e^{-i\phi}m^3[[(\beta\gamma(1+\gamma-\Delta_f)(\beta\gamma+\Delta_i\cos\theta_i-\Delta_f\cos\theta_f)+(\gamma-1-\Delta_f)(1+\gamma)(1+\Delta_i-\Delta_f)) \\
&- (\beta\gamma(1+\gamma+\Delta_i-\Delta_f)+(\beta\gamma+\Delta_i\cos\theta_i-\Delta_f\cos\theta_f)(1+\gamma))(\beta\gamma-\Delta_f\sin\theta_f)]\Delta_f\sin^2\theta_f\cos\theta_i \\
&+ (\beta\gamma(1+\gamma+\Delta_i-\Delta_f)+(\beta\gamma+\Delta_i\cos\theta_i-\Delta_f\cos\theta_i)(1+\gamma)\Delta_i\sin^2\theta_i\cos\theta_f)] \qquad (3.2.97)
\end{aligned}
$$

and the fourth square bracket can be solved as:

$$
\begin{aligned}
[(A'+B(P_i-\omega_f\cos\theta_f))e_{fz}e_{iz}] &= [[(E_i-\omega_f+m)P_iP_f+(E_i-\omega_f-m)(E_i+m)(E_f+m)] \\
&+ (P_i(E_f+m)+P_f(E_i+m))(P_i-\omega_f\cos\theta_i)]\sin\theta_i\sin\theta_f
\end{aligned}
$$

$=m^3[a'+b(\beta\gamma-\Delta_f\cos\theta_f)]\sin\theta_i\sin\theta_f$

Thus substituting this back in to $|Y|$gives $Y(1i\to 1f)$ thus:

$$
\begin{aligned}
Y(1_i\to 1_f) \quad = \quad & m[(A_-\cos\theta_f-B_1)\cos\theta_i-B_2\cos\theta_f]\sum_{n=0}^{\infty}\frac{1}{n!}\xi^n S_{i,n+1}e^{i(n+1)\phi} \\
-m \quad & [(A'_-\cos\theta_f+B_1)\cos\theta_i+B_2\cos\theta_f]\sum_{n=0}^{\infty}\frac{1}{n!}S_{f,n+1}e^{-i[(n+1)\phi-\eta\eta]} \\
+m \quad & \sin\theta_i\sin\theta_f[A_+\sum_{n=0}^{\infty}\frac{1}{n!}\xi^n S_{i,n}e^{in\phi}-A'_+\sum_{n=0}^{\infty}\frac{1}{n!}\xi^n S_{f,n}e^{-i(n\phi-\eta)}]
\end{aligned}
$$

Where $A_\pm = a\pm b(\beta\gamma+\Delta\Delta_i\cos\theta_i, A'_\pm = a'\pm b(\beta\gamma-\Delta_f\cos\theta_f), B_1 = b\Delta_f\sin^2\theta_f \& B_2 = b\Delta_i\sin^2\theta_i$

in which

$$a = \beta\gamma(1+\gamma+\Delta_i)(\beta\gamma+\Delta_i\cos\theta_i-\Delta_f\cos\theta_f)+(\gamma-1+\Delta_i)(1+\gamma)(1+\gamma+\Delta_i-\Delta_f)$$

$$a' = \beta\gamma(1+\gamma-\Delta_f)(\beta\gamma+\Delta_i\cos\theta_i-\Delta_f\cos\theta_f)+(\gamma-1+\Delta_f)(1+\gamma)(1+\gamma+\Delta_i-\Delta_f)$$

$$b = \beta\gamma(1+\gamma+\Delta_i-\Delta_f)(\beta\gamma+\Delta_i\cos\theta_i-\Delta_f\cos\theta_f)(1+\gamma)$$

which can be written again as:

$$
\begin{aligned}
Y(1_i \rightarrow 1_f) &= mD_1 \sum_{n=0}^{\infty} \frac{1}{n!}\xi^n S_{i,n+1} e^{i(n+1)\phi} \\
&- mD_2 \sum_{n=0}^{\infty} \frac{1}{n!}\xi^n S_{f,n+1} e^{-i[(n+1)\phi-\eta]} \\
+m &\sin\theta_i \sin\theta_f [A_+ \sum_{n=0}^{\infty} \frac{1}{n!}\xi^n S_{i,n} e^{in\phi} \\
&- A'_+ \sum_{n=0}^{\infty} \frac{1}{n!}\xi^n S_{f,n} e^{-i(n\phi-\eta)}]
\end{aligned}
\tag{3.2.98}
$$

Following similar procedures for $Y(1_i \rightarrow 2_f), Y(2_i \rightarrow 1_f)$and$Y(2_i \rightarrow 2_f)$, that is using the respective polarization components:

$$
\begin{aligned}
Y(1_i \rightarrow 2_f) &= im(A_- \cos\theta_i - B_2) \sum_{n=0}^{\infty} \frac{1}{n!}\zeta^n S_{i,n+1} e^{i(n+1)\phi} \\
&+ im[A'_- \cos\theta_i + B_2] \sum_{n=0}^{\infty} \frac{1}{n!}\zeta^n S_{f,n+1} e^{-i(n+1)\phi+i\eta}
\end{aligned}
\tag{3.2.99}
$$

$$
\begin{aligned}
Y(2_i \rightarrow 1_f) &= -mi(A_- \cos\theta_f - B_1) \sum_{n=0}^{\infty} \frac{1}{n!}\zeta^n S_{i,n+1} e^{i(n+1)\phi} \\
&+ -mi[A'_- \cos\theta_f + B_1] \sum_{n=0}^{\infty} \frac{1}{n!}\zeta^n S_{f,n+1} e^{-i(n+1)\phi+i\eta}
\end{aligned}
\tag{3.2.100}
$$

$$
\begin{aligned}
Y(2_i \rightarrow 2_f) &= mA_- \sum_{n=0}^{\infty} \frac{1}{n!}\zeta^n S_{i,n+1} e^{i(n+1)\phi} \\
&- mA'_- \sum_{n=0}^{\infty} \frac{1}{n!}\zeta^n S_{f,n+1} e^{-i(n+1)\phi+i\eta}
\end{aligned}
\tag{3.2.101}
$$

For $n = 0$:

$$\begin{aligned}
Y(1_i \to 1_f) &= mD_1 S_{i,1} e^{i\phi} - mD_2 S_{f,1} e^{-i(\phi-\eta)} + m \sin\theta_i \sin\theta_f [A_f S_{i,0} - A'_f S_{f,0} e^{i\eta}] \\
Y(1_i \to 2_f) &= im(A_- \cos\theta_i - B_2) S_{i,1} e^{i\phi} + im(A'_- \cos\theta_i + B_2) S_{f,1} e^{-i(\phi-\eta)} \\
Y'(2_i \to 1_f) &= -im(A_- \cos\theta_f - B_1) S_{i,1} e^{i\phi} - im(A'_- \cos\theta_f + B_1) S_{f,1} e^{-i(\phi-\eta)} \\
Y'(2_i \to 2_f) &= mA_- S_{i,1} e^{i\phi} - mA'_- S_{f,1} e^{-i(\phi-\eta)} \qquad (3.2.102)
\end{aligned}$$

$$\begin{aligned}
\frac{|Y|^2}{m^2} &= \{D_1^2 S_{i,1}^2 e^{2i\phi} + D_2^2 S_{f,1}^2 e^{-2i(\phi-\eta)} + \sin^2\theta_i \sin^2\theta_f \\
&\times [A_+ S_{i,0} - A'_+ S_{f,0} e^{i\eta}]^2 \\
&+ 2(D_1 S_{i,1} e^{i\phi} - D_2 S_{f,1} e^{-i(\phi-\eta)}) \sin\theta_i \sin\theta_f \\
&\times [A_+ S_{i,0} - A'_+ S_{f,0} e^{i\eta}] \\
&- 2D_1 D_2 S_{i,1} e^{i\phi} S_{f,1}^2 e^{-i(\phi-\eta)}\} + | + (A_- \cos\theta_i - B_2)^2 S_{i,1}^2 e^{2i\phi} \\
&+ (A'_- \cos\theta_i + B_2)^2 S_{f,1}^2 e^{-2i(\phi-\eta)} + 2\big((A_- \cos\theta_i - B_2) \\
&\times (A'_- \cos\theta_i + B_2) S_{i,1} S_{f,1} e^{i\phi} e^{i(\phi-\eta)}\big)| \\
&+ | + (A_- \cos\theta_f - B_1)^2 S_{i,1}^2 e^{2i\phi} - (A'_- \cos\theta_f + B_1)^2 S_{f,1}^2 e^{-2i(\phi-\eta)} \\
&+ 2(A_- \cos\theta_f - B_1)(A'_- \cos\theta_f + B_1) S_{i,1} S_{f,1} e^{i\phi} e^{-i(\phi-\eta)}| \\
&+ |A_-^2 S_{i,1}^2 e^{2i\phi} + A_-'^2 S_{f,1}^2 e^{-2i(\phi-\eta)} - 2A_- A'_- S_{i,1} e^{i\phi} S_{f,1} e^{-i(\phi-\eta)} \qquad (3.2.103)
\end{aligned}$$

taking the magnitude of $\frac{|Y|^2}{m^2}$ leads to:

$$\begin{aligned}
\frac{|Y|^2}{m^2} &= [D_1^2 + (A_- \cos\theta_i - B_2)^2 + (A_- \cos\theta_f - B_1)^2 + A_-^2] S_{i,1}^2 e^{2i\phi} \\
&+ [D_2^2 + (A_- \cos\theta_i + B_2)^2 (A'_- \cos\theta_f + B_1)^2 + A_-'^2] S_{f,1}^2 e^{-2i(\phi-\eta)} \\
&+ \sin^2\theta_i \sin^2\theta_f [A_+ S_{i,0} - A'_+ S_{f,0} e^{i\eta}]^2 \\
&+ 2[(A_- \cos\theta_i - B_2)(A'_- \cos\theta_i + B_2) + (A_- \cos\theta_f - B_1)(A'_- \cos\theta_f + B_1) \\
&- D_1 D_2 - A_- A'_-] e^{i\phi} e^{-i(\phi-\eta)} S_{i,1} S_{f,1} \\
&+ 2(D_1 S_{i,1} e^{-i\phi} - D_2 S_{f,1} e^{-i(\phi-\eta)} \sin\theta_i \sin\theta_f [A_+ S_{i,0} - A'_+ S_{f,0} e^{i\eta}] \qquad (3.2.104)
\end{aligned}$$

$$\begin{aligned}
\frac{|Y|^2}{m^2} &= C_1 S_{i,1}^2 e^{-2i\phi} + C_2 S_{f,1}^2 e^{-2i(\phi-\eta)} \sin^2\theta_i \sin^2\theta_f \\
&\times [A_+^2 S_{i,0}^2 + A_+'^2 S_{f,0}^2 e^{2i\eta} - 2A_+ A'_+ S_{i,0} S_{f,0} e^{i\eta}] \\
&+ 2[(A_- \cos\theta_i)(A'_- \cos\theta_f + B_2) \\
&+ (A_- \cos\theta_f - B_1)(A'_- \cos\theta_f - B_1) \\
&- D_1 D_2 - A_- A'_-] e^{i\phi} e^{-i(\phi-\eta)} S_{i,1} S_{f,1} \\
&+ 2(D_1 A_+ S_{i,1} S_{f,0} e^{i(\phi+\eta)} - D_2 A_+ S_{i,0} S_{f,1} e^{-i(\phi-\eta)}) \\
&+ D_1 A_+ S_{i,1} S_{i,0} e^{i\phi} + D_2 A'_+ S_{f,1} S_{f,0} e^{-i(\phi-\eta)}) e^{i\eta} \qquad (3.2.105)
\end{aligned}$$

Now integrating over $\phi$ the above equation gives the following:

$$\begin{aligned}
\frac{|Y|^2}{m^2} &= C_1 S_{i,1}^2 \int e^{2i\phi} d\phi + C_2 S_{f,1}^2 \int e^{-2i(\phi-\eta)} + \sin^2\theta_i \sin^2\theta_f [A_+^2 S_{i,0}^2 + A_+'^2 S_{f,0}^2 \int e^{2i\eta} d\phi \\
&- 2A_+ A'_+ S_{i,0} S_{f,0} \int e^{i\eta} d\phi] + 2[(A_- \cos\theta_i - B_2)(A'_- \cos\theta_i + B_2) + (A_- \cos\theta_f - B_1) \\
&\times (A'_- \cos\theta_f + B_1) - D_1 D_2 - A_- A'_-] S_{i,1} S_{f,1} \int e^{i\phi} e^{-i(\phi-\eta)} d\phi - 2(D_1 A'_+ S_{i,1} S_{f,0} \int e^{i(\phi+\eta)} d\phi \\
&+ D_2 A_+ S_{i,0} S_{f,1} \int e^{-i(\phi-\eta)}) \sin\theta_i \sin\theta_f + 2(D_1 A_+ S_{i,1} S_{i,0} \int e^{i\phi} d\phi + D_2 A'_+ S_{f,1} S_{f,0} \int e^{-i(\phi-\eta)}
\end{aligned}$$

Since$\int e^{2i\phi} d\phi = \int e^{-2i(\phi-\eta)} d\phi = \int e^{2i\eta} = 2\pi$ and $\int e^{i\eta} = 2\pi J_0(\xi)$=$2\pi \frac{1}{2\pi} \int_0^{2\pi} d\phi \cos\eta =$ $2\pi \frac{1}{2\pi} \int_0^{2\pi} d\phi \cos(-\xi \sin\pi)$ $\int e^{i\phi} e^{-i(\phi-\eta)} d\phi = 2\pi J_2(-\xi)$

Let $Q = [(A_- \cos\theta_i - B_2)(A'_- \cos\theta_i + B_2) + (A_- \cos\theta_f - B_1)(A'_- \cos\theta_f + B_1) - D_1 D_2 - A_- A'_-]$

$$\begin{aligned}
\frac{|Y|^2}{m^2} &= 2\pi C_1 S_{i,1}^2 + 2\pi C_2 S_{f,1}^2 + \sin^2\theta_i \sin^2\theta_f [2\pi A_+^2 S_{i,0}^2 - 2A_+ A'_+ S_{i,0} S_{f,0} 2\pi J_0(\xi) \\
&+ 2A_+ A'_+ S_{i,0} S_{f,0} 2\pi - 2A_+ A'_+ S_{i,0} S_{f,0} 2\Pi] \\
&= 2[Q S_{i,1} S_{f,1} 2\pi J_2(\xi)] \\
&- 2[D_1 A'_+ S_{i,1} S_{f,0} 2\pi J_1(-\xi) + D_2 A'_+ S_{i,0} S_{f,1} 2\pi J_1(-\xi)] \sin\theta_i \sin\theta_f \\
&+ 2[D_1 A_+ S_{i,1} S_{i,0} 2\pi + D_2 A'_+ S_{f,1} S_{f,0} 2\pi] \qquad (3.2.106)
\end{aligned}$$

$$\begin{aligned}
\frac{|Y|^2}{2\pi m^2} &= C_1 S_{i,1}^2 + C_2 S_{f,1}^2 + \sin^2\theta_i \sin^2\theta_f [(A_+ S_{i,0} - A'_+ S_{f,0})^2 + 2A_+ A'_+ S_{i,0} S_{f,0}(1 - J_0(\xi))] \\
&+ 2Q S_{i,1} S_{f,1} J_2(\xi) - 2[D_1 A'_+ S_{i,1} S_{f,0} + D_2 A_+ S_{i,0} S_{f,1}] \sin\theta_i \sin\theta_f J_1(-\xi) \\
&+ 2[D_1 A_+ S_{i,1} S_{i,0} + D_2 A'_+ S_{f,1} S_{f,0}] \qquad (3.2.107)
\end{aligned}$$

$S_{i,1} S_{i,0}$ and $S_{f,1} S_{f,0}$ can be neglected since both are $S_{i,n} S_{f,n}$ terms

$$\begin{aligned}
&= C_1 S_{i,1}^2 + C_2 S_{f,1}^2 + [(A_+ S_{i,0} - A'_+ S_{f,0})^2 + 2(1 - J_0(\xi)) A_+ A'_+ S_{i,0} S_{f,0}](\sin\theta_i \sin\theta_f)^2 \\
&+ 2(Q) J_2(\xi) S_{i,1} S_{f,1} - 2[D_1 A'_+ S_{i,1} S_{f,0} + D_2 A_+ S_{i,0} S_{f,1}] \sin\theta_i \sin\theta_f J_1(-\xi) \quad (3.2.108)
\end{aligned}$$

Now substituting back the value of $Q$ we will get

$$\begin{aligned}
\frac{|Y|^2}{2\pi m^2} &= C_1 S_{i,1}^2 + C_2 S_{f,1}^2 + [(A_+ S_{i,0} - A'_+ S_{f,0})^2 + 2(1 - J_0(\xi)) A_+ A'_+ S_{i,0} S_{f,0}](\sin\theta_i \sin\theta_f)^2 \\
&+ 2(\xi) J_2([(A_- \cos\theta_i - B_2)(A'_- \cos\theta_i + B_2) + (A_- \cos\theta_f - B_1)(A'_- \cos\theta_f + B_1) \\
&- D_1 D_2 - A_- A'_-]) S_{i,1} S_{f,1} - 2[D_1 A'_+ S_{i,1} S_{f,0} + D_2 A_+ S_{i,0} S_{f,1}] \sin\theta_i \sin\theta_f J_1(-\xi)
\end{aligned}$$

$$\begin{aligned}
D_1 &= (A_- \cos\theta_f - B_1) \cos\theta_i - B_2 \cos\theta_f \\
&= 4\Delta_i \cos\theta_i \cos\theta_f
\end{aligned}$$

$$
\begin{aligned}
C_1 &= D_1^2 + (A_- \cos\theta_i - B_2)^2 + (A_- \cos\theta_f - B_1)^2 + A_-^2 \\
&= D_1^2 + (4\Delta_i \cos\theta_i)^2 + (4\Delta_i \cos\theta_f)^2 + (4\Delta_i)^2 \\
&= (4\Delta_i \cos\theta_i \cos\theta_f)^2 + (4\Delta_i \cos\theta_i)^2 + (4\Delta_i \cos\theta_i)^2 + (4\Delta_i)^2
\end{aligned}
$$

$$
\begin{aligned}
D_2 &= (A_-^{'} \cos\theta_f + B_1)\cos\theta_i + B_2 \cos\theta_f \\
&= -4\Delta_i \cos\theta_i \cos\theta_f
\end{aligned}
$$

$$
\begin{aligned}
C_2 &= D_2^2 + (A_-^{'} \cos\theta_i + B_2)^2 + (A_-^{'} \cos\theta_f + B_1)^2 + A_-^{'2} \\
&= (-4\Delta_i \cos\theta_i \cos\theta_f)^2 + (-4\Delta_i \cos\theta_i)^2 + (-4\Delta_i \cos\theta_f)^2 + (-4\Delta_i)^2
\end{aligned}
$$

The differential crossection given in Eq.(3.2.79) is reduced to

$$
d\sigma = \sigma(\Delta_i, \theta_i, \gamma, \theta_f) \sin\theta_f d\theta_f, \tag{3.2.109}
$$

where

$$
\begin{aligned}
\sigma(\Delta_i, \theta_i, \gamma, \theta_f) &= \frac{\pi r_0^2}{4} \frac{\Delta_f}{\Delta_{ir}(\gamma+1)(1+\gamma+\Delta_i-\Delta_f)} \\
&\times \frac{e^{\frac{-B}{2B_c}\Delta_f^2 \sin^2\theta_f} Y_r}{[\gamma(1-\beta\cos\theta_f) + \Delta_i(1-\cos\theta_i\cos\theta_f) - \Delta_f^2 \sin^2\theta_f]}
\end{aligned} \tag{3.2.110}
$$

Let us denote $Y_r$ for $\frac{|Y|^2}{2\pi m^2}$,then

$$
\begin{aligned}
Y_r &= C_1 S_{i,1}^2 + C_2 S_{f,1}^2 + [(A_+ S_{i,0} - A_+^{'} S_{f,0})]^2 (\sin\theta_i \sin\theta_f)^2 \\
&+ [2\big(1 - J_0(\zeta)\big) A_+ A_+^{'} S_{i,0} S_{f,0}] (\sin\theta_i \sin\theta_f)^2
\end{aligned}
$$

Under Thomson limit ($\Delta_f \approx \Delta_i \ll 1$) it is easy to see that $J_0(\zeta) \approx 1$, $A_- = -A_-' \approx$

$4\Delta_i$, $B_1 = B_2 \approx 0$ and $A_+ = -A'_+ \approx 4\Delta_i$.Up on using this we proceed as follows

$$\begin{aligned}
Y_r &= C_1 S_{i,1}^2 + C_2 S_{f,1}^2 + [(A_+ S_{i,0} - A'_+ S_{f,0})]^2 (\sin\theta_i \sin\theta_f)^2 \\
&= [(4\Delta_i \cos\theta_i \cos\theta_f)^2 + (4\Delta_i \cos\theta_i)^2 + (4\Delta_i \cos\theta_f)^2 + (4\Delta_i)^2](\frac{1}{2(\Delta_i - \Delta_0)})^2 \\
&+ [(-4\Delta_i \cos\theta_i \cos\theta_f)^2 + (-4\Delta_i \cos\theta_i)^2 + (-4\Delta_i \cos\theta_f)^2 + (-4\Delta_i)^2](\frac{1}{2(\Delta_i + \Delta_0)})^2 \\
&+ [(4\Delta_i)(\frac{1}{2\Delta_i}) - (-4\Delta_i)(\frac{1}{2\Delta_i})^2 (\sin\theta_i \sin\theta_f)^2 \\
&= 4\Delta_i^2 [\cos^2\theta_i \cos^2\theta_f + \cos^2\theta_i + \cos^2\theta_f + 1][\frac{1}{(\Delta_i - \Delta_0)^2} + \frac{1}{(\Delta_i + \Delta_0)^2}] \\
&+ 16\sin^2\theta_i \sin^2\theta_f \\
&= 4[\cos^2\theta_f(\cos^2\theta_i + 1) + 1(\cos^2\theta_i + 1)][\frac{\Delta_i^2}{(\Delta_i - \Delta_0)^2} + \frac{\Delta_i^2}{(\Delta_i + \Delta_0)^2}] + 16\sin^2\theta_i \sin^2\theta_f \\
&= 16\sin^2\theta_i \sin^2\theta_f + 4(1 + \cos^2\theta_i)(1 + \cos^2\theta_f)[\frac{\Delta_i^2}{(\Delta_i - \Delta_0)^2} + \frac{\Delta_i^2}{(\Delta_i + \Delta_0)^2}]
\end{aligned}$$

$$\frac{\Delta_f}{\Delta_{ir}(\gamma+1)(1+\gamma+\Delta_i-\Delta_f)} \quad \times \quad \frac{e^{\frac{-B}{2B_c}\Delta_f^2 \sin^2\theta_f}}{[\gamma(1-\beta\cos\theta_f) + \Delta_i(1-\cos\theta_i\cos\theta_f) - \Delta_f^2\sin^2\theta_f]}$$

in Eq.(3.2.110) is $\frac{1}{4}$ under this limit, hence

$$\begin{aligned}
\sigma(\Delta_i, \theta_i, \theta_f) &= \frac{\pi r_0^2}{4}\frac{Y_r}{4} \\
&= \frac{\pi r_0^2}{16}\{16\sin^2\theta_i \sin^2\theta_f + 4(1 + \cos^2\theta_i)(1 + \cos^2\theta_f)[\frac{\Delta_i^2}{(\Delta_i - \Delta_0)^2} + \frac{\Delta_i^2}{(\Delta_i + \Delta_0)^2}]\}
\end{aligned}$$

$$\frac{\sigma(\Delta_i, \theta_i, \theta_f)}{\pi r_0^2} = \sin^2\theta_i \sin^2\theta_f + \frac{1}{4}(1 + \cos^2\theta_i)(1 + \cos^2\theta_f)[\frac{\Delta_i^2}{(\Delta_i - \Delta_0)^2} + \frac{\Delta_i^2}{(\Delta_i + \Delta_0)^2}],$$

which is just Herold's nonrelativistic result[18].

# Chapter 4

# Gamma Ray Production Through Inverse Compton Scattering

## 4.1 Introduction

Inverse compton scattering has attracted more and more attention and a great deal of work has been done on this subject (Daugherty & Harding 1989;Dermer 1990;Zhang & Qiao 1997;Harding & Muslimov 1998).Daugherty & Harding (1989) studied the gamma ray generation by Monte Carlo simulation based on Herold's crossection (Herold 1979) of magnetic compton scatering in the electron rest frame(ERF).

In this chapter we will give analytical study of Gamma-ray production resulted from scattering of lower frequency photon (soft x-ray) by a beam of relativistic electron on the surface of an accreted neutron star based on our calculated crossection in chapter 3 Where we have derived the lab frame version of Herold's crossection in the electron rest frame(ERF).Where we calculate the spectrum function of gamma-ray.

## 4.2 Calculation Of Spectrum Function Of Gamma-ray

In the following we use the simplified crossection to calculate the spectrum function of the magnetic inverse compton scattering. Consider the scattering by a monochromatic ($\gamma$) electron beam. The density of scattered photons per unit time is

$$\frac{dN(\gamma)}{dt} = n_e \int \sin\theta_i d\theta_i \int \sin\theta_f d\theta_f \int n(\Delta_i) d\Delta_i (1 - \beta\cos\theta_i) \sigma(\Delta_i, \theta_i, \gamma, \theta_f) f(\cos\theta_i) \tag{4.2.1}$$

Where $n_e$ is the density of the electron beam and $f(\cos\theta_i)$ an anisotropic factor for the incident photons. If the incident photons are isotropic, then $f(\cos\theta_i) = 1$. Taking in to account energy conservation and the condition $\Delta_i \ll 1$, it is easy to drive

$$\sin\theta_f d\theta_f = -\frac{\gamma(1 - \beta\cos\theta_f) + \Delta_i(1 - \cos\theta_i\cos\theta_f) - \Delta_f \sin^2\theta_f}{\Delta_f(\gamma - \Delta_f\cos\theta_f)} d\Delta_f \tag{4.2.2}$$

with substitution of this variable transformation in $Eq.(4.1.1)$, the spectrum of power density per unit scattered photon energy of a low frequency photon gas scattered by the monochromatic electron beam can be derived: Since

$$\begin{aligned} \frac{dN(\gamma, \Delta_f)}{dt} &= n_e \int \sin\theta_i d\theta_i \int \sin\theta_f d\theta_f \int n(\Delta_i) d\Delta_i (1 - \beta\cos\theta_i) \\ &\times \sigma(\Delta_i, \theta_i, \gamma, \theta_f) f(\cos\theta_i) \end{aligned}$$

Now we will substitute $\sin\theta_f d\theta_f$given in equation $Eq.(4.1.2)$ and $\sigma(\Delta_i, \theta_i, \gamma, \theta_f)$in equation$Eq.$(3.2.110)in the above equation we will get the following:

$$\begin{aligned}
\frac{dN(\gamma,\Delta_f)}{dt} &= n_e \int \sin\theta_i d\theta_i \int [\frac{\gamma(1-\beta\cos\theta_f + \Delta_i(1-\cos\theta_i\cos\theta_f) - \Delta_f \sin^2\theta_f)}{\Delta_f(\gamma - \Delta_f\cos\theta_f)}] \\
&\times\ d\Delta_f \int n(\Delta_i) d\Delta_i (1-\beta\cos\theta_i) \frac{\pi r_0^2}{4} \frac{\Delta_f}{\Delta_{ir}(\gamma+1)(1+\gamma+\Delta_i-\Delta_f)} \\
&\times\ \frac{e^{\frac{-B}{2B_c}\Delta_f^2\sin^2\theta_f} Y_r f(\cos\theta_i)}{[\gamma(1-\beta\cos\theta_f)+\Delta_i(1-\cos\theta_i\cos\theta_f) - \Delta_f^2\sin^2\theta_f]} \qquad (4.2.3)
\end{aligned}$$

Multiplying this equation with $\frac{\Delta_f}{d\Delta_f}$ using$\Delta_{ir} = \gamma\Delta_i(1-\beta\cos\theta_i)$and

$$\gamma(\gamma+1)(1+\gamma+\Delta_i-\Delta_f) = \gamma^2(\gamma-\Delta_f)$$

for$\Delta_i \ll 1$and$\gamma \gg 1$We obtain

$$\begin{aligned}
\frac{\Delta_f dN(\gamma,\Delta_f)}{dtd\Delta_f} &= 8\pi r_0^2 n_e \int n(\Delta_i)d\Delta_i \int \sin\theta_i d\theta_i \frac{\Delta_f}{\Delta_i} \\
&\times\ \frac{Y_r e^{\frac{-B}{B_c}\Delta_f^2\sin^2\theta_f} f(\cos\theta_i)}{32\gamma^2(\gamma-\Delta_f)(\gamma-\Delta_f\cos\theta_f)} \qquad (4.2.4)
\end{aligned}$$

$$\frac{\Delta_f dN(\gamma,\Delta_f)}{dtd\Delta_f} = 8\pi r_0^2 n_e \int n(\Delta_i)d\Delta_i F(\gamma,\Delta_i,\Delta_f)$$

$$\frac{\Delta_f dN(\gamma,\Delta_f)}{dtd\Delta_f} = 8\pi r_0^2 n_e \int n(\Delta_i)d\Delta_i F(\gamma,\Delta_i,\Delta_f), \qquad (4.2.5)$$

where

$$F(\gamma,\Delta_i,\Delta_f) = \int \sin\theta_i d\theta_i \frac{\Delta_f}{\Delta_i} \frac{Y_r \exp[\frac{-B_c}{2B}\Delta_f^2\sin^2\theta_f]}{32\gamma^2(\gamma-\Delta_f)(\gamma-\Delta_f\cos\theta_f)} f(\cos\theta_f), \qquad (4.2.6)$$

is just the desired spectrum function. Now we take in to account the condition $\gamma \gg 1$, under this condition the constants $B_1, B_2, A_+, A'_+, A_-, A'_-$ will be reduced to their results given from Eq.(4.1.7)to Eq.(4.1.12)as follows;

$$\begin{aligned}
B_1 &= b\Delta_f \sin^2\theta_f \\
&= [\beta\gamma(1+\gamma+\Delta_i-\Delta_f)+(\beta\gamma+\Delta_i\cos\theta_i-\Delta_f\cos\theta_f)(1+\gamma)]\Delta_f\sin^2\theta_f \\
&= [\beta\gamma+\beta\gamma^2+\beta\gamma\Delta_i-\beta\gamma\Delta_f+\beta\gamma+\beta\gamma^2+\Delta_i\cos\theta_i+\beta\Delta_i\cos\theta_i-\Delta_f\cos\theta_f \\
&- \gamma\Delta_f\cos\theta_f]\Delta_f\sin^2\theta_f \\
&= \gamma[2\beta+2\beta\gamma-\Delta_f(\beta+\cos\theta_f)]\Delta_f(1-\cos\theta_f)(1+\cos\theta_f) \\
&= (2\gamma-\Delta_f(1+\cos\theta_f)]\gamma(1-\cos\theta_f)(1+\cos\theta_f) \\
&= [2\gamma-\Delta_f(1+\cos\theta_f)](1+\cos\theta_f)\Delta_{fr}.
\end{aligned}$$

similarly

$$\begin{aligned}
B_2 &= b\Delta_i\sin^2\theta_i \\
&= [\beta\gamma(1+\gamma+\Delta_i-\Delta_f)+(\beta\gamma+\Delta_i\cos\theta_i-\Delta_f\cos\theta_f)(1+\gamma)]\Delta_i\sin^2\theta_i \\
&= [2\gamma-\Delta_i(1+\cos\theta_i)]\Delta_i(1-\cos\theta_i)(1+\cos\theta_i) \\
&= [2\gamma-\Delta_i(1+\cos\theta_i)](1+\cos\theta_i)\Delta_i(1-\cos\theta_i) \\
&= [2\gamma-\Delta_i(1+\cos\theta_i)](1+\cos\theta_i)\Delta_{ir}
\end{aligned}$$

and

$$\begin{aligned}
A_+ &= a + b(\beta\gamma + \Delta_i \cos\theta_i) \\
&= \beta\gamma(1+\gamma+\Delta_i)(\beta\gamma + \Delta_i\cos\theta_i - \Delta_f\cos\theta_f) + (\gamma - 1 + \Delta_i)(1+\gamma)(1+\gamma+\Delta_i - \Delta_f) \\
&+ [2\gamma - \Delta_f(1+\cos\theta_f)]\gamma(\beta\gamma + \Delta_i\cos\theta_i) \\
&= \beta\gamma^2(\beta\gamma - \Delta_f\cos\theta_f) + \gamma^2(\gamma - \Delta_f) + [2\gamma - \Delta_f(1+\cos\theta_f)]\gamma(\beta\gamma + \Delta_i\cos\theta_i) \\
&= 2\gamma^3 - \gamma^2\Delta_f(\cos\theta_f + 1) + [2\gamma - \Delta_f(1+\cos\theta_f)]\gamma(\beta\gamma + \Delta_i\cos\theta_i) \\
&= \gamma^2(2\gamma - \Delta_f(\cos\theta_f + 1)) + [2\gamma - \Delta_f(1+\cos\theta_f)]\gamma^2 \\
&= [2\gamma - \Delta_f(1+\cos\theta_f)]2\gamma^2
\end{aligned}$$

similarly

$$
\begin{aligned}
A'_+ &= a' + b(\beta\gamma - \Delta_f \cos\theta_f \\
&= \beta\gamma(1+\gamma-\Delta_f)(\beta\gamma+\Delta_i\cos\theta_i - \Delta_f\cos\theta_f) + (\gamma - 1 - \Delta_f)(1+\gamma)(1+\gamma\Delta_i - \Delta_f) \\
&+ \gamma[2\gamma - \Delta_f(1+\cos\theta_f)](\beta\gamma - \Delta_f\cos\theta_f) \\
&= \beta\gamma(\gamma-\Delta_f)(\beta\gamma - \Delta_f\cos\theta_f) + (\gamma-\Delta_f)\gamma(\gamma-\Delta_f) \\
&+ \gamma[2\gamma - \Delta_f(1+\cos\theta_f)](\beta\gamma - \Delta_f\cos\theta_f) \\
&= [\beta\gamma(\beta\gamma - \Delta_f\cos\theta_f) + \gamma(\gamma-\Delta_f)](\gamma-\Delta_f) \\
&+ \gamma[2\gamma - \Delta_f(1+\cos\theta_f)](\beta\gamma - \Delta_f\cos\theta_f) \\
&= [\beta\gamma^2 - \beta\gamma\Delta_f\cos\theta_f + \gamma^2 - \gamma\Delta_f](\gamma-\Delta_f) \\
&+ \gamma[2\gamma - \Delta_f(1+\cos\theta_f)](\beta\gamma - \Delta_f\cos\theta_f) \\
&= [2\gamma - \Delta_f(1+\cos\theta_f) - [2\gamma - \Delta_f(1+\cos\theta_f)]\gamma\Delta_f) \\
&+ \gamma[2\gamma - \Delta_f(1+\cos\theta_f)](\beta\gamma) - [2\gamma - \Delta_f(1+\cos\theta_f)](\Delta_f\cos\theta_f) \\
&= [2\gamma - \Delta_f(1+\cos\theta_f)]2\gamma^2 - [2\gamma - \Delta_f(1+\cos\theta_f)][\gamma\Delta_f\cos\theta_f + \gamma\Delta_f] \\
&= [2\gamma - \Delta_f(1+\cos\theta_f)][2\gamma^2 - \gamma\Delta_f(1+\cos\theta_f)] \\
A'_+ &= [2\gamma - \Delta_f(1+\cos\theta_f)][2\gamma^2 - \gamma\Delta_f\cos\theta_f)]
\end{aligned}
$$

similarly,

$$\begin{aligned}
A_- &= a - b(\beta\gamma + \Delta_i \cos\theta_i) \\
&= \gamma^2[2\gamma - \Delta_f(1+\cos\theta_f)] - \gamma[2\gamma - \Delta_f \\
&\times (1+\cos\theta_f)](\beta\gamma + \Delta_i\cos\theta_i) \\
&= [2\gamma - \Delta_f(1+\cos\theta_f)](\gamma^2 - \gamma^2\beta - \gamma\Delta_i\cos\theta_i) \\
&= [2\gamma - \Delta_f(1+\cos\theta_f)][-\gamma\Delta_i\cos\theta_i] \\
A_- &= [2\gamma - \Delta_f(1+\cos\theta_f)]\Delta_{ir}
\end{aligned}$$

Similarly $A'_- = a' - b(\beta\gamma - \Delta_f\cos\theta_f)$

$$\begin{aligned}
&= [2\gamma - \Delta_f(1+\cos\theta_f)][\gamma^2 - \gamma\Delta_f] - [2\gamma - \Delta_f(1+\cos\theta_f)] \\
&\times \gamma(\beta\gamma^2 - \gamma\Delta_f\cos\theta_f) \\
&= -[2\gamma - \Delta_f(1+\cos\theta_f)](\gamma\Delta_f - \gamma^2 + \beta\gamma^2 - \gamma\Delta_f\cos\theta_f) \\
A'_- &= -[2\gamma - \Delta_f(1+\cos\theta_f)]\gamma\Delta_f(1-\cos\theta_f) \\
A'_- &= -[2\gamma - \Delta_f(1+\cos\theta_f)]\Delta_{fr}
\end{aligned}$$

$$A_- = [2\gamma - \Delta_f(1+\cos\theta_f)]\Delta_{ir}, \tag{4.2.7}$$

$$A'_- = -[2\gamma - \Delta_f(1+\cos\theta_f)]\Delta_{fr}, \tag{4.2.8}$$

$$A_+ = [2\gamma - \Delta_f(1+\cos\theta_f)]2\gamma^2, \tag{4.2.9}$$

$$A'_+ = [2\gamma - \Delta_f(1+\cos\theta_f)](2\gamma^2 - \Delta_{fr}), \tag{4.2.10}$$

$$B_1 = [2\gamma - \Delta_f(1+\cos\theta_f)](1+\cos\theta_f)\Delta_{fr}, \tag{4.2.11}$$

$$B_2 = [2\gamma - \Delta_f(1+\cos\theta_f)](1+\cos\theta_i)\Delta_{ir}, \tag{4.2.12}$$

and the spectrum function can be derived as follows:For $\gamma \gg 1$ We can calculate$C_1$and$C_2$as follows:

$$C_1 = (A_- \cos\theta_f - B_1)^2(1+\cos^2\theta_i) + (A_- \cos\theta_f - B_2)^2 + (B_2\cos\theta_f)^2 + A_-^2$$

$$\begin{aligned} C_1 &= ([2\gamma - \Delta_f(1+\cos\theta_f)]\Delta_{ir}\cos\theta_f - [2\gamma - \Delta_f(1+\cos\theta_f)](1+\cos\theta_f\Delta_{fr}))^2(1+\cos^2\theta_i) \\ &+ ([2\gamma - \Delta_f(1+\cos\theta_f)]\Delta_{ir}\cos\theta_i - [2\gamma - \Delta_f(1+\cos\theta_f)](1+\cos\theta_i\Delta_{ir}))^2 \\ &+ ([2\gamma - \Delta_f(1+\cos\theta_f)](1+\cos\theta_i)\Delta_{ir}\cos\theta_f)^2 + ([2\gamma - \Delta_f(1+\cos\theta_f)]\Delta_{ir}))^2 \end{aligned}$$

$$\begin{aligned} C_1 &= (2\gamma - \Delta_f(1+\cos\theta_f))^2[(\Delta_{ir}\cos\theta_f - \Delta_{fr}(1+\cos\theta_f))^2(1+\cos^2\theta_i) \\ &+ [\Delta_{ir}\cos\theta_i - \Delta_{ir}(1+\cos\theta_i)]^2 + [(1+\cos^2\theta_i)\Delta_{ir}\cos\theta_f]^2 + \Delta_{ir}^2] \end{aligned}$$

$$\begin{aligned} C_1 &= (2\gamma - \Delta_f(1+\cos\theta_f))^2[(\Delta_{ir}\cos\theta_f - \Delta_{fr}(1+\cos\theta_f))^2(1+\cos^2\theta_i) \\ &+ \Delta_{ir}^2(1+\cos\theta_i)^2\cos^2\theta_f + \Delta_{ir}^2] \end{aligned}$$

$$\begin{aligned} \frac{C_1}{(2\gamma - \Delta_f(1+\cos\theta_f))^2} &= (\Delta_{ir}\cos\theta_f - \Delta_{fr}(1+\cos\theta_f))^2(1+\cos^2\theta_i) \\ &+ \Delta_{ir}^2[2 + (1+\cos\theta_i)^2\cos^2\theta_f] \end{aligned}$$

Let let $D_1$ stands for the LHS.

$$\begin{aligned} D_1 &= [\Delta_{ir}\cos\theta_f - \Delta_{fr}(1+\cos\theta_f)]^2(1+\cos^2\theta_i) \\ &+ \Delta_{ir}^2[2 + (1+\cos\theta_i)]^2\cos^2\theta_f \end{aligned}$$

Similarly

$$C_2 = (A'_- \cos\theta_f + B_1)^2(1+\cos^2\theta_i) + (A'_- \cos\theta_i + B_2)^2 + (B_2\cos\theta_f)^2 + A'^2_-$$

$$
\begin{aligned}
C_2 &= (-[2\gamma-\Delta_f(1+\cos\theta_f)]\Delta_{fr}\cos\theta_f+[2\gamma-\Delta_f(1+\cos\theta_f)](1+\cos\theta_f)\Delta_{fr}))^2(1+\cos^2\theta_i)\\
&+ (-[2\gamma-\Delta_f(1+\cos\theta_f)]\Delta_{fr}\cos\theta_i+[2\gamma-\Delta_f(1+\cos\theta_f)](1+\cos\theta_i\Delta_{ir}))^2\\
&+ ([2\gamma-\Delta_f(1+\cos\theta_f)](1+\cos\theta_i)\Delta_{ir}\cos\theta_f)^2+(-[2\gamma-\Delta_f(1+\cos\theta_f)]\Delta_{fr}))^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{C_2}{[2\gamma-\Delta_f(1+\cos\theta_f)]^2} &= \Delta_{fr}^2(1+\cos^2\theta_i)+[\Delta_{ir}(1+\cos\theta_i)-\Delta_{fr}\cos\theta_i]^2\\
&+ \Delta_{ir}^2((1+\cos\theta_i)^2\cos^2\theta_f)+\Delta_{fr}^2
\end{aligned}
$$

Here also let $D_2$ stands for the LHS.

$$D_2=[\Delta_{ir}(1+\cos\theta_i)-\Delta_{fr}\cos\theta_i]^2+\Delta_{fr}^2(2+\cos^2\theta_i)+\Delta_{ir}^2(1+\cos\theta_i)^2\cos^2\theta_f$$

Now

$$F(\gamma,\Delta_i,\Delta_f)=\int\sin\theta_i d\theta_i\frac{\Delta_f}{\Delta_i}\frac{Y_r}{32\gamma^2}\frac{e^{\frac{-B}{2B_c}\Delta_f^2\sin^2\theta_f}}{(\gamma-\Delta_f)(\gamma-\Delta_f\cos\theta_f)}f(\cos\theta_i)$$

can be rewritten as

$$F(\gamma,\Delta_i,\Delta_f)=\int\sin\theta_i d\theta_i\frac{\Delta_f}{\Delta_i}\frac{Y_r'}{32\gamma^2}e^{\frac{-B}{2B_c}\Delta_f^2\sin^2\theta_f}f(\cos\theta_i)$$

Where$Y_r'=\frac{Y_r}{(\gamma-\Delta_f)(\gamma-\Delta_f\cos\theta_f)}$

$$
\begin{aligned}
Y_r' &= \frac{1}{(\gamma-\Delta_f)(\gamma-\Delta_f\cos\theta_f)}[C_1S_{i,1}^2+C_2S_{f,1}^2+[(A_+S_{i,0}-A_+'S_{f,0})^2\\
&+ 2(1-J_0(\xi))A_+A_+'S_{i,0}S_{f,0}](\sin\theta_i\sin\theta_f)^2]
\end{aligned}
$$

Now let us substitute the values of$C_1, C_2$and$S_{i,n}, S_{f,n}$

$$C_1=[2\gamma-\Delta_f(1+\cos\theta_f)]^2D_1$$

$$C_2=[2\gamma-\Delta_f(1+\cos\theta_f)]^2D_2$$

$$S_{i,n} = \frac{1}{n![2(\Delta_{ir} - n\Delta_0) + \Delta_i^2 \sin^2\theta_i]}$$
$$S_{f,n} = \frac{1}{n![2(\Delta_{fr} + n\Delta_0) - \Delta_f^2 \sin^2\theta_f]}$$

With$A_+$and$A'_+$given in the previous calculation

$$\begin{aligned} Y'_r &= \frac{1}{(\gamma-\Delta_f)(\gamma-\Delta_i\cos\theta_i)}[\frac{[2\gamma-\Delta_f(1+\cos\theta_f)]^2 D_1}{[2(\Delta_{ir}-\Delta_0)+\Delta_i^2\sin^2\theta_i]^2} + \frac{[2\gamma-\Delta_f(1+\cos\theta_f)]^2 D_2}{[2(\Delta_{fr}+\Delta_0)-\Delta_f^2\sin^2\theta_f]^2} \\ &+ [(\frac{[2\gamma-\Delta_f(1+\cos\theta_f)]^2 2\gamma^2}{2\Delta_{ir}+\Delta_i^2\sin^2\theta_i} - \frac{[2\gamma-\Delta_f(1+\cos\theta_f)](2\gamma^2-\Delta_{fr})}{2\Delta_{fr}-\Delta_{fr}^2\sin^2\theta_f})^2 \\ &+ \frac{2(1-J_0(\xi))[2\gamma-\Delta_f(1+\cos\theta_f)]2\gamma^2[2\gamma-\Delta_f(1+\cos\theta_f)](2\gamma^2-\Delta_{fr})}{(2\Delta_{ir}+\Delta_i^2\sin^2\theta_i)(2\Delta_{fr}-\Delta_f^2\sin^2\theta_f)}]\sin^2\theta_i\sin^2\theta_f] \end{aligned}$$

$$\begin{aligned} Y'_r &= \frac{[2\gamma-\Delta_f(1+\cos\theta_f)]^2}{4(\gamma-\Delta_f)(\gamma-\Delta_f\cos\theta_f)}[\frac{D_1}{(\Delta_{ir}-\Delta_0)^2+(\Delta_{ir}-\Delta_0)\Delta_i^2\sin^2\theta_i+4\Delta_i^4\sin^4\theta_i} \\ &+ \frac{D_2}{(\Delta_{fr}+\Delta_0-\frac{1}{2}\Delta_f^2\sin^2\theta_f)^2} \\ &+ (\frac{2\gamma^2}{\Delta_{ir}+\frac{1}{2}\Delta_i^2\sin^2\theta_i} - \frac{(2\gamma^2-\Delta_{fr})}{\Delta_{fr}-\frac{1}{2}\Delta_f^2\sin^2\theta_f})^2\sin^2\theta_i\sin^2\theta_f \\ &+ \frac{4(1-J_0(\xi))\gamma^2(2\gamma^2-\Delta_{fr})\sin^2\theta_i\sin^2\theta_f}{(\Delta_{fr}-\frac{1}{2}\Delta_f^2\sin^2\theta_f)}] \end{aligned}$$

Up on using the approximation

$$\frac{[2\gamma-\Delta_f(1+\cos\theta_f)]^2}{4(\gamma-\Delta_f)(\gamma-\Delta_f\cos\theta_f)} \simeq 1$$

and$\Delta_i \ll 1$ This will reduce to:

$$\begin{aligned} Y'_r &= \frac{D_1}{(\Delta_{ir}-\Delta_0)^2+\Gamma_0^2} + \frac{D_2}{(\Delta_{fr}+\Delta_0-0.5\Delta_f^2\sin^2\theta_f)^2} + (\frac{2\gamma^2}{\Delta_{ir}} - \frac{(2\gamma^2-\Delta_{fr})}{\Delta_{fr}-0.5\Delta_f^2\sin^2\theta_f})^2 \\ &\times \sin^2\theta_i\sin^2\theta_f + \frac{4(1-J_0(\xi))\gamma^2(2\gamma^2-\Delta_{fr})\sin^2\theta_i\sin^2\theta_f}{\Delta_{fr}(\Delta_{fr}-0.5\Delta_f^2\sin^2\theta_f)} \end{aligned}$$

Where$\Gamma_0^2 = (\Delta_{ir}-\Delta_0)\Delta_i^2\sin^2\theta_i + 4\Delta_i^4\sin^4\theta_i$

$$F(\gamma,\Delta_i,\Delta_f) = \int \sin\theta_i d\theta_i \frac{\Delta_f}{\Delta_i}\frac{Y'_r}{32\gamma^2}\exp[\frac{-B_c}{2B}\Delta_f^2\sin^2\theta_f]f(\cos\theta_f), \qquad (4.2.13)$$

where

$$
\begin{aligned}
Y_r' &= \frac{D_1}{(\Delta_{ir}-\Delta_o)^2+\Gamma_0^2}+\frac{D_2}{(\Delta_{fr}+\Delta_o-0.5\Delta_f^2\sin^2\theta_f)^2} \\
&+ [\frac{2\gamma^2}{\Delta_{ir}}-\frac{2\gamma^2-\Delta_{fr}}{\Delta_{fr}-0.5\Delta_f^2\sin^2\theta_f}]^2\sin^2\theta_i\sin^2\theta_f \\
&+ \frac{4[1-J_o(\zeta)^2]\gamma^2(2\gamma^2-\Delta_{fr})\sin^2\theta_i\sin^2\theta_f}{\Delta_{ir}(\Delta_{fr}-0.5\Delta_f^2\sin^2\theta_f)}
\end{aligned}
\tag{4.2.14}
$$

in which

$$D_1=[\Delta_{ir}\cos\theta_f-\Delta_{fr}(1+\cos\theta_f)]^2(1+\cos^2\theta_i)+\Delta_{ir}^2[2+(1+\cos\theta_i)^2\cos^2\theta_f], \tag{4.2.15}$$

$$D_2=[\Delta_{ir}(1+\cos\theta_i)-\Delta_{fr}\cos\theta_i]^2+\Delta_{fr}^2(2+\cos^2\theta_i)+\Delta_{ir}^2(1+\cos\theta_i)^2\cos^2\theta_f \tag{4.2.16}$$

and $\Gamma_0$ is related to the inverse life time of an electron intermediate states which is usually estimated according to the transition rate of an electron from the first landau level to the ground state (Daughery and Vantura 1978);that is ,$\Gamma_0=\frac{2}{3}\alpha(\frac{B}{B_c})^2$ with $\alpha$ the fine structure constant. In obtaining Eq.(4.2.13), use has been made of the following approximation.

$$\frac{[2\gamma-\Delta_f(1+\cos\theta_f)]^2}{4(\gamma-\Delta_f)(\gamma-\Delta_f\cos\theta_f)}\approx 1 \tag{4.2.17}$$

To see this we note that $\Delta_f$ reaches maximum only at $\theta_f=0$ and $\theta_i=\pi, so$ $\Delta_f\ll\gamma$ if $\theta_f$ is not close to zero , then it is clear that Eq.(4.2.17) is also valid, thus the approximation is justified. It can be shown that $\Delta_f\sin\theta_f$ can also be neglected if there are other dominant terms. Using again the energy conservation, we get the following approximate expression, From conservation of energy we have

$$E_f=E_i+\omega_i-\omega_f$$

but

$$E_f - P_f \cos\theta_f = E_i(1 - \tfrac{P_i}{E_i \cos\theta_f}) + \omega_i(1 - \cos\theta_i \cos\theta_f) - \omega_f \sin^2\theta_f$$

$$= m\gamma(1 - \beta\cos\theta_f) + m\Delta_i(1 - \cos\theta_i\cos\theta_f) - m\Delta_f \sin^2\theta_f$$

$$\frac{E_f}{m} - \frac{P_f}{m}\cos\theta_f = \gamma(1 - \beta\cos\theta_f) + \Delta_i(1 - \cos\theta_i\cos\theta_f) - \Delta_f\sin^2\theta_f$$

$$\frac{1}{m\gamma}(E_f - P_f\cos\theta_f) = 1 - \beta\cos\theta_f + \frac{\Delta_i}{\gamma}(1 - \cos\theta_i\cos\theta_f) - \frac{\Delta_f}{\gamma}\sin^2\theta_f$$

Since$\Delta_f \ll \gamma$Where we have used$\frac{E_f}{E_i} = \frac{E_i+\omega_i-\omega_f}{E_i} = 1 + \frac{m(\Delta_i-\Delta_f)}{m\gamma} \simeq 1$ and$\frac{P_f}{E_i} = \frac{P_i+\omega_i\cos\theta_i-\omega_f\cos\theta_f}{E_f} = \beta + \frac{m\Delta_i\cos\theta_i}{m\gamma} - \frac{m\Delta_f\cos\theta_f}{m\gamma} \simeq \beta$

$$1 - \beta\cos\theta_f = [1 - \beta\cos\theta_f + \frac{\Delta_i}{\gamma}(1 - \cos\theta_i\cos\theta_f)]$$

But

$$1 - \beta\cos\theta_f \simeq \frac{\Delta_i(1 - \beta\cos\theta_i)(1 - \beta\cos\theta_f)}{\Delta_f(1 - \beta\cos\theta_f)}$$

$1 - \beta\cos\theta_f = \frac{\Delta_i(1-\beta\cos\theta_i)}{\Delta_f}$ Substituting this back:

$$\frac{\Delta_i(1 - \beta\cos\theta_i)}{\Delta_f} = [1 - \beta\cos\theta_f + \frac{\Delta_i}{\gamma}(1 - \cos\theta_i\cos\theta_f)]$$

$$\Delta_i(1 - \beta\cos\theta_i) = [1 - \beta\cos\theta_f + \frac{\Delta_i}{\gamma}(1 - \cos\theta_i\cos\theta_f)]\Delta_f$$

$$[1 - \beta\cos\theta_f + \frac{\Delta_i}{\gamma}(1 - \cos\theta_i\cos\theta_f)]\Delta_f = (1 - \beta\cos\theta_i)\Delta_i. \tag{4.2.18}$$

For further simplification, we consider the case where $\theta_f$ is not close to zero, then Eq.(4.2.18)can be simplified to

$$(1 - \beta\cos\theta_f)\Delta_f = (1 - \beta\cos\theta_i)\Delta_i \tag{4.2.19}$$

.

$$[1 - \beta\cos\theta_f + \frac{\Delta_i}{\gamma}(1 - \cos\theta_i\cos\theta_f)]\Delta_f = (1 - \beta\cos\theta_i)\Delta_i \tag{4.2.20}$$

$\Delta_f = \frac{(1+\beta)\Delta_i}{1-\beta\cos\theta_f+\frac{\Delta_i}{\gamma}(1+\cos\theta_f)}$ For maximum scattered photon frequency $\cos\theta_i = \pi$ and $\cos\theta_f = 0$

$$\Delta_f = \frac{\gamma^2(1+\beta)\Delta_i(1+\beta)}{\gamma^2(1-\beta^2)+\gamma\Delta_i 2(1+\beta)}$$

$$\omega_f = \frac{\gamma^2(1+\beta)^2\omega_i}{1+2(1+\beta)\gamma\Delta_i}$$

$$\omega_f = \frac{(1+\beta)^2\gamma^2\omega_i}{1+2(1+\beta)\gamma\Delta_i}$$

$$\simeq \frac{4\gamma^2\omega_i}{1+4\gamma\Delta_i}$$

Where$\Delta_i = \frac{\omega_i}{m}$and$\gamma = \frac{1}{\sqrt{1-\beta^2}}$

This means that the Doppler frequencies of the incident and scattered photons are equal $\Delta_{ir} = \Delta_{fr}$, which is just the Thomson limit in the LF.Equation(4.2.18) tells us also that the highest scattered photon energy is of magnitude $\frac{(1+\beta)^2\gamma^2\omega_i}{(1+2(1+\beta)\gamma\Delta_i)} \approx \frac{4\gamma^2\omega_i}{(1+4\gamma\Delta_i)}$,which becomes $4\gamma^2\omega_i$ for $\gamma\Delta_i \ll 1$,which is derived earlier. This is a well-known feature of inverse compton scattering.Applying the Thomson limit Eq.(4.2.19) and expanding $J_0(\zeta)$ up to $\zeta^2$ ,Eq.(4.2.14) is reduced to

$$D_1 = [\Delta_{ir}\cos\theta_f - \Delta_{fr}(1+\cos\theta_f)]^2(1+\cos^2\theta_i) + \Delta_{ir}^2[2+(1+\cos\theta_i)^2\cos^2\theta_f]$$

Since$\Delta_{ir} \simeq \Delta_{fr}$

$$D_1 = \Delta_{ir}^2[(\cos\theta_f - 1 - \cos\theta_f)^2(1+\cos^2\theta_i) + 2 + (1+\cos\theta_i)^2\cos^2\theta_f]$$

$$D_1 = \Delta_{ir}^2[(1+\cos^2\theta_i)(1+\cos^2\theta_f) + 2(1+\cos\theta_i\cos^2\theta_f)]$$

$$D_2 = [\Delta_{ir}(1+\cos\theta_i) - \Delta_{fr}\cos\theta_i]^2 + \Delta_{fr}^2(2+\cos^2\theta_i) + \Delta_{ir}^2(1+\cos\theta_i)^2\cos^2\theta_f$$

$$= \Delta_{ir}^2[1 + 2 + \cos^2\theta_i + (1 + \cos^2\theta_i + 2\cos\theta_i)\cos^2\theta_f]$$

$$= \Delta_{ir}^2[(1 + \cos^2\theta_i) + 2 + (1 + \cos^2\theta_i)\cos^2\theta_f + 2\cos\theta_i\cos^2\theta_f]$$

$$D_2 = \Delta_{ir}^2[(1 + \cos^2\theta_i)(1 + \cos^2\theta_f) + 2(1 + \cos\theta_i\cos^2\theta_f)]$$

Thus$D_1 = D_2 = C\Delta_{ir}^2$where$C = (1 + \cos^2\theta_i)(1 + \cos^2\theta_f) + 2(1 + \cos\theta_i\cos^2\theta_f)$ Now

$$\begin{aligned} Y_r' &= \frac{C\Delta_{ir}^2}{(\Delta_{ir} - \Delta_0)^2 + \Gamma_0^2} + \frac{C\Delta_{ir}^2}{(\Delta_{ir} + \Delta_0)^2} + [\frac{2\gamma^2}{\Delta_{ir}} - \frac{(2\gamma^2 - \Delta_{ir})}{\Delta_{ir}}]^2 \sin^2\theta_i \sin^2\theta_f \\ &+ \frac{4(1 - 1 + \frac{1}{4}\xi^2)\gamma^2(2\gamma^2 - \Delta_{fr})\sin^2\theta_i\sin^2\theta_f}{\Delta_{ir}\Delta_{fr}} \end{aligned} \quad (4.2.21)$$

Where we have neglected$\Delta_f^2\sin^2\theta_f$ Since the first two terms will not change through the calculation we only solve the last two terms:

$$[\frac{2\gamma^2\Delta_{ir} - 2\gamma^2\Delta_{ir} - \Delta_{ir}}{\Delta_{ir}}]^2 \sin^2\theta_i\sin^2\theta_f + \frac{2\xi^2\gamma^2\gamma^2\sin^2\theta_i\sin^2\theta_f}{\Delta_{ir}\Delta_{fr}}$$

Where$2\gamma^2 - \Delta_{fr} \simeq 2\gamma^2$and since$\xi = \frac{B_c}{2B}\Delta_i\Delta_f\sin^2\theta_i\sin^2\theta_f$ then this will lead us to:

$$\sin^2\theta_i\sin^2\theta_f + \frac{2[\frac{B^2}{4B_c^2}\sin^4\theta_i\sin^4\theta_f\gamma^4]}{\gamma\Delta_i\gamma\Delta_f(1 - \beta\cos\theta_i)(1 - \beta\cos\theta_f)}$$

$$\sin^2\theta_i\sin^2\theta_f + 0.5(\frac{B_c}{B})^2\gamma\Delta_i\gamma\Delta_f(1 - \cos\theta_i)(1 - \cos\theta_f)(1 + \cos\theta_i)^2(1 + \cos\theta_f)^2$$

$$\sin^2\theta_i\sin^2\theta_f + 0.5(\frac{B_c}{B})^2[\gamma\Delta_i(1 - \cos\theta_i)][\gamma\Delta_f(1 - \cos\theta_f)](1 + \cos\theta_i)^2(1 + \cos\theta_f)^2$$

$$\sin^2\theta_i\sin^2\theta_f + 0.5(\frac{B_c}{B})^2(\Delta_{ir})(\Delta_{fr})(1 + \cos\theta_i)^2(1 + \cos\theta_f)^2$$

Then collecting terms left before to get the required formula

$$\begin{aligned} Y_r' &= \frac{C\Delta_{ir}^2}{(\Delta_{ir}-\Delta_0)^2+\Gamma_0^2}+\frac{C\Delta_{ir}^2}{(\Delta_{ir}+\Delta_0)^2}+\sin^2\theta_i\sin^2\theta_f \\ &+ 0.5[\Delta_{ir}(1+\cos\theta_i)(1+\cos\theta_f)(\frac{B_c}{B})]^2 \end{aligned}$$

$$\begin{aligned} Y_r' &= \frac{C\Delta_{ir}^2}{(\Delta_{ir}-\Delta_o)^2+\Gamma_0^2}+\frac{C\Delta_{ir}^2}{(\Delta_{ir}+\Delta_o)^2} \\ &+ (\sin\theta_i\sin\theta_f)^2+0.5[\Delta_{ir}(1+\cos\theta_i)(1+\cos\theta_f)\frac{B_c}{B}]^2, \end{aligned} \quad (4.2.22)$$

where the coefficient C is defined by

$$C=(1+\cos^2\theta_i)(1+\cos^2\theta_f)+2(1+\cos\theta_i\cos^2\theta_f). \quad (4.2.23)$$

We can express this in other form

$C(x)=(1+2x+x^2)(2-\frac{2x}{\omega_r}+\frac{x^2}{\omega_r^2})+2[1+(1-x)(1-\frac{x}{\omega_r}{}^2)]$ Let $x=(1-\cos\theta_i\simeq(1-\beta)\cos\theta_i)$ Then

$1+\cos^2\theta_f=1+(1-\frac{x}{\omega_r})^2=1+1-\frac{2x}{\omega_r}+\frac{x^2}{\omega_r^2}=2-\frac{2x}{\omega_r}+\frac{x^2}{\omega_r^2}$

Since $\cos\theta_f=1-\frac{x}{\omega_r}$ Because

$(1-\beta\cos\theta_f)\Delta_f=(1-\beta\cos\theta_i)\Delta_i)$

$(1-\beta\cos\theta_f)\frac{\omega_f}{\omega_i}=1-\beta\cos\theta_i$ $(1-\beta\cos\theta_f)\omega_r=x$

$1-\beta\cos\theta_f=\frac{x}{\omega_r}\simeq1-\cos\theta_f=\frac{x}{\omega_r}$ hence $\cos\theta_f=1-\frac{x}{\omega_r}$

$1+\cos^2\theta_i=1+(\cos\theta_i)^2$,but $x=1-\cos\theta_i\Rightarrow\cos\theta_i=1-x$ $1+(1-x)^2=1+1-2x+x^2=2-2x+x^2$, $2(1+\cos\theta_i\cos\theta_f^2)=2\big(1+(1-x)(1-\frac{x}{\omega_r})\big)$

then up on substituting this we can get $C=(1+\cos^2\theta_i)(1+\cos^2\theta_f)+2(1+\cos\theta_i\cos^2\theta_f)$

Again for $x=(1-\cos\theta_i)\approx(\beta-\cos\theta_i)$, with its lower limit being determined by Eq.(4.2.20)

$$\gamma\Delta_f(1-\beta\cos\theta_f)=\gamma(1-\beta\cos\theta_f)\Delta_i-\Delta_i(1-\cos\theta_i\cos\theta_f)$$

But

$$\Delta_f(1-\beta\cos\theta_f) = (1-\beta\cos\theta_i)\Delta_i(1-\frac{\Delta_f}{\gamma})$$

$$x_{min} = \frac{\Delta_f(1-\beta\cos\theta_f)}{\Delta_i(1-\frac{\Delta_f}{\gamma})}(\frac{(1+\beta\cos\theta_f)}{(1+\beta\cos\theta_f)})$$

Where we have used,$1-\beta\cos\theta_f \approx 2$ & $1-\beta\cos^2\theta_f \approx 1-\beta^2 = \frac{1}{\gamma^2}$

$$x_{min} = \frac{\Delta_f(1-\beta^2)}{2\Delta_i(1-\frac{\Delta_f}{\gamma})}$$

$$x_{min} = \frac{\Delta_f}{2\gamma^2\Delta_i(1-\frac{\Delta_f}{\gamma})}$$

$$x_{min} \approx \frac{\Delta_f}{2\gamma^2\Delta_i(1-\frac{\Delta_f}{\gamma})} \tag{4.2.24}$$

Up on substituting Eq.(4.2.22) back into Eq.(4.2.13) we get three integrals.

$$F = F_1 + F_2 + F_3 \tag{4.2.25}$$

where $F_1$,$F_2$&$F_3$ are the first ,second & third part of the integral after the above substitution. In the following we will derive for $F_1$,$F_2$&$F_3$ Using $\Delta_{ir} = \gamma(1-\beta\cos\theta_i)\Delta_i$ we get

$$\Delta_{fr}^2 = x^2\gamma^2\Delta_i^2$$

We know that $\Delta_0 = \frac{\omega_0}{m} = \frac{B}{B_c}$,$\Delta_i = \frac{\omega_i}{m}$ then we define $a_0$ to be,

$$\frac{1}{\gamma}\frac{\Delta_0}{\Delta_i} = \frac{1}{\gamma}\frac{\omega_0}{\omega_i} = a_0$$

which can be rewritten as $\Delta_0 = a_0\gamma\Delta_i$ Now from above, the first part of the integral is

$$F_1 = \frac{\omega_r}{32\gamma^2}\int_{x_{min}}^{2} dx[\frac{C\Delta_{ir}^2}{(\Delta_{ir}-\Delta_0)^2+\Gamma^2} + \frac{C\Delta_{ir}^2}{(\Delta_{ir}+\Delta_0)^2}]f(\cos\theta_i)$$

then we get

$$F_1 \quad = \quad \frac{\omega_r}{32\gamma^2}\int_{x_{min}}^{2} dx[\frac{C(x)\gamma^2\Delta_i^2x^2}{(\gamma\Delta_i x-\gamma\Delta_i a_0)^2+\Gamma_0^2}+\frac{C(x)\gamma^2\Delta_i^2x^2}{(\gamma\Delta_i x-\gamma\Delta_i a_0)^2}]f(1-x)$$

$$F_1 \quad = \quad \frac{\omega_r}{32\gamma^2}\int_{x_{min}}^{2} dx[\frac{C(x)x^2}{[(x-a_0)^2+\Gamma^2]}+\frac{C(x)x^2}{(x+a_0)^2}]f(1-x) \qquad (4.2.26)$$

where we have used $\Gamma = \frac{\Gamma_0}{\gamma\Delta_i}$ And the second part of the integral is

$$F_2 \quad = \quad \int_{x_{min}}^{2} \sin\theta_i \frac{\Delta_f}{\Delta_i}\frac{1}{32\gamma^2}[\sin\theta_i\sin\theta_f]^2 f(\cos\theta_i)$$

which can be rewritten as $F_2 = \int_{x_{min}}^{2} dx\frac{\omega_r}{32\gamma^2}(1-\cos^2\theta_i)(1-\cos^2\theta_f)f(\cos\theta_i)$ However,

$$(1-\cos^2\theta_i) = (1-(1-x)^2)$$

since $1-x=\cos\theta_i$ by definition

$$= 1-1+2x-x^2$$

$$(1-\cos^2\theta_f) = (1-(1-\frac{x}{\omega_r})^2)$$

since $\cos\theta_f = 1-\frac{x}{\omega_r}$

$$1-1+\frac{2x}{\omega_r}-\frac{x^2}{\omega_r^2}$$

Using the above two relation $F_2$ can be rewritten as

$$F_2 = \int_{x_{min}}^{2} dx\frac{\omega_r}{32\gamma_2}(2x-x^2)(\frac{2x}{\omega_r}-\frac{x^2}{\omega_r^2})f(1-x)$$

Simplifying leads to

$$F_2 \quad = \quad \frac{1}{32\gamma^2}\int_{x_{min}}^{2} dx(2-x)(2-\frac{x}{\omega_r})x^2 f(1-x) \qquad (4.2.27)$$

And the third part of the integral is

$$F_3 \quad = \quad \int_{x_{min}}^{2} \sin\theta_i d\theta_i \frac{\Delta_f}{\Delta_i} \frac{1}{32\gamma_2} (\frac{1}{2}[\Delta_{ir}(1+\cos\theta_i)(1+\cos\theta_f)\frac{B_c}{B}]^2)$$

Which can be rewritten as $F_3 = \frac{\omega_r}{64}(\frac{B_c}{B})^2 \int_{x_{min}}^{2} dx \frac{\Delta_{ir}^2}{\gamma^2}[(2-x)(2-\frac{x}{\omega_r})]^2 f(1-x)$ But

$$\frac{\Delta_{ir}}{\gamma^2} \quad = \quad x^2\Delta_i^2 \tag{4.2.28}$$

$$F_3 = \frac{\omega_r \Delta_i^2}{64}(\frac{B_c}{B})^2 \int_{x_{min}}^{2} dx[x(2-x)(2-\frac{x}{\omega_r})]^2 f(1-x) \tag{4.2.29}$$

Recall the spectrum function is $F = F_1 + F_2 + F_3$. Thus $F_1$,$F_2$&$F_3$ can be summarized as

$$F_1 = \frac{\omega_r}{32\gamma^2} \int_{x_{min}}^{2} dx[\frac{x^2 c(x)}{(x-a_o)^2+\Gamma^2} + \frac{x^2 c(x)}{(x+a_o)^2}]f(1-x), \tag{4.2.30}$$

$$F_2 = \frac{1}{32\gamma^2} \int_{x_{min}}^{2} dx(2-x)(2-\frac{x}{\omega_r})x^2 f(1-x), \tag{4.2.31}$$

$$F_3 = \frac{\omega_r \Delta_i^2}{64}(\frac{B_c}{B}) \int_{x_{min}}^{2} dx[x(2-x)](2-\frac{x}{\omega_r})]^2 f(1-x), \tag{4.2.32}$$

where,$\omega_r = \frac{\omega_f}{\omega_i}$,$a_o = \frac{\omega_o}{\gamma\omega_i}$,$\Gamma = 2\alpha(B/B_c)^2/3\gamma\Delta_i$,and $c(x)$ can be read from Eq.(4.2.23),

$$c(x) = (2-2x+x^2)(2-\frac{2x}{\omega_r}+\frac{x^2}{\omega_r^2}) + 2[1+(1-x)(1-\frac{x}{\omega_r})^2]. \tag{4.2.33}$$

A simple investigation will show that $F_2$ is much smaller than $F_1$ and $F_3$.Resonant scattering occurs when $x_{min} \leq a_0 \leq 2$ is satisfied where $F_1$ becomes very large. This means that the resonance scattering becomes operative when the following conditions are satisfied.

$$\frac{\omega_0}{\gamma\omega_i} \leq 2$$

and

$$\frac{\omega_0}{\gamma\omega_i} \geq \frac{\omega_f}{2\gamma^2\omega_i(1-\frac{\Delta_f}{\gamma})}$$

which can be expressed alternatively by

$$\omega_0 \geq \frac{\omega_0}{2\gamma}$$

and

$$\omega_f \leq \frac{2\gamma\omega_0}{1+2\Delta_0} \tag{4.2.34}$$

# Chapter 5

# DISSCUSION AND CONCLUSSION

## 5.1 Introduction

Due to the high temperature near the cap of a pulsar ,$T \sim 10^6 k$, there is a large number of photons resulting from thermal radiation. Producing x-ray photon spectrum ranges from $124eV - 12.4keV$ as given in the table below. There are also high energy electron beams in the magnetosphere produced either by electrostatic acceleration or accretion. According to $Eq.$(4.1.34), these two conditions can produce Gamma-ray on the surface of strongly magnetized neutron stars. In actual calculation we use typical values for pulsars. We consider first the highest scattered photon energy which is seen to depend both on the incident electron energy and the incident photon energy,we will also consider resonance frequency of the scattered photon which depend on the incident electron energy and the magnetic field, but is independent of the incident photon energy.

| Class of Radiation | Frequency (hertz, Hz) | Wavelength (meters, m) | Energy (in eV) |
|---|---|---|---|
| Gamma | 300 - 30 Ehz | 1 - 10 pm | 1.24 MeV - 124 keV |
| Hard X-rays | 30-3 Ehz | 10-100 pm | 124-12.4 keV |
| Soft X-rays | 3 Ehz-30 PHz | 100 pm-10 nm | 12.4 keV-124 eV |
| ... | ... | ... | ... |

Table 1:Spectrum of soft Gamma-ray,Hard x-ray,and soft x-ray.

## 5.2 Highest scattered photon energy

The highest scattered photon energy which was derived earlier is $\omega_{fmax} = \frac{4\gamma^2\omega_i}{1+4\gamma\Delta_i}$ reduces to $4\gamma^2\omega_i$ for $\gamma\Delta_i \ll 1$ since in inverse compton scattering we deal with the scattering of lower frequency photon(soft x-ray) by a beam of relativistic electron. Recall that $\omega_{fmax} = 4\gamma^2\omega_i$ On varying $\omega_i$ we will get a corresponding scattered frequency. In doing so, using $\gamma = 100$ which is the typical value for most pulsars. We have generated the following figure (Fig: 5.1).

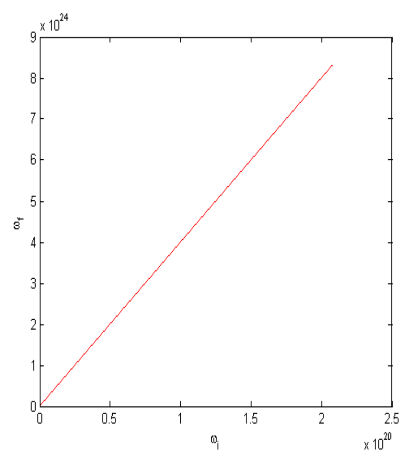


Figure 5.1: Highest scattered photon energy.

## 5.3 Resonance Frequency of the Scattered Photon

The resonance scattering becomes operative when the following conditions are satisfied.

$$\frac{\omega_0}{\gamma\omega_i} \leq 2$$

and

$$\frac{\omega_0}{\gamma\omega_i} \geq \frac{\omega_f}{2\gamma^2\omega_i(1-\frac{\Delta_f}{\gamma})}$$

This can be expressed in a useful way for $\frac{\omega_0}{\gamma\omega_i} = 2$ which it self may be rewritten as $2m\gamma\frac{\omega_0}{m} = \omega_f$. This by definitions given in the literatures reduces to $2\Delta_o = \frac{\Delta_f}{\gamma}$. Thus we finally get for the resonance scattered frequency

$$\omega_f \quad \leq \quad \frac{2\gamma\omega_0}{1+2\Delta_0} \tag{5.3.1}$$

It is seen from the last equation that the maximum scattered photon resulting from magnetic resonance scattering is given by

$$\omega_{res} \quad = \quad \frac{2\gamma\omega_0}{1+2\Delta_0}$$

which depends on the incident electron energy and the magnetic field, but is independent of the incident photon energy. Since $\omega_0 = \frac{eB}{m}$ and $\Delta_0 = \frac{\omega_0}{m} = \frac{B}{B_c}$ The above relation will give

$$\omega_{res} \quad = \quad \frac{2\gamma eB}{m(1+2\frac{B}{B_c})}$$

Note that $\frac{eB}{m^2} = \frac{B}{B_c}$. Thus

$$\omega_{res} \quad = \quad (\frac{2\gamma\frac{B}{B_c}}{1+2\frac{B}{B_c}})m$$

Now in the graph below we have varied the magnetic field from $2 \times 10^{10} - 2 \times 10^{13}$ Gauss which is the typical value for gamma-ray pulsars; $\gamma$ is taken to be 100 .Thus we have plotted $\omega_{res}$ Vs $\frac{B}{B_c}$ as shown below.

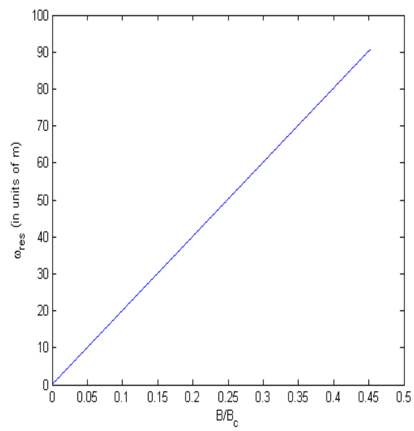


Figure 5.2: resonance frequency of the scattered photon.

## 5.4 Conclusion

To conclude , we have found an important relation in Eq.(4.1.34). The equation clearly suggests that the resonance scattering of relatively soft photons with high energy electron beams normally found in neutron star magnetospheres makes the production possible on the surface of these stars in the presence of high magnetic fields. This fields for example could be generated by an acquired angular momentum from the basic field as suggested in kebede, 2002.

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# Declaration

I hereby declare that this thesis is my original work and has not been presented for a degree in any other university. All sources of material used for the thesis have been duly acknowledged.

Name: Remudin Reshid

Signature: ...................

This thesis has been submitted for the examination with my approval as university advisor.

Name: Dr. Legesse Wetro

Signature: .....................

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